Formulations of the shallow-water equations

Martin Schreiber <M.Schreiber@exeter.ac.uk>
Pedro S. Peixoto <pedrosp@ime.usp.br>

November 10, 2015

There are a variety of different formulations for the shallow-water equations available. This document serves as an overview of these different formulations and how they are approximated.

1 Overview

We will adopt the following terminology.

Symbol	Description
η	Fluid layer depth/height
$\bar{\eta}$	Mean water depth
$\vec{u}$	2D velocity
u	Velocity in x direction
v	Velocity in y direction
ηu	Momentum in x direction
ηv	Momentum in y direction
U	Vector with (η, u, v)
ζ_R	Relative vorticity
ζ_A	Absolute vorticity $(\zeta_R + f)$
q	Potential vorticity $(\frac{\zeta_A}{\eta})$
g	Gravity
Φ	Bernoulli potential
f	Coriolis force $2\Omega sin(\phi)$
$\vec{k}$	Vector pointing outward of the plane

We will also assume no bottom topography (orography/bathymetry), so the fluid layer height η is the total height. Most of what is described here is as well described in [1].

2 Flux formulation (momentum, conservative formulation)

The conservative formulation uses the momentum (ηu, ηv) as the conserved quantities. In this form, the equations are given as

$$\eta_t + (\eta u)_x + (\eta v)_y = 0$$

$$(\eta u)_t + (\eta u^2 + \frac{1}{2} g\eta^2)_x + (\eta uv)_y - fv = 0$$

$$(\eta v)_t + (\eta v^2 + \frac{1}{2} g\eta^2)_y + (\eta uv)_x + fu = 0$$

Since this formulation is conserving the momentum, it is also referred as the conservative form.

In vector notation we have it as,

$$\eta_t + \nabla \cdot (\eta \vec{u}) = 0$$

$$(\eta \vec{u})_t + \nabla \cdot (\vec{u} \eta \vec{u}) + f\vec{k} \times (\eta \vec{v}) + gh\nabla\eta = 0$$

Please note, that this formulation contains several non-linear terms.

3 Advective formulation (non-conservative formulation)

This is a formulation which makes it more challenging to get conservative properties such as for the momentum, hence it is also called non-conservative formulation. By using the produce rule, we can reformulate the equations above to make the velocities conserved quantities. Here, we first apply the produce rule to $(\eta u)_t$

$$\eta u_t + \eta_t u + \eta_x u^2 + 2\eta u u_x + g\eta\eta_x + (\eta u)_y v + \eta u v_y - fv = 0$$

$$\eta u_t + (-(\eta u)_x - (\eta v)_y)u + \eta_x u^2 + 2\eta u u_x + g\eta\eta_x + (\eta v)_y u + \eta v u_y - fv = 0$$

$$\eta u_t + (-(\eta u)_x)u + \eta_x u^2 + 2\eta u u_x + g\eta\eta_x + \eta v u_y - fv = 0$$

$$\eta u_t - \eta_x u u - \eta u u_x + \eta_x u^2 + 2\eta u u_x + g\eta\eta_x + \eta v u_y - fv = 0$$

$$u_t + g\eta_x + u u_x + v u_y - fv = 0$$

Altogether, we get the following non-conservative formulation:

$$\eta_t + u\eta_x + v\eta_y + \eta u_x + \eta v_y = 0$$

$$u_t + uu_x + vu_y - fv + g\eta_x = 0$$

$$v_t + uv_x + vv_y + fu + g\eta_y = 0$$

We can also split these equations in linear and non-linear parts

$$U_t = L(U) + N(U)$$

$$L(U) := \begin{pmatrix} & & \\ -g\partial_x & & f \\ -g\partial_y & -f & \end{pmatrix} U$$

$$N(U) := \begin{pmatrix} -(\eta u)_x - (\eta v)_y \\ -uu_x - vu_y \\ -uv_x - vv_y \end{pmatrix}$$

with $U := (\eta, u, v)^T$. Defining the material derivative as

$$\frac{d}{dt} := \frac{\partial}{\partial t} + \vec{u} \cdot \nabla$$

we may write the equations in vector notation,

$$\frac{d\eta}{dt} + \eta \nabla \cdot \vec{u} = 0,$$

$$\frac{d\vec{u}}{dt} + f\vec{k} \times \vec{v} + g\nabla\eta = 0.$$

4 Vector invariant formulation (vorticity formulation)

The vector invariant formulation is used in models such as NEMO, GungHo, MPAS, etc. The discretization of it strongly differs between the different models.

For certain reasons such as energy and enstropy conservation, the equations are reformulated and split into the vorticity and the Bernoulli potential Φ component, defined as

$$\Phi := g\eta + \frac{u^2 + v^2}{2} = g\eta + \frac{\vec{u} \cdot \vec{u}}{2}.$$

We first add and subtract an artificial term vv_x

$$u_t - vv_x + g\eta_x + uu_x + vv_x + vu_y - fv = 0,$$

which leads to,

$$u_t - vv_x + vu_y + (g\eta + \frac{u^2+v^2}{2})_x - fv = 0,$$

with $\zeta_A := v_x - u_y + f$, yielding

$$u_t - v\zeta_A + \Phi_x = 0.$$

Similarly, we can reformulate v_t,

$$v_t + g\eta_y + uv_x + vv_y + uu_y - uu_y + fu = 0$$

$$v_t + uv_x - uu_y + fu + (g\eta + \frac{v^2+u^2}{2}) = 0$$

This yields the vorticity formulation,

$$\eta_t + (\eta u)_x + (\eta v)_y = 0,$$

$$u_t - v\zeta_A + \Phi_x = 0,$$

$$v_t + u\zeta_A + \Phi_y = 0.$$

In vector notation we have

$$\eta_t + \nabla \cdot (\eta \vec{u}) = 0,$$

$$\vec{u}_t + \zeta_A \vec{k} \times \vec{u} + \nabla\Phi = 0.$$

5 Fluid depth perturbation

It is convenient to split the fluid depth into a mean height and a perturbation, such that we express $\eta = \bar{\eta} + \eta'$, with $\bar{\eta}$ being the assumed constant mean water depth. Since the gradient of η will be the same as the gradient of η', the only change happens in the continuity equation,

$$\eta_t + \nabla \cdot ((\bar{\eta} + \eta')\vec{u}) = 0,$$
$$\eta'_t = -\bar{\eta}\nabla \cdot \vec{u} - \nabla \cdot (\eta'\vec{u})$$

and the momentum equations can be adopted with your preferred formulation but now considering η' instead of η.

We can write these equations, in any of the above formulations, as a linear operator and a non linear one

$$L(U) := \begin{pmatrix} 0 & -\bar{\eta}\partial_x & -\bar{\eta}\partial_y \\ -g\partial_x & 0 & f \\ -g\partial_y & -f & 0 \end{pmatrix} U,$$

$$U_t = L(U) + N(U)$$

where the nonlinear part $N(U)$ will depend on the equation formulation.

References

[1] Williamson, David L. and Drake, John B. and Hack, James J. and Jakob, Rüdiger and Swarztrauber, Paul N. A standard test set for numerical approximations to the shallow water equations in spherical geometry, J. Comput. Phys, 1992.