Formulations of the shallow-water equations

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There are a variety of different formulations for the shallow-water equations available. This document serves as an overview of these different formulations and how they are approximated.

1 Overview

h	Water height
u	Velocity in x direction
v	Velocity in y direction
hu	Momentum in x direction
hv	Momentum in y direction
η	Potential vorticity
g	Gravity
H	Awesome quantity
f	Coriolis force
p_0	Steady state of quantity
p'	Perturbation of p_0

2 Momentum formulation

The conservative formulation uses the momentum (hu, hv) as the conserved quantities. In this form, the equations are given as

$$h_t + (hu)_x + (hv)_y = 0$$

$$(hu)_t + (hu^2 + \frac{1}{2}gh^2)_x + (huv)_y - fv = 0$$

$$(hv)_t + (hv^2 + \frac{1}{2}gh^2)_y + (huv)_x + fu = 0$$

Since this formulation is conserving the momentum, it is also refered as the conservative form.

Please note, that this formulation contains several non-linear terms!

3 Non-conservative formulation

By using the produce rule, we can reformulate the equations above to make the velocities conserved quantities. Here, we first apply the produce rule to $(hu)_t$

$$hu_t + h_t u + h_x u^2 + 2huu_x + ghh_x + (hu)_y v + huv_y - fv = 0$$

$$hu_t + (-(hu)_x - (hv)_y)u + h_x u^2 + 2huu_x + ghh_x + (hv)_y u + hvu_y - fv = 0$$

$$hu_t + (-(hu)_x)u + h_x u^2 + 2huu_x + ghh_x + hvu_y - fv = 0$$

$$hu_t - h_x uu - huu_x + h_x u^2 + 2huu_x + ghh_x + hvu_y - fv = 0$$

$$u_t + gh_x + uu_x + vu_y - fv = 0$$

Alltogether, we get the following non-conservative formulation:

$$h_t + (hu)_x + (hv)_y = 0$$

$$u_t + gh_x + uu_x + vu_y - fv = 0$$

$$v_t + gh_y + uv_x + vv_y + fu = 0$$

We can also split these equations in linear and non-linear parts:

$$L(U) := \begin{pmatrix} & & \\ g\delta_x & & -f \\ g\delta_y & f & \end{pmatrix} U$$

$$N(U) := \begin{pmatrix} (hu)_x + (hv)_y \\ uu_x + vu_y \\ uv_x + vv_y \end{pmatrix}$$

4 Vorticity formulation

For certain reasons such as energy and enstropy conservation, the equations are reformulated and split into the vorticity and H component. We first add an artificial term vv_x

$$u_t - vv_x + gh_x + uu_x + vv_x + vu_y - fv = 0$$

$$u_t - vv_x + vu_y + (gh + u^2 + v^2)_x - fv = 0$$

and set $H := gh + u^2 + v^2$ and $\eta := v_x - u_y + f$, yielding

$$u_t - v\eta + H_x = 0$$

Similarly, we can reformulate v_t:

$$v_t + gh_y + uv_x + vv_y + uu_y - uu_y + fu = 0$$

$$v_t + uv_x - uu_y + fu + (gh + v^2 + u^2) = 0$$

This yields the vorticity formulation:

$$h_t + (hu)_x + (hv)_y = 0$$

$$u_t - v\eta + H_x = 0$$

$$v_t + u\eta + H_y = 0$$

5 Full linearization with perturbation

The equations can be linearized around a perturbation $p = p_0 + p'$, with $p \in \{h, u, v\}$with $p_0$denoting a constant steady state. Assuming $a'b' < O(\epsilon)$, we can use

$(ab)_x = ((a_0+a')(b_0+b'))_z = (a'b_0)_z + (b'a_0)_z + (a'b')_z \approx (a'b_0)_z + (b'a_0)_z = b_0 a'_z + a_0 b'_z$

to simplify the non-conservative formulation to

$$h'_t + h_0(u'_x + v'_x) + h'_x(u_0 + v_0) = 0$$

Further, we assume a low velocity $\{|u_0|, |v_0|\} < O(\epsilon)$, yielding

$$h'_t + h_0 u'_x + h_0 v'_x = 0$$

Using the approximation $ab_z = (a_0 + a')(b_0 + b')_z = a_0 b'_z + a' b'_z \approx a_0 b'_z$, we can then simplify the non-conservative formulation to

$$u'_t + gh'_x + u_0 u'_x + v_0 u'_y - fv' = 0$$

$$v'_t + gh'_y + u_0 v'_x + v_0 v'_y + fu' = 0$$

and again by assuming a low velocity, get the following formulation:

$$h'_t + h_0 u'_x + h_0 v'_x = 0$$

$$u'_t + gh'_x - fv' = 0$$

$$v'_t + gh'_y + fu' = 0$$

Setting $U := (h', u', v')$, we can write these equations as a linear operator

$$L(U) := \begin{pmatrix} & h_0\delta_x & h_0\delta_y \\ g\delta_x & & -f \\ g\delta_y & f & \end{pmatrix} U$$

$$U_t + L(U) = 0$$

6 Linear/Non-linear version:

In the equation above, we neglected also the non-linear velocities. Assuming, that their influence is of high importance, we can write the equations in a form which is not neglecting these equations:

$$U_t + L(U) + N(U) = 0$$

with

$$N(U) := \begin{pmatrix} (hu)_x - h_0 u_x + (hv)_y - h_0 v_y \\ uu_x + vu_y \\ uv_x + vv_y \end{pmatrix}$$

or with the assumption of negligible perturbations of the water height, we get

$$N(U) := \begin{pmatrix} 0 \\ uu_x + vu_y \\ uv_x + vv_y \end{pmatrix}$$