Formulations of the shallow-water equations

Martin Schreiber <M.Schreiber@exeter.ac.uk>

2015-07-13

There are a variety of different formulations for the shallow-water equations available. This document serves as an overview of these different formulations and how they are approximated.

1 Overview

Symbol	**Description**
h	Water height
h_0	Mean water depth
$\underline{u}$	Velocity in x direction
v	Velocity in y direction
hu	Momentum in x direction
hv	Momentum in y direction
ζ_R	Relative vorticity
ζ_A	Absolute vorticity ($\zeta_R + f$)
q	Potential vorticity ($\frac{\zeta_A}{h}$)
g	Gravity
Φ	Bernoulli potential
f	Coriolis force $2\Omega sin(\phi)$
p_0	Steady state of a quantity p
p'	Perturbation of p_0

2 Flux formulation (momentum, conservative formulation)

The conservative formulation uses the momentum (hu, hv) as the conserved quantities. In this form, the equations are given as

$$h_t + (hu)_x + (hv)_y = 0$$

$$(hu)_t + (hu^2 + \frac{1}{2}gh^2)_x + (huv)_y - fv = 0$$

$$(hv)_t + (hv^2 + \frac{1}{2}gh^2)_y + (huv)_x + fu = 0$$

Since this formulation is conserving the momentum, it is also refered as the conservative form.

Please note, that this formulation contains several non-linear terms!

3 Advective formulation

This is a formulation which makes it more challenging to get conservative properties. By using the produce rule, we can reformulate the equations above to make the velocities conserved quantities. Here, we first apply the produce rule to $(hu)_t$

$$hu_t + h_t u + h_x u^2 + 2huu_x + ghh_x + (hu)_y v + huv_y - fv = 0$$

$$hu_t + (-(hu)_x - (hv)_y)u + h_x u^2 + 2huu_x + ghh_x + (hv)_y u + hvu_y - fv = 0$$

$$hu_t + (-(hu)_x)u + h_x u^2 + 2huu_x + ghh_x + hvu_y - fv = 0$$

$$hu_t - h_x uu - huu_x + h_x u^2 + 2huu_x + ghh_x + hvu_y - fv = 0$$

$$u_t + gh_x + uu_x + vu_y - fv = 0$$

Alltogether, we get the following non-conservative formulation:

$$h_t + (hu)_x + (hv)_y = 0$$

$$u_t + gh_x + uu_x + vu_y - fv = 0$$

$$v_t + gh_y + uv_x + vv_y + fu = 0$$

We can also split these equations in linear and non-linear parts:

$$L(U) := \begin{pmatrix} & & \\ g\delta_x & & -f \\ g\delta_y & f & \end{pmatrix} U$$

$$N(U) := \begin{pmatrix} (hu)_x + (hv)_y \\ uu_x + vu_y \\ uv_x + vv_y \end{pmatrix}$$

4 Vector invariant formulation (vorticity formulation)

The vector invariant formulation is used in models such as NEMO, GungHo, MPAS, etc. The discretization of it strongly differs between the different models.

For certain reasons such as energy and enstropy conservation, the equations are reformulated and split into the vorticity and H component. We first add an artificial term vv_x

$$u_t - vv_x + gh_x + uu_x + vv_x + vu_y - fv = 0$$

$$u_t - vv_x + vu_y + (gh + u^2 + v^2)_x - fv = 0$$

and set $\Phi := gh + u^2 + v^2$ and $\zeta_A := v_x - u_y + f$, yielding

$$u_t - v\zeta_A + \Phi_x = 0$$

Similarly, we can reformulate v_t:

$$v_t + gh_y + uv_x + vv_y + uu_y - uu_y + fu = 0$$

$$v_t + uv_x - uu_y + fu + (gh + v^2 + u^2) = 0$$

This yields the vorticity formulation:

$$h_t + (hu)_x + (hv)_y = 0$$

$$u_t - v\zeta_A + \Phi_x = 0$$

$$v_t + u\zeta_A + \Phi_y = 0$$

5 Full linearization with perturbation

The equations can be linearized around a perturbation $p = p_0 + p'$, with $p \in \{h, u, v\}$with $p_0$denoting a constant steady state. Assuming $a'b' < O(\epsilon)$, we can use

$(ab)_x = ((a_0 + a')(b_0 + b'))_z = (a'b_0)_z + (b'a_0)_z + (a'b')_z \approx (a'b_0)_z + (b'a_0)_z = b_0 a'_z + a_0 b'_z$

to simplify the non-conservative formulation to

$$h'_t + h_0(u'_x + v'_x) + h'_x(u_0 + v_0) = 0$$

Further, we assume a low velocity $\{|u_0|, |v_0|\} < O(\epsilon)$ and hence *neglect the non-linear* parts. This yields

$$h'_t + h_0 u'_x + h_0 v'_x = 0$$

Using the approximation $ab_z = (a_0 + a')(b_0 + b')_z = a_0 b'_z + a' b'_z \approx a_0 b'_z$, we can then simplify the non-conservative formulation to

$$u'_t + g h'_x + u_0 u'_x + v_0 u'_y - f v' = 0$$

$$v'_t + g h'_y + u_0 v'_x + v_0 v'_y + f u' = 0$$

and again by assuming a low velocity, get the following formulation:

$$h'_t + h_0 u'_x + h_0 v'_x = 0$$

$$u'_t + g h'_x - f v' = 0$$

$$v'_t + g h'_y + f u' = 0$$

Setting $U := (h', u', v')$, we can write these equations as a linear operator

$$L(U) := \begin{pmatrix} & h_0 \delta_x & h_0 \delta_y \\ g \delta_x & & -f \\ g \delta_y & f & \end{pmatrix} U$$

$$U_t + L(U) = 0$$

6 Linear/Non-linear version:

In the equation above, we neglected also the non-linear velocities by assuming close-to-zero velocities. Assuming, that their influence is of high importance, we can write the equations in a form which is not neglecting these equations:

$$U_t + L(U) + N(U) = 0$$

with

$$N(U) := \begin{pmatrix} (hu)_x - h_0 u_x + (hv)_y - h_0 v_y \\ u u_x + v u_y \\ u v_x + v v_y \end{pmatrix}$$

or with the assumption of negligible perturbations of the water height, we get

$$N(U) := \begin{pmatrix} 0 \\ u u_x + v u_y \\ u v_x + v v_y \end{pmatrix}$$