

《高等数学 B (下)》期中试卷参考解答

一、填空题 (4 分*5= 20 分)

1. 设 $w(x, y, z) = \ln(y^2 + z^2 - x)$, 则 $dw|_{(1,1,-1)} = \underline{-dx + 2dy - 2z}$
2. 函数 $f(x, y, z) = (x - y)(y - z)(z - x)$ 在点 $(0, 1, -1)$ 处下降最快的方向为 $(0, -1, 1)$ 或 $(0, -3, 3)$, 其方向导数是 $-3\sqrt{2}$ 。
3. 曲面 $z - e^z + xyz + 1 = 0$ 在点 $(2, -1, 0)$ 处的切平面方程为 $z = 0$ 。
4. 设有 $\begin{cases} uv = x^2 + y^2 \\ xy = u^2 + v^2 \end{cases}$, 如果 $|u| \neq |v|$, 则 $\frac{\partial v}{\partial y} = \frac{4yu - xv}{2(u^2 - v^2)}$
5. $x + y = 5$ 和 $\sqrt{x} + \sqrt{y} = \sqrt{5}$ 所围区域面积是 $\frac{25}{3}$ 。

二、计算和问答题 (10 分*8=80 分)

1. 三元函数 $f(x, y, z) = \begin{cases} \frac{xyz}{x^2+y^2+z^2} & x^2 + y^2 + z^2 \neq 0 \\ 0 & x^2 + y^2 + z^2 = 0 \end{cases}$ 在 $(0, 0, 0)$ 是否连续、可偏导、可微?

解: 1) 写出 $\left| \frac{xyz}{x^2+y^2+z^2} - 0 \right| < g(x, y, z)$, 例如: $\left| \frac{xyz}{x^2+y^2+z^2} - 0 \right| < \frac{1}{9}(|x| + |y| + |z|)$

指出: $f(x, y, z) \rightarrow 0$ 当 $(x, y, z) \rightarrow (0, 0, 0)$,

结论: 连续

$$2) \frac{\partial f}{\partial x} \Big|_{(0,0,0)} = \frac{\partial f}{\partial y} \Big|_{(0,0,0)} = \frac{\partial f}{\partial z} \Big|_{(0,0,0)} = 0,$$

结论: 可偏导

$$3) \text{ 写出 } \left| \frac{xyz}{x^2+y^2+z^2} - 0 - f'_x(0,0,0)x - f'_y(0,0,0)y - f'_z(0,0,0)z \right| = \left| \frac{xyz}{x^2+y^2+z^2} \right|$$

指出 $\frac{\frac{xyz}{x^2+y^2+z^2}}{\sqrt{x^2+y^2+z^2}} = \frac{xyz}{(x^2+y^2+z^2)^{\frac{3}{2}}}$ 当 $(x, y, z) \rightarrow (0, 0, 0)$ 时极限不存在

结论: 不可微

2. 设 $xz = \ln(yz)$, 求 $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ 和 $\frac{\partial^2 z}{\partial x \partial y}$ 。

解 1: 对原方程关于 x 求导, 利用链式法则得出关于 $\frac{\partial z}{\partial x}$ 的方程 $z + x \frac{\partial z}{\partial x} = \frac{1}{z} \frac{\partial z}{\partial x}$

$$\text{解方程得 } \frac{\partial z}{\partial x} = \frac{z^2}{1 - xz} \quad (*)$$

同样, $x \frac{\partial z}{\partial y} = \frac{1}{y} + \frac{1}{z} \frac{\partial z}{\partial y}$

解方程得 $\frac{\partial z}{\partial x} = \frac{z}{y(xz-1)}$ (**)

对 (*) 关于 y 求偏导得 $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = \frac{2z \frac{\partial z}{\partial y} (1-xz) - z^2 (-x) \frac{\partial z}{\partial y}}{(1-xz)^2}$

将 (**) 代入, 化简得 $\frac{\partial^2 z}{\partial x \partial y} = \frac{z^2 (2-xz)}{y(xz-1)^3}$

解 2: 设 $F(x, y, z) = xz - \ln(yz)$,

$$F'_x = z, F'_y = -\frac{1}{y}, F'_z = \frac{xz-1}{z}$$

$$\frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z} = \frac{z^2}{1-xz}, \quad \frac{\partial z}{\partial y} = -\frac{F'_y}{F'_z} = \frac{z}{y(1-xz)},$$

以下同解 1

3. 求 $x^2 = 2xyz - 1$ 和 $z^3 = 3xy - z(x+y)$ 的交线在点 $P(1,1,1)$ 处的切线方程和法平面方程。

解: 设 $F(x, y, z) = x^2 - 2xyz + 1$, $G(x, y, z) = z^3 - 3xy + z(x+y)$

$$(F'_x, F'_y, F'_z)|_P = (2x - 2yz, -2xz, -2yz)|_P = (0, -2, -2)$$

$$(G'_x, G'_y, G'_z)|_P = (-3y + z, -3x + z, 3z^2 + (x+y))|_P = (-2, -2, 5)$$

$$\text{切线方向: } (0, 1, 1) \times (-2, -2, 5) = (7, -2, 2)$$

$$\text{切线方程: } \frac{x-1}{7} = \frac{y-1}{-2} = \frac{z-1}{2}$$

$$\text{法平面方程: } 7(x-1) - 2(y-1) + 2(z-1) = 0, \quad 7x - 2y - 2z = 7$$

4. 求函数 $f(x, y, z) = x^3 + y^2 + \sqrt{6}z - 2xyz$ 的极值。

$$\begin{aligned} \text{解: } F'_x = 3x^2 - 2yz = 0 & \Rightarrow 3x^2 = 2yz \\ F'_y = 2y - 2xz = 0 & \Rightarrow y = xz \end{aligned}$$

$$f'_z = \sqrt{6} - 2xy = 0 \Rightarrow xy = \frac{\sqrt{6}}{2}$$

$$\text{解方程组得: } x = 1, y = z = \frac{\sqrt{6}}{2}$$

$$\text{Hessian 阵 } H = \begin{pmatrix} 6 & -\sqrt{6} & -\sqrt{6} \\ -\sqrt{6} & 2 & -2 \\ -\sqrt{6} & -2 & 0 \end{pmatrix}, \quad \Delta_1 = 6, \Delta_2 = 6, \Delta_3 = -60, H \text{ 非正定亦非负定}$$

$f(x, y, z)$ 没有极值

5. 求曲面 $2x^2 + y^2 + z = 0$ 与平面 $x + y - 2z + 1 = 0$ 间的最近距离。

解: 设球面上任意一点 (x, y, z) 到平面的距离为 d , 则

$$d^2 = \left(\frac{|x+y-2z+1|}{\sqrt{1+1+4}} \right)^2 = \frac{(x+y-2z+1)^2}{6}$$

$$\text{点 } (x, y, z) \text{ 满足 } 2x^2 + y^2 + z = 0$$

$$L(x, y, z, \lambda) = \frac{(x+y-2z+1)^2}{6} + \lambda(2x^2 + y^2 + z)$$

$$L'_x = (x + y - 2z + 1)/3 + 4\lambda x = 0$$

$$L'_y = (x + y - 2z + 1)/3 + 2\lambda y = 0$$

$$L'_z = -2(x + y - 2z + 1)/3 + \lambda = 0$$

$$L'_\lambda = 2x^2 + y^2 + z = 0$$

解得: $x=-1/8, y=-1/4, z=-3/32$

代入 d 得最近距离为 $\frac{13}{16\sqrt{6}}$

6. 计算 $\iint_{\Omega} |x^2 + \frac{y^2}{a^2} - 1| dx dy$, 其中 $\Omega = \{(x, y) | (x-1)^2 + \frac{y^2}{a^2} \leq 1, a > 0\}$ 。

解: $\Omega = \Omega_1 \cup \Omega_2$

$$\Omega_1 = \left\{ (x, y) \left| (x-1)^2 + \frac{y^2}{a^2} \leq 1, x^2 + \frac{y^2}{a^2} - 1 \leq 0, a > 0 \right. \right\}$$

$$\Omega_2 = \left\{ (x, y) \left| (x-1)^2 + \frac{y^2}{a^2} \leq 1, x^2 + \frac{y^2}{a^2} - 1 > 0, a > 0 \right. \right\}$$

设: $x = r \cos \theta, y = a \sin \theta$,

$$\Omega'_1 = \left\{ (r, \theta) \left| -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}, 0 \leq r \leq 1; -\frac{\pi}{2} \leq \theta \leq -\frac{\pi}{3}, \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}, 0 \leq r \leq 2 \cos \theta \right. \right\}$$

$$\Omega'_2 = \left\{ (r, \theta) \left| -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}, 1 \leq r \leq 2 \cos \theta \right. \right\},$$

$$\iint_{\Omega} \left| x^2 + \frac{y^2}{a^2} - 1 \right| dx dy$$

$$= 2 \left[\int_0^{\frac{\pi}{3}} d\theta \int_0^1 (1 - r^2) a r dr + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\theta \int_0^{2 \cos \theta} (1 - r^2) a r dr + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\theta \int_1^{2 \cos \theta} (r^2 - 1) a r dr \right]$$

$$= \left(\frac{\pi}{2} + \frac{3\sqrt{3}}{4} \right) a$$

7. 求由抛物面: $y = x^2 + z^2, y = 4x^2 + 2z^2$ 和柱面: $xz = a, xz = b$ ($b > a > 0$), $z = 27x^2, x^2 = 8z$ 所围区域的体积。

$$\text{解: } V = \iint_{\Omega} dx dz \int_{x^2+z^2}^{4x^2+2z^2} dy = \iint_{\Omega} (3x^2 + z^2) dx dz$$

设 $u = xz, v = \frac{z}{x^2}$, 则 $\Omega' = [a, b] \times [\frac{1}{8}, 27]$,

$$\left| \frac{D(u, v)}{D(x, y)} \right| = \frac{3z}{x^2} = 3v$$

$$V = \int_a^b du \int_{\frac{1}{8}}^{27} \left(3u^{\frac{2}{3}} v^{-\left(\frac{2}{3}\right)} + u^{\frac{4}{3}} v^{\frac{2}{3}} \right) \frac{1}{3v} dv$$

$$= \frac{7}{2} \left(b^{\frac{5}{3}} - a^{\frac{5}{3}} \right) + \frac{15}{8} \left(b^{\frac{7}{3}} - a^{\frac{7}{3}} \right)$$

8. 一均质物体占据区域 Ω , Ω 由单叶双曲面 $x^2 + y^2 - z^2 = R^2$ 与平面 $z = 2R$ 、 $z = R$ 所围成, 设物体密度的密度为常数 C , 求物体绕 y 轴旋转的转动惯量。

$$\text{解: } I_y = \iiint_{\Omega} C(x^2 + z^2) dx dy dz$$

$$= C \int_R^{2R} dz \iint_{x^2 + y^2 \leq R^2 + z^2} (x^2 + z^2) dx dy$$

$$= C \left[\int_R^{2R} dz \iint_{x^2 + y^2 \leq R^2 + z^2} x^2 dx dy + C \int_R^{2R} dz \iint_{x^2 + y^2 \leq R^2 + z^2} z^2 dx dy \right]$$

$$= C \left[\int_R^{2R} dz \int_0^{2\pi} d\theta \int_0^{\sqrt{R^2 + z^2}} r^2 \cos^2 \theta r dr + \int_R^{2R} z^2 \pi (R^2 + z^2) dz \right]$$

$$= C \left[\pi \int_R^{2R} \frac{1}{4} (R^2 + z^2)^2 dz + \int_R^{2R} z^2 \pi (R^2 + z^2) dz \right]$$

$$= \frac{\pi C}{4} \left[\int_R^{2R} (R^4 + 6R^2 z^2 + 5z^4) dz \right] = \frac{23}{2} \pi C R^5$$