

# 参考答案

$$1. \frac{\partial z}{\partial x} = \sin(xy) \cdot (x+y^2)^{\sin xy - 1} + (x+y^2)^{\sin xy} \ln(x+y^2) (\cos xy) \cdot y$$

2. 设切点  $(x_0, y_0, z_0)$

$$\text{则切平面 } x_0(x-x_0) + y_0(y-y_0) - (z-z_0) = 0$$

$$\text{由切平面平行于 } 2x + 2y - z = 0$$

$$\text{则有 } x_0 = 2, y_0 = 1 \text{ 代入曲面方程得 } z_0 = 1$$

$$\text{从而切平面为 } 2x + 2y - z = 5$$

3. 设所求直线与直线  $\frac{x}{1} = \frac{y-5}{-1} = \frac{z-5}{-4}$  的交点为

$$\begin{cases} x_0 = 4+t \\ y_0 = 5-t \\ z_0 = 5-4t \end{cases} \quad \text{其中 } t \text{ 是待定系数}$$

$$\text{则所求直线为 } \frac{x}{4+t} = \frac{y}{5-t} = \frac{z}{5-4t}$$

根据直线与平面  $x+y+z=0$  平行可得

$$(4+t) \cdot 1 + (5-t) \cdot 1 - (5-4t) = 0$$

$$\text{求得 } t = -1, \text{ 则所求直线为 } \frac{x}{1} = \frac{y}{2} = \frac{z}{3}$$

4. 求解  $\min x^2 + y^2 + z^2$

$$\text{s.t. } z^2 = xy + 4$$

构造 Lagrang 函数  $x^2 + y^2 + z^2 + \lambda(xy + 4 - z^2)$

$$\text{求偏导} \quad \begin{cases} 2x + \lambda y = 0 \\ 2y + \lambda x = 0 \\ 2z - 2\lambda z = 0 \\ xy + 4 - z^2 = 0 \end{cases}$$

$$\text{解之可得 } x = 2, y = -2, \lambda = 2, z = 0$$

$$5. \quad \frac{\partial g}{\partial x} = -f'(\frac{1}{r}) (\frac{1}{r^2}) \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial^2 g}{\partial x^2} = f''(\frac{1}{r}) \left[ (\frac{1}{r^2}) \frac{x}{\sqrt{x^2+y^2}} \right]^2 + f'(\frac{1}{r}) (\frac{2}{r^3}) \left( \frac{x}{\sqrt{x^2+y^2}} \right)^2 - f'(\frac{1}{r}) \frac{1}{r^2} \cdot \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}$$

由对称性可得

$$\frac{\partial^2 g}{\partial y^2} = f''(\frac{1}{r}) \frac{1}{r^4} \frac{y^2}{x^2+y^2} + f'(\frac{1}{r}) (\frac{2}{r^3}) (\frac{y^2}{x^2+y^2}) - f'(\frac{1}{r}) \frac{1}{r^2} \cdot \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}}$$

$$\text{所以 } \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = f''(\frac{1}{r}) \cdot \frac{1}{r^4} + f'(\frac{1}{r}) \cdot \frac{1}{r^3}$$

$$6. \quad \lim_{n \rightarrow \infty} \sqrt[n]{\left( \frac{n+2}{2n+\sqrt{n} \sin n} \right)^{2n}} = \lim_{n \rightarrow \infty} \left( \frac{n+2}{2n+\sqrt{n} \sin n} \right)^2 = \frac{1}{4}$$

所以收敛半径为 4 收敛域为  $(-4, 4)$

$$7. \quad \sum_{n=0}^{\infty} y^n = \frac{1}{1-y} \quad \sum_{n=1}^{\infty} n y^{n-1} = \frac{1}{(1-y)^2} \quad \sum_{n=2}^{\infty} n(n-1) y^{n-2} = \frac{2}{(1-y)^3}$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} (n^2 - n) &= \sum_{n=2}^{\infty} \frac{(-1)^n}{2^n} (n^2 - n) = \sum_{n=2}^{\infty} n(n-1) y^n \Big|_{y=-\frac{1}{2}} \\ &= \frac{2y^2}{(1-y)^3} \Big|_{y=-\frac{1}{2}} = \frac{4}{27} \end{aligned}$$

8. 将  $f$  的表达式从  $(0, 2)$  延拓成  $\mathbb{R}$  上的偶函数

求  $f$  的 Fourier 级数

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} f(x) dx \\ &= \frac{2}{\pi} \left[ \int_0^{\sqrt{\pi}} -x dx + \int_{\sqrt{\pi}}^{\pi} x dx \right] = 2(\pi - 2\sqrt{\pi}) \end{aligned}$$

$$a_n = \frac{2}{\pi} \left[ \int_0^{\sqrt{\pi}} -\pi \cos nx \, dx + \int_{\sqrt{\pi}}^{\pi} \pi \cos nx \, dx \right]$$

$$= -\frac{4}{n} \sin(n\sqrt{\pi})$$

所以  $f(x) \sim (\pi - 2\sqrt{\pi}) - \sum_{n=1}^{\infty} \frac{4}{n} \sin(n\sqrt{\pi}) \cos nx$   
 记和函数为  $S(x)$

则  $S(\frac{\pi}{2}) = f(\frac{\pi}{2}) = -\pi$

(由  $S(0) = -\pi$  得  $\sum_{n=1}^{\infty} \frac{\sin n\sqrt{\pi}}{n} = \frac{\pi - \sqrt{\pi}}{2}$

(由  $S(\sqrt{\pi}) = 0$  得  $\sum_{n=1}^{\infty} \frac{\sin 2n\sqrt{\pi}}{n} = \frac{\pi - 2\sqrt{\pi}}{2}$  .