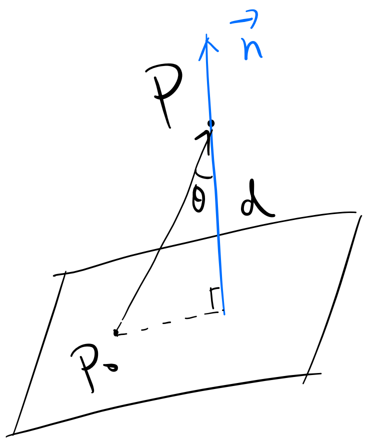
$$(1+k)+(1-k)+(-1+k)=0 \qquad k=-1,$$

平面： $(x+y-z-1)-(x-y+z+1)=0$

$$2y-2z-2=0$$

$$y-z-1=0$$

直线： $\begin{cases} y-z-1=0 \\ x+y+z=0 \end{cases}$



$\theta \in [0, \frac{\pi}{2}]$

$$\theta = \widehat{(\overrightarrow{P_0P}, \vec{n})} \text{ 或 } \widehat{(\overrightarrow{P_0P}, -\vec{n})}$$

$$d = \|\overrightarrow{P_0P}\| \cos\theta = \|\overrightarrow{P_0P}\| \frac{|\overrightarrow{P_0P} \cdot \vec{n}|}{\|\overrightarrow{P_0P}\| \|\vec{n}\|}$$

$$= \frac{|\overrightarrow{P_0P} \cdot \vec{n}|}{\|\vec{n}\|} = |\overrightarrow{P_0P} \cdot \vec{n}_0|$$

$$\left(\vec{n}_0 = \frac{\vec{n}}{\|\vec{n}\|}\right)$$

$$P_0 = (x_0, y_0, z_0), \quad P(x, y, z)$$

$$\vec{n} = (A, B, C)$$

$$d = \frac{1}{\sqrt{A^2+B^2+C^2}} \left| A(x-x_0)+B(y-y_0)+C(z-z_0) \right|$$

例：求平面 $\pi_1: 2x-3y-6z-7=0$

$\pi_2: 2x-3y-6z+1=0$

之间的距离

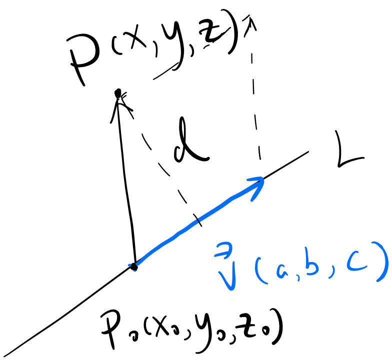
$$P_1=\left(\frac{7}{2},0,0\right)\in\pi_1$$

$$P_2=\left(-\frac{1}{2},0,0\right)\in\pi_2 \qquad \vec{n}=(2,-3,-6)$$

$$\overrightarrow{P_1P_2}=(4,0,0)$$

$$d=\frac{|\overrightarrow{P_1P_2}\cdot\vec{n}|}{\|\vec{n}\|}=\frac{8}{\sqrt{2^2+(-3)^2+(-6)^2}}=\frac{8}{7}$$

直线 L：$\dfrac{x-x_0}{a}=\dfrac{y-y_0}{b}=\dfrac{z-z_0}{c}$



$$d\,\|\vec{v}\|=\|\overrightarrow{P_0P}\times\vec{v}\|$$

$$d=\frac{\|\overrightarrow{P_0P}\times\vec{v}\|}{\|\vec{v}\|}$$

$$= \frac{\|(x-x_0, y-y_0, z-z_0)\times(a,b,c)\|}{\sqrt{a^2+b^2+c^2}}$$

例：求点 $(5,-2,3)$ 与直线 $\frac{x-1}{1}=\frac{y+1}{2}=\frac{z}{3}$ 的距离

$P=(5,-2,3)$

$P_0=(1,-1,0)$ $\qquad\qquad$ $\vec{v}=(1,2,3)$

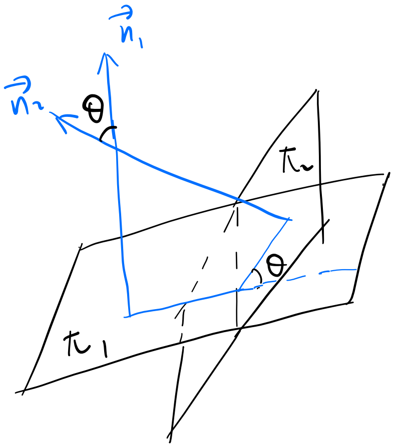
$\overrightarrow{P_0P}=(5,-2,3)-(1,-1,0)=(4,-1,3)$

$\overrightarrow{P_0P}\times\vec{v}=(4,-1,3)\times(1,2,3)$

$=(-1\times3-3\times2,\ 3\times1-4\times3,\ 4\times2-(-1)\times1)$

$=(-9,-9,9)$

$$d=\frac{\|\overrightarrow{P_0P}\times\vec{v}\|}{\|\vec{v}\|}=\frac{\sqrt{(-9)^2+(-9)^2+9^2}}{\sqrt{1+2^2+3^2}}=\frac{9\sqrt{3}}{\sqrt{14}}$$



$$0 \le \theta \le \frac{\pi}{2}$$

$$\theta = \widehat{(\vec{n}_1, \vec{n}_2)} \text{ 或 } \widehat{(-\vec{n}_1, \vec{n}_2)}$$

$$\vec{n}_1 = (A_1, B_1, C_1)$$

$$\vec{n}_2 = (A_2, B_2, C_2)$$

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\| \, \|\vec{n}_2\|}$$

$$= \frac{|A_1A_2 + B_1B_2 + C_1C_2|}{\sqrt{A_1^2+B_1^2+C_1^2}\,\sqrt{A_2^2+B_2^2+C_2^2}}$$

(1) $A_1A_2 + B_1B_2 + C_1C_2 = 0$：两平面垂直

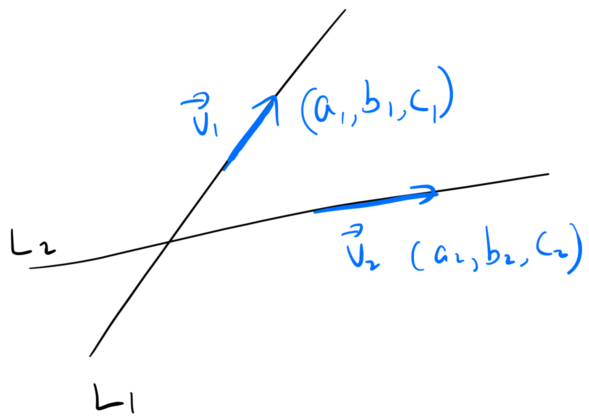
(2) $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} \neq \frac{D_1}{D_2}$：两平面平行 不重合

(3) $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \frac{D_1}{D_2}$：两平面重合

例1. 求两平面 $x-y+2z-6=0$ 和 $2x+y+z-5$ 的夹角

$$\cos\theta = \frac{|(1,-1,2)\cdot(2,1,1)|}{\sqrt{1^2+(-1)^2+2^2}\cdot\sqrt{2^2+1^2+1^2}} = \frac{3}{6} = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$



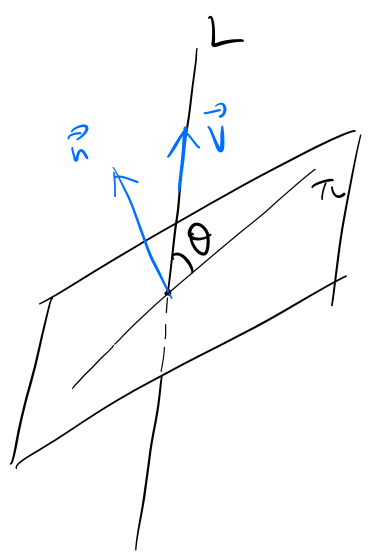
$$L_1: \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$$

$$L_2: \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$$

$$\cos\theta = \frac{|\vec{v}_1 \cdot \vec{v}_2|}{\|\vec{v}_1\| \|\vec{v}_2\|} = \frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

(1) $a_1a_2+b_1b_2+c_1c_2=0$：两直线垂直

(2) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$：两直线平行或重合



$$0 \le \theta \le \frac{\pi}{2}$$

$$\pi: Ax+By+Cz+D=0$$

$$L: \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

$\vec{n} = (A, B, C)$. $\vec{v} = (a, b, c)$

$$\sin\theta = \cos\left(\frac{\pi}{2}-\theta\right)$$

$$= \frac{|\vec{n}\cdot\vec{v}|}{\|\vec{n}\|\|\vec{v}\|}$$

$$= \frac{|Aa+Bb+Cc|}{\sqrt{A^2+B^2+C^2}\sqrt{a^2+b^2+c^2}}$$

(1). $\frac{A}{a}=\frac{B}{b}=\frac{C}{c}$: 平面与直线垂直

(2) $Aa+Bb+Cc=0$; 平面与直线平行. 或 直线在平面上

例: 平面 $x+y+z-3=0$.

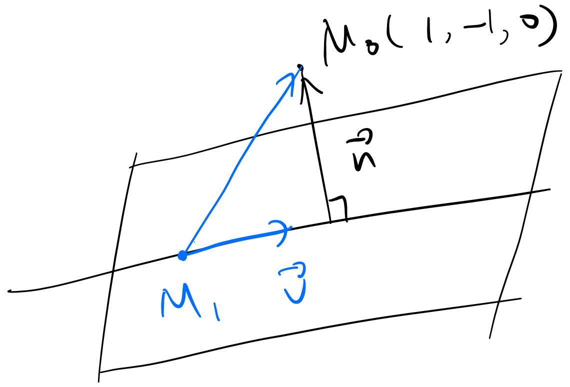
直线 $\frac{x-2}{3}=\frac{y+2}{1}=\frac{z-3}{-4}$

$$\sin\theta = \frac{|(1,1,1)\cdot(3,1,-4)|}{\sqrt{1^2+1^2+1^2}\sqrt{3^2+1^2+(-4)^2}} = 0$$

$(2,-2,3)$ 在平面上 $\Rightarrow$ 直线在平面上

求过直线 $L:\begin{cases}2x-y-z+1=0\\ x+y-z-1=0\end{cases}$ 且与点 $M_0(1,-1,0)$ 距离最远

的平面 Π 的一般方程



方法Ⅰ：

$$\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & -1 & -1\\ 1 & 1 & -1\end{vmatrix}$$

$$=\vec{i}(1+1)+\vec{j}(-1+2)+\vec{k}(2+1)$$

$$=2\vec{i}+\vec{j}+3\vec{k}$$

平面 $\quad 2x-y-z+1+k(x+y-z-1)=0$

$$\vec{n}=(2+k,\ -1+k,\ -1-k)$$

$M_1(0,1,0) \qquad \overrightarrow{M_1M_0}=(1,-2,0)$

$\vec{n}, \vec{v}, \overrightarrow{M_1M_0}$ 共面

$$\begin{vmatrix} 2+k & -1+k & -1-k \\ 2 & 1 & 3 \\ 1 & -2 & 0 \end{vmatrix} = 0$$

$$3(-1+k) - 4(-1-k) - (-1-k) + 6(2+k) = 0$$

$$-3+3k+4+4k+1+k+12+6k=0$$

$$14k+14=0$$

$$k=-1$$

平面方程 $2x-y-z+1-(x+y-z-1)=0$

$$x-2y+2=0$$

方法 II: $2x-y-z+1+k(x+y-z-1)=0$

$$(k+2)x+(k-1)y-(k+1)z+1-k=0$$

经过 M_0 和 L 的平面 π_0

$$(2-(-1)-0+1)+k(1-1-0-1)=0$$

$$4-k=0$$

$$k=4$$

$$2x-y-z+1+4(x+y-z-1)=0$$

π_0 $$6x+3y-5z-3=0$$

π_0 和 π 垂直

$$(k+2,\ k-1,\ -k-1)\cdot(6,\ 3,\ -5)=0$$

$$6(k+2)+3(k-1)+5(k+1)=0$$

$$14k+14=0$$

$$k=-1$$

π: $$x-2y+2=0$$

方法III：　M_0 到平面 $(k+2)x+(k-1)y-(k+1)z+1-k=0$

而距离　$\overrightarrow{M_1M_0}=(1,-2,0)$

$$D(k)=\frac{|(k+2)-2(k-1)+0|}{\sqrt{(k+2)^2+(k-1)^2+(k+1)^2}}$$

$$=\frac{|k-4|}{\sqrt{3k^2+4k+6}}$$

$$\frac{dD^2(k)}{dk}=\frac{d}{dk}\left(\frac{(k-4)^2}{3k^2+4k+6}\right)$$

$$=\frac{2(k-4)\cdot(3k^2+4k+6)-(k-4)^2\cdot(6k+4)}{(3k^2+4k+6)^2}$$

$$=\frac{(k-4)\left[6k^2+8k+12-(k-4)(6k+4)\right]}{(3k^2+4k+6)^2}$$

$$=\frac{(k-4)(6k^2+8k+12-6k^2+20k+16)}{(3k^2+4k+6)^2}$$

$$=\frac{(k-4)(28k+28)}{(3k^2+4k+6)^2}$$

$$\frac{dD^2(k)}{dk}=0 \Rightarrow k=-1 \text{ 或 } 4$$

$k=4$, $D=0$

$k=-1$ D最大

以 $P_0(x_0,y_0,z_0)$ 为中心，半径为 r 的球面的方程

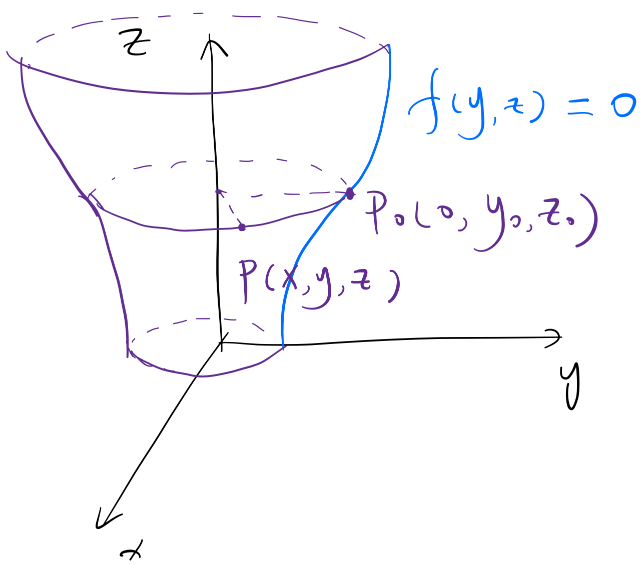
$P(x,y,z)$ 在球面上

$\|\overrightarrow{P_0P}\| = r$

$$\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} = r$$

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = r^2$$

$F(x,y,z)=0$：空间中的曲面



$$x^2+y^2 = y_0^2 \quad , z = z_0$$

$$y_0 = \pm\sqrt{x^2+y^2}$$

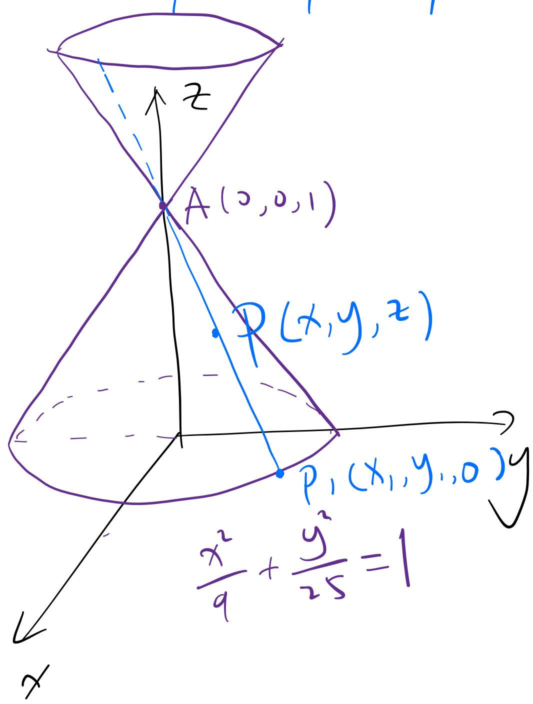
$$f(\pm\sqrt{x^2+y^2},\ z)=0$$

例 Oxy 平面上 $\frac{x^2}{4}+\frac{y^2}{9}=1$ 绕 x 轴旋转一周所生成的旋转曲面的方程

$$\frac{x^2}{4}+\frac{\left(\pm\sqrt{y^2+z^2}\right)^2}{9}=1$$

$$\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{9}=1$$

例:



$$\overrightarrow{AP} \parallel \overrightarrow{AP_1}$$

$$(x,\ y,\ z-1) \parallel (x_1,\ y_1,\ -1)$$

$$\frac{x}{x_1} = \frac{y}{y_1} = \frac{z-1}{-1}$$

$$\frac{x_1^2}{9} + \frac{y_1^2}{25} = 1$$

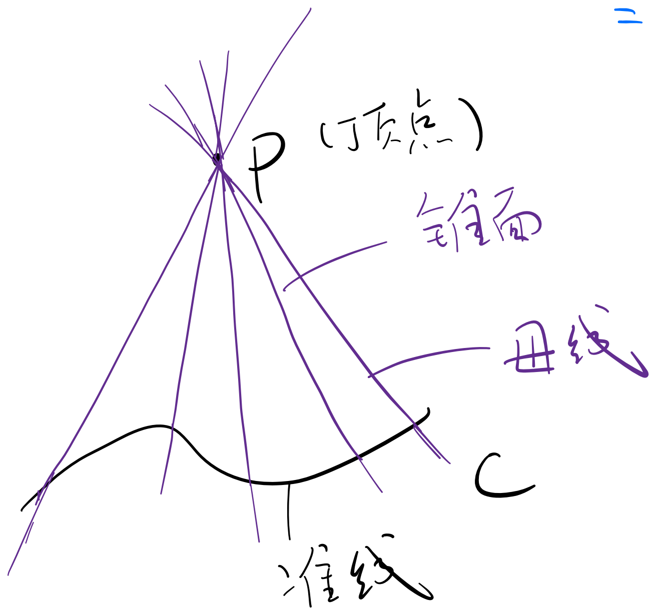
$x = t x_1$, $y = t y_1$, $z - 1 = -t$

$$\frac{x^2}{9} + \frac{y^2}{25} = \frac{t^2 x_1^2}{9} + \frac{t^2 y_1^2}{25}$$

$$= t^2\left(\frac{x_1^2}{9} + \frac{y_1^2}{25}\right)$$

$$= t^2$$

$$= (z-1)^2$$



例 $y = x^2$