例：$\vec{a}=(1,3,4)$，$\vec{b}=(2,7,-5)$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 3 & 4\\ 2 & 7 & -5\end{vmatrix}$$

$$=\begin{vmatrix}3 & 4\\ 7 & -5\end{vmatrix}\vec{i}-\begin{vmatrix}1 & 4\\ 2 & -5\end{vmatrix}\vec{j}+\begin{vmatrix}1 & 3\\ 2 & 7\end{vmatrix}\vec{k}$$

$$=(-15-28)\vec{i}-(-5-8)\vec{j}+(7-6)\vec{k}$$

$$=-43\vec{i}+13\vec{j}+\vec{k}$$

$\vec{x}\times\vec{y}$ 和 $\vec{x}$ 以及 $\vec{y}$ 垂直

$$(\vec{x}\times\vec{y})\cdot\vec{x}=(x_2y_3-x_3y_2,\ x_3y_1-x_1y_3,\ x_1y_2-x_2y_1)\cdot(x_1,\ x_2,\ x_3)$$

$$=x_1(x_2y_3-x_3y_2)+x_2(x_3y_1-x_1y_3)+x_3(x_1y_2-x_2y_1)$$

$$=x_1x_2y_3-x_1x_3y_2+x_2x_3y_1-x_1x_2y_3+x_1x_3y_2-x_2x_3y_1$$

$$=0$$

$$(\vec{x}\times\vec{y})\cdot\vec{y}=0$$

$$\|\vec{x}\times\vec{y}\|^2=(x_2y_3-x_3y_2)^2+(x_3y_1-x_1y_3)^2+(x_1y_2-x_2y_1)^2$$

$$\begin{aligned}
&=x_2^2y_3^2-2x_2x_3y_2y_3+x_3^2y_2^2\\
&\quad+x_3^2y_1^2-2x_1x_3y_1y_3+x_1^2y_3^2\\
&\quad+x_1^2y_2^2-2x_1x_2y_1y_2+x_2^2y_1^2
\end{aligned}$$

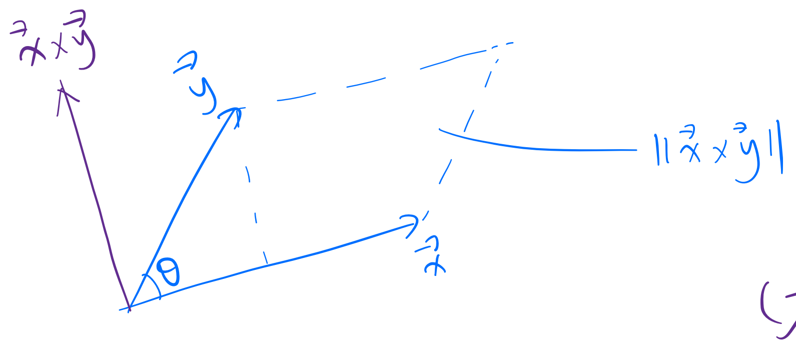
$$=(x_1^2+x_2^2+x_3^2)(y_1^2+y_2^2+y_3^2)-(x_1y_1+x_2y_2+x_3y_3)^2$$

$$=\|\vec{x}\|^2\|\vec{y}\|^2-(\|\vec{x}\|\|\vec{y}\|\cos\theta)^2$$

$$=\|\vec{x}\|^2\|\vec{y}\|^2(1-\cos^2\theta)$$

$$=\|\vec{x}\|^2\|\vec{y}\|^2\sin^2\theta$$

$$\|\vec{x}\times\vec{y}\|=\|\vec{x}\|\|\vec{y}\|\sin\theta \qquad 0<\theta<\pi$$



(右手定则)

$\vec{x} \times \vec{y} = -\vec{y} \times \vec{x}$

$(\lambda\vec{x} + \mu\vec{y}) \times \vec{z} = \lambda(\vec{x} \times \vec{z}) + \mu(\vec{y} \times \vec{z})$

$\vec{z} \times (\lambda\vec{x} + \mu\vec{y}) = \lambda(\vec{z} \times \vec{x}) + \mu(\vec{z} \times \vec{y})$

$\vec{x} \parallel \vec{y} \Longleftrightarrow \vec{x} \times \vec{y} = \vec{0} \Longleftrightarrow (x_1, x_2, x_3) = \lambda(y_1, y_2, y_3)$

($\theta = 0$ 或 π)

$\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$, $\vec{k} \times \vec{i} = \vec{j}$

$\vec{j} \times \vec{i} = -\vec{k}$ $\vec{k} \times \vec{j} = -\vec{i}$, $\vec{i} \times \vec{k} = -\vec{j}$

$\vec{i} \times (\vec{i} \times \vec{j}) = \vec{i} \times \vec{k} = -\vec{j}$

$(\vec{i} \times \vec{i}) \times \vec{j} = \vec{0} \times \vec{j} = \vec{0}$

混合积

$\vec{x} = (x_1, x_2, x_3)$, $\vec{y} = (y_1, y_2, y_3)$, $\vec{z} = (z_1, z_2, z_3)$

$$(\vec{x},\vec{y},\vec{z})$$

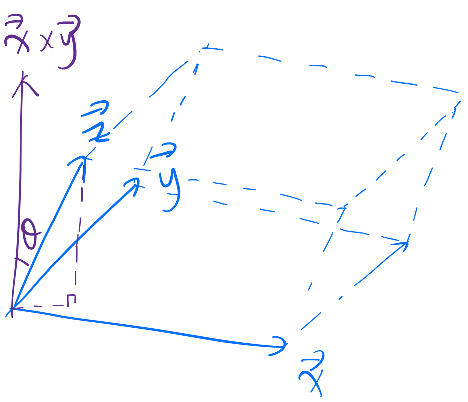
$$= (\vec{x}\times\vec{y})\cdot\vec{z}$$

$$= (x_2y_3 - x_3y_2,\ x_3y_1 - x_1y_3,\ x_1y_2 - x_2y_1)\cdot(z_1, z_2, z_3)$$

$$= (x_2y_3 - x_3y_2)z_1 + (x_3y_1 - x_1y_3)z_2 + (x_1y_2 - x_2y_1)z_3$$

$$= \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = \begin{vmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$$(\vec{x},\vec{y},\vec{z}) = (\vec{y},\vec{z},\vec{x}) = (\vec{z},\vec{x},\vec{y})$$



$$\left|(\vec{x}\times\vec{y})\cdot\vec{z}\right|$$

$$= \|\vec{x}\times\vec{y}\|\cdot\|\vec{z}\|\cos\theta$$

$$= V$$

$(\vec{u},\vec{v},\vec{w})$ 共面 $\Leftrightarrow$ $(\vec{u},\vec{v},\vec{w})=0$

例: $A(a_1, a_2, a_3)$ $B(b_1, b_2, b_3)$ $C(c_1, c_2, c_3)$

$D(d_1, d_2, d_3)$

四面体 ABCD 体积

$$\frac{1}{6}\left|(\vec{AB}\times\vec{AC})\cdot\vec{AD}\right|$$

$$= \pm\frac{1}{6}\begin{vmatrix} b_1-a_1 & b_2-a_2 & b_3-a_3 \\ c_1-a_1 & c_2-a_2 & c_3-a_3 \\ d_1-a_1 & d_2-a_2 & d_3-a_3 \end{vmatrix}$$

例: $\vec{a}=(1, 4, -7)$, $\vec{b}=(2, -1, 4)$. $\vec{c}=(0, -9, 18)$

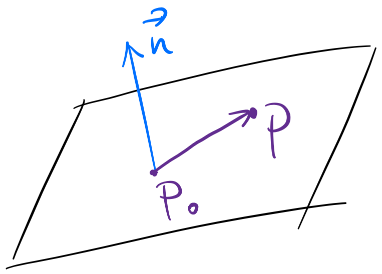
$$(\vec{a}\times\vec{b})\cdot\vec{c} = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & 4 \\ -9 & 18 \end{vmatrix} -2\begin{vmatrix} 4 & -7 \\ -9 & 18 \end{vmatrix}$$

$= -18+36 -2(72-63)$

$= 0$

$\vec{c} = \vec{b} - 2\vec{a}$



$\vec{n} = (A, B, C) \neq 0$, 法向量

$P_0(x_0, y_0, z_0)$

$P(x, y, z)$

$\overrightarrow{P_0P} = (x-x_0, y-y_0, z-z_0)$

$\vec{n} \cdot \overrightarrow{P_0P} = 0$

$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$

$Ax + By + Cz + D = 0$ （平面一般方程）

$D = -(Ax_0 + By_0 + Cz_0)$

任取$P_0(x_0,y_0,z_0)$满足

$$Ax_0+By_0+Cz_0+D=0$$

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0$$

例：过原点$O(0,0,0)$和点$P(6,-3,2)$，与平面 $4x-y+2z=8$ 垂直的平面：

法向量 $\vec{n}$， $\vec{n}\perp\overrightarrow{OP}\ (6,-3,2)$

$\vec{n}\perp(4,-1,2)$

$$\vec{n}=(6,-3,2)\times(4,-1,2)$$

$$=(-6+2,\ 8-12,\ -6+12)$$

$$=(-4,\ -4,\ 6)$$

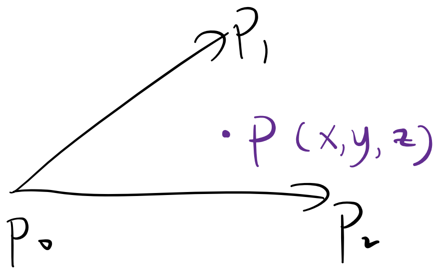
$$-4(x-0)-4(y-0)+6(z-0)=0$$

$$2x+2y-3z=0$$

经过三点（不共线）

$P_0(x_0, y_0, z_0)$，$P_1(x_1, y_1, z_1)$，$P_2(x_2, y_2, z_2)$

求平面方程



$$\vec{n} = \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}$$

$$\vec{n} \cdot \overrightarrow{P_0P} = 0$$

$$(\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}) \cdot \overrightarrow{P_0P} = 0$$

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ x_1-x_0 & y_1-y_0 & z_1-z_0 \\ x_2-x_0 & y_2-y_0 & z_2-z_0 \end{vmatrix} = 0$$

$Ax+By+Cz+D=0$，代入 P_0, P_1, P_2 坐标