

参考答案

$$1. \frac{\partial z}{\partial x} = \sin(xy) \cdot (x+y)^{\sin xy - 1} + (x+y)^{\sin xy} \ln(x+y^2)(\cos xy) \cdot y$$

2. 设切点 (x_0, y_0, z_0)

$$(1) \text{ 切平面 } x_0(x-x_0) + y_0(y-y_0) - (z-z_0) = 0$$

$$\text{由切平面平行于 } 2x+2y-z=0$$

$$\text{则有 } x_0 = 2, y_0 = 1 \text{ 代入切面方程得 } z_0 = 1$$

$$\text{从而切平面为 } 2x+2y-z=5$$

3. 设所求直线与直线 $\frac{x}{1} = \frac{y-5}{-1} = \frac{z-5}{-4}$ 的交点为

$$\begin{cases} x_0 = 4+t \\ y_0 = 5-t \\ z_0 = 5-4t \end{cases} \quad \text{其中 } t \text{ 是待定系数}$$

$$\text{则所求直线为 } \frac{x}{4+t} = \frac{y}{5-t} = \frac{z}{5-4t}$$

根据直线与平面 $x+y+z=0$ 平行 可得

$$(4+t) \cdot 1 + (5-t) \cdot 1 - (5-4t) = 0$$

$$\text{求得 } t = -1, \text{ 则所求直线为 } \frac{x}{1} = \frac{y}{2} = \frac{z}{3}$$

4. 求解 $\min x^2 + y^2 + z^2$

$$\text{s.t. } z^2 = xy + 4$$

$$\text{构造 Lagrange 乘数 } x^2 + y^2 + z^2 + \lambda(xy + 4 - z^2)$$

$$\text{求解} \quad \begin{cases} 2x + \lambda y = 0 \\ 2y + \lambda x = 0 \\ 2z - 2\lambda z = 0 \\ xy + 4 - z^2 = 0 \end{cases}$$

$$\text{解之可得 } x = 2, y = -2, \lambda = 2, z = 0$$

$$5. \frac{\partial g}{\partial x} = -f'(\frac{1}{r})(\frac{1}{r^2})\frac{x}{\sqrt{x^2+y^2}}$$

$$\begin{aligned}\frac{\partial^2 g}{\partial x^2} &= f''(\frac{1}{r}) \left[(\frac{1}{r^2}) \frac{x}{\sqrt{x^2+y^2}} \right]^2 + f'(\frac{1}{r}) \left(\frac{2}{r^3} \right) \left(\frac{x}{\sqrt{x^2+y^2}} \right)^2 \\ &\quad - f'(\frac{1}{r}) \frac{1}{r^2} \cdot \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}\end{aligned}$$

(由对称性可得)

$$\begin{aligned}\frac{\partial^2 g}{\partial y^2} &= f''(\frac{1}{r}) \frac{1}{r^4} \frac{y^2}{x^2+y^2} + f'(\frac{1}{r}) \left(\frac{2}{r^3} \right) \left(\frac{y^2}{x^2+y^2} \right) \\ &\quad - f'(\frac{1}{r}) \frac{1}{r^2} \cdot \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}}.\end{aligned}$$

$$\text{从而 } \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = f''(\frac{1}{r}) \cdot \frac{1}{r^4} + f'(\frac{1}{r}) \cdot \frac{1}{r^3}$$

$$6. \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{n+2}{2n+\sqrt{n}\sin n} \right)^{2n}} = \lim_{n \rightarrow \infty} \left(\frac{n+2}{2n+\sqrt{n}\sin n} \right)^2 = \frac{1}{4}$$

Ans 收敛半径为 4 收敛域为 (-4, 4)

$$7. \sum_{n=0}^{\infty} y^n = \frac{1}{1-y} \quad \sum_{n=1}^{\infty} ny^{n-1} = \frac{1}{(1-y)^2} \quad \sum_{n=2}^{\infty} n(n-1)y^{n-2} = \frac{2}{(1-y)^3}$$

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} (n^2-n) &= \sum_{n=2}^{\infty} \frac{(-1)^n}{2^n} (n^2-n) = \sum_{n=2}^{\infty} n(n-1)y^n \Big|_{y=-\frac{1}{2}} \\ &= \frac{2y^2}{(1-y)^3} \Big|_{y=-\frac{1}{2}} = \frac{4}{9}\end{aligned}$$

8. 将 f 的定义域从 $(0, \infty)$ 延展成 \mathbb{R} 上的奇函数

求 f 的 Fourier 级数

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi} -x dx + \int_{\pi}^{\pi} x dx \right] = 2(\pi - 2\sqrt{\pi})$$

$$a_n = \frac{2}{\pi} \left[\int_0^{\sqrt{2}} -x \cos nx dx + \int_{\sqrt{2}}^{\pi} x \cos nx dx \right]$$

$$= -\frac{4}{n} \sin(n\sqrt{2})$$

故 $f(x) \sim (\pi - 2\sqrt{2}) - \sum_{n=1}^{\infty} \frac{4}{n} \sin(n\sqrt{2}) \cos nx$
 之和為 $s(x)$

(1) $s(\frac{\pi}{2}) = f(\frac{\pi}{2}) = -\pi$

(由 $s(0) = -\pi$ 得 $\sum_{n=1}^{\infty} \frac{\sin n\sqrt{2}}{n} = \frac{\pi - 2\sqrt{2}}{2}$

(由 $s(\pi) = 0$ 得 $\sum_{n=1}^{\infty} \frac{\sin 2n\sqrt{2}}{n} = \frac{\pi - 2\sqrt{2}}{2}$