CSCI 391: SINDy Mathematical Representation Report

Kevin Bedoya

September 2024

Consider the second order D.E modeling the mechanics of a simple pendulum:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

Data was collected from equations 1 (angular displacement) and 2 (angular velocity), sampled for 20 discrete time-steps for functions $\theta(t)$.

1.
$$\theta(t) = \theta_0 \cos(t\omega)$$
 $\omega = \sqrt{\frac{g}{L}}$

$$2. \quad \theta(t)\frac{d}{dt} = -w\theta_0 \sin(t\omega)$$

 $\theta_0 = \text{maximum angular displacement} \quad \theta_0 \ll 1.$

 $\omega = \text{angular frequency}$

 $g = 9.80665 \ m/s^2$ (gravitational acceleration constant)

L = length of the rod (in our procedure we use length 3)

The data-matrix X, column 1: angular displacement, column 2: angular velocity.

$$X = \begin{bmatrix} 1.0 & -0.0 \\ 0.5804 & -1.4723 \\ -0.3263 & -1.7091 \\ -0.9591 & -0.5115 \\ -0.7871 & 1.1153 \\ 0.0455 & 1.8061 \\ 0.8399 & 0.9813 \\ 0.9294 & -0.6671 \\ -0.6520 & -1.3708 \\ -0.9959 & 0.1644 \\ -0.5040 & 1.5616 \\ 0.4109 & 1.6483 \\ 0.9809 & 0.3518 \\ 0.7277 & -1.2400 \\ -0.1361 & -1.7912 \\ -0.8858 & -0.8392 \\ -0.8921 & 0.8171 \\ -0.1497 & 1.7876 \\ 0.7183 & 1.2580 \end{bmatrix}$$

$$\begin{bmatrix} -0.3345 & -3.9712^{\circ} \\ -1.2600 & -1.6236 \\ -1.4626 & 0.9127 \\ -0.4378 & 2.6831 \\ 0.9954 & 2.26831 \\ 0.9954 & 2.26831 \\ 0.9954 & 2.26831 \\ 0.9954 & 2.26831 \\ 0.8397 & -2.3495 \\ -0.5709 & -2.6000 \\ -1.5709 & -2.6000 \\ -1.5709 & -2.6000 \\ -1.5709 & -2.6000 \\ -1.7731 & 1.8240 \\ 0.1407 & 2.7858 \\ 1.3364 & 1.4098 \\ 1.4106 & -1.1494 \\ 0.3010 & -2.7440 \\ -1.0612 & -2.0358 \\ -1.5328 & 0.3808 \\ -0.7181 & 2.4779 \\ 0.6992 & 2.4954 \\ 1.5298 & 0.4188 \\ 1.7685 & -2.4315 \end{bmatrix}$$

The following libraries are used: Polynomial (degree 2) and Fourier (frequency 1) as candidate functions to our system of equations.

$$\Theta(\mathbf{X}) = \begin{bmatrix} 1.0 & x_0 & x_1 & x_0^2 & x_0x_1 & x_1^2 & \sin(x_0) & \cos(x_0) & \sin(x_1) & \cos(x_1) \\ | & | & | & | & | & | & | & | & | \\ \mathbf{X} & \mathbf{X} \end{bmatrix}$$

We would like to solve the following:

$\Gamma - 0.3345$	-3.97127	Г	1.0	x_0	x_1	x_0^2	x_0x_1	x_1^2	$\sin(x_0)$	$\cos(x_0)$	$\sin(x_1)$	$\cos(x_1)$		
	I		1.0	1.0	-0.0	1.0	-0.0	0.0	0.841	0.54	-0.0	1.0		
-1.2600	-1.6236		1.0	0.58	-1.472	0.337	-0.855	2.168	0.548	0.836	-0.995	0.098		
-1.4626	0.9127		1.0	-0.326	-1.709	0.106	0.558	2.921	-0.321	0.947	-0.99	-0.138		β_0
-0.4378	2.6831		1.0	-0.959	-0.512	0.92	0.491	0.262	-0.819	0.574	-0.49	0.872		
0.9544	2.2018		1.0	-0.787	1.115	0.619	-0.878	1.244	-0.708	0.706	0.898	0.44	-	
1.5456	-0.1273		1.0	0.046	1.806	0.002	0.082	3.262	0.045	0.999	0.972	-0.233	α_0	
0.8397	-2.3495		1.0	0.84	0.981	0.705	0.824	0.963	0.745	0.668	0.831	0.556	α_1	β_1
-0.5709	-2.6000		1.0	0.929	-0.667	0.864	-0.62	0.445	0.801	0.598	-0.619	0.786	α_2	β_2
-1.5024	-0.6686		1.0	0.239	-1.756	0.057	-0.42	3.082	0.237	0.972	-0.983	-0.184	α_3	β_3
-1.1731	1.8240	_	1.0	-0.652	-1.371	0.425	0.894	1.879	-0.607	0.795	-0.98	0.199	α_4	β_4
0.1407	2.7858		1.0	-0.996	0.164	0.992	-0.164	0.027	-0.839	0.544	0.164	0.987	α_5	β_5
1.3364	1.4098		1.0	-0.504	1.562	0.254	-0.787	2.439	-0.483	0.876	1.0	0.009	α_6	β_6
1.4106	-1.1494		1.0	0.411	1.648	0.169	0.677	2.717	0.399	0.917	0.997	-0.077	α_7	β_7
0.3010	-2.7440		1.0	0.981	0.352	0.962	0.345	0.124	0.831	0.556	0.345	0.939	α_8	β_8
-1.0612	I											- 1	α_9	β_9
-1.5328		1												
-0.7181	2.4779											- 1		
0.6992	2.4954													
1.5298	0.4188													
1.7685	-2.4315	- 1										- 1		
$ \begin{array}{r} -0.7181 \\ 0.6992 \\ 1.5298 \end{array} $	2.4954 0.4188	- 1	1.0 1.0 1.0 1.0 1.0 1.0	0.981 0.728 -0.136 -0.886 -0.892 -0.15 0.718	0.352 -1.24 -1.791 -0.839 0.817 1.788 1.258	0.962 0.53 0.019 0.785 0.796 0.022 0.516	0.345 -0.902 0.244 0.743 -0.729 -0.268 0.904	0.124 1.538 3.208 0.704 0.668 3.196 1.582	0.831 0.665 -0.136 -0.774 -0.778 -0.149 0.658	0.556 0.747 0.991 0.633 0.628 0.989 0.753	0.345 -0.946 -0.976 -0.744 0.729 0.977 0.951	$\begin{bmatrix} 0.939 \\ 0.325 \\ -0.219 \\ 0.668 \\ 0.684 \\ -0.215 \\ 0.308 \end{bmatrix}$		β_9

 $\dot{x_0} = \Theta(\mathbf{X})\xi_0$

-0.33457	Γ1.0	x_0	x_1	x_0^2	$x_0 x_1$	x_{1}^{2}	$\sin(x_0)$	$\cos(x_0)$	$\sin(x_1)$	$\cos(x_1)$
-1.2600	1.0	1.0	-0.0	1.0	-0.0	0.0	0.841	0.54	-0.0	1.0
-1.4626	1.0	0.58	-1.472	0.337	-0.855	2.168	0.548	0.836	-0.995	0.098
-0.4378	1.0	-0.326	-1.709	0.106	0.558	2.921	-0.321	0.947	-0.99	-0.138
0.4378	1.0	-0.959	-0.512	0.92	0.491	0.262	-0.819	0.574	-0.49	0.872
.5456	1.0	-0.787	1.115	0.619	-0.878	1.244	-0.708	0.706	0.898	0.44
1	1.0	0.046	1.806	0.002	0.082	3.262	0.045	0.999	0.972	-0.233
.8397	1.0	0.84	0.981	0.705	0.824	0.963	0.745	0.668	0.831	0.556
0.5709	1.0	0.929	-0.667	0.864	-0.62	0.445	0.801	0.598	-0.619	0.786
1.5024 1.1731	1.0	0.239	-1.756	0.057	-0.42	3.082	0.237	0.972	-0.983	-0.184
	= 1.0	-0.652	-1.371	0.425	0.894	1.879	-0.607	0.795	-0.98	0.199
.3364	1.0	-0.996	0.164	0.992	-0.164	0.027	-0.839	0.544	0.164	0.987
1	1.0	-0.504	1.562	0.254	-0.787	2.439	-0.483	0.876	1.0	0.009
$\frac{4106}{3010}$	1.0	0.411	1.648	0.169	0.677	2.717	0.399	0.917	0.997	-0.077
.0612	1.0	0.981	0.352	0.962	0.345	0.124	0.831	0.556	0.345	0.939
	1.0	0.728	-1.24	0.53	-0.902	1.538	0.665	0.747	-0.946	0.325
.5328	1.0	-0.136	-1.791	0.019	0.244	3.208	-0.136	0.991	-0.976	-0.219
.7181	1.0	-0.886	-0.839	0.785	0.743	0.704	-0.774	0.633	-0.744	0.668
6992	1.0	-0.892	0.817	0.796	-0.729	0.668	-0.778	0.628	0.729	0.684
5298	1.0	-0.15	1.788	0.022	-0.268	3.196	-0.149	0.989	0.977	-0.215
.7685]	1.0	0.718	1.258	0.516	0.904	1.582	0.658	0.753	0.951	0.308

 $\dot{x_1} = \Theta(\mathbf{X})\xi_1$

 $\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \\ \beta_9 \end{bmatrix}$

Γ-3.97127	Γ1.0	x_0	x_1	x_{0}^{2}	x_0x_1	x_{1}^{2}	$\sin(x_0)$	$\cos(x_0)$	$\sin(x1)$	$\cos(x_1)$	
$\begin{bmatrix} -3.9712 \\ -1.6236 \end{bmatrix}$	1.0	1.0	-0.0	1.0	-0.0	0.0	0.841	0.54	-0.0°	1.0	
1 1	1.0	0.58	-1.472	0.337	-0.855	2.168	0.548	0.836	-0.995	0.098	
$\begin{vmatrix} -0.9127 \\ 2.6831 \end{vmatrix}$	1.0	-0.326	-1.709	0.106	0.558	2.921	-0.321	0.947	-0.99	-0.138	
1 1	1.0	-0.959	-0.512	0.92	0.491	0.262	-0.819	0.574	-0.49	0.872	
2.2018	1.0	-0.787	1.115	0.619	-0.878	1.244	-0.708	0.706	0.898	0.44	
-0.1273	1.0	0.046	1.806	0.002	0.082	3.262	0.045	0.999	0.972	-0.233	
$\begin{vmatrix} -2.3495 \\ 2.6000 \end{vmatrix}$	1.0	0.84	0.981	0.705	0.824	0.963	0.745	0.668	0.831	0.556	
-2.6000	1.0	0.929	-0.667	0.864	-0.62	0.445	0.801	0.598	-0.619	0.786	
-0.6686	1.0	0.239	-1.756	0.057	-0.42	3.082	0.237	0.972	-0.983	-0.184	
1.8240	= 1.0	-0.652	-1.371	0.425	0.894	1.879	-0.607	0.795	-0.98	0.199	
2.7858	1.0	-0.996	0.164	0.992	-0.164	0.027	-0.839	0.544	0.164	0.987	
1.4098	1.0	-0.504	1.562	0.254	-0.787	2.439	-0.483	0.876	1.0	0.009	
-1.1494	1.0	0.411	1.648	0.169	0.677	2.717	0.399	0.917	0.997	-0.077	
$\begin{vmatrix} -2.7440 \\ 2.0250 \end{vmatrix}$	1.0	0.981	0.352	0.962	0.345	0.124	0.831	0.556	0.345	0.939	
$\begin{vmatrix} -2.0358 \\ 0.3898 \end{vmatrix}$	1.0	0.728	-1.24	0.53	-0.902	1.538	0.665	0.747	-0.946	0.325	
0.3808	1.0	-0.136	-1.791	0.019	0.244	3.208	-0.136	0.991	-0.976	-0.219	
2.4779	1.0	-0.886	-0.839	0.785	0.743	0.704	-0.774	0.633	-0.744	0.668	
2.4954	1.0	-0.892	0.817	0.796	-0.729	0.668	-0.778	0.628	0.729	0.684	
0.4188	1.0	-0.15	1.788	0.022	-0.268	3.196	-0.149	0.989	0.977	-0.215	
$\lfloor -2.4315 \rfloor$	1.0	0.718	1.258	0.516	0.904	1.582	0.658	0.753	0.951	0.308	

After applying sparse-regression the resulting matrix is sparse, leaving the best-fit terms for the equations. More specifically the optimizer used in the STLSQ optimizer. The STLSQ optimizer was leveraged by default, and from the PySINDy API, print(f'Optimizer used: model.optimizer") was written in the program to uncover this detail. The GitHub repository containing csv, png graphs, and python files are included on the final page.

$$\begin{bmatrix} \textbf{1.0} & \textbf{x}_0 & \textbf{x}_1 & \textbf{x}_0^2 & \textbf{x}_0\textbf{x}_1 & \textbf{x}_1^2 & \sin(x_0) & \cos(x_0) & \sin(x_1) & \cos(x_1) \\ 0.0 & -1.9452 & 0.7185 & 0.2201 & 0.0 & 0.0 & 2.2275 & 0.0 & 0.2557 & -0.2561 \\ 0.0 & -4.9747 & 0.0 & 0.0 & 0.0 & 0.0 & 2.3120 & 0.0 & 0.0 & -0.2087 \\ \end{bmatrix}$$

The resulting equations for $\dot{x_0}$ and $\dot{x_1}$ are constructed by taking a linear combination of the non-zero terms from the library Θ , corresponding to the rows denoting $\dot{x_0}$ and $\dot{x_1}$ respectively.

$$\dot{x}_0 = -1.9452x_0 + 0.7185x_1 + 0.2201x_0^2 + 2.2275\sin(x_0) + 0.2557\sin(x_1) - 0.2561\cos(x_1)$$

$$\dot{x}_1 = -4.9747x_0 + 2.312\sin(x_0) - 0.2087\cos(x_1)$$

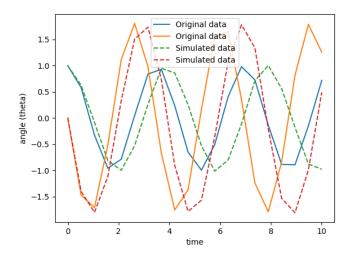


Figure 1: The following graphic depicts a comparison between the simulated and original data. The discrepancy between the data is clear, however at very early times the curves are nearly aligned.

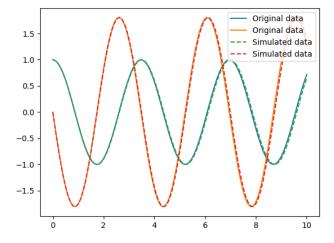


Figure 2: The following graphic depicts a simulation from the equations discovered from pendulum mechanics run for a time-series data of **100** time steps. The same parameters from the original **20** time series data were used to produce the **100** time series data. The governing equations discovered from the data were as follows: **Discovered Equations**.