CSCI 391: SINDy Mathematical Representation Report

Kevin Bedoya

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Consider the second order D.E modeling the mechanics of a **simple pendulum:**

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

Data was collected from equations 1 **(angular displacement)** and 2 **(angular velocity)**, sampled for **20** discrete time-steps for functions $\theta(t)$.

$$1. \quad \theta(t) = \theta_0 \cos(t\omega) \quad \omega = \sqrt{\frac{g}{L}}$$

$$2. \quad \theta(t)\frac{d}{dt} = -w\theta_0 \sin(t\omega)$$

θ_0 = maximum angular displacement $\quad \theta_0 \ll 1$.
ω = angular frequency
g = 9.80665 m/s^2 (**gravitational acceleration constant**)
L = length of the rod (in our procedure we use length **3)**

The data-matrix **X**, **column 1:** angular displacement, **column 2:** angular velocity.

$$X = \begin{bmatrix} 1.0 & -0.0 \\ 0.5804 & -1.4723 \\ -0.3263 & -1.7091 \\ -0.9591 & -0.5115 \\ -0.7871 & 1.1153 \\ 0.0455 & 1.8061 \\ 0.8399 & 0.9813 \\ 0.9294 & -0.6671 \\ 0.2390 & -1.7556 \\ -0.6520 & -1.3708 \\ -0.9959 & 0.1644 \\ -0.5040 & 1.5616 \\ 0.4109 & 1.6483 \\ 0.9809 & 0.3518 \\ 0.7277 & -1.2400 \\ -0.1361 & -1.7912 \\ -0.8858 & -0.8392 \\ -0.8921 & 0.8171 \\ -0.1497 & 1.7876 \\ 0.7183 & 1.2580 \end{bmatrix} \qquad \dot{X} = \begin{bmatrix} -0.3345 & -3.9712 \\ -1.2600 & -1.6236 \\ -1.4626 & 0.9127 \\ -0.4378 & 2.6831 \\ 0.9544 & 2.2018 \\ 1.5456 & -0.1273 \\ 0.8397 & -2.3495 \\ -0.5709 & -2.6000 \\ -1.5024 & -0.6686 \\ -1.1731 & 1.8240 \\ 0.1407 & 2.7858 \\ 1.3364 & 1.4098 \\ 1.4106 & -1.1494 \\ 0.3010 & -2.7440 \\ -1.0612 & -2.0358 \\ -1.5328 & 0.3808 \\ -0.7181 & 2.4779 \\ 0.6992 & 2.4954 \\ 1.5298 & 0.4188 \\ 1.7685 & -2.4315 \end{bmatrix}$$

The following libraries are used: **Polynomial (degree 2)** and **Fourier (frequency 1)** as candidate functions to our system of equations.

$$\Theta(\mathbf{X}) = \begin{bmatrix} 1.0 & x_0 & x_1 & x_0^2 & x_0x_1 & x_1^2 & \sin(x_0) & \cos(x_0) & \sin(x_1) & \cos(x_1) \\ | & | & | & | & | & | & | & | & | & | \\ \mathbf{X} & \mathbf{X} & \mathbf{X} & \mathbf{X} & \mathbf{X} & \mathbf{X} & \mathbf{X} & \mathbf{X} & \mathbf{X} & \mathbf{X} \end{bmatrix}$$

We would like to solve the following:

$$\dot{X} = \Theta(\mathbf{X})\Xi$$

$$
\begin{bmatrix}
-0.3345 & -3.9712\\
-1.2600 & -1.6236\\
-1.4626 & 0.9127\\
-0.4378 & 2.6831\\
0.9544 & 2.2018\\
1.5456 & -0.1273\\
0.8397 & -2.3495\\
-0.5709 & -2.6000\\
-1.5024 & -0.6686\\
-1.1731 & 1.8240\\
0.1407 & 2.7858\\
1.3364 & 1.4098\\
1.4106 & -1.1494\\
0.3010 & -2.7440\\
-1.0612 & -2.0358\\
-1.5328 & 0.3808\\
-0.7181 & 2.4779\\
0.6992 & 2.4954\\
1.5298 & 0.4188\\
1.7685 & -2.4315
\end{bmatrix}
=
\begin{bmatrix}
1.0 & x_0 & x_1 & x_0^2 & x_0x_1 & x_1^2 & \sin(x_0) & \cos(x_0) & \sin(x_1) & \cos(x_1)\\
1.0 & 1.0 & -0.0 & 1.0 & -0.0 & 0.0 & 0.841 & 0.54 & -0.0 & 1.0\\
1.0 & 0.58 & -1.472 & 0.337 & -0.855 & 2.168 & 0.548 & 0.836 & -0.995 & 0.098\\
1.0 & -0.326 & -1.709 & 0.106 & 0.558 & 2.921 & -0.321 & 0.947 & -0.99 & -0.138\\
1.0 & -0.959 & -0.512 & 0.92 & 0.491 & 0.262 & -0.819 & 0.574 & -0.49 & 0.872\\
1.0 & -0.787 & 1.115 & 0.619 & -0.878 & 1.244 & -0.708 & 0.706 & 0.898 & 0.44\\
1.0 & 0.046 & 1.806 & 0.002 & 0.082 & 3.262 & 0.045 & 0.999 & 0.972 & -0.233\\
1.0 & 0.84 & 0.981 & 0.705 & 0.824 & 0.963 & 0.745 & 0.668 & 0.831 & 0.556\\
1.0 & 0.929 & -0.667 & 0.864 & -0.62 & 0.445 & 0.801 & 0.598 & -0.619 & 0.786\\
1.0 & 0.239 & -1.756 & 0.057 & -0.42 & 3.082 & 0.237 & 0.972 & -0.983 & -0.184\\
1.0 & -0.652 & -1.371 & 0.425 & 0.894 & 1.879 & -0.607 & 0.795 & -0.98 & 0.199\\
1.0 & -0.996 & 0.164 & 0.992 & -0.164 & 0.027 & -0.839 & 0.544 & 0.164 & 0.987\\
1.0 & -0.504 & 1.562 & 0.254 & -0.787 & 2.439 & -0.483 & 0.876 & 1.0 & 0.009\\
1.0 & 0.411 & 1.648 & 0.169 & 0.677 & 2.717 & 0.399 & 0.917 & 0.997 & -0.077\\
1.0 & 0.981 & 0.352 & 0.962 & 0.345 & 0.124 & 0.831 & 0.556 & 0.345 & 0.939\\
1.0 & 0.728 & -1.24 & 0.53 & -0.902 & 1.538 & 0.665 & 0.747 & -0.946 & 0.325\\
1.0 & -0.136 & -1.791 & 0.019 & 0.244 & 3.208 & -0.136 & 0.991 & -0.976 & -0.219\\
1.0 & -0.886 & -0.839 & 0.785 & 0.743 & 0.704 & -0.774 & 0.633 & -0.744 & 0.668\\
1.0 & -0.892 & 0.817 & 0.796 & -0.729 & 0.668 & -0.778 & 0.628 & 0.729 & 0.684\\
1.0 & -0.15 & 1.788 & 0.022 & -0.268 & 3.196 & -0.149 & 0.989 & 0.977 & -0.215\\
1.0 & 0.718 & 1.258 & 0.516 & 0.904 & 1.582 & 0.658 & 0.753 & 0.951 & 0.308
\end{bmatrix}
\begin{bmatrix}
\alpha_0 & \beta_0\\
\alpha_1 & \beta_1\\
\alpha_2 & \beta_2\\
\alpha_3 & \beta_3\\
\alpha_4 & \beta_4\\
\alpha_5 & \beta_5\\
\alpha_6 & \beta_6\\
\alpha_7 & \beta_7\\
\alpha_8 & \beta_8\\
\alpha_9 & \beta_9
\end{bmatrix}
$$

$$\dot{x_0} = \Theta(\mathbf{X})\xi_0$$

$$
\begin{bmatrix}
-0.3345\\
-1.2600\\
-1.4626\\
-0.4378\\
0.9544\\
1.5456\\
0.8397\\
-0.5709\\
-1.5024\\
-1.1731\\
0.1407\\
1.3364\\
1.4106\\
0.3010\\
-1.0612\\
-1.5328\\
-0.7181\\
0.6992\\
1.5298\\
1.7685
\end{bmatrix}
=
\begin{bmatrix}
1.0 & x_0 & x_1 & x_0^2 & x_0x_1 & x_1^2 & \sin(x_0) & \cos(x_0) & \sin(x_1) & \cos(x_1)\\
1.0 & 1.0 & -0.0 & 1.0 & -0.0 & 0.0 & 0.841 & 0.54 & -0.0 & 1.0\\
1.0 & 0.58 & -1.472 & 0.337 & -0.855 & 2.168 & 0.548 & 0.836 & -0.995 & 0.098\\
1.0 & -0.326 & -1.709 & 0.106 & 0.558 & 2.921 & -0.321 & 0.947 & -0.99 & -0.138\\
1.0 & -0.959 & -0.512 & 0.92 & 0.491 & 0.262 & -0.819 & 0.574 & -0.49 & 0.872\\
1.0 & -0.787 & 1.115 & 0.619 & -0.878 & 1.244 & -0.708 & 0.706 & 0.898 & 0.44\\
1.0 & 0.046 & 1.806 & 0.002 & 0.082 & 3.262 & 0.045 & 0.999 & 0.972 & -0.233\\
1.0 & 0.84 & 0.981 & 0.705 & 0.824 & 0.963 & 0.745 & 0.668 & 0.831 & 0.556\\
1.0 & 0.929 & -0.667 & 0.864 & -0.62 & 0.445 & 0.801 & 0.598 & -0.619 & 0.786\\
1.0 & 0.239 & -1.756 & 0.057 & -0.42 & 3.082 & 0.237 & 0.972 & -0.983 & -0.184\\
1.0 & -0.652 & -1.371 & 0.425 & 0.894 & 1.879 & -0.607 & 0.795 & -0.98 & 0.199\\
1.0 & -0.996 & 0.164 & 0.992 & -0.164 & 0.027 & -0.839 & 0.544 & 0.164 & 0.987\\
1.0 & -0.504 & 1.562 & 0.254 & -0.787 & 2.439 & -0.483 & 0.876 & 1.0 & 0.009\\
1.0 & 0.411 & 1.648 & 0.169 & 0.677 & 2.717 & 0.399 & 0.917 & 0.997 & -0.077\\
1.0 & 0.981 & 0.352 & 0.962 & 0.345 & 0.124 & 0.831 & 0.556 & 0.345 & 0.939\\
1.0 & 0.728 & -1.24 & 0.53 & -0.902 & 1.538 & 0.665 & 0.747 & -0.946 & 0.325\\
1.0 & -0.136 & -1.791 & 0.019 & 0.244 & 3.208 & -0.136 & 0.991 & -0.976 & -0.219\\
1.0 & -0.886 & -0.839 & 0.785 & 0.743 & 0.704 & -0.774 & 0.633 & -0.744 & 0.668\\
1.0 & -0.892 & 0.817 & 0.796 & -0.729 & 0.668 & -0.778 & 0.628 & 0.729 & 0.684\\
1.0 & -0.15 & 1.788 & 0.022 & -0.268 & 3.196 & -0.149 & 0.989 & 0.977 & -0.215\\
1.0 & 0.718 & 1.258 & 0.516 & 0.904 & 1.582 & 0.658 & 0.753 & 0.951 & 0.308
\end{bmatrix}
\begin{bmatrix}
\alpha_0\\
\alpha_1\\
\alpha_2\\
\alpha_3\\
\alpha_4\\
\alpha_5\\
\alpha_6\\
\alpha_7\\
\alpha_8\\
\alpha_9
\end{bmatrix}
$$

$$\dot{x_1} = \Theta(\mathbf{X})\xi_1$$

$$
\begin{bmatrix}
-3.9712\\
-1.6236\\
-0.9127\\
2.6831\\
2.2018\\
-0.1273\\
-2.3495\\
-2.6000\\
-0.6686\\
1.8240\\
2.7858\\
1.4098\\
-1.1494\\
-2.7440\\
-2.0358\\
0.3808\\
2.4779\\
2.4954\\
0.4188\\
-2.4315
\end{bmatrix}
=
\begin{bmatrix}
1.0 & x_0 & x_1 & x_0^2 & x_0x_1 & x_1^2 & \sin(x_0) & \cos(x_0) & \sin(x1_{)} & \cos(x_1)\\
1.0 & 1.0 & -0.0 & 1.0 & -0.0 & 0.0 & 0.841 & 0.54 & -0.0 & 1.0\\
1.0 & 0.58 & -1.472 & 0.337 & -0.855 & 2.168 & 0.548 & 0.836 & -0.995 & 0.098\\
1.0 & -0.326 & -1.709 & 0.106 & 0.558 & 2.921 & -0.321 & 0.947 & -0.99 & -0.138\\
1.0 & -0.959 & -0.512 & 0.92 & 0.491 & 0.262 & -0.819 & 0.574 & -0.49 & 0.872\\
1.0 & -0.787 & 1.115 & 0.619 & -0.878 & 1.244 & -0.708 & 0.706 & 0.898 & 0.44\\
1.0 & 0.046 & 1.806 & 0.002 & 0.082 & 3.262 & 0.045 & 0.999 & 0.972 & -0.233\\
1.0 & 0.84 & 0.981 & 0.705 & 0.824 & 0.963 & 0.745 & 0.668 & 0.831 & 0.556\\
1.0 & 0.929 & -0.667 & 0.864 & -0.62 & 0.445 & 0.801 & 0.598 & -0.619 & 0.786\\
1.0 & 0.239 & -1.756 & 0.057 & -0.42 & 3.082 & 0.237 & 0.972 & -0.983 & -0.184\\
1.0 & -0.652 & -1.371 & 0.425 & 0.894 & 1.879 & -0.607 & 0.795 & -0.98 & 0.199\\
1.0 & -0.996 & 0.164 & 0.992 & -0.164 & 0.027 & -0.839 & 0.544 & 0.164 & 0.987\\
1.0 & -0.504 & 1.562 & 0.254 & -0.787 & 2.439 & -0.483 & 0.876 & 1.0 & 0.009\\
1.0 & 0.411 & 1.648 & 0.169 & 0.677 & 2.717 & 0.399 & 0.917 & 0.997 & -0.077\\
1.0 & 0.981 & 0.352 & 0.962 & 0.345 & 0.124 & 0.831 & 0.556 & 0.345 & 0.939\\
1.0 & 0.728 & -1.24 & 0.53 & -0.902 & 1.538 & 0.665 & 0.747 & -0.946 & 0.325\\
1.0 & -0.136 & -1.791 & 0.019 & 0.244 & 3.208 & -0.136 & 0.991 & -0.976 & -0.219\\
1.0 & -0.886 & -0.839 & 0.785 & 0.743 & 0.704 & -0.774 & 0.633 & -0.744 & 0.668\\
1.0 & -0.892 & 0.817 & 0.796 & -0.729 & 0.668 & -0.778 & 0.628 & 0.729 & 0.684\\
1.0 & -0.15 & 1.788 & 0.022 & -0.268 & 3.196 & -0.149 & 0.989 & 0.977 & -0.215\\
1.0 & 0.718 & 1.258 & 0.516 & 0.904 & 1.582 & 0.658 & 0.753 & 0.951 & 0.308
\end{bmatrix}
\begin{bmatrix}
\beta_0\\
\beta_1\\
\beta_2\\
\beta_3\\
\beta_4\\
\beta_5\\
\beta_6\\
\beta_7\\
\beta_8\\
\beta_9
\end{bmatrix}
$$

After applying sparse-regression the resulting matrix is sparse, leaving the best-fit terms for the equations. More specifically the optimizer used in the **STLSQ** optimizer. The STLSQ optimizer was leveraged by default, and from the PySINDy API, **print(f"Optimizer used: model.optimizer")** was written in the program to uncover this detail. The GitHub repository containing csv, png graphs, and python files are included on the **final page**.

$$\begin{bmatrix} \mathbf{1.0} & \mathbf{x}_0 & \mathbf{x}_1 & \mathbf{x}_0^2 & \mathbf{x}_0\mathbf{x}_1 & \mathbf{x}_1^2 & \sin(x_0) & \cos(x_0) & \sin(x_1) & \cos(x_1) \\ 0.0 & -1.9452 & 0.7185 & 0.2201 & 0.0 & 0.0 & 2.2275 & 0.0 & 0.2557 & -0.2561 \\ 0.0 & -4.9747 & 0.0 & 0.0 & 0.0 & 0.0 & 2.3120 & 0.0 & 0.0 & -0.2087 \end{bmatrix}$$

The resulting equations for $\dot{x_0}$ and $\dot{x_1}$ are constructed by taking a linear combination of the non-zero terms from the library Θ, corresponding to the rows denoting $\dot{x_0}$ and $\dot{x_1}$ respectively.

$$\dot{x}_0 = -1.9452x_0 + 0.7185x_1 + 0.2201x_0^2 + 2.2275\sin(x_0) + 0.2557\sin(x_1) - 0.2561\cos(x_1)$$

$$\dot{x}_1 = -4.9747x_0 + 2.312\sin(x_0) - 0.2087\cos(x_1)$$

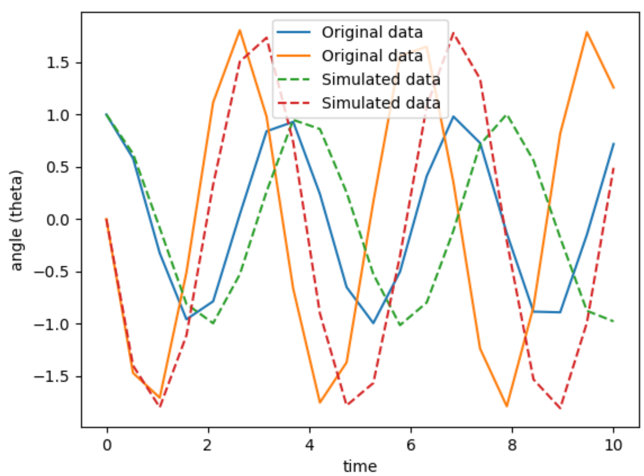

Figure 1: The following graphic depicts a comparison between the simulated and original data. The discrepancy between the data is clear, however at very early times the curves are nearly aligned.

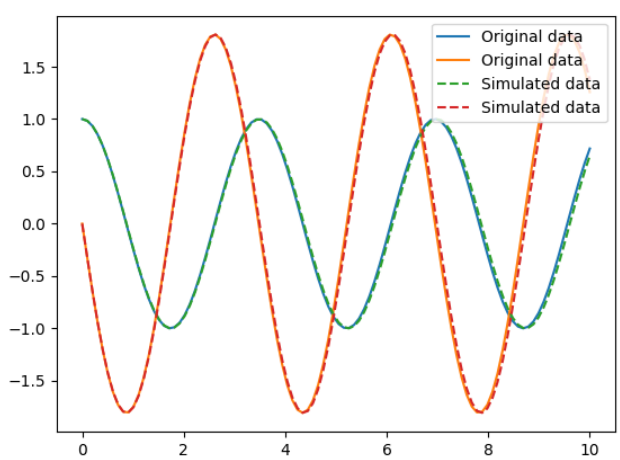

Figure 2: The following graphic depicts a simulation from the equations discovered from pendulum mechanics run for a time-series data of **100** time steps. The same parameters from the original **20** time series data were used to produce the **100** time series data. The governing equations discovered from the data were as follows: **Discovered Equations.**

GitHub Repository for Pendulum Mechanics Discovery