Elaborato di Sistemi Dinamici per l'Intelligenza Artificiale

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AIDA 2023

Anharmonic Oscillator

Equations of Motion

The equations of motion for a Hamiltonian system are derived from:

$$\begin{cases} \dot{x} = & -\frac{dH}{dp} \\ \dot{p} = & -\frac{dH}{dx} \end{cases}$$

With the Hamiltonian given by

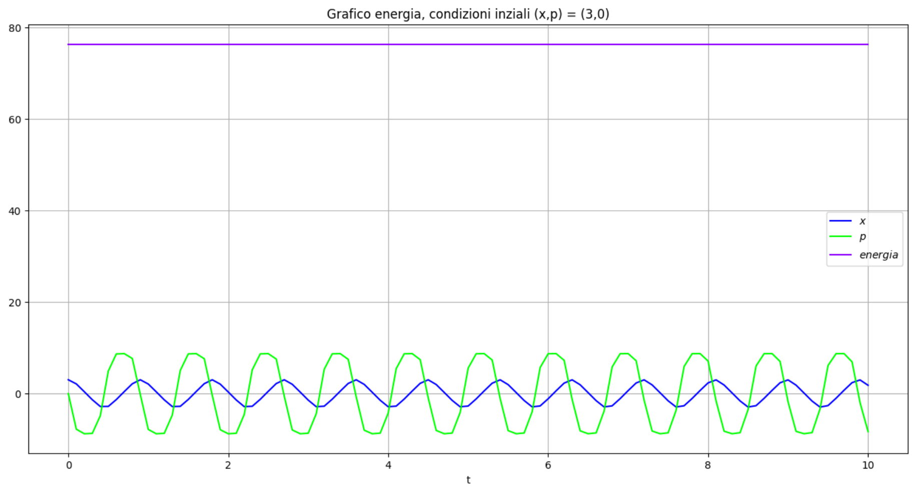
$$H = p^2 + ax^2 + x^4$$

We obtain the equations of motion:

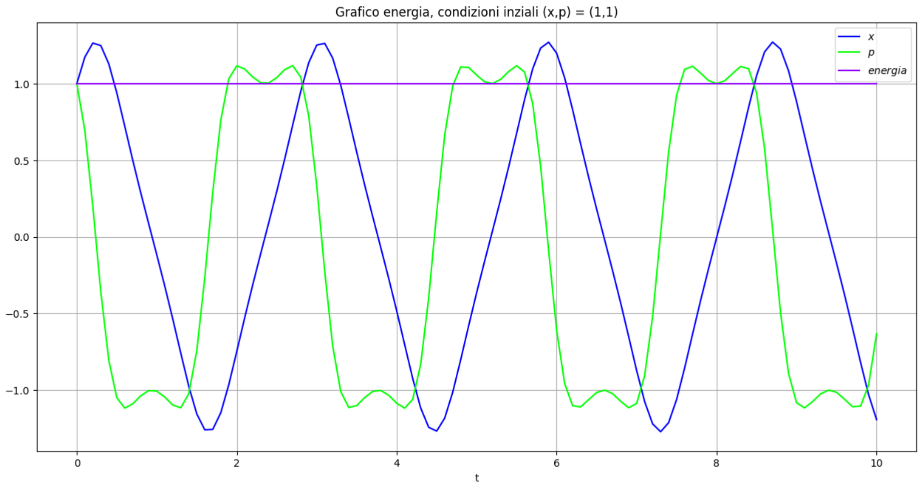
$$\begin{cases} \dot{x} = & 2p \\ \dot{p} = & -2ax - 4x^3 \end{cases}$$

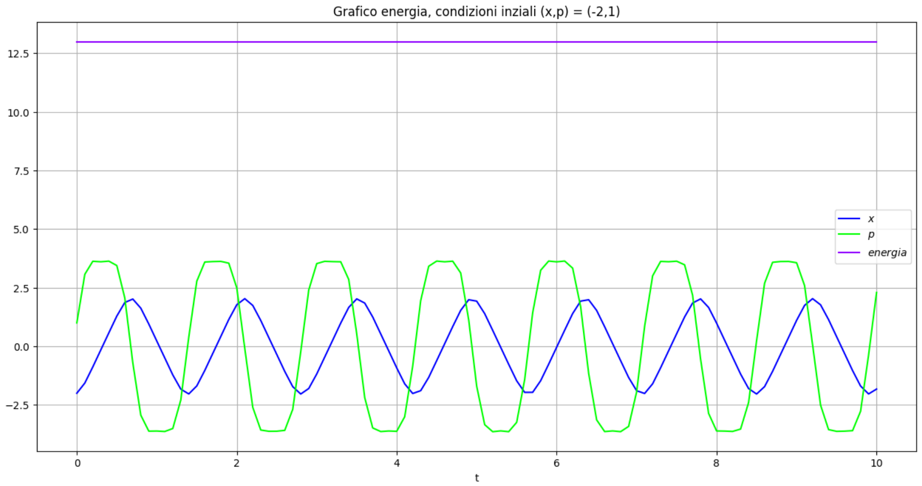
Energy Conservation for Different Initial Conditions

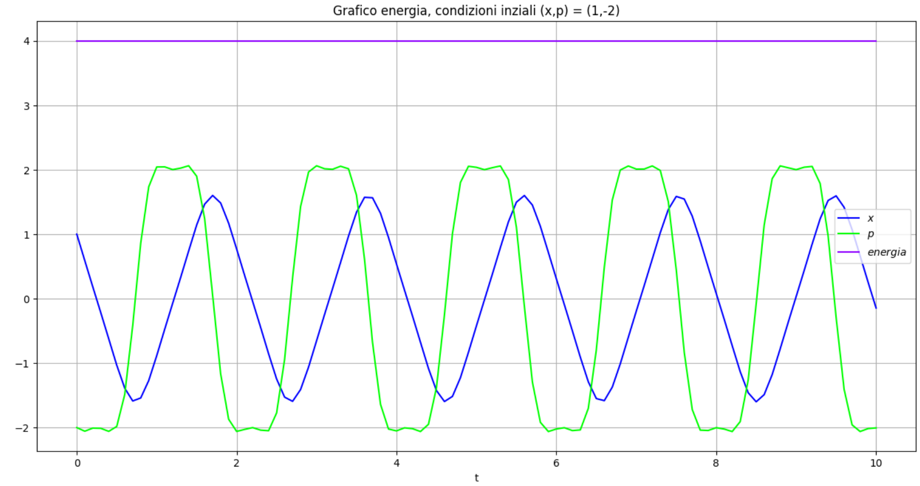
In the following graph, we plotted the evolution of x, p, and energy over time. On the ordinate axis, we have the values of energy, while on the abscissa axis, we have the time value. From the graph, we can infer that the variables p and x, which describe momentum and coordinates and consequently compose kinetic and potential energy, oscillate between maximum and minimum values, alternating in such a way that when one of them is zero, the opposite variable is at the maximum value in modulus. This balance in oscillations allows us to deduce that energy is a conserved quantity. To verify this, it suffices to substitute the values of p and x into the Hamiltonian equation $H = p^2 – x^2 + x^4$ This allows us to visualize that energy is conserved, as a straight line is plotted, in this case with a value of 76.

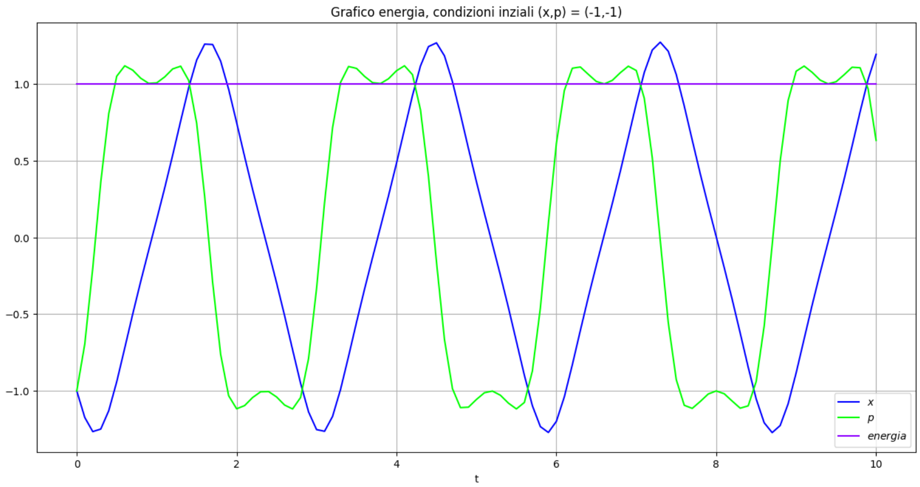


Various graphs are now presented as x and p vary:









More formally, to verify energy conservation, we can evaluate $\frac{dH}{dt}$ and verify that this quantity is equal to 0 $\forall x, p$. Thus, we obtain:

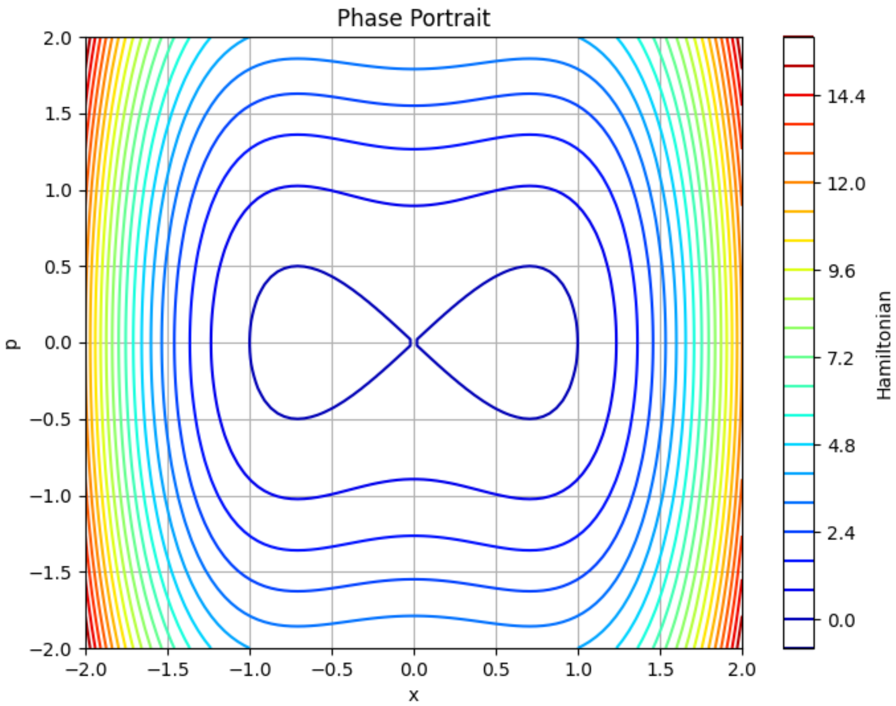
$$\frac{dH}{dt} = \frac{dH}{dp} * \dot{p} + \frac{dH}{dx} * \dot{x}$$

$$-\dot{x} * \dot{p} + \dot{p} * \dot{x} = 0$$

Phase Portrait of the System

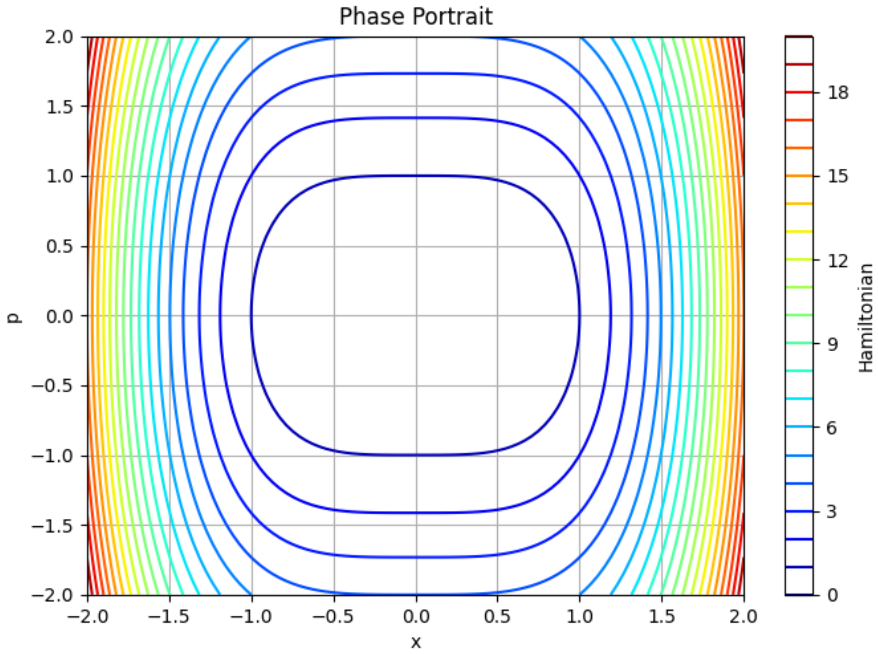
The phase portrait of the system can be obtained by studying the potential (see ”Stability Study through Potential Analysis”). Since varying a yields qualitatively different values, we study the system with a¡0, a=0 and a¿0. Additionally, we can observe from the energy graph that as a varies in the intervals $(-\infty, 0)$ e $(0, \infty)$, the graph does not qualitatively change. Thus, we can take 3 different values of a to distinguish each class qualitatively.

For a=-1:

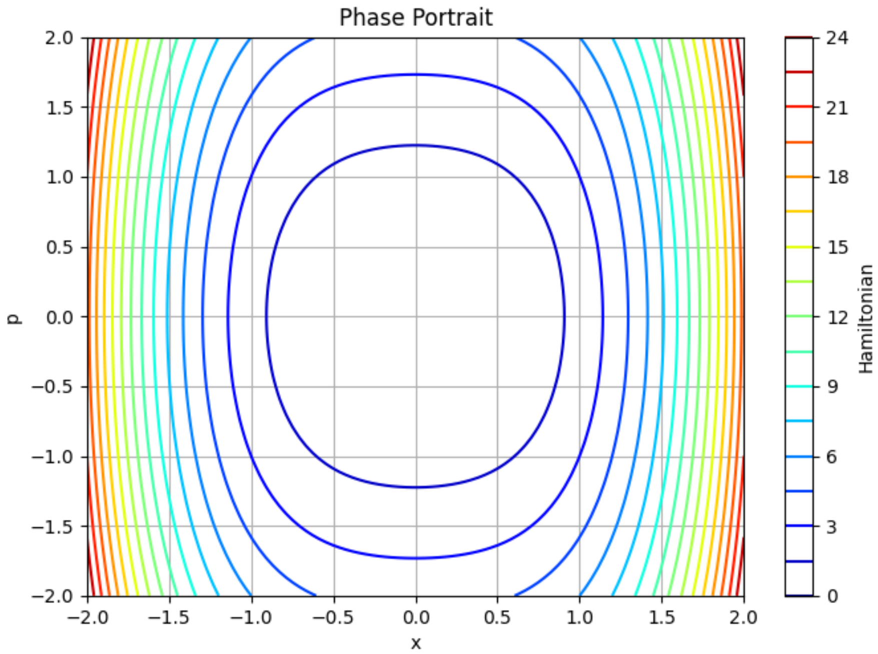


Phase Portrait
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
p
−2.0 −1.5 −1.0 −0.5 0.0 0.5 1.0 1.5 2.0
x
14.4
12.0
9.6
7.2
4.8
2.4
0.0
Hamiltonian

For a=0:



For a=1:



The lines represented are system orbits for various initial conditions and are also the level curves of the Hamiltonian.

Fixed Points and Stability

To evaluate fixed points, we need to set

$$\begin{cases} \dot{x} = & 2p = 0 \\ \dot{p} = & -2ax - 4x^3 = 0 \end{cases}$$

From which we obtain points of the form (p,x):
per a<0:

$$(0,0) \tag{1}$$

$$(0, \sqrt{\frac{-a}{2}}) \tag{2}$$

$$(0, -\sqrt{\frac{-a}{2}}) \tag{3}$$

per a≥0:

$$(0,0) \tag{4}$$

Stability

Stability Analysis through Jacobian Matrix Study

To assess the stability of fixed points, we need to find the Jacobian matrix defined as:

$$J = \begin{pmatrix} \frac{\partial f(x,p)}{\partial x} & \frac{\partial f(x,p)}{\partial p} \\ \frac{\partial g(x,p)}{\partial x} & \frac{\partial g(x,p)}{\partial p} \end{pmatrix}$$

Where

$$f(x,p) = 2p$$

$$g(x,p) = -2ax - 4x^3$$

And then evaluate the trace and determinant of the matrix.

1. For a< 0:

$$J = \begin{pmatrix} 0 & 2 \\ -2a - 12x^2 & 0 \end{pmatrix}$$

   - Evaluated at (1), we obtain $\tau = 0$ e $\Delta = 4a < 0$ from which we conclude that it is a saddle point with eigenvectors $v_1 = (-\sqrt{\frac{1}{-a}}, 1), v_2 = (\sqrt{\frac{1}{-a}}, 1)$.
   - Evaluated at (2), we obtain $\tau = 0$ and $\Delta = -a + 3|-a| > 0$, from which we conclude that it is a center with eigenvectors $v_1 = (i\sqrt{\frac{2}{3a}}, 1)$, $v_2 = (i\sqrt{\frac{2}{3a}}, 1)$.
   - Evaluated at (3), we obtain $\tau = 0$ and $\Delta = -a - 3|-a| > 0$, from which we conclude that it is a center with eigenvectors $v_1 = (i\sqrt{\frac{2}{3a}}, 1)$, $v_2 = (-i\sqrt{\frac{2}{3a}}, 1)$.

2. For a=0:

$$J = \begin{pmatrix} 0 & 2 \\ -12x^2 & 0 \end{pmatrix}$$

- valuated at (4), we obtain $\tau = 0$ e $\Delta = 0$ from which we cannot conclude anything, so we defer this study further.

3. For a>0:

$$J = \begin{pmatrix} 0 & 2 \\ -2a - 12x^2 & 0 \end{pmatrix}$$

- Evaluated at (5), we obtain $\tau = 0$ and $\Delta = \sqrt{-4a} < 0$ from which we conclude that it is an unstable node with eigenvectors: $v_1 = (\frac{i}{\sqrt{a}}, 1)$, $v_2 = (-\frac{i}{\sqrt{a}}, 1)$

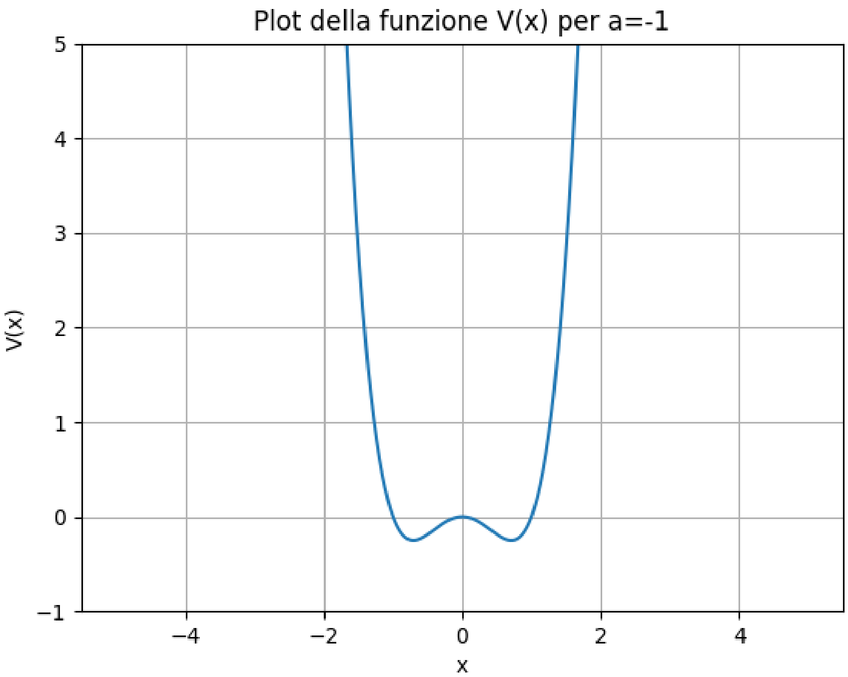
Stability Analysis through Potential Study

Since we know that

$$H = V(x) + T$$

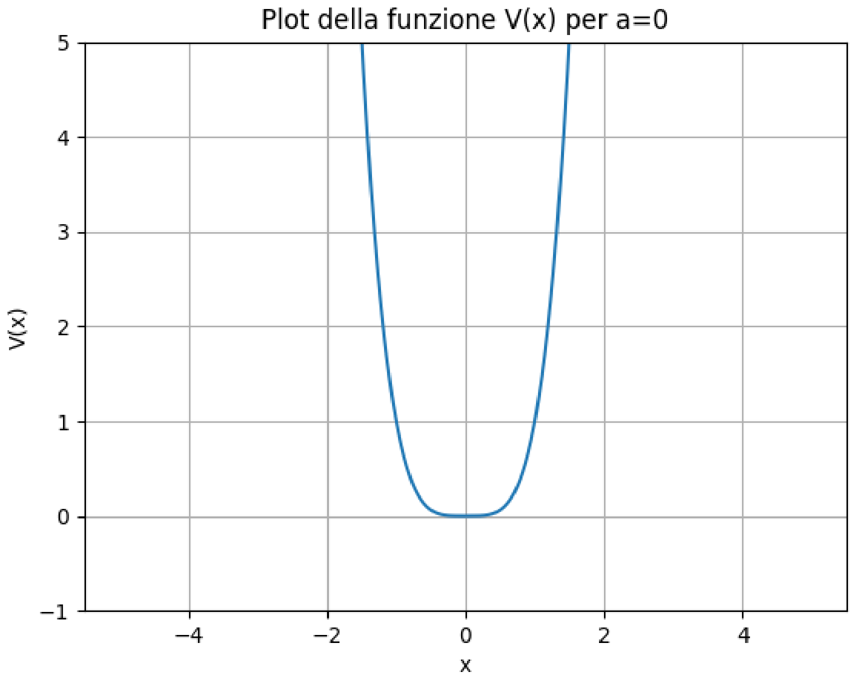
with T=$\frac{p^2}{2m}$, it follows that V(x)=$ax^2 + x^4$, by plotting V(x) as function of x:

For a=-1 V(x)=$-x^2 + x^4$:



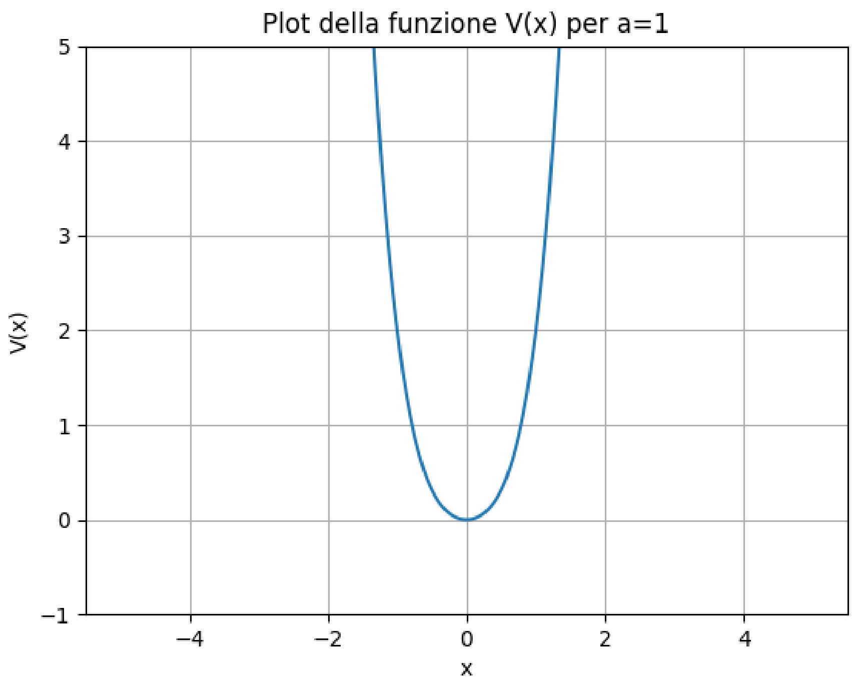
From the graph, we can observe that point (1) is an unstable fixed point as it is a relative maximum of the function, while points (2) and (3), being relative minima, will be unstable fixed points. Note that the phase portrait graph can be obtained from this graph, as we can observe that points (2) and (3) form centers, while point (1) forms a saddle point, and thus, we can trace the level curves based on the energy we want to represent.

For a=0 V(x)= x^4:



In this case, we can observe that point (4) is a stable fixed point, particularly a center. Moreover, we can affirm that the level curves tend to "linger" longer on points of maximum and minimum since there is only a contribution to the fourth power, so in the vicinity of 0, the function remains very flat.

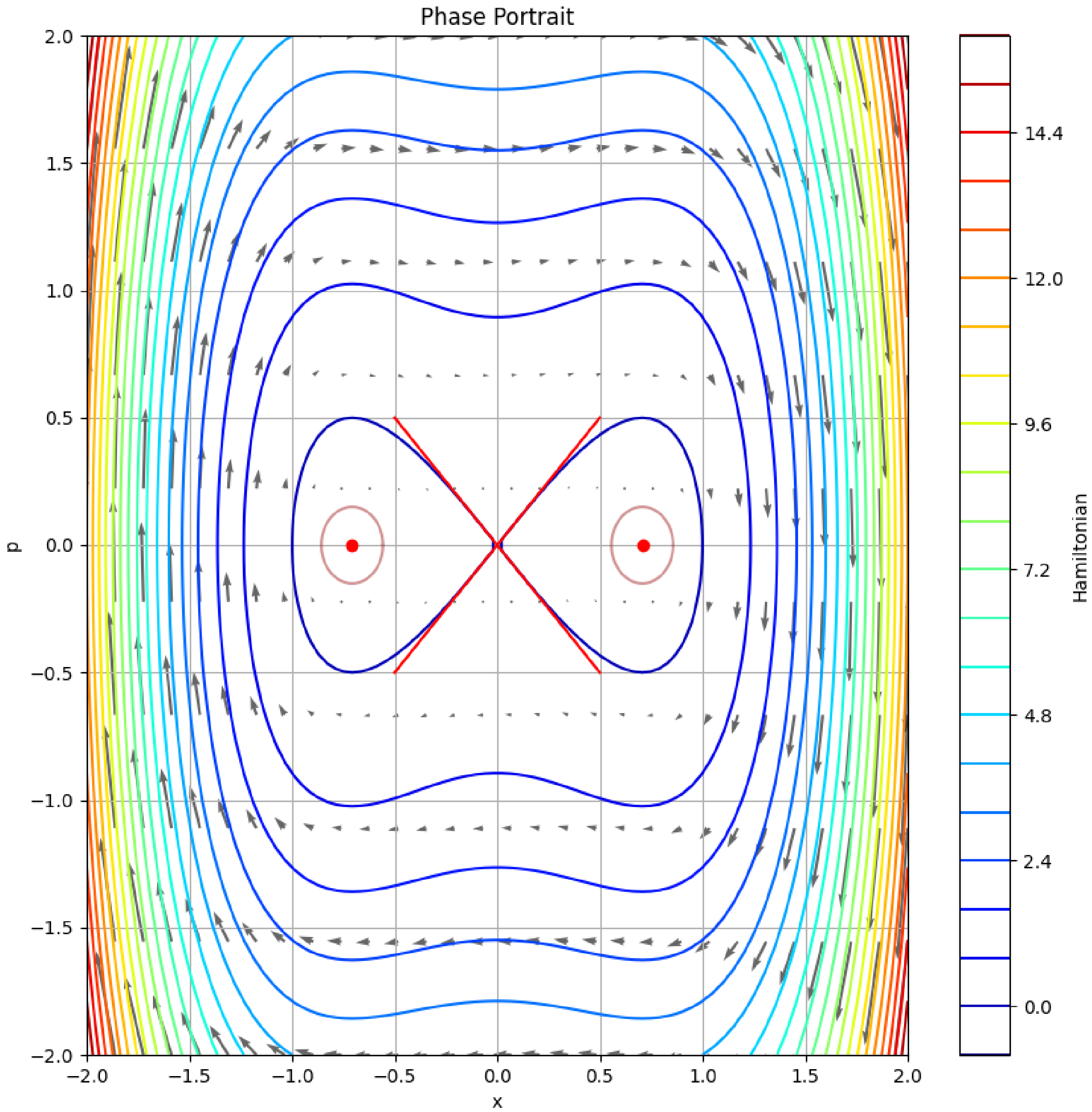
Per a=1 otteniamo V(x)= $x^2 + x^4$:



From the graph, we again obtain that point (5) is a stable fixed point, also a center, but since there is also a quadratic contribution, it will reach 0 less rapidly, so the ellipses will be more concentric.

Separatrices and Dynamical Regimes

For the dynamical regimes as a varies, refer to the phase portrait. The separatrices for $a \geq 0$ are the ellipses found through potential analysis. For a$<$ 0 however, it presents two separatrices found through the study of the Jacobian at point 1, thus the eigenvectors of the fixed point (0,0). . Thus, for a Hamiltonian less than $|\sqrt{\frac{-a}{2}}|$ the level curves will be ellipses around the two fixed points, for $|H| = |\sqrt{\frac{-a}{2}}|$, the level curve will follow for $|x| < \sqrt{\frac{-a}{2}}|$ the trajectory of a straight line with equation derived from the eigenvectors, while for$|x| > \sqrt{\frac{-a}{2}}|$, it will be an ellipse. Finally, for $H > \sqrt{\frac{-a}{2}}|$ the level curves will follow a trend dictated by the previously found curves. Thus, the graph will be:


Phase Portrait
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
p
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
x
14.4
12.0
9.6
7.2
4.8
2.4
0.0
Hamiltonian