UNIVERSITY OF TRIESTE

Department of Mathematics, Informatics and Geosciences

Bachelor Degree in
Artificial Intelligence and Data Analytics

Summability of Fourier Series and convergence to nowhere differentiable functions

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What is rational is real;
and what is real is rational

- Georg Wilhelm Friedrich Hegel

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Chapter 1

Fourier series

Introduction

Fourier series arise from the need to represent a periodic function as a linear combination of trigonometric functions, in order to obtain a reasonably accurate approximation of the original function. This method was pioneered by Joseph Fourier while investigating solutions to the heat equation, with later applications to wave phenomena like vibrating strings.[1] In this context, the objective is to examine the movement of a string anchored at its endpoints and allowed to oscillate freely. The wave equation describes how waves propagate through a medium. For a vibrating string, the wave equation takes the form:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1.0.1}$$

Where: $u(x,t)$ represents the displacement of the string at position x and time t, c is the velocity of the motion, $\frac{\partial^2 u}{\partial t^2}$ represents the acceleration of the string with respect to time and $\frac{\partial^2 u}{\partial x^2}$ represents the curvature of the string with respect to position. One way to solve the eq. (1.0.1) is by using the superposition of standing waves which is a composite of dual waves propagating in opposing directions, both characterized by identical amplitude and frequency. In other words we search special solution of the form $u(x,t) = \phi(x)\psi(t)$ where $\phi(x)$ represent the wave at time $t = 0$ and $\psi(t)$ is an amplifying factor depending on t. On this way the eq. (1.0.1), with $c = 1$, becomes

$$\phi(x)\psi''(t) = \phi''(x)\psi(t)$$

and the solution are of the form

$$\psi(t) = A\cos(mt) + B\sin(mt), \quad \phi(x) = \tilde{A}\cos(mx) + \tilde{B}\sin(mx)$$

Where m is a constant. Suppose that the string is attached at $x = 0$ and $x = \pi$ we obtain that $\tilde{A} = 0$ and if $\tilde{B} \neq 0$ then m must be an integer.

- if $m = 0$ then $u(x,t) = 0$ for all x, t
- if $m \leq -1$ we can rename the constants and reduce to $m \geq 1$
- if $m \geq 1$ then for each m the function

  $$u_m(x,t) = (A_m \cos(mt) + B_m \sin(mt)) \sin(mx)$$

  which is a standing wave, is a solution for eq. (1.0.1).

Noticing that the wave equation is linear, we can construct multiple solutions by taking a linear combination of the standing waves u_m and we obtain

$$u(x,t) = \sum_{m=1}^{\infty} (A_m \cos(mt) + B_m \sin(mt)) \sin(mx)$$

If we set $f(x) = u(x,0)$ then f is defined in $[0,\pi]$ and

$$f(x) = \sum_{m=0}^{\infty} A_m \sin(mx)$$

Given that the function $u(x,0)$ can take any form, we may question whether any function can be expressed as a linear combination of sines and cosine. In particular we want find the coefficients A_m that make, a generic $f(x)$, equal to this sum, so we need to explore the conditions under which this representation holds true for any given function f. [1]

1.1 L^p space

The motivation behind introducing Lebesgue measure and Lebesgue integral lies in the fact that it allows us to handle a wider range of functions that are not necessarily continuous or finite at certain points, by extending the concept of convergence and integration.

Definition 1.1. (Lebesgue measure zero)
Let $d \geq 1$ and $T \subset \mathbb{R}^d$, we say that T has Lebesgue measure zero if for every $\epsilon > 0$ there exists a sequence of rectangles $(R_n)_{n\in\mathbb{N}}$ such that

$$T \subset \bigcup_{n\in\mathbb{N}} R_n \quad \text{and} \quad \sum_{n\in\mathbb{N}} |R_n| < \epsilon$$

[1] Stein, E. M. (2009). *Princeton Lectures in Analysis: Fourier Analysis*. Princeton University Press, pp. 1-14.

Definition 1.2. (Property satisfied almost everywhere)
a property (or predicate) p is said to be satisfied almost everywhere ($a.e.$) in $E \subset \mathbb{R}^d$ if $p(x)$ is satisfied for every $x \in E' \subset E$ such that $|E \backslash E'| = 0$

Definition 1.3. ($a.e.$ pointwise convergence)
Let $d \geq 1$ and let $(f_n)_{n\in\mathbb{N}}$ a sequence of functions $f_n : E \subset \mathbb{R}^d \to \mathbb{C}$ and let $f : E \subset \mathbb{R}^d \to \mathbb{C}$ we say that f_n converge to f $a.e.$ if

$$\lim_{n\to\infty} f_n(x) = f(x) \quad a.e. \text{ in } E$$

Definition 1.4. (Indicator function)
The indicator function of a subset A of a set B is a function $\mathbb{1}_A : B \to \{0,1\}$ defined as

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Definition 1.5. (Simple function)
Let $d \geq 1, E \subset \mathbb{R}^d$ and $\mathbb{X} \in \{\mathbb{R}, \mathbb{C}\}$, we define with $\mathrm{SF}(E, \mathbb{X})$ the set of simple function from E to $\mathbb{X}$ the functions of the form

$$\phi(x) = \sum_{n=1}^{N} \alpha_n \mathbb{1}_{R_n}(x), \quad \alpha_n \in \mathbb{X}$$

With $(R_n)_n \subset E$ rectangles disjointed.

Definition 1.6. (Lebesgue integrable functions)
Let $d \geq 1$, we say that $f : \mathbb{R}^d \to \mathbb{C}$ is Lebesgue integrable in $\mathbb{R}^d$ if there exists a sequence $(\phi_n)_{n\in\mathbb{N}} \in \mathrm{SF}(\mathbb{R}^d; \mathbb{C})$ such that

1. $\phi_n(x) \xrightarrow{n\to\infty} f(x)$ L-a.e. in $\mathbb{R}^d$;
2. $\int_{\mathbb{R}^d} |\phi_n - \phi_m| \xrightarrow{n,m\to\infty} 0$ (integral Cauchy condition);

In that case, we define

$$\int_{\mathbb{R}^d} f := \lim_n \int_{\mathbb{R}^d} \phi_n.$$

Definition 1.7. (L-measurable functions in $\mathbb{R}^d$)
We say that $f : \mathbb{R}^d \to \mathbb{C}$ is L-measurable if there exists a sequence $(\phi_n)_{n\in\mathbb{N}} \subset \mathrm{SF}(\mathbb{R}^d; \mathbb{C})$ such that $\phi_n(x) \xrightarrow{n\to\infty} f(x)$ L-a.e.

Definition 1.8. (L-measurable sets in $\mathbb{R}^d$)
We say that $E \subset \mathbb{R}^d$ is L-measurable if the function $\mathbb{1}_E$ is L-measurable. In this case, we have $|E| := \int_{\mathbb{R}^d} \mathbb{1}_E$.

Definition 1.9. (L-integrable functions on subsets of $\mathbb{R}^d$)
Let $f : E \subset \mathbb{R}^d \to \mathbb{C}$ with E L-measurable. We say that f is L-integrable on E if the extension

$$f_0(x) := \begin{cases} f(x) & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases} \tag{1.1.1}$$

is L-integrable

Definition 1.10. (Lebesgue spaces)
Let $d \geq 1, p \in [1, \infty)$ and let $E \subset \mathbb{R}^d$ a set of Lebesgue measurable. We define

$$L^p(E;\mathbb{C}) := \left\{ f : E \to \mathbb{C} \quad \text{L-measurable} \quad \text{s.t.} \quad \left(\int_E |f|^p \right)^{\frac{1}{p}} < \infty \right\}$$

The space $L^p(E;\mathbb{C})$ is endowed with the norm

$$\|f\|_{L^p(E;\mathbb{C})} := \left(\int_E |f|^p \right)^{\frac{1}{p}}$$

Is a Banach space

Definition 1.11. (Space L^∞)
With the same hypothesis in definition 1.10 we define

$$L^\infty(E;\mathbb{C}) := \left\{ f : E \to \mathbb{C} \quad \text{L-measurable} \middle| \exists K > 0 \quad \text{s.t.} \quad |f(x)| \leq K \text{ a.e. in } E \right\}$$

The space $L^\infty(E;\mathbb{C})$ is endowed with the norm

$$\|f\|_{L^\infty(E;\mathbb{C})} := \inf \left\{ K > 0 \middle| |f(x)| \leq K \text{ a.e. in E} \right\} := \text{ess sup}_{x \in E} |f(x)|$$

Is a Banach space

Remark. We observe that, only in the case $p = 2$, we can define the following:

$$\langle f, g \rangle := \langle f, g \rangle_{L^2(E;\mathbb{C})} := \int_E f(x)\overline{g(x)}dx$$

that is a scalar product since:

1. **Positivity:**
$$\langle f, f \rangle = \int_E f(x)\overline{f(x)}dx = \int_E |f(x)|^2 dx \geq 0$$
If $\int_E |f(x)|^2 = 0$ Then $f(x) = 0$.

2. **Conjugate symmetry:** We notice that $f(x), g(x)$ are of the form $u(x) + i\mu(x)$,with u, μ functions of real variables, then $\overline{\overline{f(x)}} = \overline{\overline{u(x) + i\mu(x)}} = \overline{u(x) - i\mu(x)} = u(x) + i\mu(x) = f(x)$. We also note the fact that $\overline{f(x) * g(x)} = \overline{f(x)} * \overline{g(x)}$[2] then

$$\begin{aligned}\overline{\langle g, f\rangle} &= \overline{\int_E g(x)\overline{f(x)}dx} \\ &= \int_E \overline{\overline{f(x)}g(x)}dx \\ &= \langle f, g\rangle\end{aligned}$$

3. **Linearity:**

$$\begin{aligned}\langle f + \lambda g, z\rangle &= \int_E (f(x) + \lambda g(x))\overline{z(x)}dx \\ &= \int_E f(x)\overline{z(x)} + \lambda g(x)\overline{z(x)}dx \\ &= \int_E f(x)\overline{z(x)} + \lambda \int_E g(x)\overline{z(x)}dx \\ &= \langle f, g\rangle + \lambda \langle g, z\rangle .\end{aligned}$$

The norm induced by the scalar product is:

$$\|f\|_{L^2} := \sqrt{\langle f, f\rangle} = \left(\int_E |f|^2\right)^{\frac{1}{2}}$$

Observing that this space is complete, we conclude that the space $L^2(E; C)$ is a Hilbert space, as it is equipped with a norm induced by an inner product.

Lemma 1.1. (Hölder's inequality) Let $p, p' \in [1, \infty]$ s.t. $\frac{1}{p} + \frac{1}{p'} = 1$, then

$$\langle f, g\rangle_{L^2(E;\mathbb{C})} \leq \|f\|_{L^p(E;\mathbb{C})}\|g\|_{L^{p'}(E;\mathbb{C})}$$

Lemma 1.2. Let $1 \leq p \leq q \leq \infty$ and let E L-measurable and with finite measure, then $L^q(E; \mathbb{C}) \subset L^p(E; \mathbb{C})$ and we have that

$$\|f\|_{L^p}(E, \mathbb{C}) \leq |E|^{\frac{q-p}{pq}}\|f\|_{L^q(E;\mathbb{C})} \tag{1.1.2}$$

[2] The operator * is used solely in this case to highlight the difference between the two expressions, from now on, it will be omitted as previously done

Proof. First, observe that $\|f\|^p_{L^p(E;\mathbb{C})} = \langle |f|^p, 1\rangle_{L^1(E;\mathbb{C})}$. Now, we apply Hölder's inequality with exponents $r = \frac{q}{p}$ and $r' = \frac{q}{q-p}$. This yields:

$$\langle |f|^p, 1\rangle_{L^1(E;\mathbb{C})} \leq \left(\int_E |f|^{rp} dL\right)^{\frac{1}{r}} \left(\int_E 1^{r'} dL\right)^{\frac{1}{r'}}$$

We notice that $\left(\int_E 1^{r'} dL\right)^{\frac{1}{r'}} = |E|^{\frac{1}{r'}} = |E|^{\frac{q-p}{q}}$. Moreover, by the definition of r, $\left(\int_E |f|^{pr} dL\right)^{\frac{1}{r}} = \left(\int_E |f|^q dL\right)^{\frac{1}{q}} = \|f\|_{L^q(E;\mathbb{C})}$. Thus, we obtain:

$$\|f\|^p_{L^p(E;\mathbb{C})} \leq \|f\|_{L^q(E;\mathbb{C})} \cdot |E|^{\frac{q-p}{q}}$$

Raising both sides to the power of $\frac{1}{p}$, we arrive at eq. (1.1.2) □

We will now demonstrate a typical example illustrating how a function can be Lebesgue-integrable but not Riemann-integrable:

Lemma 1.3. Let

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{if } x \in ([0,1] \setminus \mathbb{Q}) \end{cases}$$

then f is not Riemann-integrable but is Lebesgue-integrable.

Proof. Consider the set of points $I_i := [\frac{i-1}{N}, \frac{i}{N}]$ for $i = 1, 2, ..., N$. Note that these intervals cover the interval $[0,1]$ since $[0,1] = \bigcup_{i=1}^N I_i$. We notice that for each interval I_i, there will be at least one rational point and at least one irrational point. Now, consider the:

$$\sum_{i=1}^N |I_i| \sup_{x \in I_i} f(x)$$

So $\sup_{x\in I_i} f(x) = 1$ for each I_i, and thus the upper sum will be 1 . Similarly:

$$\sum_{i=1}^N |I_i| \inf_{x \in I_i} f(x)$$

Since $\inf_{x\in I_i} f(x) = 0$ for each I_i the lower sum will be 0. So the function is not Riemann-integrable. However since $\mathbb{Q}$ is a set of measure zero according to Lebesgue measure, $\mathbb{Q} \cap [0,1]$ also has measure zero. Consequently, $f(x) = 0$ almost everywhere in $[0,1]$, making $f(x)$ Lebesgue-integrable. □

1.2 Fourier Coefficients

Definition 1.12. (Trigonometric series) We define as Trigonometric series an expression of the form:

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos(n\omega x)+b_n\sin(n\omega x)) \tag{1.2.1}$$

$$\sum_{n=-\infty}^{\infty} c_n e^{in\omega x} \tag{1.2.2}$$

with $a_0, a_n, b_n \in \mathbb{R}$, $c_n \in \mathbb{C}$ and $\omega := \frac{2\pi}{T}$ is the angular frequency where T is the period

Definition 1.13. (N-th truncated)
Let $N \in \mathbb{N}$, we define as N-th truncated of eq. (1.2.1) the trigonometric polynomial

$$S_N(x) := \frac{a_0}{2}+\sum_{n=1}^{N}(a_n\cos(n\omega x)+b_n\sin(n\omega x)) \tag{1.2.3}$$

similarly, for eq. (1.2.2) we define

$$S_N(x) := \sum_{n=-N}^{n=N} c_n e^{in\omega x} \tag{1.2.4}$$

Remark. We notice that we can relate to eq. (1.2.2) starting from eq. (1.2.1) (and vice versa) through Euler's formulas

$$a_n\cos(n\omega x)+b_n\sin(n\omega x)=a_n\frac{e^{in\omega x}+e^{-in\omega x}}{2}+b_n\frac{e^{in\omega x}-e^{-in\omega x}}{2i}$$

and by defining $c_n := \frac{a_n-ib_n}{2}e^{in\omega x}$ and $c_{-n} := \frac{a_n+ib_n}{2}e^{in\omega x}$, if $c_{-n}=\overline{c_n}$ ($n \in \mathbb{N}$), then we obtain the desired result, since we require the coefficients c_n to be real. To go from eq. (1.2.1) to eq. (1.2.2) we can set

$$c_0 := \frac{a}{2}; \quad c_n := \frac{1}{2}(a_n-ib_n); \quad c_{-n} := \frac{1}{2}(a_n+ib_n) \quad (n \in \mathbb{N})$$

In this way eq. (1.2.1), takes the form of eq. (1.2.2)

Lemma 1.4. (Integration term by term)
Let $f_n : [a,b] \subset \mathbb{R} \to \mathbb{C}$ be Riemann integrable on $[a,b]$ for all n. If $\sum_{n\in\mathbb{N}} f_n$ converges uniformly to f on $[a,b]$ then f is Riemann Integrable on $[a,b]$ and

$$\int_a^b f(x)dx = \int_a^b \sum_{n\in\mathbb{N}} f_n(x)dx = \sum_{n\in\mathbb{N}}\int_a^b f_n(x)dx$$

Proposition 1.1. Let a trigonometric series as eq. (1.2.1) and suppose it converges uniformly to a target function f, then it follows that:

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(x)e^{-in\omega x}dx$$

Proof. Since we suppose the uniform convergence, we have

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{in\omega x}$$

We now fix $k \in \mathbb{Z}$ and calculate the following

$$\langle f, e^{ik\omega\bullet}\rangle = \int_{-T/2}^{T/2} f(x)\overline{e^{ik\omega x}}dx = \int_{-T/2}^{T/2}\sum_{n\in\mathbb{Z}} c_n e^{i(n-k)\omega x}dx$$

Since we suppose the uniform convergence, for lemma 1.4 we have

$$\int_{-T/2}^{T/2}\sum_{n\in\mathbb{Z}} e^{i(n-k)\omega x}dx = \sum_{n\in\mathbb{Z}} c_n T\delta_{nk} = Tc_k$$

We substitute the last equation into the first one, yielding

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} f(x)e^{-ik\omega x}dx$$

□

through the previously made observation, we deduce that

$$a_n = c_n + c_{-n} = \frac{1}{T}\int_{-T/2}^{T/2} f(x)(e^{-in\omega x} + e^{in\omega x})dx = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos(n\omega x)dx$$

similarly

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(x)dx$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin(n\omega x)dx$$

Some basic properties of Fourier coefficients

1. (Translation in x) Let $x_0 \in \mathbb{R}$ and $g(x) := f(x - x_0)$ then

$$\begin{aligned}
c_n(g(x)) &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} g(x)e^{-in\omega x}\,dx \\
&= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x-x_0)e^{-in\omega(x-x_0)}\,dx \\
&= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(y)e^{-in\omega(y+x_0)}\,dy \\
&= e^{-in\omega x_0}c_n(f(x))
\end{aligned}$$

   Where we have used the fact that y varies over a time interval T.

2. (Linearity) Let $f(x), g(x) \in L^1(T;\mathbb{C})$, then the function $z(x) := \alpha f(x) + \beta g(x)$, with $\alpha, \beta \in \mathbb{R}$ is a periodic function, with period T, in addition $c_n(z(x)) = \alpha c_n(f(x)) + \beta c_n(g(x))$

   Proof. The function $z(x)$ is periodic with period T since $z(x+T) = \alpha f(x+T) + \beta g(x+T) = \alpha f(x) + \beta g(x)$. Since f, g admit the Fourier series, $c_n(f), c_n(g)$ are defined as proposition 1.1, then

$$\begin{aligned}
c_n(z(x)) &= c_n(\alpha f(x) + \beta g(x)) \\
&= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} (\alpha f(x) + \beta g(x))e^{-in\omega x}dx \\
&= \alpha\left(\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)e^{-in\omega x}dx\right) + \beta\left(\int_{-\frac{T}{2}}^{\frac{T}{2}} g(x)e^{-in\omega x}dx\right) \\
&= \alpha c_n(f(x)) + \beta c_n(g(x))
\end{aligned}$$

   □

3. (Riemann–Lebesgue lemma) Let $f \in L^1(T;\mathbb{C})$ and $c_n = c_n(f)$ then $c_n(f) \xrightarrow{|n|\to\infty} 0$

4. Let $k \in L^1(T;\mathbb{C})$ differentiable k times and such that $f^{(j)} \in L^1(T;\mathbb{C})$ for all $j = 0, \ldots, k$ then

$$c_n\left(f^{(j)}\right) = (i\omega n)^j c_n(f) \quad \forall n \in \mathbb{Z}, j \in \{0, ..., k\}$$

Example 1.1. Let

$$f(x) = \begin{cases} -1 & \text{if} \quad -\pi \leq x < 0 \\ 1 & \text{if} \quad 0 \leq x < \pi \end{cases}$$

with f periodic with period 2π then the coefficient as eq. (1.2.2) are well define since

$$|c_n| \leq \frac{1}{2\pi}\int_{-\pi}^{0} |-1|dx + \frac{1}{2\pi}\int_{0}^{\pi} |1|dx < \infty$$

then we calculate the coefficients (we find it advantageous to utilize the form eq. (1.2.1) and utilize the fact that $f(x)$ is an odd function)

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin(nx)dx = 2\frac{(1-\cos(n\pi))}{n\pi}$$

then

$$b_n = \begin{cases} 0 & \text{if n is even} \\ \frac{4}{n\pi} & \text{if n is odd} \end{cases}$$

then we conclude writing the Fourier Series

$$\sum_{n=1}^{\infty} \frac{4}{(2n+1)\pi}\sin(nx)$$

However, we cannot conclude at this point that such series converges to the original function.

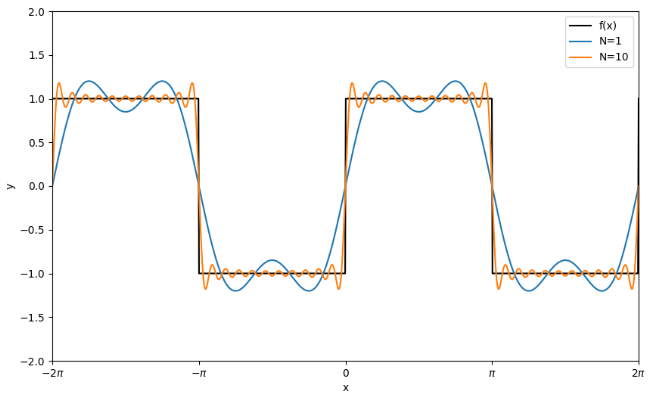

Figure 1.1: Plot of the N-th truncated series and $f(x)$

We want to investigate under which conditions the Fourier series converges to the target function, that is, when $S_n(f) \xrightarrow{n\to\infty} f$. This analysis is called Fourier synthesis

1.3 Convergence

Definition 1.14. (Pointwise convergence)
Let $\mathbb{X} \in \{\mathbb{R}, \mathbb{C}\}$, E a subset of $\mathbb{X}$ and $(f_n)_n$, a sequence of functions, we say the f_n converge pointwise on f, where, $f : E \to \mathbb{X}$, if for all $x \in E$

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \forall x \in \mathbb{R}$$

Definition 1.15. (Uniform convergence)
Let $\mathbb{X} \in \{\mathbb{R}, \mathbb{C}\}$, E a subset of $\mathbb{X}$ and $(f_n)_n$, a sequence of functions, we say the f_n converge uniformly on f, where, $f : E \to \mathbb{X}$, if for all $x \in E$

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} (f_n(x) - f(x)) = 0 \quad \forall x \in \mathbb{R}$$

It is noted that, in the case of Fourier series, the problem of rewriting a function f persists only if the Fourier coefficients are well defined. In this case, we can ask when $S_N(x)$, defined as eq. (1.2.3) or eq. (1.2.4), converges pointwise or uniformly to the original function.

Remark. We notice that uniform convergence implies pointwise convergence, but the reverse is not true. We can see it with the following example: Let $E =]0, 1[$ and $f_n(x) = x^n$ then

$$\lim_{n\to\infty} f_n(x) = x^n = 0 \quad \forall x \in E$$

so the sequence f_n converges pointwise to $f(x) = 0$. However, if we consider the following:

$$\sup_{x\in E} |f_n(x) - f(x)| = \sup_{E} x^n = 1$$

Therefore f_n does not converge uniformly to f

Lemma 1.5. (Weierstrass M-test)
Let (f_n) a sequence of real- or complex-valued functions defined on a set A, and that there is a sequence of non-negative numbers (M_n) satisfying the conditions

$$|f_n(x)| \leq M_n \quad \text{for all } n \geq 1 \text{ and all } x \in A$$

and

$$\sum_{n=1}^{\infty} M_n < \infty$$

Then the series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges uniformly on A.

Lemma 1.6. If $f_k : I \to \mathbb{R}$ is a continuous function for every $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly to $S(x)$ on I, then S is a continuous function on I.

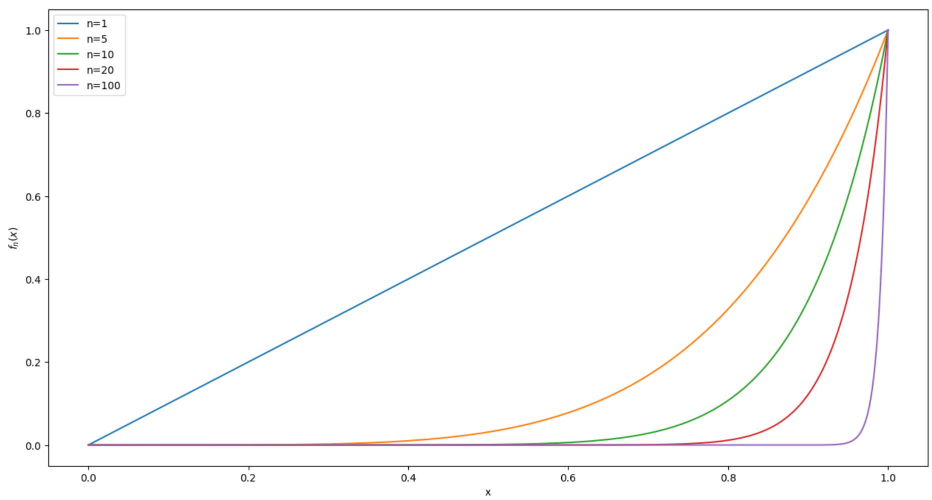

Figure 1.2: Plot of the function $f_n(x) = x^n$

Chapter 2

Pointwise convergence of Fourier series

Theorem 2.1. (Dirichelet-Weierstrass)
Let $f \in L^1(T;\mathbb{C})$ and $x_0 \in \mathbb{R}$. Suppose that the four unilateral limits exist, and are finite:

$$\lim_{x \to x_0^{\pm}} f(x) =: f\left(x_0^{\pm}\right), \quad \lim_{x \to x_0^{\pm}} \frac{f(x) - f\left(x_0^{\pm}\right)}{x - x_0} =: f'\left(x_0^{\pm}\right) \tag{2.0.1}$$

then

$$\lim_{n \to \infty} S_n(f)(x_0) = \frac{f\left(x_0^+\right) + f\left(x_0^-\right)}{2}$$

Remark. If we take in consideration the aforementioned example 1.1, we observe that $f(x) \in L^1([-\pi,\pi];[-1,1])$ and the limits as eq. (2.0.1) exist for all $x \in [-\pi,\pi]$, so we can use the theorem 2.1 and notice that for every $x_0 \neq 0$ we have $f\left(x_0^-\right) = f\left(x_0^+\right)$. Therefore

$$\lim_{n \to \infty} S_n(f)(x_0) = f(x_0)$$

When $x_0 = 0$ we have $f\left(0^-\right) = -1$ and $f\left(0^+\right) = 1$ so

$$\lim_{n \to \infty} S_n(f)(0) = \frac{f\left(0^-\right) + f\left(0^+\right)}{2} = 0$$

Therefore, the Fourier series does not converge to the original function for $x = 0$. We can see it from fig. 1.1

From theorem 2.1, we can observe that if we consider a function f periodic with period T, and $f \in C^1([0;T];\mathbb{C})$,[1] for notational semplicity we write $C^k(T) :=$

[1] The function f is said to be of differentiability class $C^k(E;\mathbb{C})$ if in every point of E the derivatives $f', f'', \ldots, f^{(k)}$ exist and are continuous, if $k = 0$ then f is only continuous.

$C^k([0;T],\mathbb{C})$, then

$$\lim_{n\to\infty} S_n(f)(x_0) = f(x_0) \quad \forall x_0 \in T$$

Furthermore, continuity of the function f is not even required, but rather unilateral differentiability is. Therefore, we can state that differentiability at a point is a sufficient condition for the pointwise convergence of $S_n(f)$ to f at that point. Thus, we wonder if by relaxing the conditions of the theorem and only requiring continuity, we can still assert that $S_n(f)$ converges pointwise to the original function, we will verify that this is not true through the following theorem.

Theorem 2.2. (Du Boys Reymond) There is a $f \in C^0(T)$ s.t.

$$\sup_n |S_n(f)(0)| = \infty$$

In particular, the Fourier series of f diverges at $x = 0$

Definition 2.1. Let $n \in \mathbb{N}$, then the function

$$D_n(x) = \sum_{k=-n}^{k} e^{ikx}$$

is called Dirichlet Kernel of order n

Lemma 2.1. for each $n \in \mathbb{N}$

$$D_n(x) = \sum_{k=-n}^{n} e^{ikx} = \frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)}$$

Proof. Since D_n is a trigonometric series we have

$$\begin{aligned} D_n(t) &= e^{-int}\left(\frac{e^{i(2n+1)t}-1}{e^{it}-1}\right) \\ &= \frac{e^{i(n+1/2)t}-e^{-i(n+1/2)t}}{\left(e^{it/2}-e^{-it/2}\right)} \\ &= \frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)} \end{aligned}$$

Where we use the Euler's identity $e^{ix} = \cos(x) + i\sin(x)$ [2] □

Lemma 2.2. Let $T = [0, 2\pi]$ and $f \in L^1(T;\mathbb{C})$, then

$$S_n(f)(x) = \frac{1}{2}\int_0^{2\pi} f(x-t)D_n(t)dt$$

Proof. we notice that $\omega = \frac{2\pi}{T} = 1$ and

$$\begin{aligned} S_n(f)(x) &= \sum_{k=-n}^{n} c_k(f)e^{ikx} \\ &= \sum_{k=-n}^{n} \left(\frac{1}{2\pi} \int_0^{2\pi} f(t)e^{-ikt}\,dt \right) e^{ikx} \end{aligned}$$

Since we know that $D_n = \frac{\sin\left((n+\frac{1}{2})x\right)}{\sin\left(\frac{x}{2}\right)}$ we can use the lemma 1.4 and obtain

$$\begin{aligned} S_n(f)(x) &= \frac{1}{2\pi} \int_0^{2\pi} f(t) \sum_{k=-n}^{n} e^{ik(x-t)}\,dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(x-t) \sum_{k=-n}^{n} e^{ikt}\,dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(x-t)D_n(t)\,dt \end{aligned}$$

□

Lemma 2.3.

$$\int_{-\pi}^{\pi} |D_n(t)| dt \to \infty$$

for $n \to \infty$

Proof. Since we know that $|x| \geq |\sin(x)|$ and $D_n(x)$ is an even function, we can write

$$\int_{-\pi}^{\pi} |D_n(t)|\,dt = 2 \int_0^{2\pi} \left| \frac{\sin((n+1/2)t)}{\sin(t/2)} \right| dt \geq 2 \int_0^{2\pi} \frac{|\sin((n+1/2)t)|}{t/2}\,dt$$

We can perform a change of variables by setting $y = (n+\frac{1}{2})t$ and obtain

$$2 \int_0^{2\pi} \frac{|\sin((n+1/2)t)|}{t/2}\,dt = 4 \int_0^{(n+1/2)\pi} \frac{|\sin y|}{y}\,dy$$

Since $\frac{|\sin y|}{y} > 0$ in $[0, n\pi]$ we can write

$$\begin{aligned}
\int_{-\pi}^{\pi} |D_n(t)| dt \geq & 4 \int_0^{(n+1/2)\pi} \frac{|\sin y|}{y} dy \\
& \geq 4 \int_0^{n\pi} \frac{|\sin y|}{y} \, dy \\
& = 4 \sum_{j=1}^{n} \int_{(j-1)\pi}^{j\pi} \frac{|\sin y|}{y} \, dy \\
& \geq \frac{4}{\pi} \sum_{j=1}^{n} \frac{1}{j} \int_0^{\pi} \sin y \, dy \\
& = \frac{8}{\pi} \sum_{j=1}^{n} \frac{1}{j} \to \infty \quad (m \to \infty)
\end{aligned}$$

Where we use the fact that $\int_{(j-1)\pi}^{j\pi} |\sin(y)| dy = 0$ for all $j = 2, \ldots, n$ $\square$

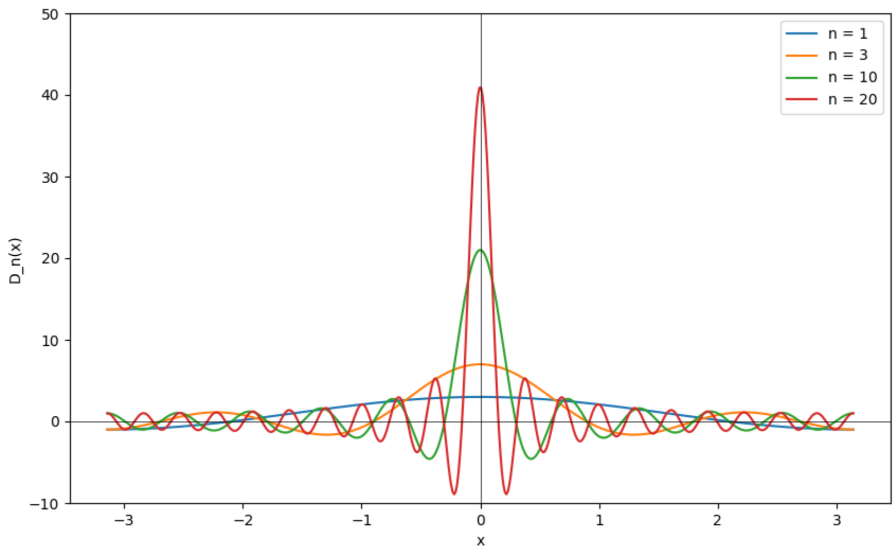

Figure 2.1: Plot of the function $D_n(x)$ for different values of n

Lemma 2.4.

$$|S_n(f)(0)| \leq (2n+1)\|f\|_\infty$$

Proof.

$$|S_n(f)(0)| = \left| \sum_{k=-n}^{n} c_k(f) e^{in\omega x} \right| \leq \sum_{k=-n}^{n} |c_k(f) e^{in\omega x}| \leq \sum_{k=-n}^{n} |c_k(f)| \leq (2n+1)\|f\|_\infty$$

Where

$$\|f\|_\infty := \sup_x |f(x)|$$

□

Lemma 2.5. Let $\epsilon > 0, k > 0$, then there exists $n \geq 1$ and $f \in C^0(T)$ s.t.

$$\|f\|_\infty \leq \epsilon \quad \text{and} \quad |S_n(f)(0)| \geq k$$

Proof. We take

$$f_n(t) := \frac{\epsilon \overline{D_n(-t)}}{1 + |D_n(-t)|}$$

Where n is to be chosen. Then $f \in C^0(T)$ and $\|f_n\|_\infty \leq \epsilon$, for lemma 2.2

$$\begin{aligned} S_n f_n(0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f_n(0-t) D_n(t)\, dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\epsilon |D_n(t)|^2}{1 + |D_n(t)|}\, dt \end{aligned}$$

Since we know that $\int_{-\pi}^{\pi} |D_n(t)| dt \to \infty$ for $n \to \infty$, we can therefore try to bring back to the conditions of lemma 2.4 and write

$$\epsilon \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)|\, dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|D_n(t)|}{1 + |D_n(t)|}\, dt \right)$$

and utilizing the lemma 2.4 we can conclude that choosing n large enough we have $|S_n(f_n)(0)| \geq k$ □

Finally we can give the proof of theorem 2.2

Proof. using lemma 2.5 we can choose recursively $f_k \in C^0(T)$ and $n_k \in \mathbb{N}$ s.t. $\|f_k\|_\infty \leq \epsilon$ and $|S_{n_k}(f_k)(0)| \geq k$ with

$$\epsilon = \begin{cases} 2^{-k} \min_{1 \leq j \leq k-1} \left(\frac{1}{2n_j + 1} \right) & \text{if } k \neq 1 \\ 2^{-1} & \text{if } k = 1 \end{cases}$$

Since the Fourier series is a linear operator, from item 2, we can write:

$$\left| S_{n_k}\left(\sum_{j=1}^{k} f_j\right)(0)\right| = \left|\sum_{j=1}^{k} S_{n_k}(f_j)(0)\right| \geq |S_{n_k}(f_k)(0)| \geq k$$

Since $\|f_j\|_\infty \leq 2^{-j}$ for all j and $\sum_{j\in\mathbb{N}} 2^{-j} = 1$ then, for lemma 1.5, $\sum_{j\in\mathbb{N}} f_j$ converges uniformly to a target function. We can set $f := \sum_{j\in\mathbb{N}} f_j$ and for all k

$$S_{n_k} f(0) = S_{n_k}\left(\sum_{j=1}^{k} f_j\right)(0) + S_{n_k}\left(\sum_{j=k+1}^{\infty} f_j\right)(0)$$

But we've seen that

$$\left| S_{n_k}\left(\sum_{j=1}^{k} f_j\right)(0)\right| \geq k$$

and by lemma 2.4

$$\begin{aligned}
\left| S_{n_k}\left(\sum_{j=k+1}^{\infty} f_j\right)(0)\right| &\leq (2n_k+1)\left\|\sum_{j=k+1}^{\infty} f_j\right\|_\infty \\
&\leq (2n_k+1)\sum_{j=k+1}^{\infty} \|f_j\|_\infty \\
&\leq (2n_k+1)\sum_{j=k+1}^{\infty} \frac{2^{-j}}{2n_k+1} \\
&= \sum_{j=k+1}^{\infty} 2^{-j} \\
&\leq 1
\end{aligned}$$

finally $|S_{n_k}(f)(0)| \geq k-1 \to \infty$ for $k \to \infty$ [3]

□

Chapter 3

Continuous Nowhere Differentiable Functions

Intuitively, we are inclined to think of continuous functions as rather regular and smooth. However, we will see that there exist continuous functions that are not differentiable at any point and exhibit infinitesimal oscillations everywhere. We will also discuss the presence of continuous but non-differentiable functions within the set of continuous functions. Then we will observe how pathological functions are much more numerous compared to the set of continuous and differentiable functions within the set of continuous functions. In particular, we will demonstrate that the set of continuous but nowhere differentiable functions is an uncountable set. Finally, these functions will be introduced within a historical context, discussing the first pathological examples.

Theorem 3.1. Let $g : \mathbb{R} \to \mathbb{R}$, periodic whith period 4 and s.t.

$$g(x) = \begin{cases} 1 - x & \text{if } x \in [0, 2] \\ 1 + x & \text{if } x \in [-2, 0] \end{cases}$$

then

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} g(2^{2^n} x)$$

is continuous but nowhere differentiable

Proof. $f(x)$ is uniformly convergent, for lemma 1.5, once we observe

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} g(2^{2^n} x) \leq \sum_{n=1}^{\infty} 2^{-n} = 1$$

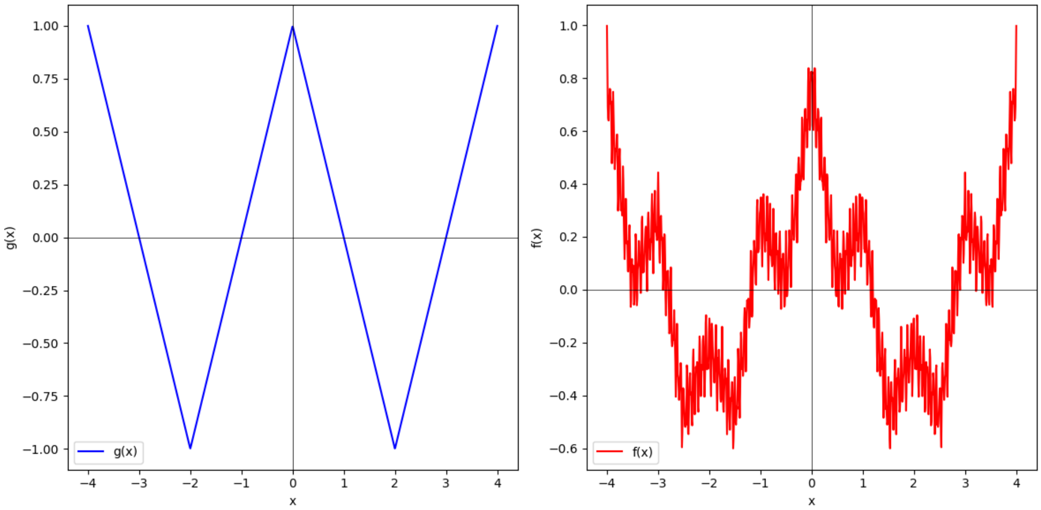

Figure 3.1: Plot of the function $g(x)$ and $f(x)$

in addition since all the functions $x \to 2^{-n}g(2^{2^n}x)$ is continuous , for lemma 1.6, we conclude that $f(x)$ is continuous . Now we want to prove that f is nowhere differentiable. Let $\overline{x} \in \mathbb{R}, k \in \mathbb{N}$. Consider $2^{2^k}\overline{x}$. We have $2^{2^k} \in [2m, 2m+2]$ for some $m \in \mathbb{Z}$ and we choose $h = 2^{-2^k}$ if

$$2^{2^k}\overline{x}, 2^{2^k}\overline{x} + 1 \in [2m, 2m+2]$$

or $h = -2^{-2^k}$ if

$$2^{2^k}\overline{x}, 2^{2^k}\overline{x} - 1 \in [2m, 2m+2]$$

We suppose we are in the first case, then

$$|g(2^{2^n}(\overline{x}+h)) - g(2^{2^n})\overline{x}| = |g(2^{2^n}\overline{x} + 2^{2^n-2^k}) - g(2^{2^n}\overline{x})| = \begin{cases} 0 & \text{if } n > k \\ 1 & \text{if } n = k \\ \leq 2^{2^n-2^k} & \text{if } n < k \end{cases}$$

where

- for the first case, we notice that $2^{2^n-2^k} = 4q$ with $q \in \mathbb{N}$, since $n > k$ and $2^n - 2^k \geq 2$. Since $g(x)$ is $4-$periodic, then

$$|g(2^{2^n}\overline{x} + 2^{2^n-2^k}) - g(2^{2^n}\overline{x})| = |g(2^{2^n}x) - g(2^{2^n}x)| = 0$$

- for the second case, since we choose $2^{2^k}\overline{x}+h$ such that are on the same linear segment of $g(2^{2^k})$, then, we assume that we are in the segment of $1+x$ (the other case is similar),

$$|g(2^{2^n}\overline{x}+1) - g(2^{2^n}\overline{x})| = |(1 + 2^{2^n}\overline{x} + 1) - (1 + 2^{2^n})| = 1$$

- for the last case, since $2^{2^k}\overline{x} + h$ are on the same linear segment of $g(2^{2^n})$, if we suppose we are in the segment $1 + x$, then for $n < k$

$$|g(2^{2^n}\overline{x} + 2^{2^n - 2^k}) - g(2^{2^n}\overline{x})| = |1 + (2^{2^n}\overline{x} + 2^{2^{n-k}}) - (1 + 2^{2^n}\overline{x})| = 2^{2^n - 2^k}$$

Consequently

$$f(\overline{x} + h) - f(\overline{x}) = \sum_{n=1}^{k} 2^{-n}(g(2^{2^n}(\overline{x} + h)) - g(2^{2^n}\overline{x})),$$

according to the previous observation

$$\begin{aligned} |f(\overline{x} + h) - f(\overline{x})| &\geq 2^{-k} - \sum_{n=1}^{k-1} 2^{-n} 2^{2^n - 2^k} \\ &\geq 2^{-k} - (k-1)2^{2^{k-1} - 2^k} \\ &\geq 2^{-k} - (k-1)2^{-2^{k-1}} \end{aligned}$$

and finally

$$\left| \frac{f(\overline{x} + h) - f(\overline{x})}{h} \right| \geq 2^{-k + 2^k} - (k-1)2^{2^{k-1}}.$$

In this way, for all $\overline{x} \in \mathbb{R}$, we construct a sequence $(h_k)_k$ in $\mathbb{R}$, such that

$$\lim_{k\to\infty} h_k = 0 \quad \text{and} \quad \lim_{k\to\infty} \left| \frac{f(\overline{x} + h_k) - f(\overline{x})}{h_k} \right| = +\infty.$$

This implies that f is not differentiable at $\overline{x}$. $\square$

Theorem 3.2. If $0 < \alpha < 1$, then the function $f_\alpha(x) = f(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} e^{i2^n x}$ is continuous but nowhere differentiable.

Before giving the proof, we note that $f(x)$ can be viewed as the Fourier series of a 2π-periodic function $g(x)$. Therefore, let $E_N := \left\{2^k < N \quad \text{s.t. } k, N \in \mathbb{N}\right\}$. For $n \in \mathbb{N}$,

$$S_N(g)(x) = \sum_{n \in E_N} c_n e^{inx} \quad \text{with} \quad c_n = 2^{-\log_2(n)\alpha} = n^{-\alpha}$$

Therefore, as $N \to \infty$, $S_N(g)(x) = f(x)$. Notice that the Fourier series has non-zero coefficients only for $n \in E_N$, so if we consider its N-th reduction, it will have only $\log_2(N)$ non-zero coefficients. This series is defined as Lacunary because many terms are zero, requiring a very large N to approximate the function well enough. Therefore, new methods are introduced to compute the Fourier series.

3.1 Alternative methods of summing a Fourier Series

3.1.1 Cesàro summability

Definition 3.1. (Cesàro Summable Series)
let $(a_i)_{i\in\mathbb{N}}$, and $s_n = \sum_{i=0}^{n} a_i$, if the sum

$$\sigma_N := \frac{1}{N}\sum_{n=0}^{N-1} s_i$$

for $n \to \infty$ converges to A, then $\sum_{i\in\mathbb{N}} a_i$ is defined as Cesàro summable.

Lemma 3.1. (Fejer Theorem)
Let $f : \mathbb{R} \to \mathbb{C}$ and $f \in C^0(\mathbb{C})$ periodic with period 2π and let

$$\sigma_N(f)(x) = \frac{1}{N}\sum_{i=0}^{N-1} S_i(f)(x)$$

Where $S_i(f)(x)$ is defined as eq. (1.2.4) then, for $N \to \infty$, $\sigma_N \to f$ uniformly in $\mathbb{C}$

Lemma 3.2.

$$\sigma_N(f)(x) = (f * F_N)(x)$$

Where $F_N(x)$ is the Fejer Kernel, i.e.

$$F_N(x) = \frac{1}{N}\sum_{n=0}^{N-1} D_n = \frac{1}{n}\frac{\sin^2(nx/2)}{\sin^2(x/2)}$$

with D_n the Dirichlet Kernel defined as definition 2.1

Remark. if $g(x)$ admit a Fourier Series and $g(x) = \sum_{n\in\mathbb{N}} c_n e^{inx}$ then $\sigma_N(g)(x)$ can be arises by

$$\sum_{|n|<N}\left(1 - \frac{|n|}{N}\right) c_n e^{inx}$$

since

$$\begin{aligned}
\sigma_N(g)(x) &= \frac{1}{N}\left(S_0(g)(x) + S_1(g)(x) + \cdots + S_{N-1}(g)(x)\right) \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{|k|\le a} e^{ikx} \\
&= \frac{1}{N}\sum_{|n|\le N}(N - |n|)c_n e^{inx}
\end{aligned}$$

3.1.2 Delayed means

Another method, of summing a Fourier series, is by $\Delta_N := 2\sigma_{2N} - \sigma_N$, that for the lemma 3.1 and for $N \to \infty$, converges uniformly to $f(x)$, since

$$\lim_{N\to\infty} 2\sigma_{2N} - \sigma_N = 2\lim_{N\to\infty} \sigma_{2N} - \lim \sigma_N = 2f(x) - f(x) = f(x)$$

Lemma 3.3.

$$\Delta_N(f)(x) = (f * [2F_{2N} - F_N])(x)$$

Remark. Since we know that $\sigma_N(f)(x) = \sum_{|n|<N} \left(1 - \frac{|n|}{N}\right) c_n e^{inx}$, for section 3.1.1, then $\Delta_N(f)(x)$ can be arises by multiplying $c_n e^{inx}$ by k, with:

$$k = \begin{cases} 1 & \text{if } |n| \le N, \\ 2\left(1 - \frac{|n|}{2N}\right) & \text{if } N < |n| \le 2N, \\ 0 & \text{if } |n| > 2N. \end{cases}$$

Since

$$\begin{aligned} \Delta_N(f)(x) &= 2\sum_{|n|\le 2N} \left(1 - \frac{|n|}{2N}\right) c_n e^{inx} - \sum_{|n|\le N} \left(1 - \frac{|n|}{N}\right) c_n e^{inx} \\ &= 2\sum_{N\le|n|\le 2N} \left(1 - \frac{|n|}{2N}\right) c_n e^{inx} - \sum_{|n|\le N} c_n e^{inx} \left(2 - \frac{|n|}{N} - 1 + \frac{|n|}{N}\right) \\ &= 2\sum_{N\le|n|\le 2N} c_n e^{inx} \left(1 - \frac{|n|}{2N}\right) - \sum_{|n|\le N} c_n e^{inx} \end{aligned}$$

3.1.3 Proof of theorem 3.2

At this point, returning to the original function $f(x)$, defined in the theorem, we can state that

$$\Delta_{N'}(f)(x) = S_N(f)(x)$$

with N' being the largest integer of the form 2^k such that $N' \le N$, since, let $\widetilde{E}_N := \left\{N \le 2^k \le 2N \quad \text{such that } k, N \in \mathbb{N}\right\}$

$$\begin{aligned} \Delta_{N'}(f)(x) &= \sum_{n\in E_{N'}} n^{-\alpha} e^{inx} + 2\sum_{n\in \widetilde{E}_{N'}} n^{-\alpha} e^{inx} \left(1 - \frac{|n|}{2N'}\right) \\ &= \sum_{n\in E_{N'}} n^{-\alpha} e^{inx} + 2(2N')^{-\alpha} e^{i2N'x} \left(1 - \frac{2N'}{2N'}\right) \\ &= \sum_{n\in E_{N'}} n^{-\alpha} e^{inx} = S_N(f)(x) \end{aligned}$$

Furthermore, from the same observations, we can also state that

$$\Delta_{2N}(f) - \Delta_N(f) = n^{-\alpha} e^{inx} \quad \text{for } n = 2^{2N} \tag{3.1.1}$$

through lemma 3.1, we can affirm that as $n \to \infty$, $\sigma_N(S_N(g)(x)) = f(x)$. Therefore, we will give the proof of the theorem by contradiction, assuming thus that $f'(x_0)$ exists for some x_0.

Lemma 3.4.

$$(f * g)' = f' * g = f * g'$$

Lemma 3.5. Let g be any continuous function that is differentiable at x_0. Then, the Cesàro means satisfy $\sigma_N(g)'(x_0) = O(\log N)$, therefore

$$\Delta_N(g)'(x_0) = O(\log N)$$

Proof. If we demonstrate that σ_N' is $O(\log(n))$, then it will immediately follow that Δ_N is $O(\log(n))$, since

$$\Delta_N' = 2\sigma_{2N} - \sigma_N$$

So we proceed to demonstrate that σ_N' is $O(\log(n))$. For lemma 3.4,

$$\sigma_N(g)'(x_0) = \int_{-\pi}^{\pi} F_N'(t) g(x_0 - t)\, dt$$

We observe that

$$F_N'(t) = \frac{\sin(Nt/2)\cos(Nt/2)}{\sin^2(t/2)} - \frac{1}{N} \frac{\cos(t/2)\sin^2(tN/2)}{\sin^3(t/2)}$$

and since an odd function times an even function is an odd function then $F_N'(t)$ is an odd function we have

$$\int_{-\pi}^{\pi} F_N'(t) dt = 0$$

and

$$\sigma_N(g)'(x_0) = \int_{-\pi}^{\pi} F_N'(t)[g(x_0 - t) - g(x_0)] dt$$

From the assumption that g is differentiable at x_0 and continuous , we get

$$|\sigma_N(g)'(x_0)| \le C \int_{-\pi}^{\pi} |F_N'(t)||t| dt$$

Now observe that F_N' satisfies the two estimates

$$|F_N'(t)| \le AN^2 \quad \text{and} \quad |F_N'(t)| \le \frac{A}{|t|^2}.$$

Since

- for the first inequality

$$|F_N'| \leq (2N+1)N \leq AN^2$$

- The second inequality arises by

$$\frac{\sin(Nt/2)\cos(Nt/2)}{\sin^2(t/2)} \leq \frac{1}{\sin^2(t/2)} \leq \frac{A}{t^2}$$

  and

$$\left|\frac{1}{N}\frac{\cos(t/2)\sin^2(Nt/2)}{\sin^3(t/2)}\right| \leq \left|\frac{\sin(Nt/2)}{N}\right|\left|\frac{\sin(Nt/2)}{\sin^3(t/2)}\right| \leq A|t|\left|\frac{1}{t^3}\right| = \frac{A}{t^2}$$

Using all of these estimates, we find that

$$\begin{aligned}
|\sigma_N(g)'(x_0)| &\leq C\int_{|t|\geq 1/N} |F_N'(t)||t|dt + C\int_{|t|\leq 1/N} |F_N'(t)||t|dt \\
&\leq CA\int_{|t|\geq 1/N}\frac{dt}{|t|} + CAN\int_{|t|\leq 1/N} dt \\
&= O(\log N) + O(1) \\
&= O(\log N)
\end{aligned}$$

□

Now we conclude by giving the proof of the theorem

Proof. The function $f(x)$ converges uniformly, for lemma 1.5, given that $f(x) \leq 1$. In addition, $f(x)$ is continuous, since $x \to 2^{-n\alpha}e^{i2^n x}$ is continuous and thanks to lemma 1.6. Since we know that

$$\Delta_{2N}(f)'(x_0) - \Delta_N(f)'(x_0) = O(\log(n))$$

and thanks eq. (3.1.1) we know that

$$|\Delta_{2N}(f)'(x_0) - \Delta_N(f)'(x_0)| = 2^{n(1-\alpha)} \geq cN^{1-\alpha}$$

this is the desired contradiction, since $N^{1-\alpha}$ grows faster than $\log N$. □

3.2 Patological Functions

We now want to investigate the cardinality of the set of non-differentiable functions within the set of continuous functions. Therefore, we want to understand whether continuous but nowhere differentiable functions are exceptions or rather the norm.

Definition 3.2. (metric space)
A metric d on a set X is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y \in X$:

1. $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ (symmetry);
3. $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

A metric space (X, d) is a set X with a metric d defined on X.

Definition 3.3. (Topological space)
Let X a nonempty set, a topolgy on X is a family $\mathcal{F}$ of X which contains $\emptyset$ and X. In addition is closed under arbitrary unions and finite intersection i.e.

$$\text{if } \{U_n\}_{n\in\mathbb{N}} \subset \mathcal{F} \quad \text{then} \quad \bigcup_{n\in\mathbb{N}} U_n \in \mathcal{F}$$

and

$$\text{if } U_1, \ldots, U_n \in \mathcal{F} \quad \text{then} \quad \bigcap_{j\in\mathbb{N}} U_j \in \mathcal{F}$$

The pair $(X, \mathcal{F})$ is called a topological space.

Definition 3.4. (Closure of set)
Let (X, T) be a topological space, and let $A \subseteq X$. We define the closure of A in (X, T), which we denote by $\overline{A}$, by:

$$\overline{A} = \{x \in X : \forall U \in T \text{ such that } x \in U, U \cap A \neq \emptyset\}$$

Definition 3.5. (Nowhere dense)
Let X be a metric space. A subset $A \subseteq X$ is called nowhere dense in X if the interior of the closure of A is empty, i.e., $(\overline{A})^\circ = \emptyset$. Otherwise put, A is nowhere dense iff it is contained in a closed set with empty interior.

Definition 3.6. (Meager set and residual set)
if X is a topological space, a set $E \subset X$ is meager (or of the firts category), according to Baire, if E is a countable union of nowhere dense sets; Otherwhise E is residual (or of the second category).

Theorem 3.3. (Baire Category Theorem [4])
Let X be a complete metric space.

1. if $\{U_n\}_{n\in\mathbb{N}}$ is a sequence of open dense subset of X, then $\bigcap_{n=1}^{\infty} U_n$ is dense in X

2. X is not a countable union of nowhere dense sets.

Now we want to show that the set of nowhere differentiable function is a residual set compare to the set of continuous function. In other words, we will prove that is the complement of a set, contained in the countable union of closed sets with empty interior, in the metric space of continuous functions with the sup-distance.

Lemma 3.6. (Theorem of Stone- Weierstrass)
Suppose f is a continuous real-valued function defined on the real interval $[a, b]$. For every $\epsilon > 0$, there exists a polynomial p such that for all x in $[a, b]$, we have $|f(x) - p(x)| < \epsilon$, or equivalently, $\|f - g\|_\infty < \epsilon$

Definition 3.7. Let $\phi : I \to \mathbb{R}$, with I an open interval in $\mathbb{R}$, and let $x_0 \in I$. We define

$$\liminf_{x\to x_0^+} \phi(x) = \sup_{t>0} \left\{ \inf_{x_0<x<x_0+t} \phi(x) \right\},$$

$$\limsup_{x\to x_0^+} \phi(x) = \inf_{t>0} \left\{ \sup_{x_0<x<x_0+t} \phi(x) \right\},$$

$$\liminf_{x\to x_0^-} \phi(x) = \sup_{t>0} \left\{ \inf_{x_0-t<x<x_0} \phi(x) \right\},$$

$$\limsup_{x\to x_0^-} \phi(x) = \inf_{t>0} \left\{ \sup_{x_0-t<x<x_0} \phi(x) \right\}.$$

Definition 3.8. Let $f : I \to \mathbb{R}$, with I an open interval in $\mathbb{R}$, and let $x_0 \in I$. We define

$$D^+f(x_0) = \limsup_{x\to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}, \quad D_+f(x_0) = \liminf_{x\to x_0^+} \frac{f(x) - f(x_0)}{x - x_0},$$

$$D^-f(x_0) = \limsup_{x\to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}, \quad D_-f(x_0) = \liminf_{x\to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}.$$

$D^+f(x_0)$, $D^+f(x_0)$, $D^-f(x_0)$, $D^-f(x_0)$ are the so-called Dini's derivatives of the function f at the point x_0.

Remark. A function $f(x)$ is differentiable at point x_0 if and only if

$$D_-f(x_0) = D^-f(x_0) = D_+f(x_0) = D^+f(x_0) \in \mathbb{R}$$

Furthermore, Dini derivatives can be seen as a generalization of ordinary derivatives. Indeed, if we consider a function $f(x)$ s.t.

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x\sin(\frac{1}{x}) & \text{if } x \neq 0 \end{cases}$$

Then the Dini's derivatives for $x = 0$ are well defined, unlike the ordinary derivative

$$f'(0) = \lim_{h\to 0} \frac{f(h) - 0}{h} = \lim_{x\to 0} \sin(\frac{1}{x})$$

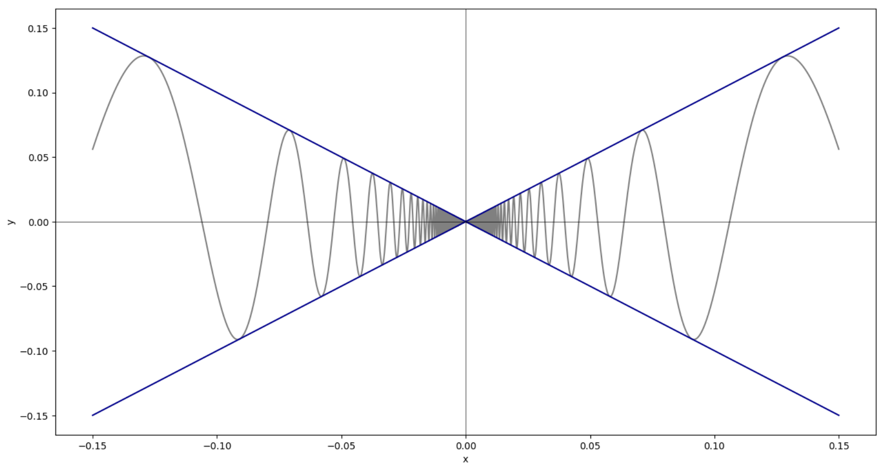

Figure 3.2: Plot of the function $f(x)$ with it's boundaries D^+, D^-, D_+, D_-

Theorem 3.4. Let

$$D = \{f \in C^0([0,1], \mathbb{R}) \mid \exists x \in [0, 1[: D^+f(x), D_+f(x) \in \mathbb{R}\}$$

Then D is contained in the union of a sequence of closed sets with empty interior.

Proof. We want demonstrate that D is contained in the union of a sequence of closed sets with empty interior, since, for theorem 3.3, if we demonstrate that, then D is meager and the complement of D is a residual set. In particular we will have shown that the set of nowhere differentiable function is a residual set.

We notice that the set of functions which are differentiable (from the right) at least in one point, is contained in D so that the set of continuous nowhere differentiable functions contains the complement of D. Let

$$C_n = \{f \in C^0([0,1],\mathbb{R}) \mid \exists x \in [0, 1-\frac{1}{n}] : \forall h \in]0, \frac{1}{n}], \left|\frac{f(x+h)-f(x)}{h}\right| \leq n\}.$$

then $C_n \subseteq D$. We prove that $D \subseteq \bigcup_{n=1}^{\infty} C_n$. Let $f \in D$ then there exists $\overline{x} \in [0,1[$ and there exists $C, C_0 \in \mathbb{R}$, with $C_0 < C$, such that

$$\inf_{t>0}\left\{\sup_{x<\overline{x}<x+t} \frac{f(x)-f(\overline{x})}{x-\overline{x}}\right\} < C$$

and

$$\sup_{s>0}\left\{\inf_{x<\overline{x}<x+s} \frac{f(x)-f(\overline{x})}{x-\overline{x}}\right\} > C_0.$$

In particular, there exists $t > 0$ and $s > 0$ s.t.

$$C_0 < \frac{f(x)-f(\overline{x})}{x-\overline{x}} < C$$

Consequently, there exists $\alpha, \delta > 0$ such that, for all $h \in (0, \delta]$

$$\left|\frac{f(\overline{x}+h)-f(\overline{x})}{h}\right| \leq \alpha,$$

and this implies that $f \in C_n$ for some n and $D \subseteq \bigcup_{n\in\mathbb{N}} C_n$. Now we prove that C_n is closed. Let n be fixed and let $f \in \overline{C}_n$ where $\overline{C}_n$ denotes the closure of C_n in the space $C^0([0,1],\mathbb{R})$. Since $\overline{C}_n$ is closed then we can take a sequence $(f_k)_k$ in C_n which converges uniformly to f. We have that, for each k, there exists a point x_k such that

$$x_k \in [0, 1-\frac{1}{n}] \text{ and for all } h \in]0, \frac{1}{n}], \left|\frac{f_k(x_k+h)-f_k(x_k)}{h}\right| \leq n.$$

Passing to a subsequence, we can suppose that there exists $\overline{x} \in [0, 1-\frac{1}{n}]$ such that $x_k \to \overline{x}$. We fix now $h \in]0, \frac{1}{n}]$, we fix $\epsilon > 0$ and we choose k in such a way that

$$\|f - f_k\|_\infty \leq \frac{\epsilon h}{4}, \quad |f(x_k) - f(\overline{x})| \leq \frac{\epsilon h}{4}, \quad |f(\overline{x}+h) - f(x_k+h)| \leq \frac{\epsilon h}{4}.$$

Consequently

$$\begin{aligned}\left|\frac{f(\overline{x}+h)-f(\overline{x})}{h}\right| &\leq \left|\frac{f(\overline{x}+h)-f(x_k+h)}{h}\right| + \left|\frac{f_k(x_k+h)-f_k(x_k)}{h}\right| + \\ &+ |f_k(x_k) - f(x_k)| + |f(x_k) - f(\overline{x})| \\ &\leq nh + \epsilon h\end{aligned}$$

finally

$$\inf_{\epsilon}\left|\frac{f(\overline{x}+h)-f(\overline{x})}{h}\right| \geq \inf_{\epsilon}(nh+\epsilon h) \Rightarrow \left|\frac{f(\overline{x}+h)-f(\overline{x})}{h}\right| \leq nh$$

and this prove that C_n is a closed set. now we prove that C_n has an empty interior. By contradiction suppose that there exists n, there exists $f \in C_n$ and there exists $\epsilon > 0$ such that the ball $B(f,\epsilon) = \{g \in C^0([0,1],\mathbb{R}) \mid \|g-f\|_\infty < \epsilon\}$ is contained in C_n. Using lemma 3.6 there exists a polynomial p on $[0,1]$ s.t.

$$\|f-p\|_\infty < \epsilon.$$

Let $\delta = \epsilon - \|f-p\|_\infty$. As a consequence, since $\delta < \epsilon$

$$B(p,\delta) \subseteq B(f,\epsilon) \subseteq C_n.$$

We construct now a function $g \in C^0([0,1],\mathbb{R})$ such that $\|g\|_\infty < \delta$, g has a finite right derivative $g'_+(x)$ at each point x of $[0,1[$ and, for all $x \in [0,1[$,

$$|g'_+(x)| > n + \|p'_+\|_\infty,$$

(to find such a function g it is sufficient to take a suitable sawtooth function). Then we have $p+g \in C_n$, $(p+g)'_+ = p'_+ + g'_+$ and, for all $x \in [0,1[$,

$$|(p+g)'_+(x)| \geq |g'_+(x)| - \|p'\|_\infty > n,$$

which is a contradiction. This completes the proof [5] □

3.3 Riemann ad Weierstrass function

The first nowhere differentiable function was proposed by Riemann in 1861, who asserted that the function

$$R(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2}$$

is a continuous but nowhere differentiable function. It was only, in 1916 that Hardy published the article [6] in which he demonstrated that the function $R(x)$ has no finite derivative at all irrational points and in the points of the form $\frac{2x}{4x+1}$ or $\frac{2x+1}{4x+2}$. In 1969, Gerver [7] proved that g has a derivative -2 at points of the form $\pi\frac{2p+1}{2q+1}$ with $p,q \in \mathbb{Z}$. He also proved that g has no finite derivative at points of the form $\frac{2p+1}{2^n}$ with $n \geq 1$. Lastly A. Smith [8] extend the results to the remaining cases, showing also the existence of finite left- and right-hand derivatives at certain rationals, and proving that these derivatives exist at all rationals if we allow the values $\pm\infty$.

The first nowhere differentiable function was introduced and demonstrated by K. Weierstrass in 1872, July 18th, on the first official (oral) presentation, at the Berlin Academy of Sciences. The function he proposed is the following

$$W(x) = \sum_{n=1}^{\infty} b^n \cos(a^n \pi x)$$

where $0 < b < 1$ and a is an integer > 1. The initial documentation of this function dates back to a letter penned by Karl Weierstrass to Paul Gustave du Bois-Reymond in 1873, where Weierstrass wrote:

> "Dear Colleague,
>
> In your last paper, published by Borchardt, you expose my proof showing that the function (...) was everywhere non-differentiable under the conditions I gave. I agree with everything."[9]

in accordance with Weierstrass, the first pubblication of the proof, was proposed by Du Boys-Reymond in 1875 [10].

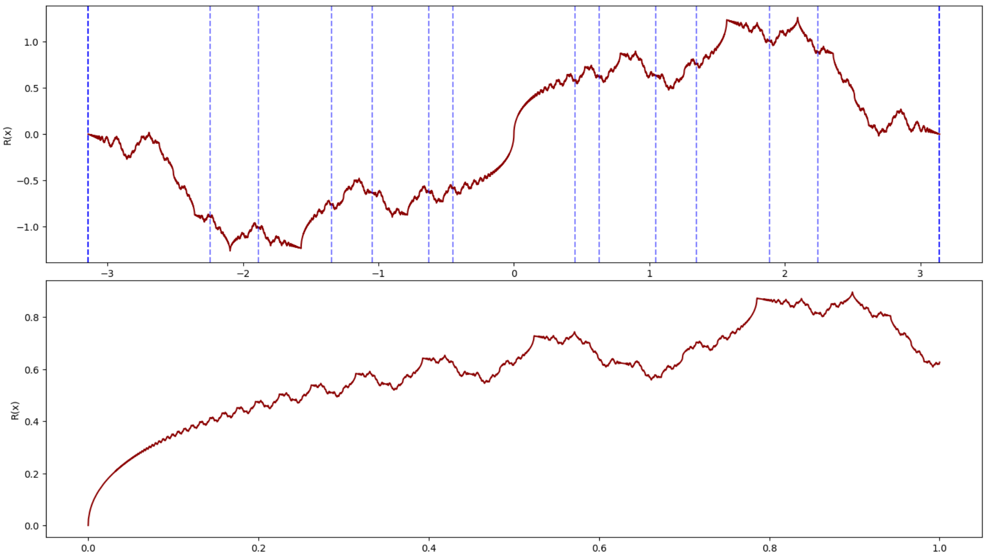

Figure 3.3: Plot of the function $R(x)$ for different values of x

From figure 3.3, we can observe that the function $R(x)$ exhibits self-similarity, as evident from the second plot where certain points show similar characteristics but on a different scale (the range of x in the second plot is $[\pi, \pi]$). Additionally, in the first plot, some points of differentiability are plotted (dashed lines); notably, we can observe that the Riemann function is a.e. non differentiable (the range of x in the first plot is $[0, 1]$).

From figure 3.4, we can observe the fractal nature of the function $W(x)$. In fact, as we decrease the scale more and more (in the second plot, transitioning from an interval of $[-2, 2]$ to an interval $[0.999, 1.001]$), we continue to see the typical jaggedness of the function. In particular, we can notice that the function is constructed at each point with a frequency that tends to $+\infty$, hence at every point, one can find a continuous oscillating behavior.

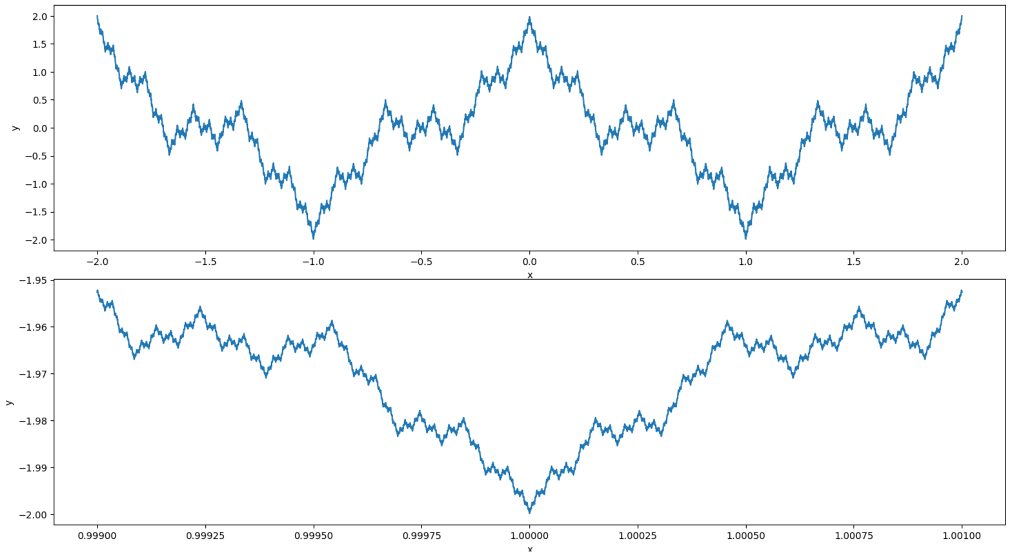

Figure 3.4: Plot of the function $W(x)$ for different values of x

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