Singularity Formation in the Incompressible Porous Medium Equation without Boundary Mass

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University of California San Diego, Analysis Seminar
November 18th, 2025

Active Scalars

An *active scalar equation* is a transport equation of the form

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0 \\ u = \nabla^{\perp} \mathcal{K}(\rho). \end{cases}$$

- ρ denotes a scalar such as density of temperature being transported by an incompressible velocity field u.
- $\mathcal{K}$ is an operator which determines a coupling between u, ρ.

The IPM Equation

An important example of active scalar transport is the incompressible porous medium (IPM) equation:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ u + \nabla p = (0, -\rho) \\ \nabla \cdot u = 0 \end{cases}$$

with $\rho(0,x) = \rho_0(x)$. This system models fluid flow through a porous medium.

- The IPM system is an active scalar with $\mathcal{K} = (-\Delta)^{-1}\partial_1 \rho$
- We consider the system on domains $\Omega \subset \mathbb{R}^2$ with the no-penetration boundary condition $u \cdot n = 0$ on $\partial\Omega$

Past Results

- It remains open whether smooth solutions are global on $\mathbb{R}^2$
- Kiselev and Yao (2023) used a virial argument to prove infinite in time growth in H^s for a wide class of unstably stratified states
- Córdoba and Martinez-Zoroa (2025) constructed smooth solutions to the *forced* equation which blow-up in finite time
- Zlatoš (2024) proved smooth singularity formation for the 2D Muksat equation which models the interface of 2D fluids of different densities evolving according to the IPM equation

Transport vs Stretching

- Taking $\nabla^{\perp}$ we have

$$\partial_t \nabla^{\perp}\rho + \underbrace{u \cdot \nabla\nabla^{\perp}\rho}_{\text{transport}} = \underbrace{\nabla u \nabla^{\perp}\rho}_{\text{stretching}}$$

- $\nabla^{\perp}\rho$ is stretched and transported by the same velocity field u
- The stretching works towards blow-up however the transport rapidly ejects particles from regions of high stretching working against blow-up
- Effects are in a seemingly perfect balance making problem of global existence subtle

Weakening Transport

There are two main strategies which can weaken the effect of transport:

1. Working in low regularity
   - Considering $\rho \in C^{1,\alpha}$ for $\alpha < 1$, allows for mass to be more concentrated near the origin
   - If the mass is highly concentrated enough, the transport will not be strong enough to eject the mass before a singularity forms
2. Adding a boundary
   - Due to the no-penetration boundary condition, any mass initially on the boundary must remain there for all time
   - Transport cannot eject boundary mass so stretching dominates

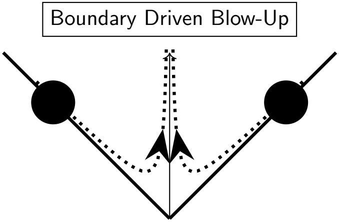
There is no known scenario to weaken advection for smooth solutions on the whole plane.

The Setup

Goal: Study singularity formation in a setting in which the full effect of transport is present.

- IPM has the following scaling symmetry: if ρ is a solution then $\rho_\lambda(t,x) := \lambda^{-1}\rho(t,\lambda x)$ is also a solution
- This leads us to consider 1-homogeneous solutions $\rho(t,r,\theta) = rP(t,\theta)$
- To have a suitable local well-posedness theory we consider domains which are wedges strictly smaller than the half-plane and impose even symmetry of ρ about the y-axis

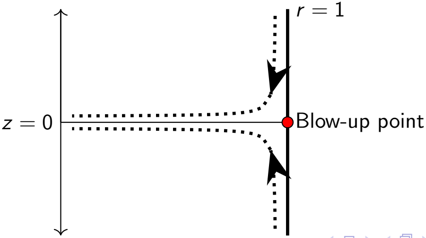
Blow-Up Scenario



- We consider a scenario in which there is a hyperbolic flow with particles flowing from the interior towards the boundary and escaping tangent to the boundary
- The ability for particles to escape is a fundamental obstacle to proving blow-up which is not present in the boundary driven case

Comparison With Previous Blow-Up Scenarios

- In previous blow-up scenarios for incompressible fluids, the flow was directed from the boundary towards the symmetry axis
- Elgindi and Jeong (2016) proved blow-up in the Boussinesq system for the same types of solutions considered here but using the boundary driven scenario
- The Hou-Luo scenario for the axisymmetric 3D incompressible Euler equations also features a flow in which particles flow along the boundary of the cylinder towards the $z = 0$ plane



Comparison With Previous Blow-Up Scenarios

- We consider a blow-up scenario which is distinct from previously considered boundary driven scenarios
- The direction of the hyperbolic flow is reversed with particles flowing towards the boundary not away from it
- The geometry is analogous to that of Elgindi and Pasqualotto (2023) where they prove $C^{1,\alpha}$ blow-up (smooth in the angular variable) for 2D Boussinesq and 3D axisymmetric Euler without boundary
- This scenario is not driven by the boundary and could conceivably be used to prove blow-up in the bulk

Scale-Invariant Hölder Spaces

We consider scale-invariant Hölder spaces defined by the norm

$$\|f\|_{\mathring{C}^\alpha} = \|f\|_{L^\infty} + \sup_{x \neq x'} \frac{\left| |x|^\alpha f(x) - |x'|^\alpha f(x') \right|}{|x - x'|^\alpha}$$

$$\|f\|_{\mathring{C}^{k,\alpha}} = \|f\|_{\mathring{C}^{k-1,\alpha}} + \sup_{x \neq x'} \frac{\left| |x|^{k+\alpha} \nabla^k f(x) - |x'|^{k+\alpha} \nabla^k f(x') \right|}{|x - x'|^\alpha}.$$

- $\mathring{C}^\alpha$ scales like L^∞ however singular integral operators are bounded on $\mathring{C}^\alpha$ in suitable settings
- For 0-homogeneous functions, the $\mathring{C}^\alpha$ norm is equivalent to the C^α norm in the angular variable
- $\nabla f \in \mathring{C}^{k,\alpha}$ implies f is Lipschitz continuous

The Main Result

Theorem (D. 2025)

For any $k \geq 0$ and $0 < \alpha < 1$, there exist $\nabla u_0, \nabla \rho_0 \in \mathring{C}^{k,\alpha}(\Omega)$ with ρ_0 compactly supported such that the unique local in time solution to the IPM equation satisfies

$$\limsup_{t \to 1} \int_0^t \|\nabla \rho(s)\|_{L^\infty} ds = +\infty.$$

Here, $\Omega = \{(r, \theta) : -\beta\pi < \theta < \beta\pi\}$ for some $\beta < 1/2$ where (r, θ) denote the standard polar coordinates on $\mathbb{R}^2$. Moreover, the initial density ρ_0 can be chosen to be compactly supported in the angular variable.

Discussion of Proof

Inserting the ansatz $\rho(t, r, \theta) = rP(t, \theta), u = \nabla^{\perp}(r^2 G(t, \theta))$ gives a 1D system

$$\begin{cases} \partial_t P + 2G\partial_\theta P = (\partial_\theta G)P \\ \partial_\theta^2 G + 4G = P \sin\theta + \partial_\theta P \cos\theta \\ G(0) = G(L) = 0. \end{cases}$$

where $L < \pi/2$ is the endpoint of the domain. Here P is even in θ and G is then odd.

There are two main steps:

1. Construct a profile $P(t, \theta) = \frac{1}{1-t} P_*(\theta)$
2. Perturb around the profile to truncate near the boundary

The Profile Equation

Setting $P = (1-t)^{-1}P_*(\theta)$, the profile satisfies

$$\begin{cases} P_* + 2G_*P_*' = G_*'P_* \\ G_*'' + 4G_* = P_*\sin\theta + P_*'\cos\theta \\ G_*(0) = G_*(L) = 0 \end{cases}$$

where $L < \pi/2$ is the endpoint of the domain. The system is:

- **Singular** since $G_*(0) = 0$
- Highly **nonlocal** since G_* solves a boundary value problem and depends on the global values of P_*

Key Observation # 1: Evaluating at zero the first equation implies $G_*'(0) = 1$ is fixed.

The Initial Value Profile Equation

We can then search for a solution to the *initial value problem*

$$\begin{cases} P_* + 2G_* P_*' = G_*' P_* \\ G_*'' + 4G_* = P_* \sin\theta + P_*' \cos\theta \\ G_*(0) = 0, \quad {\color{red} G_*'(0) = 1}. \end{cases}$$

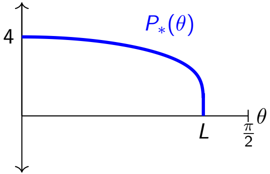
for which $G_*(L) = 0$.

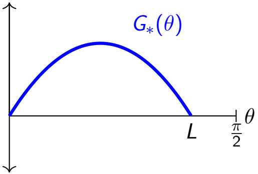
- Replacing BVP with IVP removes most of the nonlocality
- Now, we wish to solve locally via Taylor expansion and extend

Key Observation #2: It can be shown that if $M_* := \dfrac{P_*(\theta)}{\cos\theta}$ is initially decreasing then it remains decreasing.

Existence of Profile

- It can then be seen that any solution for which M_* is not constant G_*, P_* must hit zero at some point $L < \pi/2$.
- The goal is therefore to find a solution for which M_* is decreasing in a neighbourhood of zero.
- There is a unique choice $P_*(0) = 4$ such that $P_*''(0)$ is free

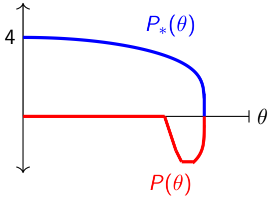




- Choosing $P_*''(0) < -4$ ensures $M_*''(0) < 0$ so M_* is non-constant, decreasing
- Thus fixing $P_*''(0) = -4 - A$ for $A > 0$ we obtain a domain endpoint $L(A) < \pi/2$
- Sending $A \to 0$ we show that $L(A) \to \pi/2$ so we can achieve domains arbitrarily close to the half-plane
- Since the flow is outgoing towards the boundary, this causes particles to concentrate near the boundary and thus $P_* \in C^{\frac{1}{2}+\epsilon}([0, L]) \cap C^\infty([0, L))$

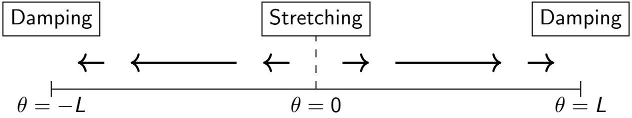
Perturbing the Profile

We now wish to add a perturbation capable of truncating P_* near the boundary:



Heuristic Stability Picture

The flow generated by the profile is as below:



- Particles are transported away from regions of stretching towards regions of damping
- The only concern is near zero where the velocity field is weak
- Work in a weighted space with a strong weight near zero to discourage mass near the origin

Stability

- Defining $s = -\log(1-t)$ we want to prove (finite-codimension) stability as $s \to \infty$ of P_* in a weighted Sobolev space
- The finite number of unstable directions comes from imposing vanishing conditions near the origin
- We prove the linearized operator $\mathcal{L}$ about P_* can be decomposed into a coercive part and a finite-rank smoothing part

Linearization

- Linearizing about P_* we have that a perturbation P satisfies

$$\partial_s P + \mathcal{L}(P) = N(M, M)$$

- The linearized operator is given by

$$\mathcal{L}(P) = 2G_* P' + 2GP_*' - G_*' P - G' P_*$$

- $N(M, M)$ is a quadratic nonlinearity which satisfies good energy estimates

Sketch of Coercivity

- The region of concern is near $\theta = 0$ so we focus there
- By adding a finite-rank operator, we can assume P vanishes to degree 4 at zero.
- We can also localize G with a finite-rank operator to solve the IVP

$$G'' + 4G = P\sin\theta + P'\cos\theta$$

with $G(0) = G'(0) = 0$ so $G'' \approx P' \implies G(\theta) \approx \int_0^\theta P(\phi)d\phi$

- We have

$$\mathcal{L}(P) = P + \underbrace{2G_*P'}_{G_*\approx\theta} + \underbrace{2GP'_*}_{\text{lower order}} - \underbrace{G'_*P}_{G'_*\approx 1} - \underbrace{G'P_*}_{G'\approx P, P_*\approx 4}$$

- Therefore,

$$\overline{\mathcal{L}}(P) \approx 2\theta P' - 4P$$

Sketch of Coercivity (Cont'd)

- Since P vanishes to fourth order, we can put a weight of θ^{-8} near zero and consider an inner product like

$$\langle f, g\rangle_X = \int_0^{\epsilon} f(\theta)g(\theta)\theta^{-8}d\theta$$

- Then,

$$\langle P, \overline{\mathcal{L}}(P)\rangle \approx \int_0^{\epsilon}(2\theta PP' - 4P^2)\theta^{-8}d\theta = \int_0^{\epsilon}(\theta(P^2)' - 4P^2)\theta^{-8}d\theta$$

- Integrating by parts gives

$$\langle P, \overline{\mathcal{L}}(P)\rangle \approx (7-4)\int_0^{\epsilon} P^2\theta^{-8}d\theta = 3\|P\|_X^2$$

Coercivity

- In the bulk of the domain, we can use the same strategy however now we can put a weight θ^{-K} for any $K > 0$
- This gives a coercive term of order K in the bulk
- This estimate degenerates at the endpoint $\theta = L$; in this region however we can rely on the damping
- Near L we recall that G, G_*, P_* vanish and $G_*'(L) < 0$ so we have

$$\begin{aligned}\mathcal{L}(P) &= P + \underbrace{2G_* P'}_{\text{small}} + \underbrace{2GP_*'}_{\text{small}} - \underbrace{G_*' P}_{G_*'<0} - \underbrace{G' P_*}_{\text{small}} \\ &\approx (1 - G_*'(L))P\end{aligned}$$

Truncation

- After proving such a decomposition exists standard semigroup theory implies there are finitely many unstable directions
- Using an unstable manifold theorem argument we are able to modulate these modes to avoid any instabilities obtaining a perturbation P_0 of P_* such that $P_0 + P_*$ is smooth and supported away from the boundary and the solution converges to P_* at the blow-up time
- Finally, we prove that blow-up in the 1D system implies blow-up in the full 2D system and employ a truncation technique developed by Elgindi and Jeong to ensure the solutions are compactly supported at $r = +\infty$ and hence finite-energy

Summary and Future Directions

- We provide a new scenario in which blow-up in the IPM equation can occur where particles are able to escape along the boundary.
- This suggests it may be possible to prove blow-up in the bulk of the domain
- It would be natural to consider if this scenario can be lifted to the 2D Boussinesq equation and 3D axisymmetric Euler equation