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Trajectory generation for mobile robots

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Abstract

In this paper, first we present different curves used in path generation. We show the advantages and drawbacks of each curve. Next, we propose a method of calculation of a complex smooth path. This smooth path minimizes the integral-square-jerk. Finally, we introduce a variable velocity along the path. The velocity of the robot is a function of path curvature so as to control the centrifugal acceleration.

1. Introduction

An autonomous mobile robot has to be able to analyze sensor information and plan its tasks. When the realization of a task requires a displacement, the robot plans the path and follows it.

In this paper, we deal with the navigation problem. The robot has to be able to reach any configuration in the world frame. The nature of the path followed by a mobile robot, is essential in trajectory generation. If the path presents strong variations of the curvature in some points, it can bring about a slip movement of the wheels, since the speed of the wheels directly depends on curvature. In order to avoid this problem, the path must have a continuous curvature. That is why several researchers have studied different types of curves and their implementation for robotics. Among these curves, there are polynomial curves [5,6], clothoids and cubic spirals [4,3]. We compare these different curves. As a consequence of this study, we have chosen the cubic spiral for our path generator. The main reason is that it provides an optimally smooth path, in the sense that it minimizes the integral-square-jerk when travelled at constant speed. The jerk is the time derivative of the centrifugal acceleration.

A trajectory is obtained by associating a velocity law to a curve. Usually the linear velocity is taken constant along the curve. In this paper, we introduce variable velocity. This velocity is a function of both the curvature of the path and the physical constraints of the robot. Indeed, when the curvature is low, the robot should be allowed a greater speed than when the curvature is high. This new trajectory provides a lower centrifugal acceleration than a trajectory with a constant velocity.

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This method has been successfully implemented on the mobile robot "MELODY" (Mobile d'Exterieur à LOcalisation DYnamique) which has been developed at the LAN (Laboratoire d'Automatique de Nantes).

2. Problem formulation

During a displacement task, the mobile robot has to follow a path, defined by a set of robot's postures, with a specific velocity law and these postures depend on its position and its orientation. If the robot moves in a plane, its posture P is defined, in a world frame (Fig. 1), by three variables (x, y, θ).

2.1. Path generation

The first step is to find a geometric curve which connects the initial and final postures of the robot. The curves which are used here are defined in a 2D Cartesian coordinate system. The orientation θ is defined as the tangent direction of the point $(x(s), y(s))$, where s is a curvilinear abscissa. The curvature is defined as the derivative of $\theta(s)$ with respect to s:

$$\theta(s) = \tan^{-1}\left[\frac{\mathrm{d}x}{\mathrm{d}y}\right], \qquad K(s) = \frac{\mathrm{d}\theta(s)}{\mathrm{d}s}. \tag{1}$$

If the curvature function $K(s)$ and the initial values x_0, y_0, θ_0 are given, then the curve is calculated by

$$\begin{aligned} x(s) &= x_0 + \int_0^s \cos\left[\theta_0 + \int_0^v K(u)\,\mathrm{d}u\right]\mathrm{d}v, \\ y(s) &= y_0 + \int_0^s \sin\left[\theta_0 + \int_0^v K(u)\,\mathrm{d}u\right]\mathrm{d}v. \end{aligned} \tag{2}$$

2.2. Trajectory generation

Once the curve is known, the second step is to define the linear velocity law with which the robot will follow this curve. To obtain it, we take into account the geometric properties of the curve and the physical constraints of the robot (maximum velocity of the wheels).

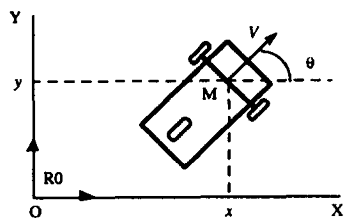

Fig. 1.

3. Common curves used in the path generation

Path finding problem. Let $P_0 = (x_0, y_0, \theta_0)$ and $P_1 = (x_1, y_1, \theta_1)$ be two postures, find a curve starting from (x_0, y_0) with orientation θ_0 and ending at (x_1, y_1) with orientation θ_1. This curve must have a continuous curvature at any point. The curvature is always assumed to be zero at the start and end points of the curve. The disposition of the postures is essential in path finding. An important case has to be considered: *Symmetric postures.* P_0 and P_1 are symmetric if the orientation β of the line defined by (x_0, y_0) and (x_1, y_1) (Fig. 2) satisfies the following condition [3]:

$$\beta = \tan^{-1}\left(\frac{y_1 - y_0}{x_1 - x_0}\right) = \frac{\theta_0 + \theta_1}{2}. \tag{3}$$

If the line starting at (x_0, y_0) with a direction θ_0 and the line which goes through (x_1, y_1) with a direction θ_1 have an intersection point Q (Fig. 2), then these two lines are symmetric with respect to QA. The magnitude of the change of direction α (deflection) is defined as

$$\alpha = \theta_1 - \theta_0. \tag{4}$$

When the postures are symmetric, the curve is defined by a symmetric curvature function. The path is called *simple*.

There are several types of curves used in path generation. In this section, we compare some of them.

3.1. Straight lines

The path is composed by several straight lines [2]. The calculation is easy and requires only the choice of intermediate points. There are several paths depending on the choice of these points. However in all cases, the orientation is discontinuous. The robot must stop and change its direction.

3.2. Circular arcs

The change of direction of the robot can be made by following a circular arc of radius R [7]. The drawback of this method is that the path presents a discontinuous curvature at junction points. This means that the speed of each wheel of the robot is not continuous at these points.

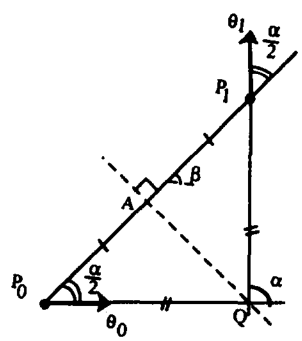

Fig. 2. Symmetric postures.

3.3. *Polynomial functions*

Polynomial curves permit to avoid the discontinuous curvature. Three different polynomials are presented here:
– Polar polynomials to connect symmetric postures;
– Cartesian and Bezier's polynomials to connect non-symmetric postures.

3.3.1. *Polar polynomials* [5]

We replace the circular arc by a polynomial arc with continuous curvature. We look for a curve which satisfies the zero curvature constraints at the start and end points and is close to a circle (Fig. 3). This curve is defined in the polar coordinates (o, x_p, y_p):

$$x(\phi) = r\cos(\phi) \quad \text{and} \quad y(\phi) = r\sin(\phi). \tag{5}$$

The polar radius r is a polynomial function of the angle ϕ with $\phi \in [0, \alpha]$:

$$r(\Phi) = R\left(1 + \frac{\Phi^2}{2} - \frac{\Phi^3}{\alpha} + \frac{\Phi^4}{2\alpha^2}\right). \tag{6}$$

The curve coordinates (5) will be converted to the world frame. When the angle α is greater than 90°, the distance between the polar polynomial and the circular arc may be large. In this case, a piecewise polynomial function consisting of two polar-polynomials separated by a circular arc is used instead of a single curve. The drawback of this method is that the radius R must be fixed and that it is only adapted to symmetric cases.

3.3.2. *Cartesian polynomials* [5]

This type of curves only allows to connect parallel postures $(\theta_0 = \theta_1 = \theta)$. The Cartesian polynomial is a fifth degree polynomial. The six coefficients are chosen to satisfy the position, the orientation and the zero curvature constraints at the boundary points of the curve. Nelson [5] used this curve for the particular case $\theta = 0$. Here we present a general form of a Cartesian polynomial. The curve is parametrized by $x \in [0, \Delta x]$:

$$y(x) = x\tan(\theta) + (\Delta y - \Delta x\tan(\theta))\left[10\left(\frac{x}{\Delta x}\right)^3 - 15\left(\frac{x}{\Delta x}\right)^4 + 6\left(\frac{x}{\Delta x}\right)^5\right], \tag{7}$$

where $\Delta x = x_1 - x_0$ and $\Delta y = y_1 - y_0$. This method is very handy, since the curve is directly calculated in the world frame.

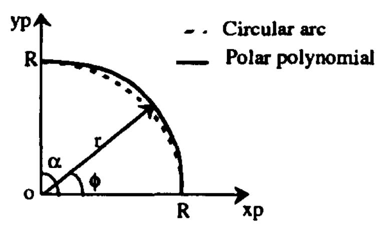

Fig. 3. Cartesian polynomials.

3.3.3. Bezier's polynomials

A Bezier curve is defined by a set of points $B_i = [x_i, y_i]^{\mathrm{T}}$ which forms the "characteristic polygon" [6]. The curve is parametrized by $u \in [0, 1]$ under a polynomial form. For the path generation a third degree polynomial is used. In our case Bezier's polynomial is characterized by 3 points (B_0, B_1, B_c):

$$B(u) = B_0 + 3u(B_1 - B_0) + 3u^2(B_0 - B_c) + u^3(B_1 - B_0), \tag{8}$$

where B_0, B_1 are the start and end points and B_c is the intersection of the line starting from B_0 with directions θ_0 and ending at B_1 with direction θ_1. The curve is calculated in the world frame. This method has no solution when the point B_c does not exist or is equal to B_0 or B_1.

3.4. Clothoids (Cornu spirals)

In order to eliminate the discontinuity of the curvature at the junction points, the circular arc can be replaced by a curve which has a piecewise linear curvature function. These curves are called "clothoids" [4]. The curvature function of a clothoid curve of length l (with $s \in [-l/2, l/2]$) is symmetric with respect to $s = 0$. Its expression for $s \in [0, l/2]$ is

$$K(s) = -k\left(s - \tfrac{1}{2}l\right), \tag{9}$$

where k is the coefficient of the curvature (sharpness). The coordinates (x, y) of the clothoid are given by (2). Its length l is

$$l = 2(\alpha/k)^{1/2}. \tag{10}$$

For a pair of postures (α is a constant), there is a minimum value $k_{\min}$ (corresponding to the maximum length). So, we have to choose a coefficient k higher than $k_{\min}$. The problem of this method is to choose k.

3.5. Cubic spirals

We choose a path with a minimum curvature variation. This type of path is a smooth path [3]. The criterion to be minimized is defined as the integral of the square of the jerk (integral-square-jerk). The jerk is the time derivative of the centrifugal acceleration $\Gamma_c = V^2 K$. If the linear velocity is constant along the curve of length l, then the criterion becomes

$$\text{cost (path)} = \int_0^l (K')^2\,\mathrm{d}s = \int_0^l (\theta'')^2\,\mathrm{d}s. \tag{11}$$

This performance index is the cost of path. Cubic spirals minimize this cost. Their orientation is described by a cubic function of path distance s. For a cubic spiral of length l ($s \in [0, l]$), the curvature function is continuous and symmetric with respect to $s = l/2$. Its expression is

$$K(s) = (6\alpha/l^3)s(1 - s). \tag{12}$$

The spiral coordinates are given by (2). Fig. 4 shows a standard cubic spiral with a unit length for a constant deflection α. $D(\alpha)$ denotes the Euclidean distance (size) between the boundary points of the standard spiral.

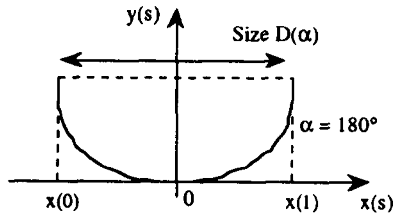

Fig. 4. Standard cubic spiral.

$$D(\alpha) = 2\int_0^{1/2} \cos\left(6\alpha\left(-\frac{u^3}{3}+\frac{u^2}{2}-\frac{1}{12}\right)\right)\mathrm{d}u. \tag{13}$$

All cubic spirals are similar to each other. So the spiral of size d with the same angle α is similar to the standard spiral, its length is

$$l = d/D(\alpha). \tag{14}$$

The solution of the path finding problem is simple when the postures are symmetric. In this case the path is made up of a single cubic spiral [3]. The cost of the path is

$$\text{cost (path)} = (12\alpha^2)/l^3. \tag{15}$$

In the non-symmetric case we cannot use a single cubic spiral because the curve symmetry is not respected. So, we define an intermediate posture P_{int} in the plane, which is symmetric with respect to both P_0 and P_1. The path is made up of two cubic spirals. Two different cases may occur:

– (P_0, P_1) *are parallel.* The intermediate posture P_{int} belongs to the line defined by (x_0, y_0) and (x_1, y_1).
– (P_0, P_1) *are not parallel.* In that case P_{int} belongs to the circle $O_c(x_c, y_c, R_c)$ which goes through (x_0, y_0) and (x_1, y_1). The path is called *complex.* Where

$$(x_c, y_c) = ((x_1 + x_0 - c(y_1 - y_0))/2, (y_1 + y_0 + c(x_1 - x_0))/2)$$

and

$$R_c^2 = x_c^2 + y_c^2 - x_1 x_0 - y_1 y_0 + c(x_0 y_1 - x_1 y_0) \tag{16}$$

with

$$c = \operatorname{cotg}\left(\frac{1}{2}(\theta_1 - \theta_0)\right).$$

P_{int} will be chosen so as to minimize the total cost of the path: $\text{cost}(P_0, P_{\text{int}}) + \text{cost}(P_{\text{int}}, P_1)$. In the parallel case the optimal solution is

$$P_{\text{int}} = \left(\frac{x_1 + x_0}{2}, \frac{y_1 + y_0}{2}, 2\beta - \theta_0\right). \tag{17}$$

There is no analytical method to find the minimum of the total costs in the non-parallel case. So we use an iterative method based on the Golden Search algorithm. The cost function is parametrized as a function of

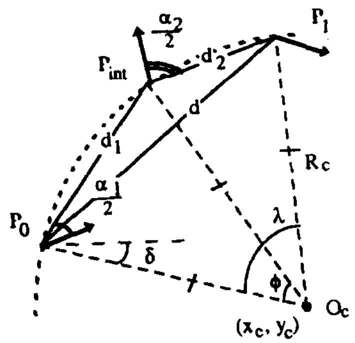

Fig. 5.

the polar angle ϕ (Fig. 5). Let α_1 and α_2 be the deflections of (P_0, P_{int}) and (P_{int}, P_1). Using (14) and (15) we can write

$$\mathrm{cost}(P_0, P_1) = \frac{12\alpha_1^2 D(\alpha_1)}{d_1^3} + \frac{12\alpha_2^2 D(\alpha_2)}{d_2^3}, \tag{18}$$

where d_1, d_2 and d are the sizes of (P_0, P_{int}), (P_{int}, P_1) and (P_0, P_1). Let us define λ as the angle between $P_0 O_c$ and $P_1 O_c$, and δ as the orientation of the line $P_0 O_c$ (Fig. 5),

$$\lambda = \sin^{-1}\left(\frac{d}{R_c}\right), \qquad \delta = \tan^{-1}\left(\frac{y_c - y_0}{x_c - x_0}\right). \tag{19}$$

Using the properties of the isosceles triangle $P_0 P_{int} O_c$, we can write

$$\begin{aligned} &\alpha_1 = C_1 - \phi, \qquad \alpha_2 = C_2 + \phi, \\ &d_1 = 2R_c \sin\left(\tfrac{1}{2}\phi\right), \qquad d_2 = 2R_c \sin\left(\tfrac{1}{2}(\lambda - \phi)\right), \end{aligned} \tag{20}$$

with $C_1 = (\pi - 2\theta_0 - 2\delta)$, $C_2 = (\theta_1 + \theta_0 - \pi + 2\delta)$. Then Eq. (20) in (18) give the cost as a function of the polar angle ϕ, with $\phi \in [0, \lambda]$:

$$\mathrm{cost}(P_0, P_1) = F(\phi).$$

To minimize $F(\phi)$ we use the Golden Search algorithm [1]. When the solution $\hat{\phi}$ is found then the intermediate posture is defined by

$$\begin{aligned} x_{int} &= x_0 + 2R_c \sin\left(\tfrac{1}{2}\hat{\phi}\right)\cos\left(\tfrac{1}{2}(\theta_{int} + \theta_0)\right), \\ y_{int} &= y_0 + 2R_c \sin\left(\tfrac{1}{2}\hat{\phi}\right)\sin\left(\tfrac{1}{2}(\theta_{int} + \theta_0)\right), \\ \theta_{int} &= \alpha_1 + \theta_0. \end{aligned} \tag{21}$$

In conclusion, cubic spirals can connect any two postures in the plane. These curves show a minimum curvature variation and their lengths are close to those of circular arcs.

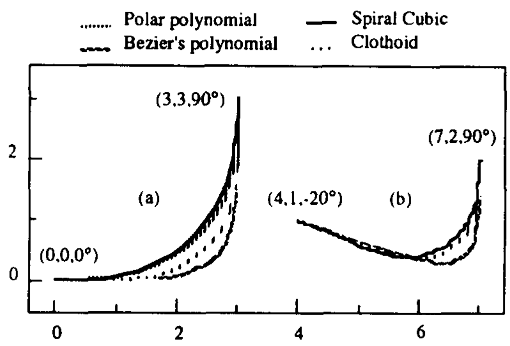

Fig. 6.

3.6. Discussion

Fig. 6 shows the previous paths connecting two pairs of postures: symmetric postures (Fig. 6(a)) and non-symmetric postures (Fig. 6(b)). To use cubic spirals or Bezier curves, we only need to know the initial and final postures. However, to use the other curves, more knowledge is necessary. For circular arcs and polar polynomials the radius R must be known. Clothoid curves require the sharpness k. In the symmetric cases polar polynomials are close to cubic spirals. When comparing cubic spirals and Bezier curves we find that Bezier curves are longer. That is why we prefer the cubic spirals which have, besides the fact they are shorter, a minimum curvature variation.

4. Trajectory generation using a variable velocity law

In Section 2.2, we defined the trajectory as an association of a path and a velocity. Generally the robot follows the path with a constant velocity. For the robot MELODY we use a variable velocity function. The velocity has to be maximum when the robot moves on a straight line, but it has to decrease when there is a turn. This velocity may be defined by

$$V(K(s)) = \frac{V_{\max}}{\left(1 + (K(s)/K_0)^2\right)^{1/2}}, \tag{22}$$

$V_{\max}$ is the maximum vehicle linear speed, K_0 is a coefficient which fixes the minimum value of the velocity and $K(s)$ is the path curvature function. This velocity must be adapted to the navigation constraints (moving off and stop). For the robot MELODY we use a kinematic control of the wheels. Its characteristic kinematic equations are

$$\dot{x} = \frac{\omega_{\mathrm{d}} + \omega_{\mathrm{g}}}{2} r \cos(\theta), \qquad \dot{y} = \frac{\omega_{\mathrm{d}} + \omega_{\mathrm{g}}}{2} r \sin(\theta), \qquad \dot{\theta} = \frac{\omega_{\mathrm{d}} - \omega_{\mathrm{g}}}{2e} r, \tag{23}$$

where $2e$ is the distance between the wheels, r is the radius of the wheels, ω_{d} and ω_{g} are the rotation speeds of the right and left wheels. The space–time relation $s(t)$(t denotes time) is obtained from the velocity function. From these data and using (24), we find the desired speeds:

$$\omega_{\mathrm{d}} = (1 + eK[s(t)])(V[s(t)])/r, \qquad \omega_{\mathrm{g}} = (1 - eK[s(t)])(V[s(t)])/r. \tag{24}$$

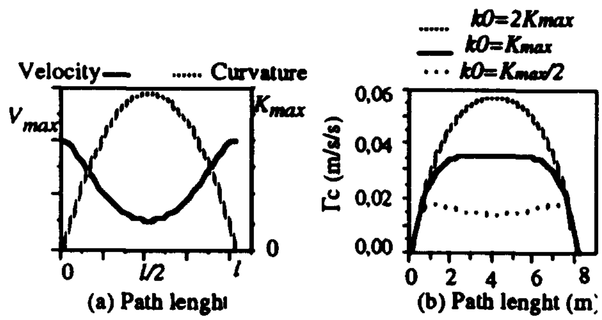

Fig. 7.

In the following discussion we use a path made up of two straight lines connected by a cubic spiral of length l. The graphics and results presented here correspond to the spiral's portion connecting the postures $(0, 0, 0°)$ and $(5, 5, 90°)$. The curvature has a maximum value $K_{\max}$ at the middle of the turn (Fig. 7(a)). This means that the velocity is always minimum at this point and that its value is fixed by K_0. The centrifugal acceleration at this point varies with respect to K_0 (Fig. 7(b)).

For $K_0 = K_{\max}$ the centrifugal acceleration is constant during the turn, because the derivative of Γ_c with respect to s has three zeros at $s = l/2$:

$$\frac{\mathrm{d}\Gamma_c}{\mathrm{d}s} = \frac{\mathrm{Cste}\,(l - 2s)^3(l^2 + 4sl - 4s^2)}{(l^2 + (4s(l - s))^2)^2}. \tag{25}$$

Cste is a constant depending on α and l. The jerk and the velocity along the path are

$$\mathrm{jerk} = V(s)\mathrm{d}\Gamma_c/\mathrm{d}s, \tag{26}$$

$$V(K(s)) = \frac{V_{\max}}{\left(1 + ((4/l^2)s(l - s))^2\right)^{1/2}}. \tag{27}$$

Now, we compare two robots following the path defined previously. The first one uses a constant velocity along the path, the second one uses a velocity defined by (27).We choose two cases:
Case 1: the duration of the movements are equal.
Case 2: the robots have the same maximum velocity (i.e. they have the same velocity when following straight lines).
If the total duration is not important (case 2) then the use of a variable velocity presents a lower Γ_c (Fig. 8(c)) and a minimum integral-square-jerk (Fig. 8(d)). If the time optimal trajectory is required (case 1) this method gives a higher jerk, due to the variation of the linear acceleration during the turn (Fig. 8(b)). The centrifugal acceleration is always lower (Fig. 8(a)).

5. Conclusion

We have presented some curves used in the path generation problem. We have chosen cubic spirals because, in addition to a continuous curvature, they provide the smoothest path. We have proposed an iterative method to solve the problem of connecting complex postures. In this paper we introduce a variable velocity. The robot velocity must be adapted to the path following. That is why velocity is a function of

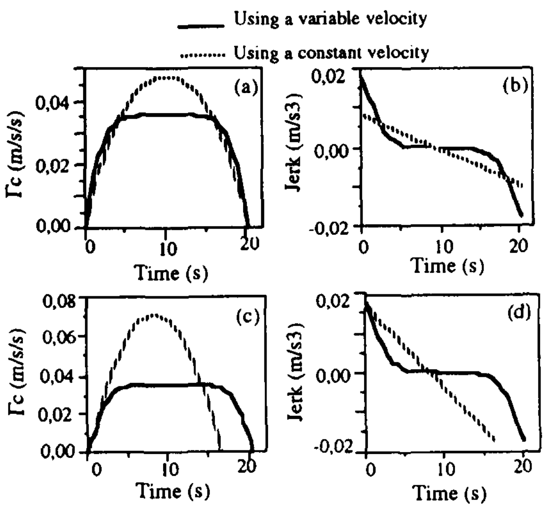

Fig. 8.

curvature. When we associate variable velocity to cubic spirals we obtain a trajectory with a minimum jerk (case 2). This new trajectory permits to bound the centrifugal force applied to the robot during any turn of the path.

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