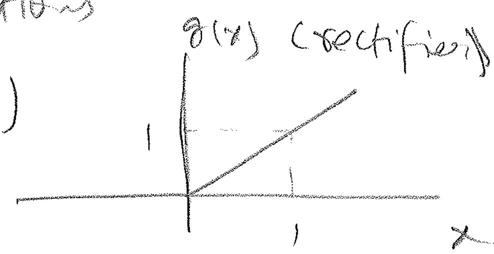
ECE 302 HW 4 Solutions

(1) Input $X \sim N(0, \sigma^2)$ $\quad Y = g(X)$



(a) current $= \dfrac{Y}{R}$

$$\Pr(\{\frac{Y}{R} > 1\}) = \Pr(\{Y > R\}) = \Pr(\{X > R\}) \text{ since R>0}$$

$$\Pr(\{X > R\}) = \int_R^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx = Q(\frac{R}{\sigma})$$

where $Q(\alpha) = \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx = 1 - \Phi(\alpha)$

(b) power$= \dfrac{Y^2}{R}$

$$\Pr(\{\frac{Y^2}{R} > 1\}) = \Pr(\{Y^2 > R\}) = \Pr(\{X > \sqrt{R}\}) = Q(\frac{\sqrt{R}}{\sigma})$$

(c) $\mu_I = E[\frac{Y}{R}] = \frac{1}{R} E[Xu(X)] = \frac{1}{R}\int_0^{\infty} x f_X(x) dx$

$$= \frac{1}{R}\int_0^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi}\sigma R}(-\sigma^2) e^{-\frac{x^2}{2\sigma^2}} \Big|_0^{\infty} = \frac{\sigma}{\sqrt{2\pi} R}$$

$$E[\frac{Y^2}{R^2}] = \frac{1}{R^2}\int_0^{\infty} x^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi}\sigma R^2}\left\{\left[-x\sigma^2 e^{-\frac{x^2}{2\sigma^2}} \Big|_0^{\infty}\right] + \sigma^2 \int_0^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx\right\}$$

$$= \frac{\sigma}{\sqrt{2\pi} R^2}\left(\int_0^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx\right)$$

Suppose $A = \int_0^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx$

$$A^2 = \int_0^{\infty}\int_0^{\infty} e^{-\frac{x^2+y^2}{2\sigma^2}} dx dy = \int_0^{\infty}\int_0^{\pi/2} e^{-\frac{r^2}{2\sigma^2}} r d\theta dr = \frac{\sigma^2 \pi}{2}$$

$$E[\frac{Y^2}{R^2}] = \frac{\sigma}{\sqrt{2\pi} R^2} \times \sigma\sqrt{\frac{\pi}{2}} = \frac{\sigma^2}{2R^2}$$

$$\sigma^2{}_I = Var[\frac{Y^2}{R^2}] = E[\frac{Y^2}{R^2}] - \left(E[\frac{Y}{R}]\right)^2 = \frac{\sigma^2}{2R^2}(1 - \frac{1}{\pi})$$

(d) $\mu_P = E[\frac{Y^2}{R}] = \frac{1}{R}\int_0^{\infty} x^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx = \frac{\sigma^2}{2R}$

From $E[\left(X - \overline{X}\right)^4] = 3\sigma^4$ and $\overline{X} = 0$

$$\int_{-\infty}^{\infty} x^4 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx = 3\sigma^4$$

$$\Rightarrow \int_0^\infty x^4 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx = \frac{3}{2}\sigma^4$$

$$E[\frac{Y^4}{R^2}] = \frac{1}{R^2}\int_0^\infty x^4 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx = \frac{3\sigma^4}{2R^2}$$

$$\sigma^2{}_P = Var[\frac{Y^4}{R^2}] = E[\frac{Y^4}{R^2}] - \left(E[\frac{Y^2}{R}]\right)^2 = \frac{3\sigma^4}{2R^2} - \frac{\sigma^4}{4R^2} = \frac{5\sigma^4}{4R^2}$$

(2)

(a) $E[X] = E[X|G_1]\Pr(G_1) + E[X|G_2]\cdot \Pr(G_2)$

$= 80 \times 0.6 + 20 \times 0.4$

$= 48 + 8 = 56$

$\mathrm{var}(X) = E[\mathrm{var}(X|G)] + \mathrm{var}[E[X|G]]$

$= 20 \times 0.6 + 10 \times 0.4 + (80-56)^2(0.6) + (20-56)^2 \times 0.4$

$= 880$

where G denotes the groups.

$\sigma_X = 29.66$

(b) $f_X(x) = 0.6\left(\frac{1}{\sqrt{2\pi}\sqrt{20}} e^{-\left(\frac{(x-80)^2}{2\times 20}\right)}\right) + 0.4\frac{1}{\sqrt{2\pi}\sqrt{10}} e^{-\left(\frac{(x-20)^2}{2\times 10}\right)}$.

X is NOT Gaussian here

$\Pr(X > 56 + 29.66) = 0.6\left(1 - \phi\left(\frac{56+29.66-80}{\sqrt{20}}\right)\right) + 0.4\left(1 - \phi\left(\frac{56+29.66-20}{\sqrt{10}}\right)\right)$

$\Pr(A)$

$= 0.6(1 - \phi(1.265)) + 0.4(1 - \phi(20.76))$

≈ 0.0617

$\Pr(B) = \Pr(\bar{x} < X \leq \bar{x} + \sigma_X)$

$= 0.6(\phi(1.265) - \phi(-5.366)) + 0.4(\phi(20.76) - \phi(11.38))$

≈ 0.53826

$\Pr(C) = \Pr(\bar{x} - \sigma_X < X \leq \bar{x})\ldots$

Proceed further on similar lines.

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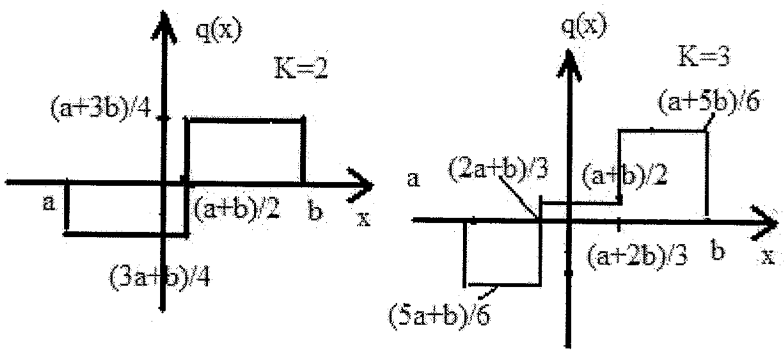
(a) If the quantizer has K levels, then the ith interval will be

$$x \in \left(a + (i-1) \times \frac{(b-a)}{K}, a + i \times \frac{(b-a)}{K} \right), i = 1, \ldots, K$$

When x is in that interval, the output $Y = q(X)$ is equal to the midpoint of the interval.

$$y = a + \frac{(b-a)(2i-1)}{2K}$$

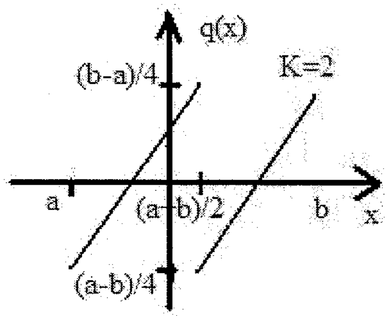


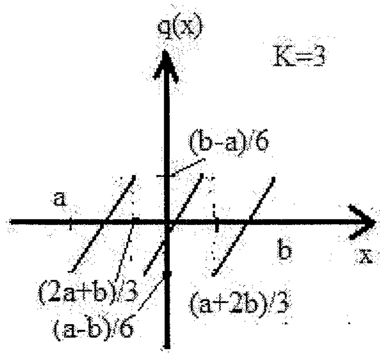
(b) $g(x) = x - q(x)$

In the i[th] interval,

$$x \in \left(a + (i-1) \times \frac{(b-a)}{K}, a + i \times \frac{(b-a)}{K} \right), i = 1, \ldots, K$$

$$g(x) = x - \left[a + \frac{(b-a)(2i-1)}{2K} \right]$$





(c) for general K

$Y = q(X)$

$$f_Y(y) = \sum_{i=1}^{K} \frac{1}{K} \delta\left(y - \left[a + \frac{(b-a)(2i-1)}{2K} \right] \right)$$

$E = g(X)$

$$f_E(e) = f_X(x)\left|\frac{dx_1}{de}\right| + \ldots + f_X(x)\left|\frac{dx_K}{de}\right|, \ E \in [\frac{(a-b)}{2K}, \frac{(b-a)}{2K}]$$

$$= 0 \text{ ,else}$$

$$f_E(e) = \begin{cases} \dfrac{K}{(b-a)} & , E \in [\dfrac{(a-b)}{2K}, \dfrac{(b-a)}{2K}] \\ 0 & , else \end{cases}$$

(d) $$E[E^2] = \int_a^b g(x)^2 f_X(x)dx = K \int_a^{a+\frac{1}{K}(b-a)} \left(x - \left(a + \frac{b-a}{2K} \right) \right)^2 \frac{1}{(b-a)} dx$$

$$= \frac{K}{(b-a)} \frac{1}{3} \left[x - \left(\frac{(2K-1)a+b}{2K} \right) \right]^3 \Bigg|_a^{a+\frac{b-a}{K}}$$

$$= \frac{K}{3(b-a)} \left\{ \left[a + \frac{b-a}{K} - \left(\frac{(2K-1)a+b}{2K} \right) \right]^3 - \left[a - \left(\frac{(2K-1)a+b}{2K} \right) \right]^3 \right\}$$

$$= \frac{K}{3(b-a)} \left[\left(\frac{b-a}{2K} \right)^3 - \left(\frac{a-b}{2K} \right)^3 \right]$$

$$= \frac{(b-a)^2}{12K^2}$$

Alternatively, E is uniform in $\left[\frac{a-b}{2k}, \frac{b-a}{2k}\right]$

$E[E] = 0$

$E[E^2] = \text{var}[E] = \frac{(b-a)^2}{12K}$

(4)

(i) Poisson Process.

(a) $Pr(T \geq 5) = \int_5^\infty \frac{1}{5} e^{-\frac{t}{5}} dt$

$= -\int_5^\infty e^{-\frac{t}{5}} d(-\frac{t}{5})$

$= -e^{-\frac{t}{5}} \Big|_5^\infty$

$= e^{-1}$

Alternatively, $Pr(T \geq 5) = 1 - Pr(T \leq 5)$

$= 1 - F_T(5)$

$= 1 - (1 - e^{-\frac{1}{5} \cdot 5})$

$= e^{-1}$

(b) $Pr(T_2 > 10) = \int_{10}^\infty \frac{1}{5}(\frac{t}{5}) e^{-\frac{t}{5}} dt$

$= \frac{1}{5} \int_{10}^\infty t e^{-\frac{t}{5}} d(-\frac{t}{5})$

$= -\frac{1}{5} \int_{10}^\infty t \, d e^{-\frac{t}{5}} = -\frac{1}{5} t e^{-\frac{t}{5}} \Big|_{10}^\infty + \frac{1}{5} \int_{10}^\infty e^{-\frac{t}{5}} dt$

$= 2e^{-2} - \int_{10}^\infty e^{-\frac{t}{5}} d(-\frac{t}{5})$

$= 2e^{-2} - e^{-\frac{t}{5}} \Big|_{10}^\infty$

$= 3e^{-2}$

(c) $Pr(T > 10 \mid T > 5) = Pr(T > 5) = e^{-1}$ Memoryless Property.

Alternatively $Pr(T > 10 \mid T > 5) = \dfrac{Pr(T > 10)}{Pr(T > 5)} = \dfrac{\int_{10}^\infty \frac{1}{5} e^{-\frac{t}{5}} dt}{e^{-1}}$

$= e^{-1}$

(d) $Pr(T_2 > 20 \mid T_2 > 10) = \dfrac{Pr(T_2 > 20)}{Pr(T_2 > 10)} = \dfrac{\int_{20}^\infty \frac{1}{5}(\frac{t}{5}) e^{-\frac{t}{5}} dt}{3e^{-2}}$

$= \dfrac{5e^{-4}}{3e^{-2}} = \frac{5}{3} e^{-2}$

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(e) $E[T \mid T \geq 5] = 5 + E[T] = 10$ Memoryless Property.

Alternatively, $f_T(t \mid T>5) = \dfrac{f_T(t)}{Pr(T>5)} = \dfrac{\frac{1}{5}e^{-\frac{t}{5}}}{e^{-1}} \quad t>5$

$$
\begin{aligned}
E[T \mid T>5] &= \int_5^\infty t f_T(t \mid T>5)\, dt \\
&= \int_5^\infty e^{+1} \frac{t}{5} e^{-\frac{t}{5}}\, dt \\
&= -e \int_5^\infty t e^{-\frac{t}{5}}\, d(-\tfrac{t}{5}) \\
&= -e \int_5^\infty t\, d(e^{-\frac{t}{5}}) \\
&= -e t e^{-\frac{t}{5}}\Big|_5^\infty + e\int_5^\infty e^{-\frac{t}{5}} dt \\
&= 5 + 5 = 10 .
\end{aligned}
$$

(f) $E[T_2 \mid T_2 > 10]$

$f_{T_2}(t \mid T_2>10) = \dfrac{f_{T_2}(t)}{P_r(T_2>10)} = \dfrac{\frac{1}{5}(\frac{t}{5})e^{-\frac{t}{5}}}{3e^{-2}} \quad t>10$

$$
\begin{aligned}
E[T_2 \mid T_2>10] &= \int_{10}^\infty t f_{T_2}(t \mid T_2>10)\, dt \\
&= \frac{e^2}{75}\int_{10}^\infty t^2 e^{-\frac{t}{5}}\, dt \\
&= \frac{e^2}{75}\left[-5\int_{10}^\infty t^2 e^{-\frac{t}{5}}\, d(-\tfrac{t}{5})\right] \\
&= \frac{e^2}{75}\left[-5\int_{10}^\infty t^2\, de^{-\frac{t}{5}}\right] \\
&= \frac{e^2}{75}\left[-5t^2 e^{-\frac{t}{5}}\Big|_{10}^\infty + 5\int_{10}^\infty e^{-\frac{t}{5}} 2t\, dt\right] \\
&= \frac{e^2}{75}\left[500e^{-2} - 50\int_{10}^\infty t e^{-\frac{t}{5}}\, d(-\tfrac{t}{5})\right] \\
&= \frac{e^2}{75}\left[500e^{-2} - 50te^{-\frac{t}{5}}\Big|_{10}^\infty + 50\int_{10}^\infty e^{-\frac{t}{5}} dt\right] \\
&= \frac{e^2}{75}\left[500e^{-2} + 500e^{-2} - 250\int_{10}^\infty e^{-\frac{t}{5}}\, d(-\tfrac{t}{5})\right] \\
&= \frac{e^2}{75}(1250e^{-2}) = \frac{50}{3}.
\end{aligned}
$$

(ii) Bernoulli Process

(a) $Pr(T \geq 5) = 1 - Pr(T \leq 4)$ $\quad p = \frac{1}{5}$

$$= 1 - \sum_{t=1}^{4} p(1-p)^{t-1} = 0.41$$

Alternatively, $Pr(T \geq 5) = \sum_{t=5}^{\infty} p(1-p)^{t-1} = \dfrac{1-5p+10p^2-10p^3+5p^4-p^5}{1-p} = 0.41$

(b) Let T_2 be a 2nd order negative binomial r.v.

$$Pr(T_2 \geq 10) = 1 - Pr(T_2 \leq 9)$$

$$= 1 - \sum_{t_2=2}^{9} \binom{t_2-1}{2-1} (1-p)^{t_2-2} p^2 = 0.44$$

Alternatively, $Pr(T_2 \geq 10) = \sum_{t_2=10}^{\infty} \binom{t_2-1}{2-1} (1-p)^{t_2-2} p^2 = p^2 \sum_{t_2=10}^{\infty} (t_2-1)(1-p)^{t_2-2}$

$$= \frac{(-1+p)^{10}(1+8p)}{(1-p)^2} = 0.44$$

(c) $Pr(T \geq 10 \mid T \geq 5) = Pr(T \geq 10 \mid T > 4) = Pr(T \geq 6)$ by Memoryless Property.

$$= 1 - \sum_{t=1}^{5} p(1-p)^{t-1} = 0.33$$

Alternatively $Pr(T \geq 10 \mid T \geq 5) = \dfrac{Pr(T \geq 10)}{Pr(T \geq 5)} = \dfrac{\sum_{t=10}^{\infty} p(1-p)^{t-1}}{\sum_{t=5}^{\infty} p(1-p)^{t-1}} = 0.33$

(d) $Pr(T_2 \geq 20 \mid T_2 \geq 10) = \dfrac{Pr(T_2 \geq 20)}{Pr(T_2 \geq 10)} = \dfrac{1 - \sum_{t_2=2}^{19} \binom{t_2-1}{2-1}(1-p)^{t_2-2} p^2}{1 - \sum_{t_2=2}^{9} \binom{t_2-1}{2-1}(1-p)^{t_2-2} p^2}$

$$= 0.19$$

(e) $E[T \mid T \geq 5] = E[T \mid T > 4] = E[T] + 4 = \frac{1}{p} + 4 = 9$. by Memoryless Property

Alternatively, $P_r(T=t \mid T \geq 5) = \frac{P_r(T=t)}{P_r(T \geq 5)} = \frac{p(1-p)^{t-1}}{\sum_{t=5}^{\infty} p(1-p)^{t-1}}$

$$= \frac{(1-p)^t p}{1-5p+10p^2-10p^3+5p^4-p^5}$$

$$E[T \mid T \geq 5] = \sum_{t=5}^{\infty} t P_r(T=t \mid T \geq 5) = \frac{1+4p}{p} = 9$$

(f)

$$P_r(T_2 = t_2 \mid T_2 \geq 10) = \frac{P_r(T_2 = t_2)}{P_r(T_2 \geq 10)} = \frac{\binom{t_2-1}{2-1}(1-p)^{t_2-2}p^2}{1 - \sum_{t_2=2}^{9}\binom{t_2-1}{2-1}(1-p)^{t_2-2}p^2} \quad t_2 \geq 10$$

$$= (t_2-1) \cdot 0.09 \cdot 0.8^{t_2-2}$$

$$E[T_2 \mid T_2 \geq 10] = \sum_{t_2=10}^{\infty} t_2 P_r(T_2 = t_2 \mid T_2 \geq 10) = 15.54.$$