Discrete Math, 375 (Spring 2015): Mid-term 3.

Do all of the problems below. Please write your name and your class (the class time of 1030 or 1200) on blank cover sheet (the backside should be blank as well). Write neatly and be clear. The $1/n$ rule for collaboration will be employed if I feel n people collaborated.

Don't think about all of those things you fear. Just be glad you are here! After all, you are made of matter that once resided in a star.

Problem 1. [Happy Birthday: 10 points] State the definition for the following:

(a) A complete graph on a set V

(b) A 2–coloring of a graph $\Gamma(V,E)$.

Solution:

(a) A complete graph on a set V is the graph $\Gamma(V,E)$ where the edge set E is given by

$$E = \{e \in E \ : \ e = [v,w], \ v,w \in V, \ v \neq w\}.$$

(b) A 2–coloring of a graph $\Gamma(V,E)$ is a function

$$C \colon E \longrightarrow \{\text{popper-mint green}, \ \text{luscious lavender}\}.$$

Problem 2. [Tortoise or Hare: 10 points] State whether or not the following is true or false. You do not need to justify your answer.

(a) If $\Gamma(V,E)$ is a tree and $\Gamma(V',E')$ is a subgraph of $\Gamma(V,E)$, then $\Gamma(V',E')$ is a tree.

(b) If $\Gamma(V,E)$ is a bipartite graph and $\Gamma(V',E')$ is a subgraph of $\Gamma(V,E)$, then $\Gamma(V',E')$ is bipartite.

(c) If $\Gamma(V,E)$ is a bipartite graph, $\Gamma(V',E')$ is a subgraph of $\Gamma(V,E)$, and $\Gamma(V,E)$ admits a complete matching, then $\Gamma(V',E')$ admits a complete matching.

(d) If $\Gamma(V,E)$ is a tree and $|V| = 13$, then $|E| = 12$.

Solution:

(a). False. Every subgraph of a tree is a forest but the subgraph is in general not connected.

(b). True. Remember, bipartite is very weak. It just means that the vertex set V can be partitioned into two sets V_1, V_2 such that every edge has an endpoint in each. If the big graph satisfies that condition, every subgraph will.

(c) False. Every bipartite graph is a subgraph of the corresponding complete bipartite graph for the partition sets. That has lots of complete matches. Not every bipartite graph has a matching.

(d) True. See Problem 6.

Problem 3. [Dr. van der Waerden: 10 points] Let k, r be positive integers. Prove that there exists an $n_{k,r}$ such that for any $n \geq n_{k,r}$, if the set $\{1, \dots, n\}$ is r–colored, there exists a k–term monochromatic arithmetic progression $\{s_j\}_{j=1}^{k}$ such that $s_{j+1} - s_j$ is the same color for all $j \in \{1, \dots, k-1\}$.

Solution: By van der Waerden's theorem, we can find an $n_{k,r}$ such that for $n \geq n_{k,r}$, if the set $\{1, \dots, n\}$ is r–colored, there exists a k–term monochromatic arithmetic progression $\{s_j\}_{j=1}^{k}$. Note that if $s_j = aj + b$, then $s_j - s_{j-1} = a$ for all j and so the above $n_{k,r}$ suffices for the "upgrade".

Problem 4. [Connections: 10 points]

(a) Given a connected graph $\Gamma(V,E)$ with $|V| \geq 2$, prove that there are at least two vertices x,y such that subgraph on the vertex sets $V-\{x\}, V-\{y\}$ are connected.

(b) Let $\Gamma(V,E)$ be a connected graph with $|V| = n$. Prove that for each $1 \leq k \leq n$, there is a subset $A \subset V_1$ with $|A| = k$ such that the associated subgraph for A is connected.

Solution:

(a). There are a number of ways of proving this result. I will give three solutions.

The birdcage ball solution: On the graph Γ, let

$$M_\Gamma = \max\{d(v,w) \ : \ v,w \in V\},$$

where d is the path metric with edges weight to have length one. By definition, there exist two vertices $x,y \in V$ such that

$$d(x,y) = d_\Gamma.$$

I claim that $\Gamma(V-\{x\}, E-E_x)$ is a connected graph where

$$E_x = \{e \in E \ : \ e = [x,\star], \ \star \in V\}.$$

If this graph were not connected, there would be some vertex $z \in V-\{x\}$ that could not be connected to y by an edge path. In particular, if we took a path from z to y in the full graph Γ, that edge path would have to pass through x. Now, we see that

$$d(z,y) > d(x,y) = M_\Gamma.$$

By definition of M_Γ, that is impossible and so $\Gamma(V-\{x\}, E-E_x)$ is connected. An identical argument shows that $\Gamma(V-\{y\}, E-E_y)$ is connected.

Spanning tree solution: On Γ, take a maximal spanning tree T and note that it suffices to prove it T. Using problem 6, there exist (at least) two vertices in the tree T that have precisely 1 edge where they arise as endpoints, provided $|V| \geq 2$. Remove those points.

Induction Method: We prove it by induction noting the case $|V| = 2$ is straightforward. We assume the result holds for all $j < n$ and seek to verify if for n. If the graph on the vertex set $V-\{x\}$ is connected for a vertex $x \in V$, we cheer hooray and more on to find another. Otherwise, the graph on the vertex set $V-\{x\}$ is not connected. By induction, we win on any connected component provided the vertex set of that component is at least 2. There is a remaining case when there is not path components with at least two vertex sets. This means all of the path components are a single point with no edges. That means our graph looks like the skeleton of an umbrella with x at the pointed top. You see that every other vertex in the graph works in this case.

(b) Pick any point $x \in V$ and let

$$N_i(x) = \{y \in V \ : \ d(x,y) = j\}$$

and set

$$N_i(x) = \left\{x_{i,j}\right\}_{j=1}^{n_i}.$$

Further, set i_x to be the greatest i such that $N_i(x) \neq \emptyset$. It follows that

$$\sum_{i=1}^{i_x} n_i = n.$$

For each $x_{i,j}$, there is an edge path of length j starting at x and ending at $x_{i,j}$. In particular, there is an edge with endpoints $e = [x_{i-1,j'}, x_{i,j}]$ for some $1 \leq j' \leq n_{i-1}$. We build now the subgraphs inductively. For $1 \leq k \leq n_1$, we set $\Gamma_{1,k}$ to be the graph with vertex set $V_{1,k} = \{x, x_{1,1}, x_{1,2}, \ldots, x_{1,k}\}$. Each vertex $x_{1,j} \in V_{1,k}$ is connected to x by an edge $e_{1,j}$ with endpoints $e_{1,j} = [x, x_{1,j}]$. We set

$$E_{1,j} = \{e_{1,j}\}_{j=1}^{k}.$$

The graphs $\Gamma(V_{1,k}, E_{1,k})$ are connected subgraphs with vertex size k. Ranging from 1 to n_1 takes care of all of the "first level". At the "second level", for $n_1 < k < n_2$, we take

$$V_{2,k} = V_{1,n_1} \cup \{x_{2,1}, \ldots, x_{2,k-n_1}\}.$$

For the edge set, for each $x_{2,j}$ there is an edge $e_{2,j}$ with endpoints $e_{2,j} = [x_{1,j'}, x_{2,j}]$ for some $1 \leq j' \leq n_1$. We define

$$E_{2,k} = E_{1,n_1} \cup \{e_{2,1}, \ldots, e_{2,k-n_1}\}.$$

The graphs $\Gamma(V_{2,k}, E_{2,k})$ are connected for all $n_1 < k \leq n_2$. We now proceed via as above in the final range when $n_{i_x-1} < k \leq n_{i_x}$. We repeat as before. These gives a family of connected graphs with vertex set size ranging from 1 to n. Note also that the final subgraph is a spanning tree.

Alternative solution that proves less:

By (a), we can remove a vertex x so that the graph associated to $V - \{x\}$ is connected. That solves $|A| = |V| - 1$. Now iterate.

Problem 5. [Which Hall was that again?: 10 points]

(a) Let $\Gamma(V,E)$ be a bipartite graph with partition $V = V_1 \cup V_2$. Prove that if $|V_1| = |V_2| = n > 1$ and for each $v \in V$, we have

$$|\{e \in E \ : \ e = [v,w]\}| \geq \frac{n}{2},$$

then $\Gamma(V,E)$ has a complete matching.

(b) Let $\Gamma(V,E)$ be a k–regular bipartite graph and let E_0 be a set of edges of size $k-1$. Prove that $\Gamma(V, E - E_0)$ has a complete matching.

Solution:

(a) By Hall's Theorem, it suffices to show for each $A \subset V_1$ that

$$|N(A)| \geq |A|.$$

Since $|N(x)| \geq n/2$ for each $x \in V_1$, we can assume $|A| > n/2$. If

$$|N(A)| < |A|$$

then for $y \in V_2 - N(A)$, we have

$$N(y) \subset V_1 - A.$$

In particular,

$$|N(y)| \leq |V_1 - A| < n/2.$$

However, that contradicts our assumption that $|N(y)| \geq n/2$.

(b) Since $\Gamma(V,E)$ is k–regular and bipartite, we have a partitioning $V = V_1 \cup V_2$ with $|V_1| = |V_2| = n$. We are given a subset E_0 of the edge set with $|E_0| = k-1$. No one solved this. I will leave this one as a point of mystery. (a) took longer to solve for me.

Problem 6. [Repeat: 10 points] Let $\Gamma(V,E)$ be a graph.

(a) Prove that if $\Gamma(V,E)$ is connected, then $|E| \geq |V| - 1$.

(b) Prove that if $\Gamma(V,E)$ is a tree, then $|E| \leq |V| - 1$.

(c) Prove that if $\Gamma(V,E)$ is connected and $|E| = |V| - 1$, then $\Gamma(V,E)$ is a tree.

Solution:

(a) In the construction of the subgraphs in Problem 4 (b), the final subgraph has $|V|$ vertices and $|V| - 1$ edges. This implies that $|E| \geq |V| - 1$.

Analogy that shows it is trivial: Each vertex is a city or location. Each edge represents a road. If the graph is connected, you can drive to any city. So there must be at least the number of cities minus 1 as there are roads since you have to drive on a road to move to a new city. (this is not a mathematical proof).

(b) We prove this via induction. It is trivial for the case of $|V| = 1$. Assuming it holds for all $|V| < n$, we now prove it holds for $|V| = n$. Take any vertex $x \in V$ and let $\Gamma(V_{x,i}, E_{x,i})$ be the path components if the subgraph of our tree $\Gamma(V - \{x\}, E - E_x)$. It follows that each subgraph is also a tree since any non-trivial cycle would produce a non-trivial cycle in the big tree $\Gamma(V,E)$. So by induction, we have

$$|E_{x,i}| \leq |V_{x,i}| - 1.$$

If for each i, there is more than one $y \in V_{x,i}$ that is connected to x path a path, we would have a non-trivial cycle. So we know that

$$\sum_i (|E_{x,i}| + 1) = |E|.$$

It is straightforward to see that this implies $|E| \leq |V| - 1$.

(c) Take our maximal subtree T from Problem 4 (b) inside of $\Gamma(V,E)$. If Γ is not a tree, then

$$|E_T| < |E| \leq |V| - 1,$$

where E_T is the edge set for T. This is impossible since T is tree and thus we contradict (a).