Math 453 Abstract Algebra sample

1. Find $gcd(123, 745)$
2. Find the inverse of 23 in $\mathbb{Z}_{71}$
3. Find the last two digits of 15^{100}.
4. Prove that the set of all 2×2 matrices with entries from $\mathbb{R}$ and determinant 1 is a group under matrix multiplication.
5. Prove that a group G is Abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for any $a, b \in G$.
6. Prove that in any group, an element and its inverse have the same order.
7. Suppose that H is a proper subgroup of $\mathbb{Z}$ under addition and H contains $18, 30$ and 40, Determine H.
8. Suppose a and b are elements in a group such that $|a| = 4, |b| = 2$, and $a^3b = ba$. Find $|ab|$.
9. Determine the subgroup lattice of $\mathbb{Z}_{48}$.
10. let G be a group and let a be an element of G.
    (1) if $a^{12} = e$, what can we say about the order of a?
    (2) if $a^m = e$, what can we say about the order of a?
    (3) suppose that $|G| = 24$ and that G is cyclic. If $a^8 \neq e$ and $a^{12} \neq e$, show that $G = \langle a \rangle$.
11. Consider $\sigma = (13256)(23)(46512)$.

    (a)Express σ as a product of disjoint cycles.

    (b) Find its order.
12. (a) Let $\alpha = (1, 3, 5, 7, 9, 8, 6)(2, 4, 10)$. What is the smallest positive integer n such that $\alpha^n = \alpha^{-5}$?

    (b) Let $\beta = (1, 3, 5, 7, 9)(2, 4, 6)(8, 10)$ If β^m is a 5-cycle, what can you say about m?
13. In S_7 show that $x^2 = (1, 2, 3, 4)$ has no solutions, but $x^3 = (1, 2, 3, 4)$ has at least two.
14. Let G be a group. Show that $\phi : G \to G$ defined by $\phi(g) = g^{-1}$ is an isomorphism if and only if G is Abelian.
15. Show that $\phi : (\mathbb{Z}/16\mathbb{Z})^* \to (\mathbb{Z}/16\mathbb{Z})^*$ defined by $\phi(x) = x^3$ is an isomorphism (automorphism). What about the maps $x \mapsto x^5$ and $x \mapsto x^7$?
16. Suppose x is an element of a cyclic group of order 15 and $x^3 = x^5 \neq x^9$. Determine $|x^{13}|$.
17. Find all generators for $\mathbb{Z}/49\mathbb{Z}$.

18. Find the number of generators for $\mathbb{Z}/49000\mathbb{Z}$. The generators are the positive integers relatively prime to 49000, and which relatively prime to 49000. Its number is

$$\phi(49000) = \phi(7^2 \cdot 5^4 \cdot 2^4) = \phi(7^2)\phi(5^4)\phi(2^4) = (7^2 - 7)(5^4 - 5^3)(2^4 - 2^3) = ...$$

Finish...

19. Prove that the following groups are not cyclic:

    (a) $\mathbb{Z}_2 \times \mathbb{Z}_2$

    (b) $\mathbb{Z}_2 \times \mathbb{Z}$

    (c) $\mathbb{Z} \times \mathbb{Z}$

    (d) $\mathbb{Q} \times \mathbb{Q}$

    Solution: (c) Suppose (a, b) is a generator of $\mathbb{Z} \times \mathbb{Z}$. Then, in particular,

$$(1, 0) \in \mathbb{Z} \times \mathbb{Z} =< (a, b) >,$$

    and $(1, 0) = c(a, b)$ for some integer c , with $ca = 1$, $cb = 0$. By the above $c \neq 0$ $a \neq 0$ and $b = 0$.

    Similarly $(0, 1) = d(a, b)$ for some integer d , with $da = 0$, $db = 1$. By the above $d \neq 0$ $b \neq 0$ and $a = 0$, a contradiction.

20. Assume $|x| = n$ and $|y| = m$. Suppose that x and y commute: $xy = yx$. Prove that $|xy|$ divides the least common multiple of m and n.

    Solution: Since $|x| = n$ and $|y| = m$ we have $x^m = e$, and $y^n = e$. If $l := lcm(m, n)$ then $l = kn$, and $l = sm$ for some k, s. Thus $x^l = (x^n)^k = e$ and similarly $y^l = e$.

    Thus $(xy)^l = xyxy \ldots xy = x^l y^l = e$. which implies that $|xy|$ divides l.

21. Let G be a cyclic group of order n, and let k be an integer relatively prime to n. Prove that the map $\phi : x \to x^k$ is an isomomorphism.

    Solution: ϕ is a homorphism: $\phi(xy) = (xy)^k = xyxy \ldots xy = x^k y^k = \phi(x)\phi(y)$.

    Note that $x^n = e$ for any $x \in G$. Let l be an integer such that $lk \equiv 1(modn)$. Then for any $x \in G$ we have $x^{kl} = x$. Thus for ψ such that $\psi(x) = x^l$, we have $\phi\psi(x) = \psi\phi(x) = x^{kl} = x$. This means that ϕ has its inverse $\phi^{-1} = \psi$, and is bijective. Since ϕ is a homorphism which is bijective it is an isomorphism.

22. Let G be an abelian group. Prove that the map $\phi : x \to x^k$ is a homomorphism.

    Solution:

    $\phi(xy) = (xy)^k = xyxy \ldots xy = x^k y^k = \phi(x)\phi(y)$

23. Prove that $\phi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ by $\phi(a, b) = a - b$ is a homomorphism.

24. Determine all homomorphisms and isomorphisms from $\mathbb{Z}_n$ to itself

25. Find the center of D_{12}, and $Gl_2(F_2)$.

    Solution (for D_{12}):

    $D_{12} = \{e, r, r^2, r^3, r^4, r^5, s, sr, sr^2, sr^3, sr^4, sr^5, \}$.

    The center $Z(D_{12})$ consists of elements commuting with all the elements in D_{12}.

    Commutativity of r^i with s. Since $r^i s = sr^{-i}$, and $r^i \neq r^{-i}$ for $i = 1, 2, 4, 5$ then these elements are not in $Z(D_{12})$.

    Commutativity of sr^i with r. $r(sr^i) = sr^{-1}r^i = sr^{i-1}$. But $(sr^i)r = sr^{i+1}$. Since $r^2 \neq e$ we get that $sr^{i-1} \neq sr^{i+1}$ and $r(sr^i) \neq (sr^i)r$. This implies that none of the elments sr^i $i = 0, \dots, 5$ is in $Z(D_{12})$.

    It suffices to show commutativity with r^3.

    It commutes with s: $sr^3 = r^3 s$ and with r: $r^3 r = rr^3 = r^4$. Since r, s generate $Z(D_12)$. Thus r^3 commutes with all the elements $Z(D_{12})$.

26. Find the centralizers of the elements r, r^3, sr in D_{12}. Find the normalizer of $\{r, r^5\}$ in D_{12}.

    Solution: (The centralizer for sr): We check commutativity of sr with all kinds of elements.

    Commutativity of r^i with sr:

    Since $r^i sr = sr^{-i+1}$, and $sr^i r = sr^{i+1}$ $r^i \neq r^{-i}$ for $i = 1, 2, 4, 5$ then these elements are not in $C_{sr}(D_{12})$. If $i = 0, 3$ then $r^i = r^{-i}$ and e, r^3 are in $C_{sr}(D_{12})$.

    Commutativity of sr^i with sr:

    $sr(sr^i) = ssr^{-1}r^i = r^{i-1}$. But $(sr^i)sr = ssr^{-i+1} = r^{-i+1}$. The elemnts sr and sr^4 are in $C_{sr}(D_{12})$,

    Finally $C_{sr}(D_{12}) = \{e, r^3, sr, sr^4\}$.

    Normalizer of $\{r, r^5\}$.

    Normalization by r:

    $rrr^{-1} = r$

    $rr^5 r^{-1} = r^5$.

    Thus $r\{r, r^5\}r^{-1} = \{r, r^5\}$ and $r \in N_{\{}r, r^5\}$.

    Normalization by s:

    $srs^{-1} = r^{-1}ss^{-1} = r^{-1} = r^5$

    $sr^5 s^{-1} = r^{-5}ss^{-1} = r^{-5} = r$

    Thus $s\{r, r^5\}s^{-1} = \{r, r^5\}$.

$s \in N_{\{}r, r^5\}$.

Since $r, s \in N_{\{}r, r^5\}$ the element generated by r, s are in $N_{\{r,r^5\}}$, and consequently $G = N_{\{r,r^5\}}$.