Math 425, fall 2015, First midterm solutions

NAME: A. Eremenko

1. Evaluate
$$2i\mathrm{Log}\,\frac{1-i}{1+i}.$$
The answer must be in the form $a+ib$ where a and b are real.

Solution
$$2i\mathrm{Log}\,\frac{1-i}{1+i}=2i\left(\mathrm{Log}\left|\frac{1-i}{1+i}\right|+i\mathrm{Arg}\,\frac{1-i}{1+i}\right)=2i(0+i(-\pi/4-\pi/4))=\pi.$$

2. Solve the equation $\sin z = i$, and sketch the solutions.

Solution. Set $w = e^{iz}$. Then

$$\frac{1}{2i}(w - w^{-1}) = i,$$

or

$$w^2 + 2w - 1 = 0,$$

which has solutions $w_1 = -1 + \sqrt{2}$ and $w_2 = -1 - \sqrt{2}$. To the first solution corresponds the series

$$z_{1,k} = -i\mathrm{Log}\,(\sqrt{2} - 1) + 2\pi k,$$

and to the second corresponds the series

$$z_{2,k} = -i\mathrm{Log}\,(1 + \sqrt{2}) + (2k + 1)\pi.$$

To make a correct picture, one should notice that $\sqrt{2} - 1 = |w_1|$ is between zero and one, therefore its logarithm is *negative*, and also $|w_1||w_2| = 1$, so their logarithms differ only by signs.

The picture consists of two horizontal rows with equal spacing 2π, the upper row has a point on the positive imaginary axis $(-i\mathrm{Log}\,(\sqrt{2} - 1))$, while the lower row contains a point

$$\pi - i\mathrm{Log}\,(\sqrt{2} + 1) = \pi + i\mathrm{Log}\,(\sqrt{2} - 1).$$

3. Let f be a non-constant analytic function. Can $|f|^2$ be harmonic? (Either give an example or prove that it cannot).

Solution. If we write $f = u + iv$, then the functions u and v are harmonic, that is

$$u_{xx} + u_{yy} = 0, \quad v_{xx} + v_{yy} = 0. \tag{1}$$

To check whether $|f|^2 = u^2 + v^2$ can be harmonic, we differentiate:

$$(\partial/\partial x)|f|^2 = 2uu_x + 2vv_x,$$

$$(\partial/\partial x)^2|f|^2 = 2u_x^2 + 2uu_{x,x} + 2v_x^2 + 2vv_{x,x}.$$

Differentiation with respect to y gives a similar result. When we add these second derivatives, all second derivatives of u and v will cancel because of (1). What will remain is

$$\Delta|f|^2 = 2(u_x^2 + v_x^2 + u_y^2 + v_y^2).$$

So $|f|^2$ can be harmonic only if this sum of the squares is zero, but sum of the squares of real numbers is zero only when all of them are zero, so we conclude that all first partials of u and v are zero, which implies that f is constant.

This constitutes a proof that $|f|^2$ cannot be harmonic for non-constant analytic f. Notice that we did not use the Cauchy–Riemann equations, but only (1). So for every harmonic functions, if $u^2 + v^2$ is harmonic then they both must be constant.

4. Evaluate the integral

$$\int_{|z|=1} (\operatorname{Re} z)\, dz,$$

where the circle is oriented counterclockwise.

Solution. Parametrize the circle we get $z = e^{it} = \cos t + i \sin t,\ 0 \le t \le 2\pi$, so $\operatorname{Re} z = \cos t$ and $dz = ie^{it} = -\sin t + i \cos t$. So one needs to compute the integral

$$\int_0^{2\pi} \cos t\,(ie^{it})dt = \int_0^{2\pi} (-\cos t \sin t + i \cos^2 t)dt.$$

If you remember trigonometric formulas, or can derive them on the spot, it is fine. But there is a way to compute this without knowing any trigonometric formulas: just express everything in terms of exp:

$$\frac{i}{2} \int_0^{2\pi} (e^{it} + e^{-it})e^{it} dt = \frac{i}{2} \int_0^{2\pi} (e^{2it} + 1)dt = \pi i.$$

5. Find the image of the circle $\{z : |z-1| = 1\}$ under the function $f(z) = 1/z$. Sketch it.

(Hint: notice that this function is inverse to itself, so it does not matter whether you find the image or preimage. This may help).

Solution 1. Parametrize the circle $z = 1 + e^{it}$, $-\pi \le t \le \pi$. Then the image is

$$1/z = \frac{1}{1+e^{it}} = \frac{1+e^{it}}{(1+e^{it})(1+e^{-it}))} = \frac{1+\cos t}{2+2\cos t} + i\frac{\sin t}{2+2\cos t} = \frac{1}{2} + i\frac{\sin t}{2+2\cos t}.$$

So the image belongs to the vertical line $\operatorname{Re} w = 1/2$. As t changes from $-\pi$ to π the imaginary part covers the whole line.

6. Show that solutions of the equation $z^2 - 2cz + 1$, where c is real, are either all real or all belong to the unit circle. For which c the first possibility holds and for which c the second?

Solution. By quadratic formula

$$z_{1,2} = c \pm \sqrt{c^2 - 1}.$$

When $|c| > 1$, both roots are real; when $-1 < c < 1$, their absolute value is

$$c^2 - c^2 + 1 = 1,$$

so they both lie on the unit circle.