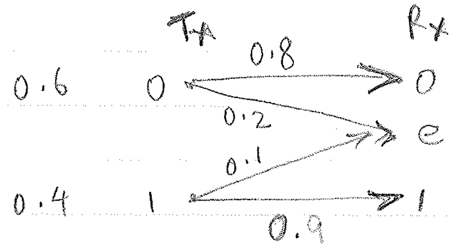
ECE302 H·W #2

(1) The channel can be visualized as follows



$Pr(R_x|T_x)$ = weight on the arrow $T_x \to R_x$

denote 0 is transmitted as 0T, 1 is received as 1R } similarly denote other cases.

(a) $Pr(eR) = Pr(eR|0T)\cdot Pr(0T) + Pr(eR|1T)\cdot Pr(1T)$

0T & 1T are mutually exclusive & collectively exhaustive which lets us use total probability rule

$P(eR) = 0.2 \times 0.6 + 0.1 \times 0.4 = 0.12 + 0.04 = 0.16$

$Pr(\text{erasure occurs}) = 0.16$

(b) $Pr(1T|eR) = \dfrac{Pr(eR|1T)\cdot Pr(1T)}{Pr(eR)}$ [Bayes theorem]

$= \dfrac{0.1 \times 0.4}{0.16} = 1/4$

(c) A & B are independent iff $Pr(A|B) = Pr(A)$
ie, $Pr(A \text{ occurs})$ does not depend on whether B occurs or not.

$Pr(eR|1T) = 0.1$; $Pr(eR) = 0.16$
clearly, these 2 events are NOT independent

Problem 2.

a. The error process described is Bernoulli. Let K be the number of errors out of n operations. We have then

$$P[K = k] = \binom{n}{k} p^k (1-p)^{n-k},$$

for $k = 1, \ldots, n$.

b. We have that type 1 errors occur with probability αp. Thus, type 1 errors also form a Bernoulli process with parameter αp. If we let K_1 be the number of type 1 errors out of n operations, then

$$P[K_1 = k_1] = \binom{n}{k_1} (\alpha p)^k (1 - \alpha p)^{n-k},$$

for $k_1 = 1, \ldots, n$.

c. Similar to above. Let K_2 be the number of type 2 errors out of n operations.

$$P[K_2 = k_2] = \binom{n}{k_2} ((1-\alpha)p)^k (1 - (1-\alpha)p)^{n-k},$$

for $k_2 = 1, \ldots, n$.

d. We first rewrite the joint probability

$$\begin{aligned} P[K_1 = k_1 \cap K_2 = k_2] &= P[K_1 = k_1 \cap K = k_1 + k_2] \\ &= P[K = k_1 | K = k_1 + k_2] P[K = k_1 + k_2]. \end{aligned}$$

As mentioned in the hint, we will have that if we are given that $K = k = k_1 + k_2$ errors occur in n operations, then each of the k errors are either type 1 (with probability α) or type 2 (with probability $1-\alpha$). Thus, type 1 errors *amoung* errors are Bernoulli trials. We then have that:

$$P[K = k_1 | K = k_1 + k_2] = \binom{k_1 + k_2}{k_1} \alpha^{k_1} (1-\alpha)^{k_2},$$

and

$$\begin{aligned} P[K_1 = k_1 \cap K_2 = k_2] &= P[K = k_1 | K = k_1 + k_2] P[K = k_1 + k_2] \\ &= \binom{k_1 + k_2}{k_1} \alpha^{k_1} (1-\alpha)^{k_2} \binom{n}{k_1 + k_2} p^{k_1+k_2} (1-p)^{n-k_1-k_2} \end{aligned}$$

Problem 3

a. We have that the pmf

$$p_X(k) = P(X = k) = P(\text{first } k-1 \text{ guesses are wrong}) P(k\text{−th guess is correct}).$$

When the key is replaced in the pocket, we will have that the probability of picking the right key will be $1/n$ on each try, since we are picking uniformly random from n keys on each attempt. Therefore.

$$p_X(k) = \left(1 - \frac{1}{n}\right)^{k-1} \left(\frac{1}{n}\right) = \frac{(n-1)^{k-1}}{n^k},$$

for $k = 1, 2, 3, \ldots$.

b. When the attempted keys are removed from the pocket, we will have diminishing probability of choosing incorrectly as more and more keys are chosen. In other words, there remains to be a single correct key in the pocket, but more and more keys are taken away. On the first attempt, we have that there are n keys, of which $n-1$ are incorrect. In the event that the first key chosen is incorrect, there now remain $n-1$ keys in the pocket, of which $n-2$ are incorrect. Therefore:

$$p_X(k) = P(X = k) = \left(\frac{n-1}{n}\right)\left(\frac{n-2}{n-1}\right)\cdots\left(\frac{n-(k-1)}{n-(k-2)}\right)\left(\frac{1}{n-(k-1)}\right)$$
$$= \frac{(n-1)!/(n-k)!}{n!/(n-k)!}$$
$$= \frac{1}{n},$$

for $k = 1, 2, \ldots, n$.

Problem 84

a. Since $\int_{-\infty}^{\infty} f_X(x)dx = 1$, we have

$$\int_{-1}^{5} ke^{-x}dx = k(e - e^{-5}) = 1$$
$$\Rightarrow k = \frac{1}{(e - e^{-5})}$$

b.

$$P[1 \leq X < 3] = \int_{1}^{3} f_X(x)dx = \int_{1}^{3} \frac{e^{-x}}{(e - e^{-5})}dx$$
$$= \frac{e^{-1} - e^{-3}}{(e - e^{-5})}$$
$$P[1 \leq X < 5] = \int_{1}^{5} f_X(x)dx = \int_{1}^{5} \frac{e^{-x}}{(e - e^{-5})}dx$$
$$= \frac{e^{-1} - e^{-5}}{(e - e^{-5})}$$

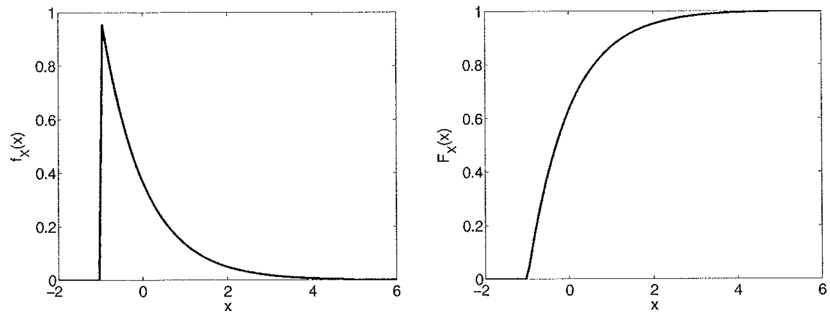
c. $X^3 < 5 \Rightarrow X < 5^{1/3}$.

$$P[X^3 < 5] = P[X < 5^{1/3}] = \int_{-1}^{5^{1/3}} \frac{e^{-x}}{(e - e^{-5})}dx = \frac{e^{-1} - e^{-5^{1/3}}}{(e - e^{-5})}.$$

d.

$$F_X(x) \triangleq \int_{-\infty}^{x} \frac{e^{-\lambda}}{(e - e^{-5})}d\lambda = \begin{cases} 0, & x \leq -1 \\ \frac{e - e^{-x}}{(e - e^{-5})}, & -1 < x \leq 5 \\ 1, & x > 5 \end{cases}$$

The PDF and CDF are plotted in Figure 1.



pdf $f_X(x)$ CDF $F_X(x)$

Figure 1: Problem 3(d)

Problem ~~4~~ 5

a. Since $\sum_{i=-\infty}^{\infty} p_X(x_i)dx = 1$, we have

$$\begin{aligned}
&\sum_{x_i=-1}^{5} k\left(\frac{1}{2}\right)^{x_i} \\
=\;& k\left(\frac{1}{2}\right)^{-1} + k\left(\frac{1}{2}\right)^{0} + k\left(\frac{1}{2}\right)^{1} + k\left(\frac{1}{2}\right)^{2} + k\left(\frac{1}{2}\right)^{3} + k\left(\frac{1}{2}\right)^{4} + k\left(\frac{1}{2}\right)^{5} = 1 \\
\Rightarrow\;& k = \frac{32}{127}
\end{aligned}$$

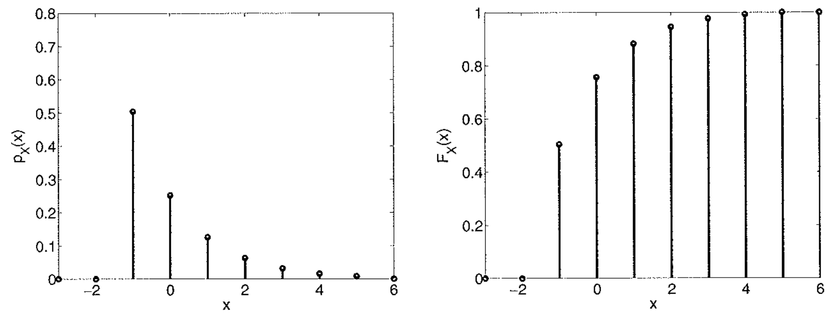
b.

$$P[1 \le X < 3] = p_X(1) + p_X(2)\frac{32}{127}\left(\frac{1}{2}\right)^{1} + \frac{32}{127}\left(\frac{1}{2}\right)^{2} = \frac{24}{127}$$

$$\begin{aligned}
P[1 \le X < 5] &= p_X(2) + p_X(3) + p_X(4) + p_X(5) \\
&= \frac{32}{127}\left(\frac{1}{2}\right)^{2} + \frac{32}{127}\left(\frac{1}{2}\right)^{3} + \frac{32}{127}\left(\frac{1}{2}\right)^{4} + \frac{32}{127}\left(\frac{1}{2}\right)^{5} = \frac{15}{127}
\end{aligned}$$

c.

$$\begin{aligned}
P[X^3 < 5] &= P[X < 5^{1/3}] \\
&= p_X(-1) + p_X(0) + p_X(1) = \frac{32}{127}\left(\frac{1}{2}\right)^{-1} + \frac{32}{127} + \frac{32}{127}\left(\frac{1}{2}\right) = \frac{112}{127}
\end{aligned}$$



PMF $p_X(x)$ CDF $F_X(x)$

Figure 2: Problem 4(d)

d.

$$
\begin{aligned}
F_X(x) \triangleq P[X \leq x] &= \sum_{x_i=\infty}^{x} p_X(x_i) \\
&= \sum_{x_i=-1}^{x} \frac{32}{127}\left(\frac{1}{2}\right)^{x_i} = \frac{32}{127}\left(4-\left(\frac{1}{2}\right)^{x}\right),
\end{aligned}
$$

for $x \in [-1,5]$. $F_X(x) = 0$ for $x < -1$ and $F_X(x) = 1$ for $x > 5$. The PMF and CDF are plotted in Figure 2.

Problem 6

a. **True.**

$$P[A|B \cap C] = \frac{P[A \cap B \cap C]}{P[B \cap C]} = \frac{P[A]P[B]P[C]}{P[B]P[C]} = P[A].$$

b. **True.**

$$
\begin{aligned}
P[S \cap \emptyset] &= P[\emptyset] = 0 = P[S]P[\emptyset] = 0 \\
P[S \cap A] &= P[A] = P[S]P[A] = 1 \cdot P[A] = P[A] \\
P[\emptyset \cap A] &= P[\emptyset] = 0 = P[\emptyset]P[A] = 0 \\
P[A \cap S \cap \emptyset] &= P[\emptyset] = 0 = P[A]P[S]P[\emptyset] = 0.
\end{aligned}
$$

c. **True.** $P[A \cap B] = P[\emptyset] = 0$, but $P[A]P[B] > 0$ since $P[A] > 0$ and $P[B] > 0$. Therefore $P[A \cap B] \neq P[A]P[B]$.

d. **True**.

$$\begin{aligned} & P[A_n]P[A_{n-1}|A_n]P[A_{n-2}|A_n \cap A_{n-1}] \ldots P[A_1|A_n \cap \ldots \cap A_2] \\ = \; & P[A_n \cap A_{n-1}]P[A_{n-2}|A_n \cap A_{n-1}] \ldots P[A_1|A_n \cap \ldots \cap A_2] \\ = \; & P[A_n \cap A_{n-1} \cap A_{n-2}] \ldots P[A_1|A_n \cap \ldots \cap A_2] \\ = \; & P[A_n \cap A_{n-1} \cap A_{n-2} \cap \ldots \cap A_1]. \end{aligned}$$

e. **False.** By the binomial theorem:

$$1 = (p + (1-p))^n = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}.$$

Therefore,

$$\sum_{k=1}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1 - (1-p)^n$$

f. **False.**

$$f_X(x_0) = \left[\frac{d}{dx} F_X(x)\right]_{x=x_0} \left[\frac{d}{dx} P[X \le x]\right]_{x=x_0} \neq P[X = x_0].$$

g. **True.**

$$P[a \le X \le b] = P[a \le X < b] = P[a < X < b] = P[a < X \le b] = \int_a^b f_X(x)dx.$$

h. **True.** By definition, $p_X(x_0) = P[X = x_0]$.

i. **True.** By definition, $P[a \le X \le b] = \sum_{x_i \in [a,b]} p_X(x_i)$.

j. **False.** Strictly speaking, we have

$$\begin{aligned} P[a < X \le b] &= F_X(b) - F_X(a) \\ \text{and } P[a \le X \le b] &= P[X = a] + F_X(b) - F_X(a). \end{aligned}$$

If X is a discrete R.V. at $X = a$, then $P[X = a] \neq 0$. Therefore, in general, $P[a \le X \le b] \neq F_X(b) - F_X(a)$.