ECE 302 [redacted]

Homework solution
3

Problem 1

3.25 without replacement

a) $E[X] = 2 \cdot \frac{4}{5} + 51 \frac{1}{5} = \frac{59}{5} = 11.80$

$E[X^2] = 4 \cdot \frac{4}{5} + 51^2 \cdot \frac{1}{5} = \frac{2617}{5}$

$VAR[X] = \frac{2617}{5} - \left(\frac{59}{5}\right)^2 = \frac{9604}{25} = 384.16$

b) with replacement:

$E[X] = 2 \cdot \frac{81}{100} + 51 \frac{18}{100} + 100 \cdot \frac{1}{100} = \frac{1180}{100} = 11.80$

$E[X^2] = 4 \cdot \frac{81}{100} + 51^2 \cdot \frac{18}{100} + 10^4 \cdot \frac{1}{100} = \frac{57142}{100}$

$VAR[X] = \frac{57142}{100} - \left(\frac{1180}{100}\right)^2 = \frac{43218}{100} = 432.18$

Means in both draws is the same!

Problem 2

3.39

(a) $P[X=k \mid X>1] = \left(\frac{1}{2}\right)^{k-1} \quad k=2,3,\ldots$

$$E[X \mid X>1] = \sum_{k=2}^{\infty} k\left(\frac{1}{2}\right)^{k-1} = \sum_{k'=1}^{\infty} (k'+1)\left(\frac{1}{2}\right)^{k'} \quad \text{where } k'=k-1$$

$$= \sum_{k'=0}^{\infty} k'\left(\frac{1}{2}\right)^{k'} + \underbrace{\sum_{k'=1}^{\infty}\left(\frac{1}{2}\right)^{k'}}$$

$$= \underbrace{\boxed{E[X]}}_{\text{avg. \# as if starting from scratch}} + \underbrace{1}_{\text{1 transmission is certain}} = 3$$

(3.37b) "message gets through in 1st transmit" $\equiv X=1$

$$P[X=k \mid X=1] = \begin{cases} 0 & k>1 \\ 1 & k=1 \end{cases}$$

$$E[X \mid X=1] = 1 \cdot P[X=1] = 1$$

(c) ~~Let~~ $A=\{X=1\}$ $B=\{X>1\}$ then A & B form a partition

$$E[X] = E[X|A]P[A] + E[X|B]P[B]$$

$$= 1 \cdot \tfrac{1}{2} + 3 \cdot \tfrac{1}{2} = 2$$

note we can use the result of part a to find $E[X]$:

$$E[X] = 1 \cdot \tfrac{1}{2} + (E[X]+1)\tfrac{1}{2} \Rightarrow E[X]=2.$$

(d) $$E[X^2 \mid X>1] = \sum_{k=2}^{\infty} k^2 \left(\tfrac{1}{2}\right)^{k-1} = \sum_{k'=1}^{\infty} (k'+1)^2 \left(\tfrac{1}{2}\right)^{k'}$$

$$= \sum_{k'=1}^{\infty} k'^2 \left(\tfrac{1}{2}\right)^{k'} + 2\sum_{k'=1}^{\infty} k' \left(\tfrac{1}{2}\right)^{k'} + \sum_{k'=1}^{\infty} \left(\tfrac{1}{2}\right)^{k'}$$

$$= E[X^2] + 2E[X] + 1$$

$$= E[X^2] + 5$$

$$E[X^2 \mid X=1] = 1$$

$$\therefore \quad E[X^2] = E[X^2 \mid X=1]\tfrac{1}{2} + E[X^2 \mid X>1]\tfrac{1}{2}$$

$$= \tfrac{1}{2} + [E[X^2]+5]\tfrac{1}{2}$$

$$\Rightarrow E[X^2] = 6$$

$$VAR[X] = E[X^2] - E[X]^2 = 6 - 2^2 = 2$$

$VAR(X \mid X>1)$

$= E(X^2 \mid X>1) - [E(X \mid X>1)]^2$

$= E[X^2] + 5 - 3^2$

$= 6 + 5 - 9 = 2$

Similarly $Var(X \mid X=1)$ will come out to 0.

$Var(X) = E(X^2) - E(X)^2$

$= \sum E(X^2 \mid A_i) P(A_i) - [\sum E(X \mid A_i) P(A_i)]^2$

$\neq \sum [E(X^2 / A_i) - E(X \mid A_i)^2] P(A_i)$

Hence $Var(X)$ cannot be found using conditional variance.

Problem 4

ⓐ $f_X(x) = \begin{cases} 2e^{-2x} & 0<x<\infty \\ 0 & \text{else} \end{cases}$

$Y = X^3$

$\Rightarrow X = \sqrt[3]{Y}$

$f_Y(y) = f_X(x) \left|\frac{dx}{dy}\right|$

$= 2e^{-2x} \frac{1}{3} y^{-\frac{2}{3}}$

$= 2e^{-2y^{\frac{1}{3}}} \frac{1}{3} y^{-\frac{2}{3}}$

$= \frac{2}{3} y^{-\frac{2}{3}} e^{-2y^{\frac{1}{3}}}$ $\Rightarrow f_Y(y) = \begin{cases} \frac{2}{3} y^{-\frac{2}{3}} e^{-2y^{\frac{1}{3}}} & y>0 \\ 0 & y\le 0 \end{cases}$

Alternatively,

$F_X(x) = \int_{-\infty}^{x} f_X(x')\,dx'$

$= \int_0^x 2e^{-2x'}\,dx'$

$= -e^{-2x'}\Big|_0^x$

$= 1-e^{-2x}$

$F_Y(y) = P(Y\le y)$ $= \int_{-\infty}^{\sqrt[3]{y}} f_X(x)\,dx$

$= P(X^3 \le y)$ $= \int_0^{\sqrt[3]{y}} 2e^{-2x}\,dx$

$= P(X \le \sqrt[3]{y})$ $= 1- e^{-2\sqrt[3]{y}}$ $y>0$

$= F_X(\sqrt[3]{y})$ $(= 0$ $y\le 0)$

$f_Y(y) = \frac{dF_Y(y)}{dy} = -e^{-2\sqrt[3]{y}} \cdot (-\frac{2}{3}) y^{-\frac{2}{3}} = \frac{2}{3} y^{-\frac{2}{3}} e^{-2\sqrt[3]{y}}$

$$\Rightarrow f_Y(y) = \begin{cases} \frac{2}{3} y^{-\frac{2}{3}} e^{-2\sqrt[3]{y}} & y > 0 \\ 0 & y \le 0 \end{cases}$$

③ the probability that $Y > 2X$

$Y = X^3 > 2X$

$\Rightarrow X > \sqrt{2}$ (for $X > 0$)

$P(Y > 2X)$

$= P_X(X > \sqrt{2})$

$= 1 - F_X(\sqrt{2})$

$= 1 - \int_{-\infty}^{\sqrt{2}} f_X(x)dx$

$= 1 - \int_0^{\sqrt{2}} 2e^{-2x} dX$

$= 1 + \int_0^{\sqrt{2}} e^{-2x} d(-2x)$

$= 1 + e^{-2x} \Big|_0^{\sqrt{2}}$

$= e^{-2\sqrt{2}}$

(b) ① $Y = X^3$

$X = \sqrt[3]{Y}$

Since $P_X(x_i) = (\frac{1}{2})^{x_i+1}$

$\Rightarrow P_Y(y) = \begin{cases} (\frac{1}{2})^{\sqrt[3]{y}+1} & y = 0, 1, 8, 27, \cdots \\ 0 & \text{else.} \end{cases}$

② $Y = X^3 > 2X$

$X^3 - 2X > 0$

$X > \sqrt{2}$ (since $X \geq 0$)

$P(X > \sqrt{2}) = P_X(2) + P_X(3) + \cdots$, Or $P(X > \sqrt{2}) = 1 - P(X=1) - P(X=0)$

$= (\frac{1}{2})^3 + (\frac{1}{2})^4 + \cdots \qquad = 1 - (\frac{1}{2})^{1+1} - (\frac{1}{2})^1$

$= \dfrac{(\frac{1}{2})^3}{1 - \frac{1}{2}} \qquad = \frac{1}{4}$

$= \frac{1}{4}$

(c) We now let

$$Y = |X|^3 = \begin{cases} X^3, & X \geq 0 \\ -(X^3), & X < 0 \end{cases}$$

Since $P[X < 0] = 0$ in both cases for the distribution of X in parts (a) and (b), then we find that $Y = |X|^3 = X^3$ *almost always* (i.e. for all X with nonzero probability), and so the answers will remain the same when $Y = |X|^3$ as when $Y = X^3$ above.

(5) 4.53 from text

$Y = A\cos(\omega t) + c$ [similar to $aX+b$ where X is random, a & b are constants]

$E[Y] = E[A]\cos(\omega t) + c = m\cos(\omega t) + c$

$\text{var}[Y] = \cos^2(\omega t)\,\text{var}(A) = \sigma^2\cos^2(\omega t)$

[using the relation for $aX+b$]

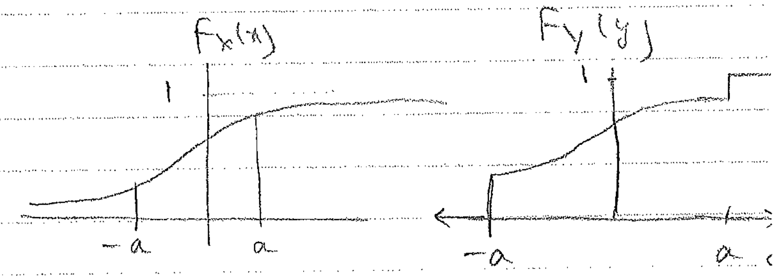
4.15 had a random phase so the mean went to 0 & variance went to the mean-squared value

In this case the amplitude is random. So mean is non-zero.

(5) (8)
(only part (c) required)

4.54

$Y = g(X)$



Find cdf of Y

$$\Pr(Y \le y) = \begin{cases} 0 & y < -a \\ 1 & y \geq a \\ F_X(y) & -a \le y < a \quad (\because\ g(x) = y \text{ when } -a \le y < a) \end{cases}$$

$\therefore$ pdf $(y) = f_Y(y) = F_X(-a)\,\delta(y+a) + f_X(y)[u(x+a) - u(x-a)] + (1 - F_X(a))\,\delta(y-a)$

(derivative of $F_Y(y)$)

$\therefore\ E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy$

$\text{var}[Y] = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy - (E[Y])^2$

$$E[Y] = F_X(-a)(-a) + \int_{-a}^{a} y f_X(y)\,dy + (1 - F_X(a))(a) \quad \text{—} \; 1$$

$$E[Y^2] = a^2 F_X(-a) + \int_{-a}^{a} y^2 f_X(y)\,dy + (1 - F_X(a))a^2 \quad \text{—} \; 2$$

Density of Laplacian $f_X(x) = \frac{\lambda}{2} e^{-\lambda|x|}$ $-\infty < x < \infty$, $\lambda > 0$

(optional) (b) $\lambda = 1$, $a = 1$

since Laplacian is symmetric about 0, $E[Y] = 0$

$$V = \int_{-a}^{a} y^2 f_X(y)\,dy \quad \because F_X(-a) = 1 - F_X(a) \text{ symmetry}$$

$$= 2\int_0^a y^2 \frac{\lambda}{2} e^{-\lambda y}\,dy = \frac{-\lambda}{\lambda}\int_0^a y^2 d(e^{-\lambda y})$$

$$= -(y^2 e^{-\lambda y})_0^a + 2\int_0^a e^{-\lambda y} y\,dy = -(a^2 e^{-\lambda a}) + \frac{2}{(-\lambda)}\int_0^a y\, d(e^{-\lambda y})$$

$$= -a^2 e^{-\lambda a} - \frac{2}{\lambda}\left((e^{-\lambda y} \cdot y)_0^a + \frac{2}{\lambda}\int_0^a e^{-\lambda y}\,dy \right)$$

$$= -a^2 e^{-\lambda a} - \frac{2a}{\lambda} e^{-\lambda a} + \frac{2}{\lambda}\left(\frac{e^{-\lambda y}}{-\lambda}\right)_0^a$$

$$= -a^2 e^{-\lambda a} - \frac{2a}{\lambda} e^{-\lambda a} - \frac{2}{\lambda^2} e^{-\lambda a} + \frac{2}{\lambda^2}$$

$$\therefore E[Y^2] = V + 2a^2 F_X(-a)$$

$$= \cancel{-a^2 e^{-\lambda a}} + \frac{2a}{\lambda} e^{-\lambda a} - \frac{2}{\lambda^2} e^{-\lambda a} + \frac{2}{\lambda^2} + \cancel{2a^2 \times \frac{1}{2} e^{-a}}$$

$$E[Y^2] = \frac{2a}{\lambda} e^{-\lambda a} - \frac{2}{\lambda^2} e^{-\lambda a} + \frac{2}{\lambda^2} = \mathrm{Var}(Y) \quad \because E[Y] = 0$$

$a = \lambda = 1 \Rightarrow \mathrm{Var}(Y) = 2$;

(optional) (d) $X = U^3 \quad U \sim [-1,1],\ a = 1/2 \qquad u = x^{1/3}$

$$f_X(x) = f_U(u)\left|\frac{du}{dx}\right|$$

$$f_X(x) = \frac{1}{2}\cdot\frac{1}{3}x^{-2/3} \qquad -1 \le x \le 1$$

$E[Y] = 0 \because f_X(x)$ is an even function.

$$E[Y^2] = \int_{-a}^{a} \frac{1}{6}x^{-2/3}\cdot x^2\,dx = \frac{1}{6}\int_{-a}^{a} x^{4/3}\,dx = \frac{1}{6_2}\left(x^{7/3}\right)_{-a}^{a} \times 3/7$$

(from ②) $\qquad = \frac{1}{7}\left(\underline{a^{7/3}}\right)$

$$\mathrm{var}(Y) = \frac{1}{7}a^{1/3} = \frac{1}{7}\left(\frac{1}{2}\right)^{1/3}$$

(e) From the normalization, $c = 3/4$

Again from symmetry, $E[Y] = 0$

$$f_X(x) = \frac{3}{4}(1-x^2) \quad -1 \le x \le 1 \ ; \ F_X(x) = \begin{cases} \frac{3}{4}\left(x - \frac{x^3}{3} + \frac{2}{3}\right) & 1 \ge x \ge -1 \\ 1 & x > 1 \\ 0 & x < -1 \end{cases}$$

$$\therefore \mathrm{var}(Y) = E[Y^2] = \int_{-\frac{1}{2}}^{\frac{1}{2}} y^2\cdot\frac{3}{4}(1-y^2)\,dy + 2\times\frac{1}{4}\times\frac{3}{4}\left(-\frac{1}{2}+\frac{1}{24}+\frac{2}{3}\right)$$

$$= \frac{3}{4}\int_{-\frac{1}{2}}^{\frac{1}{2}}(y^2 - y^4)\,dy + \frac{3}{8}\left(-\frac{1}{2}+\frac{1}{24}+\frac{2}{3}\right)$$

$$= \frac{21}{160}$$

(6) (8) (only part (c) required)

4.82

pdf for part b.

$$\frac{1}{2}e^{-a\lambda}\delta(y+a)+\frac{1}{2}e^{-a\lambda}\delta(y-a)+\frac{\lambda}{2}e^{-\lambda|y|}(u(y+a)-u(y-a))$$

(optional) cdf:

$$= \begin{cases} 0 & y<a \\ \frac{1}{2}e^{y\lambda} & -a\le y<0 \\ \frac{1}{2}e^{-y\lambda} & 0\le y<a \\ 1 & y\ge a \end{cases} \qquad (a=\tfrac{1}{2})$$

cdf has to be right continuous.

(optional) (d) pdf:

$$f_Y(y)=\frac{1}{6}y^{-2/3}(u(y+\tfrac{1}{2})-u(y-\tfrac{1}{2}))-\frac{1}{2}(1-(\tfrac{1}{2})^{1/3})\,\delta(y+\tfrac{1}{2})-\frac{1}{2}(1-(\tfrac{1}{2})^{1/3})\,\delta(y-\tfrac{1}{2})$$

cdf:

$$\begin{cases} 0 & y<-\frac{1}{2} \\ 1 & y\ge\frac{1}{2} \\ \frac{1}{2}(1+y^{1/3}) & -\frac{1}{2}\le y<\frac{1}{2} \end{cases}$$

(c) pdf:

$$f_Y(y)=\frac{5}{32}\delta(y+\tfrac{1}{2})+\frac{3}{4}(1-y^2)(u(y+\tfrac{1}{2})-u(y-\tfrac{1}{2}))+\frac{5}{32}\delta(y-\tfrac{1}{2})$$

$$F_Y(y)=\begin{cases} 0 & y<\frac{1}{2} \\ 1 & y\ge-\frac{1}{2} \\ \frac{3}{4}(x-\frac{x^3}{3}+\frac{2}{3}) & -\frac{1}{2}\le y<\frac{1}{2} \end{cases}$$

(7) (8) (a) $\Pr(Y=y_j) = \Pr(g(X)=y_j) = \Pr(X=g^{-1}(y_j))$ ↳ inverse image of y_j

$$= \sum_{x_i : g(x_i)=y_j} \Pr(X=x_i) = \sum_{x_i : g(x_i)=y_j} p_X(x_i)$$

True

(b) $F_Y(y) = \Pr(g(X) \le y) = \int_{x: g(x) \le y} f_X(x)\, dx \Rightarrow$

$f_Y(y) = \frac{d}{dy} \int_{x: g(x) \le y} f_X(x)\, dx$; this is true.

(c) True (consult notes)
this is the direct pdf method

(d) True (consult notes)

(e) True - from linearity of expectation.

(f) True - notes

(g) False if X is some constant x_0, the variance is still 0

(h) False if $f_X(x) \neq 0$ for some points outside of A, then $f_X(x) I_A(x)$ is not even a valid pdf while $f_X(x | X \in A)$ is a pdf if $\Pr(X \in A) \neq 0$

(i) False eg: X is a bernoulli r.v
M_1 = success
M_2 = failure