MATH 425, Fall 2015, second midterm answers and solutions

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1. Find the radii of convergence of the following series:

$$a) \qquad \sum_{n=0}^{\infty} n^4 2^n z^n,$$

$$b) \qquad \sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$$

$$c) \qquad \sum_{n=1}^{\infty} (3+(-1)^n)^n z^n,$$

$$d) \qquad \sum_{n=1}^{\infty} 3^n z^{n^2},$$

e) Taylor series at 0 of the function

$$\frac{\mathrm{Log}(2+z)}{1+z}.$$

Solution.
a) 1/2,
b) e. I used ratio test:

$$\frac{(n+1)!n^n}{n!(n+1)^{n+1}} = \left(\frac{n}{n+1}\right)^n = (1+1/n)^{-n}$$

and this tends to $1/e$ by calculus. If you don't remember this from calculus, take log.
c) 1/4,
d) 1
e) 2. The singularity at $z=-1$ is removable.

2. Suppose that the series

$$\sum_{n=0}^{\infty} c_n(z-1-i)^n$$

converges at the point $z = 1/2 + i$.

a) What conclusion can be made from this about convergence at the point $z = (5/4)(1+i)$? (Converges, diverges, no conclusion).

b) Same question about the point $z = 1 + i/2$.

As the series with center at $1 + i$ is convergent at $1/2 + i$, the radius of convergence satisfies $R \geq 1/2$. So at the point $(5/4)(1+i)$ which is closer to $1 + i$ the series converges.

But the point $1 + i/2$ is at the distance $1/2$ from $1 + i$, so no conclusion can be made.

3. Find and classify all isolated singularities in $\mathbf{C}$ of the following functions. For poles determine their multiplicities.

$$a) \qquad \frac{z-1}{z^{2\pi i/z}-1},$$

$$b) \qquad \frac{z}{1-\cos z},$$

$$c) \qquad \frac{1}{z}\sin\frac{z-\pi}{1+z},$$

$$d) \qquad z^2\exp(1/z),$$

$$e) \qquad (\cot z) - 1/z.$$

Solution:

a) This part has a misprint in it: as it is written the function is not defined. I gave 2 points to everyone for it, independently on the answer. (It was intended with $e^{2\pi i/z} - 1$ in denominator.

b) All zeros of the denominator are *double*. But at $z = 0$ there is a cancellation. So the pole at 0 is simple and at $2\pi n$ are double, for $n = \pm 1, \pm 2, \ldots$.

c) $z = -1$ is an essential singularity, $z = 0$ is removable because $\sin(-\pi) = 0$.

d) $z = 0$ is an essential singularity

e) Cotangent has simple poles at πn with resudie 1. So $1/z$ cancels, $z = 0$ is removable and $z = \pi n,\ n = \pm 1, \pm 2, \ldots$ are simple poles.

4. Let f be an analytic function whose Taylor series begins with

$$f(z) = c_2 z^2 + c_3 z^3 + c_4 z^4 + \dots.$$

Find the residue of $1/f$ at 0.

Answer: $-c_3/c_2^2$.

5. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{e^{2ix}}{1+x^4} dx.$$

Answer: This integral is badly divergent at $+\infty$ and as an integral over a positive function equals $+\infty$.