Discrete Math, 375 (Spring 2015): Mid-term 1.

Do all of the problems below. Please write your name and your class (the class time of 1030 or 1200) on blank cover sheet. Write neatly and be clear. Remember, this is just an exam. The universe does not care about the outcome of this exam and nor should you. You should enjoy the challenge. You should remember that 10 years from now, your memory of this exam will be quite vague if not entirely nonexistent.

Problem 1. [The gift: 10 points]

(a) State the definition of an injective function $f\colon X \to Y$.

(b) State the definition of a surjective function $f\colon X \to Y$.

Problem 2. [The coin flip: 10 points] State whether or not the following is true or false. You do not need to justify your answer.

(a) For sets $A, B \subset X$ and a function $f\colon X \to Y$, we have

$$f(A \cap B) = f(A) \cap f(B).$$

(b) There exists an injective function $f\colon \mathbf{Z} \to \mathbf{N}$.

(c) ”If P, then Q” and ”if Q, then P” are logically equivalent.

(d) If $|A| = |B|$ and $f\colon A \to B$ is an injective function, then f is surjective.

Problem 3. [Almighty Inductor: 10 points]

(a) Prove that for each positive integer n, the equality

$$\sum_{j=1}^{n} j \cdot j! = (n+1)! - 1.$$

(b) Prove that for any set of distinct positive integers $A = \{x_1, \dots, x_{n+1}\}$ such that $x_j \leq 2n$ for all j, there exist $x_{i_1}, x_{i_2} \in A$ such that x_{i_1} divides x_{i_2}.

Problem 4. [What is truth: 10 points]

(a) Prove that $(P \to Q) \vee (P \to R)$ and $P \to (Q \vee R)$ are logically equivalent.

(b) Show that if P, Q, and R are compound statements such that P, Q are logically equivalent and Q, R are logically equivalent, then P, R are logically equivalent.

Problem 5. [Compose that: 10 points] Suppose that $f\colon X \to Y$ and $g\colon Y \to Z$ are functions.

(a) Prove that if f, g are injective, then so is $g \circ f$.

(b) Prove that if f, g are surjective, then so is $g \circ f$.

(c) Prove or disprove that if $g \circ f$ is surjective, then so are f, g.

Problem 6. [Pascal, you beautiful bast***d: 10 points]

(a) Give the coefficient for $x^{101}y^{99}$ in the expansion of $(2x - 3y)^{200}$.

(b) Prove that if $A = \{x_1, \ldots, x_{n+1}\}$ are positive integers, then there exist $x_i, x_j \in A$ such that n divides $x_i - x_j$.