CS381 **Homework 6 Sketch of Solution** **Fall 2013**

Question 1.

1. $f(j) = j-2$ because the string consists of the repetition of its first two symbols: $ababababababababa \cdots ab$
2. The fact that $f(m) = 3m/4$ implies that the string consists of 4 repetitions of its first $m/4$ symbols, and the fact that $f(m/4) = 0$ implies that no proper prefix of the first $m/4$ symbols is a suffix of them. This implies that $f(m/2) = m/4$ and $f(3m/4) = m/2$.

Question 2. The bottleneck is the smallest number of edges from any S_i to S_{i+1}:

$$\beta = \min\{|S_1| * |S_2|, |S_2| * |S_3|, |S_3| * |S_4|, \ldots, |S_{L-1}| * |S_L|\}.$$

That the bottleneck can be achieved is seen by noting that if a flow of size β enters any S_j ($2 \leq j \leq L-1$) then surely it can be pushed out of S_j into S_{j+1} (because otherwise $|S_j| * |S_{j+1}|$ would be smaller than β, contradicting the definition of β).

Question 3.

1. π is the identity permutation: $\pi(i) = i$ for all i. The proof is by induction on n. The basis ($n = 1$) is trivial. For the induction step, assume that the claim holds for $n-1$. To prove that it holds for n, it suffices to prove that there is always an optimal solution in which r_1 is matched to b_1, because it implies that removing the matched r_1, b_1 pair leaves a problem of size $n-1$ to which the induction hypothesis applies. Let π' be an optimal permutation in in which r_1 is matched to b_i ($i > 1$) and b_1 is matched to r_j ($j > 1$). Let us examine the impact of replacing these two matched pairs by the two matched pairs r_1, b_1 and r_j, b_i (we henceforth refers to this replacement as *the swap*). Without loss of generality, we assume that $r_1 < b_1$ (otherwise interchange the roles of red and blue in what follows). Since $b_1 < b_i$, the cases we need to consider depend on where r_j fits in the ordering $r_1 < b_1 < b_i$. The case of

$$r_1 < b_1 < b_i < r_j$$

cannot hold because it would contradict the optimality of π': The swap would decrease the summation being minimized by twice the distance between b_1 and b_i. The case of

$$r_1 < b_1 < r_j < b_i$$

also cannot hold because it too would contradict the optimality of π': The swap would decrease the summation being minimized by twice the distance between b_1 and r_j. The only case that does not contradict the optimality of π' is

$$r_1 < r_j < b_1 < b_i$$

but in that case the swap leaves the summation unchanged and results in a permutation just as good as π' but in which r_1 is matched with b_1.

2. π is the identity permutation: $\pi(i) = i$ for all i. A proof by induction similar to the one for the summation case would work, except that the two cases of

$$r_1 < b_1 < b_i < r_j$$

and

$$r_1 < b_1 < r_j < b_i$$

can no longer be said to contradict the optimality of π', rather, they would be similar to the case

$$r_1 < r_j < b_1 < b_i$$

in the sense that the swap can be applied without any damage to the optimality of the solution (the resulting permutation would be just as good as π' except that it would match r_1 with b_1).