

# smmc mock test sols

Cyril Subramanian

October 2023

## 1 Q1

From inspection of the question, it can be theorised that  $A^n$  must eventually be equal and stay equal to its limiting value, kind of like how a sequence of integers that converges to a number must eventually become that number.

We can prove this formally with the definition of convergence: define the limiting matrix to be  $A^\infty$ , and  $f_{i,j}(n) = A_{i,j}^n$  ( $i, j$  entry of  $A^n$ ). Then, pick  $\epsilon = \frac{1}{2}|A_{i,j}^\infty - \text{round}(A_{i,j}^\infty)|$ , where  $\text{round}(x)$  returns the closest integer to  $x$ . Then, if  $\epsilon > 0$ , this means from the epsilon-delta definition of convergence, there exists  $N$  where for  $n > N$ ,  $|A_{i,j}^n - A_{i,j}^\infty| < \epsilon$ . However, note that the smallest value of the LHS is exactly  $2\epsilon$ , since this is the distance to the closest integer, thus we have  $2\epsilon < \epsilon$ , which is a contradiction.

This means  $\epsilon = 0$ , and  $A^\infty$  is an integer matrix. Furthermore, we can generate a set of  $N_{i,j}$  where, after picking  $\epsilon = \frac{1}{2}$ , we deduce  $n > N_{i,j} \implies |A_{i,j}^n - A_{i,j}^\infty| < \frac{1}{2}$ . However, as established before,  $A_{i,j}^n$  and  $A_{i,j}^\infty$  are integers; if they were different, the difference between them would be at least 1. So, instead they must be equal for all  $n > N_{i,j}$ . If we take  $N = \max\{N_{i,j} : 1 \leq i, j \leq m\}$ , then we note for all  $n > N$ ,  $A^n = A^\infty$  (since all entries are equal). This condition is exactly equivalent to  $A^N = A^{N+1}$ , which we can see by multiplying both sides by  $A$  repeatedly.

Now, we want to show that  $A^m = A^{n+1}$ . This can be done nicely if you understand Jordan forms theory, but we can do this using just MATH1141/MATH1241 knowledge. Rearranging our equation (replacing  $N$  with  $n$ ), we obtain  $A^{n+1} - A^n = O = A^n(A - I)$ . From here, we can see that  $\ker A^k \leq A^{k+1}$ , but a more interesting result is that if  $\ker A^k = \ker A^{k+1}$ ,  $\ker A^k = \ker A^t$  for all  $t \geq k$ .

We do this by letting a vector  $\vec{v}$  be in the kernel of  $A^{k+2}$ . Then,  $A^{k+2}\vec{v} = A^{k+1}(A\vec{v}) = \vec{0}$ , so  $\vec{v} \in \ker(A^{k+1})$ , and thus also  $\ker(A^k)$ . So,  $A^{k+1}\vec{v} = A^k(A\vec{v}) = \vec{0}$ , so  $v \in \ker(A^k)$ . Repeating this process for  $t \geq k$  gives the desired result. This means that the kernel must always be strictly increasing in size until it stops growing entirely. Since the maximum dimension of any subspace of an  $m$ -dimensional vector space is  $m$ ,  $A^m$  will always have the largest kernel out of

all powers of  $A$ . Thus,  $\ker(A^n) \leq \ker(A^m)$ , and since from the original equation we deduce  $A^n$  sends all vectors  $(A - I)\vec{v}$  to  $\vec{0}$  (as  $A^n(A - I)$  is the zero matrix), so must  $A^m$ , from which it follows  $A^m(A - I) = O$ , so  $A^m = A^{m+1}$ , and we are done.

## 2 Q2

From a tree, adding an edge to a leaf node will keep the number of leaf nodes the same (the leaf node becomes a non-leaf node and we have a new leaf node,  $-1 + 1$ ), and adding an edge to a non-leaf node will increase the number of leaf nodes by exactly 1 (no leaf nodes removed,  $+1$ ). This is true except for the case with one vertex, so we ignore this. If we let the expected number of leaf nodes after  $n$  steps be  $f(n)$ , using linearity of expectations and the fact that choosing a non-leaf node has probability  $\frac{n-f(n)}{n}$ , we obtain  $f(2) = 2$ , and

$$f(n+1) = f(n) + \frac{n-f(n)}{n} = \frac{n-1}{n}f(n) + 1.$$

From here, the most common approach to solving the question was to construct a recurrence relation and find its limiting value, but we can make our life easier by constructing a nice inequality (which can be observed by noticing  $f(n)$  grows approximately by  $\frac{1}{2}$  every time):

$$\frac{n}{2} \leq f(n) \leq \frac{n+1}{2}.$$

This can be proved inductively. Skipping the base case and assuming the induction hypothesis for  $n = k$ , we find

$$f(k+1) = \frac{k-1}{k}f(k) + 1 \geq \frac{k-1}{k} \frac{k+1}{2} + 1 = \frac{k+1}{2},$$

and

$$f(k+1) = \frac{k-1}{k}f(k) + 1 \leq \frac{k-1}{k} \frac{k+1}{2} + 1 = \frac{k^2-1}{2k} + 1 \leq \frac{k^2}{2k} + 1 = \frac{k+2}{2}.$$

Now, dividing our inequality through by  $n$ , we obtain  $\frac{1}{2} \leq \frac{f(n)}{n} \leq \frac{1}{2} + \frac{1}{2n}$ , so by the pinching theorem,  $\frac{f(n)}{n} \rightarrow \frac{1}{2}$  as  $n \rightarrow \infty$ .

## 3 Q3

We will prove the contrapositive: let  $S$  be a set of numbers greater than 1 where  $\gcd(s, t)$ , for any  $s, t \in S$ , is not prime. We wish to construct  $n$  where  $\gcd(n, s) \notin \{1, s\}$  for all  $s \in S$ .

Firstly, note that no number  $s \in S$  can be prime as that would imply  $\gcd(s, s) =$

$s$  is prime. Now, for every non-coprime distinct pair of  $s, t \in S$ , we take some  $p \mid \gcd(s, t)$  and multiply them together, *ignoring duplicates of prime factors*. To this, we also multiply a single prime factor  $p \mid t$  for all  $t$  that is coprime to every *other* element of  $S$  (that is, we are now taking prime factors from numbers that weren't covered in the previous step). Now, we have  $\gcd(n, s) \neq 1$  for all  $s \in S$ .

Note that every number  $s \in S$  coprime to all  $t \in S$  where  $t \neq s$  must be such that  $\gcd(s, n)$  is prime, as that's how we chose  $n$ , meaning  $\gcd(s, n) \neq s$  as  $s$  is not prime. We will now try to "correct" our value of  $n$  to achieve  $\gcd(n, s) \neq s$  for all  $s \in S$ .

If  $\gcd(s, n) = s$ , then pick some prime  $p \mid s$ . For all  $t \in S$  which is divisible by  $p$ , since  $p \mid \gcd(t, s)$  and we cannot have  $\gcd(t, s)$  be equal to  $p$  (a prime), there must exist some other  $q \mid \gcd(t, s)$  (note that if this was not the case, then there must exist another factor of  $p$  i.e. that  $p^2 \mid \gcd(t, s)$ . This means  $p^2 \mid s$ , however only a single factor of  $p$  was used in the multiplication of  $n$ : specifically,  $n$  is not divisible by  $p^2$ , so  $\gcd(n, s) \neq s$ , which is a contradiction).  $q$  must divide  $n$  as we've established  $q \mid s = \gcd(s, n)$ , so if we reassign  $n$  to be  $\frac{n}{p}$ , we maintain that  $q \mid \gcd(n, t)$ , so  $\gcd(n, t) \neq 1$  and  $\gcd(n, s) \neq 1$ . This relabelling is the same for all  $t$ , and reassigning in this way has given us  $\gcd(n, s) \neq s$ . Filtering out a single prime in this way for all other such  $s$  gives us an  $n$  where  $\gcd(n, s) \notin \{1, s\}$  for all  $s \in S$ .

## 4 Q4

We direct you to "Narrowing the Bounds" in [this](#) wonderful article :).

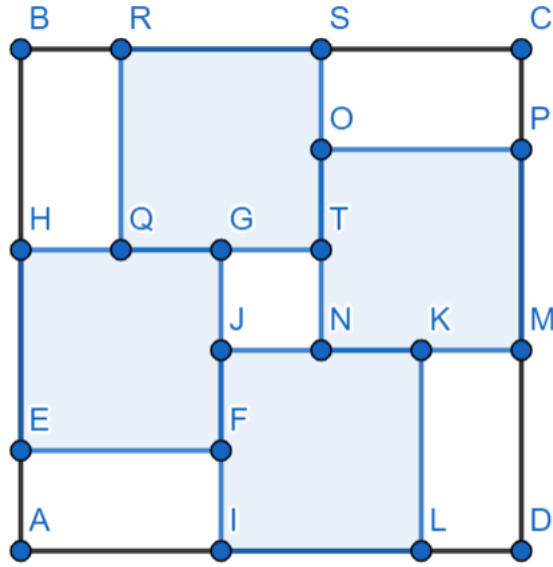
## 5 Q5

For part a), we essentially want to pack as many points as possible with the Manhattan distance between any two no less than  $2d$  on an  $n \times n$  lattice whose sides wrap around. We can model this by having a "zone" around each point, which is the set of points at most  $d$  manhattan distance away from it. Now, points are packed with distance no less than  $2d$  if and only if zones overlap by no more than their diagonal. Note that every zone has the same, fixed number of points.

This is actually equivalent to packing a bunch of  $d$ -radius diamonds in a square overlapping on both sides, with corners centered at integer coordinates! This is because the overlapping of diagonals corresponds exactly to the sides of diamonds touching. So, if the total area of a packing of diamonds is the square itself, it must be optimal. Consider, then, the points:  $\{(2di, 2dj) : i, j \in \mathbb{Z}\} \cup \{(2di + d, 2dj + d) : i, j \in \mathbb{Z}\}$ . Taking the subset of these points that lie in  $\mathcal{A}$  gives us  $2 \cdot \frac{n^2}{4d^2} = \frac{n^2}{2d^2}$  points (calculate by taking two separate "square"

lattices each with  $\frac{n^2}{4d^2}$  points) in  $\mathcal{A}$ , and since the area of each diamond is  $2d^2$ , the total area is  $n^2$ , which is the entire square and thus proves optimality.

For part b), the modular bounds of the square make packing annoying and sometimes non-intuitive. Even the problem of packing equivalent, axis-aligned squares is annoying: consider doing this, for example, with  $2 \times 2$  squares in a  $5 \times 5$  larger square. Though it may seem optimal to pack 4 all fitted to the top-left corner, here's a packing with 4 squares where a 5th can be placed by piecing together 4  $1 \times 1$  squares in each of the corners:



One way to see this optimal is to imagine moving a vertical line through the square: each square is either completely outside the line or completely covers  $s$  of the line ( $s$  being every small square's side length), and there must thus be at least  $n \bmod s$  of the line not covered by any squares: "integrating" this over the entire square gives a minimum total area of  $n(n \bmod s)$ . This can also be obtained from a horizontal line, and if we look at our configuration, no two "negative-space-squares" overlap, thus creating the maximum amount of overlap with the missing horizontal area and missing vertical area (attaining a missing area of  $5(5 \bmod 2)$ ). And that's a good place to stop.