Computational Tools

Project 1: Solving Kepler's Equation

Kepler's equation

$$M = E - e\sin E \tag{1}$$

is transcendental in E, i.e. given M a solution in E cannot be expressed in closed form. However, during the years there have been developed a variety of numerical and analytical methods to compute the solutions.

• **TASK 1:** Write a root finding algorithm for Kepler's equation using the Newton-Raphson method. Compute the values of $E(M)$ with $M \in [0, 2\pi]$ for eccentricities $e = 0, 0.25, 0.5, 0.75, 0.99$ and make a plot to present your results.

In 1770, Lagrange's work on Kepler's equation led to a generally useful expansion theorem. Given an equation of the form $y = x + \alpha\phi(y)$, its solution is approximated by the series expansion:

$$y = x + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \frac{d^{n-1}}{dx^{n-1}} \phi(x)^n \tag{2}$$

• **TASK 2:** Apply Langrage's theorem to Kepler's equation and write a code that computes the series solution. Use your program to compute the series expression $E(M)$ by keeping terms up to order $n = 5$ and $n = 10$. Do a trigonometric reduction of the outcome, such that the series takes the form $E = M + \sum_n \Pi_n(e)\sin(nM)$, with $\Pi_n(e)$ polynomials of e. Evaluate the order five ($n = 5$) and order ten ($n = 10$) series for $e = 0.5$ and compare it with the numerical results of TASK 1. Do the same for $e = 0.99$. Discuss your results.

In fact Lagrange's series reversion theorem is equivalent to analytically applying the $x = g(x)$ method to find the root of an algebraic equation and then Taylor-expanding the composition $g^N(x)$ with respect to the small parameter.

• **TASK 3:** Apply analytically the $x = g(x)$ method to find $E(M)$. To do this you need to bring Kepler's equation to the form $E = g(E)$ and compute the N-times composition of $g^N(E)$, where $N > n$. Then compute the Taylor series of $g^N(E)$ with respect to the eccentricity up to order $n = 5$ and $n = 10$ to obtain $E(M)$. Compare your results with the Lagrange series computed in TASK 2. What do you observe?

Another representation of the solution in E is given by re-arranging the terms as a Fourier-Bessel series expansion:

$$E = M + \sum_{j=1}^{\infty} \frac{2}{j} J_j(je)\sin jM \tag{3}$$

where the coefficients J_n are Bessel functions of the first kind, defined by the infinite series

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{n+2k} \tag{4}$$

• **TASK 4:** Write a program that computes the Bessel functions $J_n(x)$ and make a plot of the functions $J_1, J_2, \ldots J_5$ for $x \in [0, 15]$. Use your Bessel functions to implement the Fourier-Bessel expansion of E and compute the series keeping terms up to $j = 5$ and $j = 10$. For $e = 0.5$ evaluate $E(M)$ from the order five ($j = 5$) and order ten ($j = 10$) series and compare with the numerical results of TASK 1. Do the same for $e = 0.99$. What do you observe? Compare with the results of TASK 2 and discuss.