**Computational Tools**

# Project 2: Rigid-body dynamics

The free rotation of a rigid body is governed by a set of three first order differential equations (Euler's equations):

$$I_1\dot{\omega_1}(t) = (I_2 - I_3)\omega_2(t)\omega_3(t)$$
$$I_2\dot{\omega_2}(t) = (I_3 - I_1)\omega_3(t)\omega_1(t)$$
$$I_3\dot{\omega_3}(t) = (I_1 - I_2)\omega_1(t)\omega_2(t)$$

Following the laws of conservation for the energy and the angular momentum, two integrals of Euler's equations exist:

$$I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = 2E$$
$$I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = M^2$$

where $E$ is the rotational kinetic energy, and $M$ the magnitude of the angular momentum. Using the two integrals, Jacobi managed to solve Euler's equations. The solution is given in terms of the Jacobi elliptic functions:

$$\omega_1(t) = \sqrt{\frac{2EI_3 - M^2}{I_1(I_3 - I_1)}}cn(\tau, k^2)$$

$$\omega_2(t) = \sqrt{\frac{2EI_3 - M^2}{I_2(I_3 - I_2)}}sn(\tau, k^2)$$

$$\omega_3(t) = \sqrt{\frac{M^2 - 2EI_1}{I_3(I_3 - I_1)}}dn(\tau, k^2)$$

where

$$\tau = t\sqrt{\frac{(I_3 - I_2)(M^2 - 2EI_1)}{I_1 I_2 I_3}}, \quad k^2 = \frac{(I_2 - I_1)(2EI_3 - M^2)}{(I_3 - I_2)(M^2 - 2EI_1)}$$

- **TASK 1:** Write a program that computes the analytical solution for the free rotation. Draw the solution $\omega_1(t), \omega_2(t), \omega_3(t)$ for a body with $I_1 = 0.8, I_2 = 0.9, I_3 = 1.0$ and initial conditions $\omega_1(0) = 1.0, \omega_2(0) = 0.0, \omega_3(0) = 2.0$ up to $t_{max} = 100$. Compute the relative error in the energy $log(|(E(t) - E(0))/E(0)|)$ for the analytic solution.

Another way to compute the solutions of Euler's equation is via numerical integration of the first order differential equations. One commonly used method is the Runge-Kutta of 4th order.

- **TASK 2:** Write a program that solves Euler's equations with the Runge-Kutta method of 4th order. Draw the solution $\omega_1(t), \omega_2(t), \omega_3(t)$ for the same parameters and initial condition as in TASK 1, up to $t_{max} = 100$ with a time-step of $dt = 0.1$. Compute the relative error $log(|(E(t) - E(0))/E(0)|)$ for the RK4 numerical solution. What do you observe?

Finally, there exist splitting methods for the solution of the free rigid body motion. In these methods the total problem can be split into components that can be solved exactly. A simple splitting for the Euler's problem is the following:

$$H = \frac{1}{2}\left(\frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3}\right) = H_{M_1} + H_{M_2} + H_{M_3}, \quad \text{with} \quad H_{M_1} = \frac{M_1^2}{2I_1}, H_{M_2} = \frac{M_2^2}{2I_2}, H_{M_3} = \frac{M_3^2}{2I_3}$$

where $\mathbf{M} = (M_1, M_2, M_3) = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$ is the angular momentum vector. Each of $H_{M_i}$ can be solved exactly and the solutions are given from:

$$\Phi^{H_{M_1}} : \mathbf{M}(t) = R_x(M_1(0)t/I_1)\mathbf{M}(0)$$
$$\Phi^{H_{M_2}} : \mathbf{M}(t) = R_y(M_2(0)t/I_2)\mathbf{M}(0)$$
$$\Phi^{H_{M_3}} : \mathbf{M}(t) = R_z(M_3(0)t/I_3)\mathbf{M}(0)$$

where $R_x, R_y, R_z$ are the clockwise rotation matrices about the $x, y$ and $z$-axis respectively. A step $dt$ of the splitting method consists of combining the exact solution for each $H_{M_i}$ with the following order:

$$\Phi^H(dt) = \Phi^{H_{M_1}}(dt/2) \circ \Phi^{H_{M_2}}(dt/2) \circ \Phi^{H_{M_3}}(dt) \circ \Phi^{H_{M_2}}(dt/2) \circ \Phi^{H_{M_1}}(dt/2) + \mathcal{O}(dt^3).$$

• **TASK 3:** Write a program that solves Euler's equations with the splitting method. Draw the solution $\omega_1(t), \omega_2(t), \omega_3(t)$ for the same parameters and initial condition as in TASK 1, up to $t_{max} = 100$ with a time-step of $dt = 0.02$. Compute the relative error $log(|(E(t) - E(0))/E(0)|)$ for the splitting method solution. What do you observe? Compare with the results of the other tasks and discuss.