ACDDA BBDAC BCDCD DCEDC BCCBB DABAC

1. A $$f'(x) = 2x + 2x + 3x^2 \to f'(1) = 7\ (A)$$

2. C We use L'Hopital since $\frac{0}{0}$ is an indeterminate form. $\lim_{x\to 0}\frac{\sin(6x)}{2x} = \lim_{x\to 0}\frac{6\cos(6x)}{2} = 3\ (C)$

3. D $$f'(x) = \frac{d}{dx}(x)\left(e^x\cos\left(\frac{x}{3}\right)\right) = \left(e^x\cos\left(\frac{x}{3}\right)\right) \to f'(\pi) = e^\pi\cos\left(\frac{\pi}{3}\right) = \frac{e^\pi}{2}\ (D)$$

4. D We use L'Hopital since $\frac{0}{0}$ is an indeterminate form. $\lim_{x\to 0}\frac{e^{2x}-1}{x} = \lim_{x\to 0}\frac{2e^{2x}}{1} = 2\ (D)$

5. A We use L'Hopital since $\frac{0}{0}$ is an indeterminate form. $\lim_{x\to 4}\frac{x^2-16}{x^2-2x-8} = \lim_{x\to 4}\frac{2x}{2x-2} = \frac{4}{3}\ (A)$

6. B We use L'Hopital twice since $\frac{0}{0}$ is an indeterminate form. $\lim_{x\to 0}\frac{\ln(cos(x))}{x^2} = \lim_{x\to 0}\frac{-\tan(x)}{2x} = \lim_{x\to 0}\frac{-\sec^2(x)}{2} = -\frac{1}{2}\ (B)$

7. B We use L'Hopital since $\frac{0}{0}$ is an indeterminate form. $\lim_{x\to 0}\frac{1-\cos(x)}{\tan^2(x)} = \lim_{x\to 0}\frac{\sin(x)}{2\tan(x)\sec^2(x)} = \lim_{x\to 0}\frac{\cos^3(x)}{2} = \frac{1}{2}\ (B)$. Note that we could've also used Taylor series as the first term of the numerator is $1-\left(1-\frac{x^2}{2}\right) = \frac{x^2}{2}$ and the first term of the denominator is $\left(x+\frac{x^3}{3}\right)^2 = x^2$ so the limit is $\frac{\frac{x^2}{2}}{x^2} = \frac{1}{2}$

8. D $$g'\left(\frac{\pi}{4}\right) = \frac{1}{f'\left(g\left(\frac{\pi}{4}\right)\right)} = \frac{1}{f'(e)} = \frac{1}{\frac{1}{e(1+(\ln(e))^2)}} = 2e\ (D)$$

   Note that $f'(x) = \frac{1}{x(1+(\ln(x))^2)}$ and $g\left(\frac{\pi}{4}\right) = e$ since $\arctan(ln(e)) = \arctan(1) = \frac{\pi}{4}$

9. A We can turn this summation into an integral by letting $\frac{1}{n}$ be dx and $\frac{i}{n}$ be x with the bounds from 0 to 1:

$$\int_0^1 xsin(\pi x)dx$$

Now we use tabular with x on the derivative side and $\sin(\pi x)$ on the integral side to get an expression we evaluate from $x = 0$ to $x = 1$:

$$-\frac{x}{\pi}\cos(\pi x) + \frac{1}{\pi^2}\sin(\pi x)$$

$$x = 1 \to \frac{1}{\pi}$$

$$x = 0 \to 0$$

Our answer is $\frac{1}{\pi} - 0 = \frac{1}{\pi}$ (A)

10. C Let $d = \sqrt{x^2 + y^2} \to d^2 = x^2 + y^2 \to \frac{dd}{dt}d = \frac{dx}{dt}x + \frac{dy}{dt}y$. At $t = 2$, we have $\frac{dx}{dt} = 2(2) - 8 = -4$ and $\frac{dy}{dt} = 3(2)^2 = 12$. We also have $x = (2)^2 - 8(2) = -12$ and $y = (2)^3 - 3 = 5$ at $t = 2$. With $x = -12$ and $y = 5$, we have $d = 13$. $\frac{dd}{dt}(13) = (-4)(-12) + (12)(5) = 108 \to \frac{dd}{dt} = \frac{108}{13}$ (C)

11. B

$$\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$\frac{dx}{dt} = 2t + 6$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-3t^2 + 4}{2t + 6}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{(2t + 6)(-6t) - (-3t^2 + 4)(2)}{(2t + 6)^2} = \frac{-6t^2 - 36t - 8}{(2t + 6)^2}$$

At $t = 1$, we have $\frac{\frac{-50}{64}}{8} = -\frac{25}{256}$ (B)

12. C We use Taylor Series. $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ so $\ln^2(1 + x) = x^2 - x^3 + \frac{11}{12}x^4 + \cdots$

$$\lim_{x \to 0} \frac{\ln^2(1 + x) - x^2}{x^3} = \lim_{x \to 0} \frac{x^2 - x^3 + \cdots - x^2}{x^3} = -1\ (C)$$

13. D We isolate $y' = xy + 2x + y + 2 = (x + 1)(y + 2) \to \frac{dy}{dx} = (x + 1)(y + 2) \to \frac{1}{y+2}dy = (x + 1)dx \to \ln(y + 2) = x + \frac{1}{2}x^2 + C$

With $f(0) = 1$, $\ln(3) = C$ so $\ln(y + 2) = x + \frac{1}{2}x^2 + \ln(3)$. At $x = 2$, we have $\ln(y + 2) = 2 + 2 + \ln(3) \to y + 2 = 3e^4 \to y = 3e^4 - 2$ (D)

14. C We look at the graph of $4-x^2$. As x approaches 0, $g(x)$ approaches 4 from the negative direction. Thus, we're taking the floor function of a value that's barely less than 4, so the answer is 3 (C)

15. D

$$\frac{dy}{dx}=\frac{r'sin(\theta)+rcos(\theta)}{r'\cos(\theta)-rsin(\theta)}$$

$$r'=\cos(\theta)-\sin(2\theta)$$

$$\theta=\frac{3\pi}{4}\rightarrow r=\frac{1+\sqrt{2}}{2}$$

$$r'=\frac{2-\sqrt{2}}{2}$$

$$\frac{dy}{dx}=\frac{\left(\frac{2-\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)+\left(\frac{1+\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)}{\left(\frac{2-\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)-\left(\frac{1+\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}=\frac{\frac{1-2\sqrt{2}}{2}}{\frac{-3}{2}}=\frac{2\sqrt{2}-1}{3}\ (D)$$

16. D Let this cubic be $f(x)=ax^3+bx^2+cx+d$. $f(0)=d$

$$f'(0)=c$$

$$f''(0)=2b$$

$$f'''(0)=6a$$

Since $d=c=2b=6a$, we can write $f(x)=ax^3+3ax^2+6ax+6a$. Note that $f(x)=a(x-r_1)(x-r_2)(x-r_3)$ so $\frac{f'(x)}{f(x)}=\frac{1}{x-r_1}+\frac{1}{x-r_2}+\frac{1}{x-r_3}$. The question asks for $\frac{f'(2)}{f(2)}=\frac{a(3(2)^2+6(2)+6)}{a(8+12+12+6)}=\frac{30}{38}=\frac{15}{19}$

$$15+19=34\ (D)$$

17. C We get $\frac{0}{0}$ when we plug in $h=0$ so we can use L'Hopital where we take the derivate of top and bottom with respect to h. $\lim_{h\to 0}\frac{f(x+5h)-f(x-3h)}{2h}=\lim_{h\to 0}\frac{5f'(x+5h)+3f'(x-3h)}{2}=4f'(x)=4(3x^2)=12x^2\ (C)$

18. E

$$f'(x)=\frac{-\left((x-5)^2+(x-8)(2)(x-5)\right)}{(x-5)^4(x-8)^2}=-\frac{3x^2-36x+105}{(x-5)^4(x-8)^2}$$
$$=-\frac{3(x-5)(x-7)}{(x-5)^4(x-8)^2}=\frac{-3(x-7)}{(x-8)^2(x-5)^3}$$

We let this equal to 0 and then we get $x = 7$. $x = 8$ and $x = 5$ makes $f'(x)$ undefined but they're not critical points because they're not even in the domain of $f(x)$. Thus, our answer is 7 (E)

19. D

$$f'(c) = \frac{\sqrt{a}}{a} \rightarrow \frac{1}{2\sqrt{c}} = \frac{1}{\sqrt{a}} \rightarrow a = 4c$$

Since a and c are positive integers and we want to minimize $a^2 + c^2$, we let $a = 4$ and $c = 1$ so that $a^2 + c^2 = 17$ and the sum of the digits is $1 + 7 = 8$ (D)

20. C

$$f'(x) = \sum_{n=2}^{\infty} \frac{x^{n-1}}{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} x^{n-2} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

We now take anti-derivatives and get $f'(x) = -\ln(1-x) + C$. From the summation, $f'(0) = 0$ so $C = 0$. $f'(x) = -\ln(1-x)$. We take the anti-derivative and calculate the integral on the right using tabular to get $f(x) = -xln(1-x) + x + \ln(1-x) + C$. The summation tells us $f(0) = 0$ so $C = 0$. $f\left(\frac{1}{2}\right) = -\frac{1}{2}\ln\left(\frac{1}{2}\right) + \frac{1}{2} + \ln\left(\frac{1}{2}\right) = \frac{1}{2}\ln(2) + \frac{1}{2} - \ln(2) = \frac{1-\ln(2)}{2} \rightarrow 100f\left(\frac{1}{2}\right) = 50 - 50\ln(2) \approx 50 - 50(.69) = 50(.31) = 15.5$. The floor function gives us 15 (C)

21. B We use $a_n = c^2$ $a_{n-1} = c$ and $a_{n-2} = 1$ so that $c^2 = 2c + 3 \rightarrow c^2 - 2c - 3 = 0 \rightarrow (c-3)(c+1) = 0 \rightarrow c = 3, -1$. So $a_n = a(3)^n + b(-1)^n$. $a_0 = 10$ means that $a + b = 10$ and $a_1 = 14$ means that $3a - b = 14$. We add the two equations and get $4a = 24 \rightarrow a = 6$ and $b = 4$. So $a_n = 2(3)^{n+1} + 4(-1)^n$. As n goes to infinity, the $(-1)^n$ term is negligible so $\lim\limits_{n\to\infty} \frac{a_{n+2}}{a_n} = \frac{2(3)^{n+3}}{2(3)^{n+1}} = 9$ (B)

22. C We let $v(t) = \frac{dx}{dt}$ so that $\frac{\frac{dx}{dt}}{x} = karctan(t) \rightarrow \frac{1}{x}dx = karctan(t)dt \rightarrow \ln(x) = k\left(tarctan(t) - \frac{1}{2}\ln(1+t^2)\right) + C$. At $t = 0$, we have $x = 1$ so $0 = k(0) + C$ and $C = 0$. At $t = 1$, we have $x = \frac{e^{\pi}}{4}$ and $\ln\left(\frac{e^{\pi}}{4}\right) = k\left(\frac{\pi}{4} - \frac{1}{2}\ln(2)\right)$. Note that $k = 4$ satisfies this. When $t = \sqrt{3}$, we have $\ln(x) = k\left(\frac{\pi\sqrt{3}}{3} - \ln(2)\right) = \frac{4\pi\sqrt{3}}{3} - 4\ln(2) \rightarrow x = \frac{e^{\frac{4\pi\sqrt{3}}{3}}}{16}$ so $a = \frac{4\sqrt{3}}{3}$ and $b = 16$ so that $\left(\frac{b}{a}\right)^2 = \left(\frac{16}{\frac{4}{\sqrt{3}}}\right)^2 = 48$ (C)

23. C We use Taylor series so $\left(1-\frac{x^2}{2}\right)^4 = 1+(1)^3\left(-\frac{x^2}{2}\right)(4) = 1-2x^2$. It remains to find the area bounded by $y=1-2x^2$ and the x-axis. This is a parabola so we can use the area formula $\frac{2}{3}bh = \frac{2}{3}(\sqrt{2})(1) = \frac{2\sqrt{2}}{3}$ (C)

24. B We have $V=\frac{\pi r^2 h}{3} \to r^2 = \frac{3V}{h\pi}$. The lateral surface area of the cone is $\pi r l = \pi r\sqrt{r^2+h^2} = \pi\sqrt{r^2(r^2+h^2)} = \pi\sqrt{\left(\frac{3V}{h\pi}\right)\left(\frac{3V}{h\pi}+h^2\right)} = \pi\sqrt{\frac{9V^2}{h^2\pi^2}+\frac{3Vh}{\pi}} = \sqrt{\frac{9V^2}{h^2}+3Vh\pi}$. We now take the derivative with respect to h and get $\frac{\frac{-18V^2}{h^3}+3V\pi}{2\sqrt{\frac{9V^2}{h^2}+3Vh\pi}} = 0 \to 3V\pi = \frac{18V^2}{h^3} \to 3V\pi h^3 = 18V^2 \to \pi h^3 = 6V \to \frac{3V}{h\pi} = \frac{h^2}{2} = r^2 \to \frac{h^2}{r^2} = 2 \to \frac{h}{r} = \sqrt{2}$ (B)

25. B We will use Taylor Series. Note that we only care about the x^6 term. $\tan^{-1}(x) = x-\frac{x^3}{3}+\frac{x^5}{5}+\cdots$ is the Maclaurin expansion. We'll also use the fact that $e^x = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots$. $e^{\arctan(x)} = 1+\arctan(x)+\frac{\arctan^2(x)}{2}+\frac{\arctan^3(x)}{6}+\frac{\arctan^4(x)}{24}+\frac{\arctan^5(x)}{120}+\frac{\arctan^6(x)}{720}+\cdots$. Note that $\frac{\arctan^n(x)}{n!}$ doesn't contribute any x^6 terms for $n\geq 7$ because the smallest exponent in that case would be x^n where $n \geq 7 > 6$.

The 1 doesn't contribute and note that $\arctan(x)$, $\frac{\arctan^3(x)}{6}$, and $\frac{\arctan^5(x)}{120}$ only contribute x terms with odd exponents since all of them are odd functions ($\arctan(x)$ is an odd function). Thus, the only terms that could contribute x^6 terms are $\frac{\arctan^2(x)}{2}+\frac{\arctan^4(x)}{24}+\frac{\arctan^6(x)}{720}$. $\arctan^2(x) = \left(x-\frac{x^3}{3}+\frac{x^5}{5}\right)^2 = 2(x)\left(\frac{x^5}{5}\right)+\left(-\frac{x^3}{3}\right)^2 = \left(\frac{2}{5}+\frac{1}{9}\right)x^6 = \frac{23}{45}x^6$ so $\frac{\arctan^2(x)}{2}$ contributes a $\frac{23}{90}$. $\arctan^4(x) = \left(x-\frac{x^3}{3}+\frac{x^5}{5}\right)^4 = 4(x)^3\left(-\frac{x^3}{3}\right)^1 = -\frac{4}{3}$ so $\frac{\arctan^4(x)}{24}$ contributes a $-\frac{1}{18}$. $\arctan^6(x) = \left(x-\frac{x^3}{3}+\frac{x^5}{5}\right)^6 = x^6$ so $\frac{\arctan^6(x)}{720}$ contributes a $\frac{1}{720}$.

$$\frac{23}{90}-\frac{1}{18}+\frac{1}{720} = \frac{29}{144} \to 29+144 = 173\ (B)$$

26. D We use Stirling's Approximation that $n! \sim \frac{n^n}{e^n}\sqrt{2\pi n}$. We also write $\binom{an}{n} = \frac{(an)!}{n!((a-1)n)!}$

This way, we can denote $(an)! = \frac{(an)^{an}}{e^{an}}\sqrt{2\pi an}$ and the rest of the factorials using Stirling's. After lots of careful computation, we arrive at the following limit:

$$\lim_{n\to\infty}\left(\frac{a^a}{b(a-1)^{a-1}}\right)^n\sqrt{\frac{a}{2\pi(a-1)n}}$$

It's clear that if $\frac{a^a}{b(a-1)^{a-1}}$ is greater than 1, then the limit will go to infinity because the exponential function when the base is greater than 1 will always overpower the $\frac{1}{\sqrt{n}}$ when they're multiplied together. Note that a is treated as a fixed constant where all we can assume about it is that $a > 1$. Logically, if b gets extremely large then $\frac{a^a}{b(a-1)^{a-1}}$ will be closer to 0 and raising that to n where $n \to \infty$ makes the limit zero. Thus, our minimum value of b must be $\frac{a^a}{b(a-1)^{a-1}} = 1 \to b = \frac{a^a}{(a-1)^{a-1}}$

When $a = 4$, we have $b = \frac{4^4}{3^3} \approx 9.7$ so the floor function becomes 9 (D)

27. A We can write $b = \frac{a^a}{(a-1)^{a-1}} = \left(1+\frac{1}{a-1}\right)^a(a-1)$. In the limit as $a \to \infty$ this is approximately $e(a-1)$ since $\lim_{a\to\infty}\left(1+\frac{1}{a-1}\right)^a = e$. Thus, $\frac{db}{da}$ approaches $\frac{d}{da}(ea-e) = e$ as $a \to \infty$ and our answer is e (A)

28. B We use our finding from number 26 and thus $\left(\frac{a^a}{b(a-1)^{a-1}}\right)^n\sqrt{\frac{a}{2\pi(a-1)n}} = (1)^n\sqrt{\frac{a}{2\pi(a-1)n}} = \sqrt{\frac{a}{2\pi(a-1)n}}$. Thus, we just need to multiply by $\sqrt{n}$ and we arrive at $\lim_{n\to\infty}\frac{\binom{an}{n}\sqrt{n}}{b^n} = \lim_{n\to\infty}\sqrt{\frac{a}{2\pi(a-1)}} = \sqrt{\frac{a}{2\pi(a-1)}} = k$. Now we set $a = k$:

$$\sqrt{\frac{k}{2\pi(k-1)}} = k \to 2\pi k(k-1) = 1 \to k = \frac{\pi+\sqrt{\pi^2+2\pi}}{2\pi}$$

We take the positive value of k. Thus, $2k = \frac{\pi+\sqrt{\pi^2+2\pi}}{\pi} = 1+\sqrt{1+\frac{2}{\pi}}$ and our answer is 2 (B)

29. A We use quotient rule and get that $g(x) = \frac{x^2(x^2+2bx+3c)}{(x^2+bx+c)^2}$ so $g(x) = 0$ has a solution at $x = 0$ and the other solution must come from $x^2+2bx+3c$. Thus, the quadratic only has 1 root and thus the discriminant is equal to zero. $(2b)^2 - 4(3c) = 0 \to c = \frac{b^2}{3}$. Now we can write $b - c = b - \frac{b^2}{3}$ which has a maximum value at the vertex of the parabola which is $b = \frac{3}{2}$. Plugging it back in, we get $b - c = \frac{3}{4} \to 3 + 4 = 7$ (A)

30. C If $g(x)$ has no vertical asymptotes, that means that the denominator can't ever equal 0. This means that the discriminant of x^2+bx+c is strictly less than 0:

$$b^2 - 4c < 0 \rightarrow c > \frac{b^2}{4}$$

If $g(x)$ crosses the axis at 3 points, that means that $x = 0$ is one of those points and that the quadratic $x^2 + 2bx + 3c = 0$ has 2 distinct solutions. Thus, the discriminant is strictly greater than 0. $4b^2 - 12c > 0 \rightarrow c < \frac{b^2}{3}$. Now we use geometric probability. Imagine the y-axis as c and imagine the x-axis as b. We now graph $y < \frac{x^2}{3}$ and $y > \frac{x^2}{4}$ from $x = 0$ to $x = 16$ and from $y = 0$ to $y = 16$. The area that this bounds is:

$$\int_0^{16} \sqrt{4c} - \sqrt{3c}dc = \frac{128}{3}(2 - \sqrt{3})$$

To make this a probability, we divide by the area of the 16 x 16 square and get a final answer of

$$\frac{1}{256}\left(\frac{128}{3}\right)(2 - \sqrt{3}) = \frac{2 - \sqrt{3}}{6} \rightarrow 2 + 3 + 6 = 11\ (C)$$