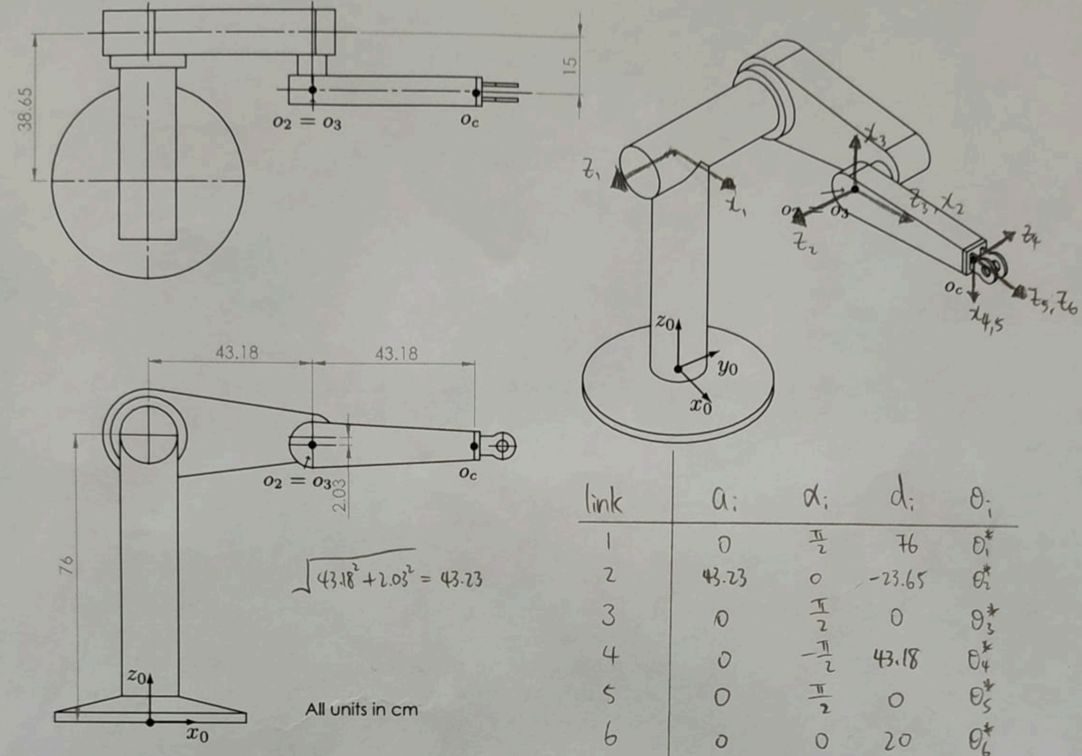

link	a_i	α_i	d_i	θ_i
1	0	$\frac{\pi}{2}$	76	θ_1^*
2	43.23	0	-23.65	θ_2^*
3	0	$\frac{\pi}{2}$	0	θ_3^*
4	0	$-\frac{\pi}{2}$	43.18	θ_4^*
5	0	$\frac{\pi}{2}$	0	θ_5^*
6	0	0	20	θ_6^*

Figure 1: Schematics of the PUMA560 robot

By extracting the 3×3 matrix R_6^0 and the 3×1 vector o_6^0 from H_6^0, the forward kinematics computation is complete. Recall the geometric meaning of R_6^0 and o_6^0: the columns of R_6^0 are the unit axes of the end effector frame 6 represented in the coordinates of frame 0: $R_6^0 = [x_6^0 \mid y_6^0 \mid z_6^0]$; the vector o_n^0 is the origin of frame 6 expressed in the coordinates of frame 0.

2.2 Inverse Kinematics

Given a desired position $o_d^0 \in \mathbb{R}^3$ and desired orientation $R_d \in \mathrm{SO}(3)$ of the end effector, the inverse kinematics problem is to find $(\theta_1, \ldots, \theta_6)$ such that $R_6^0(\theta_1, \ldots, \theta_6) = R_d$ and $o_6^0(\theta_1, \ldots, \theta_6) = o_d^0$.

Since the PUMA 560 has six links and a spherical wrist, one can solve the inverse kinematics problem by the technique of kinematic decoupling. In kinematic decoupling, the problem is divided in two parts: inverse position and inverse orientation.

Inverse position. The position of the wrist centre o_c only depends on the angles of the first three joints, $(\theta_1, \theta_2, \theta_3)$. The idea is to compute the desired location of the wrist centre and then find $(\theta_1, \theta_2, \theta_3)$ accordingly.

- Compute the vector

$$o_d^0 - R_d \begin{bmatrix} 0 \\ 0 \\ d_6 \end{bmatrix}.$$

With $\theta_1, \theta_2, \theta_3$, we have R_3^0

$$R_6^3 = (R_3^0)^T R_d$$

$$R_6^3 = \begin{bmatrix} * & * & C_4 S_5 \\ * & * & S_4 S_5 \\ -S_5 C_6 & S_5 S_6 & C_5 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

Using ZYZ Euler angles:

Case 1 if $M_{13}^2 + M_{23}^2 \neq 0$

① $S_5 > 0$

$\theta_4 = \text{atan2}(M_{23}, M_{13})$

$\theta_5 = \text{atan2}(\sqrt{1 - M_{33}^2}, M_{33})$

$\theta_6 = \text{atan2}(M_{32}, -M_{31})$

② $S_5 < 0$

$\theta_4 = \text{atan2}(-M_{23}, -M_{13})$

$\theta_5 = \text{atan2}(-\sqrt{1 - M_{33}^2}, M_{33})$

$\theta_6 = \text{atan2}(-M_{32}, M_{31})$

Case 2 if $M_{13}^2 + M_{23}^2 = 0$

① $M_{33} = 1$

$\theta_5 = 0$

$\theta_4 + \theta_6 = \text{atan2}(M_{21}, M_{11})$

② $M_{33} = -1$

$\theta_6 = \pi$

$\theta_4 + \theta_6 = \text{atan2}(-M_{12}, -M_{11})$

$$H_3^0 = H_1^0(\theta_1) H_2^1(\theta_2) H_3^2(\theta_3)$$

$$H_n^{n-1} = \begin{bmatrix} C\theta_n & -S\theta_n C\alpha_n & S\theta_n S\alpha_n & a_n C\theta_n \\ S\theta_n & C\theta_n C\alpha_n & -C\theta_n S\alpha_n & a_n S\theta_n \\ 0 & S\alpha_n & C\alpha_n & d_n \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_3^0 = \begin{bmatrix} C_1 & 0 & S_1 & 0 \\ S_1 & 0 & -C_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_2 & -S_2 & 0 & a_2 C_2 \\ S_2 & C_2 & 0 & a_2 S_2 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_3 & 0 & S_3 & 0 \\ S_3 & 0 & -C_3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_3^0 = \begin{bmatrix} C_1C_2C_3 - C_1S_2S_3 & S_1 & C_1C_2S_3 + C_1C_3S_2 & d_2S_1 + a_2C_1C_2 \\ C_2C_3S_1 - S_1S_2S_3 & -C_1 & C_2S_1S_3 + C_3S_1S_2 & a_2C_2S_1 - C_1d_2 \\ C_2S_3 + C_3S_2 & 0 & S_2S_3 - C_2C_3 & d_1 + a_2S_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ From Matlab

$$R_6^3 = \underbrace{\begin{bmatrix} C_1C_2C_3 - C_1S_2S_3 & C_2C_3S_1 - S_1S_2S_3 & C_2S_3 + C_3S_2 \\ S_1 & -C_1 & 0 \\ C_1C_2C_3 + C_1C_3S_2 & C_2S_1S_3 + C_3S_1S_2 & S_2S_3 - C_2C_3 \end{bmatrix}}_{R_3^{0\,T}} R_d$$

Inverse Kinematics

Using hints diagrams:

$$\boxed{\theta_1 = \operatorname{atan2}(y_c, x_c) - \operatorname{atan2}\left(d_2, \sqrt{x_c^2+y_c^2-d_2^2}\right)}$$

$$PQ^2 + QR^2 - PR^2 = 2(PQ)(QR)\cos\left(\pi - \left(\theta_3 - \tfrac{\pi}{2}\right)\right)$$

$$a_2^2 + d_4^2 - (z_c-d_1)^2 - x_c^2 - y_c^2 + d_2^2 = -2a_2d_4\sin(\theta_3)$$

$$\sin\theta_3 = \frac{a_2^2+d_4^2-(z_c-d_1)^2-x_c^2-y_c^2+d_2^2}{-2a_2d_4}$$

$$\sin\theta_3 = \frac{(z_c-d_1)^2+x_c^2+y_c^2-d_2^2-a_2^2-d_4^2}{2a_2d_4} = D$$

$$\boxed{\theta_3 = \operatorname{atan2}\left(D, \pm\sqrt{1-D^2}\right)}$$

$$\phi_2 = \operatorname{atan2}(RT, PT)$$

$$\phi_2 = \operatorname{atan2}\left(z_c-d_1, \sqrt{x_c^2+y_c^2-d_2^2}\right)$$

$$\phi_1 = \operatorname{atan2}(RS, PS)$$

$$\phi_1 = \operatorname{atan2}\left(d_4\sin\left(\theta_3-\tfrac{\pi}{2}\right), a_2+d_4\cos\left(\theta_3-\tfrac{\pi}{2}\right)\right)$$

$$\phi_1 = \operatorname{atan2}\left(-d_4\cos\theta_3, a_2+d_4\sin\theta_3\right)$$

$$\theta_2 = \phi_2 - \phi_1$$

$$\boxed{\theta_2 = \operatorname{atan2}\left(z_c-d_1, \sqrt{x_c^2+y_c^2-d_2^2}\right) - \operatorname{atan2}\left(-d_4\cos\theta_3, a_2+d_4\sin\theta_3\right)}$$