AER1516 Assignment 1 Report

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1 Problem 1

Gimbal lock occurs when Y = $\frac{\pi}{2}$ rads or when Y = $\frac{-\pi}{2}$. Gimbal lock is a concern because it causes two of the three axes of rotation to align, resulting in a loss of one degree of freedom and causing an object to get stuck in a fixed orientation. There is also no way to differentiate between the Z and X Euler angles which is an issue if you want to determine the Z-Y-X Euler angle sequence.

2 Problem 2

Let the point (x,y) = (0,1) be the "north pole" of the unit circle and the point (x,y) = (0,-1) be the "south pole" of the unit circle. Draw a horizontal real number line for distance, where 0 is located at the center of the circle and the line stretches to positive and negative infinity in each direction and is in the same units of distance as the unit circle. Let a point P(x,y) be located somewhere on the unit circle excluding the north pole. Draw a line from the north pole through the point P and extend the line until it reaches the real number line. The distance where the line intersects the real number line is the coordinates of point P in the coordinate map. This is called a stereographic projection from the north pole which is a coordinate chart for the circle. Another coordinate chart for the circle is the stereographic projection from the south pole, which is the same thing except that the line is drawn from the south pole and the south pole is excluded for the point P. In other words, $U = \{(x,y) \in S^1\}\{\backslash(0,1)\}$ and $V = \{(x,y) \in S^1\}\{\backslash(0,-1)\}$. The map $\phi : U \to \mathbb{R}$ is $\phi(x,y) = \frac{x}{1-y}$ and $\psi : V \to \mathbb{R}$ is $\psi(x,y) = \frac{x}{1+y}$ which both can be derived using similar triangles. These two charts clearly cover the whole unit circle. The map $\phi \circ \psi^{-1}$ is a diffeomorphism because it is bijective and the inverse is a smooth function. Note that what the map is doing is taking the domain of function ψ and applying it to function ϕ. Bijectivity means that the function is both one-to-one and onto. The function $\phi(x,y) = \frac{x}{1-y}$ is one-to-one for the domain of $U = \{(x,y) \in S^1\}\{\backslash(0,-1)\}$ as every point (x,y) in the domain maps to a different $\phi(x,y)$ which can be shown by substituting in the expression for y for the unit circle. The expression $x^2 + y^2 = 1$ can be written as $y = \sqrt{1-x^2}$ and substituting into ϕ we get $\phi = \frac{x}{1-\sqrt{1-x^2}}$ which is clearly a one-to-one function. Using the substituted expression we can also see that the function is onto or for each ϕ in the range there is at least one x in the domain that maps to it. The inverse of the map $\phi \circ \psi^{-1}$ is a smooth function because the inverse of the map represents the circle without the south pole which is clearly a smooth function.

3 Problem 3

Since we are dealing with a unit sphere it makes sense to use spherical coordinates. Recall that converting from Cartesian coordinates to spherical coordinates, we have $x = rsin\theta cos\phi$, $y = rsin\theta cos\phi$, and $z = rcos\theta$. Since it is a unit sphere, $r = 1$ and so $x = sin\theta cos\phi$, $y = sin\theta cos\phi$, and $z = cos\theta$. Let there be two vectors to two points on the unit sphere where each vector/point is represented by the subscripts 1 and 2. So we have $\mathbf{v_1} = (\mathbf{sin\theta_1 cos\phi_1, sin\theta_1 cos\phi_1, cos\theta_1})$ and $\mathbf{v_2} = (\mathbf{sin\theta_2 cos\phi_2, sin\theta_2 cos\phi_2, cos\theta_2})$. The distance between the two points on the surface of the sphere can be represented as an arclength created from the angle between the two vectors. The angle ψ between the two vectors can be calculated using the dot product, where $\mathbf{v_1} \cdot \mathbf{v_2} = \mathbf{cos\theta_1 cos\theta_2 + sin\theta_1 sin\theta_2 cos(\phi_1 - \phi_2) = cos\psi}$. So the arclength is $s = r\psi = \psi = cos^{-1}(cos\theta_1 cos\theta_2 + sin\theta_1 sin\theta_2 cos(\phi_1 - \phi_2))$. So an appropriate distance metric for the unit sphere in $\mathbb{R}^3$ is the arclength which is $s = cos^{-1}(cos\theta_1 cos\theta_2 + sin\theta_1 sin\theta_2 cos(\phi_1 - \phi_2))$.

4 Problem 4

4.1 a

Two mobile robots rotating and translating in the plane is SE(2) x SE(2) or 6 dimensional because the configuration space for one robot rotating and translating in the plane is SE(2) and 3 dimensional, 2 D0Fs for translation and 1 DOF for orientation.

4.2 b

Two mobile robots tied together by a rope rotating and translating in the plane is still SE(2) x SE(2) or 6 dimensional because it is still possible for the two robots to move independently and have different states if the rope is not taut. The rope may just add a constraint on the configuration states.

4.3 c

The configuration space of a train on train tracks including wheel angles is $\mathbb{S}^n$ x $\mathbb{R}$, or $n + 1$ dimensional where n is the number of wheels. One dimension is used to measure the distance the train has travelled on the train tracks and the other dimensions are used to measure the angle of each wheel. The reason only one dimension needs to be used to measure the state of the train is because the train tracks constrain the location of the train to one degree of freedom.

4.4 d

The configuration space of legs pedalling a bicycle is $\mathbb{S}^4$ or 4 dimensional, since the state of each leg can be specified by the angle of the hip joint and the angle of the knee and there are two legs.

5 Problem 5

We want to prove that $\mathcal{C}(\mathcal{WO}_i \bigcup \mathcal{WO}_j) = \mathcal{CO}_i \bigcup \mathcal{CO}_j$. Logically for any configuration q, if it is blocked by an obstacle in the configuration space $\mathcal{WO}_i \bigcup \mathcal{WO}_j$, there is a point in the workspace where at least one of $\mathcal{WO}_i$ or $\mathcal{WO}_j$ obstructs the manipulator in that configuration. It means that q is blocked in $\mathcal{CO}_i$ or $\mathcal{CO}_j$ and thus blocked in $\mathcal{CO}_i \bigcup \mathcal{CO}_j$. Conversely, if q is blocked in $\mathcal{CO}_i \bigcup \mathcal{CO}_j$, then there is a configuration where at least one of $\mathcal{CO}_i$ or $\mathcal{CO}_j$ blocks the manipulator in that configuration. It means that q is blocked in $\mathcal{WO}_i$ or $\mathcal{WO}_j$ and also $\mathcal{WO}_i \bigcup \mathcal{WO}_j$. Thus, $\mathcal{C}(\mathcal{WO}_i \bigcup \mathcal{WO}_j) = \mathcal{CO}_i \bigcup \mathcal{CO}_j$ and we have shown that the union operator propagates from the workspace to the configuration space.

6 Problem 6

The wavefront planner on a discrete grid yields the shortest path to the goal because it searches for all possible endpoints of paths from the goal of a certain length and increases it by one until it reaches the start, guaranteeing the path is the shortest length possible from the start to the goal. The wavefront planner is equivalent to breadth-first search starting from the goal which also provides the shortest possible path. The metric the wavefront planner is Manhattan distance assuming the four-point pattern is used as in the lecture slide pictures.