Assignment 3

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Question 1

The unicycle system may be written as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \cos(x_3) \\ \sin(x_3) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \tag{1}$$

Where $y = h(x) = x_1^2 + x_2^2 - 1$. Taking the derivative of y with respect to time:

$$\begin{aligned} \dot{y} &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 && (2) \\ &= 2x_1(\cos(x_3)) + 2x_2(\sin(x_3)) && (3) \end{aligned}$$

The control input does not appear in $\dot{y}$. Therefore, the derivative may be taken again.

$$\begin{aligned} \ddot{y} &= 2\cos(x_3) + 2\sin(x_3) + 2x_1(-\sin(x_3))\dot{x}_3 + 2x_2(\cos(x_3))\dot{x}_3 && (4) \\ &= 2\cos(x_3) + 2\sin(x_3) - (2x_1\sin(x_3))u + (2x_2\cos(x_3))u && (5) \\ &= 2\cos(x_3) + 2\sin(x_3) + (-2x_1\sin(x_3) + 2x_2\cos(x_3))u && (6) \end{aligned}$$

The control input appears with coefficient $(-2x_1\sin(x_3) + 2x_2\cos(x_3))$. Therefore, the system has relative degree 2 except on the set $S = \{x \in \chi : -2x_1\sin(x_3) + 2x_2\cos(x_3) = 0\}$.

This set may be interpreted as follows. Rearranging the equation:

$$\begin{aligned} -2x_1\sin(x_3) + 2x_2\cos(x_3) &= 0 && (7) \\ x_1\sin(x_3) &= x_2\cos(x_3) && (8) \\ \frac{x_1}{x_2} &= \frac{\cos(x_3)}{\sin(x_3)} && (9) \end{aligned}$$

This means that the slope of the line from the origin to the unicycle position is equal to the slope of the unicycle orientation. This corresponds to states where the unicycle is pointed radially away from the origin or radially towards the origin as follows. Note that it also includes the origin.

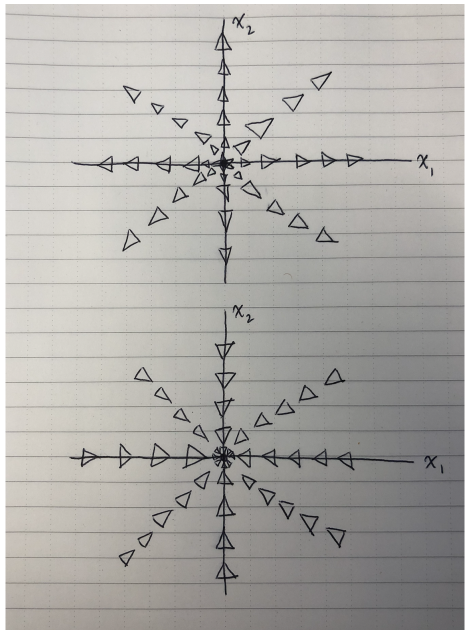

Figure 1: The set S

Question 2

The zero dynamics manifold for a system of relative degree 2 may be found according to the following set:

$$Z : \{x \in \chi : h(x) = L_f h(x) = 0\} \tag{10}$$

Note that $h(x) = x_1^2 + x_2^2 - 1$ and $L_f h(x) = 2x_1 \cos(x_3) + 2x_2 \sin(x_3)$. Therefore, it can be seen that $S \cap Z =$, as $h(x) = 0$ implies that the unicycle is on a circle of unit radius and $L_f h(x) = 0$ implies that the orientation of the unicycle is tangent to the circle. Therefore, the unicycle cannot have a state simultaneously in Z and S because all states in Z are tangent to the unit circle and all states in S point radially away from the origin or radially towards the origin.

Let Z_1 be the curve where the unicycle is tangent to the circle pointed in the counterclockwise direction and Z_2 be the curve where the unicycle is tangent to the circle pointed in the clockwise direction.

Question 3

Let the embedding for Z_1 be denoted as $T_1 : ([\mathbb{R}]_{2\pi}) \to \chi$:

$$T_1(\angle(x_1, x_2)) = \begin{bmatrix} \cos(\angle(x_1, x_2)) \\ \sin(\angle(x_1, x_2) \\ \angle(x_1, x_2) + \frac{\pi}{2} \end{bmatrix} \in Z_1 \tag{11}$$

Let the embedding for Z_2 be denoted as $T_2 : ([\mathbb{R}]_{2\pi}) \to \chi$:

$$T_2(\angle(x_1, x_2)) = \begin{bmatrix} \cos(\angle(x_1, x_2)) \\ \sin(\angle(x_1, x_2) \\ \angle(x_1, x_2) - \frac{\pi}{2} \end{bmatrix} \in Z_2 \tag{12}$$

Taking the derivative of each shows that $\forall \angle(x_1, x_2) \in [\mathbb{R}]_{2\pi}$:

$$dT_1(\angle(x_1, x_2)) = \begin{bmatrix} -\sin(\angle(x_1, x_2)) \\ \cos(\angle(x_1, x_2) \\ 1 \end{bmatrix} \neq 0 \tag{13}$$

$$dT_2(\angle(x_1, x_2)) = \begin{bmatrix} -\sin(\angle(x_1, x_2)) \\ \cos(\angle(x_1, x_2) \\ 1 \end{bmatrix} \neq 0 \tag{14}$$

Also note that $dT_1(\theta_1) \neq dT_1(\theta_2) \Rightarrow \theta_1 \neq \theta_2$ and $dT_2(\theta_1) \neq dT_2(\theta_2) \Rightarrow \theta_1 \neq \theta_2$ as $\sin(\theta)$ and $\cos(\theta)$ take on unique values for each angle $0 \leq \theta < 2\pi$.

Question 4

For the retractions, let the retraction on set U_1 be denoted $p_1 : U_1 \to Z_1$ and the retraction on set U_2 be denoted $p_2 : U_2 \to Z_2$. Let:

$$U_1 = \mathbb{R}\backslash\{0\} \times \mathbb{R}\backslash\{0\} \times \{[\mathbb{R} : \angle(x_1, x_2) + \pi < \theta < \angle(x_1, x_2)]_{2\pi}\} \tag{15}$$

$$U_2 = \mathbb{R}\backslash\{0\} \times \mathbb{R}\backslash\{0\} \times \{[\mathbb{R} : \angle(x_1, x_2) < \theta < \angle(x_1, x_2) + \pi]_{2\pi}\} \tag{16}$$

Therefore, the retraction p_1 may be written as:

$$p_1 = \begin{bmatrix} \frac{x_1}{||x||} \\ \frac{x_2}{||x||} \\ \angle(x_1, x_2) + \frac{\pi}{2} \end{bmatrix} \tag{17}$$

The retraction p_2 may be written as:

$$p_2 = \begin{bmatrix} \frac{x_1}{||x||} \\ \frac{x_2}{||x||} \\ \angle(x_1, x_2) - \frac{\pi}{2} \end{bmatrix} \tag{18}$$

Note that $U_1 \cup U_2 = \chi \backslash S$ as expected.

Question 5

First find the system with the feedback transformation $u = (L_g L_f h(x))^{-1}(-L_f^2 h(x) + v)$. Using $f(x) = \begin{bmatrix}\cos(x_3) & \sin(x_3) & 0\end{bmatrix}^\top$, $g(x) = \begin{bmatrix}0 & 0 & 0\end{bmatrix}$, and $h(x) = x_1^2 + x_2^2 - 1$ we have:

$$\begin{aligned} L_f h(x) &= \begin{bmatrix}2x_1 & 2x_2 & 0\end{bmatrix}\begin{bmatrix}\cos(x_3)\\ \sin(x_3)\\ 0\end{bmatrix} \\ &= 2x_1\cos(x_3) + 2x_2\sin(x_3) \end{aligned}$$

$$\begin{aligned} L_g L_f h(x) &= \begin{bmatrix}2\cos(x_3) & 2\sin(x_3) & -2x_1\sin(x_3) + 2x_2\cos(x_3)\end{bmatrix}\begin{bmatrix}0\\ 0\\ 1\end{bmatrix} \\ &= 2x_2\cos(x_3) - 2x_1\sin(x_3) \end{aligned}$$

$$\begin{aligned} L_f^2 h(x) &= \begin{bmatrix}2cos(x_3) & 2sin(x_3) & -2x_1\sin(x_3) + 2x_2\cos(x_3)\end{bmatrix}\begin{bmatrix}\cos(x_3)\\ \sin(x_3)\\ 0\end{bmatrix} \\ &= 2cos^2(x_3) + 2sin^2(x_3) \end{aligned}$$

So the system with the feedback transformation can be simplified as:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}\cos(x_3)\\ \sin(x_3)\\ \frac{1}{x_1\sin(x_3) - x_2\cos(x_3)}\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \frac{1}{2x_2\cos(x_3) - 2x_1\sin(x_3)}v\end{bmatrix}$$

Now we determine the new states $F_1(x) = (z, \xi) = (T_1^{-1}(p_1(x)), H(x))$ and $F_2(x) = (z, \xi) = (T_2^{-1}(p_2(x)), H(x))$, for each of the disjoint curves Z_i making up the zero dynamics manifold.

First, we determine the inverse transformations $T_1^{-1}(x)$ and $T_2^{-1}(x)$. From looking at the equations 11 and 12 from Question 3, we can determine $T_1^{-1}(x) = x_3 - \frac{\pi}{2}$ and $T_2^{-1}(x) = x_3 + \frac{\pi}{2}$.

Applying these inverse transformations to points p_1 and p_2 respectively, specified by equations 17 and 18, we get $T_1^{-1}(p_1(x)) = \angle(x_1, x_2) + \frac{\pi}{2} - \frac{\pi}{2} = \angle(x_1, x_2)$ and $T_2^{-1}(p_2(x)) = \angle(x_1, x_2) - \frac{\pi}{2} + \frac{\pi}{2} = \angle(x_1, x_2)$.

For $H(x)$, we can use the known expressions $h(x)$ and $L_f h(x)$, so $H(x) = col(h(x), L_f h(x)) = \begin{bmatrix}x_1^2 + x_2^2 - 1\\ 2x_1\cos(x_3) + 2x_2\sin(x_3)\end{bmatrix}$.

Thus, the new states are $(z, \xi) = (\angle(x_1, x_2), \begin{bmatrix}x_1^2 + x_2^2 - 1\\ 2x_1\cos(x_3) + 2x_2\sin(x_3)\end{bmatrix})$ for both Z_1 and Z_2.

Using the hint given by the question, we can determine the z dynamics since $z = (\angle(x_1, x_2))$ and $y : \mathbb{R} \to \mathbb{R}\backslash\{0\}$ where $y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is clearly a smooth function. From the hint, we can find that the z dynamics are:

$$\dot{z} = \frac{x_1\dot{x_2} - \dot{x_1}x_2}{x_1^2 + x_2^2} \tag{19}$$

We can express x_1 and x_2 as the following by inspection:

$$x_1 = \cos(z) \tag{20}$$
$$x_2 = \sin(z) \tag{21}$$

In order to write x_3 in terms of ξ_2, the expression may be rearranged:

$$\frac{\xi_2}{2} = \cos(z)\cos(x_3) + \sin(z)\sin(x_3) \tag{22}$$

Using the following trigonometric identity [1]:

$$\cos A \cos B + \sin A \sin B = \cos(A - B) \tag{23}$$

Therefore, the expression becomes:

$$\frac{\xi_2}{2} = \cos(z - x_3) \tag{24}$$
$$x_3 = z - \cos^{-1}(\frac{\xi_2}{2}) \tag{25}$$

Therefore, substituting into Equation 19 yields the following:

$$\dot{z} = \frac{\cos(z)\sin(x_3) - \sin(z)\cos(x_3)}{x_1^2 + x_2^2} \tag{26}$$

Using the following identity [1]:

$$\sin A \cos B - \cos A \sin B = \sin(A - B) \tag{27}$$

The following simplification may be made:

$$\dot{z} = \frac{\sin(x_3 - z)}{x_1^2 + x_2^2} \tag{28}$$
$$= \frac{\sin(-\cos^{-1}(\frac{\xi_2}{2}))}{\xi_1 + 1} \tag{29}$$

We can simplify $\sin(-\cos^{-1}(\frac{\xi_2}{2}))$ by considering a right triangle with angle $\theta = \cos^{-1}(\frac{\xi_2}{2})$. Since $\cos(\theta) = \frac{adjacent}{hypotenuse}$, and the triangle is a right triangle, we use the Pythagorean theorem to find *opposite* $= \sqrt{4 - \xi_2^2}$. Since

$\sin(\theta) = \frac{opposite}{hypotenuse}$, $\sin(\theta) = \frac{\sqrt{4-\xi_2^2}}{\xi_2}$. But here we have $\sin(-\theta)$ so we use the trigonometric identity that $\sin(-\theta) = -\sin(\theta)$ so $\sin(-\cos^{-1}(\frac{\xi_2}{2})) = -\frac{\sqrt{4-\xi_2^2}}{\xi_2}$.

As a result the z dynamics can be written as:

$$\dot{z} = -\frac{\sqrt{4-\xi_2^2}}{\xi_2(\xi_1+1)} \tag{30}$$

Once we get the z dynamics we know that the rest of the normal form consists of a double integrator with state ξ_1 and ξ_2.

So the normal equations are:

$$\dot{z} = \frac{\sin(-\cos^{-1}(\frac{\xi_2}{2}))}{\xi_1+1} \tag{31}$$

or

$$\dot{z} = -\frac{\sqrt{4-\xi_2^2}}{\xi_2(\xi_1+1)} \tag{32}$$

$$\dot{\xi_1} = \xi_2 \tag{33}$$

$$\dot{\xi_2} = v \tag{34}$$

Question 6

Setting ξ_1 and ξ_2 equal to zero in $\dot{z} = \frac{\sin(-\cos^{-1}(\frac{\xi_2}{2}))}{\xi_1+1}$ we get that $\dot{z} = 0$ which indicates movement at a constant speed around the circle.

Question 7

For this problem, we use K = $\begin{bmatrix} -10 & -10 \end{bmatrix}$ which allow the solutions to converge. The number of solutions that converge to Z_1 and the number of solutions that converge to Z_2 are approximately equal. In the figure below, out of ten trials, four solutions converge to Z_1 or are counterclockwise and six solutions converge to Z_2 or are clockwise. There are no solutions that get interrupted by the singularity because in that case, the initial condition would have to be exactly on the set S and that is extremely unlikely. However, if you manually set the initial condition to be on S, the integration fails as expected. For 1000 trials, the number of solutions that converged to Z_1 was 498, the number of solutions that converged to Z_2 was 501, and 1 solution did not converge, landing on a singularity.

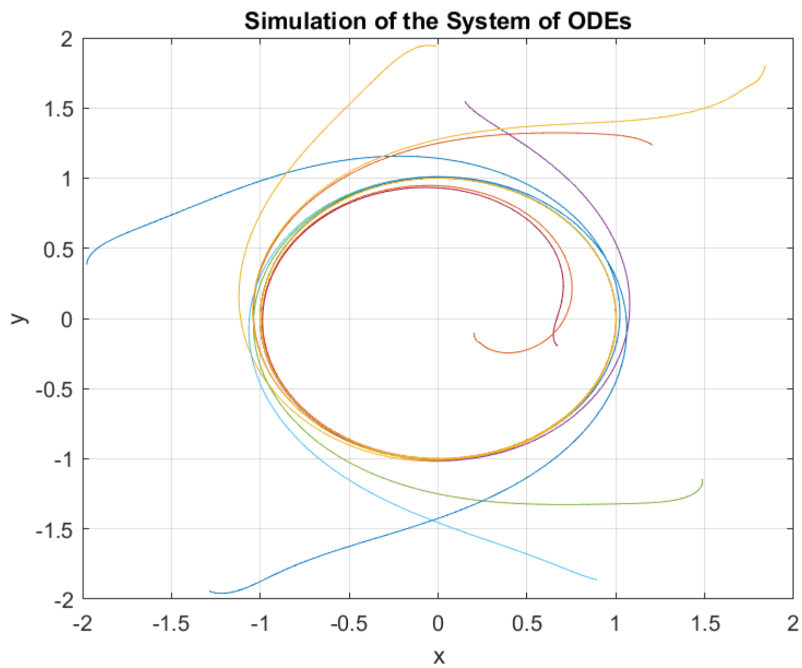

Figure 2: Simulation of 10 trials with Random Uniform Distribution

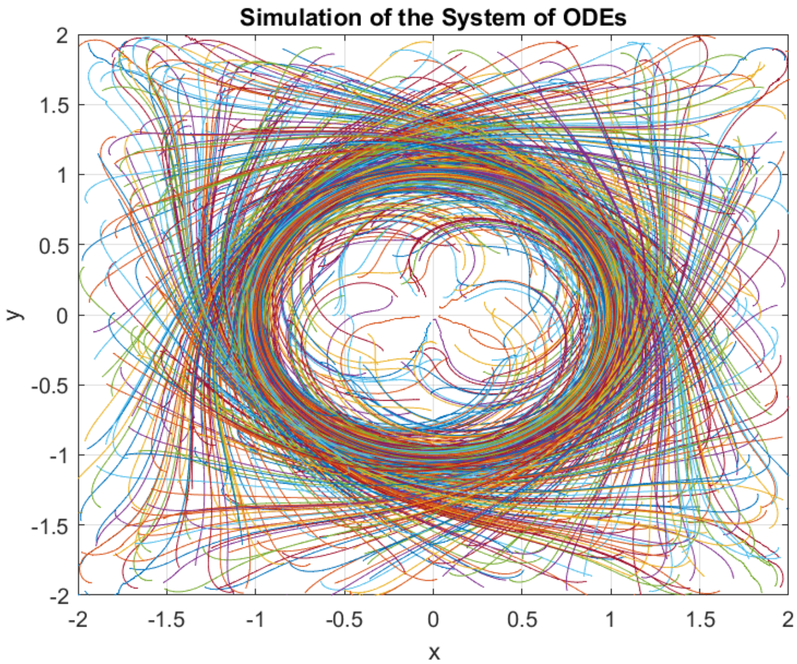

Figure 3: Simulation of 1000 trials with Random Uniform Distribution

Appendix

Code for Question 7

```
% Set tolerance
options = odeset('RelTol',1e-50,'AbsTol',1e-50);

% Define the system of ODEs
x1_dot = @(t, x) cos(x(3));
x2_dot = @(t, x) sin(x(3));
x3_dot = @(t, x) 1/round(x(1)*sin(x(3))-x(2)*cos(x(3)),10) + (-10*(x(1)^2+x(2)^2-1) +↙
-10*(2*x(1)*cos(x(3))+2*x(2)*sin(x(3))))/round(2*x(2)*cos(x(3))-2*x(1)*sin(x(3)),10);

% Define the time span for simulation
t_span = [0 500]; % From t=0 to t=10

%Plot single solution
%initial_condition = [10;10;pi/4-pi];
%[t, sol] = ode45(@(t, x) [x1_dot(t, x); x2_dot(t, x); x3_dot(t, x)], t_span,↙
initial_condition);
%plot(sol(:, 1),sol(:, 2))

%Number of trials
N = 1000;
x1_min = -2; % Lower bound of the range
x1_max = 2; % Upper bound of the range
x1_init = x1_min + (x1_max - x1_min) * rand([1, N]); %Uniform Distribution

x2_min = -2; % Lower bound of the range
x2_max = 2; % Upper bound of the range
x2_init = x2_min + (x2_max - x2_min) * rand([1, N]); %Uniform Distribution

x3_min = 0; % Lower bound of the range
x3_max = 2*pi; % Upper bound of the range
x3_init = x3_min + (x3_max - x3_min) * rand([1, N]); %Uniform Distribution

initial_conditions = [x1_init;x2_init;x3_init];

% Solutions that converge to Z1 or Z2
Z_1_counter = 0;
Z_2_counter = 0;

for i = 1:N
    % Define the initial condition
    initial_condition = initial_conditions(:,i); % Initial values of x and y
    % Call ode45 to solve the system of ODEs
    [t, sol] = ode45(@(t, x) [x1_dot(t, x); x2_dot(t, x); x3_dot(t, x)], t_span,↙
initial_condition);

    if isnan(sol(end,end))
        fprintf('Integration interrupted')
    else
        % Plot the solution
        x1 = sol(:, 1);
```

```
        x2 = sol(:, 2);
        x3 = sol(:, 3);

        plot(x1, x2)
        hold on

        size(sol);
        % Find solution type
        if x3(end)-x3(end-1)>0
            'clockwise - Z_2'
            Z_2_counter = Z_2_counter + 1;

        elseif x3(end)-x3(end-1)<0
            'counterclockwise - Z_1'
            Z_1_counter = Z_1_counter + 1;
        else
            'error'
        end
    end

end
%Label figure
xlabel('x');
ylabel('y');
xlim([-2 2]);
ylim([-2 2]);
title('Simulation of the System of ODEs');
grid on;
```

References

[1] D. E. Joyce, “Summary of trigonometric identities,” Clark University, 1996. Available: https://www2.clarku.edu/faculty/djoyce/trig/identities.html.