Daily Homework 2-16

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Introduction to Abstract Math

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Exercise 3.50. Rewrite the following statements in words and determine whether each is true or false. Make sure you justify your answer

1. $(\forall n \in \mathbb{N})(n^2 \geq 5)$
2. $(\exists n \in \mathbb{N})(n^2 - 1 = 0)$
3. $(\exists N \in \mathbb{N})(\forall n > N)(\frac{1}{n} < 0.01)$
4. $(\forall m, n \in \mathbb{Z})(2|m \wedge 2|n \implies 2|(m + n))$
5. $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(x - 2y = 0)$
6. $(\exists x \in \mathbb{N})(\forall y \in \mathbb{N})(y \leq x)$

Solution.

1. For all n in the Natural numbers, $n^2 \geq 5$.
   This is false, 1 is a natural number and $1^2 = 1$, which is less than 5.
2. For some n in the Natural numbers, $(n^2 - 1) = 0$. This is true, 1 is a natural number and $1^2 = 1$, so $1^2 - 1 = 0$
3. For some N in the natural numbers, for all n greater than N, $\frac{1}{n}$ is less than 0.01. This is true when N is 100 or greater.
4. For all m and n in the Integers, if 2 divides m and 2 divides n, then 2 divides $m + n$. This is true by Theorem 2.19.
5. For all natural numbers x, there exists a natural number y such that $x - 2y = 0$.
   This is false. If we add 2y to both sides of the equation we get $x = 2y$. By definition 2.1, x must be an even number, which shows us that x cannot be an odd number.
6. For some natural number x, every natural number y is less than or equal to x. This is false, if we take any natural number x we can say y is defined as $x + 1$ and then $y \nleq x$.

$\square$

Exercise 3.52. Consider the propositions $(\exists x)(x^2 - 4 = 0)$ and $(\exists x)(x^2 - 2 = 0)$.

1. Are these propositions equivalent if the universe of discourse is the set of real numbers?
2. Give two different universes of discourse that yield different truth values for these propositions.
3. What can you conclude about the equivalence of these statements?

Solution. Let A := $(\exists x)(x^2 - 4 = 0)$ and B := $(\exists x)(x^2 - 2 = 0)$.

1. A and B are equivalent if the universe of discourse is the Real numbers. Proposition A is satisfied by $x = \pm 2$ and proposition B is satisfied by $x = \pm\sqrt{2}$.
2. If we consider the universe of discourse of proposition A to be the integers and the universe of discourse of proposition B to be the natural numbers, then A is true and B is false. The only possible solutions for B in any universe of discourse are $x = \sqrt{2}$ or $x = -\sqrt{2}$, neither of which are natural numbers.
3. By Definition 3.51, the statements are not equivalent, but they are equivalent in a given universe (i.e. the real numbers).

$\square$

Theorem 3.53. Let P(x) be a predicate. Then

1. $\neg(\forall x)P(x)$ is equivalent to $(\exists x)(\neg P(x))$
2. $\neg(\exists x)P(x)$ is equivalent to $(\forall x)(\neg P(x))$

Proof.

1. We will prove this by contradiction. We assume that not all x are such that $P(x)$ and that there is not an x such that $\neg P(x)$. If there is not an x such that $\neg P(x)$, then all x must be such that $P(x)$. However, this contradicts our assumption that not all x are such that $P(x)$. Therefore $\neg(\forall x)P(x) \implies (\exists x)(\neg P(x))$.
   Conversely, we use contradiction the other direction. We assume that there exists some x such that $\neg P(x)$ and that for all x, $P(x)$. If there is some x for which P(x) is not true, then not all x can satisfy $P(x)$. So, this contradicts our assumption that all values of x satisfy P(x). Therefore $(\exists x)(\neg P(x)) \implies \neg(\forall x)(P(x))$.
   By definition 3.3, $\neg(\forall x)P(x) \iff (\exists x)(\neg P(x))$.

2. We proceed with proof by contradiction. So we assume that there does not exist an x such that $P(x)$ and not all x are such that, $\neg P(x)$. If not all x are such that $\neg P(x)$, then there is at least one x that satisfies $P(x)$ which contradicts our assumption that there does not exist an x such that $P(x)$. Therefore $\neg(\exists x)P(x) \implies (\forall x)(\neg P(x))$.
   Conversely, we proceed with contradiction the other direction, assuming that all x are such that $\neg P(x)$ and that it is false that there does not exist an x such that $P(x)$. If it is false that there does not exist an x such that P(x), then there must exist an x such that $P(x)$ which is a contradiction with our assumption that all x are such that $\neg P(x)$. Therefore $(\forall x)(\neg P(x)) \implies \neg(\exists x)P(x)$. By Definition 3.3, $\neg(\exists x)P(x) \iff (\forall x)(\neg P(x))$.

□

Exercise 3.54. Negate each of the following. Disregard the truth value and the universe of discourse.

1. $(\forall x)(x > 3)$
2. $(\exists x)(x \text{ is prime} \wedge x \text{ is even})$.
3. All cars are red
4. Every Wookiee is named Chewbacca
5. Some hippies are republican
6. For all $x \in \mathbb{N}, x^2 + x + 41$ is prime.
7. There exists $x \in \mathbb{Z}$ such that $1/x \notin \mathbb{Z}$.
8. There is no function f such that if f is continuous, then f is not differentiable.

Solution.

1. The negation of the statement is $(\exists x)(x \leq 3)$.
2. The negation of the statements is $(\forall x)(x \text{ is not prime} \vee x \text{ is odd})$.
3. The negation of the statement is "There exists a car that is not red"
4. The negation of the statement is "There exists a Wookiee such that that Wookiee is not named Chewbacca"
5. The negation of the statement is "All hippies are not Republicans"
6. The negation of the statement is "For some $x \in \mathbb{N}$, $x^2 + x + 41$ is not prime."
7. The negation of the statement is "All $x \in \mathbb{Z}$ are such that $1/x \in \mathbb{Z}$"
8. All functions f are such that if f is continuous, then f is differentiable.

$\square$

Exercise 3.55. Negate each of the following. Disregard the truth value.

1. $(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})(m < n)$.
2. $(\forall x, y, z \in \mathbb{Z})((xy \text{ is even} \wedge yz \text{ is even}) \implies xz \text{ is even})$.
3. For all positive real numbers x, there exists a real number y such that $y^2 = x$.
4. There exists a married person x such that for all married people y, x is married to y.

Solution.

1. $(\exists n \in \mathbb{N})(\forall m \in \mathbb{N})(m \geq n)$
2. $(\exists x, y, z \in \mathbb{Z})((xy \text{ is even} \wedge yz \text{ is even }) \implies xz \text{ is odd})$.
3. There exists a positive real number x such that all real numbers y satisfy the equation $y^2 \neq x$.
4. For all married persons x, there exists a married person y such that x is not married to y.

$\square$

Problem 3.60. For each of the following statements, determine its truth value. If the statement is false, provide a counterexample. Prove at least two of the true statements.

1. For all $n \in \mathbb{N}, n^2 \geq 5$.
2. There exists $n \in \mathbb{N}$ such that $n^2 - 1 = 0$.
3. There exists $x \in \mathbb{N}$ such that for all $y \in \mathbb{N}, y \leq x$.
4. For all $x \in \mathbb{Z}, x^3 \geq x$.
5. For all $n \in \mathbb{Z}$, there exists $m \in \mathbb{Z}$ such that $n + m = 0$.
6. There exists integers a and b such that $2a + 7b = 1$.
7. There do not exist integers m and n such that $2m + 4n = 7$.
8. For all integers a, b, c, if a divides bc, then either a divides b or a divides c.

Solution.

1. False. If we take $n = 1$, then $n^2 = 1$, which is less than 5.
2. True.

Proof. Let $n = 1$. Compute that $n^2 - 1 = (1)^2 - 1 = 0$. Therefore, because 1 is a natural number, There exists $n \in \mathbb{N}$ such that $n^2 - 1 = 0$. $\square$

3. False. If we set $y = x + 1$, then y will be greater than x for every possible x.

4. False. If we take $x = -3$, then $x^3 = -27$, which is less than -3.

5. True.

6. True.

7. True.

   Proof. By Theorem 3.53 the statement "there do not exist integers m and n such that $2m + 4n = 7$" is equivalent to "for all integers m and n, $2m + 4n \neq 7$". Notice that 7 is odd by Definition 2.2 because $7 = 2(3) + 1$. Also, notice that the left side of the equation can be rewritten as 2(m+2n) making it even by Definition 2.1. By Fact 2.3.1 a number is either even or odd, but not both, making the statement true that $2m + 4n \neq 7$. Therefore, our original statement is true. $\square$

8. False. Set $a = 6, b = 3$, and $c = 2$. Then $6|2(3)$, but $6 \not| 2$ and $6 \not| 3$

$\square$