Weekly Homework 3-15

Kevin Christensen, Ailey Robinson
Introduction to Abstract Math

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Theorem 5.2. Let $P(1), P(2), P(3), ...$ be a sequence of statements, one for each natural number. Assume

(i) $P(1)$ is true and

(ii) if $P(k)$ is true, then $P(k+1)$ is true

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof. Let $S := \{n \in \mathbb{N} \mid P(n) \text{ is true}\}$. By our assumptions, we know that $P(1) \in S$ and that if $P(k) \in S$, then $P(k+1) \in S$. By Axiom 5.1, we know that $S = \mathbb{N}$, therefore $P(n)$ is true for all $n \in \mathbb{N}$.

□

Theorem 5.4. For all $n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Proof. We proceed by induction.

(i) Base step: Assume $n = 1$. Then, the left hand side of the equation is 1, and the right hand side of the equation is $\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = 1$. Therefore, the equality is true for $n = 1$.

(ii) Inductive step: Let $k \in \mathbb{N}$ and assume that $\displaystyle\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ is true. Consider the $k+1$ case.
The right hand side of the equation is now

$$\begin{aligned}
\sum_{i=1}^{k+1} i &= \sum_{i=1}^{k} i + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{k^2 + 3k + 2}{2}
\end{aligned}$$

The left hand side of the equation is $\dfrac{(k+1)((k+1)+1)}{2} = \dfrac{k^2+3k+2}{2}$.

Notice that the right hand side equals the left hand side. Therefore the equation is true for $k+1$. Thus, by the PMI, $P(n)$ is true for all $n \in N$. $\square$