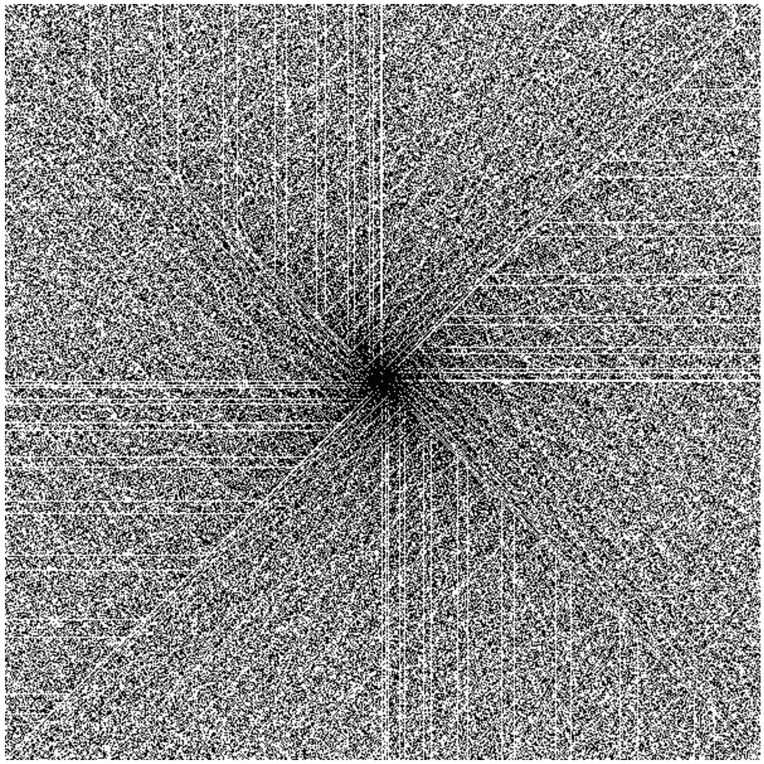
M 307 Intro to Abstract Math



The Ulam spiral of prime numbers.

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These notes are closely adapted from *An introduction to proof via inquiry-based learning* by Dana Ernst (NAU). Much of the text herein is copied directly from Prof. Ernst's notes. We thank Prof. Ernst profusely for his hard work and decision to openly share his beautiful notes with the broader community under a Creative Commons license.

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Preface

You are the creators. This book is a guide.

This book will not show you how to solve all the problems that are presented, but it should *enable* you to find solutions, on your own and working together. The material you are about to study did not come together fully formed at a single moment in history. It was composed gradually over the course of centuries, with various mathematicians building on the work of others, improving the subject while increasing its breadth and depth.

Mathematics is essentially a human endeavor. Whatever you may believe about the true nature of mathematics—does it exist eternally in a transcendent Platonic realm, or is it contingent upon our shared human consciousness?—our *experience* of mathematics is temporal, personal, and communal. Like music, mathematics that is encountered only as symbols on a page remains inert. Like music, mathematics must be created in the moment, and it takes time and practice to master each piece. The creation of mathematics takes place in writing, in conversations, in explanations, and most profoundly in the mental construction of its edifices on the basis of reason and observation.

To continue the musical analogy, you might think of these notes like a performer's score. Much is included to direct you towards particular ideas, but much is missing that can only be supplied by you: participation in the creative process that will make those ideas come alive. Moreover, your success will depend on the pursuit of both *individual* excellence and *collective* achievement. Like a musician in an orchestra, you should bring your best work and be prepared to blend it with others' contributions.

In any act of creation, there must be room for experimentation, and thus allowance for mistakes, even failure. A key goal of our community is that we support each other—sharpening each other's thinking but also bolstering each other's confidence—so that we can make failure a *productive* experience. Mistakes are inevitable, and they should not be an obstacle to further progress. It's normal to struggle and be confused as you work through new material. Accepting that means you can keep working even while feeling stuck, until you overcome and reach even greater accomplishments.

This book is a guide. You are the creators.

Chapter 1

Introduction

1.1 What is mathematics?

What does "math" mean to you? What makes math different than physics or chemistry? than computer science? than a language class?

I would argue that the lowest common denominator of math is logic. In math, there is always an answer to "But why is that true?" Especially in pure math, one's job as a mathematician is to (1) figure out why something is true, and (2) convince other mathematicians that it's true.

One can argue that this is the job of any scientist. The difference with (pure) math is that the "But why?" investigation has to happen in our own brains. It doesn't happen in an external space, like a lab or a computer, where other people can look in and see what's going on. That means that, regarding point (1), we have to be extra careful, inside our own brains, to be detailed and thorough when figuring out why something is true. Regarding point (2), the arguments we use to convince other mathematicians of the truth we've seen in our own brains have to be very precise. Those arguments have to communicate, to someone else's brain, exactly the same reasoning that we used in our own brain to figure out the truth of a statement.

The focus of this class will be on developing your skills at (1) and (2) above – logical thinking, and mathematical communication. This is a rather different approach to math than you may have taken before. You may love it; you may hate it; you will probably find it frustrating at times. I hope you will also find it exhilarating and satisfying.

1.2 This Class: Logistics

The majority of the development of your "logical thinking" skills in M 307 will happen outside the classroom. In class, you'll practice your mathematical communication skills. In doing so, you'll sometimes discover that there was a gap in your logical thinking, and so we'll also work on fixing those gaps in class (and on having fewer gaps in future). You'll also learn, in this class, the standard conventions of mathematical writing. For example, mathematicians almost always write in the third person plural: "We now show that the integral of the sum is the sum of the integrals."

1.2.1 Class structure

Every day, before class, I will ask you to work in pairs to solve a few exercises – you'll turn in one set of solutions per pair at the beginning of class. Each exercise will consist of a statement about mathematical objects; your solution should answer the question "But why is that statement true?" In keeping with mathematical convention, these solutions should start with the word *Proof:* and end with a box □.

In class, I will randomly choose people to present their solutions to the homework exercises. As a class, we'll discuss the presented solutions, and revise them as needed, until everyone agrees that the logic of each solution is correct, and that the solution is explained clearly and in accordance with the conventions for mathematical writing.

This means that, both in class and when preparing for class, you should be asking lots and lots of **questions**. "Why is that true? Can we say that more cleanly? I don't see the connection here. What if you did this instead?" Anything that occurs to you that could make the proof better is fair game! After all, the goal of the time spent in class on revising these presented solutions is to develop everyone's ability to communicate math.

But it is equally important to be supportive with your questions. You are all building your mathematical communication skills together. You will all make mistakes – I will too – and the important thing is always to figure out why it was a mistake, and to do better next time. There are often many different correct solutions, and sometimes it's a matter of opinion about which way is "best."

1.2.2 Inquiry-based learning

The main goals for this class are not for you to learn new procedures for solving math problems, but to learn new habits of thinking and communicating. To that end, the structure of this class is inspired by an educational philosophy called *Inquiry-based learning* or IBL. Loosely speaking, IBL is a student-centered method of teaching mathematics, where the instructor guides the students to construct their own understanding of the subject matter via carefully structured activities, rather than lecture. According to Laursen and Rasmussen (2019), the Four Pillars of IBL are:

- Students engage deeply with coherent and meaningful mathematical tasks.
- Students collaboratively process mathematical ideas.
- Instructors inquire into student thinking.
- Instructors foster equity in their design and facilitation choices.

Because of the IBL philosophy underlying this class, the daily homework in this class will usually not be a simple or straightforward application of concepts that have previously been explained to you. Rather, you will have to think hard and creatively to see how the tools youve developed can be used to solve the homework problems. Often, you will learn new concepts by doing the homework.

1.3 Some advice

Because you will be "learning by doing" throughout this class, you will make mistakes. You will, not infrequently, be wrong, or confused, or frustrated. Making mistakes is a central part of learning – if you do everything right the first time, you haven't really learned anything new that you didn't know before at some level. Rather than "do it right," make it your goal to "make new mistakes," so that you're always learning something new.

Try to be patient with yourself, and your classmates, as you learn and make mistakes. Try not to give up when you're feeling stuck and frustrated. Learn to persevere even when you're not sure what the next step should be. Try something! If it doesn't work, figure out why, then try something else.

Solving genuine problems is difficult and takes time. You shouldn't expect to complete each problem in 10 minutes or less. Sometimes, you might have to stare at the problem for an hour before even understanding how to get started.

This class requires both time and persistence, but if you put in the time and the work, it will pay off. The feeling of pride and exhilaration when you've solved a really challenging problem yourself is unbeatable. So, hang tough through the confusion; the harder you have to work to figure it out, the more satisfaction you'll feel when you do. This process is an important part of both everyday life and mathematics.

Try to remember that it is not just the end (the reason why something is true), but also the **process** of finding that answer that is our focus. Be patient and enjoy the luxury of thought. Set aside some time to sit in a comfy chair with a pile of scratch paper and a cup of tea and soak in the experience.

I am here to guide you as you learn for yourself. I don't want to spoil the challenge, and I do want you to develop your own "mental muscles" of logical deduction. This means that, while I will happily give you hints if you're stuck on an exercise, I will always try to give the weakest hint possible that allows you to move forward.

1.4 Your Toolbox, Questions, and Observations

Throughout the semester, we will develop a list of *tools* that will help you understand and do mathematics. Your job is to keep a list of these tools, and it is suggested that you keep a running list someplace.

Next, it is of utmost importance that you work to understand every proof. (Every!) Questions are often your best tool for determining whether you understand a proof. Therefore, here are some sample questions that apply to any proof that you should be prepared to ask of yourself or the presenter:

- What method(s) of proof are you using?
- What form will the conclusion take?
- How did you know to set up that [equation, set, whatever]?
- How did you figure out what the problem was asking?

- Was this the first thing you tried?
- Can you explain how you went from this line to the next one?
- What were you thinking when you introduced this?
- Could we have ... instead?
- Would it be possible to ...?
- What if ...?

Another way to help you process and understand proofs is to try and make observations and connections between different ideas, proof statements and methods, and to compare approaches used by different people. Observations might sound like some of the following:

- When I tried this proof, I thought I needed to ... But I didn't need that, because ...
- I think that ... is important to this proof, because ...
- When I read the statement of this theorem, it seemed similar to this earlier theorem. Now I see that it [is/isn't] because ...

1.5 Rules of the Game

You should *not* look to resources outside the context of this course for help. That is, you should not be consulting the Internet, other texts, other faculty, or students outside of our course. On the other hand, you may use each other, the course notes, me, and your own intuition. In this class, earnest failure outweighs counterfeit success; you need not feel pressure to hunt for solutions outside your own creative and intellectual reserves. For more details, check out the Syllabus.

1.6 Structure of the Notes

As you read the notes, you will be required to digest the material in a meaningful way. It is your responsibility to read and understand new definitions and their related concepts. However, you will be supported in this sometimes difficult endeavor. In addition, you will be asked to complete exercises aimed at solidifying your understanding of the material. Most importantly, you will be asked to make conjectures, produce counterexamples, and prove theorems.

Most items in the notes are labelled with a number. The items labelled as **Definition** and **Example** are meant to be read and digested. However, the items labelled as **Exercise**, **Question**, **Theorem**, **Corollary**, and **Problem** require action on your part. In particular, items labelled as **Exercise** are typically computational in nature and are aimed at improving your understanding of a particular concept. There are very few items in the notes labelled as **Question**, but in each case it should be obvious what is required of you. Items with the **Theorem** and **Corollary** designation are mathematical facts and the intention is for you to produce a valid proof of the given statement. The main difference between a **Theorem** and **Corollary** is that corollaries are typically statements that follow quickly from a previous theorem. In general, you should expect corollaries to have very short proofs. However, that doesn't mean that you can't produce a more lengthy yet valid proof of a corollary. The items labelled as **Problem** are sort of a mixed bag. In many circumstances, I ask you to provide a counterexample for a statement if it is false or to provide a proof if the statement is true. Usually, I have left it to you to determine the truth value. If the statement for a problem is true, one could relabel it as a theorem.

It is important to point out that there are very few examples in the notes. This is intentional. One of the goals of the items labelled as **Exercise** is for you to produce the examples.

Lastly, there are many situations where you will want to refer to an earlier definition or theorem/corollary/problem. In this case, you should reference the statement by number. For example, you might write something like, "By Theorem 2.14, we see that. . . ."

1.7 Some Minimal Guidance

Especially in the opening sections, it won't be clear what facts from your prior experience in mathematics we are "allowed" to use. Unfortunately, addressing this issue is difficult and is something we will sort out along the way. However, in general, here are some minimal and vague guidelines to keep in mind.

First, there are times when we will need to do some basic algebraic manipulations. You should feel free to do this whenever the need arises. But you should show sufficient work along the way. You do not need to write down justifications for basic algebraic manipulations (e.g., adding 1 to both sides of an equation, adding and subtracting the same amount on the same side of an equation, adding like terms, factoring, basic simplification, etc.).

On the other hand, you do need to make explicit justification of the logical steps in a proof. When necessary, you should cite a previous definition, theorem, etc. by number.

Unlike the experience many of you had writing proofs in geometry, our proofs will be written in complete sentences. You should break sections of a proof into paragraphs and use proper grammar. There are some pedantic conventions for doing this that I will point out along the way. Initially, this will be an issue that most students will struggle with, but after a few weeks everyone will get the hang of it.

Ideally, you should rewrite the statements of theorems before you start the proof. Moreover, for your sake and mine, you should label the statement with the appropriate number. I will expect you to indicate where the proof begins by writing "*Proof.*" at the beginning. Also, we will conclude our proofs with the standard "proof box" (i.e., $\square$ or $\blacksquare$), which is typically right-justified.

Lastly, every time you write a proof, you need to make sure that you are making your assumptions crystal clear. Sometimes there will be some implicit assumptions that we can omit, but at least in the beginning, you should get in the habit of stating your assumptions up front. Typically, these statements will start off "Assume..." or "Let...".

This should get you started. We will discuss more as the semester progresses. Now, go have fun and kick some butt!

Chapter 2

A Taste of Number Theory

In this section, we will work with the set of integers, $\mathbb{Z} = \{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$. The purpose of this section is to get started with proving some theorems about numbers and study the properties of $\mathbb{Z}$. While you are all very familiar with the integers, in this chapter as well as the rest of the course, as far as possible, *please use only the definitions and theorems provided in the notes.* You might use your prior knowledge for intuition, but intuition is a squirrely creature, very hard to pin down in proofs. Definitions, on the other hand, are carefully and precisely written down, so everyone can agree on exactly what they mean.

It is important to note that we are diving in head first here. There are going to be some subtle issues that you will bump into and our goal will be to see what those issues are, and then we will take a step back and start again. See what you can do!

We use the symbol "$\in$" as an abbreviation for the phrase "is an element of" or sometimes simply "in." For example, the mathematical expression "$n \in \mathbb{Z}$" means "n is an element of the integers."

Definition 2.1. *An integer n is **even** if $n = 2k$ for some $k \in \mathbb{Z}$.*

Definition 2.2. *An integer n is **odd** if $n = 2k+1$ for some $k \in \mathbb{Z}$.*

Notice that we framed the definition of "even" in terms of multiplication as opposed to division. When tackling theorems and problems involving even or odd, be sure to make use of our formal definitions and not some of the well-known divisibility properties. For now, you should avoid arguments that involve statements like, "even numbers have no remainder when divided by 2 while odd numbers do have a remainder."

Our first issue comes up already. Most would agree with the following fact.

Fact 2.3.

1. *Every integer is either even or odd but never both.*

2. *Sums and products of integers are integers.*

The problem is that it is not easy to prove this rule from the definition of $\mathbb{Z}$ we've taken. In fact, our definition is not a very good one. It requires that you already know what integers are.

More formal definitions for $\mathbb{Z}$ have the disadvantage that they are surprisingly abstract, complicated, and counterintuitive. For this reason, we will stick to our definition. The problem now is that we are unable to prove some of the most basic facts about the integers and instead must take them for granted. Fact 2.3 can be proved easily from the various formal definitions of $\mathbb{Z}$, but not from ours.

You may assume henceforth that Fact 2.3 is true. You do not need to prove it and you may use it freely, referring to it by its number.

Theorem 2.4. *If $n \in \mathbb{Z}$ then the sum of n and $n+1$ is odd.*

Proof. Suppose $n \in \mathbb{Z}$. We know that $n + (n+1) = 2n+1$. Since n is an integer, Definition 2.2 implies that $n + (n+1)$ is odd. □

Theorem 2.5. *If n is an even integer, then n^2 is an even integer.*

Proof. Suppose $n \in \mathbb{Z}$ is even. Then $n = 2k$ for some $k \in \mathbb{Z}$. We have

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2).$$

Fact 2.3 implies that $2k^2$ is an integer. Thus, n^2 is even. □

For a mathematician, because we need to be as precise as possible with our reasoning and our language, a statement is only true if it is true *all the time.* For example, the statement "I always want chocolate" is not a true statement if there is a single moment in time when I don't want chocolate. In other words, to establish that a statement is false, all you have to do is find one $\boxed{\textit{counterexample}}$ – one single setting where your statement is not true.

One common proof strategy that you will want to employ in (some of) the following exercises, and throughout the semester, is *proof by contradiction*, also sometimes called *reductio ad absurdum*. The idea is that you start out by assuming that a statement (that you want to prove true) is false. Then, by using mathematical logic and the theorems you've already proved, you obtain a patently false statement. For example, you might end up with the statement $0 = 1$. Since we know $0 \neq 1$, if all of your logical steps were correct, it must be that your original assumption (that the statement was false) is wrong. Consequently, the original statement must be true.

Problem 2.6. *Either prove, or provide a counterexample to, the statement "The sum of an even integer and an odd integer is odd."*

This statement is true.

Proof. Suppose that $m, n \in \mathbb{Z}$ are even and odd, respectively. Then there exist $s, t \in \mathbb{Z}$ such that $m = 2s$ and $n = 2t+1$. We have

$$m + n = 2s + (2t+1) = 2(s+t) + 1.$$

Fact 2.3 implies that $s + t \in \mathbb{Z}$, so $m + n$ is odd. □

Question 2.7. *Did Theorem 2.4 need to come before Problem 2.6? Could we have used Problem 2.6 to prove Theorem 2.4? If so, outline how this alternate proof would go. Perhaps your original proof utilized the approach I'm hinting at. If this is true, can you think of a proof that does not rely directly on Problem 2.6? Is one approach better than the other?*

Problem 2.8. *Either prove, or provide a counterexample to, the statement "The product of an odd integer and an even integer is odd."*

This is false.

Proof. For a counterexample, consider the odd number $1(= 2 \cdot 0 + 1)$ and the even number $2(= 2 \cdot 1)$. Their product $1 \cdot 2 = 2$ is even, so by Fact 2.3, it is not odd. □

Problem 2.9. *Either prove, or provide a counterexample to, the statement "The product of an odd integer and an odd integer is odd."*

Problem 2.10. *Either prove, or provide a counterexample to, the statement "If at least one of a pair of integers is even then their product is even."*

Definition 2.11. *An integer n* ***divides*** *the integer m, written $n|m$, if and only if there exists $k \in \mathbb{Z}$ such that $m = nk$. We may also say that m* ***is divisible by*** *n.*

Question 2.12. *For integers n and m, how are following mathematical expressions different?*

(a) $m|n$ *(b)* m/n

Among other things, Fact 2.3 implies that addition, subtraction, and multiplication of integers yields integers. We know that this is not true for division of integers. In other words, it is not always true that $m/n \in \mathbb{Z}$ when $m, n \in \mathbb{Z}$. The upshot: when working with $\mathbb{Z}$ it is a good idea to avoid using m/n. When you feel the urge to divide, switch to an equivalent formulation using multiplication. This will make your life easier when proving statements involving divisibility.

Problem 2.13. *Let $n \in \mathbb{Z}$. Either prove, or provide a counterexample to, the statement "If 6 divides n, then 3 divides n."*

Problem 2.14. *Let $n \in \mathbb{Z}$. Either prove, or provide a counterexample to, the statement "If 6 divides n, then 4 divides n."*

Theorem 2.15. *Assume $n, m, a \in \mathbb{Z}$. If $a|n$, then $a|mn$.*

A theorem that follows very easily from another theorem is called a **corollary**. Try to quickly prove the next result using the previous theorem. Be sure to cite the theorem in your proof.

Corollary 2.16. *Assume $n, a \in \mathbb{Z}$. If a divides n, then a divides n^2.*

Problem 2.17. *Assume $n, a \in \mathbb{Z}$. Consider the statement:*

$$\boxed{\text{If } a \text{ divides } n^2 \text{ then } a \text{ divides } n.}$$

Is this statement always true? Is it always false? Prove that your answers are correct by giving particular examples and/or general arguments.

It is considered good form to make your arguments as simple as possible. This usually means that if you can establish a fact by giving a specific example then this is preferable to giving a general argument.

Theorem 2.18. *Assume $n, a \in \mathbb{Z}$. If a divides n, then a divides $-n$.*

Theorem 2.19. *Assume $n, m, a \in \mathbb{Z}$. If a divides m and a divides n, then a divides $m + n$.*

It is considered good form to use our theorems to prove results rather that resorting to first principles. There's no point in reinventing the wheel if we don't have to. If, while writing a proof, you feel like you've argued this way before, then it is likely that you could quote a theorem instead. Try to use previous results to prove the next theorem.

Theorem 2.20. *Assume $n, m, a \in \mathbb{Z}$. If a divides m and a divides n, then a divides $m - n$.*

Problem 2.21. *Assume $a, b, m \in \mathbb{Z}$. Determine whether the following statement holds sometimes, always, or never.*

$$\boxed{\text{If } ab \text{ divides } m\text{, then } a \text{ divides } m \text{ and } b \text{ divides } m.}$$

Justify with a proof or counterexample.

Theorem 2.22. *If $a, b, c \in \mathbb{Z}$ are such that a divides b and b divides c, then a divides c.*

The previous theorem is referred to as **transitivity of division of integers**.

Theorem 2.23. *If $n \in \mathbb{Z}$ then 3 divides $n + (n+1) + (n+2)$.*

Chapter 3

Introduction to Logic

After diving in head first in the last chapter, we'll take a step back and do a more careful examination of what it is we are actually doing.

Definition 3.1. *A **proposition** (or **statement**) is a sentence that is either true or false.*

For example, the sentence "All dogs have four legs" is a false proposition. However, the perfectly good sentence "$x = 1$" is *not* a proposition all by itself. Until we specify what x is, we can't say categorically "This sentence is true" or "This sentence is false."

Exercise 3.2. *Determine whether the following are propositions or not. Explain.*

(a) *All cars are red.*

(b) *Every person whose name begins with J has the name Joe.*

(c) $x^2 = 4$.

(d) *There exists an x such that* $x^2 = 4$.

(e) *For all real numbers* x, $x^2 = 4$.

(f) $\sqrt{2}$ *is an irrational number.*

(g) p *is prime.*

(h) *Led Zeppelin is the best band of all time.*

Given two propositions, we can form more complicated propositions using logical connectives. Perhaps the most important one (although it doesn't really help us form new propositions from old ones) is the phrase "if and only if," which is often abbreviated iff.

Definition 3.3. *Given two propositions A, B, we say A is true* if and only if *B is true (or "A iff B" or "$A \iff B$") if A is true exactly when B is true. That is, $A \iff B$ means that if A is true then B is true, and if A is false then B is false.*

Definition 3.4. *Let A and B be propositions.*

*(a) The proposition "**not** A" is true if and only if$A$ is false; this is expressed symbolically as* $\boxed{\neg A}$ *and called the **negation** of A.*

*(b) The proposition "A **and** B" is true if and only if both A and B are true; this is expressed symbolically as* $\boxed{A \wedge B}$ *and called the **conjunction** of A and B.*

*(c) The proposition "A **or** B" is true if and only if at least one of A or B is true; this is expressed symbolically as* $\boxed{A \vee B}$ *and called the **disjunction** of A and B.*

*(d) The proposition "**If** A**, then** B" is true if and only if whenever A is true, B is also true; this is expressed symbolically as* $\boxed{A \implies B}$ *and called an **implication** or **conditional statement**. Note that $A \implies B$ may also be read as "A implies B" or "A only if B".*

Notice the way we used the word "or" above, to mean that either A or B *or both* was true. This is the standard mathematical meaning of "or." In this class, and in most of your future math classes, you should always interpret "or" to mean "either/or."

Question 3.5. *What's the difference between "$A \implies B$" and "$A \iff B$"? Can you give an example of two statements A, B where $A \implies B$ is true but $A \iff B$ is false? Can you give an example of two statements A, B where $A \iff B$ is true but $A \implies B$ is false?*

Exercise 3.6. *Describe the meaning of $\neg(A \wedge B)$ and $\neg(A \vee B)$.*

Exercise 3.7. *Let A represent "6 is an even number" and B represent "6 is a multiple of 4." First, explain why A and B are statements. Then, express each of the following statements in ordinary English sentences. Which of these statements are true? Why?*

(a) $A \wedge B$

(b) $A \vee B$

(c) $\neg A$

(d) $\neg B$

(e) $\neg(A \wedge B)$

(f) $\neg(A \vee B)$

(g) $A \implies B$

Each of the theorems that we proved in Chapter 2 are examples of conditional statements. However, some of the statements were disguised as such. For example, Theorem 2.4 states, "The sum of two consecutive integers is odd." We can reword this theorem as, "If $n \in \mathbb{Z}$, then $n + (n + 1)$ is odd."

Exercise 3.8. *Reword Theorem 2.23 so that it explicitly reads as a conditional statement.*

Whoops, sorry, this theorem was already stated as a conditional statement!

The proofs of each of the theorems in Section 2 had the same format, which we refer to as a **direct proof**.

Skeleton Proof 3.9 (Proof of $A \implies B$ by direct proof)**.** *If you want to prove the implication $A \implies B$ via a direct proof, then the structure of the proof is as follows.*

> *Proof.* Assume A.
>
> ... *[Use definitions and known results to derive B]* ...
>
> Therefore, B. □

Definition 3.10. *A **truth table** is a table that illustrates all possible truth values for a proposition.*

Example 3.11. *Let A and B be propositions. Then the truth table for the conjunction $A \wedge B$ is given by the following.*

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

We read truth tables horizontally, line by line. For example, the second line of the truth table above reads as "If A is true and B is false then $A \wedge B$ is false."

Notice that we have columns for each of A and B. The rows for these two columns correspond to all possible combinations of truth values for A and B. The third column gives us the truth value of $A \wedge B$ given the possible truth values for A and B.

Note that each proposition has two possible truth values: true or false. Thus, if a compound proposition P is built from n propositions, then the truth table for P will require 2^n rows.

Exercise 3.12. *Create a truth table for each of $A \vee B$, $\neg A$, $\neg(A \wedge B)$, and $\neg A \wedge \neg B$. Feel free to add additional columns to your tables to assist you with intermediate steps. (For example, for the last one, you might want to include columns for $\neg A$ and $\neg B$.)*

Problem 3.13. *A coach promises, "If we win tonight, then I will buy you pizza tomorrow." Determine the case(s) in which the players can rightly claim to have been lied to. Use this to help create a truth table for the proposition $A \implies B$.*

We can use truth tables to give another equivalent formulation of Definition 3.3, which is sometimes easier to work with.

Definition 3.14. *Two statements A and B are (**logically**) **equivalent**, expressed symbolically as $\boxed{A \iff B}$, if and only if they have the same truth table.*

Exercise 3.15. *Explain why Definition 3.14 and Definition 3.3 both assign the same meaning to the symbol $\iff$.*

Theorem 3.16 (DeMorgan's Law)**.** *If A and B are propositions, then $\neg(A \wedge B) \iff \neg A \vee \neg B$.*

Problem 3.17. *Let A and B be propositions. Conjecture a statement similar to Theorem 3.16 for the proposition $\neg(A \vee B)$ and then prove it. This is also called DeMorgan's Law.*

Definition 3.18. *The **converse** of $A \implies B$ is $B \implies A$.*

Exercise 3.19. *Provide an example of a true conditional proposition whose converse is false.*

Definition 3.20. *The **inverse** of $A \implies B$ is $\neg A \implies \neg B$.*

Exercise 3.21. *Provide an example of a true conditional proposition whose inverse is false.*

Definition 3.22. *The **contrapositive** of $A \implies B$ is $\neg B \implies \neg A$.*

Exercise 3.23. *Let A and B represent the statements from Exercise 3.7. Express the following in ordinary English sentences.*

(a) The converse of $A \implies B$.

(b) The contrapositive of $A \implies B$.

Exercise 3.24. *Find the converse and the contrapositive of the following statement: "If a person lives in Flagstaff, then that person lives in Arizona."*

Use a truth table to prove the following theorem.

Theorem 3.25. *The implication $A \implies B$ is equivalent to its contrapositive.*

The upshot of Theorem 3.25 is that if you want to prove a conditional proposition, you can prove its contrapositive instead, called **proof by contrapositive**.

Skeleton Proof 3.26 (Proof of $A \implies B$ by contrapositive)**.** *If you want to prove the implication $A \implies B$ by proving its contrapositive $\neg B \implies \neg A$ instead, then the structure of the proof is as follows.*

Proof. We will prove that $A \implies B$ by proving its contrapositive. Assume $\neg B$.

… *[Use definitions and known results to derive $\neg A$]* …

This proves that if $\neg B$, then $\neg A$. Therefore, by Theorem 3.25, $A \implies B$ is true. □

Problem 3.27. *Consider the following statement:*

Assume $n \in \mathbb{Z}$. If n^2 is odd, then n is an odd integer.

The items below can be assembled to form a proof of this statement, but they are currently out of order. Put them in the proper order.

1. *Thus, we assume that n is an even integer.*
2. *We will prove this by contrapositive.*
3. *Since $n = 2k$, we have that $n^2 = (2k)^2 = 4k^2$.*

4. *Since k is an integer, $2k^2$ is also an integer by Fact 2.3.*

5. *By Definition 2.1, there is an integer k such that $n = 2k$.*

6. *Since the contrapositive is equivalent to the original statement and we have proved the contrapositive, the original statement is true.*

7. *By Definition 2.1, n^2 is an even integer.*

8. *The contrapositive of the statement "If n^2 is odd then n is odd" is "If n is an even integer, then n^2 is an even integer."*

9. *Notice that $n^2 = 2(2k^2)$.*

Try proving each of the next three theorems by proving the contrapositive of the given statement.

Theorem 3.28. *Assume $n \in \mathbb{Z}$. If n^2 is even, then n is even.*

Theorem 3.29. *Assume $n, m \in \mathbb{Z}$. If nm is odd, then both n and m are odd.*

Theorem 3.30. *Assume $n, m \in \mathbb{Z}$. If nm is even, then either n or m is even.*

3.1 Proofs by contradiction

A **proof by contradiction** is any proof where the negation of a given statement is shown to be false. It is a very useful proof technique.

Recall that an implication $A \implies B$ is only false when A is true and B is false. So, if we want to prove by contradiction that the implication $A \implies B$ is true, (ie we want to assume that $A \implies B$ is false and work logically to get a contradiction), we should start by assuming that A is true but B is false.

Here's an example of the proof technique.

Theorem 3.31. *Suppose that x is an integer. If x is odd, then 2 does not divide x.*

Proof. Suppose that x is odd and 2 divides x. By Fact 2.3, x is not even. However, 2 divides x, so $x = 2k$ for some $k \in \mathbb{Z}$. By Definition 2.1, this means x is even, contradicting the fact that x is odd. □

The following is the converse to Theorem 2.5. Use contradiction to prove it.

Theorem 3.32. *Suppose $n \in \mathbb{Z}$. If n^2 is even then n is even.*

Use contradiction to prove the next two theorems.

Theorem 3.33. *If x is a real number in $[0, \pi/2]$ then*

$$\sin x + \cos x \geq 1.$$

Definition 3.34. *We write $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ for the set of* natural numbers *or* positive integers.

Theorem 3.35. *Assume $x, y \in \mathbb{N}$. If x divides y then $x \leq y$.*

Question 3.36. *Is Theorem 3.35 true if we only assume $x, y \in \mathbb{Z}$? Give a proof or a counterexample.*

Appendix A

Elements of Style for Proofs

Years of elementary school math taught us incorrectly that the answer to a math problem is just a single number, "the right answer." It is time to unlearn those lessons; those days are over. From here on out, mathematics is about discovering proofs and writing them clearly and compellingly.

The following rules apply whenever you write a proof. I may refer to them, by number, in my comments on your homework and exams. Keep these rules handy so that you may refer to them as you write your proofs.

1. **The burden of communication lies on you, not on your reader.** It is your job to explain your thoughts; it is not your reader's job to guess them from a few hints. You are trying to convince a skeptical reader who doesn't believe you, so you need to argue with airtight logic in crystal clear language; otherwise the reader will continue to doubt. If you didn't write something on the paper, then (a) you didn't communicate it, (b) the reader didn't learn it, and (c) the grader has to assume you didn't know it in the first place.

2. **Tell the reader what you're proving.** The reader doesn't necessarily know or remember what "Theorem 2.13" is. Even a professor grading a stack of papers might lose track from time to time. Therefore, the statement you are proving should be on the same page as the beginning of your proof. For an exam this won't be a problem, of course, but on your homework, recopy the claim you are proving. This has the additional advantage that when you study for exams by reviewing your homework, you won't have to flip back in the notes/textbook to know what you were proving.

3. **Use English words.** Although there will usually be equations or mathematical statements in your proofs, use English sentences to connect them and display their logical relationships. If you look in your notes/textbook, you'll see that each proof consists mostly of English words.

4. **Use complete sentences.** If you wrote a history essay in sentence fragments, the reader would not understand what you meant; likewise in mathematics you must use complete sentences, with verbs, to convey your logical train of thought.

Some complete sentences can be written purely in mathematical symbols, such as equations (e.g., $a^3 = b^{-1}$), inequalities (e.g., $x < 5$), and other relations (like $5 \big| 10$ or $7 \in \mathbb{Z}$). These statements usually express a relationship between two mathematical *objects*, like numbers or sets. However, it is considered bad style to begin a sentence with symbols. A common phrase to use to avoid starting a sentence with mathematical symbols is "We see that..."

5. **Show the logical connections among your sentences.** Use phrases like "Therefore" or "because" or "if..., then..." or "if and only if" to connect your sentences.

6. **Know the difference between statements and objects.** A mathematical object is a *thing*, a noun, such as a group, an element, a vector space, a number, an ordered pair, etc. Objects either exist or don't exist. Statements, on the other hand, are mathematical *sentences*: they can be true or false.

   When you see or write a cluster of math symbols, be sure you know whether it's an object (e.g., "$x^2 + 3$") or a statement (e.g., "$x^2 + 3 < 7$"). One way to tell is that every mathematical statement includes a verb, such as $=$, $\leq$, "divides", etc.

7. **"=" means equals.** Don't write $A = B$ unless you mean that A actually equals B. This rule seems obvious, but there is a great temptation to be sloppy. In calculus, for example, some people might write $f(x) = x^2 = 2x$ (which is false), when they really mean that "if $f(x) = x^2$, then $f'(x) = 2x$."

8. **Don't interchange $=$ and $\implies$.** The equals sign connects two *objects*, as in "$x^2 = b$"; the symbol "$\implies$" is an abbreviation for "implies" and connects two *statements*, as in "$a + b = a \implies b = 0$." You should avoid using $\implies$ in your formal write-ups.

9. **Say exactly what you mean.** Just as the $=$ is sometimes abused, so too people sometimes write $A \in B$ when they mean $A \subseteq B$, or write $a_{ij} \in A$ when they mean that a_{ij} is an entry in matrix A. Mathematics is a very precise language, and there is a way to say exactly what you mean; find it and use it.

10. **Don't write anything unproven.** Every statement on your paper should be something you *know* to be true. The reader expects your proof to be a series of statements, each proven by the statements that came before it. If you ever need to write something you don't yet know is true, you *must* preface it with words like "assume," "suppose," or "if" (if you are temporarily assuming it), or with words like "we need to show that" or "we claim that" (if it is your goal). Otherwise the reader will think they have missed part of your proof.

11. **Write strings of equalities (or inequalities) in the proper order.** When your reader sees something like

$$A = B \leq C = D,$$

   he/she expects to understand easily why $A = B$, why $B \leq C$, and why $C = D$, and he/she expects the *point* of the entire line to be the more complicated fact that $A \leq$

D. For example, if you were computing the distance d of the point $(12,5)$ from the origin, you could write

$$d = \sqrt{12^2 + 5^2} = 13.$$

In this string of equalities, the first equals sign is true by the Pythagorean theorem, the second is just arithmetic, and the *point* is that the first item equals the last item: $d = 13$.

A common error is to write strings of equations in the wrong order. For example, if you were to write "$\sqrt{12^2 + 5^2} = 13 = d$", your reader would understand the first equals sign, would be baffled as to how we know $d = 13$, and would be utterly perplexed as to why you wanted or needed to go through 13 to prove that $\sqrt{12^2 + 5^2} = d$.

12. **Avoid circularity.** Be sure that no step in your proof makes use of the conclusion!

13. **Don't write the proof backwards.** Beginning students often attempt to write "proofs" like the following, which attempts to prove that $\tan^2(x) = \sec^2(x) - 1$:

$$\begin{aligned}
\tan^2(x) &= \sec^2(x) - 1 \\
\left(\frac{\sin(x)}{\cos(x)}\right)^2 &= \frac{1}{\cos^2(x)} - 1 \\
\frac{\sin^2(x)}{\cos^2(x)} &= \frac{1 - \cos^2(x)}{\cos^2(x)} \\
\sin^2(x) &= 1 - \cos^2(x) \\
\sin^2(x) + \cos^2(x) &= 1 \\
1 &= 1
\end{aligned}$$

Notice what has happened here: the student *started* with the conclusion, and deduced the true statement "$1 = 1$." In other words, he/she has proved "If $\tan^2(x) = \sec^2(x) - 1$, then $1 = 1$," which is true but highly uninteresting.

Now this isn't a bad way of *finding* a proof. Working backwards from your goal often is a good strategy *on your scratch paper*, but when it's time to *write* your proof, you have to start with the hypotheses and work to the conclusion.

14. **Be concise.** Most students err by writing their proofs too short, so that the reader can't understand their logic. It is nevertheless quite possible to be too wordy, and if you find yourself writing a full-page essay, it's probably because you don't really have a proof, but just an intuition. When you find a way to turn that intuition into a formal proof, it will be much shorter.

15. **Introduce every symbol you use.** If you use the letter "k," the reader should know exactly what k is. Good phrases for introducing symbols include "Let $n \in \mathbb{N}$," "Let k be the least integer such that...," "For every real number a...," and "Suppose that X is a counterexample."

16. **Use appropriate quantifiers (once).** When you introduce a variable $x \in S$, it must be clear to your reader whether you mean "for all $x \in S$" or just "for some $x \in S$." If you just say something like "$y = x^2$ where $x \in S$," the word "where" doesn't indicate whether you mean "for all" or "some".

    Phrases indicating the quantifier "for all" include "Let $x \in S$"; "for all $x \in S$"; "for every $x \in S$"; "for each $x \in S$"; etc. Phrases indicating the quantifier "some" (or "there exists") include "for some $x \in S$"; "there exists an $x \in S$"; "for a suitable choice of $x \in S$"; etc.

    On the other hand, don't introduce a variable more than once! Once you have said "Let $x \in S$," the letter x has its meaning defined. You don't *need* to say "for all $x \in S$" again, and you definitely should *not* say "let $x \in S$" again.

17. **Use a symbol to mean only one thing.** Once you use the letter x once, its meaning is fixed for the duration of your proof. You cannot use x to mean anything else.

18. **Don't "prove by example."** Most problems ask you to prove that something is true "for all"—You *cannot* prove this by giving a single example, or even a hundred. Your answer will need to be a logical argument that holds for *every example there possibly could be*.

19. **Write "Let $x = \ldots$," not "Let $\cdots = x$."** When you have an existing expression, say a^2, and you want to give it a new, simpler name like b, you should write "Let $b = a^2$," which means, "Let the new symbol b mean a^2." This convention makes it clear to the reader that b is the brand-new symbol and a^2 is the old expression he/she already understands.

    If you were to write it backwards, saying "Let $a^2 = b$," then your startled reader would ask, "What if $a^2 \neq b$?"

20. **Make your counterexamples concrete and specific.** Proofs need to be entirely general, but counterexamples should be absolutely concrete. When you provide an example or counterexample, make it as specific as possible. For a set, for example, you must name its elements, and for a function you must give its rule. Do not say things like "θ could be one-to-one but not onto"; instead, provide an actual function θ that *is* one-to-one but not onto.

21. **Don't include examples in proofs.** Including an example very rarely adds anything to your proof. If your logic is sound, then it doesn't need an example to back it up. If your logic is bad, a dozen examples won't help it (see rule 18). There are only two valid reasons to include an example in a proof: if it is a *counterexample* disproving something, or if you are performing complicated manipulations in a general setting and the example is just to help the reader understand what you are saying.

22. **Use scratch paper.** Finding your proof will be a long, potentially messy process, full of false starts and dead ends. Do all that on scratch paper until you find a real proof, and only then break out your clean paper to write your final proof carefully. *Do not hand in your scratch work!*

Only sentences that actually contribute to your proof should be part of the proof. Do not just perform a “brain dump,” throwing everything you know onto the paper before showing the logical steps that prove the conclusion. *That is what scratch paper is for.*

Appendix B

Fancy Mathematical Terms

Here are some important mathematical terms that you will encounter in this course and throughout your mathematical career.

1. **Definition**—a precise and unambiguous description of the meaning of a mathematical term. It characterizes the meaning of a word by giving all the properties and only those properties that must be true.

2. **Theorem**—a mathematical statement that is proved using rigorous mathematical reasoning. In a mathematical paper, the term theorem is often reserved for the most important results.

3. **Lemma**—a minor result whose sole purpose is to help in proving a theorem. It is a stepping stone on the path to proving a theorem. Very occasionally lemmas can take on a life of their own (Zorn's lemma, Urysohn's lemma, Burnside's lemma, Sperner's lemma).

4. **Corollary**—a result in which the (usually short) proof relies heavily on a given theorem (we often say that "this is a corollary of Theorem A").

5. **Proposition**—a proved and often interesting result, but generally less important than a theorem.

6. **Conjecture**—a statement that is unproved, but is believed to be true (Collatz conjecture, Goldbach conjecture, twin prime conjecture).

7. **Claim**—an assertion that is then proved. It is often used like an informal lemma.

8. **Axiom/Postulate**—a statement that is assumed to be true without proof. These are the basic building blocks from which all theorems are proved (Euclid's five postulates, Zermelo-Frankel axioms, Peano axioms).

9. **Identity**—a mathematical expression giving the equality of two (often variable) quantities (trigonometric identities, Euler's identity).

10. **Paradox**—a statement that can be shown, using a given set of axioms and definitions, to be both true and false. Paradoxes are often used to show the inconsistencies in a flawed theory (Russell's paradox). The term paradox is often used informally to describe a surprising or counterintuitive result that follows from a given set of rules (Banach-Tarski paradox, Alabama paradox, Gabriel's horn).

Appendix C

Definitions in Mathematics

It is difficult to overstate the importance of definitions in mathematics. Definitions play a different role in mathematics than they do in everyday life.

Suppose you give your friend a piece of paper containing the definition of the rarely-used word **rodomontade**. According to the Oxford English Dictionary[1] (OED) it is:

> A vainglorious brag or boast; an extravagantly boastful, arrogant, or bombastic speech or piece of writing; an arrogant act.

Give your friend some time to study the definition. Then take away the paper. Ten minutes later ask her to define rodomontade. Most likely she will be able to give a reasonably accurate definition. Maybe she'd say something like, "It is a speech or act or piece of writing created by a pompous or egotistical person who wants to show off how great they are." It is unlikely that she will have quoted the OED word-for-word. In everyday English that is fine—you would probably agree that your friend knows the meaning of the rodomontade. This is because most definitions are *descriptive*. They describe the common usage of a word.

Let us take a mathematical example. The OED[2] gives this definition of *continuous*.

> Characterized by continuity; extending in space without interruption of substance; having no interstices or breaks; having its parts in immediate connection; connected, unbroken.

Likewise, we often hear calculus students speak of a continuous function as one whose graph can be drawn "without picking up the pencil." This definition is descriptive. (As we learned in calculus the picking-up-the-pencil description is not a perfect description of continuous functions.) This is not a mathematical definition.

Mathematical definitions are *prescriptive*. The definition must prescribe the exact and correct meaning of a word. Contrast the OED's descriptive definition of continuous with the the definition of continuous found in a real analysis textbook.

> A function $f : A \to \mathbb{R}$ is **continuous at a point** $c \in A$ if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|x-c| < \delta$ (and $x \in A$) it follows that $|f(x)-f(c)| < \varepsilon$. If f

[1] http://www.oed.com/view/Entry/166837
[2] http://www.oed.com/view/Entry/40280

is continuous at every point in the domain A, then we say that f is **continuous on** A.[3]

In mathematics there is very little freedom in definitions. Mathematics is a deductive theory; it is impossible to state and prove theorems without clear definitions of the mathematical terms. The definition of a term must completely, accurately, and unambiguously describe the term. Each word is chosen very carefully and the order of the words is critical. In the definition of continuity changing "there exists" to "for all," changing the orders of quantifiers, changing $<$ to $\leq$ or $>$, or changing $\mathbb{R}$ to $\mathbb{Z}$ would completely change the meaning of the definition.

What does this mean for you, the student? Our recommendation is that at this stage you memorize the definitions word-for-word. It is the safest way to guarantee that you have it correct. As you gain confidence and familiarity with the subject you may be ready to modify the wording. You may want to change "for all" to "given any" or you may want to change $|x-c| < \delta$ to $-\delta < x - c < \delta$ or to "the distance between x and c is less than δ."

Of course, memorization is not enough; you must have a conceptual understanding of the term, you must see how the formal definition matches up with your conceptual understanding, and you must know how to work with the definition. It is perhaps with the first of these that descriptive definitions are useful. They are useful for building intuition and for painting the "big picture." Only after days (weeks, months, years?) of experience does one get an intuitive feel for the ε,δ-definition of continuity; most mathematicians have the "picking-up-the-pencil" definitions in their head. This is fine as long as we know that it is imperfect, and that when we prove theorems about continuous functions in mathematics we use the mathematical definition.

We end this discussion with an amusing real-life example in which a descriptive definition was not sufficient. In 2003 the German version of the game show *Who wants to be a millionaire?* contained the following question: "Every rectangle is: (a) a rhombus, (b) a trapezoid, (c) a square, (d) a parallelogram."

The confused contestant decided to skip the question and left with €4000. Afterward the show received letters from irate viewers. Why were the contestant and the viewers upset with this problem? Clearly a rectangle is a parallelogram, so (d) is the answer. But what about (b)? Is a rectangle a trapezoid? We would describe a trapezoid as a quadrilateral with a pair of parallel sides. But this leaves open the question: can a trapezoid have *two* pairs of parallel sides or must there only be *one* pair? The viewers said two pairs is allowed, the producers of the television show said it is not. This is a case in which a clear, precise, mathematical definition is required.

[3]This definition is taken from page 109 of Stephen Abbott's *Understanding Analysis*, but the definition would be essentially the same in any modern real analysis textbook.