Weekly Homework 5

Kevin Christensen
Introduction to Abstract Math

February 24, 2021

Exercise 3.50. Rewrite the following statements in words and determine whether each is true or false. Make sure you justify your answer

1. $(\forall n \in \mathbb{N})(n^2 \geq 5)$
2. $(\exists n \in \mathbb{N})(n^2 - 1 = 0)$
3. $(\exists N \in \mathbb{N})(\forall n > N)(\frac{1}{n} < 0.01)$
4. $(\forall m, n \in \mathbb{Z})(2|m \wedge 2|n \implies 2|(m+n))$
5. $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(x - 2y = 0)$
6. $(\exists x \in \mathbb{N})(\forall y \in \mathbb{N})(y \leq x)$

Solution.

1. For all n in the Natural numbers, $n^2 \geq 5$.
   This is false, 1 is a natural number and $1^2 = 1$, which is less than 5.
2. For some n in the Natural numbers, $(n^2 - 1) = 0$. This is true, 1 is a natural number and $1^2 = 1$, so $1^2 - 1 = 0$.
3. For some N in the natural numbers, for all n greater than N, $\frac{1}{n}$ is less than 0.01. This is true because N can take on a value of 100 or greater.
4. For all m and n in the Integers, if 2 divides m and 2 divides n, then 2 divides $m + n$. This is true by Theorem 2.19.
5. For all natural numbers x, there exists a natural number y such that $x - 2y = 0$.
   This is false. If we add 2y to both sides of the equation we get $x = 2y$. By definition 2.1, x must be an even number in order for $x - 2y = 0$ to be true. However, not all natural numbers are even.
6. For some natural number x, every natural number y is less than or equal to x. This is false, if we take any natural number x we can say y is defined as $x + 1$ and then $y \not\leq x$.

$\square$

Theorem 3.53. Let P(x) be a predicate. Then

1. $\neg(\forall x)P(x)$ is equivalent to $(\exists x)(\neg P(x))$
2. $\neg(\exists x)P(x)$ is equivalent to $(\forall x)(\neg P(x))$

Proof.

1. We will prove this by contradiction. We assume that not all x are such that $P(x)$ and that there is not an x such that $\neg P(x)$. If there is not an x such that $\neg P(x)$, then all x must be such that $P(x)$. However, this contradicts our assumption that not all x are such that $P(x)$. Therefore $\neg(\forall x)P(x) \implies (\exists x)(\neg P(x))$.
   Conversely, we use contradiction the other direction. We assume that there exists some x such that $\neg P(x)$ and that for all x, $P(x)$. If there is some x for which P(x) is not true, then not all x can satisfy $P(x)$. So, this contradicts our assumption that all values of x satisfy P(x). Therefore $(\exists x)(\neg P(x)) \implies \neg(\forall x)(P(x))$.
   By definition 3.3, $\neg(\forall x)P(x) \iff (\exists x)(\neg P(x))$.

2. We proceed with proof by contradiction. So we assume that there does not exist an x such that $P(x)$ and not all x are such that, $\neg P(x)$. If not all x are such that $\neg P(x)$, then there is at least one x that satisfies $P(x)$ which contradicts our assumption that there does not exist an x such that $P(x)$. Therefore $\neg(\exists x)P(x) \implies (\forall x)(\neg P(x))$.
   Conversely, we proceed with contradiction the other direction, assuming that all x are such that $\neg P(x)$ and that it is false that there does not exist an x such that $P(x)$. If it is false that there does not exist an x such that P(x), then there must exist an x such that $P(x)$ which is a contradiction with our assumption that all x are such that $\neg P(x)$. Therefore $(\forall x)(\neg P(x)) \implies \neg(\exists x)P(x)$. By Definition 3.3, $\neg(\exists x)P(x) \iff (\forall x)(\neg P(x))$.

$\square$