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ECE211 Signal Processing & Systems Analysis
Problem Set II: Signals & Spectra Part II
January 22, 2019

1. In this problem we are working in $L^2(\mathbb{R})$ with the usual inner product:

$$\langle x, y \rangle = \int_{-\infty}^{\infty} x(t)\, y(t)\, dt$$

Refer to Figure 1. Three signals $\{s_i(t)\}_{i=1}^{3}$ are shown.

   (a) Use Gram-Schmidt orthogonalization to find an orthonormal set $\{\phi_i(t)\}_{i=1}^{3}$. Show work. Sketch the signals ϕ_i, $1 \leq i \leq 3$.

   (b) Express $s_3(t)$ as an equivalent vector $\vec{s}_3 \in \mathbb{R}^3$ with respect to the basis $\{\phi_i\}_{i=1}^{3}$. Also express its projection on $span\{\phi_1, \phi_2\}$ as a vector in $\mathbb{R}^3$. Call this vector $\hat{s}_3$. This part of the problem should not require any more integration!

   (c) Compute $\|s_3\|$ using an integral, and compare it to $\|\vec{s}_3\|$ (the Euclidean length of the vector form).

   (d) Sketch the approximate signal $\hat{s}_3(t)$.

   (e) Consider the error signal $e(t) = s_3(t) - \hat{s}_3(t)$. Compute $\|e\|$ WITHOUT DOING ANY INTEGRATION!! Show how you found the answer.

2. A voltage amplifier provides an amplitude gain of 100 (on a linear scale).

   (a) Find the gain in decibels.

   (b) If the input is $-10dBm$, find the output in dBm.

   (c) Express the output power in Watts.

3. A real $20kHz$ sinewave is sampled at $50kHz$.

   (a) List all frequencies up to $200kHz$ that would alias into $20kHz$.

   (b) Specify the appropriate cutoff frequency for the filter that should be employed by the $50kHz$ A/D converter.

   (c) Specify the input signal frequency on the following scales (with units): analog radian frequency, normalized digital radian frequency, as a fraction of the sampling rate, and as a fraction of the Nyquist bandwidth.

   (d) The sampled data is passed to a D/A converter operated at a rate of $30kHz$ (neglect issues of buffer overflow or underflow). List the six lowest frequencies that emerge at the output.

   (e) Specify the cutoff frequency of the anti-imaging filter that should be used in the D/A converter (in general, not for this particular signal).

4. **Image Frequencies in Superheterodyne Systems**

A $4.0GHz$ signal is to mixed with a sinewave at f_0 to produce an output at $300MHz$. This would then ultimately be downconverted again to baseband, but we will omit that step here. The mixing process will produce a spurious high frequency signal, but that is easily eliminated by filtering.

(a) It turns out there are two valid choices for f_0. Specify both choices.

(b) For each choice of f_0, there will be another frequency (besides $4.0GHz$) that would produce an output at $300MHz$. These frequencies would need to be filtered out *before* mixing to prevent interference with the desired signal. These extraneous frequencies are called *images* in this context. For each choice of f_0 specify the image frequency.

Remark: This is the basic operation of a *superheterodyne* system. The $4.0GHz$ signal would be called the radio frequency (RF) signal and the 300MHz signal would be called the intermediate frequency (IF) signal. The key feature is that the conversion between baseband and bandpass is not done directly, but instead through one (or more) IF stages.

5. **Carson's Rule for FM Bandwidth**

Determining the bandwidth of an FM signal is complicated. For an AM signal such as $\cos(2\pi f_m t)\cos(2\pi f_c t + \phi)$ where the modulating frequency f_m is small compared with the carrier frequency f_c, the bandwidth is easily seen to be $2f_m$, since the signal has spectral components at $f_c \pm f_m$. Now consider the information signal $x(t) = A_m \cos(2\pi f_m t)$ and the FM signal:

$$s(t) = A_c \cos\left(2\pi f_c t + 2\pi K \int_0^t x(\tau)\, d\tau + \phi_0\right) = A_c \cos(2\pi f_c t + \beta \sin 2\pi f_m t + \phi_0)$$

where $\beta = KA_m/f_m$. This is called the FM modulation index; note that KA_m represents the maximum deviation of the instantaneous frequency from f_c, and β is the ratio of this frequency deviation to the information signal's frequency, f_m. Using the notion of instantaneous frequency, we see that $f_{inst}(t)$ at each time is determined by the instantaneous amplitude of $x(t)$, and takes on a range $f_{\min} \le f_{inst}(t) \le f_{\max}$. Thus, bandwidth could be estimated as $B = f_{\max} - f_{\min}$. Here the FM signal bandwidth appears to depend on the *amplitude* of x, not its frequency. However, the instantaneous frequency is changing over time, and this time-varying behavior should be expected to give rise to a signal with a bigger bandwidth. In particular, the bandwidth will turn out to depend on the *frequency* of x as well as its amplitude.

(a) Compute $f_{inst}(t)$, show that its DC value is f_c, and show that $|f_{inst}(t) - f_c| \le B/2$ for some constant B, which we can take as the bandwidth. Compute B, and observe that it depends on A_m and not f_m.

(b) Find the baseband representation $s_{BB}(t)$ of $s(t)$. Show that it is periodic with period T (find T), and hence it has spectral components at DC and $\pm f_0, \pm 2f_0, \cdots$, where the fundamental frequency is $f_0 = 1/T$. We can approximate the line spectrum by taking only DC and the fundamental frequency, that is at 0 and $\pm f_0$. This approximation assumes the spectrum falls off rapidly enough that we can ignore the second and higher order harmonics. In this case, $s_{BB}(t)$ has spectral components at DC and $\pm f_0$, and hence $s(t)$ has spectral components at f_c and $f_c \pm f_0$, and this determines a bandwidth $B = 2f_0$. What is f_0? Show that the bandwidth computed this way depends on the information signal's frequency and not its amplitude. It is based on the time-varying nature of the instantaneous frequency.

(c) In reality, a more accurate formula for the bandwidth would include several harmonics of the line spectrum of $s_{BB}(t)$, and hence would be larger than the value found in part 5b above. An important formula that is used in practice, which has been shown empirically to be useful over a wide range of values for β and f_m, is:

$$B_{FM} \approx 2f_m(1+\beta)$$

Show that this formula, called *Carson's rule,* can be found by summing the bandwidths found in parts 5a and 5b.

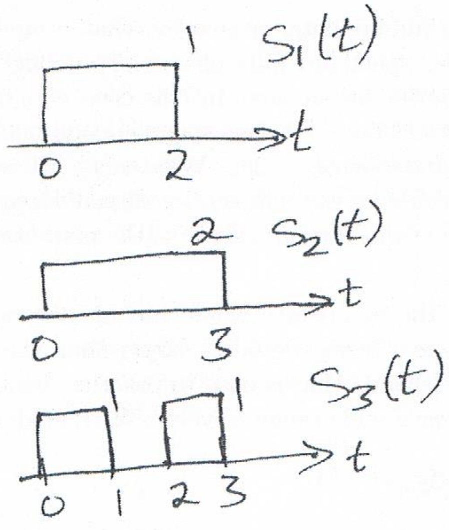

FIGURE 1:
3 SIGNALS