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ECE211 Signal Processing & Systems Analysis
Problem Set V: Transfer Functions and Frequency Response

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Note: In MATLAB, there are two common methods for representing a transfer function. One is with two vectors, often denoted b, a, to represent the numerator and denominator polynomial coefficients respectively; this is often call the *tf* form. Another method is with the triple z, p, k, where z is a vector of zeros, p is a vector of poles, and k is a constant scaling factor, for example (in the z-domain):

$$H(z) = k\frac{(z-z_1)\cdots(z-z_M)}{(z-p_1)\cdots(z-p_N)}$$

This is often called the *zp* form. Thus, for example:

$$[b, a] = zp2tf(z, p, k)$$

will convert from one form to the other (and similarly $tf2zp$).

1. Find the transfer function for the block diagram shown in **Figure 1.** Write your final answer by inspection (do not "derive" it), but do not try to simplify it.

2. Given the following transfer function:

$$H(z) = \frac{2(z-3)(3z-2)^2(2z-1)}{(4z-3)^2(2z+1)^2}$$

Let $x[n] = (0.8)^n \cos(4n\pi/5)$.

(a) Find the *tf* form for $H(z)$. Don't multiply out the numerator and denominator factors yourself! Use MATLAB to do it! Then convert to *zp* form.

(b) Use *zplane* to obtain the pole-zero plot of $H(z)$.

(c) Generate 20 samples for $x[n]$ (i.e., $0 \le n \le 19$). Use the MATLAB function *filter* to find the output $y[n]$ of the filter H with input x. On separate axes, obtain stem plots of x, y for $0 \le n \le 19$.

(d) Assume the filter is operated at a sampling rate of $20kHz$. Use *freqz* to compute the frequency response at 1000 points uniformly spaced from 0 to the Nyquist bandwidth. Then plot the magnitude response (in dB) and the phase response (unwrapped, in degrees), with the frequency axis given in Hertz. Do not use *freqz* to generate the plot, you should do it yourself. Put these as two subplots in one figure. Label the axes appropriately.

3. Look at the MATLAB documentation for *cheby1* and *ellip* to see how to use it to do the following. We want to design ANALOG Chebyshev I and elliptic LOWPASS filters (note: calling the function as *cheby1(..., 's')* or *ellip(..., 's')*) will create an analog filter; omitting the *'s'* generates a digital filter- be careful!). Also be careful: *cheby1* and *ellip* ask the specifications to be specified a bit differently: for the elliptic filter, you give both the passband ripple and stopband ripple; for the Chebyshev I, you only give the passband ripple. In each case, you give the passband edge frequency in RAD/SEC (not Hertz!). For purposes of numerical accuracy, it is best to call the function so it returns the transfer function described in *zp* form, and then convert to *tf* form. Also, if you call *freqs* specifying the NUMBER of frequency points you want, MATLAB will automatically determine the reasonable range of analog values to use; again, use the MATLAB documentation to see how to do this; you want *freqs* to return the computed frequency response and the frequencies at which it took the measurements (do not use it for plotting). Be careful- *freqs* returns the frequencies in RADIAN form.

   In any case, we want a 4^{th} order lowpass filter with passband edge frequency $50kHz$, with 3db ripple in the passband and (for the elliptic filter) 30dB ripple in the stopband.

   (a) The function *zplane* is used to graph the pole-zero plot for a digital filter. For example, it superimposes the unit circle, so it is not suitable here. Write a function called *splane* useful for analog case: plot the zeros as 'o' and poles as 'x' in the complex plane, and also superimpose the real and imaginary axes on your graph. Also use *axis('equal')* to get the aspect ratio correct (so e.g., a circle will be a circle).

   (b) Call *cheby1* and *ellip* to design the filters, specifically, obtain both the zp and tf representations of each, as described above. You will observe the poles for each filter seem to lie on a particular geometric figure in the LHP– can you guess the kind of shape this is? **Hint:** It should be obvious?

   (c) In one figure, provide the pole-zero plots for each filter as subplots (by calling *splane* for each).

   (d) In a separate figure, superimpose the magnitude responses (magnitude in decibels, frequency in kHz) for the two filters.

   (e) In a separate figure, superimpose the phase responses (phase in degrees, frequency in kHz) for the two filters.

   (f) In each case, the stopband edge frequency is defined to be the LOWEST frequency at which $30dB$ attenuation is first achieved. Compute the stopband edge frequency for each filter, and compare. These filters have the same order, and considering the properties of elliptic filters, its stopband edge should be lower (perhaps much lower?) than that of the Chebyshev I filter.

4. Let $x(t)$ have CTFT $X(\omega)$. Find the ICTFT of:

$$\exp(j\omega t_0) \cdot X^*(\omega - \omega_0)$$

   where t_0, ω_0 are constants. Do this by evaluating the inverse FT integral (not by using properties of the CTFT).

5. **Paraconjugates and All-Pass Functions**

To distinguish the transfer function from the frequency response, here we will write $H(z)$ for the z-transform, and $H_{FT}(\omega)$ for the Fourier transform, similarly $H(s)$ and $H_{FT}(\omega)$ for the analog case. We define the *paraconjugate* of a transfer function, denoted as $\tilde{H}$, via the following:

$$\begin{aligned}\tilde{H}(z) &= H^*(1/z^*)\\ \tilde{H}(s) &= H^*(-s^*)\end{aligned}$$

(a) Let $z = |z|\exp(j\theta)$. What is $1/z^*$ (in polar form)? Draw a sketch, showing z and $1/z^*$ for, say, $z = 0.5\exp(j\pi/4)$. This is defined to be the *symmetric point* of z with respect to the unit circle. (It turns out if you look at a mirror shaped as a circular arc, your image appears at the symmetric point with respect to the circle).

(b) Let $s = \sigma + j\omega$. What is $-s^*$ (in rectangular form)? Draw a sketch showing s and $-s^*$ for say $s = -1 + j2$. This is defined to be the *symmetric point* of s with respect to the imaginary axis. (It turns out if you look at a flat mirror, your image appears at this type of symmetric point).

(c) In each case, show that $\tilde{H}$ in the frequency domain is $H^*_{FT}(\omega)$.

(d) The correct transfer function associated with $H^*_{FT}(\omega)$ is NOT $H^*(z)$ or $H^*(s)$: conjugation is not analytic and all transfer functions must be analytic! NEITHER $\mathbf{H}^*(z)$ NOR $\mathbf{H}^*(s)$ ARE VALID TRANSFER FUNCTIONS! However, $\tilde{H}$ WILL be analytic because the double conjugation on z or s will cancel out. To see this, take the following two cases:

$$\begin{aligned}H(z) &= \frac{b_2z^2 + b_1z + b_0}{a_2z^2 + a_1z + a_0}\\ H(s) &= \frac{b_2s^2 + b_1s + b_0}{a_2s^2 + a_1s + a_0}\end{aligned}$$

In each case, find $\tilde{H}$; simplified enough to see that s and z appear (but not s^*, z^*); for the z-transform case, write your final answer so only positive powers of z appear. (Do you see a pattern for what happens to the polynomial coefficients in each case?)

(e) An *allpass function* has constant magnitude 1 at all frequencies:

$$|A_{FT}(\omega)| = 1 \ \ \forall\omega$$

Although it may seem perhaps only trivial cases could satisfy this condition, you can have quite complicated systems that are all-pass. Since $|A_{FT}(\omega)|^2 = A_{FT}(\omega)A^*_{FT}(\omega)$, this can be written in the transform domain as:

$$A\tilde{A} = 1$$

In the digital domain, suppose A has a pole at z_0. Since the right-hand side is just 1, then $\tilde{A}$ must have a zero at z_0 to cancel it out, but then A has a zero at

$1/z_0^*$. That is the poles and zeros have to occur in symmetric pairs, for example, if $1/2$ is a pole of A, then 2 is a zero. Similarly, in the analog case, if $A(s)$ has a pole at s, it must have a zero at $-s^*$ (so that $\tilde{A}(s)$ has a zero at s_0 to cancel the pole in $A(s)$). This gives the following formulas:

$$A(z) = cz^L \prod_{i=1}^{m} \frac{1 - \alpha_i^* z}{z - \alpha_i}$$

where c is a constant with magnitude 1, L is an integers, and α_i are complex numbers (with magnitude NOT 1). Also:

$$A(s) = c \prod_{i=1}^{m} \frac{s + \alpha_i^*}{s - \alpha_i}$$

1. The term z^L in $A(z)$ introduces pole-zero pairs WHERE? These are indeed symmetric points to the unit circle (having fun yet?)
2. By substituting in $z = e^{j\omega}$, check that:

$$\frac{1 - \alpha^* z}{z - \alpha}$$

   is all-pass. **Hint:** For any complex number $|\beta/\beta^*| = 1$. The reason it is written this way is because:

$$\frac{z - 1/\alpha^*}{z - \alpha}$$

   which has the pole-zero pairing that was promised above will have constant magnitude but it won't be 1.
3. By substituting $s = j\omega$, check that:

$$\frac{s + \alpha^*}{s - \alpha}$$

   is allpass.

Final remark on this. You may find the expressions, $H(1/z)$ and $H(-s)$, without the conjugation, in many textbooks. If the coefficients are real, as is often assumed, this will technically match the paraconjugate and in that sense is "correct". However, it hides the underlying symmetry of what is really going on (this again goes back to properties of analytic functions, and for example the pole-zero symmetry of all-pass functions is a special case of the Schwarz Reflection Principle in complex analysis).

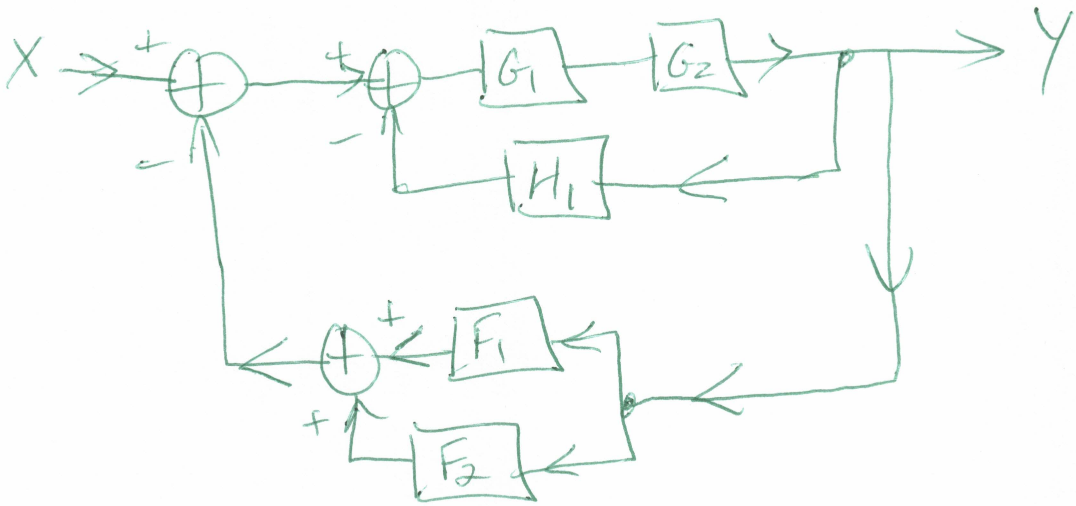

FIGURE 1: Block Diagram