The Cooper Union Department of Electrical Engineering
Prof. Fred L. Fontaine
ECE211 Signal Processing & Systems Analysis
Problem Set IV: Discrete-Time Signals & Systems
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1. Let $M \geq 2$ be an integer. For $0 \leq k \leq M-1$, the *polyphase* components of the signal $h[n]$ are defined via:
$$e_k[n] = h[Mn+k]$$
For example, for $M=2$, e_0 has the even samples of h, and e_1 has the odd samples of h. Let $e_k[n]$ have z-transform $E_k(z)$. We want to verify the following equation:
$$H(z) = \sum_{k=0}^{M-1} z^{-k} E_k\left(z^M\right)$$
Follow these steps to get the proof. First, let n be any integer. We want to show that in the above formula, there is only one term associated with z^{-n}, and that term is $h[n]$. Given n, there exist unique integers m, k with $0 \leq k \leq M-1$ such that $n = Mm+k$ (accept this as true). So now take m, k as given. Next:
   (a) Verify the coefficient z^{-n} term can ONLY occur in the $z^{-k}E_k\left(z^M\right)$ term in the above sum.
   (b) Find the UNIQUE coefficient $e_k[\cdot]$ that contributes to z^{-n} (in terms of m and/or k).
   (c) Show that this coefficient is in fact $h[n]$.

2. Given the following transfer function:
$$H(z) = \frac{2(z-3)(3z-2)^2}{(z-4)^2(2z+1)^2}$$
   (a) List all poles and zeros, with multiplicities (including any at infinity), and draw a pole-zero plot (by hand).
   (b) Identify all possible ROCs for $H(z)$.
   (c) Identify the ROC (if any) associated with: a causal system, a stable system, a system with a well-defined frequency response.
   (d) Specify the conditions on the integer L such that $z^L H(z)$ has a causal inverse transform.

3. Given the following FIR filter:

$$H(z) = 2 - 3z^{-1} + 4z^{-2} + 5z^{-3}$$

   (a) Sketch (by hand) $h[n]$.
   (b) Express $h[n]$ as a superposition of impulses.
   (c) Sketch a transversal filter realization.
   (d) Specify the length and order of the filter.

4. Given the following difference equation for an IIR filter:

$$y[n] = 3x[n] - 2x[n-1] + x[n-2] + 0.3y[n-1] - 0.4y[n-2]$$

   (a) Write $H(z)$ as a ratio of polynomials (no negative powers of z).
   (b) Sketch a direct form II realization.
   (c) Sketch a direct form II transposed realization.

5. For $M \geq 2$, let $W_M = \exp(-j2\pi/M)$. This is called the *twiddle factor*. Let $x[n]$ have z-transform $X(z)$ and DTFT $X_{DTFT}(\omega)$. Let:

$$Y(z) = X\left(zW_M^k\right)$$

   for some integer k. Find $Y_{DTFT}(\omega)$ (do this WITHOUT going to the time-domain).

6. The spectrum $X(\omega)$ of a discrete-time signal $x[n]$ is shown in **Figure 6.**

   (a) Sketch $X(3\omega)$ for $-\pi \leq \omega \leq \pi$. Is there a discrete-time signal whose DTFT is $X(3\omega)$?
   (b) Sketch $X\left(\frac{\omega}{2}\right)$, $X\left(\frac{\omega}{2} + \pi\right)$ and:

$$\frac{1}{2}\left[X\left(\frac{\omega}{2}\right) + X\left(\frac{\omega}{2} + \pi\right)\right]$$

   for $-\pi \leq \omega \leq \pi$ (all sketches by hand, on separate axes). Of these three, which (if any) correspond to a valid DTFT of a discrete-time signal?

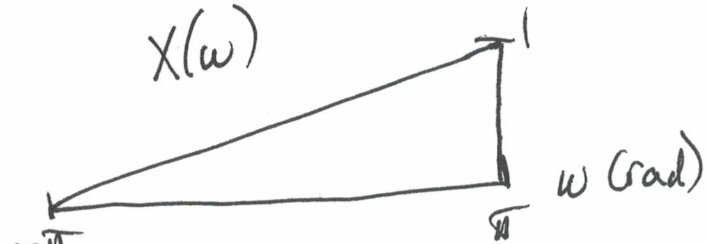

FIGURE 6: A DIGITAL SPECTRUM