FUDAN SUMMER SCHOOL INTRODUCTION TO AI

Probabilistic Reasoning and Decision Making

Day 3 – Bayes Nets and Inference on BN

Consider the following three events

- E: There was an earthquake
- B: A burglar broke into my apartment
- A: My alarm at my apartment went off

Assume that B and E are marginally independent

1. Multiple explanations for a single event

- Compare :
  - $P(B=1)$
  - $P(B=1|A=1)$
  - $P(B=1|A=1,E=1)$

Which of the following represents the most intuitive relationship between these probabilities?

$A.\ P(B=1) < P(B=1|A=1)$
$B.\ P(B=1) < P(B=1|A=1,E=1)$
$C.\ P(B=1) > P(B=1|A=1)$
$D.\ P(B=1) > P(B=1|A=1,E=1)$
E. More than one of these (which ones?)

1. Multiple explanations for a single event

- Compare :
  - $P(B = 1)$
  - $P(B = 1|A = 1)$
  - $P(B = 1|A = 1, E = 1)$

Which of the following represents the most intuitive relationship between these probabilities?

$A.$ $P(B = 1|A = 1) < P(B = 1|A = 1, E = 1)$

$B.$ $P(B = 1|A = 1) > P(B = 1|A = 1, E = 1)$

$C.$ $P(B = 1|A = 1) = P(B = 1|A = 1, E = 1)$

1. Multiple explanations for a single event

- Compare :
  - $P(B = 1)$
  - $P(B = 1|A = 1)$
  - $P(B = 1|A = 1, E = 1)$

Intuitively, we expect:
$P(B = 1) < P(B = 1|A = 1, E = 1) < P(B = 1|A = 1)$

I.e. "I generally believe there is a very low chance that a burglar breaks in. But if I get notified that my alarm is going off, then I think the alarm is probably caused by the burglar and my belief that a burglar broke in goes up. However, if I then hear about an earthquake, I no longer am as fearful that a burglar broke in, but I'm still more worried than I would have been if the alarm hadn't gone off at all"

1. Multiple explanations for a single event

$$\boxed{\begin{array}{l} P(B=1) \\ P(B=1|A=1) \\ P(B=1|A=1,E=1) \end{array}}$$

Domain knowledge:

Assumption: B and E are marginally independent: $P(B|E)=P(B)$, $P(E|B)=P(E)$

Prior probabilities:

$$P(B=1)=0.001 \quad P(B=0)=1-0.001=0.999$$
$$P(E=1)=0.005 \quad P(E=0)=1-0.005=0.995$$

Conditional probabilities (i.e. properties of the alarm):

B	E	$P(A=1\|B,E)$	$P(A=0\|B,E)$

1. Multiple explanations for a single event

$$\boxed{\begin{array}{l} P(B=1) \\ P(B=1|A=1) \\ P(B=1|A=1,E=1) \end{array}}$$

Domain knowledge:

Assumption: B and E are marginally independent: $P(B|E) = P(B)$, $P(E|B) = P(E)$

Prior probabilities:

$$P(B=1) = 0.001 \quad P(B=0) = 1 - 0.001 = 0.999$$
$$P(E=1) = 0.005 \quad P(E=0) = 1 - 0.005 = 0.995$$

Conditional probabilities (i.e. properties of the alarm):

B	E	$P(A=1\|B,E)$	$P(A=0\|B,E)$
0	0	0.002	0.998
0	1	.35	.65
1	0	.96	.04
1	1	.98	.02

2. Multiple events with a common explanation

Two new variables: J: Jamal calls, M: Maya calls

- Compare :
  - $P(A = 1)$
  - $P(A = 1|J = 1)$
  - $P(A = 1|M = 0, J = 1)$

If we assume J, M, B and E are conditionally independent given A, which of the following statements is true?

A. $P(J|A, M, B, E) = P(J|A)$
B. $P(J|A, M, B, E) = P(J)$
C. $P(J, M, B, E|A) = P(J|A)$
D. More than one of these

2. Multiple events with a common explanation

$P(A = 1) = 0.00469636$
$P(A = 1|J = 1)$
$P(A = 1|M = 0, J = 1)$

$P(A = 0) = 0.99530364$

Domain knowledge:

Assumptions:

- B and E are marginally independent: $P(B|E) = P(B), P(E|B) = P(E)$
- J, M conditionally independent from each other and from B and E, given A, e.g. $P(J|A, M, B, E) = P(J|A)$

Prior probabilities:

$P(B = 1) = 0.001 \quad P(B = 0) = 1 - 0.001 = 0.999$
$P(E = 1) = 0.005 \quad P(E = 0) = 1 - 0.005 = 0.995$

Conditional probabilities (i.e. properties of the alarm, Jamal and Maya):

B	E	$P(A = 1\|B, E)$	$P(A = 0\|B, E)$
0	0	0.002	0.998
0	1	0.35	0.65
1	0	0.96	0.04
1	1	0.98	0.02

$P(J = 1|A = 1) =$ $\quad P(J = 0|A = 1) =$
$P(J = 1|A = 0) =$ $\quad P(J = 0|A = 0) =$

$P(M = 1|A = 1) =$ $\quad P(M = 0|A = 1) =$
$P(M = 1|A = 0) =$ $\quad P(M = 0|A = 0) =$

3. Chain of Linked Events

- Compare :
  - $P(A = 1)$
  - $P(A = 1|J = 1)$
  - $P(A = 1|J = 1, B = 1)$

3. Chain of Linked Events

$P(A = 1) = 0.00469636$
$P(A = 1|J = 1) \approx 0.0420375$
$P(A = 1|J = 1, B = 1)$

$P(A = 0) = 0.99530364$

Domain knowledge:

Assumptions:

- B and E are marginally independent: $P(B|E) = P(B), P(E|B) = P(E)$
- J, M conditionally independent from each other and from B and E, given A, e.g. $P(J|A, M, B, E) = P(J|A)$

Prior probabilities:

$P(B = 1) = 0.001 \quad P(B = 0) = 1 - 0.001 = 0.999$
$P(E = 1) = 0.005 \quad P(E = 0) = 1 - 0.005 = 0.995$

Conditional probabilities (i.e. properties of the alarm, Jamal and Maya):

B	E	$P(A = 1\|B, E)$	$P(A = 0\|B, E)$
0	0	0.002	0.998
0	1	0.35	0.65
1	0	0.96	0.04
1	1	0.98	0.02

$P(J = 1|A = 1) = 0.93 \quad P(J = 0|A = 1) = 1 - 0.93 = 0.07$
$P(J = 1|A = 0) = 0.1 \quad P(J = 0|A = 0) = 1 - 0.1 = 0.9$

$P(M = 1|A = 1) = 0.65 \quad P(M = 0|A = 1) = 1 - 0.65 = 0.35$
$P(M = 1|A = 0) = 0.01 \quad P(M = 0|A = 0) = 1 - 0.01 = 0.99$

Reality check: Reasoning about multiple events

- Order the following probabilities (beliefs) from smallest to largest, using the domain knowledge from last class:

$$P(A=1)$$
$$P(A=1|J=1)$$
$$P(A=1|J=1,B=0)$$

A. $P(A=1) < P(A=1|J=1) < P(A=1|J=1,B=0)$
B. $P(A=1) < P(A=1|J=1,B=0) < P(A=1|J=1)$
C. $P(A=1|J=1,B=0) < P(A=1) < P(A=1|J=1)$
D. $P(A=1|J=1) < P(A=1|J=1,B=0) < P(A=1)$
E. None of these

Reality Check: Reasoning about multiple events

1. Multiple explanations (Burglary, Earthquake) for a single event (the Alarm):

$$P(B = 1) = 0.001$$
$$P(B = 1|A = 1) \approx 0.2044$$
$$P(B = 1|A = 1, E = 1) \approx 0.002795$$

2. Multiple events (Jamal calls, Maya calls) with a common explanation (the Alarm):

$$P(A = 1) = 0.00469636$$
$$P(A = 1|J = 1) \approx 0.0420375$$
$$P(A = 1|M = 0, J = 1) \approx 0.015277$$

3. Chain of linked events: Burglary causes Alarm causes Jamal to call

$$P(A = 1) = 0.00469636$$
$$P(A = 1|J = 1) \approx 0.0420375$$
$$P(A = 1|J = 1, B = 1) \approx 0.99555$$

Joint distribution

Assumptions about the world:

- B and E are marginally independent: $P(B|E) = P(B)$, $P(E|B) = P(E)$
- J, M conditionally independent from each other and from B and E, given A, e.g. $P(J|A, M, B, E) = P(J|A)$

$$P(B, E, A, J, M)$$

Which of the following also correctly represents the joint distribution?

A. $P(B)P(E|B)P(A|E, B)P(J|A, E, B)P(M|J, A, E, B)$

B. $P(B)P(E)P(A|E, B)P(J|A)P(M|A)$

C. $P(B)P(E)P(A)P(J)P(M)$

D. $P(B|A)P(E|A)P(A)P(J|A)P(M|A)$

E. More than one of these (which?)

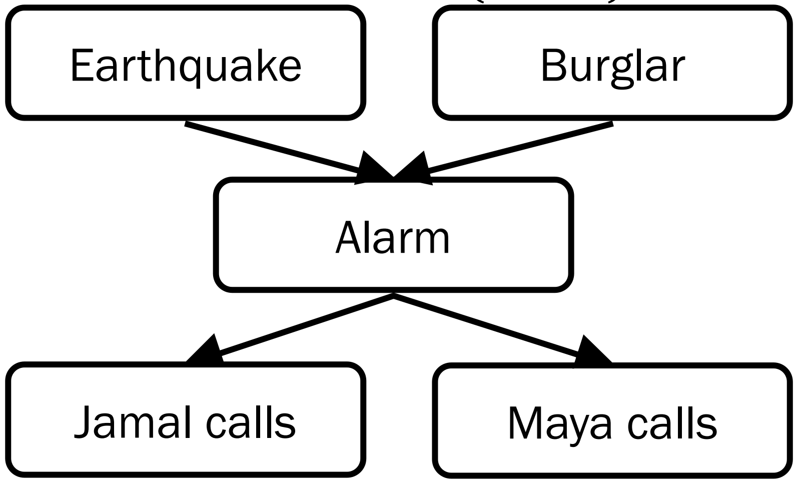
Joint distribution and Belief/Bayes Nets

Assumptions about the world:

- B and E are marginally independent: $P(B|E) = P(B)$, $P(E|B) = P(E)$
- J, M conditionally independent from each other and from B and E, given A, e.g. $P(J|A, M, B, E) = P(J|A)$

$$P(B,E,A,J,M) = P(B)P(E|B)P(A|E,B)P(J|A,E,B)P(M|J,A,E,B) = P(B)P(E)P(A|E,B)P(J|A)P(M|A)$$

$P(E = 1) = 0.001$ $P(B = 1) = 0.005$



B	E	$P(A = 1 \| B, E)$
0	0	0.002
0	1	0.35
1	0	0.96
1	1	0.98

A	$P(J = 1 \| A)$
0	0.1
1	0.93

A	$P(M = 1 \| A)$
0	0.01
1	0.65

A BN (Belief Network or Bayes Net) is a DAG in which:

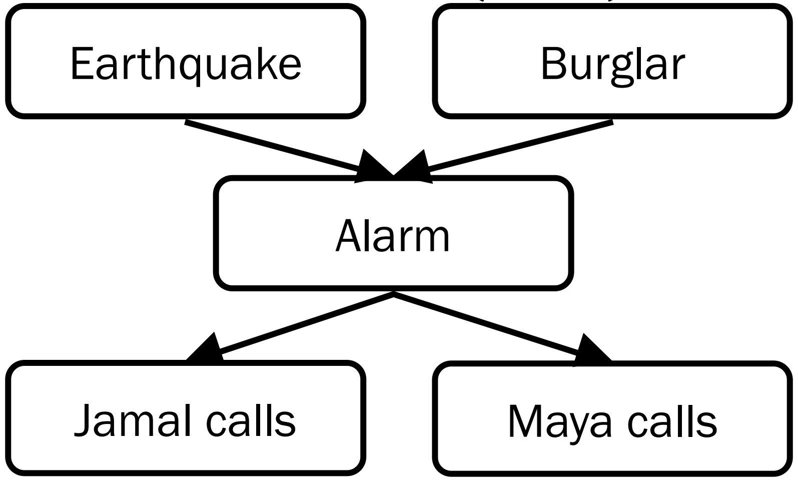
1.

2.

3.

Reasoning using Bayes Nets

$P(E = 1) = 0.001$ $P(B = 1) = 0.005$



B	E	$P(A = 1\|B, E)$
0	0	0.002
0	1	0.35
1	0	0.96
1	1	0.98

A	$P(J = 1\|A)$
0	0.1
1	0.93

A	$P(M = 1\|A)$
0	0.01
1	0.65

Which of the following probabilities is most straightforward to computing using the information in the Bayes net?

A. $P(A = 1, J = 0, M = 0)$

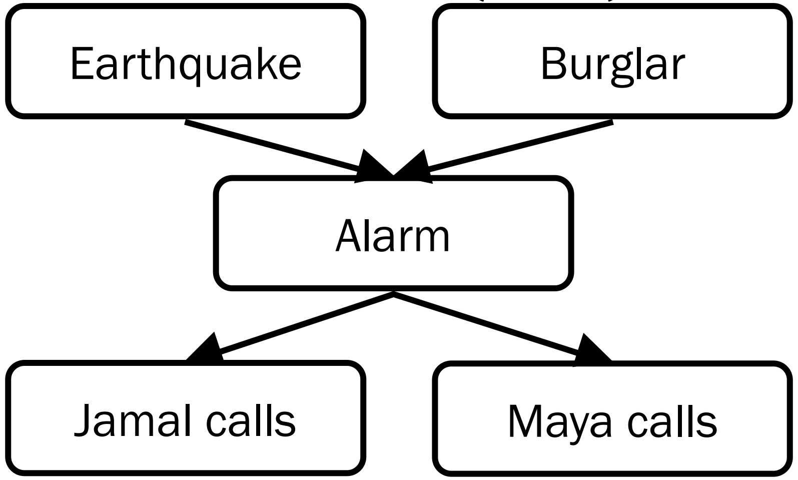
B. $P(A = 1, J = 0, M = 0 \mid B = 1, E = 0)$

C. $P(A = 1, J = 0, M = 0, B = 1, E = 0)$

D. $P(A = 1)$

Reasoning using Bayes Nets

$P(E = 1) = 0.001$ $\quad$ $P(B = 1) = 0.005$



B	E	$P(A = 1 \mid B, E)$
0	0	0.002
0	1	0.35
1	0	0.96
1	1	0.98

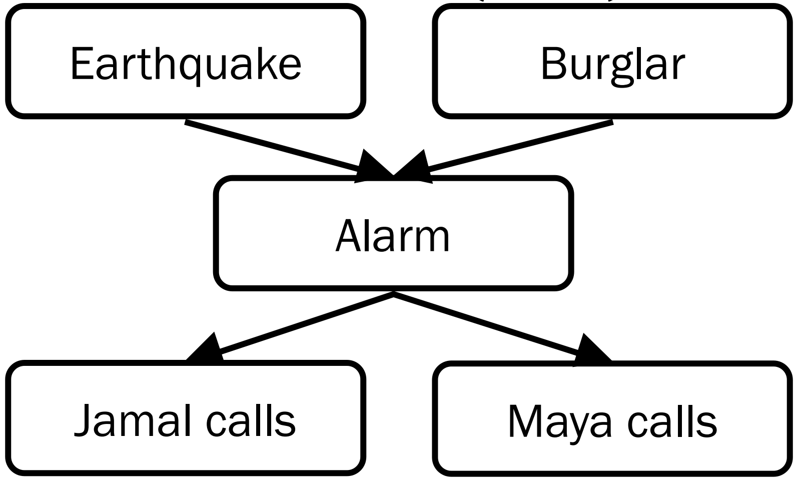
A	$P(J = 1 \mid A)$
0	0.1
1	0.93

A	$P(M = 1 \mid A)$
0	0.01
1	0.65

Bayes' Nets also give a straightforward way to calculate any probability:

Reasoning using Bayes Nets

$P(E = 1) = 0.001$ $\quad$ $P(B = 1) = 0.005$



A	$P(J = 1 \mid A)$
0	0.1
1	0.93

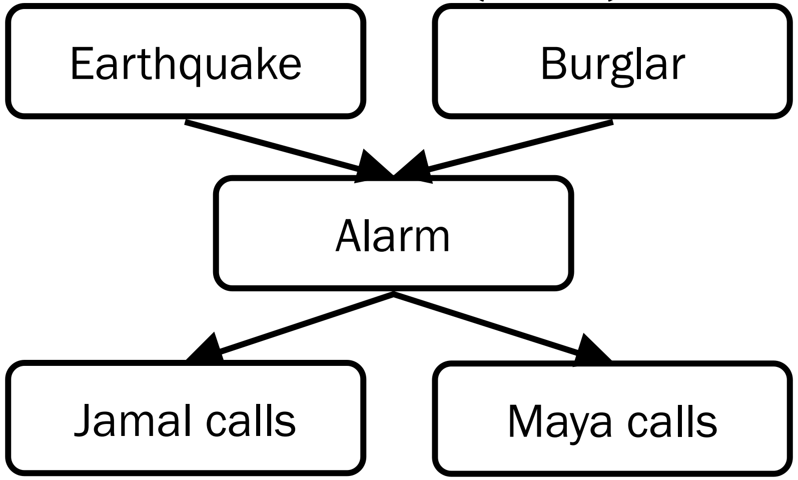
A	$P(M = 1 \mid A)$
0	0.01
1	0.65

B	E	$P(A = 1 \mid B, E)$
0	0	0.002
0	1	0.35
1	0	0.96
1	1	0.98

Bayes' Nets also give a straightforward way to calculate any probability:

Reasoning using Bayes Nets

$P(E = 1) = 0.001$ $P(B = 1) = 0.005$



B	E	$P(A = 1\|B, E)$
0	0	0.002
0	1	0.35
1	0	0.96
1	1	0.98

A	$P(J = 1\|A)$
0	0.1
1	0.93

A	$P(M = 1\|A)$
0	0.01
1	0.65

Bayes' Nets also give a straightforward way to calculate any probability:

Bayes' nets give you a compact way to represent joint probabilities

- Let $\{X_1, X_2, X_3, \ldots, X_n\}$ be n distinct binary random variables. (Approximately) how many numbers do you need to specify the full joint distribution over all X's?

  A. n
  B. $2n$
  C. 2^n
  D. n^n

Bayes' nets give you a compact way to represent joint probabilities

- Let $\{X_1, X_2, X_3, \ldots, X_n\}$ be n distinct binary random variables. (Approximately) how many values do you need to specify the full joint distribution over all X's? 2^n
- What does the number of values you need to specify in a Bayes' net depend on?
  A. The average number of parents a node has
  B. The maximum number of parents any node has
  C. The average number of children a node has
  D. The maximum number of children any node has

Conditional independence revisited

$$P(X_1, X_2, \dots, X_n) = P(X_1\ )P(X_2 | X_1\ ) \dots P(X_n | X_1, X_2, \dots, X_{n-1})$$
$$= \prod_{i=1}^{n} P(X_i | X_1, X_2, \dots, X_{i-1})$$

Suppose that in any given domain:

How to build a Bayes Net?

1. Choose random variables
2. Choose an ordering
3. While there are variables left:
    1. *Add node* X_i *to the BN*
    2. *Set parents of* X_i *to the minimal subset such that* X_i *is conditionally independent from non-parent nodes preceding i, given its parents.*
    3. *Define* $P(X_i|parent(X_i))$ *CPT*

How to build a Bayes Net?

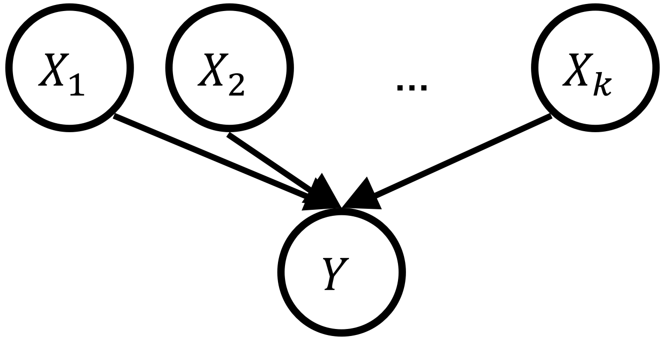
1. Choose random variables
2. Choose an ordering
3. While there are variables left:
    1. *Add node* X_i *to the BN*
    2. *Set parents of* X_i *to the minimal subset such that* X_i *is conditionally independent from non-parent nodes preceding i, given its parents.*
    3. *Define* $P(X_i|parent(X_i))$ *CPT*

Which is the better ordering to choose?

A. Root events first (i.e. causes), e.g. B, E, A, J, M
B. Child events first (i.e. consequences), e.g. M, J, A, E, B
C. It doesn't matter

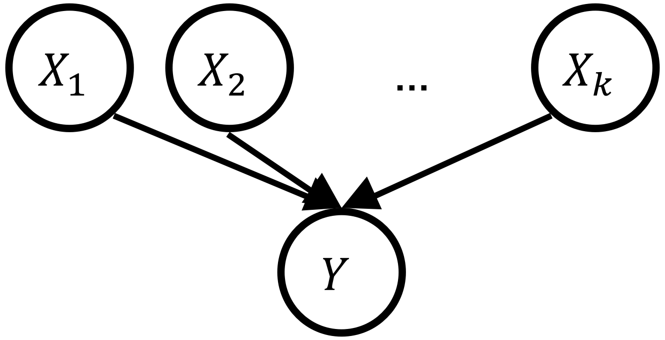
Representing CPTs

How do we represent the CPT?



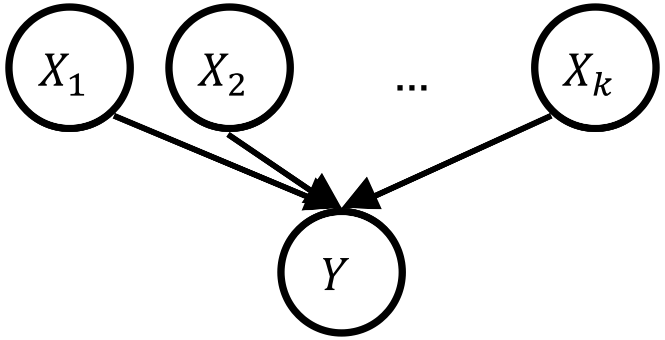
Representing CPTs

How do we represent the CPT?

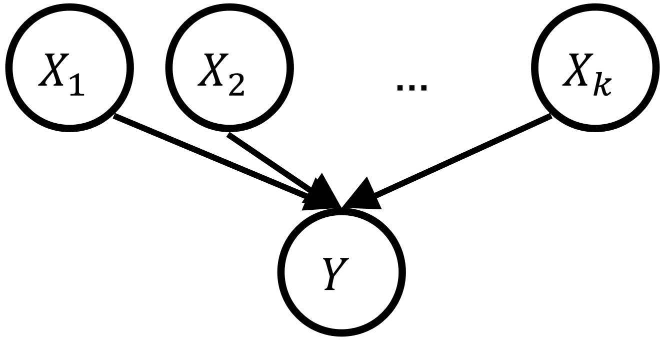


Representing CPTs

How do we represent the CPT?



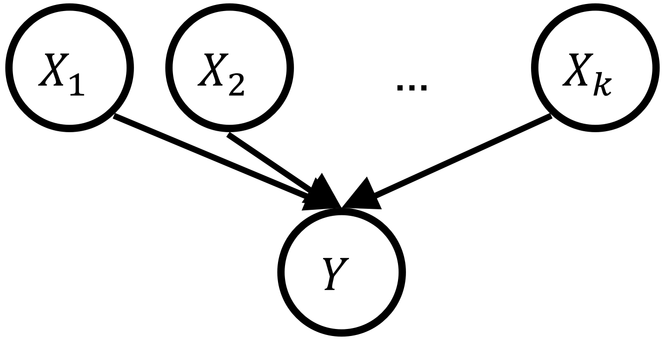
Noisy-OR



What is the value of $P(Y = 1 | X_1 = 0, X_2 = 0, \ldots, X_k = 0)$?

A. 0
B. p_i
C. 1
D. None of these
E. Not enough information given

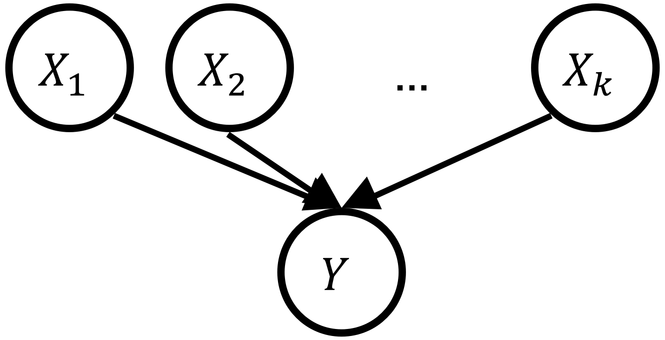
Noisy-OR



What is the value of $P(Y = 1 | X_1 = 0, X_2 = 0, \dots, X_i = 1, \dots, X_k = 0)$?

A. 0
B. p_i
C. 1
D. None of these
E. Not enough information given

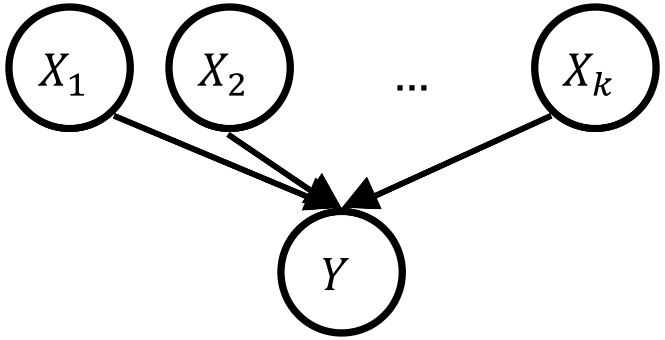
Noisy-OR



Let $P(Y = 1 | X_1 = 0, \ldots, X_i = 1, X_j = 1, \ldots, X_k = 0) = \frac{1}{3}$
and $p_i = \frac{1}{5}$
What is p_j?

A. $^1/_6$
B. $^2/_5$
C. $^1/_{15}$
D. $^2/_3$
E. I have no idea

Noisy-OR



Let $P\left(Y = 1 | X_1 = 0, \dots, X_i = 1, X_j = 1, \dots, X_k = 0\right) = \frac{1}{3}$

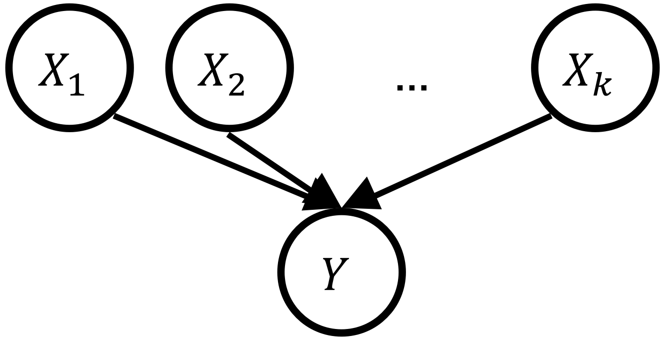
and $p_i = \frac{1}{5}$

What is p_j?

A. ${}^{1}/_{6}$
B. ${}^{2}/_{5}$
C. ${}^{1}/_{15}$
D. ${}^{2}/_{3}$
E. I have no idea

$$P(Y = 1 | X_1, \dots, X_k) = 1 - \prod_{i=1}^{k} (1 - p_i)^{X_i}$$

Noisy-OR

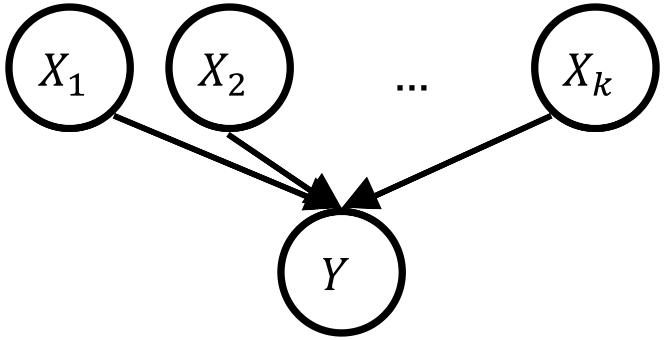


Is it true that P(Xi=1) = pi

A. Yes
B. NO

$$P(Y = 1 | X_1, \dots, X_k) = 1 - \prod_{i=1}^{k} (1 - p_i)^{X_i}$$

Noisy-OR: Intuition



$$P(Y = 1|X_1, \dots, X_k) = 1 - \prod_{i=1}^{k} (1 - p_i)^{X_i}$$

Let $P(Y = 1|X_1 = 1, X_2 = 1, \dots, X_{i-1} = 1, X_i = 0, \dots, X_k = 0) = a$
and $P(Y = 1|X_1 = 1, X_2 = 1, \dots, X_{i-1} = 1, X_i = 1, X_{i+1} = 0, \dots, X_k = 0) = b$
What is the relationship between a and b?

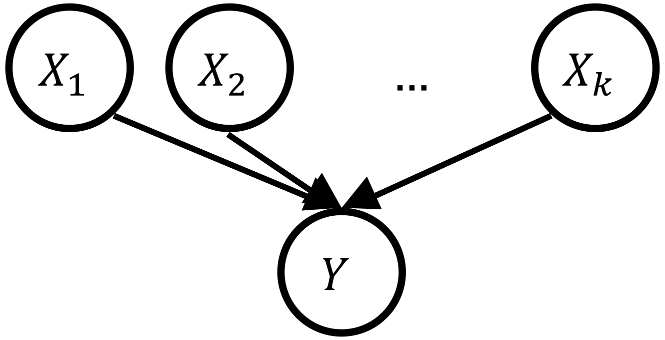
A. $a \leq b$

B. $a = b$

C. $a \geq b$

D. There is not enough information to determine the relationship

Noisy-OR: Intuition



$$P(Y = 1 | X_1, \dots, X_k) = 1 - \prod_{i=1}^{k} (1 - p_i)^{X_i}$$

- When all parents are off, child is off with certainty (just like OR).
- Each parent that turns on increases (or at least does not decrease) the probability that the child will turn on.
- When all the parents are on, it still might not be certain that the child is on, but it is probably very likely.

Bayesian Networks: Network structure and conditional independence

- By our construction of a Bayes' net, we know that a node is conditionally independent of its non-parent ancestors given its parents:

$$P(X_1, X_2, \dots, X_n) = P(X_1)P(X_2|X_1) \dots P(X_n|X_1, X_2, \dots, X_{n-1})$$

$$= \prod_{i=1}^{n} P(X_i|X_1, X_2, \dots, X_{i-1})$$

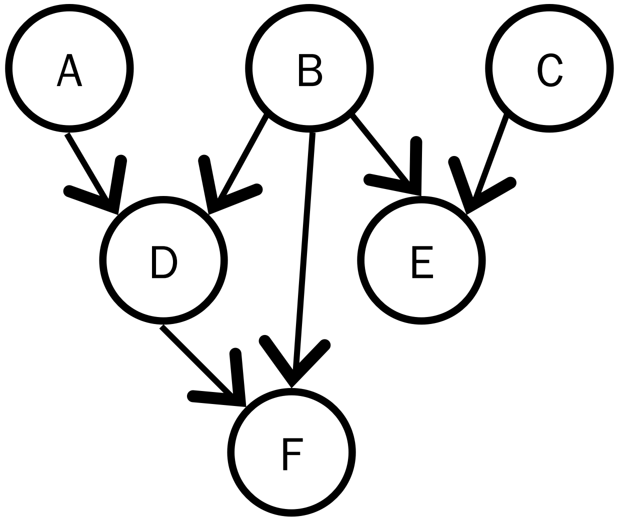
$$= \prod_{i=1}^{n} P\left(X_i|parents(X_i)\right)$$

But what about more general assertions of independence?

Bayesian Networks: Network structure and conditional independence

- Let X, Y and E refer to disjoint sets of nodes. When is X conditionally independent of Y given E (evidence)?

Write three equalities using $P(X|Y,E)$, $P(Y|X,E)$, $P(X,Y|E)$, etc, that indicate this conditional independence.

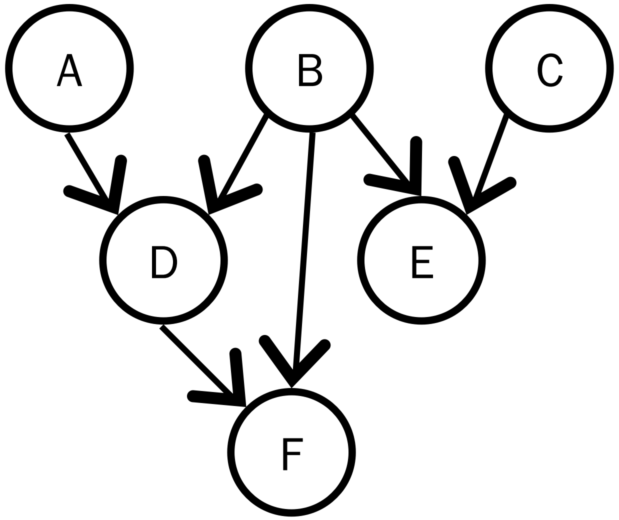


Bayesian Networks: Network structure and conditional independence

- Let X, Y and E refer to disjoint sets of nodes. When is X conditionally independent of Y given E (evidence)?

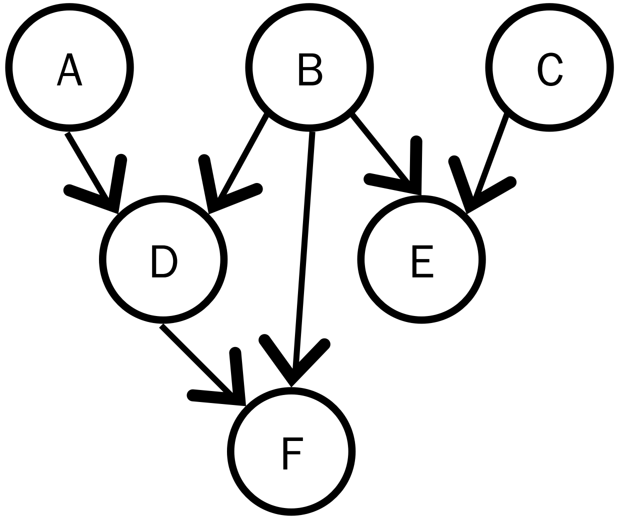
<ins>Theorem</ins>: $P(X,Y|E) = P(X|E)P(Y|E)$ if and only if every UNDIRECTED path from a node in X to a node in Y is "d-separated" ("blocked") by E.

(we will not prove this)



d-separation ("dependence" separation)

Definition: A path π (sequence of nodes connected by edges) is d-separated if there exists some node $z \in \pi$ for which one of the following three conditions hold:



Conditional Independence

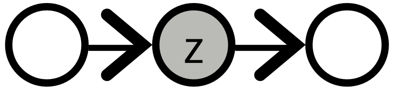
$P(B|A,M)=P(B|A)$

A. True

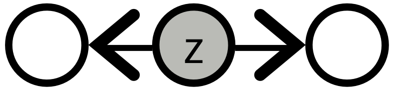
B. False

Theorem: $P(X,Y|E) = P(X|E)P(Y|E)$ if and only if **every** UNDIRECTED path from a node in X to a node in Y is "d-separated" ("blocked") by E.

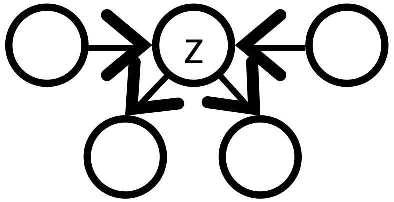
Definition: A path π (sequence of nodes connected by edges) is d-separated if there exists some node $z \in \pi$ for which **one** of the following three conditions hold:

1. 

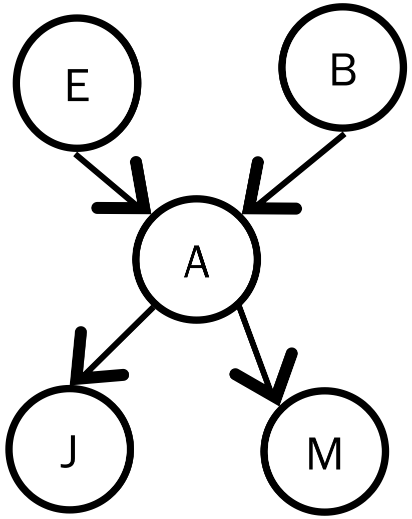
$z \in E$ (observed event in causal chain)

2. 

$z \in E$ (observed common explanation)

3. 

$z \notin E, descendants(z) \notin E$ (no observed common effect)



Conditional Independence

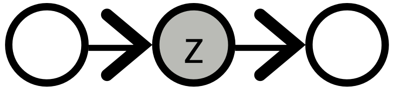
$P(J,M|A)=P(J|A)P(M|A)$

A. True

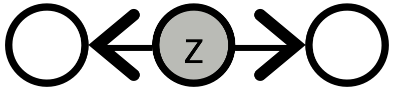
B. False

Theorem: $P(X,Y|E) = P(X|E)P(Y|E)$ if and only if **every** UNDIRECTED path from a node in X to a node in Y is "d-separated" ("blocked") by E.

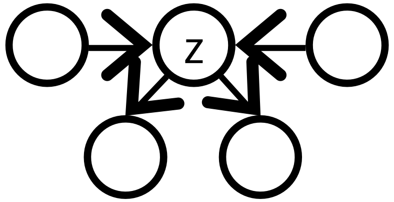
Definition: A path π (sequence of nodes connected by edges) is d-separated if there exists some node $z \in \pi$ for which **one** of the following three conditions hold:

1. 

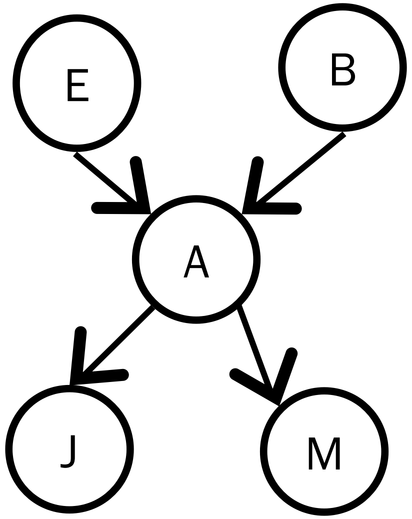
$z \in E$ (observed event in causal chain)

2. 

$z \in E$ (observed common explanation)

3. 

$z \notin E, descendants(z) \notin E$ (no observed common effect)



Conditional Independence

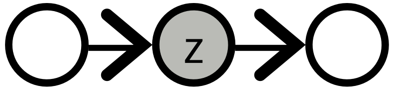
$P(B|E,J)=P(B|J)$

A. True

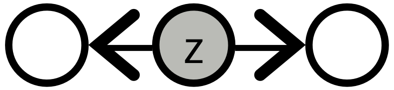
B. False

Theorem: $P(X,Y|E) = P(X|E)P(Y|E)$ if and only if **every** UNDIRECTED path from a node in X to a node in Y is "d-separated" ("blocked") by E.

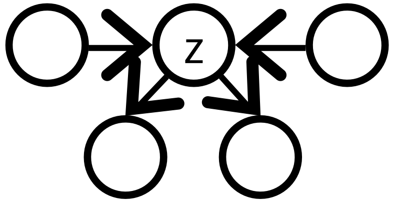
Definition: A path π (sequence of nodes connected by edges) is d-separated if there exists some node $z \in \pi$ for which **one** of the following three conditions hold:

1. 

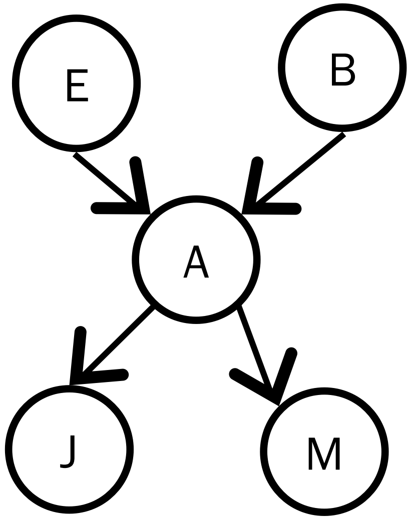
$z \in E$ (observed event in causal chain)

2. 

$z \in E$ (observed common explanation)

3. 

$z \notin E, descendants(z) \notin E$ (no observed common effect)

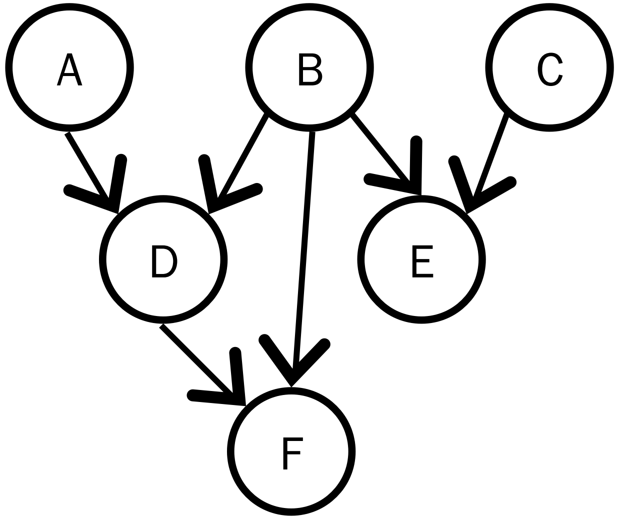


Conditional Independence

$P(A, B|D) = P(A|D)P(B|D)$?

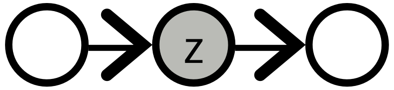
A. True

B. False

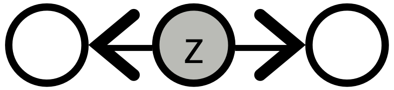


Theorem: $P(X, Y|E) = P(X|E)P(Y|E)$ if and only if **every** UNDIRECTED path from a node in X to a node in Y is "d-separated" ("blocked") by E.

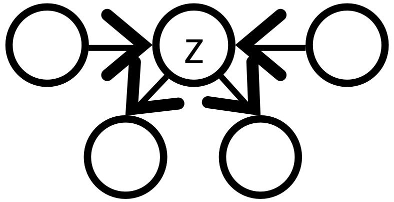
Definition: A path π (sequence of nodes connected by edges) is d-separated if there exists some node $z \in \pi$ for which **one** of the following three conditions hold:

1. 

$z \in E$ (observed event in causal chain)

2. 

$z \in E$ (observed common explanation)

3. 

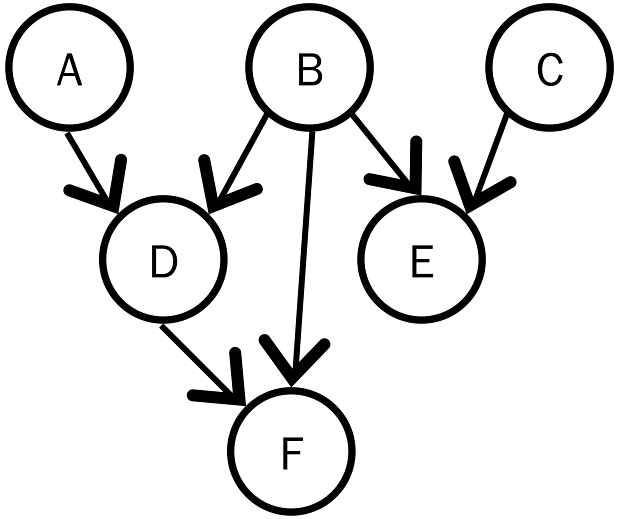
$z \notin E, descendants(z) \notin E$ (no observed common effect)

Conditional Independence

$P(D,E|B) = P(D|B)P(E|B)$?

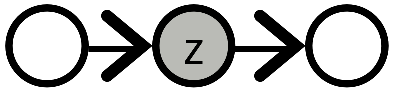
A. True

B. False

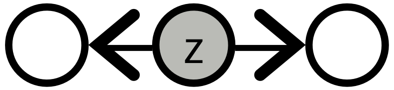


Theorem: $P(X,Y|E) = P(X|E)P(Y|E)$ if and only if **every** UNDIRECTED path from a node in X to a node in Y is "d-separated" ("blocked") by E.

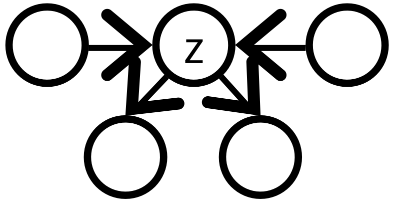
Definition: A path π (sequence of nodes connected by edges) is d-separated if there exists some node $z \in \pi$ for which **one** of the following three conditions hold:

1. 

$z \in E$ (observed event in causal chain)

2. 

$z \in E$ (observed common explanation)

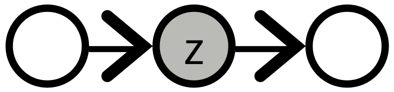
3. 

$z \notin E, descendants(z) \notin E$ (no observed common effect)

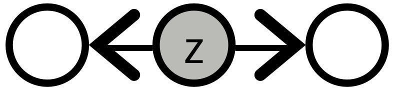
Conditional Independence

Theorem: $P(X, Y|E) = P(X|E)P(Y|E)$ if and only if **every** UNDIRECTED path from a node in X to a node in Y is "d-separated" ("blocked") by E.

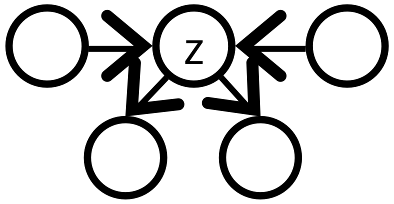
Definition: A path π (sequence of nodes connected by edges) is d-separated if there exists some node $z \in \pi$ for which **one** of the following three conditions hold:

1. 

$z \in E$ (observed event in causal chain)

2. 

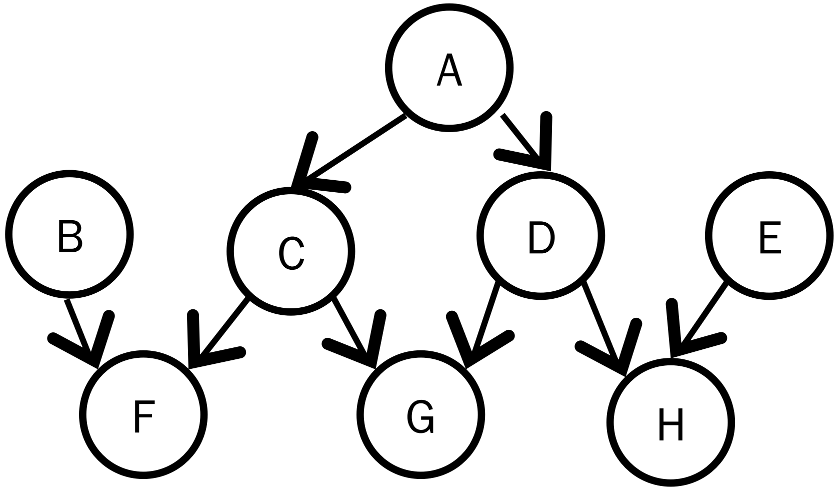
$z \in E$ (observed common explanation)

3. 

$z \notin E, descendants(z) \notin E$ (no observed common effect)

$\boxed{P(F, H|A, E) = P(F|A)P(H|A, E)?}$

A. X={F}, Y={H}, Ev={A, E}
B. X={F, H}, Y={E}, Ev={A}
C. X={F}, Y={H, E}, Ev={A}
D. X={F, H, E}, Y={A}, Ev={}

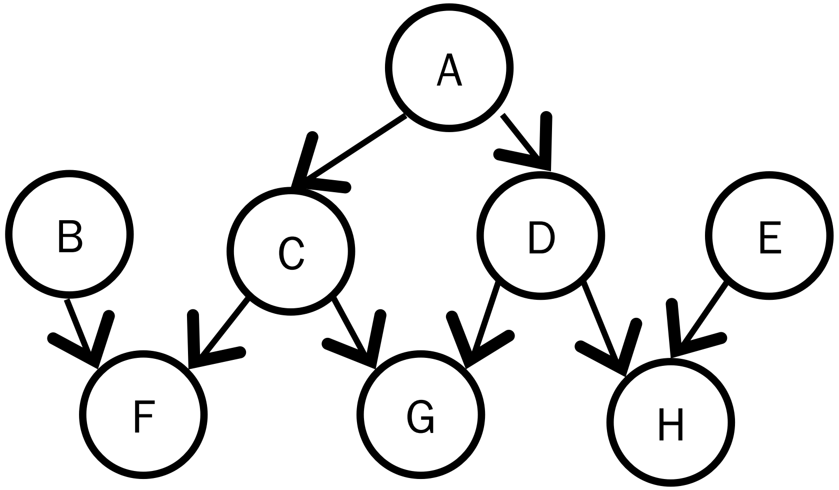


Conditional Independence

$\boxed{P(F, H|A, E) = P(F|A)P(H|A, E)?}$

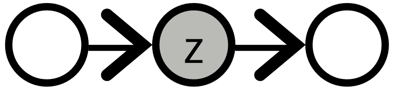
X={F}, Y={H, E}, Ev={A}

A. True
B. False

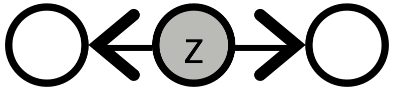


<u>Theorem</u>: $P(X, Y|E) = P(X|E)P(Y|E)$ if and only if **every** UNDIRECTED path from a node in X to a node in Y is "d-separated" ("blocked") by E.

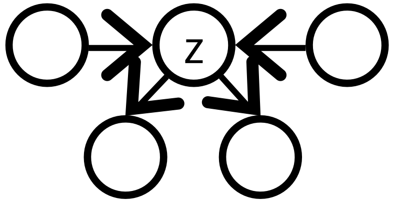
<u>Definition</u>: A path π (sequence of nodes connected by edges) is d-separated if there exists some node $z \in \pi$ for which **one** of the following three conditions hold:

1. 

$z \in E$ (observed event in causal chain)

2. 

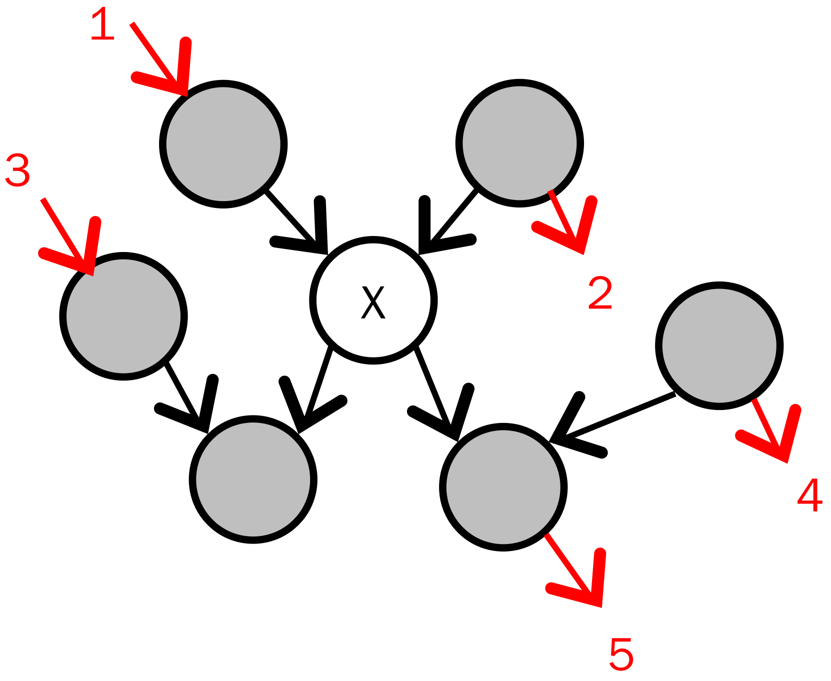
$z \in E$ (observed common explanation)

3. 

$z \notin E, descendants(z) \notin E$ (no observed common effect)

Markov Blanket

- A Markov Blanket B_x of node X consists of parents of X, children of X and "spouses" (other parents of children of X, but not X) of X.



Theorem: $P(X|B_x, Y) = P(X|B_x)$
$if\ Y \notin \{X, B_x\}$

Prove this theorem by showing that B_x d-separates each path to X, indicated by the 5 red arrows.