FUDAN SUMMER SCHOOL INTRODUCTION TO AI

Probabilistic Reasoning and Decision Making

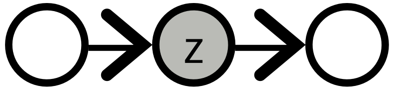
Day 5 – d-separation, variable elimination

Conditional Independence

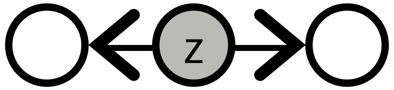
$P(J,M|A)=P(J|A)P(M|A)$
A. True
B. False

Theorem: $P(X,Y|E) = P(X|E)P(Y|E)$ if and only if **every** UNDIRECTED path from a node in X to a node in Y is "d-separated" ("blocked") by E.

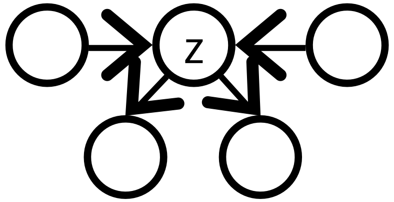
Definition: A path π (sequence of nodes connected by edges) is d-separated if there exists some node $z \in \pi$ for which **one** of the following three conditions hold:

1. 

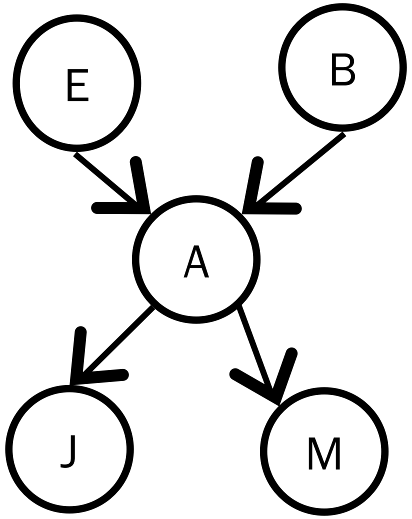
$z \in E$ (observed event in causal chain)

2. 

$z \in E$ (observed common explanation)

3. 

$z \notin E, descendants(z) \notin E$ (no observed common effect)

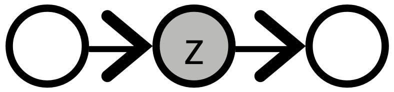
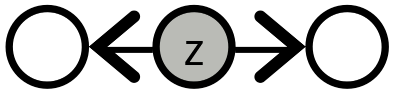
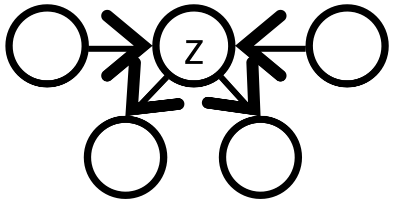


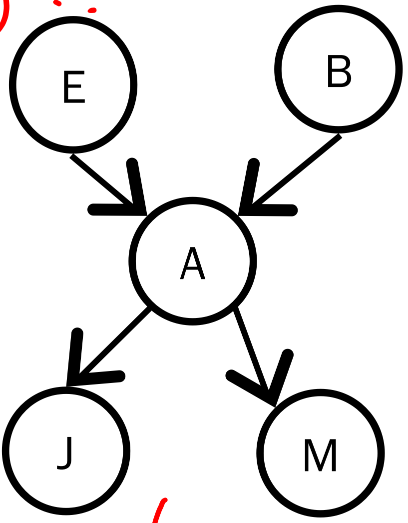
Conditional Independence

$P(B|E,J)=P(B|J)$
A. True
B. False

Theorem: $P(X,Y|E) = P(X|E)P(Y|E)$ if and only if **every** UNDIRECTED path from a node in X to a node in Y is "d-separated" ("blocked") by E.

Definition: A path π (sequence of nodes connected by edges) is d-separated if there exists some node $z \in \pi$ for which **one** of the following three conditions hold:

1.  $z \in E$ (observed event in causal chain)
2.  $z \in E$ (observed common explanation)
3.  $z \notin E, descendants(z) \notin E$ (no observed common effect)



$P(E|B)=P(E)$??
$X: \{E\}$, $Y: \{B\}$
$E: \{\phi\}$

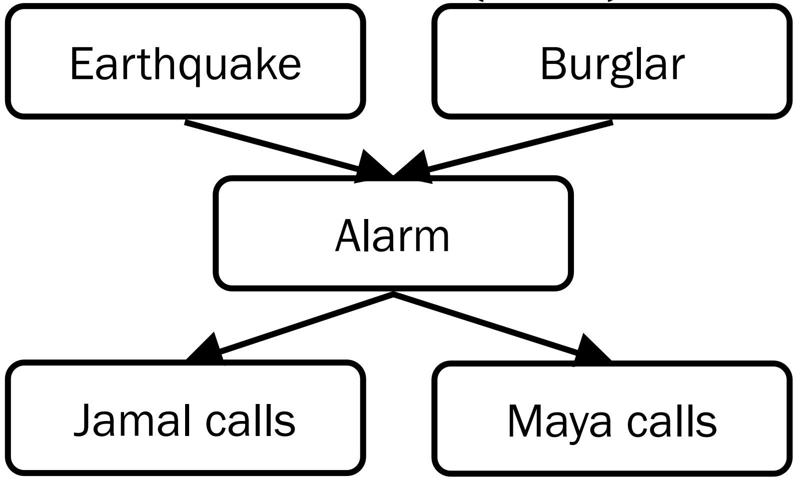
$E \rightarrow A \leftarrow B$
$A \rightarrow J$, $A \rightarrow M$
X Y

$E \rightarrow A \leftarrow B$
$A \rightarrow J$, $A \rightarrow M$
X Y
J M
Evidence
Violates Condition 3

Conditional Independence

$X: \{B\}, \; Y: \{E\}, \; C: \{J, M\}$

$P(E=1) = 0.005$ $\quad$ $P(B=1) = 0.001$



B	E	$P(A=1 \mid B,E)$
0	0	0.002
0	1	0.35
1	0	0.96
1	1	0.98

A	$P(J=1 \mid A)$
0	0.1
1	0.93

A	$P(M=1 \mid A)$
0	0.01
1	0.65

$P(B \mid E, J, M) = P(B \mid J, M)$?

A. True

B. False

Violates conditions

Conditional Independence

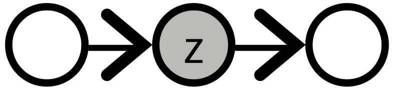
$P(A,B|D) = P(A|D)P(B|D)$?

A. True

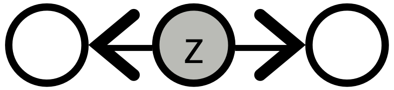
B. False

Theorem: $P(X,Y|E) = P(X|E)P(Y|E)$ if and only if **every** UNDIRECTED path from a node in X to a node in Y is "d-separated" ("blocked") by E.

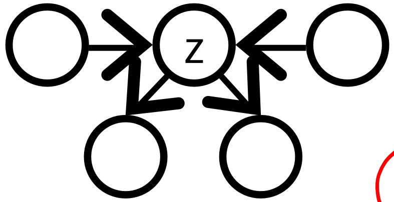
Definition: A path π (sequence of nodes connected by edges) is d-separated if there exists some node $z \in \pi$ for which **one** of the following three conditions hold:

1. 

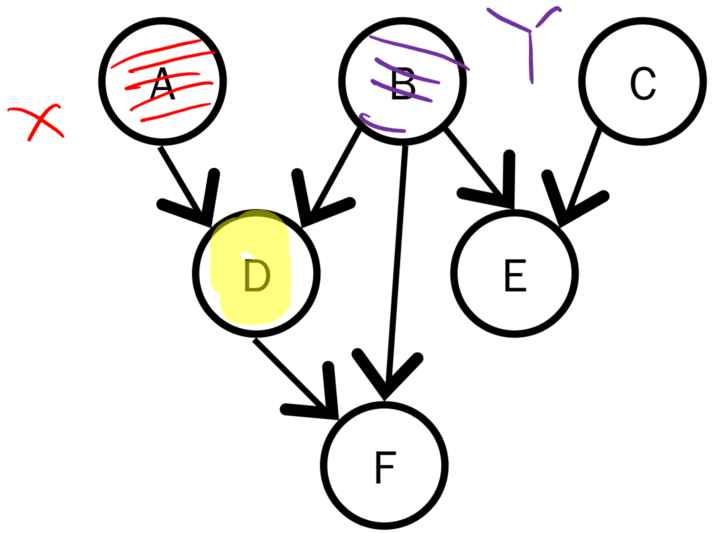
$z \in E$ (observed event in causal chain)

2. 

$z \in E$ (observed common explanation)

3. 

$z \notin E, descendants(z) \notin E$ (no observed common effect)



$X: \{A\}, Y: \{B\}, E: \{D\}$

① $A \rightarrow D \leftarrow B$ Violates 3

② $A \rightarrow D \rightarrow F \leftarrow B$ condition 1

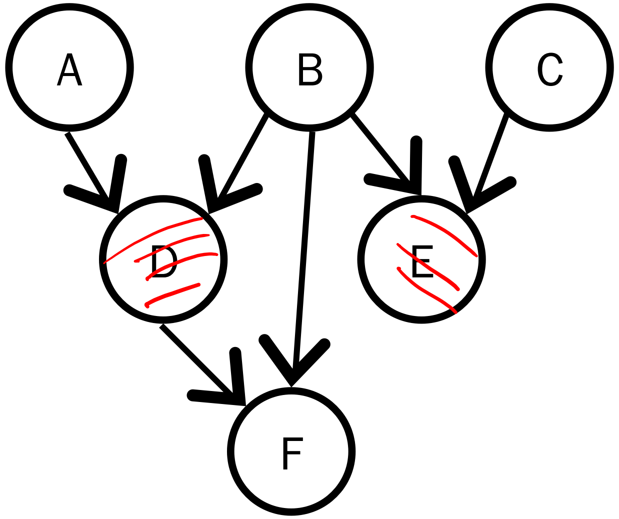
X Y

Conditional Independence

$P(D,E|B) = P(D|B)P(E|B)$?

A. True

B. False



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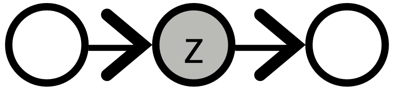
1. 

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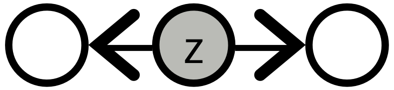
2. 

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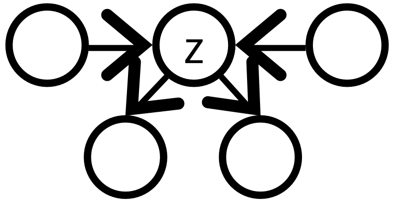
3. 

$z \notin E, descendants(z) \notin E$ (no observed common effect)

X: {D}, Y: {E}

E: {B}

$D \leftarrow B \rightarrow E$ condition 2 ✓

$D \rightarrow F \leftarrow B \rightarrow E$ condition 2 ✓ condition 3 ✓

P(F,H | A,E) = P(H | AE) P(F | HAE)

$P(F,H|A,E) = P(F|A)P(H|A,E)?$

Conditional Independence

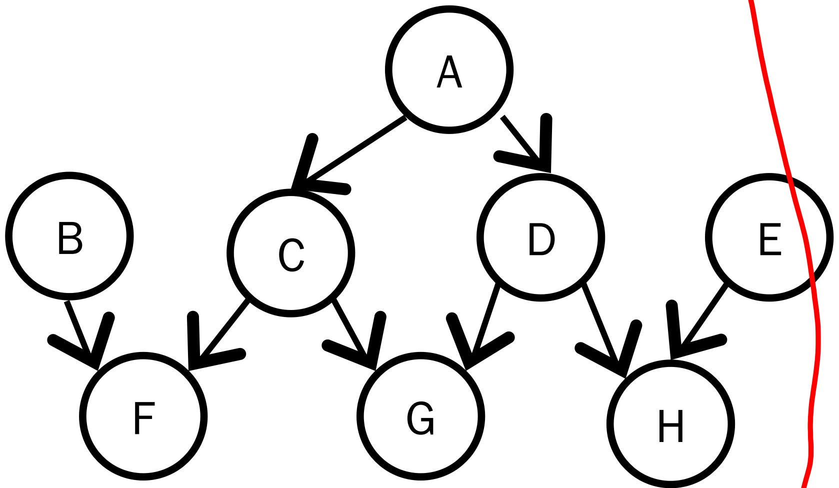
P(F | HAE) = P(F | A)

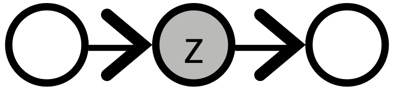
A. X={F}, Y={H}, Ev={A, E}
B. X={F, H}, Y={E}, Ev={A}
C. X={F}, Y={H, E}, Ev={A}
D. X={F, H, E}, Y={A}, Ev={}

Theorem: $P(X,Y|E) = P(X|E)P(Y|E)$ if and only if **every** UNDIRECTED path from a node in X to a node in Y is "d-separated" ("blocked") by E.

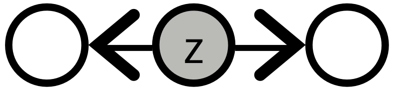
X: {F}, Y: {HE}, E: {A}

Definition: A path π (sequence of nodes connected by edges) is d-separated if there exists some node $z \in \pi$ for which **one** of the following three conditions hold:



1. 

$z \in E$ (observed event in causal chain)

2. 

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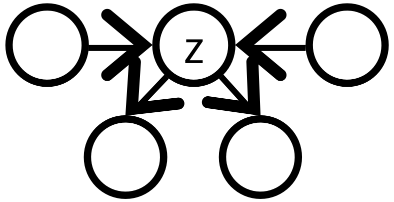
3. 

$z \notin E, descendants(z) \notin E$ (no observed common effect)

P(F,H | A,E) $\stackrel{\text{Prd}}{=}$ P(F | AE) P(H | FAE)

if P(H | AE) = P(H | FAE)

x: {H} Y: {F}, E: {A, E}

if P(F | AE) = P(F | A)

X: {F} Y: {E}

E: {A}

stnd

Conditional Independence

$\boxed{P(F,H|A,E) = P(F|A)P(H|A,E)?}$

X={F}, Y={H, E}, Ev={A}

A. True
B. False

Theorem: $P(X,Y|E) = P(X|E)P(Y|E)$ if and only if **every** UNDIRECTED path from a node in X to a node in Y is "d-separated" ("blocked") by E.

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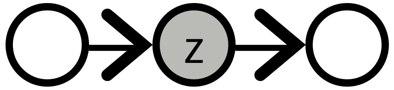
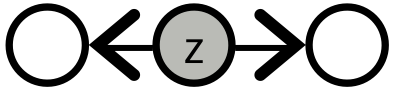
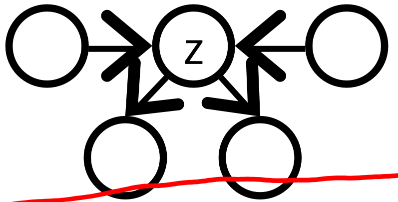
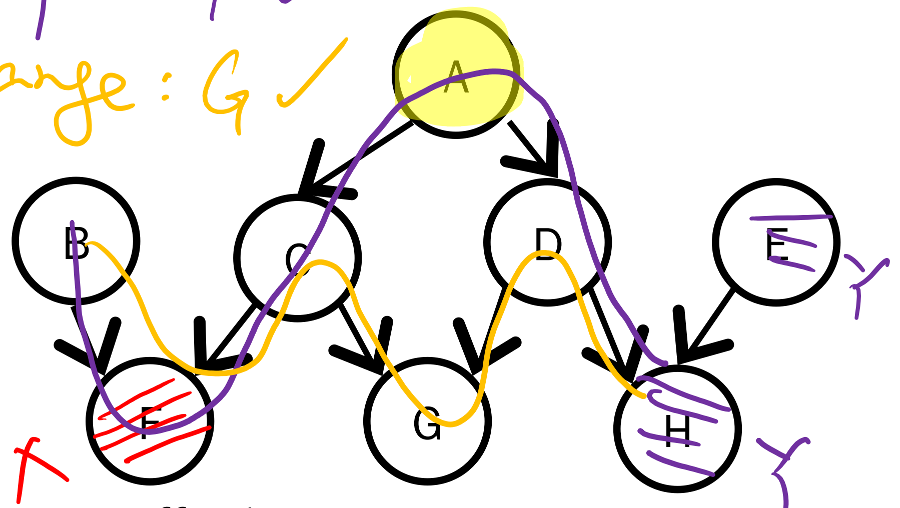
1.  $z \in E$ (observed event in causal chain)
2.  $z \in E$ (observed common explanation)
3.  $z \notin E, descendants(z) \notin E$ (no observed common effect)

purple : F ✓

orange : G ✓



$F \leftarrow C \rightarrow G \leftarrow D \rightarrow H$ ✓ condition 3

$F \leftarrow C \leftarrow A \rightarrow D \rightarrow H$ ✓ condition 2

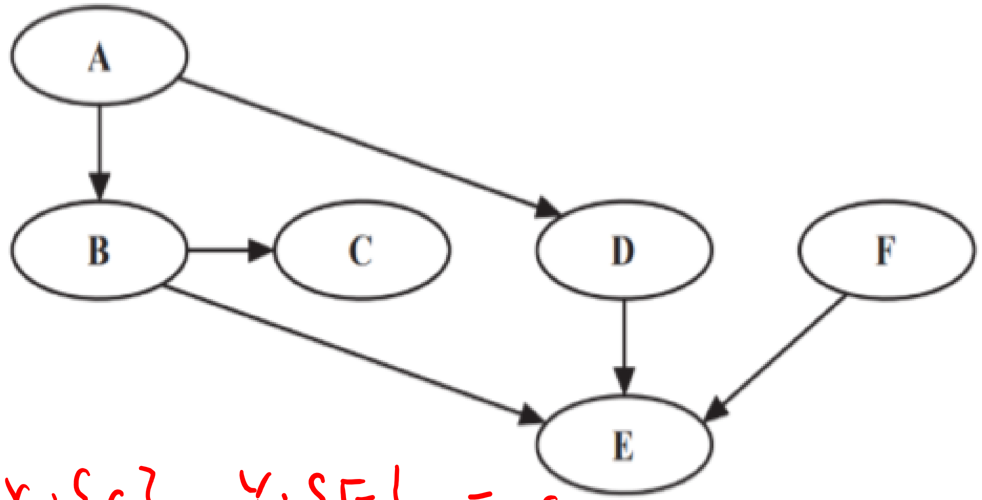
$F \leftarrow C \rightarrow G \leftarrow D \rightarrow H \leftarrow E$ ✓ condition 3

$F \leftarrow C \leftarrow A \rightarrow D \rightarrow H \leftarrow E$ ✓ condition 2

$P(B|H) = P(B)$

A: true

B: false



(a) Does D d-separate C and F? True A: yes B: no

(b) Do D and E d-separate C and F? No

(c) Write down all pairs of nodes which are independent of each other.

(d) Which pairs of nodes are independent of each other given B?

(e) $P(A, F|E) = P(A|E)P(F|E)$?

a) X: {C}, Y: {F}, E: {D}

$C \leftarrow B \leftarrow A \rightarrow D \rightarrow E \leftarrow F$: Condition 1 ✓ or 3 ✓

$C \leftarrow B \rightarrow \underline{E} \leftarrow F$: condition 3 ✓

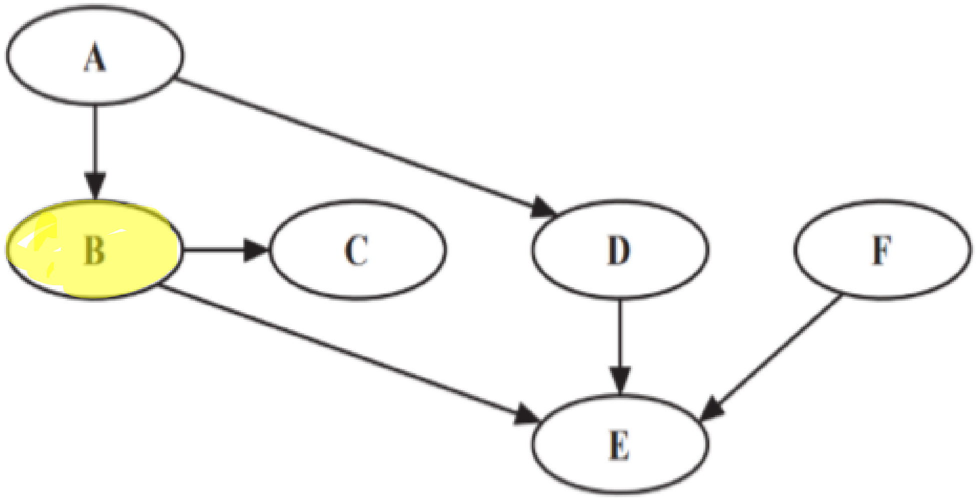
b) condition 1 ✓

$C \leftarrow B \leftarrow A \rightarrow \underline{D} \rightarrow E \leftarrow F$

X $C \leftarrow B \rightarrow E \leftarrow F$ violates 3

c) Common sink E:

AF, BF, CF, DF



(a) Does D d-separate C and F?

(b) Do D and E d-separate C and F?

(c) Write down all pairs of nodes which are independent of each other.

(d) Which pairs of nodes are independent of each other given B? E

(e) $P(A, F|E) = P(A|E)P(F|E)$?

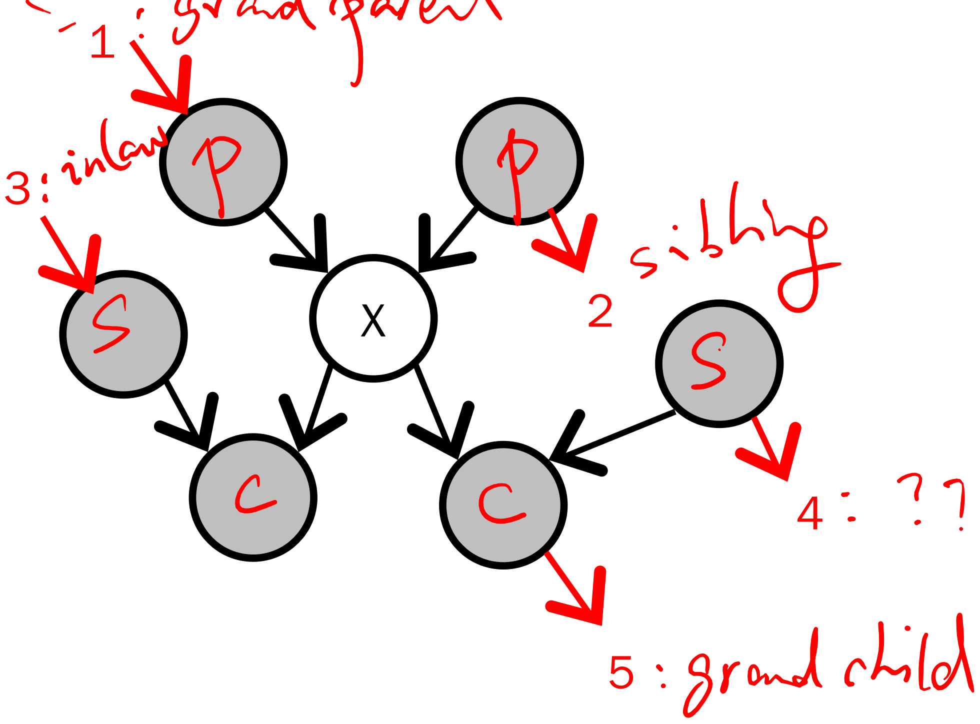
A: True

B: False

d) AF, CF, DF, AC, CD, CE

Markov Blanket

- A Markov Blanket B_x of node X consists of parents of X, children of X and "spouses" (other parents of children of X, but not X) of X.

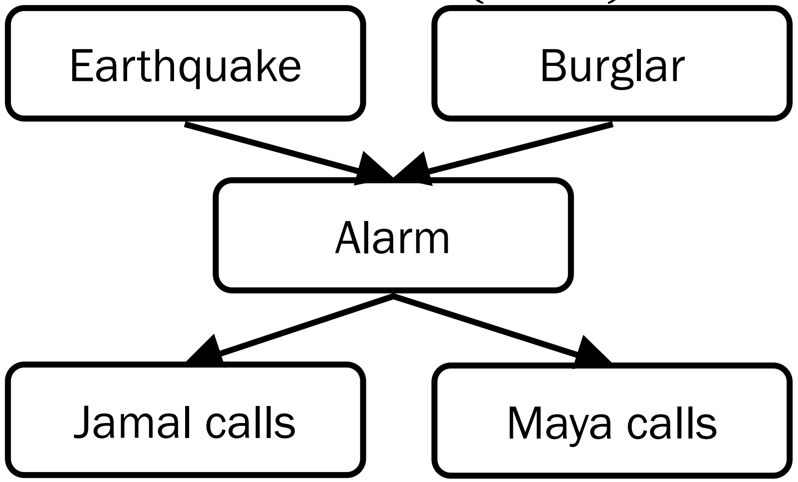


Theorem: $P(X|B_x, Y) = P(X|B_x)$
$if\ Y \notin \{X, B_x\}$

Prove this theorem by showing that B_x d-separates each path to X, indicated by the 5 red arrows.

Inference in Bayes Nets: General Approach

$P(E = 1) = 0.005$ $\quad$ $P(B = 1) = 0.001$



B	E	$P(A = 1 \mid B, E)$
0	0	0.002
0	1	0.35
1	0	0.96
1	1	0.98

A	$P(J = 1 \mid A)$
0	0.1
1	0.93

A	$P(M = 1 \mid A)$
0	0.01
1	0.65

Compute $P(B \mid J = 1, M = 1)$

$$= P(B \mid J=1, M=1)$$

$$\overset{def}{=} \frac{P(B, J=1, M=1)}{P(J=1, M=1)}$$

Margi

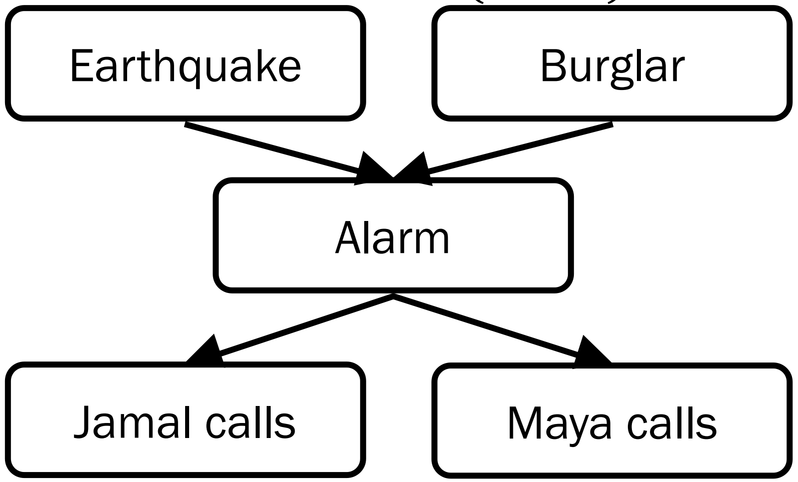
$$= \sum_{e_i} \sum_{a_j} P(B, J=1, M=1, E=e_i, A=a_j)$$

$$P(B=0, J=1, M=1) + P(B=1, J=1, M=1)$$

$$P(B \mid J=1, M=1) = \frac{P(B, J=1, M=1)}{P(B=0, J=1, M=1) + P(B=1, J=1, M=1)}$$

Inference in Bayes Nets: Enumeration

$P(E=1)=0.005$ $P(B=1)=0.001$



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Compute $P(B \mid J=1, M=1)$

Using a general approach:

$$P(B \mid J=1,M=1) = \frac{P(B,J=1,M=1)}{P(J=1,M=1)}$$

$$= \frac{\sum_a \sum_e P(B,J=1,M=1,A=a,E=e)}{\sum_b P(B=b,J=1,M=1)}$$

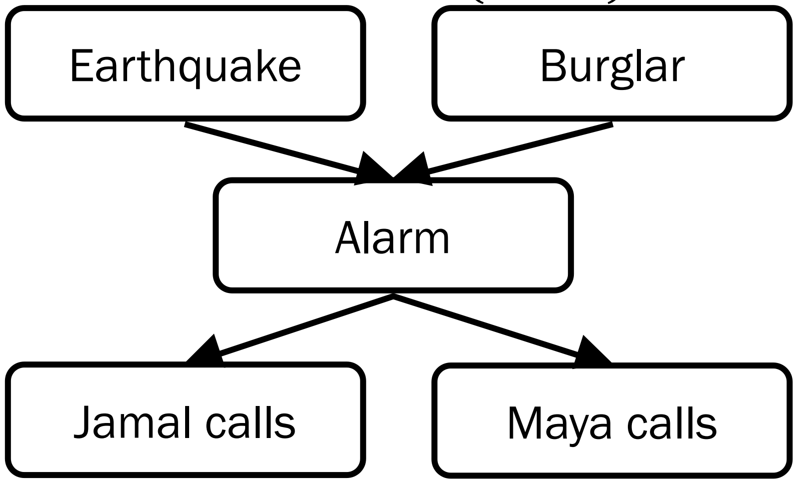
How many operations (additions, multiplications) does it take to compute $P(B=1,J=1,M=1)$?

Inference in Bayes Nets: Enumeration

16 *
3 +

$P(E=1) = 0.005$

$P(B=1) = 0.001$



B	E	$P(A=1\|B,E)$
0	0	0.002
0	1	0.35
1	0	0.96
1	1	0.98

A	$P(J=1\|A)$
0	0.1
1	0.93

A	$P(M=1\|A)$
0	0.01
1	0.65

Compute $P(B=1, J=1, M=1)$ by expanding out:

$\overset{M}{=} \sum_e \sum_a P(B=1, J=1, M=1, E=e, A=a)$

$= \sum_{e \in \{0,1\}} \sum_{a \in \{0,1\}} P(B=1)P(E=e)P(A=a|B=1, E=e) P(J=1|A=a) P(M=1|A=a)$

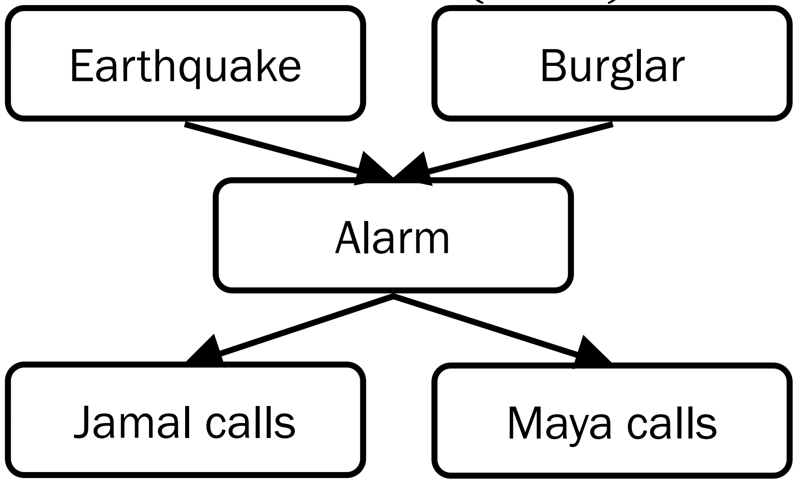
$= \underbrace{\quad}_{4*} + \underbrace{\quad}_{4*} + \underbrace{\quad}_{4*} + \underbrace{\quad}_{4*}$

3 additions

Inference in Bayes Nets: Enumeration

$P(B=1, J=1, M=1)$
16 *
3 +

$P(E=1) = 0.005$ $P(B=1) = 0.001$



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0	0.01
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How many multiplications and additions does it take to compute $P(B=1|J=1,M=1)$?

A. 16 multiplications, 3 additions
B. 18 multiplications, 4 additions
C. 32 multiplications, 6 additions
D. 32 multiplications, 7 additions
E. I have no idea

$$P(B=1|J=1,M=1) = \frac{P(B=1,J=1,M=1)}{\underbrace{P(B=1,J=1,M=1)}_{16*,\ 3+} \underbrace{+}_{1+} \underbrace{P(B=0,J=1,M=1)}_{16*,\ 3+}}$$

Inference in Bayes Nets: Enumeration

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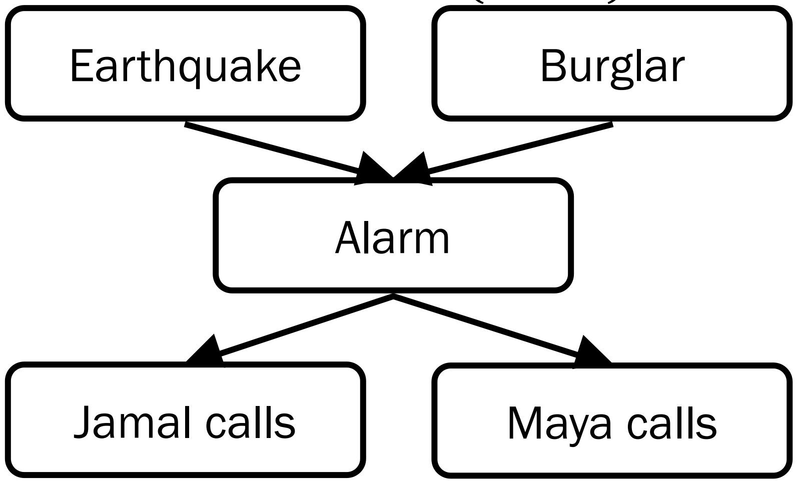

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0	0.1
1	0.93

A	$P(M=1\|A)$
0	0.01
1	0.65

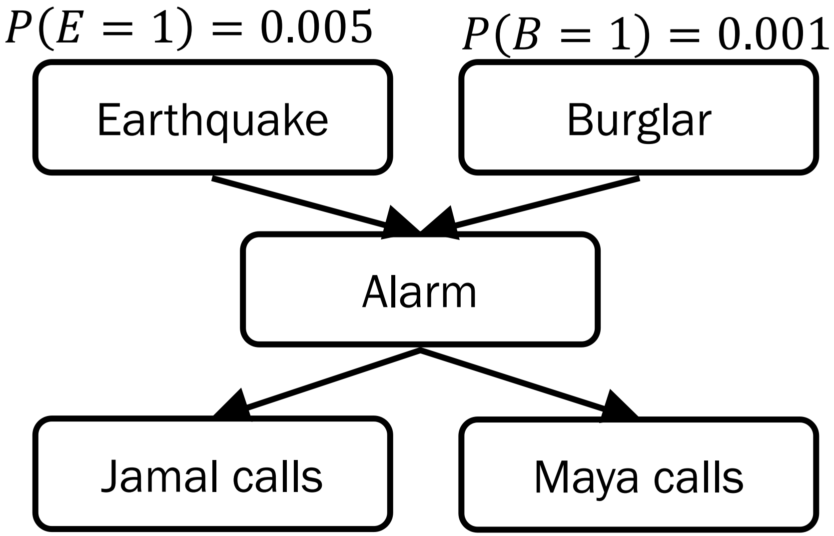
Idea: Save some intermediate results to avoid repeated calculations

(Space for speed)

repeated work

$$
\begin{aligned}
&P(B=1,J=1,M=1)\\
&= P(B=1)P(E=1)P(A=1|E=1,B=1)P(J=1|A=1)P(M=1|A=1) \rightarrow E=1,\ A=1\\
&+ P(B=1)P(E=0)P(A=1|E=0,B=1)P(J=1|A=1)P(M=1|A=1) \rightarrow E=0\ \ A=1\\
&+ P(B=1)P(E=1)P(A=0|E=1,B=1)P(J=1|A=0)P(M=1|A=0) \rightarrow E=1\ \ A=0\\
&+ P(B=1)P(E=0)P(A=0|E=0,B=1)P(J=1|A=0)P(M=1|A=0) \rightarrow E=0,\ A=0
\end{aligned}
$$

Inference in Bayes Nets: Variable Elimination



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A	$P(M = 1 \mid A)$
0	0.01
1	0.65

Eliminate redundant calculations by storing intermediate results in "factors"

A factor is:

Operations on Factors: Multiplication

$f_0(E)$

E	val
0	0.995
1	0.005

$f_1(B)$

B	val
0	0.999
1	0.001

$f_2(A, B, E)$

A	B	E	val
0	0	0	0.998
0	0	1	0.65
0	1	0	0.04
0	1	1	0.02
1	0	0	0.002
1	0	1	0.35
1	1	0	0.96
1	1	1	0.98

$f_3 * f_4$

$f_3(J, A)$

J	A	val
0	0	0.9
0	1	0.07
1	0	0.1
1	1	0.93

$f_4(M, A)$

M	A	val
0	0	0.99
0	1	0.35
1	0	0.01
1	1	0.65

Operations on Factors: Multiplication

$f_0(E)$

E	val
0	0.995
1	0.005

$f_1(B)$

B	val
0	0.999
1	0.001

$f_2(A, B, E)$

A	B	E	val
0	0	0	0.998
0	0	1	0.65
0	1	0	0.04
0	1	1	0.02
1	0	0	0.002
1	0	1	0.35
1	1	0	0.96
1	1	1	0.98

$f_3(J, A)$

J	A	val
0	0	0.9
0	1	0.07
1	0	0.1
1	1	0.93

$f_4(M, A)$

M	A	val
0	0	0.99
0	1	0.35
1	0	0.01
1	1	0.65

How many values (rows) are in the table for the factor $f_2 * f_3$

A. 4
B. 8
C. 16
D. 32

Operations on Factors: Multiplication

$f_0(E)$

E	val
0	0.995
1	0.005

$f_1(B)$

B	val
0	0.999
1	0.001

$f_2(A, B, E)$

A	B	E	val
0	0	0	0.998
0	0	1	0.65
0	1	0	0.04
0	1	1	0.02
1	0	0	0.002
1	0	1	0.35
1	1	0	0.96
1	1	1	0.98

$f_3(J, A)$

J	A	val
0	0	0.9
0	1	0.07
1	0	0.1
1	1	0.93

$f_4(M, A)$

M	A	val
0	0	0.99
0	1	0.35
1	0	0.01
1	1	0.65

How many total multiplications are needed to produce the table $f_2 * f_3$

A. 0
B. 16
C. 32
D. 64

Operations on Factors: Summing out

$f_0(E)$

E	val
0	0.995
1	0.005

$f_1(B)$

B	val
0	0.999
1	0.001

$f_2(A, B, E)$

A	B	E	val
0	0	0	0.998
0	0	1	0.65
0	1	0	0.04
0	1	1	0.02
1	0	0	0.002
1	0	1	0.35
1	1	0	0.96
1	1	1	0.98

$$\sum_e f_2$$

$f_3(J, A)$

J	A	val
0	0	0.9
0	1	0.07
1	0	0.1
1	1	0.93

$f_4(M, A)$

M	A	val
0	0	0.99
0	1	0.35
1	0	0.01
1	1	0.65

Operations on Factors: Conditioning

$f_0(E)$

E	val
0	0.995
1	0.005

$f_1(B)$

B	val
0	0.999
1	0.001

$f_2(A, B, E)$

A	B	E	val
0	0	0	0.998
0	0	1	0.65
0	1	0	0.04
0	1	1	0.02
1	0	0	0.002
1	0	1	0.35
1	1	0	0.96
1	1	1	0.98

$f_2(A, B = 1, E)$

$f_3(J, A)$

J	A	val
0	0	0.9
0	1	0.07
1	0	0.1
1	1	0.93

$f_4(M, A)$

M	A	val
0	0	0.99
0	1	0.35
1	0	0.01
1	1	0.65