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Mass and Length Dependent Chaotic Behavior of a Double Pendulum

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ABSTRACT: In this article, chaotic behavior of a double pendulum (DP) is studied numerically by varying its mass and length. The results are analyzed using time series plot, Poincare map and Lyapunov exponent plot. It is observed that the chaotic tendency of the DP increases with mass and length. However, both pendulums of the DP have varying effect on the Lyapunov exponent. The significance of the present study in the optimization and control of the dynamical systems based on the DP is also discussed.

Keywords: Chaos, Double Pendulum, Time series, Poincare Map, Lyapunov exponent

1. INTRODUCTION

A double pendulum (DP) is an interesting dynamical system since it shows chaotic motion (Baker and Gollub, 1990, Ohlhoff and Richter, 2006). Notably, experimental and numerical studies have been carried out of such a system considering the change in initial value of amplitude and angular velocity of the double pendulum (DP) (Baker and Gollub, 1990). It is observed that a DP shows periodic behaviour at low energy, transforms to quasi-periodicity at intermediate energy level and chaos at higher energy level finally again periodic motion as energy of the system increases further (Nunna, 2009). Despite significant study on the chaotic dynamics of the DP, there is no reported study in literature how the mass and length of a DP influences its chaotic behavior. Such a study is important from controlling and optimizing dynamical systems based on double pendulum for instance double arm robots (Behera and Kar, 2009). Chaos is a physical phenomenon in which a dynamical system shows random motion and the results depends on initial conditions (Baker and Gollub, 1990). Chaos is basically a manifestation of non-linearity in the dynamical system. However, all non-linear systems do not show chaotic behaviour. A condition for chaos is that the differential equations governing the dynamical system should contain non-linear terms. Moreover, it should be expressible in at least three or more phase variables(Baker and Gollub,1990). Lyapunov exponent is generally used to characterize the chaos of a dynamical system. For instance, if the average value of Lyapunov exponent is negative then the system is non chaotic. However, if Lyapunov exponent positive, then it shows chaos. Linear stability analysis of the DP shows that its two natural frequencies depend on mass and length of each pendulum in the system (Ohlhoff and Richter, 2006). Further it is to be noted that natural frequency of a pendulum also depends on the initial conditions of oscillation for instance amplitude.

Figure 1 presents the schematic sketch of double pendulum in a plane. Upper end of the top pendulum is pivoted to a fixed point while the lower end of the same is connect with the top end of the bottom pendulum.Mass and arm length of the both pendulum were changed in the numerical simulation in the present study. Moreover, mass of the each pendulum is concentrated at the end in the form of solid sphere and also arm of each pendulum is considered to be massless (Fig.1).

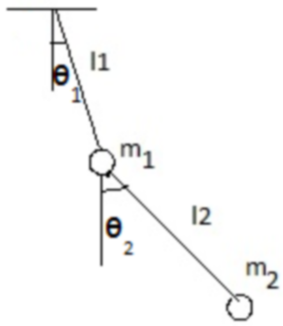

Figure 1: Schematic sketch of double pendulum under free oscillation.

Further m_1, m_2, l_1, l_2, θ_1 and θ_2 are the mass, length and angular displacement of the top and bottom pendulum respectively.

2. MODELLING OF DOUBLE PENDULUM

A similar procedure is followed to derive the system of non-linear equation as reported in the reference (Nunna, 2009). Assuming the pivot point as the origin of the coordination system with the horizontal (x-axis) and the vertical (y-axis). The DP is assumed to be located in the fourth quadrant of the coordinate system. The coordinate position of top and bottom pendulum is given by

$x_1 = l_1 \sin(\theta_1)$, $y_1 = -l_1 \cos(\theta_1)$, $x_1 = l_1 \sin(\theta_1) + l_2 \sin(\theta_2)$, $y_1 = -l_1 \cos(\theta_1) - l_2 \cos(\theta_2)$. And the corresponding angular velocity is given by the equations $\dot{x}_1 = l_1 \cos(\theta_1)\dot{\theta}_1$, $\dot{y}_1 = l_1 \sin(\theta_2)\dot{\theta}_2$, $\dot{x}_1 = l_1 \cos(\theta_1)\dot{\theta}_1 + l_2 \cos(\theta_2)\dot{\theta}_2$, $\dot{y}_1 = l_1 \sin(\theta_1)\dot{\theta}_1 + l_2 \sin(\theta_2)\dot{\theta}_2$.

Kinetic energy of the double pendulum in Fig.1 is given by the following equation

$$K_2 = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}(m_2)l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2 \theta_1 \theta_2 \cos(\theta_1 - \theta_2) \quad (1)$$

And the potential energy is

$$V_2 = -(m_1 + m_2)gl_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2 \quad (2)$$

The Lagrangian $L_2 = K_2 - V_2$ of two simply connected pendulums (Fig.1) is given by the following equation

$$\begin{aligned} L_2 &= \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + \\ &m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + (m_1 + m_2)gl_1 \cos\theta_1 \\ &+ m_2 g l_2 \cos\theta_2 \end{aligned} \quad (3)$$

Further modifying the above equations:

$$\begin{aligned} &(m_1 + m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \theta_2 \cos(\theta_1 - \theta_2) \\ &+ m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2)gl_1 \sin\theta_1 = 0 \end{aligned} \quad (4)$$

$$\begin{aligned} &m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \theta_1^2 \sin(\theta_1 - \theta_2) \\ &+ m_2 g l_2 \sin\theta_2 = 0 \end{aligned} \quad (5)$$

Converting into non-dimensional form of above equations 4& 5.

$$\begin{aligned} &\ddot{\theta}_1 + m_{2/12} l_{2/1} \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_{2/12} l_{2/1} \theta_2^2 \sin(\theta_1 - \theta_2) \\ &+ g l_1^{-1} \sin(\theta_1) = 0 \end{aligned} \quad (6)$$

$$\begin{aligned} &\ddot{\theta}_2 + l_{1/2} \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - l_{1/2} \theta_1^2 \sin(\theta_1 - \theta_2) \\ &+ g l_2^{-1} \sin(\theta_2) = 0 \end{aligned} \quad (7)$$

Where $m_{2/12} = m_2/(m_1 + m_2)$ and length of arm of the pendulum is $l_{2/1} = l_2/l_1$ and $l_{1/2} = l_1/l_2$. And the equation of momenta p_i of the DP is given by $p_i = \partial L_2 / \partial \dot{\theta}_i$ where $i = 1$ and 2

$$p_1 = (m_1 + m_2)l_1^2\dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \quad (8)$$

$$p_2 = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2) \quad (9)$$

And the Hamiltonian $H_2 = K_2 + V_2$ of the DP is given by

$$\begin{aligned} H_2 &= \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2 + m_2 l_1 l_2 \theta_1 \theta_2 \cos(\theta_1 - \theta_2) \\ &-(m_1 + m_2)gl_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2 \end{aligned} \quad (10)$$

Above differential equations (6&7) were solved numerically in MATLAB using *ode45* by following the procedures similar to the given in the reference 1. All simulations were carried out for a fixed initial condition i.e., $(\theta_1, \theta_2, p_1, p_2) = (0.2, 0.2828, 0, 0)$. Time series of angular velocity of the two pendulum, Poincare map and the average value of Lyapunov exponent are plotted to understand the mass and length dependent chaotic behavior of the DP(wolf and Swift, 1985,Zaki and Noah 2002). Poincare map (section) i.e., $(\dot{\theta}_1, p_1)$ was generated for the top pendulum under the condition $\sin(\theta_2) = 0$ and $\dot{\theta}_2 > 0$ (Rosa Nunna,2009). The results are discussed in next section.

3. RESULTS AND DISCUSSION

Fig.2 presents the plots concerning the time series (angular velocity), Poincare map and Lyapunov exponent for $m_1 = 1.0$ kg , $m_2 = 1.0$ kg , $l_1 = 1.0$ m and $l_2 = 1.0$ m and the Hamiltonian energy $E = -28.62$J . It is obvious from the plot (a) in Fig.2 that angular velocity of both pendula is periodic and both are in phase. Moreover, corresponding Poincare map is also regular as evident in plot (b) of the same Fig.2. Also the average Lyapunov exponent e is found to be $e = -0.1607$ which is negative. As mentioned earlier, dynamical system with negative exponent does not show the chaotic motion.

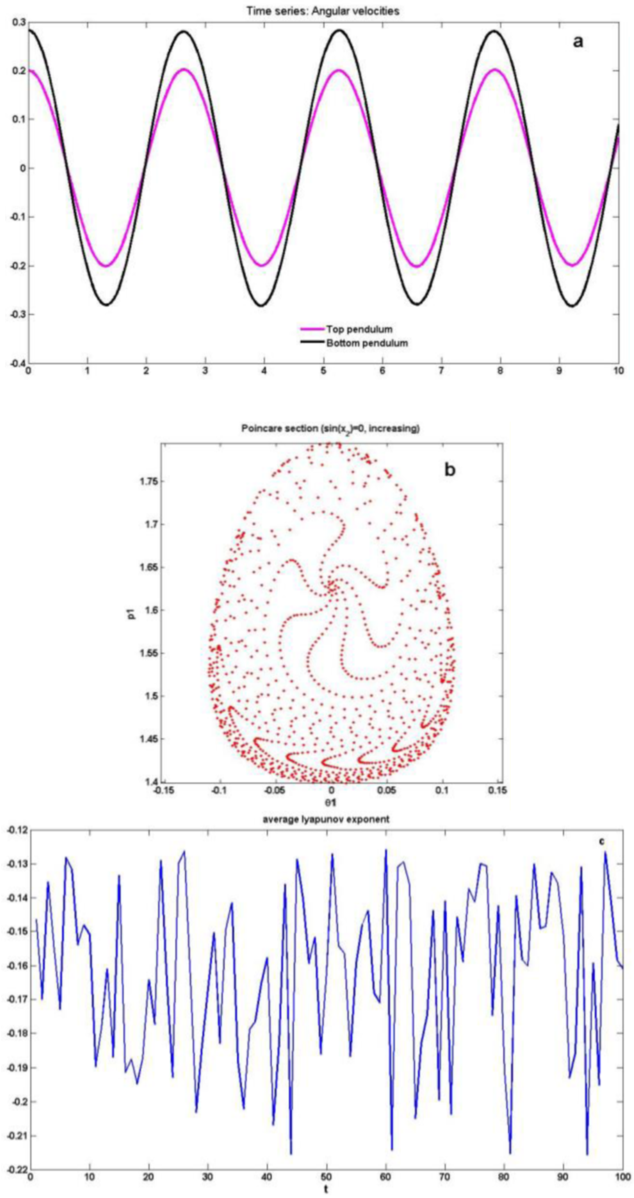

Fig.2 presents (a) Time series (b) Poincare map (c) Lyapunov exponent for $m_1 = 1$, $m_2 = 1$, $l_1 = 1$ and $l_2 = 1$, Hamiltonian $E = -28.62$.

It is now mass of the top pendulum is increased to $m_1 = 5$ while other parameters were considered the same as earlier. Now the energy corresponding to Hamiltonian increased to $E = -67.04$. Time series plot of angular velocity of the plot (a) of Fig.3 show that the motion is now quasi-periodic as there are two time period in the motion. And the corresponding Poincare map shows to be regular but with large area of null. The average Lyapunov exponent $e = -0.1086$. This value of e is larger than in Fig.1. Thus it may be concluded that chaotic tendency of the DP increased with mass of the top pendulum.

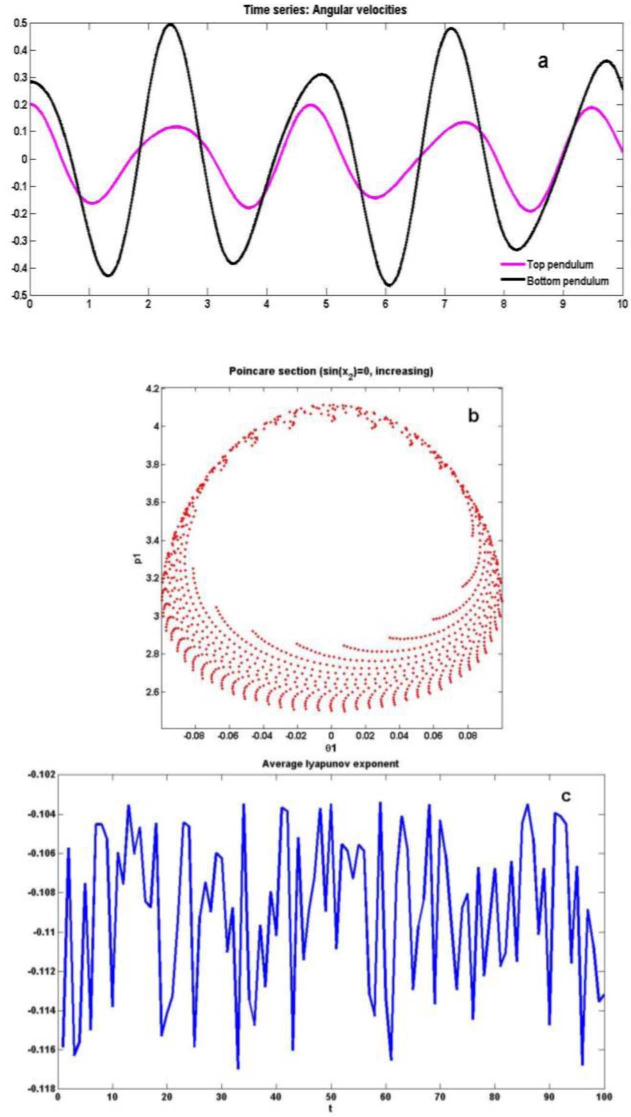

Fig.3 presents (a) Time series (b) Poincare map (c) Lyapunov exponent for $m_1 = 5$, $m_2 = 1$, $l_1 = 1$ and $l_2 = 1$, Hamiltonian $E = -67.04$.

Now in case of change in magnitude of the mass of the bottom pendulum i.e., $m_2 = 5$, the time series plot (a) in Fig.5 shows that the top pendulum is showing period oscillation while the bottom pendulum is random. The value of average Lyapunov exponent is $e = 0.0399$. As noted earlier that positive value of the average Lyapunov exponent results in chaotic oscillations. The chaotic behavior is also clear in plot (b) of Fig.4. It may be concluded that the bottom makes the DP more chaotic than the top pendulum.

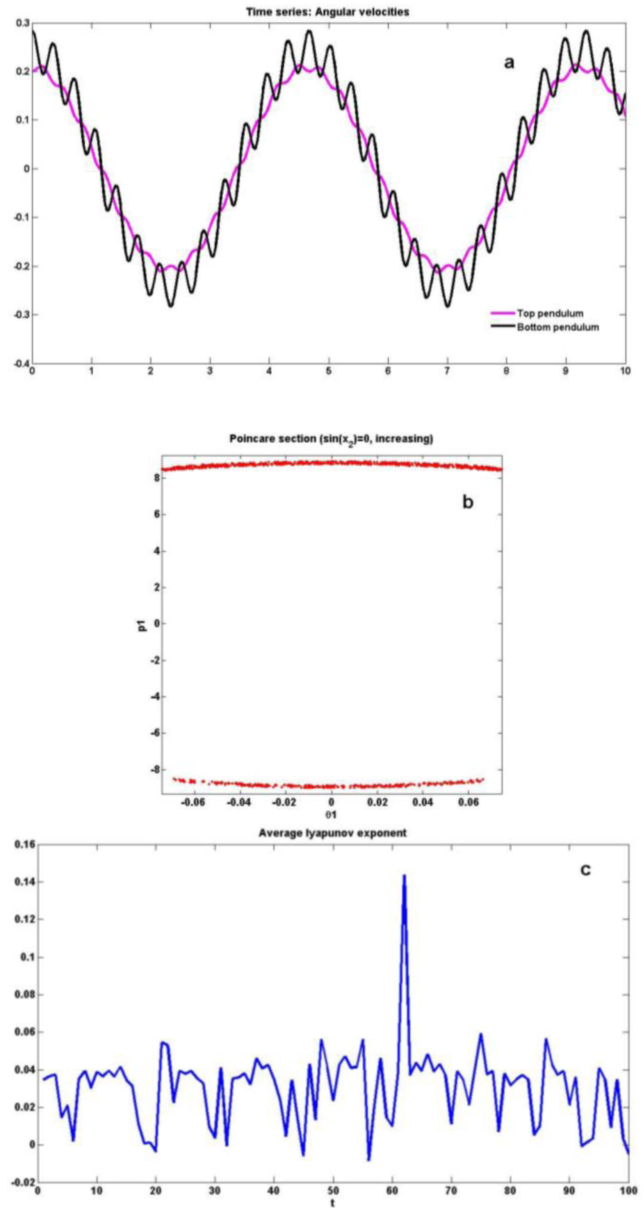

Fig.4 presents (a) Time series (b) Poincare map (c) Lyapunov exponent for $m_1 = 1$, $m_2 = 5$, $l_1 = 1$ and $l_2 = 1$ for Hamiltonian $E = -104.68$.

Motivated from the above study, it becomes interesting to know how arm length of pendula affects its chaotic dynamics. Length of the top pendulum was changed from $l_1 = 1$ to $l_1 = 3$. The result from the time series plot (a) in Fig.5 shows that the bottom pendulum is showing quasi-periodic oscillation while the top pendulum is still periodic. The value of average Lyapunov exponent is equal to -0.0991. It is concluded that increasing arm length of the top pendulum results in quasiperiodic motion.

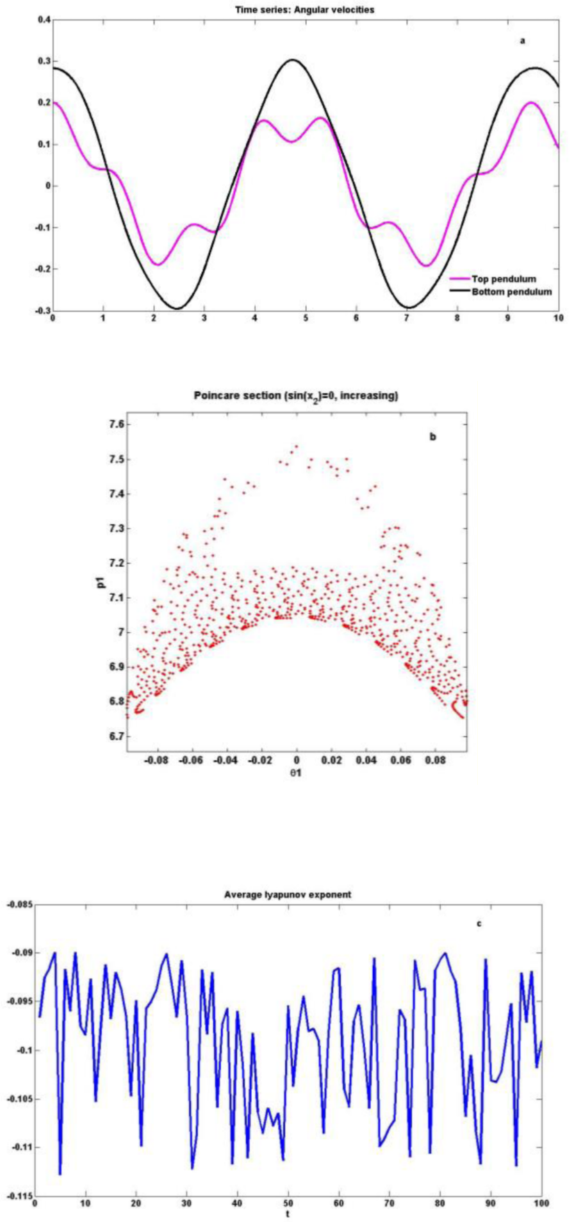

Fig.5 presents (a) Time series (b) Poincare map (c) Lyapunov exponent for $m_1 = 1$, $m_2 = 1$, $l_1 = 3$ and $l_2 = 1$, Hamiltonian $E = -67.04$.

Finally, arm length of the bottom pendulum was also changed in the simulation for $l_2 = 3$. The value of average Lyapunov exponent is $e = 0.0234$. The results in Fig.6 shows the DP is ultimately moving towards the chaos.

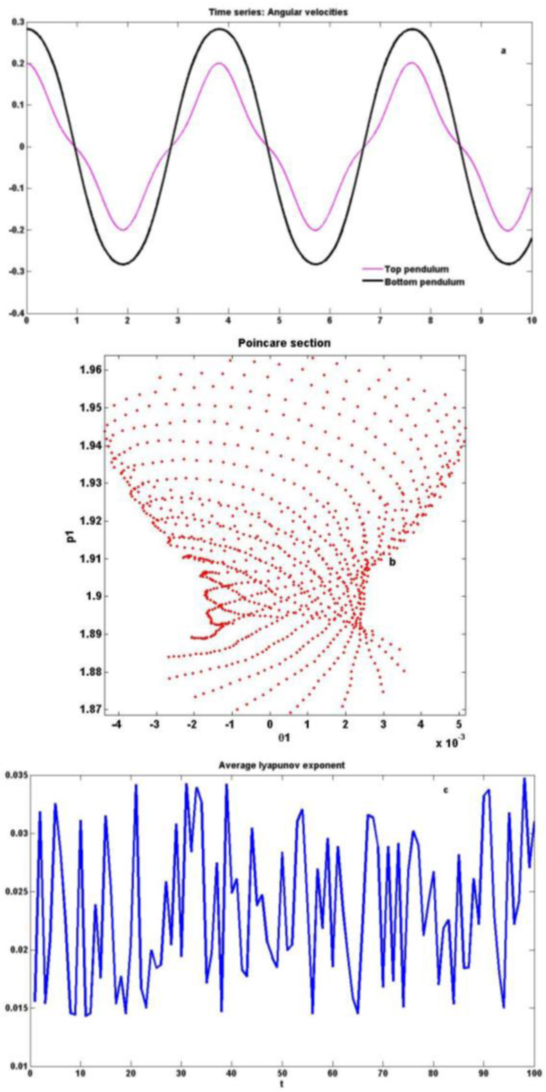

Fig.6 presents(a) Time series (b) Poincare map (c) Lyapunov exponent for $m_1 = 1$, $m_2 = 1$, $l_1 = 1$ and $l_2 = 3$, $E = -47.44J$.

The summary of the above simulations is presented in the Table 1.

Table 1: Summary of the numerical simulations

Mass (Kg) and length (m) of each pendulum	Lyapunov exponent e and Hamiltonian energy E(Joule)	Chaotic/non-chaotic behavior
$m_1 = 1$, $m_2 = 1$, $l_1 = 1$ $l_2 = 1$	$e = -0.1607$ $E = -28.62J$	Periodic
$m_1 = 5$, $m_2 = 1$, $l_1 = 1$ $l_2 = 1$	$e = -0.1086$ $E = -67.04J$	Quasi-periodic
$m_1 = 1$, $m_2 = 5$, $l_1 = 1$ $l_2 = 1$	$e = 0.0399$ $E = -104.68J$	Chaos
$m_1 = 1$, $m_2 = 1$, $l_1 = 3$ $l_2 = 1$	$e = -0.0991$ $E = -67.04J$	Quasi-periodic
$m_1 = 1$, $m_2 = 1$, $l_1 = 1$ $l_2 = 3$	$e = 0.0234$ $E = -47.44J$	Chaos

From the Table 1, it may be concluded that mass and length of the two pendulum have varying degree of effect on the average value of Lyapunov exponent. Thus there is an optimal combination of mass and arm length of the DP at which chaotic tendency of the system will be minimum. However further numerical and experimental studies are needed on this issue.

4. CONCLUSION

The chaotic behaviour of a double pendulum considering its mass and length is studied numerically. It is observed that chaotic tendency increases with mass and length. Moreover, top and bottom pendulum show varying effect on the chaos.

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