CSCI 393, Algorithm Design and Analysis, Spring 2020

HW 02 – Order of Growth (37 points)

1. For each of the following functions, indicate how much the function's value will change if its argument is increased eightfold. Use either (i) The difference between, or (ii) The ratio of, $f(8n)$ and $f(n)$, whichever is more convenient for getting a compact answer. If possible, try to get an answer that does not depend on n.

   (a) (2 points) $f(n) = \lg n$ (Recall: $\lg n$ is another of writing $\log_2 n$)

   Solution:
$$\begin{aligned} \lg 8n - \lg n &= (\lg 8 + \lg n) - \lg n \\ &= \lg 8 + \lg n - \lg n \\ &= \lg 8 \\ &= 3 \end{aligned}$$

   (b) (2 points) $f(n) = \sqrt[3]{n}$

   Solution:
$$\begin{aligned} \frac{\sqrt[3]{8n}}{\sqrt[3]{n}} &= \frac{\sqrt[3]{8}\ \cancel{\sqrt[3]{n}}}{\cancel{\sqrt[3]{n}}} \\ &= \sqrt[3]{8} \\ &= 2 \end{aligned}$$

   (c) (2 points) $f(n) = n$

   Solution:
$$\frac{8n}{n} = 8$$

   (d) (2 points) $f(n) = n^2$

   Solution:
$$\begin{aligned} \frac{(8n)^2}{n^2} &= \frac{64n^2}{n^2} \\ &= 64 \end{aligned}$$

   (e) (2 points) $f(n) = n^3$

   Solution:
$$\begin{aligned} \frac{(8n)^3}{n^3} &= \frac{512n^3}{n^3} \\ &= 512 \end{aligned}$$

   (f) (2 points) $f(n) = 2^n$

   Solution:
$$\frac{2^{8n}}{2^n} = 2^{7n}$$

2. Indicate whether the first function of each of the following pairs has a smaller, same, or larger order of growth (to within a constant multiple) than the second function. For this question you *do not* need to use the formal definitions for $Big - O$, Ω, or Θ.

   Hint: If necessary, simplify the functions to single out terms defining their orders of growth.

   (a) (2 points) $n(n+1)$ *and* $2000n^2$

   Solution:
   First has <u>SAME</u> growth rate as second:
   Both are $\in \Theta(n^2)$.

   (b) (2 points) $100n^2 + 12n + 4$ *and* $.006n^3$

   Solution:
   First has <u>SMALLER</u> growth rate than the second:
   First is $\in \Theta(n^2)$ second is $\in \Theta(n^3)$

   (c) (2 points) $\lg n$ *and* $\ln n$

   Solution:
   First has <u>SAME</u> growth rate as second:
   The bases of the logs don't matter. All logarithmic functions have are in the same complexity category.

   (d) (2 points) 2^{n-1} *and* 2^n

   Solution:
   First has <u>SAME</u> growth rate as second:
   First function: $2^{n-1} = (2^n)(2^{-1}) = (2^n)\frac{1}{2}$.
   Which is the same growth rate as the second function.

   (e) (2 points) $(n-1)!$ *and* $n!$

   Solution:
   First has <u>SMALLER</u> growth rate than the second:
   Second function: $n! = n(n-1)!$

3. (5 points) Show directly, using the definition of $Big - O$, that $6n^2 + 12n \in O(n^2)$.

Solution:

Here is the definition of "big O":

For a given complexity function $f(n)$, $O(f(n))$ is the set of complexity functions $g(n)$ for which there exists some positive real constant c and some nonnegative integer n_0 such that for all $n \geq n_0$,

$$g(n) \leq c \times f(n).$$

In this problem $g(n) = 6n^2 + 12n$ and $f(n) = n^2$. Applying the definition, we must find a positive real constant c and a nonnegative integer n_0 such that for all $n \geq n_0$, $6n^2 + 12n \leq c \times n^2$.

$$\begin{array}{rcl}
6n^2 + 12n & \leq & cn^2 \\
6n + 12 & \leq & cn \\
12 & \leq & cn - 6n \\
12 & \leq & n(c - 6) \\
\text{Letting } c = 18 & & \\
12 & \leq & n(18 - 6) \\
12 & \leq & 12n \\
1 & \leq & n \\
\text{Which is true for all } n \geq 1 & & \\
\text{Hence } n_0 = 1 & &
\end{array}$$

4. (5 points) Show directly, using the definition of Ω, that $6n^2 + 12n \in \Omega(n^2)$.

Solution:

Here is the definition of Ω:

For a given complexity function $f(n)$, $\Omega(f(n))$ is the set of complexity functions $g(n)$ for which there exists some positive real constant c and some nonnegative integer n_0 such that for all $n \geq n_0$,

$$g(n) \geq c \times f(n).$$

In this problem $t(n) = 6n^2 + 12n$ and $f(n) = n^2$. Applying the definition, we must find a positive real constant c and a nonnegative integer n_0 such that for all $n \geq n_0$, $6n^2 + 12n \geq c \times n^2$.

$$\begin{array}{rcl} 6n^2 + 12n & \geq & cn^2 \\ 12n & \geq & cn^2 - 6n^2 \\ \text{Letting } c = 6 & & \\ 12n & \geq & 6n^2 - 6n^2 \\ 12n & \geq & 0 \\ n & \geq & 0 \\ \text{Which is true for all } n \geq 0 & & \\ \text{Hence } n_0 = 0 & & \end{array}$$

Alternatively, let $c = 1$. Then

$$\begin{array}{rcl} 6n^2 + 12n & \geq & cn^2 \\ 6n^2 + 12n & \geq & n^2 \end{array}$$

Which is true for all $n \geq 0$. So we let $n_0 = 0$.

5. (5 points) Show that $6n^2 + 12n \in \Theta(n^2)$.

Solution:

The previous two questions established that $6n^2 + 12n \in O(n^2)$ and $6n^2 + 12n \in \Omega(n^2)$. Hence $6n^2 + 12n \in \Theta(n^2)$.