CSCI 393, Algorithm Design and Analysis, Spring 2020

HW 05 – Recurrences (15 points)

1. Consider the following recurrence:

$$a_k = \begin{cases} 0 & \text{if } k = 0 \\ a_{k-1} + 3k + 1 & \text{if } k > 0 \end{cases}$$

(a) (5 points) Write out the first six terms of the recurrence.

Solution:

$$\begin{aligned}
a_0 &= 0 \\
\\
a_1 &= (a_0) + 3 \cdot 1 + 1 \\
&= 0 + 3 \cdot 1 + 1 \\
&= 3 \cdot 1 + 1 \\
\\
a_2 &= (a_1) + 3 \cdot 2 + 1 \\
&= (3 \cdot 1 + 1) + 3 \cdot 2 + 1 \\
&= 3 \cdot 1 + 3 \cdot 2 + 2 \\
\\
a_3 &= (a_2) + 3 \cdot 3 + 1 \\
&= (3 \cdot 1 + 3 \cdot 2 + 2) + 3 \cdot 3 + 1 \\
&= 3 \cdot 1 + 3 \cdot 2 + 3 \cdot 3 + 3 \\
\\
a_4 &= (a_3) + 3 \cdot 4 + 1 \\
&= (3 \cdot 1 + 3 \cdot 2 + 3 \cdot 3 + 3) + 3 \cdot 4 + 1 \\
&= 3 \cdot 1 + 3 \cdot 2 + 3 \cdot 3 + 3 \cdot 4 + 4 \\
\\
a_5 &= (a_4) + 3 \cdot 5 + 1 \\
&= (3 \cdot 1 + 3 \cdot 2 + 3 \cdot 3 + 3 \cdot 4 + 4) + 3 \cdot 5 + 1 \\
&= 3 \cdot 1 + 3 \cdot 2 + 3 \cdot 3 + 3 \cdot 4 + 3 \cdot 5 + 5
\end{aligned}$$

(b) (5 points) Make a guess for the explicit formula for a_n.

Solution: Our guess is:

$$
\begin{aligned}
a_n &= (3 \cdot 1 + 3 \cdot 2 + \ldots + 3 \cdot n) + n \\
&= 3(1 + 2 + \ldots + n) + n \\
&= 3\left(\frac{n(n+1)}{2}\right) + n \\
&= 3\left(\frac{n^2+n}{2}\right) + n \\
&= \frac{3n^2+3n}{2} + \frac{2n}{2} \\
&= \frac{3n^2+5n}{2}
\end{aligned}
$$

(c) (5 points) Prove your guess is correct using induction.

Solution:

Proof. (By Induction)
Base Case (n=0)

$a_0 = 0$ Directly from the recurrence

$$a_0 = \frac{3 \cdot 0^2 + 5 \cdot 0}{2} = \frac{0+0}{2} = 0$$

Hence the guess is valid for $n = 0$.

Inductive Hypothesis ($n = k$)
Assume

$$a_k = \frac{3k^2 + 5k}{2}$$

Induction step ($n = k+1$)
To Show:

$$\begin{aligned} a_{k+1} &= \frac{3(k+1)^2 + 5(k+1)}{2} \\ &= \frac{3(k^2 + 2k + 1) + 5k + 5}{2} \\ &= \frac{3k^2 + 6k + 3 + 5k + 5}{2} \\ &= \frac{3k^2 + 11k + 8}{2} \end{aligned}$$

$a_{k+1} = a_k + 3(k+1) + 1$ From the recurrence

$= \frac{3k^2 + 5k}{2} + 3(k+1) + 1$ By Inductive Hypothesis

$$\begin{aligned} &= \frac{3k^2 + 5k}{2} + \frac{6(k+1)}{2} + \frac{2}{2} \\ &= \frac{3k^2 + 5k}{2} + \frac{6k+6}{2} + \frac{2}{2} \\ &= \frac{3k^2 + 5k + 6k + 6 + 2}{2} \\ &= \frac{3k^2 + 11k + 8}{2} \end{aligned}$$

□