Example of a Double Inclusion Proof

1. Let
   $A = \{m \in \mathbb{Z} \mid m = 2a \text{ for some } a \in \mathbb{Z}\}$
   $B = \{n \in \mathbb{Z} \mid n = 2b - 2 \text{ for some } b \in \mathbb{Z}\}$

   Show $A = B$

   Lemma 1: $A \subseteq B$

   Proof.
   Let $x \in A$. Then $x = 2a$ for some $a \in \mathbb{Z}$.

   Let $b = a + 1$ (note that $b \in \mathbb{Z}$)

   Subtracting 1 from both sides we have $a = b - 1$.

   Then

   $$\begin{aligned} x &= 2a \\ &= 2(b-1) \\ &= 2b - 2 \end{aligned}$$

   Hence $x \in B$

   Since every element in A is a element in B, $A \subseteq B$.

   □

   Lemma 2: $B \subseteq A$

   Proof.
   Let $y \in B$. Then $y = 2b - 2$ for some $b \in \mathbb{Z}$.

   Let $a = b - 1$ (note that $a \in \mathbb{Z}$)

   Adding 1 to both sides we have $b = a + 1$.

   Then

   $$\begin{aligned} y &= 2b - 2 \\ &= 2(a+1) - 2 \\ &= 2a + 2 - 2 \\ &= 2a \end{aligned}$$

   Hence $x \in A$

   Since every element in B is a element in A, $B \subseteq A$. □

   Claim: $A = B$

   Proof.
   By Lemma 1, $A \subseteq B$

   By Lemma 2, $B \subseteq A$

   $\therefore A = B$ □