Master Method

The Master Method is used to solve recurrences in the form

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

where $a \geq 1$, $b > 1$, and $f(n)$ is a positive function.

You will find that many of the recurrences you encounter fit this form and can therefore be solved using the Master Method. In particular, this form of recurrence results from an algorithm that divides a problem into a instances, each of size n/b, with $f(n)$ being the cost of dividing and recombining the subproblems.

It can be shown (but we won't do it here) that replacing the term $T\left(\frac{n}{b}\right)$ with $T\left(\lceil\frac{n}{b}\rceil\right)$ or $T\left(\lfloor\frac{n}{b}\rfloor\right)$ does not change the asymptotic behavior of the recurrence. It can also be shown that similar results hold for big-O and Ω.

To time complexity of an algorithm with this form of recurrence can be given by the *Master Therom*:

If $f(n) \in \Theta\left(n^d\right)$ with $d \geq 0$ Then

$$T(n) \in \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \lg n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

Example 1

$$T(n) = 9T\left(\frac{n}{3}\right) + n$$

Here $a = 9$, $b = 3$, and $f(n) = n$.

So $f(n) \in \Theta(n^1) \implies d = 1$

Now we need to see if any of the three cases can be applied.

Case 1

$$\begin{aligned} a &< b^d \\ 9 &< 3^1 \\ 9 &< 3 \end{aligned}$$

The inequality is invalid, so Case 1 does not apply.

Case 2

$$\begin{aligned} a &= b^d \\ 9 &= 3^1 \\ 9 &= 3 \end{aligned}$$

The equality is invalid, so Case 2 does not apply.

Case 3

$$\begin{aligned} a &> b^d \\ 9 &> 3^1 \\ 9 &>= 3 \end{aligned}$$

The inequality is valid, so Case 3 applies.

Since Case 3 applies,

$$\begin{aligned} T(n) &\in \Theta(n^{\log_b a}) \\ &= \Theta(n^{\log_3 9}) \\ &= \Theta(n^2) \end{aligned}$$

Example 2

$$T(n) = T\left(\frac{2n}{3}\right) + 1$$

Here $a = 1$, $b = \frac{3}{2}$, and $f(n) = 1$.

So $f(n) \in \Theta(n^0) \implies d = 0$

Now we need to see if any of the three cases can be applied.

Case 1

$$\begin{array}{rcl} 1 & < & b^d \\ 1 & < & \left(\frac{3}{2}\right)^0 \\ 1 & < & 1 \end{array}$$

The inequality is invalid, so Case 1 does not apply.

Case 2

$$\begin{array}{rcl} 1 & = & b^d \\ 1 & = & \left(\frac{3}{2}\right)^0 \\ 1 & = & 1 \end{array}$$

Since the equality is valid, Case 2 applies.

For Completeness, let's look at Case 3:

Case 3

$$\begin{array}{rcl} 1 & > & b^d \\ 1 & > & \left(\frac{3}{2}\right)^0 \\ 1 & > & 1 \end{array}$$

As expected, Case 3 is not valid.

Since Case 2 applies,

$$\begin{array}{rcl} T(n) & \in & \Theta(n^d \lg n) \\ & = & \Theta(n^0 \lg n) \\ & = & \Theta(1 \cdot \lg n) \\ & = & \Theta(\lg n) \end{array}$$