CSCI 281, Discrete Structures, Fall 2019

Random Variable Examples

1. Use indicator random variables to solve the following problem. Each of n customers gives a coat to a coat-check person at a restaurant. The coat-check person gives the coats back to the customers in a random order. What is the expected number of customers that get back their own coat?

Solution:

Define a random variable Y that equals the number of customers that get back their own coat. Then we need to compute $E[Y]$.

For $i = 1, 2, \ldots, n$, define the indicator random variable

$$X_i = \begin{cases} 0 & \text{if customer } i \text{ does not get his own coat back} \\ 1 & \text{if customer } i \text{ gets his own coat back} \end{cases}$$

Then $Y = \sum_{i=1}^{n} X_i$

Since the ordering of the coats is random, each customer has a probability of $1/n$ of getting his own coat back. In other words, $Pr[X_i = 1] = 1/n$.

This means

$$\begin{aligned} E[X_i] &= 1 \cdot Pr[X_i = 1] + 0 \cdot Pr[X_i = 0] \\ &= Pr[X_i = 1] \\ &= 1/n \end{aligned}$$

We now have

$$\begin{aligned} E[Y] &= E\left[\sum_{i=1}^{n} X_i\right] \\ &= \sum_{i=1}^{n} E[X_i] \text{ (linearity of expectation)} \\ &= \sum_{i=1}^{n} 1/n \\ &= 1 \end{aligned}$$

Thus we expect that exactly one person gets his own coat back.

2. Let A be an array of size n containing n distinct numbers. If $i < j$ and $A[i] > A[j]$, then the pair (i, j) is called an *inversion* of A. Suppose that each element of A is chosen uniformly at random from the range 1 through n. Use indicator random variables to compute the expected number of inversions.

Solution

Define random variable Y that equals the total number of inverted pairs in the array. The we are expected to compute $E[Y]$.

Let X_{ij} be an indicator random variable for the event where the pair $A[i], A[j]$ for $i < j$ is inverted, i.e., $A[i] > A[j]$. More precisely

$$X_{ij} = \begin{cases} 0 & \text{if } A[i] \leq A[j] \\ 1 & \text{if } A[i] > A[j] \end{cases}$$

Then

$$Y = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$$

Note that $Pr[X_{ij} = 1] = 1/2$ because given two distinct random numbers, the probability that the first is bigger is the same as the probability that the second is bigger.

This means

$$\begin{aligned} E[X_{ij}] &= 1 \cdot Pr[X_{ij} = 1] + 0 \cdot Pr[X_{ij} = 0] \\ &= Pr[X_{ij} = 1] \\ &= 1/2 \end{aligned}$$

We want the expected number of inverted pairs, so we take the expectation of both sides of the above equation and get

$$\begin{aligned} E[Y] &= E\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}\right] \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}] \text{ (linearity of expectation)} \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 1/2 \\ &= \binom{n}{2} \frac{1}{2} \\ &= \frac{n(n-1)}{2} \cdot \frac{1}{2} \\ &= \frac{n(n-1)}{4} \end{aligned}$$

Hence, the expected number of inverted pairs is $n(n-1)/4$.