Example Proof by Induction

Summations

1. Sum of the First $n-1$ Integers

   Claim: For all $n \geq 1$

   $$\sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}$$

   Proof. (By Induction)

   Base Case ($n = 1$)

   $$\sum_{i=0}^{n-1} i = \sum_{i=0}^{1-1} i = \sum_{i=0}^{0} i = 0$$

   Also

   $$\frac{n(n-1)}{2} = \frac{1(1-1)}{2} = \frac{1(0)}{2} = \frac{0}{2} = 0$$

   Inductive Hypothesis ($n = k$)

   Assume

   $$\sum_{i=0}^{k-1} i = \frac{k(k-1)}{2}$$

   Induction Step ($n = k+1$)

   To Show:

   $$\begin{aligned} \sum_{i=0}^{(k+1)-1} i &= \frac{(k+1)((k+1)-1)}{2} \\ \sum_{i=0}^{k+1-1} i &= \frac{(k+1)(k+1-1)}{2} \\ \sum_{i=0}^{k} i &= \frac{(k+1)(k)}{2} \\ \sum_{i=0}^{k} i &= \frac{k(k+1)}{2} \end{aligned}$$

$$
\begin{aligned}
\sum_{i=0}^{k} i &= \sum_{i=0}^{k-1} i + k \quad \textit{Separate last term} \\
&= \frac{k(k-1)}{2} + k \quad \textit{Inductive Hypothesis} \\
&= \frac{k(k-1)}{2} + \frac{2k}{2} \quad \textit{Get common denominators} \\
&= \frac{k(k-1)+2k}{2} \quad \textit{Add Fractions} \\
&= \frac{k((k-1)+2)}{2} \quad \textit{Factor out } k \\
&= \frac{k(k-1+2)}{2} \quad \textit{Remove parentheses} \\
&= \frac{k(k+1)}{2}
\end{aligned}
$$

By the principle of mathematical induction, for all $n \geq 1$

$$\sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}$$

$\square$