Math Facts

1 Logarithms

1.1 Definitions

If b and n are positive numbers with $b \neq 1$,

$$\log_b(n) = k \text{ iff } b^k = n$$

$$\log_b(x) = \log_b(y) \text{ iff } x = y$$

$$\log_b^i(x) = (\log_b(x))^i$$

$$\log(x) = \log_{10}(x)$$

$$\ln(x) = \log_e(x)$$

$$\lg(x) = \log_2(x)$$

1.2 Logarithmic Identities

$$\log_b(x^a) = a \cdot \log_b(x)$$

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$

$$\log_b(b) = 1$$

$$\log_b(1) = 0$$

$$\log_b(b^n) = n$$

$$\log_b(x) = \frac{\log_c(x)}{\log_c(b)}$$

$$\log_b\left(\frac{1}{x}\right) = -\log_b(x)$$

$$\log_b(x) = \frac{1}{\log_x(b)}$$

$$x^{\log_b(y)} = y^{\log_b(x)}$$

$$b^{\log_b(x)} = x$$

1.3 Iterated Log Functions

Let $\lg^{(i)}$ be defined as follows:

$$\lg^{(i)}(n) = \begin{cases} n & \text{if } i = 0 \\ \lg\left(\lg^{(i-1)}(n)\right) & \text{if } i > 0 \end{cases}$$

Then the iterated log function (read as "log star n") is defined as

$$\lg^*(n) = \min\left\{i \geq 0 \mid lg^{(i)}(n) \leq 1\right\}$$

2 Limits

$$\lim_{n\to\infty} (f(n) + g(n)) = \lim_{n\to\infty} (f(n)) + \lim_{n\to\infty} (g(n))$$

$$\lim_{n\to\infty} (f(n) - g(n)) = \lim_{n\to\infty} (f(n)) - \lim_{n\to\infty} (g(n))$$

$$\lim_{n\to\infty} (f(n) \cdot g(n)) = \lim_{n\to\infty} (f(n)) \cdot \lim_{n\to\infty} (g(n))$$

$$\lim_{n\to\infty} \left(\frac{f(n)}{g(n)}\right) = \frac{\lim_{n\to\infty} (f(n))}{\lim_{n\to\infty} (g(n))}$$

$$\lim_{x\to a} x = a$$

$$\lim_{x\to a} (mx + b) = ma + b$$

$$\lim_{x\to b} b = b$$

$$\lim_{x\to a} (k \cdot f(x)) = k \cdot \lim_{x\to a} (f(x))$$

3 Exponents

$$a^0 = 1$$

$$a^1 = a$$

$$0^0 = \text{undefined}$$

$$a^{-m} = \frac{1}{a^m}$$

$$a^m \cdot a^n = a^{m+n}$$

$$(ab)^m = a^m b^m$$

$$(a^m)^n = a^{mn}$$

$$\frac{a^m}{a^n} = a^{m-n} \text{ if } m > n$$

$$\frac{a^m}{a^n} = \frac{1}{a^{m-n}} \text{ if } n > m$$

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

$$b^{\frac{p}{q}} = \left(\sqrt[q]{b}\right)^p \equiv \sqrt[q]{b^p} \quad \text{for } b > 0 \text{ and } p \text{ and } q \text{ integers with } q > 0$$

4 Summations

4.1 Identities

If $a_1, a_2, a_3, \ldots, a_n$ and $b_1, b_2, b_3, \ldots, b_n$ are sequences, and c is a constant, then for every positive integer n,

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} (a_i + c) = \sum_{i=1+c}^{n+c} a_i$$

4.2 Closed Forms

$$\sum_{i=1}^{n} c = cn$$

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} n = n^2$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{i=1}^{n} r^i = \frac{r^{n+1}-1}{r-1}, \text{ This is the Geometric Sequence.}$$

$$\sum_{i=1}^{n} (n-i) = \frac{n(n-1)}{2}$$

5 Logical Equivalences

Given any statement variables p, q, and r, a tautology $\mathbf{t}$, and a contradiction $\mathbf{c}$, the following logical equivalences hold:

1.	Commutative laws:	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2.	Associative laws:	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3.	Distributive laws:	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4.	Identity laws:	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5.	Negation laws:	$p \vee {\sim}p \equiv \mathbf{t}$	$p \wedge {\sim}p \equiv \mathbf{c}$
6.	Double negative law:	${\sim}({\sim}p) \equiv p$	
7.	Idempotent laws:	$p \wedge p \equiv p$	$p \vee p \equiv p$
8.	Universal bound law:	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9.	De Morgan's laws:	${\sim}(p \wedge q) \equiv {\sim}p \vee {\sim}q$	${\sim}(p \vee q) \equiv {\sim}p \wedge {\sim}q$
10.	Absorption law:	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11.	Negations of $\mathbf{t}$ and $\mathbf{c}$:	${\sim}\mathbf{t} \equiv \mathbf{c}$	${\sim}\mathbf{c} \equiv \mathbf{t}$
12.	Division into cases:	$p \vee q \to r \equiv (p \to r) \wedge (q \to r)$	
13.	Implication as or:	$p \to q \equiv \sim p \vee q$	
14.	Negating a conditional:	$\sim (p \to q) \equiv p \wedge \sim q$	
15.	Contrapositive:	$p \to q \equiv \sim q \to \sim p$	
16.	iff as implications:	$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$	

6 Sets

A set is a collection of objects, called elements. The elements must be distinct (each element can appear in the set only once). A set may have a finite or an infinite number of elements. The order of the elements doesn't matter.

6.1 Symbols and Terms

$a \in A$	a is an element of set A.
$a \notin A$	a is not an element of set A.
$\{a_1, a_2, \ldots, a_n\}$	The set with elements $a_1, a_2, \ldots, a_n$.
$\{x \in D \mid \textit{condition}\}$	The set of all elements x in D that satisfy the *condition*.
$\mathbb{R}, \mathbb{R}^-, \mathbb{R}^+, \mathbb{R}^{nonneg}$	Set of all real numbers, negative reals, positive reals, and nonnegative reals.
$\mathbb{Z}, \mathbb{Z}^-, \mathbb{Z}^+, \mathbb{Z}^{nonneg}$	Set of all integers, negative integers, positive integers, and nonnegative integers.
$\mathbb{Q}, \mathbb{Q}^-, \mathbb{Q}^+, \mathbb{Q}^{nonneg}$	Set of rational numbers, negative rationals, positive rationals, and nonnegative rationals.
$\mathbb{N}$	Set of natural numbers. $\mathbb{N} = \{1, 2, \ldots\}$.
$\mathbb{W}$	Set of whole numbers. $\mathbb{W} = \{0, 1, \ldots\}$.
$\mathbb{Z}_2$	Set of binary digits. $\mathbb{Z}_2 = \{0, 1\}$.
$\lvert A \rvert$	The number of elements in A; The *cardinalty* of A.
$A \subset B$	A is a proper subset of B. Equivalently $B \supset A$.
$A \subseteq B$	A is a subset of B. Equivalently $B \supseteq A$.
$A \not\subset B$	A is a not a proper subset of B. Equivalently $B \not\supset A$.
$A \not\subseteq B$	A is not a subset of B. Equivalently $B \not\supseteq A$.
$A \supset B$	A is a proper superset of B. Equivalently $A \subset B$.
$A \supseteq B$	A is a superset of B. Equivalently $B \subseteq A$.
$A \not\supset B$	A is a not a proper superset of B. Equivalently $B \not\subset A$.
$A \not\supseteq B$	A is not a superset of B. Equivalently $B \not\subseteq A$.
$A = B$	A equals B.
$A \neq B$	A does not equal B.
$A \cup B$	A union B.
$A \cap B$	A intersect B.
$A - B$	The difference of A minus B. An alternate notation for difference is $A \setminus B$.
A^c	The complement of A. An alternate notation is $\overline{A}$ or A'.
(x, y)	Ordered pair.
$(x_1, x_2, \ldots, x_n)$	n-tuple.
$A \times B$	The Cartesian product of A and B.
$A_1 \times A_2 \times \ldots \times A_n$	The Cartesian product of $A_1, A_2, \ldots A_n$.
$\emptyset$	The empty set. An alternate notation for the empty set is $\varnothing$ or $\{\}$.
$\mathscr{P}(A)$	The power set of A.
Disjoint	A and B are disjoint if $A \cap B = \emptyset$.
Pairwise Disjoint	$A_1, A_2, \ldots A_n$ are pairwise disjoint if, for every pair A_i and A_j, $A_i \cap A_j = \emptyset$.

6.2 Selected Examples

6.2.1 Set Definitions and Cardinality

We can define a set simply by listing its elements:
$A_1 = \{2, 3, 4, 5, 6\}$.
$A_2 = \{2, 4, 6\}$.
$A_3 = \{3, 5, 7\}$.
$A_4 = \{4, 5, 6, 7, 8, 9\}$.

Sets can containing anything, not just numbers:
$A_5 = \{Dalek,\ Tardis,\ Sonic\ Screwdriver,\ Cybarman\}$.

Instead of listing every element, we can use notation with a condition:
$A_6 = \{x \in \mathbb{Z}^+ \mid x \leq 6\ and\ x\ is\ even\}$ is the set {2,4,6}.
$A_7 = \{x \in \mathbb{Z} \mid x\ is\ even\}$ is the set of all even integers.
$A_8 = \{x \in \mathbb{N} \mid x <= 1,000,000,000\}$ is the set of all counting numbers up to one billion.

Since the number 2 is an element of set A_1, we say $2 \in A_1$.
Since *Bow Tie* is not a member of set A_5, we say $Bow\ Tie \notin A_5$.

We can see that there are 5 elements in A_1. We write this as $|A_1|= 5$.
Similarly $|A_2|= 3,\ |A_3|= 3,\ |A_4|= 6,\ |A_5|= 4,\ |A_6|= 3,\ |A_7|= \infty$, and $|A_8|= 1,000,000,000$. Also, since the empty set contains no elements, $|\emptyset|= 0$.

6.2.2 Comparing Two Sets

Proper Subset
$A \subset B$ means every element in A is also in B, and there is at least one element in B that is not in A. Using the above sets, we have $A_2 \subset A_1$. However, $A_2 \not\subset A_4$ because $2 \in A_2$ but $2 \notin A_4$. Also, $A_2 \not\subset A_6$ because, while every element in A_2 is in A_6, there is no element in A_6 that is not in A_2.

Subset
$A \subseteq B$ means every element in A is also in B. We still have $A_2 \subseteq A_1$, but now we also have $A_2 \subseteq A_6$. Also, the empty set is a subset of every set. In other words, for any set A, $\emptyset \subseteq A$.

Set Equality
$A = B$ means A and B have exactly the same elements. This is often expressed as $A \subseteq B$ and $B \subseteq A$. Using the above sets we have:
$A_2 = A_6$
$\mathbb{N} = \mathbb{Z}^+$
$\mathbb{W} = \mathbb{Z}^{nonneg}$

6.2.3 Creating New Sets from Old

Union
$A \cup B$ is the set containing every element that is in A or in B. So:
$A_1 \cup A_2 = \{2, 3, 4, 5, 6\}$.
$A_1 \cup A_3 = \{2, 3, 4, 5, 6, 7\}$.

Inersection
$A \cap B$ is the set containing every element that is in A and in B. So:
$A_1 \cap A_2 = \{2, 4, 6\}$.
$A_1 \cap A_3 = \{3\}$.
$A_2 \cap A_3 = \emptyset$.

Set Difference
$A - B$ is the set containing all elements that are in A but not in B. You can think of this as starting out with the elements of A and removing everything that is also in B.
$A_1 - A_2 = \{3, 5\}$.
$A_2 - A_1 = \emptyset$.
$A_4 - A_7 = \{5, 7, 9\}$.
$\mathbb{Z}^{nonneg} - \mathbb{Z}^+ = \{0\}$.

Complement
If set A is a subset of a universal set U, then A^c is the set of all elements in U that are not in A. So if the universal set for A_1 is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, then $A_2^c = \{1, 3, 5, 7, 8, 9, 10\}$.

Cartesian Product
$A \times B$ is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.
$A_2 \times A_3 = \{(2, 3), (2, 5), (2, 7), (4, 3), (4, 5), (4, 7), (6, 3), (6, 5), (6, 7)\}$
Notice that $|A \times B| = |A| \cdot |B|$.

Power Sets
$\mathscr{P}(A)$ is the set containing all subsets of A.
$\mathscr{P}(A_2) = \{\emptyset, \{2\}, \{4\}, \{6\}, \{2, 4\}, \{2, 6\}, \{4, 6\}, \{2, 4, 6\}\}$
Note, $|\mathscr{P}(A)| = 2^{|A|}$

6.3 Set Identities

Let all sets referred to below be subsets of a universal set U.

1.	Commutative laws:	$A \cap B = B \cap A$	$A \cup B = B \cup A$
2.	Associative laws:	$(A \cap B) \cap C = A \cap (B \cap C)$	$(A \cup B) \cup C = A \cup (B \cup C)$
3.	Distributive laws:	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4.	Identity laws:	$A \cap U = A$	$A \cup \emptyset = A$
5.	Complementation laws:	$A \cup A^c = U$	$A \cap A^c = \emptyset$
6.	Double complement law:	$(A^c)^c = A$	
7.	Idempotent laws:	$A \cap A = A$	$A \cup A = A$
8.	De Morgan's laws:	$(A \cup B)^c = A^c \cap B^c$	$(A \cap B)^c = A^c \cup B^c$
9.	Universal bound law:	$A \cup U = U$	$A \cap \emptyset = \emptyset$
10.	Absorption law:	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
11.	Complements of U and $\emptyset$:	$U^c = \emptyset$	$\emptyset^c = U$
12.	Set Difference	$A - B = A \cap B^c$	

7 Graphs

A *graph* $G = (V, E)$ consists of an ordered pair of sets, V and E. The ***vertex set*** V is a non-empty set of objects called ***vertices***. The ***edge set*** $E = \{\{u, v\} \mid u, v \in V\}$ is a set of unordered pairs of vertices, called ***edges***. The vertex set of G can be written as $V(G)$ and the edge set of G can be written as $E(G)$.

The number of vertices in G is called the ***order*** of G. The order of a graph is commonly denoted as n.

The number of edges in G is called the ***size*** of G. The size of a graph is commonly denoted as m.

A ***simple graph*** is a graph that does not have any self-loops or parallel edges.

A ***complete graph*** on n vertices, denoted K_n, is a simple graph with n vertices whose set of edges contains exactly one edge for each pair of distinct vertices.

A graph is ***k edge connected*** if there does not exist a set of k edges whose removal disconnects the graph. The *maximum* edge connectivity of a given graph is the smallest degree of any vertex, since deleting these edges disconnects the graph.

A graph is ***k vertex connected*** (or simply k-connected) if there does not exist a set of k vertices whose removal disconnects the graph.

A ***complete bipartite graph*** on (m, n) vertices, denoted $K_{m,n}$, is a simple graph with vertices $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ that satisfies the following properties:

$\forall i, k = 1, 2, \ldots, m$ and $\forall j, l = 1, 2, \ldots, n$,

1. there is an edge from each vertex u_i to each vertex v_j
2. there is not an edge from any vertex u_i to any other vertex u_k
3. there is not an edge from any vertex v_j to any other vertex v_l

A simple graph is ***planar*** if it can be drawn in the plane with no intersecting edges.

A ***Delauney Triangulation*** is a triangulation win which for each $\triangle uvw$ the circumcircle $disk(u, v, w)$ does not contain any other vertices. The Delauney triangulation is a triangulation with the smallest possible edge length.

A ***Gabriel graph***, denoted $GG(V)$ contains an edge $\{u, v\}$ iff the disk with diameter $\overline{uv}$ contains no other vertices.

A ***relative neighborhood graph***, denoted $RNG(V)$, is a graph in which $\{a, b\} \in E$ iff $\forall c \in V, \|ac\| \geq \|ab\|$ or $\|bc\| \geq \|ab\|$. In other words, no vertices lie in the intersection of the disks centered at a and b.

If $\{u, v\}$ is an edge of G, then u and v are ***adjacent vertices*** in G. u and v are then called ***neighbors*** in G. Edge $\{u, v\}$ is ***incident*** to vertex u and to vertex v.

Distinct edges that share a common vertex are ***adjacent edges***.

A graph H is a ***subgraph*** of graph G iff $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. This is denoted as $H \subseteq G$.

A graph H is a ***proper subgraph*** of graph G iff $H \subseteq G$ and either $V(H) \subset V(G)$ or $E(H) \subset E(G)$. This is denoted as $H \subset G$.

A graph H is a ***spanning subgraph*** of graph G iff $H \subseteq G$ and $V(H) = V(G)$.

A subset of vertices $F \subseteq V(G)$ gives rise to the graph $\langle F \rangle$, called the ***subgraph of*** G ***induced by*** F. The vertices in the induced subgraph are $V(\langle F \rangle) = F$. The edges are $E(\langle F \rangle) = \{\{u, v\} \mid u, v \in F \textit{ and } \{u, v\} \in E(G)\}$. To emphasize it is an induced subgraph of G, $\langle F \rangle$ is often written as $G[F]$.

A subset of edges $X \subseteq E(G)$ defines the graph $\langle X \rangle$, called the ***subgraph of*** G ***induced by*** X. The edges in $\langle X \rangle$ are given by $E(\langle X \rangle) = X$. The vertices are given by $V(\langle X \rangle) = \{u \mid \{u, v\} \in X \textit{ or } \{v, u\} \in X\}$. To emphasize it is an induced subgraph of G, $\langle X \rangle$ is often written as $G[X]$.

A ***directed graph*** $G = (V, E)$ consists of an ordered pair of sets, V and E. The ***vertex set*** V is a non-empty set of objects called ***vertices***. The ***edge set*** $E = \{(u, v) \mid u \in V \textit{ and } v \in V\}$ is a set of ordered pairs of vertices.