Example of Proof by Induction using a Graph

1. **Claim:** A tree with n vertices has $n-1$ edges.

   Another way of writing this claim is: For any tree T, $|E(T)|=|V(T)|-1$.

   Yet another way is: For any tree T, $m = n-1$

   Proof. (By Induction)

   Base Case ($n = 1$)

   When $n = 1$ the graph has just a single vertex and no edges. So $n = 1$ and $m = 0$. Hence, the number of edges is one less than the number of vertices. In other words,

   $$m = n - 1$$

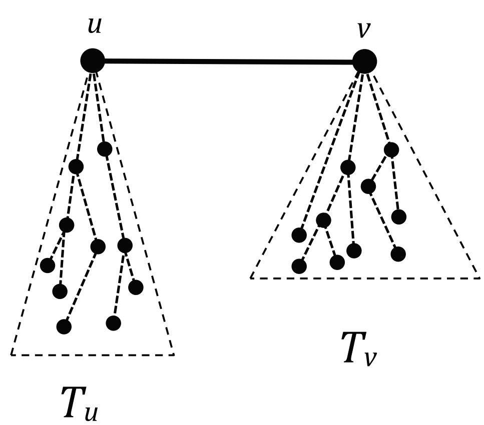
   Inductive Hypothesis

   Assume for any $j \le k$, T_j has $j - 1$ edges.

   Induction step ($n = k + 1$)

   Show that a tree with $k + 1$ vertices has k edges.

   Let T be a tree with $k+1$ vertices. Select an arbitrary edge $\{u, v\}$ in T and consider the subtree rooted at u and the subtree rooted at v. Call these subtrees T_u and T_v. These two subtrees can be drawn as follows

   Let n_u and m_u be the number of vertices and edges in tree T_u.

   Let n_v and m_v be the number of vertices and edges in tree T_v.

   Since $v \notin T_u \quad n_u \le k$

   Since $u \notin T_v \quad n_v \le k$

   So by the Inductive Hypotheses

   $$\begin{aligned} m_u &= n_u - 1 \\ m_v &= n_v - 1 \end{aligned}$$

   Notice that $V(T) = V(T_u) \cup V(T_v)$

   Which means $|V(T)| = n_u + n_v = k + 1$

Also notice that $E(T) \;=\; E(T_u) \;\;\cup\;\; E(T_v) \;\;\cup\;\; \{\{u,v\}\}$
Which means

$$\begin{aligned}
|E(T)| &= m_u + m_v + 1 \\
&= (n_u - 1) + (n_v - 1) + 1 \\
&= n_u - 1 + n_v - 1 + 1 \\
&= n_u + n_v - 1 \\
&= (n_u + n_v) - 1 \\
&= (k+1) - 1 \\
&= k
\end{aligned}$$

$\therefore$ A tree with n vertices has $n-1$ edges. $\square$