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Associação Ludus, Museu Nacional de História Natural e da Ciência,
Rua da Escola Politécnica 56 1250-102 Lisboa, Portugal

email: rmm@ludus-opuscula.org
URL: http://rmm.ludus-opuscula.org

Informations
The *Recreational Mathematics Magazine* is electronic and semiannual.
The issues are published in the exact moments of the equinox.
The magazine has the following sections (not mandatory in all issues):

Articles
Games and Puzzles
Problems
MathMagic
Mathematics and Arts
Math and Fun with Algorithms
Reviews
News

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Editorial

The *Recreational Mathematics Magazine* is published by the *Ludus Association* with the support of the *Interuniversity Centre for the History of Science and Technology*. The journal is electronic and semiannual, and focuses on results that provide amusing, witty but nonetheless original and scientifically profound mathematical nuggets. The issues are published in the exact moments of the equinox. It's a magazine for mathematicians, and other mathematics lovers, who enjoy imaginative ideas, non-standard approaches, and surprising procedures. Occasionally, some papers that do not require any mathematical background will be published.

Examples of topics to be addressed include: games and puzzles, problems, mathmagic, mathematics and arts, history of mathematics, math and fun with algorithms, reviews and news.

Recreational mathematics focuses on insight, imagination and beauty. Historically, some areas of mathematics are strongly linked to recreational mathematics - probability, graph theory, number theory, etc. Thus, recreational mathematics can also be very serious. Many professional mathematicians confessed that their love for math was gained when reading the articles by Martin Gardner in *Scientific American*.

While there are conferences related to the subject as the amazing *Gathering for Gardner*, and high-quality magazines that accept recreational papers as the *American Mathematical Monthly*, the number of initiatives related to this important subject is not large.

This context led us to launch this magazine. We will seek to provide a quality publication. Ideas are at the core of mathematics, therefore, we will try to bring in focus amazing mathematical ideas. We seek sophistication, imagination and awe.

Lisbon, 1st March, 2014,
Ludus Association
Interuniversity Centre for the History of Science and Technology

Problems

The Mutilated Chess Board (Revisited)

Colin Wright
Liverpool Mathematical Society
maths@solipsys.co.uk; @ColinTheMathmo; www.solipsys.co.uk

Abstract: *The problem of the Mutilated Chessboard is much loved and familiar to most of us. It's a wonderful example of an impossibility proof, and at the same time completely accessible to people of all ages and experience. In this work we will delve a little deeper, find an unexpected connection, and discover the true power of abstract mathematics while still seeing the concrete applications in action.*

Key-words: impossibility proofs, Marriage Theorem, matching problems.

Puzzle enthusiasts know that a really good puzzle is more than just a problem to solve. The very best problems and puzzles can provide insights that go beyond the original setting. Sometimes even classic puzzles can turn up something new and interesting.

Some time ago while re-reading one of Martin Gardner's books [1] I happened across one of these classics, the problem of the Mutilated Chess Board, and was surprised to learn something new from it. That's what I want to share with you here. The way I present it here is slightly unusual - bear with me for a moment.

So consider a chess board, and a set of dominos, each of which can cover exactly two squares. It's easy to cover the chess board completely and exactly with the dominos, and it will (rather obviously) require exactly 32 dominos to do so. We might wonder in how many ways it can be covered, but that's not where I'm going.

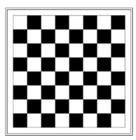


Figure 1: Standard Chessboard.

If we mutilate the chess board by removing one corner square (or any single square, for that matter) then it's clearly impossible to cover it exactly, because now it would now take $31,5$ dominos.

Something not often mentioned is that cutting off two ***adjacent*** corners leaves the remaining squares coverable, and it takes 31 dominos. It's not hard - pretty much the first attempt will succeed.

The classic problem then asks - is it possible to cover the board when two ***opposite*** corners are removed? The first attempt fails, and the second, and the third, and after a while you start to wonder if it's possible at all.

One of the key characteristics of mathematicians and puzzlers is that they don't simply give up, they try to prove that it's impossible.

Insight strikes when (if!) you realise that every attempt leaves two squares uncovered, and they are the same colour.

Why is this important? The reason is simple. Each domino must cover exactly one black and one white, and the two squares we've removed are the same colour. As we cover pairs of squares, we must cover the same number of blacks as whites. If the number of blacks and whites is unequal, it can't be done.

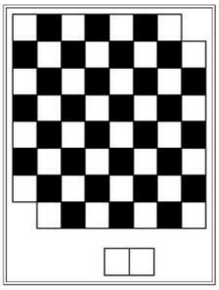


Figure 2: Mutilated Chessboard.

It's a wonderful example of being able to show that something is impossible, without having to examine all possible arrangements.

And that's where the discussion usually stops. Some people look for generalisations in different ways, and that has led to some interesting extensions of the ideas, but again, that's not where I'm going. Clearly in order to cover the mutilated board it's necessary that there be the same number of blacks as whites. Is this sufficient?

Let's start with the simple cases.

If you remove any single black and single white and try to cover the remainder it seems always to be possible. Indeed, in 1973 mathematician Ralph E. Gomory [3] showed with a beautifully simple and elegant argument that it is always possible to cover the board if just one black and one white square are removed. So that's a good start.

What about if two whites and two blacks are removed? Clearly if one corner gets detached by the process then it's no longer coverable, but let's suppose the chess board remains connected. Can it always be covered?

Yes it can, although I'll leave the details to the interested reader. I've yet to find a truly elegant argument (and would be interested in seeing one. Please.)

And so we continue. What if three whites and three blacks are removed (still leaving the board connected.) Can it then ***always*** be covered?

No.

Oh. Well, that was quick.

Challenge: Find an example of a chess board with three whites and three blacks removed, still connected, but can't be covered. There is a hint later [X].

So consider. Clearly it's necessary that the number of blacks is the same as the number of whites, but equally clearly it's not sufficient. So what condition is both necessary and sufficient? Specifically:

Given a chess board with some squares removed (but still connected), can we show that it's not possible to cover it with dominos, without having to try every possible arrangement?

Given that this is a paper for the Recreational Mathematics Magazine it won't surprise you that the answer is yes. It's an elegant and slightly surprising result concerning matchings between collections, and goes by the name of "Hall's Theorem", or sometimes "Hall's Marriage Theorem" [2]. I'll explain it briefly here, and then show an unexpected application [Z].

The original setting is this. Suppose we have a collection of men and a collection of women, and each woman is acquainted with some of the men. Our challenge is to marry them all off so that each woman is only married to a man she knows. Under what circumstances is this possible?

In another formulation, suppose we have a collection of food critics. Naturally enough, they all hate each other, and refuse to be in the same room as each other. Also unsurprisingly, each has restaurants that they won't ever eat at again. Is it possible for them all to eat at a restaurant they find acceptable, but without having to meet each other?

These are "Matching" problems - we want to match each item in one collection to an item in the other. In one case we are matching women with suitable men, in another we are matching critics to restaurants. So when can we do this?

Let's consider the marriage problem. If we do successfully create such an arrangement then we know that every woman must know at least one man, namely, her husband. If some woman is acquainted with none of the men, then clearly it's impossible. But we can go further. If a matching is possible, then any collection of, say, k women will collectively know their husbands (and possibly more). So to get a matching, any (sub-)collection of k women must between them know at least k men, otherwise it's impossible.

The surprising result is that this condition is sufficient.

The proof isn't difficult, proceeding by induction in a fairly straight-forward manner (although undergrads do seem to find it intricate and tricky, but that's probably just insufficient familiarity with the techniques) and we won't take the time here to go through it.

The result was originally stated and proved as a lemma by Philip Hall in 1935 *while proving a significant result in algebra. Later, Marshall Hall Jr. (no relation) realised that the result could be generalised, and produced a paper in which he extended and enhanced the result, but he named it after Philip Hall, thus forever producing confusion for all.*

So how does this solve our problem with the mutilated chess board?

A covering of dominos matches each white square with a black square, so we have two collections that we are trying to match. If some collection of white squares collectively are not attached to enough black squares, a covering is impossible.

Therefore:

- To show that a mutilated chess board is coverable - cover it.
- To show that it is not coverable, demonstrate a collection of white squares which are collectively attached to insufficiently many black squares (*or vice versa*).

We can now use this result to create a minimal, connected, uncoverable set of squares. Here's how.

To be uncoverable we must have a collection of black squares that collectively are connected to insufficiently many white squares. If we used just one black square then it would have to be connected to no squares at all, and so it would be disconnected, so we must use at least two black squares. They must then be attached to just one white square.

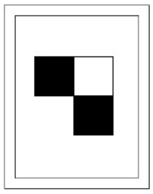


Figure 3: One of 2 configurations.

Now we need at least one more white square to balance the numbers, but that can't be connected to either of the black squares we already have. That means we need another black square. We now have a white square with three black neighbours, and so we need two more white squares to balance the numbers. These must only be attached to one of the black squares, and there's only one way to do that.

And we're done! By construction, we have a provably minimal uncoverable configuration.

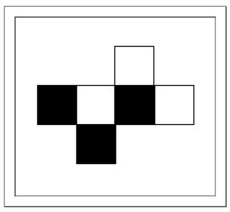


Figure 4: Uncoverable.

If you like you can see how every possible configuration of four can be covered, and every other configuration of six can also be covered. This is the unique shape. It also lets us answer the question above about the mutilated chess board with three of each colour missing. That's a hint for the puzzle [X] above.

And so finally to the surprise application I mentioned above [Z].

Take a deck of cards, shuffle (or not) as you will, then deal out 13 piles of 4 cards each. Some pile will have an ace, possibly more than one. Some pile will have at least one deuce, and so on.

But here's an interesting thing. No matter how you shuffle, or how you deal, it will be possible to draw a full straight - ace through king - taking just one card from each pile.

So remove those cards to leave thirteen piles of three. Now we can do it all a second time. And a third time.

And now we have just thirteen cards left, one of each rank, so we can do it a fourth and final time.

Alternatively, deal the cards into four piles of thirteen. It won't surprise you that it's possible to draw a card of each suit, one from each pile. What may surprise you is that you can do it again, and again, and again, and so on, right through to the end.

It's not that hard to accomplish these feats, a bit of fiddling will find a solution, but it is interesting that it's always possible. It seems plausible that some sort of magic trick could be devised that uses this principle, although I really don't see how. Maybe it can't, but I'd be fascinated to see one.

So that's a challenge - show that it's always possible to make such selections.

I'll give you a hint. It uses Hall's Theorem.

Acknowledgements:

Hat tips to Mikael Vejdemo-Johansson and David Bedford. Thank you!

References

[1] Martin Gardner. *My Best Mathematical and Logic Puzzles*. Dover Publications, 1994.

[2] Hall's Marriage Theorem: `http://en.wikipedia.org/wiki/Hall's_marriage_theorem`

[3] Honsberger, R. *Mathematical Gems I, Mathematical*. Mathematical Association of America, 1973.

Games and Puzzles

Vanishing Area Puzzles

David Singmaster
London South Bank University
zingmast@gmail.com

In Memoriam Martin Gardner

Abstract: *Martin Gardner was very fond of vanishing area puzzles and devoted two chapters to them in his first book. There are actually two distinct types. Sam Loyd's Vanishing Chinaman and similar puzzles have pictures which are reassembled so a part of the picture appears to disappear. But the physical area remains fixed. The second type cuts up an area and reassembles it to produce more or less area, as in the classic chessboard dissection which converts the* 8×8 *square into a* 5×13 *rectangle. Gardner had managed to trace such puzzles back to Hooper in* 1774*. In* 1989*, I was visiting Leipzig and reading Schwenter which referred to an error of Serlio, in his book of* 1535*. Serlio hadn't realised that his dissection and reassembly gained area, but it is clear and this seems to be the origin of the idea. I will describe the history and some other versions of the idea.*

Key-words: Sam Loyd, vanishing area puzzles.

One of Sam Loyd's most famous puzzles is "The Vanishing Chinaman" or "Get off the Earth". Martin Gardner discussed this extensively in [7] and [8].

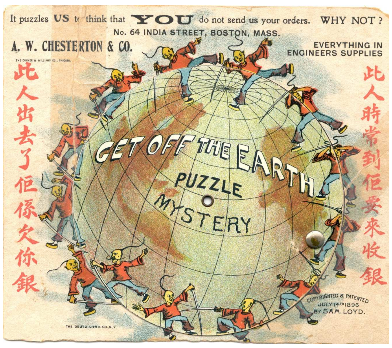
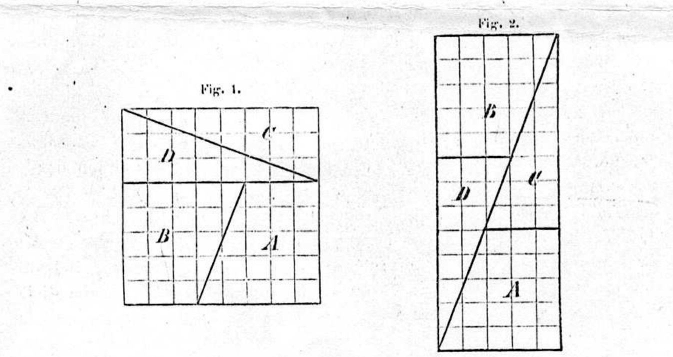


Figure 1: "Get off the Earth".

However, the term "vanishing area puzzle" is used for two different types of puzzle. In "The Vanishing Chinaman", no actual area vanishes - it is one of the figures in the picture that vanishes, so perhaps we should call this a "vanishing object puzzle". In a true "vanishing area puzzle", an area is cut into several pieces and reassembled to make an area which appears to be larger or smaller than the original area. These are considerably older than vanishing object puzzles. Martin was fond of these puzzles - indeed, his first puzzle book [6] devotes two chapters to such puzzles - still the best general survey of them - and he also wrote several columns about them [7, 8]. The most famous version of these is the "Checkerboard Paradox" where an 8×8 checkerboard is cut into four parts and reassembled into a 5×13 rectangle, with a net gain of one unit of area. This article is primarily concerned with the early history of such puzzles.

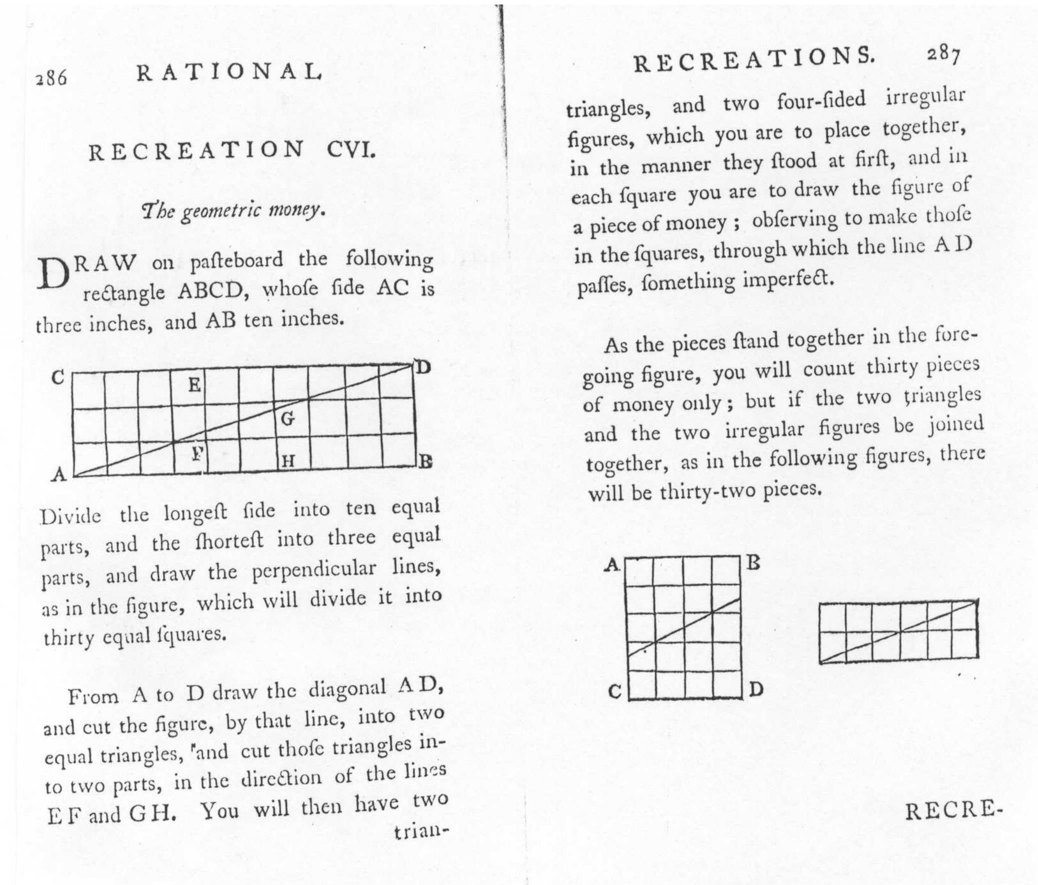
IX. Ein geometrisches Paradoxon. Um ad oculos zu demonstriren, dass das Schachbret nicht nur 64, sondern auch 65 Felder besitzt, schneide man dasselbe aus starkem Papier, zerlege es auf die in Fig. 1



angegebene Weise in vier, zu je zweien congruente Stücke *A*, *B*, *C*, *D* und setze diese zu einem Rechtecke zusammen, welches, wie Fig. 2 zeigt, die Grundlinie 5 und die Höhe 13 besitzt also 65 Felder enthält. — Wir theilen diese kleine Neckerei mit, weil die Aufsuchung des begangenen Fehlers eine hübsche Schüleraufgabe bildet und weil sich an die Vermeidung des Fehlers die Lösung und Construction einer quadratischen Gleichung knüpfen lässt. Schl.

Figure 2: Schlomilch.

Gardner describes some 18 & 19C versions of this puzzle idea, going back to Hooper (1774) [9], and a surprising connection with Fibonacci numbers discovered in 1877 [4]. In 1988, I was visiting Leipzig and looked at some obscure books in their library and discovered references going back to 1537.



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The geometric money.

DRAW on paſteboard the following rectangle ABCD, whoſe ſide AC is three inches, and AB ten inches.

Divide the longeſt ſide into ten equal parts, and the ſhorteſt into three equal parts, and draw the perpendicular lines, as in the figure, which will divide it into thirty equal ſquares.

From A to D draw the diagonal AD, and cut the figure, by that line, into two equal triangles, and cut thoſe triangles into two parts, in the direction of the lines EF and GH. You will then have two trian-

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triangles, and two four-ſided irregular figures, which you are to place together, in the manner they ſtood at firſt, and in each ſquare you are to draw the figure of a piece of money; obſerving to make thoſe in the ſquares, through which the line AD paſſes, ſomething imperfect.

As the pieces ſtand together in the foregoing figure, you will count thirty pieces of money only; but if the two triangles and the two irregular figures be joined together, as in the following figures, there will be thirty-two pieces.

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Figure 3: Hooper.

Let us look at the classic 8×8 to 5×13 of Fig. 2. The history of this particular version is obscure. It is shown in Loyd's Cyclopedia [10] (p. 288 & 378).

Tommy Riddles tells us that we need know nothing about checkers or chess to solve these puzzles. King Puzzlepate is trying to place the greatest number of men on a chess board without having three men in line in any possible direction. He has started by placing the first two men correctly; now it is up to you to assist him by adding as many men as possible without getting any three in line.

We are told that the first checkerboard ever constructed, which was made by a man by the name of Siesa, and is still preserved in the British Museum, is made of four pieces, as the one shown in the second puzzle. Now the four pieces of this board can be rearranged together so as to make three different puzzles: A square board of 64 squares, an oblong one of 65, or an odd-shaped one of but 63. It is said that Dr. Slasher won the championship by this marvelous coup of arranging the four pieces so as to reduce the board to 63 squares. See if you are able to do it. There has been so much discussion regarding this paradoxical problem that occasion is taken to say that Mr. Loyd presented it before the first American Chess Congress in 1858.

A Charade.

I am what I was, which is so much the worse,
I'm not what I was, but quite the reverse;
From morning till night I do nothing but fret,
And sigh to be what I never was yet.

A Charade.

My first, a substance hard and bright,
Is useful, morning, noon, and night;
My second, find it where you will,
Is of the same dimension still:
And by my whole, I often try,
Butchers' and grocers' honesty.

An English tourist in the wild and woolly West was informed that if he wished to walk on to Piketown the stage would only get there one mile ahead of him for although it would get to a certain wayhouse while you were walking four miles, it waits there 30 minutes, so you would catch up in time to ride on to Piketown if you wished. "But," as the host of the Pilgrim's hotel remarked, "from these facts there is a clever way of figuring out how to beat the stage by 15 minutes!" Can you tell how far it was from the hotel to Piketown?

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Figure 4: LoydCyc288.

For a long time, I tended to ignore this as it seemed smudged. I later saw that he shows that both rectangles have chessboard colouring, and he is the first to indicate this. But when I went to scan Fig. 5, I realised that the smudge is

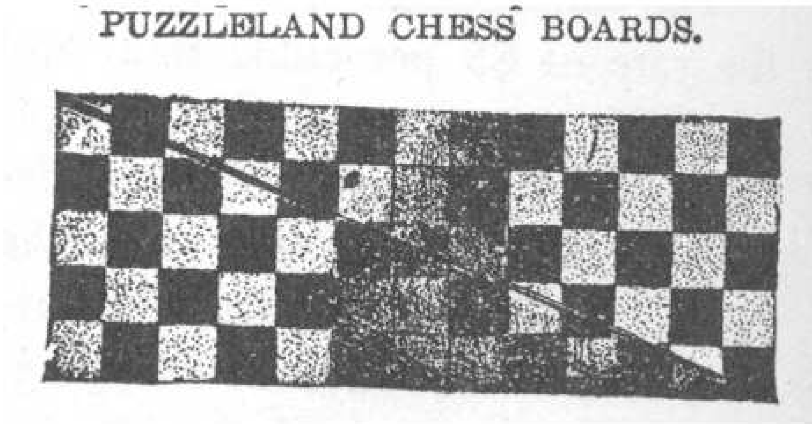


Figure 5: LoydCyc278.

deliberate to conceal the fact that the colouring does NOT match up! Indeed, the corners of a 5×13 board should all be the same colour, but two of them in the solution arise from adjacent corners of the 8×8 chessboard and have opposite colours!

Loyd also poses the related problem of arranging the four pieces to make a figure of area 63, as in Fig. 3.

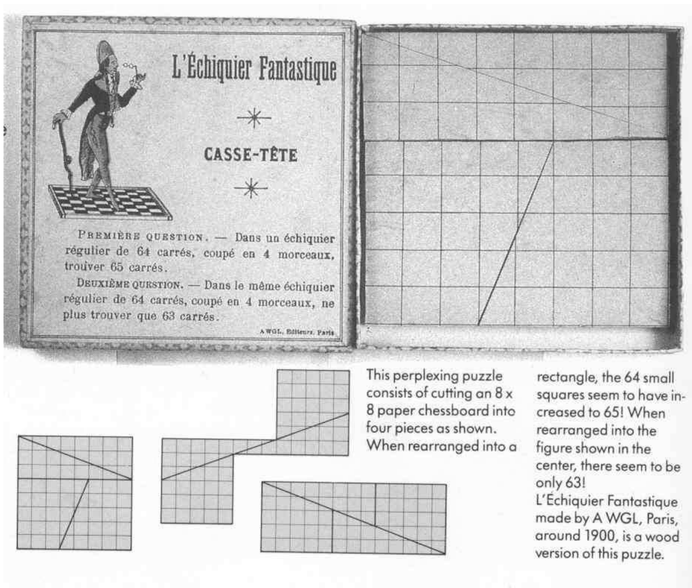


Figure 6: AWGL.

The oldest known version of this is an actual puzzle, dated c1900, [1], shown in [18] and there is a 1901 publication [5].

Loyd asserts he presented “this paradoxical problem” to the First American Chess Congress in 1858, but it is not clear if he means the area 65 version or the area 63 version. Loyd would have been 17 at the time. If this is true, he is ten years before any other appearance of the area 65 puzzle and about 42 years before any other appearance of the area 63 puzzle. I am dubious about this as Loyd did not claim this as his invention in other places where he was describing his accomplishments. In 1928, Sam Loyd Jr. [11] describes the area 63 version as something he had discovered, but makes no claims about the area 65 form, although he often claimed his father’s inventions as his own. For example, on p. 5, he says “My ”Missing Chinaman Puzzle” of 1896.

The first known publication of the 8×8 to 5×13, Fig. 2, is in 1868, in a German mathematical periodical, signed Schl. [14]. In 1938, Weaver [12] said the author was Otto Schlömilch, and this seems right as he was a co-editor of the journal at the time. In 1953, Coxeter [3] said it was V. Schlegel, but he apparently confused this with another article on the problem by Schlegel. Schlömilch doesn’t give any explanation for this “teaser”, leaving it as a student exercise!

In 1886, a writer [13] says: “We suppose all the readers . . . know this old puzzle.”

By 1877 [4], it was recognised that the paradox is related to the fact that $5 \times 13 - 8 \times 8 = 1$ and that 5, 8, 13 are three consecutive Fibonacci numbers. Taking a smaller example based on the numbers 2, 3, 5 makes the trickery clear.

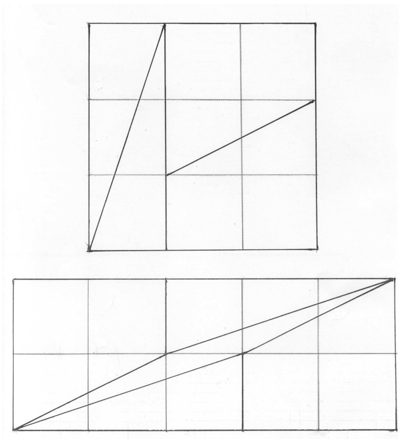


Figure 7: Nine to Ten.

One can also make a 5×5 into a 3×8, but then there is a loss of area from the square form.

However, there are other versions of vanishing area or object puzzles. Since

1900 several dozen have been devised and there are examples where both some area and an object vanish!

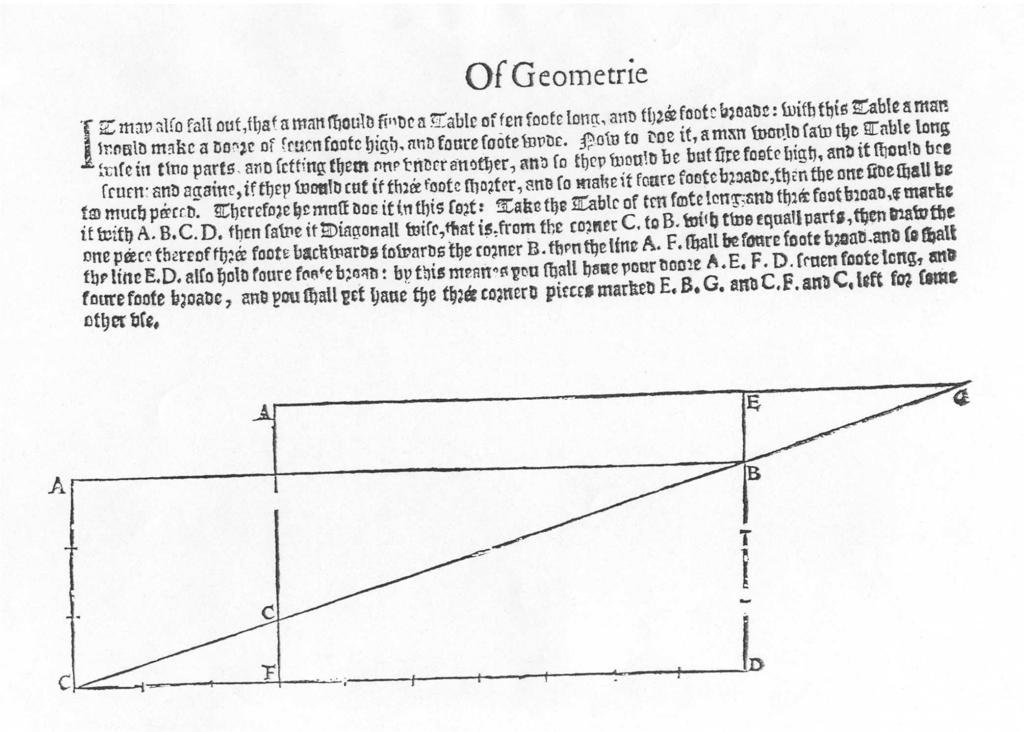
Figure 8: CreditSqueeze.

This is a fairly common magician's trick, taking a 7×7 square array of "£" signs and rearranging to get a 7×7 square array of "£" signs and an extra unit square containing an extra "£" sign. My version, called "Credit Squeeze", using "£", is attributed to Howard Gower, but Michael Tanoff kindly obtained for me an American version, using \$, called "It's Magic DOLLAR DAZE", produced by Abbott's with no inventor named. Lennart Green uses a version of this in his magic shows, but he manages to reassemble it three times, getting an extra piece out each time! Needless to say, this involves further trickery. A version of this is available on the Internet.

But there are earlier examples. Gardner and others tracked the idea back to Hooper [9] in 1783, as seen in Fig. 3. Here we have a 3×10 cut into four pieces which make a 2×6 and a 4×5. However, Hooper's first edition of 1774 erroneously has a 3×6 instead of a 2×6 rectangle and notes there are now 38 units of area. This was corrected in the second edition of 1783 and this version occurs fairly regularly in the century following Hooper.

In 1989, I visited Leipzig and was reading Gaspar Schott [15] where I found a description of a version due to the 16C architect Serlio [17]. I managed to find the Serlio reference, which is in his famous treatise in five books on architecture.

Sebastiano Serlio (1475-1554) was born in Bologna and worked in Rome in 1514-1527 with the architect Peruzzi. He went to Venice and began publishing his Treatise, which appeared in five parts in 1537-1547. This was a practical book and greatly influential. He influenced Inigo Jones and Christopher Wren. Wren's design of the Sheldonian Theatre in Oxford is based on Serlio's drawings of the Roman Theatre of Marcellus. Serlio also describes the "Chinese Lattice" method of spanning a roof using beams shorter than the width. This was studied by Wallis, leading to a system of 25 linear equations for the Sheldonian roof. In 1541, François I summoned Serlio to France and he founded the classical school of architecture in France. He designed the Château of Ancy-le-Franc and died at Fontainebleu.



Of Geometrie

It may also fall out, that a man should finde a Table of ten foote long, and three foote broade: with this Table a man would make a doore of seven foote high, and foure foote wyde. Now to doe it, a man would saw the Table long wise in two parts, and setting them one under another, and so they would be but sixe foote high, and it should bee seven: and againe, if they would cut it three foote shorter, and so make it foure foote broade, then the one side shall be too much peeced. Therefore he must doe it in this sort: Take the Table of ten foote long, and three foot broad, & marke it with A. B. C. D. then sawe it Diagonall wise, that is, from the corner C. to B. with two equall parts, then draw the one peece thereof three foote backwards towards the corner B. then the line A. F. shall be foure foote broad, and so shall the line E. D. also hold foure foote broad: by this meanes you shall have your doore A. E. F. D. seven foote long, and foure foote broade, and you shall yet have the three cornerd pieces marked E. B. G. and C. F. and C. left for some other use.

Figure 9: SerlioRot.

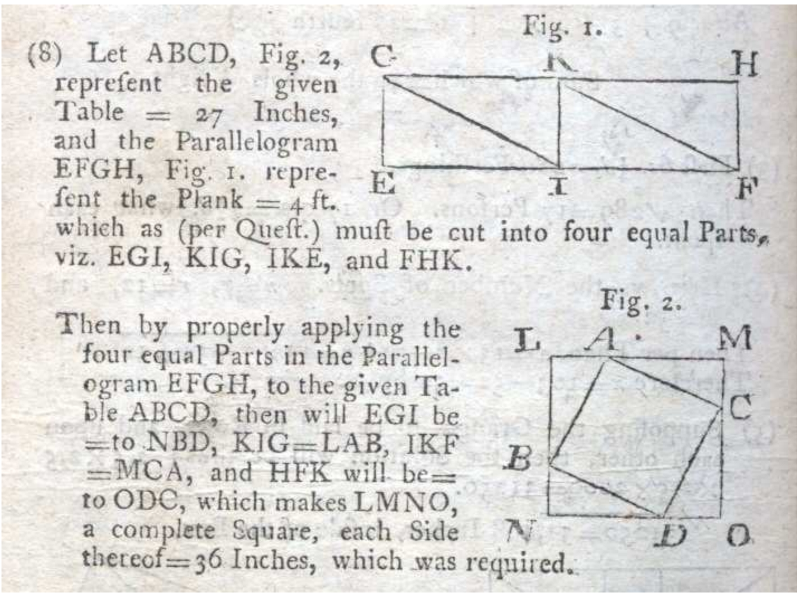
Fig. 9 comes from the 1982 Dover English edition, f. 12v. He is taking a 3×10 board and cuts it diagonally, then slides one piece by 3 to form an area 4×7 with two bits sticking out, which he then trims away. He doesn't notice that this implies that the two extra bits form a 1×3 rectangle and hence doesn't realize the change in area implied.

Already in 1567, Pietro Cataneo [2] pointed out the mistake and what the correct process would be.

I later found a discussion of this in Schwenter [16], citing another architect, so this was well known in the $16-17$C, but the knowledge disappeared despite the fact that Serlio's book has been in print in Italian, Dutch, French and English since that time and Schwenter was fairly well-known.

Since that time, I have found two other late 18C examples, possibly predating Hooper.

Charles Vyse's *The Tutor's Guide* [19], was a popular work, going through at least 16 editions, during 1770-1821. The problem is: "A Lady has a Dressing Table, each side of which is 27 Inches, but she is desirous to know how each Side of the same may = 36 Inches, by having 4 foot of Plank, superficial Measure, joined to the same. The Plan in what Manner the Plank must be cut and applied to the Table is required?" [The plank is one foot wide.] The solution is in: *The Key to the Tutor's Guide* [19] (p. 358).


(8) Let ABCD, Fig. 2, repreſent the given Table = 27 Inches, and the Parallelogram EFGH, Fig. 1. repreſent the Plank = 4 ft. which as (per Queſt.) muſt be cut into four equal Parts, viz. EGI, KIG, IKE, and FHK.

Then by properly applying the four equal Parts in the Parallelogram EFGH, to the given Table ABCD, then will EGI be =to NBD, KIG=LAB, IKF =MCA, and HFK will be= to ODC, which makes LMNO, a complete Square, each Side thereof=36 Inches, which was required.

Figure 10: Vyse's Solution.

She cuts the board into two 12" $\times$ 24" rectangles and cuts each rectangle along a diagonal. By placing the diagonals of these pieces on the sides of her table, she makes a table 36" square. But the diagonals of these triangles are $12\sqrt{5} = 26.83\ldots$".

Note that $27^2 + 12 \times 24 = 1305$ while $36^2 = 1296$. Vyse is clearly unaware that area has been lost. By dividing all lengths by 3, one gets a version where one unit of area is lost. Note that 4, 8, 9 is almost a Pythagorean triple. I have not seen the first edition of this work, but the problem is likely to occur in the first edition of 1770 and hence predates Hooper. Like Serlio, the author is unaware that some area has vanished!

The 1778 edition of Ozanam by Montucla [12] has an improvement on Hooper.

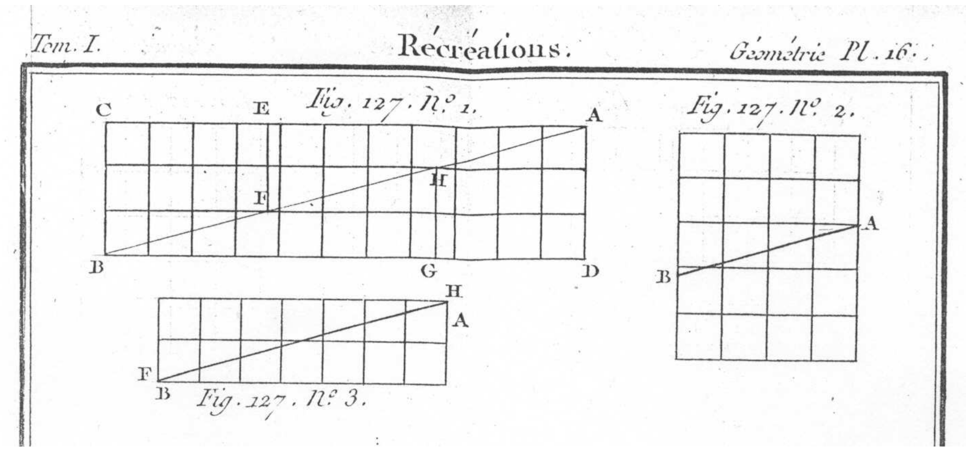


Figure 11: Ozanam.

The image is Fig. 127, plate 16, p. 363 for prob. 21, pp. 302-303 in my copy of the 1790 reissue. This has 3×11 to 2×7 and 4×5. Here just one unit of area is gained, instead of two units as in Hooper. He remarks that M. Ligier probably made some such mistake in showing $172 = 2 \times 122$ and this is discussed further on the later page.

In conclusion, we have found that vanishing area puzzles are at least two hundred years older than Gardner had found. We have also found a number of new forms of the puzzle. Who knows what may turn up as we continue to examine old texts? I think Martin would have enjoyed these results.

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[1]As Chap 3 in each of these.

[2]The image is taken from my copy of the 2nd ed.

[3]This is a reprint of Loyd's Our Puzzle Magazine, a quarterly which started in June 1907 and ran till 1908. From known issues, it appears that these problems would have appeared in Oct 1908 and Jun 1908, but I don't know if any copies of these issues exist.

[4]Numerous editions then appeared in Paris and Amsterdam, some in one volume; About 1723, the work was revised into 4 volumes, sometimes described as 3 volumes and a supplement, published by Claude Jombert, Paris, 1723. "The editor is said to be one Grandin.". In 1778, Jean Étienne Montucla revised this, under the pseudonym M. de C. G. F. [i.e. M. de Chanla, géomètre forézien], published by Claude Antoine Jombert, fils aîné, Paris, 1778, 4 volumes. The author's correct initials appear in the 1790 reissue] This is a considerable reworking of the earlier versions. In particular, the interesting material on conjuring and mechanical puzzles in Vol. IV has been omitted. The bibliography of Ozanam's book is complicated. I have prepared a detailed 7 pp. version covering the 19 (or 20) French and 10 English editions, from 1694 to 1854, as well as 15 related versions - this is part of my *The Bibliography of Some Recreational Mathematics Books*. The above is an extract.

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Articles

Lewis Carroll in Numberland

Robin Wilson
Pembroke College, Oxford University, and the Open University, UK
r.j.wilson@open.ac.uk

Abstract: *Charles Dodgson is best known for his "Alice" books, "Alice's Adventures in Wonderland" and "Through the Looking-Glass", written under his pen-name of Lewis Carroll.*

If he hadn't written them, he'd be mainly remembered as a pioneering photographer, one of the first to consider photography as an art rather than as simply a means of recording images. But if Dodgson had not written the Alice books or been a photographer, he might be remembered as a mathematician, the career he held as a lecturer at Christ Church in Oxford University.

But what mathematics did he do? How good a mathematician was he? How influential was his work?

In this illustrated paper, I'll describe his work in geometry, algebra, logic and the mathematics of voting, in the context of his other activities and, on the lighter side, I present some of the puzzles and paradoxes that he delighted in showing to his child-friends and contemporaries.

Key-words: Lewis Carroll, *Alice's Adventures in Wonderland and Through the Looking-Glass.*

1 Introduction

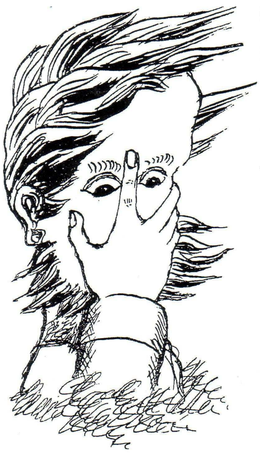
Letter written in verse to Margaret Cunnynghame from Christ Church, Oxford, on 30 January 1868.

Dear Maggie,
No carte has yet been done of me
that does real justice to my smile;
and so I hardly like, you see,
to send you one. Meanwhile,
I send you a little thing
to give you an idea of what I look like

when I'm lecturing.
The merest sketch, you will allow -
yet I still think there's something grand
in the expression of the brow
and in the action of the hand.

Your affectionate friend, C. L. Dodgson

P. S. My best love to yourself, to your Mother my kindest regards,
to your small, fat, impertinent, ignorant brother my hatred. I think that is all.



This letter to Margaret Cunnynghame shows up two aspects of Lewis Carroll - or the Revd. Charles Dodgson (his real name): his love of children and the fact that he was a teacher - in fact, a teacher of mathematics. If he hadn't written the Alice books, he would be mainly remembered as a pioneer Victorian photographer. And if he hadn't been known for that, he'd have been largely forgotten, as an Oxford mathematician who contributed very little. But is that really the case? What mathematics did he do?

Certainly, mathematics pervaded his life and works - even his Alice books (*Alice's Adventures in Wonderland and Through the Looking-Glass*) abound with mathematical language. For example, in the Mock Turtle scene, the Mock Turtle started:

We went to school in the sea.
The master was an old turtle - we used to call him Tortoise.
Why did you call him tortoise if he wasn't one?
We called him tortoise because he taught us.
I only took the regular course. Reeling and writhing, of course, to begin with.
And then the different branches of arithmetic - ambition, distraction,
uglification and derision. And how many hours a day did you do lessons?
Ten hours the first day, nine hours the next, and so on.

What a curious plan!
That's the reason they're called lessons - because they lessen from day to day.

And in *Through the Looking-Glass*, the White Queen and the Red Queen set Alice a test to see whether she should become a queen.

Can you do Addition? What's one and one and one and one and one and one and one and one and one and one?
I don't know. I lost count.
She can't do Addition. Can you do subtraction? Take nine from eight.
Nine from eight I can't, you know; but -
She can't do subtraction. Can you do division? Divide a loaf by a knife. What's the answer to that? Bread-and-butter, of course.
She can't do sums a bit!

Mathematical ideas also appear in his other children's books. In *The Hunting of the Snark*, the Butcher tries to convince the Beaver that 2 plus 1 is 3:

Two added to one - if that could be done,
It said, with one's fingers and thumbs!
Recollecting with tears how, in earlier years,
It had taken no pains with its sums.
Taking Three as the subject to reason about -
A convenient number to state -
We add Seven, and Ten, and then multiply out
By One Thousand diminished by Eight.
The result we proceed to divide, as you see,
By Nine Hundred and Ninety and Two:
Then subtract Seventeen, and the answer must be
Exactly and perfectly true.

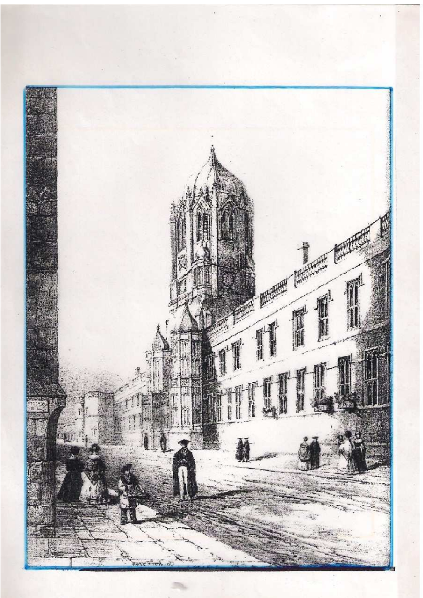
And in his last major novel, *Sylvie and Bruno concluded*, Dodgson's ability to illustrate mathematical ideas in a painless and picturesque way is used in the construction of *Fortunatus's purse* from three handkerchiefs. This purse has the form of a projective plane, with no inside or outside, and so contains all the fortune of the world. Since it cannot exist in three dimensions, he ceases just before the task becomes impossible.

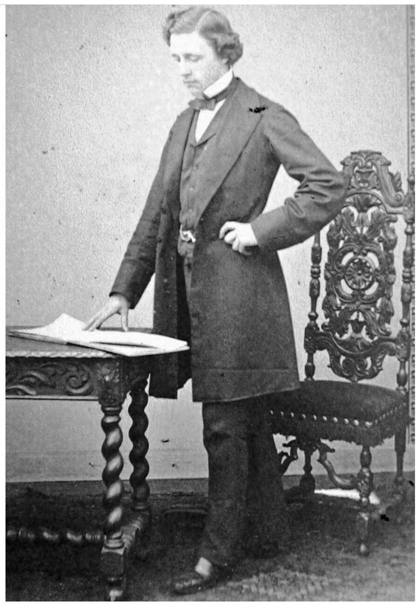
Another quirky result concerned map-making.

There's another thing we've learned from your Nation - map-making. But we've carried it much further than you.
What do you consider the largest map that would be really useful?
About six inches to the mile.
Only six inches! We very soon got to six yards to the mile. Then we tried a hundred yards to the mile. And then came the grandest idea of all!
We actually made a map of the country, on the scale of a mile to the mile!
It has never been spread out, yet. The farmers objected: they said it would cover the whole country and shut out the sunlight! So we now use the country itself, as its own map, and I assure you it does nearly as well.

2 Early Life

Charles Dodgson was born in 1832 into a "good English church family" in Daresbury in Cheshire, where his father, the Reverend Charles Dodgson, was the incumbent until 1843, when they all moved to Croft Rectory in Yorkshire. There he and his seven sisters and three brothers enjoyed a very happy childhood. When he was 14 he was sent to Rugby School, where he delighted in mathematics and the classics, but was never happy with all the rough-and-tumble.





In 1850 he was accepted at Oxford, and went up in January 1851 to Christ Church, the largest college, where he was to spend the rest of his life. His University course consisted mainly of mathematics and the classics, and involved three main examinations, starting in summer 1851 with his Responsions exams.

Letter written in old English from Christ Church, June 1851:

My beloved and thrice-respected sister,
Onne moone his daye nexte we goe yn forre Responsions,
and I amme uppe toe mine
eyes yn worke. Thine truly, Charles.

The next year he took his second Oxford examination - Moderations - gaining a First Class in Mathematics.

Whether I shall add to this any honours at Collections
I can't at present say, but

I should think it very unlikely, as I have only today
to get up the work in The Acts of the
Apostles, 2 *Greek Plays, and the Satires of Horace*
and I feel myself almost totally
unable to read at all: I am beginning to suffer
from the reaction of reading for Moderations.
I am getting quite tired of being congratulated on
various subjects: there seems to be no
end of it. If I had shot the Dean, I could hardly have had more said about it.

In the Summer of 1854, shortly before his Mathematics Finals examinations he went on a reading party to Yorkshire with the Professor of Natural Philosophy, Bartholomew Price - everyone called him "Bat" Price because his lectures were way above the audience. He was immortalized later in the Hatter's song:

Twinkle, twinkle, little bat, How I wonder what you're at ...
Up above the world you fly, Like a tea-tray in the sky,

Dodgson's Finals examinations took place in December 1854, and ranged over all areas of mathematics. Here's a question from that year, in the paper on Geometry and Algebra:

*Compare the advantages of a decimal and of a duodecimal system of notation in reference to (*1*) commerce; (*2*) pure arithmetic; and shew by duodecimals that the area of a room whose length is* 29 *feet* $7\frac{1}{2}$ *inches, and breadth is* 33 *feet* $9\frac{1}{4}$ *inches, is* 704 *feet* $30\frac{3}{8}$ *inches.*

He obtained the top mathematical First of his year:

I must also add (this is a very boastful letter) that I ought to get the Senior Scholarship next term. One thing more I will add, I find I am the next 1*st class Math. student to Faussett so that I stand next for the Lectureship.*

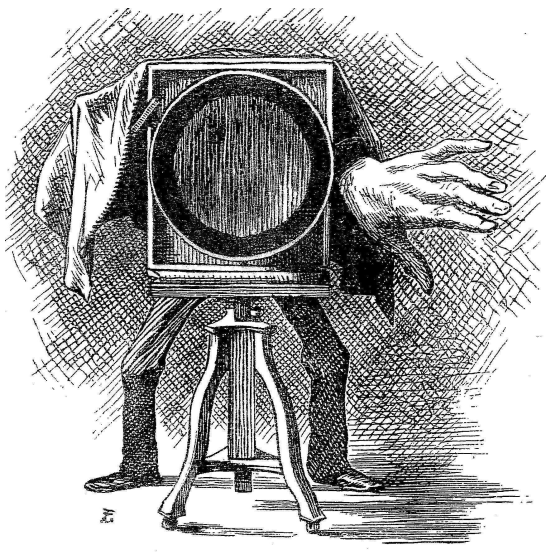
Dodgson did not win the Senior Scholarship, but he was appointed to the Mathematics Lectureship at Christ Church by the new Dean, the Rev. Henry Liddell, who was appointed in 1855 and whose daughters Alice, Edith and Lorina would soon become friends of the young Oxford don. Dodgson became the College's Sub-librarian, years later moving into a sumptuous suite of rooms for which the eminent artist William De Morgan, son of the mathematician Augustus De Morgan, designed the tiles around his fireplace.

In his early years as a Christ Church lecturer Dodgson took up writing. His pen-name Lewis Carroll derived from his real name - Carroll (or Carolus) is the Latin for Charles, and Lewis is a form of Lutwidge, his middle name and mother's maiden name. He used it when writing for children, and in particular for his *Alice* books.

Around this time he also took up the hobby of photography, using the new wet collodion process. He was one of the first to regard photography as an art, rather than as simply a means of recording images, and if he were not known for his Alice books, he would be primarily remembered as a pioneering photographer who took many hundreds of fine pictures - probably the greatest Victorian photographer of children. The Liddell daughters loved spending the afternoon with Mr Dodgson, watching him mix his chemicals, dressing up in costumes, and posing quite still for many seconds until the picture was done. A picture of Alice, dressed as a beggar girl, Alfred Tennyson described as the most beautiful photograph he had ever seen.

From *Hiawatha's photographing*:

From his shoulder Hiawatha
Took the camera of rosewood
Made of sliding, folding rosewood;
Neatly put it all together.
In its case it lay compactly,
Folded into nearly nothing;
But he opened out the hinges,
Pushed and pulled the joints and hinges,
Till it looked all squares and oblongs,
Like a complicated figure
In the Second Book of Euclid ...



3 Geometry

Mentioning Euclid brings us to Dodgson's enthusiasm for the writings of this great Greek author. Knowledge of Euclid's *Elements*, with its axiomatic structure and logical development, was required for all University candidates, as well as for the entrance examinations for the Army and the Civil Service, and dozens of new editions appeared during Dodgson's lifetime.

To help his students, Dodgson produced a *Syllabus of Plane Algebraic Geometry*, described as the 'algebraic analogue' of Euclid's pure geometry, and systematically arranged with formal definitions, postulates and axioms. A few years later he gave an algebraic treatment of the Fifth book of Euclid's *Elements* - on proportion, and possibly due to Eudoxus - taking the propositions in turn and recasting them in algebraic notation.

But in geometry he's best known for his celebrated book *Euclid and his Modern Rivals*, which appeared in 1879. Some years earlier, the Association for the Improvement of Geometrical Teaching had been formed, with the express purpose of replacing Euclid in schools by newly devised geometry books. Dodgson was bitterly opposed to these aims, and his book, dedicated to the memory of Euclid, is a detailed attempt to compare Euclid's *Elements*, favourably in every case, with the well-known geometry texts of Legendre, J. M. Wilson, Benjamin Peirce, and others of the time. It is written as a drama in four acts, with four characters - Minos and Radamanthus (two of the judges in Hades, recast as Oxford dons), Herr Niemand (the phantasm of a German professor), and Euclid himself (who appears to Minos in a dream). After demolishing each rival book in turn, Euclid approaches Minos to compare notes, and to conclude that no other book should take the place of Euclid's *Elements*.

Dodgson's love of geometry surfaced in other places, too. His *Dynamics of a Parti-cle* was a witty pamphlet concerning the parliamentary election for the Oxford University seat. Dodgson started with his definitions, parodying those of Euclid:

Euclid: *A plane angle is the inclination of two straight lines to one another, which meet together, but which are not in the same direction.*

Dodgson: *Plain anger is the inclination of two voters to one another, who meet together, but whose views are not in the same direction.*

Euclid: *When a line, meeting another line, makes the angles on one side equal to those on the other, the angle on each side is called a right angle.*

Dodgson: *When a proctor, meeting another proctor, makes the votes on one side equal to those on the other, the feeling entertained by each side is called right anger.*

Euclid: *An obtuse angle is one which is greater than a right angle.*

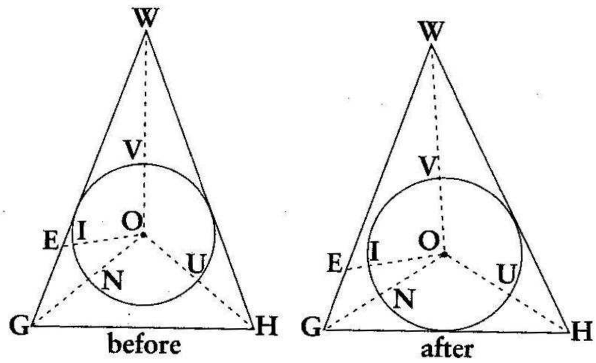
Dodgson: *Obtuse anger is that which is greater than right anger.*

He then introduced his postulates, which again parody those of Euclid:

1. *Let it be granted, that a speaker may digress from any one point to any other point.*
2. *That a finite argument (that is, one finished and disposed with) may be produced to any extent in subsequent debates.*
3. *That a controversy may be raised about any question, and at any distance from that question.*

And so he went on for several pages, leading up to a geometrical construction, where, WEG represents the sitting candidate William Ewart Gladstone (too liberal for Dodgson), GH is Gathorne-Hardy (Dodgson's preferred choice), and WH is William Heathcote, the third candidate.

Let UNIV be a large circle, and take a triangle, two of whose sides WEG and WH are in contact with the circle, while GH, the base, is not in contact with it. It is required to destroy the contact of WEG and to bring GH into contact instead . . . When this is effected, it will be found most convenient to project WEG to infinity.



4 Algebra

Dodgson was an inveterate letter-writer - in the last 35 years of his life, he sent and received some 50000 letters, cataloguing them all. Although many letters were to his brothers and sisters or to distinguished figures of the time, the most interesting ones were to his child-friends, often containing poems, puzzles, and word-games. He had a deep understanding of their minds and an appreciation of their interests, qualities that stemmed from his own happy childhood experiences.

Most of his friendships were with young girls, such as with the Liddell children, but here's a letter to a young lad of 14, Wilton Rix.

Honoured Sir,
Understanding you to be a distinguished algebraist (that is, distinguished from

other algebraists by different face, different height, etc.), I beg to submit to you a difficulty which distresses me much.
If x *and* y *are each equal to* 1, *it is plain that*
$2(x^2 - y^2) = 0$, *and also that* $5(x - y) = 0$.
Hence $2(x^2 - y^2) = 5(x - y)$.
Now divide each side of this equation by $(x - y)$. *Then* $2(x + y) = 5$.
But $x + y = 1 + 1$, *i.e.* $x + y = 2$. *So that* $2 \times 2 = 5$.
Ever since this painful fact has been forced upon me, I have not slept more than 8 *hours a night, and have not been able to eat more than* 3 *meals a day.*
I trust you will pity me and will kindly explain the difficulty to

Your obliged, Lewis Carroll.

In 1865, Dodgson wrote his only algebra book, *An Elementary Treatise on Determinants*, with their *Application to Simultaneous Linear Equations and Algebraical Geometry.* In later years the story went around, which Dodgson firmly denied, that Queen Victoria had been utterly charmed by *Alice's Adventures in Wonderland* -"Send me the next book Mr Carroll produces" - the next book being the one on determinants - "We are not amused." Unfortunately, Dodgson's book didn't catch on, because of his cumbersome terminology and notation, but it did contain the first appearance in print of a well-known result involving the solutions of simultaneous linear equations. It also included a new method of his for evaluating large determinants in terms of small ones, a method that Bartholomew Price presented on his behalf to the Royal Society of London, who subsequently published it in their *Proceedings.*

5 The theory of voting

Another interest of his was the study of voting patterns. Some of his recommendations were adopted in England, such as the rule that allows no results to be announced until *all* the voting booths have closed. Others, such as his various methods of proportional representation, were not. As the philosopher Sir Michael Dummett later remarked:

It is a matter for the deepest regret that Dodgson never completed the book he planned to write on this subject. Such was the lucidity of his exposition and mastery of this topic that it seems possible that, had he published it, the political history of Britain would have been significantly different.

The simplest example that Dodgson gave of the failure of conventional methods is that of a simple majority.

Electors	1 - 3			4 - 7				8 - 10			11
Ist Preference	a	a	a	b	b	b	b	c	c	c	d
2nd preference	c	c	c	a	a	a	a	a	a	a	a
3rd preference	d	d	d	c	c	c	c	d	d	d	c
4th preference	b	b	b	d	d	d	d	b	b	b	b

There are eleven electors, each deciding among four candidates a, b, c, d.
The first three of the electors rank them a, c, d, b; the next four rank them b, a, c, d; and so on. Which candidate, overall, is the best?
Candidate a is considered best by three electors and second-best by the remaining eight electors. But in spite of this, candidate b is selected as the winner, although he is ranked worst by over half of the electors.

Another interest of Dodgson's was the analysis of tennis tournaments:

At a lawn tennis tournament where I chanced to be a spectator, the present method of assigning prizes was brought to my notice by the lamentations of one player who had been beaten early in the contest, and who had the mortification of seeing the second prize carried off by a player whom he knew to be quite inferior to himself.

To illustrate Dodgson's irritation, let us take sixteen players, for example, ranked in order of merit, and let us organise a tournament with 1 playing 2, 3 playing 4, and so on. Then the winners of the first round are 1, 3, 5, ..., those of the second round are 1, 5, 9 and 13, and the final is between players 1 and 9, with player 1 winning first prize and with player 9 winning second prize even though he started in the lower half of the ranking. To avoid such difficulties, Dodgson managed to devise a method for re-scheduling all the rounds so that the first three prizes go to the best three players; this presaged the present system of seeding.

6 Puzzles

We now turn to more light-hearted pursuits - the puzzles he enjoyed showing to his young child-friends and to other adults. For more examples, see [4].
It was during his early years as a lecturer that he started to teach a class of young children at the school across the road. He varied the lessons with stories and puzzles, and may have been the first to use recreational mathematics as a vehicle for teaching mathematical ideas.

Here's one of his paradoxes, based on the well-known 1089 puzzle [1], but involving pounds, shillings and pence - remember that there are 12 pence in a shilling and 20 shillings in a pound.

Put down any number of pounds not more than twelve, any number of shillings under twenty, and any number of pence under twelve. Under the pounds put the number of pence, under the shillings the number of shillings, and under the pence the number of pounds, thus reversing the line.
Subtract - reverse the line again - then add.
*Answer, £*12 18*s.* 11*d., whatever numbers may have been selected.*

Another problem, hotly debated in Carroll's day, was the *Monkey on a rope puzzle.* A rope goes over a pulley - on one side is a monkey, and on the other is an equal weight. The monkey starts to climb the rope - what happens to the weight? Some people thought that it went *up*, while others said that it went down.

A puzzle book of his, *A Tangled Tale*, contains ten stories each involving mathematical problems. Here's its preface - can you guess which child-friend he dedicated it to?

Beloved Pupil! Tamed by thee,
Addish-, Subtrac-, Multiplica-tion,
Division, Fractions, Rule of Three,
Attest thy deft manipulation!
Then onward! Let the voice of Fame
From Age to Age repeat thy story,
Till thou hast won thyself a name
Exceeding even Euclid's glory!

The second letters of each line spell *Edith Rix*, the sister of Wilton Rix to whom he wrote the algebra letter earlier.

7 Logic

Throughout his life, Mr Dodgson was interested in logic. In *Through the Looking-Glass*, Tweedledum and Tweedledee are bickering as always:

I know what you're thinking about - but it isn't so, nohow.
Contrariwise - if it was so, it might be; and if it were so, it would be;
but as it isn't, it ain't. That's logic.

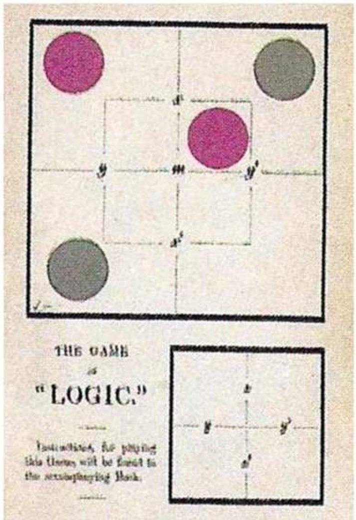
Dodgson believed that symbolic logic could be understood by his many child-friends, and devised *The Game of Logic* in order to help them sort out syllogisms. This contained a board and nine red and grey counters which are placed on sections of the board to represent true and false statements in order to sort out syllogisms.

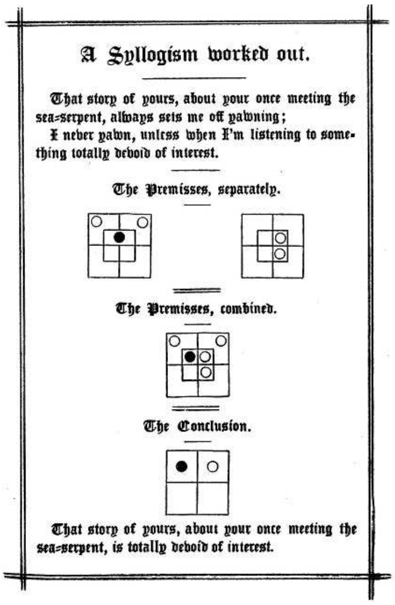
That story of yours, about your once meeting the sea-serpent, always sets me off yawning. I never yawn, unless when I'm listening to something totally devoid of interest.

Conclusion: *That story of yours, about your once meeting the sea-serpent, is totally devoid of interest.*

As he claimed:

If, dear Reader, you will faithfully observe these Rules, and so give my book a really fair trial, I promise you most confidently that you will find Symbolic logic to be one of the most, if not the most, fascinating of mental recreations.





A Syllogism worked out.

That story of yours, about your once meeting the sea-serpent, always sets me off yawning;
I never yawn, unless when I'm listening to something totally devoid of interest.

The Premisses, separately.

The Premisses, combined.

The Conclusion.

That story of yours, about your once meeting the sea-serpent, is totally devoid of interest.

In this first part he carefully avoided all difficulties which seemed to be beyond the grasp of an intelligent child of (say) twelve or fourteen years of age. He himself taught most of its contents to many children, and found them to take a real intelligent interest in the subject. Some are his examples are quite straightforward:

Babies are illogical.
Nobody is despised who can manage a crocodile.
Illogical persons are despised.
Conclusion: *Babies cannot manage crocodiles.*

Others needed more thought, but can readily be sorted out using his counters. The following example contains five statements, but the most ingenious of his examples went up to forty or more:

No kitten that loves fish is unteachable.
No kitten without a tail will play with a gorilla.
Kittens with whiskers always love fish.
No teachable kitten has green eyes.
No kittens have tails unless they have whiskers.
Conclusion: *No kitten with green eyes will play with a gorilla.*

Sadly, Dodgson died just before Volume 2 of his *Symbolic Logic* was completed, and his manuscript version did not turn up until the 1970s. If it had appeared earlier then Charles Dodgson might have been recognised as the greatest British logician of his time.

But let's leave the final word with Lewis Carroll. One night in 1857, while sitting alone in his college room listening to the music from a Christ Church ball, he composed a double acrostic, one of whose lights has often been quoted as his own whimsical self-portrait:

Yet what are all such gaieties to me
Whose thoughts are full of indices and surds?
$x^2 + 7x + 53 = \frac{11}{3}$.

Acknowledgements: An earlier version of this article appeared in the *Transactions of the Manchester Statistical Society* (2011-12), 38-48.

For biographical information about Charles Dodgson, see [2]. For his photographic work, see [3]. For information about his life as a mathematician in Oxford, see [4].

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Articles

Mathematics of Soccer

Alda Carvalho, Carlos Pereira dos Santos, Jorge Nuno Silva
ISEL & Cemapre-ISEG, ISEC & Cima, FCUL
acarvalho@adm.isel.pt, carlos.santos@isec.universitas.pt, jnsilva@cal.berkeley.edu

Abstract: *Soccer, as almost everything, presents mathematizable situations (although soccer stars are able to play wonderfully without realizing it). This paper presents some, exemplifying each with situations of official matches.*

Key-words: Magnus effect, shot angles, soccer balls.

1 Introduction

There is some work done about mathematics and soccer. We highlight the book *The Science of Soccer* [10] which forms a extensive set of non-artificial examples of soccer situations where scientific methods may be useful. Also, the articles [8] and [5] are very interesting: the first one studies optimal shot angles and the second one discusses the spatial geometry of the soccer ball and its relation to chemistry subjects. In this paper, we expose some details of these works complementing it with *some situations from practice*. We strongly recommend the reader to accompany the reading of this paper with the visualization of the video [9]. Also, the section about goal line safes presents a new approach just using elementary geometry.

2 Shot Angles

Consider a situation in which a soccer player runs straight, with the ball, towards the bottom line of the field. Intuitively, it is clear that there is an optimal point maximizing the shot angle, providing the best place to kick in order to improve the chances to score a goal. If the player chooses the bottom line, the angle is zero and his chances are just horrible; if the player is kicking far way, the angle is also too small.

The geometry of the situation was studied in [8]. Briefly, we will expose the details of this, adding some well-chosen examples from real practice. In fig. 1, the point P is the player's position. The straight line containing P is the player's path. Considering the circle passing through A, B and P, where A and B are the goalposts, it is a trivial geometric fact that the shot angle is half of the central angle determined by A and B. Therefore, maximizing the shot angle is the same as maximizing the correspondent central angle.

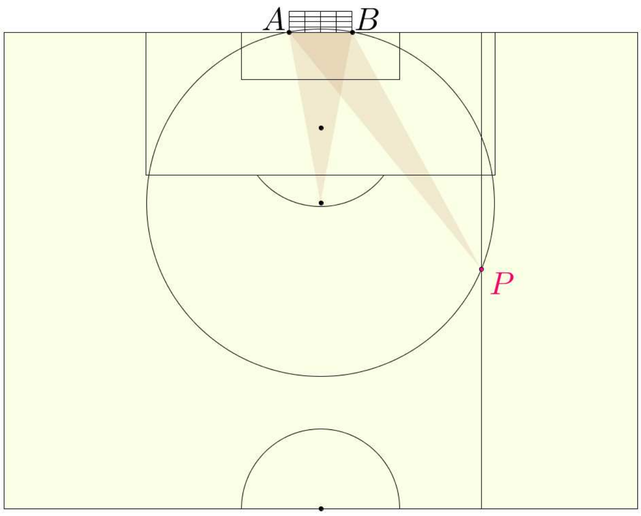

Figure 1: Related central angle.

Also, it is known from the Euclidian Geometry that the perpendicular bisector of $[AP]$ is a tangent line to the parabola whose focus is A and the directrix is the player's path (fig. 2). Because the center of the circle belongs to the perpendicular bisector and the tangent line is *"below"* the parabola, the optimal point is the intersection between the parabola and the central axis of the field.

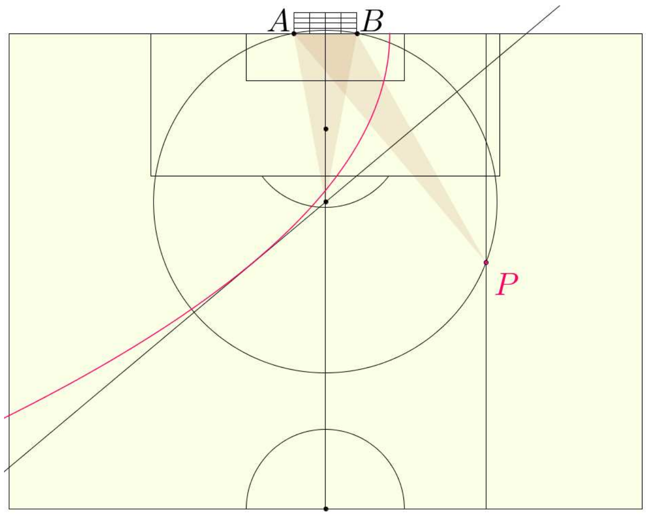

Figure 2: Fundamental parabola.

So, the distance between the optimal point and the point A is exactly equal to the distance between the central axis and the player's path (it lies on parabola). Therefore, it is possible to construct the optimal point with a simple euclidian construction (fig. 3).

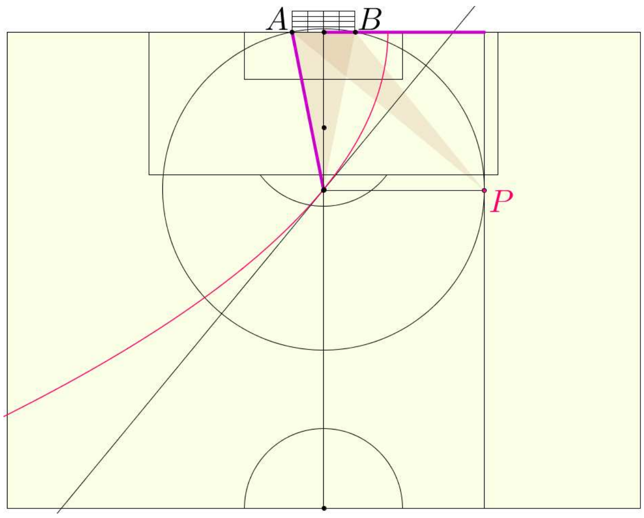

Figure 3: Optimal point.

Figure 4 plots the locus of the optimal points.

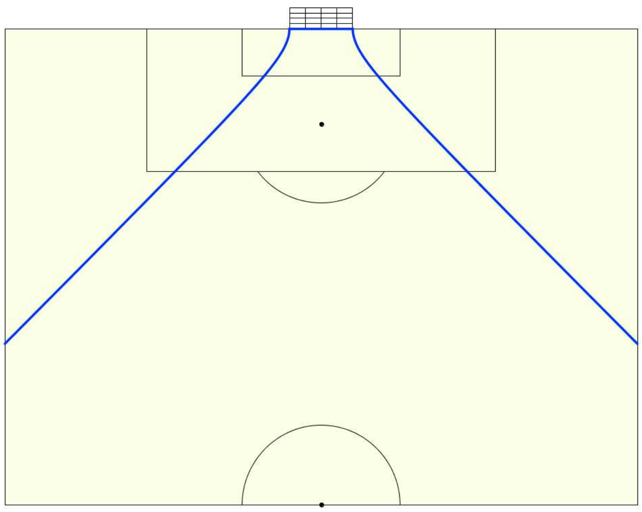

Figure 4: Locus of the optimal points.

Following, three peculiar episodes related to this geometric situation.

1. The second goal of the game Napoli 2 - Cesena 0, Serie A 1987/88, was scored by Diego Maradona [9]. It is an amazing example of the

"Maradona's feeling" about the optimal place to kick.

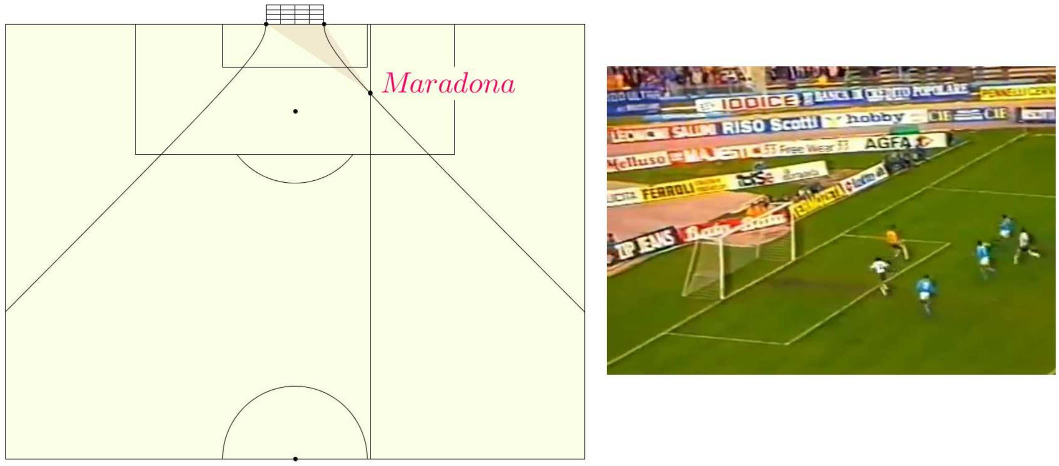

Figure 5: Maradona's kick.

2. The Van Basten's goal in the final of the Euro 88 is a classic [9]. It is a real example showing that mathematics *is not* useful for all situations. The forward kicked with a shot angle of roughly 6 degrees. Very far from the optimal!

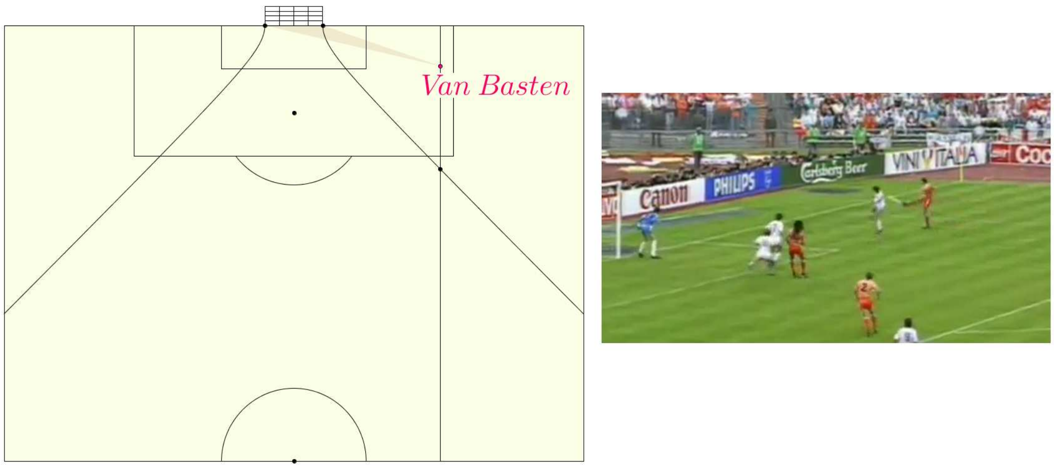

Figure 6: Van Basten's kick.

3. Some people say that Khalfan Fahad's miss was the worst ever [2]. It occurred in Qatar - Uzbekistan 0-1, Asian Games 2010 [9]. The funny kick was performed with an huge shot angle of roughly 90 degrees!

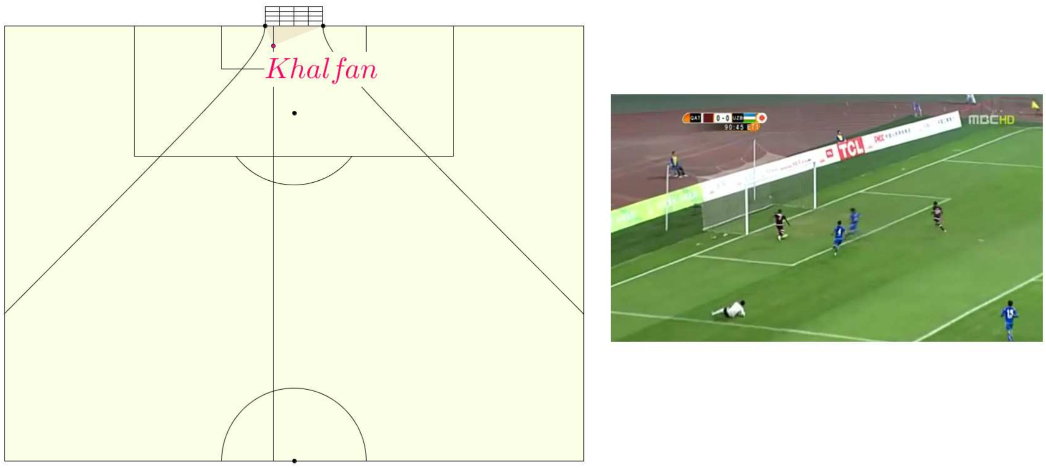

Figure 7: Khalfan Fahad's kick.

3 Goal Line Safes

In this section, we analyze the situation where a striker is kicking the ball into an empty goal and there is a defender who tries to intercept the ball in time. Consider $r = \frac{ball's\ speed}{defender's\ speed}$. In fig. 8, the point S is the position where the striker is kicking the ball and the point D is the position where the defender is when the kick is done.

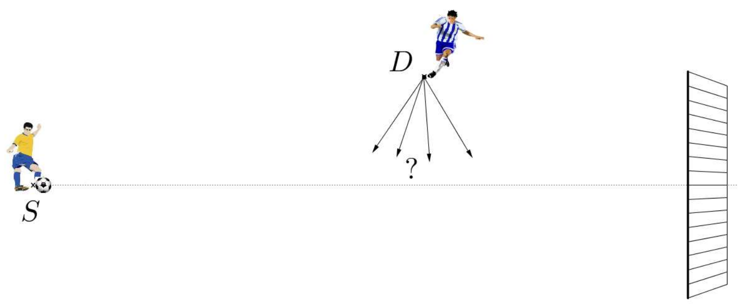

Figure 8: Goal Line Safes.

To construct geometrically the zone where the defender can intercept the ball it is possible to use similarity of triangles. First, consider arbitrarily a point U and let $\overline{SU}$ be the unit. After, construct the point R in such way that $\frac{\overline{SR}}{\overline{SU}}$ is equal to the ratio of speeds r. Constructing the unit circle with center in R, and considering A and B, the intersections between the circle and the straight line SD, we have $\frac{\overline{SR}}{\overline{RA}} = \frac{\overline{SR}}{\overline{RB}} = r$. Therefore, if we consider the points L and Q in the ball's path in such way that $DL \parallel BR$ and $DQ \parallel AR$, the conclusion is that $\frac{\overline{SL}}{\overline{DL}} = \frac{\overline{SQ}}{\overline{DQ}} = r$. The segment $[LQ]$ is the zone where the ball can be intersected by the defender (fig. 9).

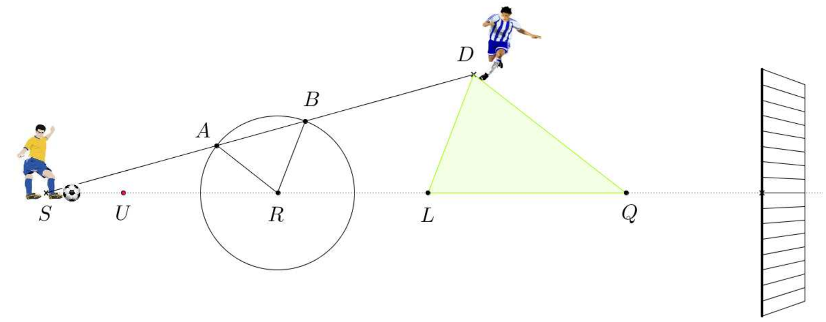

Figure 9: Intersecting the ball.

The indicated construction shows that the defender just has the possibility of success *when A and B exists*. If the defender is behind the tangent to the circle passing trough S, the intersection is just impossible (fig. 10).

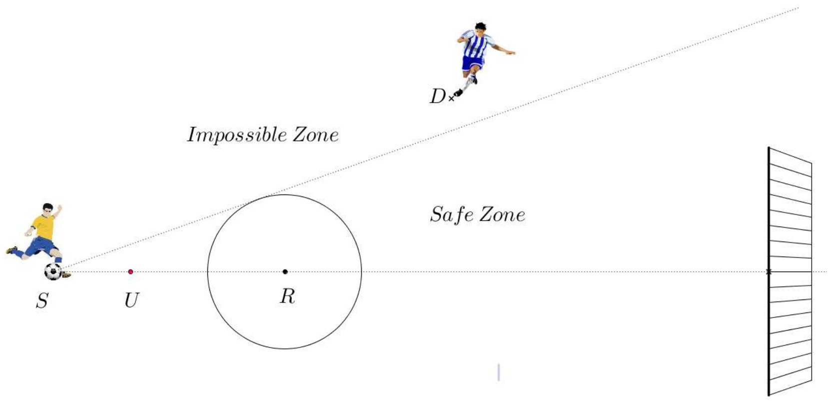

Figure 10: Safe zone.

In practice, there are several examples of nice goal line safe situations. One of the most impressive happened in a match West Ham - Aston Villa [9]. It is not an ideal situation: the ball goes through the air and we have no accurate way to assess the ratio of speeds. But it is a move that is worth seeing.

Figure 11: Great goal line safe.

4 Magnus Effect

The Magnus effect is observed when a spinning ball curves away from its flight path. It is a really important effect for the striker that charges a free kick.

For a ball spinning about an axis perpendicular to its direction of travel, the speed of the ball, relative to the air, is different on opposite sides. In fig 12 the lower side of the ball has a larger speed relative to the air than the upper side. This results in non-symmetric sideways forces on the ball. In order to have momentum conservation, we observe a sideways reaction acting downwards. German physicist Heinrich Magnus described the effect in 1852 [6].

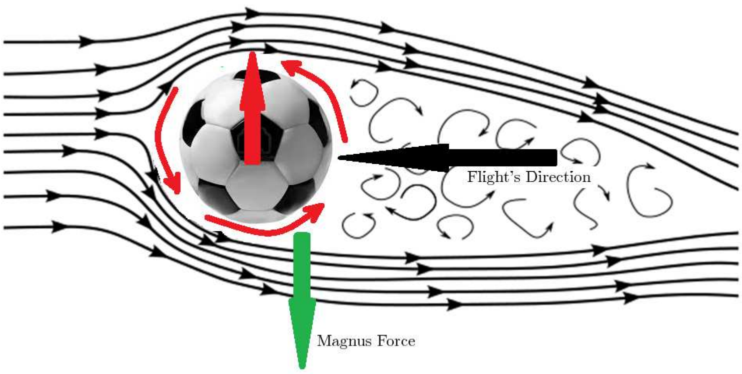

Figure 12: Magnus effect.

Suppose that a striker wants to kick as showed in fig 13. He chooses the length q, directly related to the angle relative to the straight direction, and he chooses the initial speed of the ball S. In [10] we can see a formula that nicely describes the flight path of the ball under the effect of the Magnus force. The formula uses the following physical parameters:

- a, radius (0, 11 m);
- A, cross-sectional area $(0,039\ m^2)$;
- m, mass of the ball $(0,43\ kg)$;
- ρ, air density $(1,2\ kgm^{-3})$.

The curve is described by

$$x = \frac{d\omega}{KS} y \left(1 - \frac{y}{d}\right)$$

where $K = \frac{8m}{a\rho A}$.

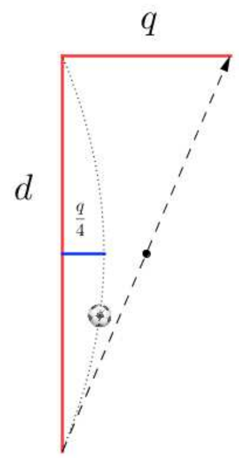

Figure 13: Free kick.

To obtain the path plotted in fig. 13, the striker must kick with the angular speed

$$\omega = \frac{KSq}{d^2}$$

in order to obtain $x = \frac{q}{d} y \left(1 - \frac{y}{d}\right)$. The number of revolutions during the complete flight is given by $\frac{d\omega}{2\pi S}$.

So, an expert must calibrate the direction (related to q), the initial speed and the angular speed (related to the number of revolutions during the flight). Of course an expert doesn't calculate anything but scores goals, a mathematician enjoys these concepts but doesn't score any goal!

One amazing example is the famous Roberto Carlos's free kick against France in 1997 [9]. This free kick was shot from a distance of 35 m. Roberto Carlos strongly hits the ball with an initial speed of 136 km/h with an angle of about 12 degrees relative to the direction of the goal [3]. With a dynamic geometry software [4] it is easy to construct a simulator of the exposed situation. Figure 14 shows an interpretation of the famous free kick. Roberto Carlos needed an angular speed about 88 rad/s. Very difficult, but plausible.

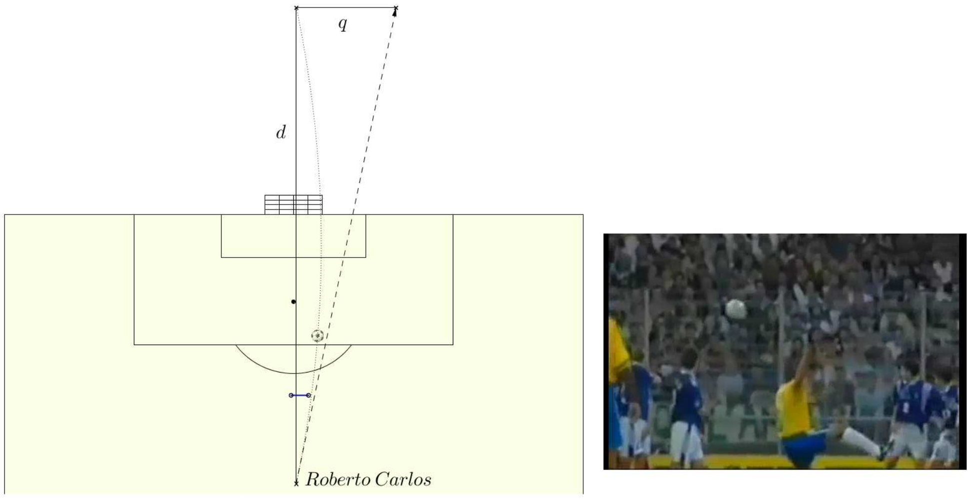

Figure 14: Roberto Carlos' free kick.

5 The Soccer Ball

The history of the geometric shape of the soccer ball has distinct phases [7]. Usually, we think in the truncated icosahedron as "the soccer ball", the design proposed by the architect Richard Buckminster Fuller. However, the first "official world cup" occurrence of this spatial shape (Adidas Telstar) was just in the Mexico, 1970. Before, the soccer balls had different shapes. Also, nowadays, the official balls have different shapes.

Figure 15: Charles Goodyear (1855), FA Cup final (1893), World Cup (1930), panel balls (1950), World Cup (1966).

Following, in the rest of this section, we will analyze some mathematics related to the Buckminster's proposal.

Figure 16: Eusébio, Portuguese player, World Cup, England, 1966 (before the Buckminster design).

An Archimedean solid is a very symmetric convex polyhedron composed of two or more types of regular polygons meeting in identical vertices. They are distinct from the Platonic solids, which are composed of only one type of polygon meeting in identical vertices. The solid is obtained truncating an icosahedron (fig. 17). The truncating process creates 12 new pentagon faces, and leaves the original 20 triangle faces of the icosahedron as regular hexagons. Thus the length of the edges is one third of that of the original icosahedron's edges.

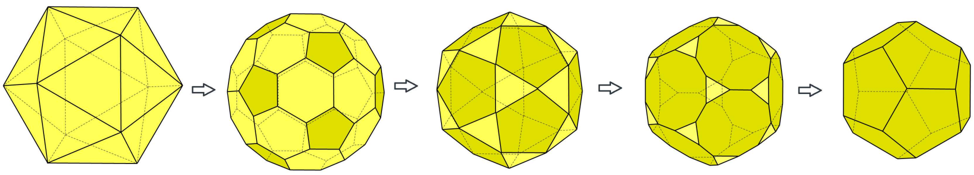

Figure 17: The "travel" from the icosahedron to dodecahedron: truncated icosahedron (1st), icosidodecahedron (2nd), truncated dodecahedron (3rd).

Figure 18: World Cup Final, Mexico 1970, Tostão with the ball, Pelé watching, (Buckminster's ball).

The Buckminster's proposal was well accepted. Just in the modernity, the soccer ball was improved (fig. 19).

Figure 19: Ball of the FIFA World Cup 2012 (South Africa), conduced by the Portuguese player Cristiano Ronaldo.

In [5], related to polyhedrons and soccer balls, because of the sphericity and aesthetical reasons, Dieter Kotschick proposed that the standard soccer ball should have three important properties:

1. The standard soccer ball should have only pentagons and hexagons (guaranteing a good sphericity) ;
2. The sides of each pentagon should meet only hexagons (isolating, for aesthetical reasons, the pentagons);
3. The sides of each hexagon should alternately meet pentagons and hexagons (for aesthetical reasons, in order to avoid groups of 3 joint hexagons).

Dieter Kotschick wrote that he first encountered the above definition in 1983, in a problem posed in *Bundeswettbewerb Mathematik* (a German mathematics competition). The definition captures the iconic image of the soccer ball.

In [5], we can see the explanation of a curious relation to a chemistry subject. In the 1980's, the 60-atom carbon molecule, the "buckyball" C_{60} was discovered. The spatial shape of C_{60} is identical to the standard soccer ball. This discover, honored by the 1996 Nobel Prize in chemistry (Kroto, Curl, and Smalley), created interest about a class of carbon molecules called fullerenes. By chemical properties, the stable fullerenes present the following properties:

1. The stable fullerenes have only pentagons and hexagons;
2. The sides of each pentagon meet only hexagons;
3. Precisely three edges meet at every vertex.

Only the third item is different in the two definitions. Some natural questions arise:

1. Are there stable fullerenes other than buckyball?
2. Are there soccer balls other than the standard one?
3. How many soccer balls are also stable fullerenes?

It is possible to use the famous and classical Euler's formula, $V-E+F=2$, to answer to these questions. About the first one, consider P and H, the number of pentagons and hexagons. Of course, $F=P+H$. Also, because each pentagon has 5 edges and each hexagon has 6 edges, and because we don't want to count each edge twice, $E=\frac{5P+6H}{2}$. By a similar argument, because precisely three edges meet at every vertex, $V=\frac{5P+6H}{3}$. Using the Euler's formula, replacing and canceling, we obtain the fundamental equality for stable fullerenes:

$$P = 12.$$

The number of pentagons must be 12, but there is an unlimited number of possibilities for the number of hexagons compatible with Euler's formula. After the C_{60}, other fullerenes were discovered and object of research. For instance, C_{70} is a fullerene molecule consisting of 70 carbon atoms. It is a structure which resembles a rugby ball, made of 25 hexagons and 12 pentagons (fig. 20).

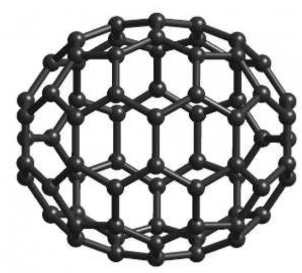

Figure 20: C_{70}

The first question is solved.

About soccer balls, we also have $F=P+H$ and $E=\frac{5P+6H}{2}$. However, the third item of the definition is different and we can have more than three edges meeting at a vertex. So, $V \leqslant \frac{5P+6H}{3}$ and $P \geqslant 12$. It is possible to use the third item of the definition of the soccer ball: exactly half of the edges of hexagons are also edges of pentagons so, $\frac{6H}{2}=5P \Leftrightarrow 3H=5P$. The fundamental conditions for Kotschick's soccer balls are

$$P \geqslant 12 \quad \textit{and} \quad 3H = 5P.$$

Again, there is an unlimited number of possibilities for Kotschick's soccer balls. In fig 17, we observe a well known process to have archimedean solids from platonic solids. In [1], it is proposed a topological way, *branched covering*, to have a new soccer ball from a previous one. Topology is the branch of mathematics that studies properties of objects that are unchanged by continuous deformation, so, a "elastic ball" is considered. In fig. 21 the process is exemplified.

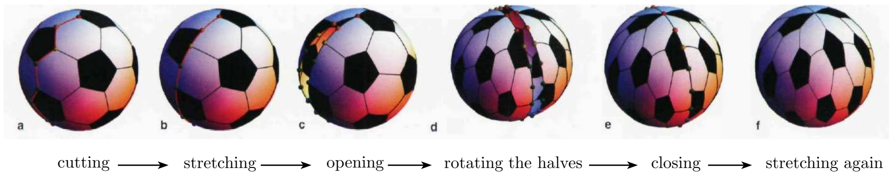

Figure 21: New soccer ball compatible with the definition and Euler's formula obtained by the branching covering process (pictures from [5]).

Therefore, the second question is solved. About the third one, for fullerenes, we have $P = 12$; for soccer balls we have $3H = 5P$. So, an object that is simultaneously a fullerene and a soccer ball must have the Buckminster's design ($P = 12$ and $H = 20$). There is something essential in the truncated icosahedron!

6 (Illogical) Rules

This final part is about the rules of competitions. Strangely, the choice of rules is not as simple as we might think. Sometimes, poorly chosen tiebreakers can generate totally bizarre situations. An impressive situation occurred in a match Barbados vs Grenada (Shell Caribbean Cup 1994). The situation after 2 games is described in the fig. 22.

Barbados needed to win by two goals the last match against Grenada to progress to the finals. But a trouble arises...

1. The organizers had stated that all games must have a winner. Also, all games drawn over 90 minutes would go to sudden death extra time;

2. There was an unusual rule which stated that in the event of a game going to sudden death extra time, *the goal would count double.*

Barbados was leading 2-0 until the 83rd minute, when Grenada scored, making it 2-1.

Approaching the final of the match, the Barbadians realized they had little chance of scoring past Grenada's mass defense in the time available, so they deliberately scored an own goal to tie the game at 2-2.

Figure 21: Shell Caribbean Cup 1994 after 2 games.

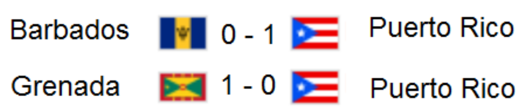

Figure 22: Shell Caribbean Cup 1994 after 2 games.

The Grenadians realized what was happening and attempted to score an own goal as well. However, the Barbados players started defending their opposition's goal to prevent this.

During the game's last five minutes, a crazy situation happened: Grenada trying to score in either goal while Barbados defended both ends of the soccer field.

After 4 minutes of extra time, Barbados scored the golden goal and qualified for the finals. The thing was unbelieving. It is possible to see a video showing Grenada's second goal [9]. If the reader has a video with the final five minutes of the regular time, please send to us!

Figure 23: The strangest goal ever.

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Articles

The Secrets of Notakto: Winning at X-only Tic-Tac-Toe

Thane E. Plambeck, Greg Whitehead
Gathering for Gardner, Counterwave
tplambeck@gmail.com

Abstract: *We analyze misere play of "impartial" tic-tac-toe—a game suggested by Bob Koca in which* **both** *players make X's on the board, and the first player to complete three-in-a-row* **loses**. *This game was recently discussed on mathoverflow.net in a thread created by Timothy Y. Chow.*

Key-words: combinatorial game theory, misere play, tic-tac-toe.

1 Introduction

Suppose tic-toe-toe is played on the usual 3×3 board, but where both players make X's on the board. The first player to complete a line of three-in-a-row *loses* the game.

Who should win? The answer for a single 3×3 board is given in a recent mathoverflow.net discussion [Chow]:

> *In the 3×3 misere game, the first player wins by playing in the center, and then wherever the second player plays, the first player plays a knight's move away from that.*

Kevin Buzzard pointed out that any other first-player move loses:

> *The reason any move other than the centre loses for [the first player to move] in the 3×3 game is that [the second player] can respond with a move diametrically opposite [the first player's] initial move. This makes the centre square unplayable, and then player two just plays the "180 degree rotation" strategy which clearly wins.*

In this note we generalize these results to give a complete analysis of multiboard impartial tic-tac-toe under the disjunctive misere-play convention.

2 Disjunctive misere play

A *disjunctive* game of 3×3 impartial tic-tac-toe is played not just with one tic-tac-toe board, but more generally with an arbitrary (finite) number of such boards forming the start position. On a player's move, he or she selects a single one of the boards, and makes an X on it (a board that already has a three-in-a-row configuration of X's is considered unavailable for further moves and out of play).

Play ends when every board has a three-in-a-row configuration. The player who completes the last three-in-a-row on the last available board is the loser.

3 The misere quotient of 3×3 impartial tic-tac-toe

We can give a succinct and complete analysis of the best misere play of an arbitrarily complicated disjunctive sum of impartial 3×3 tic-tac-toe positions by introducing a certain 18-element commutative monoid Q given by the presentation

$$Q = \langle\ a, b, c, d \mid a^2 = 1,\ b^3 = b,\ b^2c = c,\ c^3 = ac^2,\ b^2d = d,\ cd = ad,\ d^2 = c^2 \rangle. \tag{1}$$

The monoid Q has eighteen elements

$$Q = \{1, a, b, ab, b^2, ab^2, c, ac, bc, abc, c^2, ac^2, bc^2, abc^2, d, ad, bd, abd\}, \tag{2}$$

and it is called the **misere quotient** of impartial tic-tac-toe[1].

A complete discussion of the misere quotient theory (and how Q can be calculated from the rules of impartial tic-tac-toe) is outside the scope of this document. General information about misere quotients and their construction can be found in [MQ1], [MQ2], [MQ3], and [MQ4]. One way to think of Q is that it captures the misere analogue of the "nimbers" and "nim addition" that are used in normal play disjunctive impartial game analyses, but localized to the play of this particular impartial game, misere impartial 3×3 tic-tac-toe.

In the remainder of this paper, we simply take Q as given.

[1]For the cognoscenti: Q arises as the misere quotient of the hereditary closure of the sum G of two impartial misere games $G = 4 + \{2+, 0\}$. The game $\{2+, 0\}$ is the misere canonical form of the 3×3 single board start position, and "4" represents the nim-heap of size 4, which also happens to occur as a single-board position in impartial tic-tac-toe. In describing these misere canonical forms, we've used the notation of John Conway's *On Numbers and Games*, on page 141, Figure 32.

4 Outcome determination

Figure 6 (on page 53, after the References) assigns an element of Q to each of the conceivable 102 non-isomorphic positions[2] in 3×3 single-board impartial tic-tac-toe.

To determine the outcome of a multi-board position (ie, whether the position is an *N-position*—a Next player to move wins in best play, or alternatively, a *P-position*—second player to move wins), one first multiplies the corresponding elements of Q from the dictionary together. The resulting word is then reduced via the relations 1, that we started with above, necessarily eventually arriving at at one of the eighteen words (in the alphabet a, b, c, d) that make up the elements of Q.

If that word ends up being one of the four words in the set P

$$P = \{a, b^2, bc, c^2\}, \tag{3}$$

the position is P-position; otherwise, it's an N-position.

5 Example analysis

To illustrate outcome calculation for Impartial Tic-Tac-Toe, we consider the two-board start position shown in Figure 1.

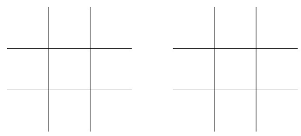

Figure 1: The two-board start position.

Consulting Figure 6, we find that the monoid-value of a single empty board is c. Since we have two such boards in our position, we multiply these two values together and obtain the monoid element

$$c^2 = c \cdot c.$$

Since c^2 is in the set P (equation (3)), the position shown in Figure 1 is a *second* player win. Supposing therefore that we helpfully encourage our opponent to make the first move, and that she moves to the center of one of the boards, we arrive at the position shown in Figure 2.

[2] We mean "non-isomorphic" under a reflection or rotation of the board. In making this count, we're including positions that couldn't be reached in actual play because they have too many completed rows of X's, but that doesn't matter since all those elements are assigned the identity element of Q.

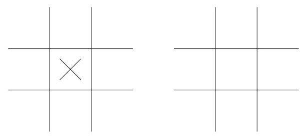

Figure 2: A doomed first move from the two-board start position.

It so happens that if we mimic our opponent's move on the other board, this happens to be a winning move. We arrive at the position shown in Figure 3, each of whose two boards is of value c^2; multiplying these two together, and simplifying via the relations shown in equation (1), we have

$$\begin{aligned} c^4 &= c^3 \cdot c \\ &= ac^2 \cdot c \\ &= ac^3 \\ &= aac^2 \\ &= c^2, \end{aligned}$$

which is a P-position, as desired.

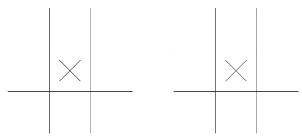

Figure 3: Mimicry works here, but not in general.

So is the general winning strategy of the two-board position simply to copy our opponent's moves on the other board? Far from it: consider what happens if our opponent should decide to complete a line on one of the boards—copying that move on the other board, we'd **lose** rather than win! For example, from the N-position shown in Figure 5, there certainly is a winning move, but it's *not* to the upper-right-hand corner of the board on the right, which loses.

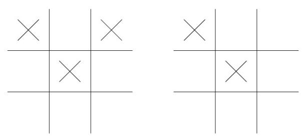

Figure 4: An N-position in which mimickry loses.

We invite our reader to find a correct reply!

6 The iPad game Notakto

Evidently the computation of general outcomes in misere tic-tac-toe is somewhat complicated, involving computations in finite monoid and looking up values from a table of all possible single-board positions.

However, we've found that a human can develop the ability to win from multi-board positions with some practice.

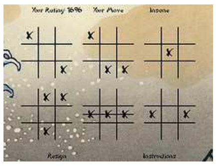

Figure 5: A six-board game of Notakto, in progress.

Notakto "No tac toe" is an iPad game that allows the user to practice playing misere X-only Tic-Tac-Toe against a computer. Impartial misere tic-tac-toe from start positions involving one up to as many as six initial tic-tac-toe boards are supported. The Notakto iPad application is available for free at `http://www.notakto.com`.

7 Final question

Does the 4×4 game have a finite misere quotient?

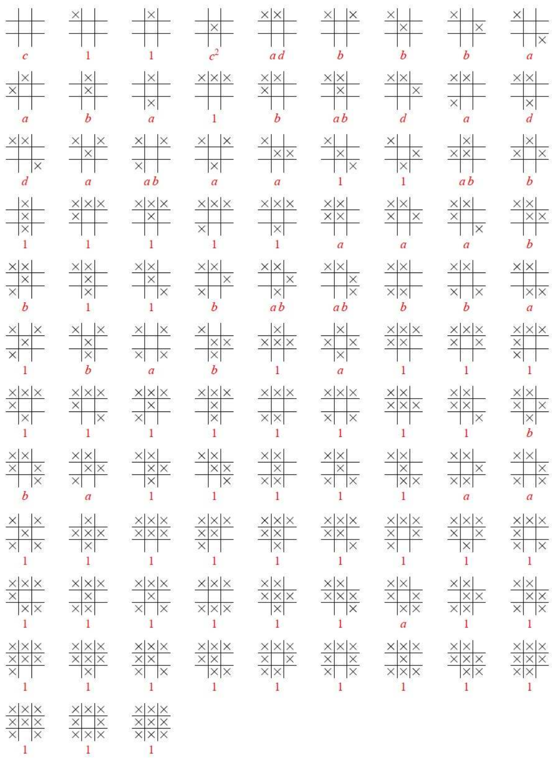

Figure 6: The 102 nonisomorphic ways of arranging zero to nine X's on a board, each shown together with its corresponding misere quotient element from Q.

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[MQ4] http://www.miseregames.org

Articles

Doppelgänger Placement Games*†

Svenja Huntemann, Richard J. Nowakowski
Dalhousie University
s.huntemann@dal.ca,r.nowakowski@dal.ca

Abstract: *Are some games Doppelgänger? Can a nice placid game, one that you'd play with your grandmother, have a nasty evil twin that invites fights? Depending on the (lack of) strength of your (mathematical) glasses, it is possible but we show that with good glasses, games with simple rules are not Doppelgänger.*

Key-words: Combinatorial games, graphs, polynomial profile, SNORT, COL.

1 Introduction

A *combinatorial game* is a game of perfect information and no chance, with two players called Left and Right (or Louise and Richard) who play alternately, for example CHESS, CHECKERS and GO. The word 'game' has many meanings in English, for us, a *game* is a set of rules and a *position* is an instance of the game on some board. We are interested in *placement* games; those in which the players place pieces on a board and thereafter the pieces are never moved or removed from play. For example, COL and SNORT are both usually played on a chessboard and in both Louise places a blue piece on a square, Richard a red. However, in COL pieces of the same colour cannot be adjacent, in SNORT two pieces of opposite colour cannot be adjacent. The question of Doppelgänger games arises in [4] where it is shown that SNORT is the rowdy twin to COL. We need a few definitions to make this question meaningful.

Definition 1.1. *Let G be a placement game played on a board (graph) B. Let n be the number of vertices of B. The pieces are placed on the vertices of B. The bi-variate polynomial profile of G on B is the bivariate polynomial*

$$P_{G,B}(x,y) = \sum_{k=0}^{n}\sum_{j=0}^{k} f_{j,k-j}x^j y^{k-j}$$

*1991 Mathematics Subject Classification. 05C

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where $f_{j,k-j}$ is the number of legal positions of G on B which have j Left pieces and $k-j$ Right pieces. The polynomial profile is obtained by putting $x = y$

$$P_{G,B}(x) = \sum_{k=0}^{n} f_k(G,B)x^k$$

where now $f_k(G,B)$ is the number of positions with exactly k pieces.

Of course, $P_{G,B}(1)$ just counts the total number of positions. Enumerating the number of positions or the number of special positions is a relatively new endeavour, see [4, 6, 7, 8, 9, 11] for connections to Bernoulli numbers of the second kind, Catalan numbers, Dyck paths, amongst others.

For COL on a strip of 3 squares, the total number of positions are

$$\cdot\cdot\cdot,\quad \cdot\cdot B,\quad \cdot B\cdot,\quad B\cdot\cdot,\quad \cdot\cdot R,\quad \cdot R\cdot,\quad R\cdot\cdot,$$

$$R\cdot B,\quad RB\cdot,\quad BR\cdot,\quad \cdot RB,\quad B\cdot R,\quad \cdot BR,\quad B\cdot B,\quad R\cdot R,$$

$$RBR,\quad BRB$$

and

$$\begin{aligned} P_{\text{COL},P_3}(x,y) &= 1+3x+3y+6xy+x^2+y^2+x^2y+xy^2; \\ P_{\text{COL},P_3}(x) &= 1+6x+8x^2+2x^3; \qquad P_{\text{COL},P_3}(1) = 17. \end{aligned}$$

For SNORT,

$$\begin{aligned} P_{\text{SNORT},P_3}(x,y) &= 1+3x+3y+2xy+3x^2+3y^2+x^3+y^3; \\ P_{\text{SNORT},P_3}(x) &= 1+6x+8x^2+2x^3; \qquad P_{\text{SNORT},P_3}(1) = 17. \end{aligned}$$

Over a class of boards $\mathcal{B}$, games G and H are called $\mathcal{B}$ *– Doppelgänger* if $P_{G,B}(x) = P_{H,B}(x)$ for all boards B in $\mathcal{B}$. Expanding upon the question from [4] we pose two questions. The focus of the paper is on the second.

The Doppelgänger questions:
1: Given two games G and H, what is the maximum set of boards $\mathcal{B}$ for which G and H are $\mathcal{B}$ – Doppelgänger?
2: Given a set of boards $\mathcal{B}$ what set of games, $\{G_1, G_2, \ldots G_n\}$, are mutually $\mathcal{B}$ – Doppelgänger?

Informally, the smaller the set of boards being considered the weaker your mathematical glasses are. A wider selection of boards allows for more chances to discriminate between the games.

If $\mathcal{B}$ is the set of $m \times n$ checkerboards then COL and SNORT are $\mathcal{G}$-Doppelgänger. Even more, they are Doppelgänger for all bipartitite graphs but no larger class ([4]) thereby answering our Question 1.

We are interested in games in which the legal, or rather the illegal, placement of pieces depends only on the distances to already played pieces. A *distance placement* game (dp-game) G is described by two sets of numbers: S_G which gives the illegal distances for two of the same pieces; and D_G is the illegal distances for two different pieces. For COL and SNORT we have $S_{\text{COL}} = \{1\} = D_{\text{SNORT}}$ and $D_{\text{COL}} = \emptyset = S_{\text{SNORT}}$.

Note that not all placement games are described by distance sets. NOCOL is COL with the extra condition that every piece must be adjacent to a non-occupied vertex. Also, DOMINEERING [2] is a non-dp but an independent placement game, and NOGO [5] (also known as *anti-atari go* [10]) is a non-independent placement game. Both games are played by "real" people.

Distance games are a subset of *independence placement* games [9]. Independence games have an associated auxiliary graph and the legal positions in the game on the board correspond to independent sets in the auxiliary graph. To show how this works for distance games, let G be a distance game and B be a board (graph) with vertices $v_1, v_2, \ldots, v_n$. We take two copies of the vertices and call them $(v_1, b), (v_2, b), \ldots, (v_n, b)$ and $(v_1, r), (v_2, r), \ldots, (v_n, r)$ and form an auxiliary board $A(B)$. We imagine that only Louise can play on a vertex with a b and Richard on a vertex with an r. We now connect two vertices $(x, y) \sim (v, w)$ in $A(B)$ as follows:

- $x = v$ — since no vertex can be played twice;
- $d(x, v) \in S_P$ and $y = w$ — illegal distance for two of the same pieces;
- $d(x, v) \in D_P$ and $y \neq w$ — illegal distance for two different pieces;

Now it should be clear that: *Let G be a dp-game on a board B. A legal position on B corresponds to an independent set on $A(B)$.* However, for the purposes of this paper, instead of focusing on the negative we will consider the positive. Given a game G and a board B with vertices $1, 2, \ldots, n$ a position will be given by the *word* $b_{i_1}, b_{1_2}, \ldots, b_{i_k}, r_{j_1}, r_{j_2}, \ldots, r_{i_l}$ where the vertices $i_1, \ldots, i_k$ are occupied by blue pieces and $j_1, \ldots, j_l$ are occupied by red.

In the next section we prove our two results. Theorem 2.1 says that, in general, there are no Doppelgänger games and the proof shows that given two dp-games, G and H, there is a relatively small and simple board that tells them apart. Any one actually playing a dp-game probably would play on a grid. We show, Theorem 2.2, that if the game contains an odd distance in its distance sets then it has a Doppelgänger on a grid as well as on bipartite graphs in general.

Aside: If G has few elements in D_G and many in S_G then it is relatively tame since a move leaves the opponent with more places to go than the player. If the $|D_G| > |S_G|$ then the reverse is true and could be called 'rowdy'. Theorem 2.2 can be used to see when a tame game will have a rowdy Doppelgänger. Although it is not required to understand this paper, for more on combinbatorial game theory please consult [1, 3].

2 Doppelgänger

Theorem 2.1. *No two different distance games are Doppelgänger on the set of all possible boards.*

Proof. Let G and H be two distance games with the distance sets $S_G = \{g_1, g_2, \ldots\}$, $S_H = \{h_1, h_2, \ldots\}$, $D_G = \{g'_1, g'_2 \ldots\}$, and $D_H = \{h'_1, h'_2, \ldots\}$. If the distance sets differ, let i and j be the smallest indices such that $g_i \neq h_i$ and $g'_j \neq h'_j$ respectively. There are several cases to consider, and we will show that, for each case, a board exists on which G and H are not Doppelgänger.

Case 1 ($S_G \neq S_H$ and $D_G = D_H$): Assume without loss of generality that $g_i < h_i$. Now consider the games G and H played on the path P_{g_i+1}. Since $diam(P_{g_i+1}) = g_i$, no two pieces will have a distance greater than g_i and we can ignore any elements in the distance sets greater than g_i. Thus essentially $S_G = S_H \cup \{g_i\}$. Then every position with two pieces that is legal in G is also legal on H. On the other hand though, $b_1 b_{g+1}$ and $r_1 r_{g+1}$ are legal positions of H, but not of G. Thus $f_2(H) = f_2(G) + 2$, proving that G and H are not Doppelgänger.

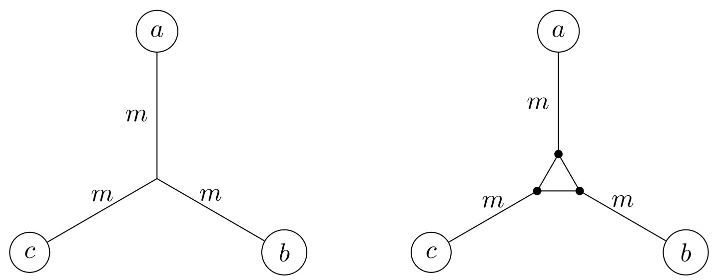
Case 2 ($S_G = S_H$ and $D_G \neq D_H$): The argument is along the same line as in the previous case.

Case 3 ($S_G \neq S_H$ and $D_G \neq D_H$): There are 3 subcases to consider:

Case 3a ($g_i < h_i$ and $g'_j < h'_j$): Let $m = \min\{g_i, g'_j\}$. Then repeat the argument of the first case on the path P_{m+1}.

Case 3b ($g_i < h_i$ and $g'_j > h'_j$ with $g_i \neq h'_j$): Let $m = \min\{g_i, h'_j\}$. Then repeat the argument of the first case on the path P_{m+1}.

Case 3c ($g_i < h_i$ and $g'_j > h'_j$ with $g_i = h'_j$): If $g_i = 2m$ is even, then we consider the board in Figure 1(A). If $g_i = 2m + 1$ is odd, we consider the board in Figure 1(B). Since the diameter of both graphs is g_i, we can again ignore all distances greater than this, thus essentially $S_G = S_H \cup \{g_i\}$ and $D_G \cup \{g_i\} = D_H$.



(A) $g_i = 2m$ (B) $g_i = 2m + 1$

Figure 1: Proof of Theorem 2.1: Boards for Case 3c.

We will look at the positions involving exactly three pieces played. Triples with at most one of the vertices a, b, or c will have all distances less than g_i, thus such a triple is a legal position of G if and only if it is a legal position of H. Let the number of triples with none of these vertices be K_0 and the number of triples involving one be K_1.

Now consider the number of triples involving two of the vertices a, b, and c. Let a_k be the vertex that has distance k from the vertex a where $k \leq m$ (i.e. a_k is on the same 'branch' as a) and similarly for b_k and c_k. For the case $g_i = 2m$, we have $a_m = b_m = c_m$. If $d_a e_b f_{a_k}$ is a legal position where $d, e, f \in \{b, r\}$, then through 'rotation' and due to symmetry of the board we also get the two legal positions $d_b e_c f_{b_k}$ and $d_c e_a f_{c_k}$, and similarly for $d_a e_b f_{b_k}$ and $d_a e_b f_{c_k}$. Thus we can partition the number of such triples into equivalence classes where two positions are equivalent if and only if they are rotations of each other. Each of these equivalence classes has size 3. Let the number of such triples in G be $3K_2$ and in H be $3K_2'$.

In G, the triple abc is illegal since no two of the vertices can have the same colour piece. In H though, the triples $b_a b_b b_c$ and $r_a r_b r_c$ are legal.

# of vertices of $\{a, b, c\}$ used	Number of triples in G	H
0	K_0	K_0
1	K_1	K_1
2	$3K_2$	$3K_2'$
3	0	2

Table 1: Proof of Theorem 2.1: Number of Triples for Case 3c.

Looking at the total number of triples in G and H, we have $f_2(G) = K_0 + K_1 + 3K_2 + 0$ and $f_2(H) = K_0 + K_1 + 3K_2' + 2$ (see Table 1). Considering these modulo 3, we have $f_2(G) \equiv_3 K_0 + K_1$ and $f_2(H) \equiv_3 K_0 + K_1 + 2$, which shows $f_2(G) \neq f_2(H)$. Thus G and H are not Doppelgänger. $\square$

2.1 Bipartite Doppelgänger games

Let G be a dp-game played on a bipartite graph B. Let $V(B) = V_1 \cup V_2$ where V_1 and V_2 are the two sets in the bipartition of the vertices of B. We refine our position notation to

$$(b_{i_1}, b_{i_2}, \ldots), (b_{j_1}, b_{j_2}, \ldots), (r_{k_1}, r_{k_2}, \ldots), (r_{l_1}, r_{l_2}, \ldots)$$

where the $\{i_1, i_2, \dots\} \cup \{k_1, k_2, \dots\} \subset V_1$ and $\{j_1, j_2, \dots\} \cup \{l_1, l_2, \dots\} \subset V_2$. The *bipartite flip* of a board B, $BF(B)$, is a map from positions to positions defined by taking a position

$$(b_{i_1}, b_{i_2}, \ldots), (b_{j_1}, b_{j_2}, \ldots), (r_{k_1}, r_{k_2}, \ldots), (r_{l_1}, r_{l_2}, \ldots)$$

and returning the position

$$(b_{i_1}, b_{1_2}, \ldots), (b_{l_1}, b_{l_2} \ldots), (r_{k_1}, r_{k_2}, \ldots), (r_{j_1}, r_{j_2}, \ldots).$$

This process changes the colour of any piece on a vertex of V_2. Clearly, taking a position and applying the bipartite flip twice returns us to the original position.

Let G be a dp-game and let G' be the dp-game defined by

$$S_{G'} = \{2i : \text{ if } 2i \in S_G\} \cup \{2i+1 : \text{ if } 2i+1 \in D_G\}$$

and

$$D_{G'} = \{2i+1 : \text{ if } 2i+1 \in S_G\} \cup \{2i : \text{ if } 2i \in D_G\}.$$

On any bipartite board B, if $(b_{i_1}, b_{1_2}, \ldots), (b_{j_1}, b_{j_2}, \ldots), (r_{k_1}, r_{k_2}, \ldots), (r_{l_1}, r_{l_2}, \ldots)$ is an *illegal* position for G , $(b_{i_1}, b_{1_2}, \ldots), (b_{l_1}, b_{l_2} \ldots), (r_{k_1}, r_{k_2}, \ldots), (r_{j_1}, r_{j_2}, \ldots)$ is an illegal position in G' and vice-versa. Thus, the bipartite flip on any bipartite board gives a correspondence between the legal positions of G to those of G'. Therefore we can refer to the *bipartite flip of the game* G, denoted $BF(G)$. Since the corresponding positions of G and $BF(G)$ on any bipartite board B have the same number of pieces then $P_{G,B}(x) = P_{BF(G),B}(x)$. This gives us the proof of Theorem 2.2 concerning bipartite Doppelgänger games.

Theorem 2.2. *On the class of bipartite boards, if G is a dp-game and $S_G \cup D_G$ contains an odd distance then G and $BF(G)$ are Doppelgänger.*

Note that the 'odd distance' constraint is necessary to ensure that G and $BF(G)$ are different.

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