LU-Decompositions of Matrices

Definition 1. *Let A be a square matrix. If there is a lower triangular matrix L with all diagonal entries equal to 1 and an upper triangular matrix U such that $A = LU$, then we say that A has an LU-decomposition.*

Suppose A is an $n \times n$ matrix and consider the linear system $A\underline{x} = \underline{b}$ of n equations in n variables. If A has an LU-decomposition, then the system $A\underline{x} = \underline{b}$ can be reduced to two simpler systems $U\underline{x} = \underline{c}$ and $L\underline{c} = \underline{b}$. Whenever the system $A\underline{x} = \underline{b}$ is consistent, we can first solve the system $L\underline{c} = \underline{b}$ by forward substitution and then the system $U\underline{x} = \underline{c}$ by backward substitution to obtain all solutions of the system $A\underline{x} = \underline{b}$. We illustrate this with an example first.

Example 2. Consider the system

$$Ax = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}.$$

Let us attempt to reduce A to its row echelon form U by the Gauss Elimination Method (GEM) as follows:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \overset{R_2 \longrightarrow R_2 - 2R_1,\ R_3 \longrightarrow R_3 - (-1)R_1}{\sim} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix} \overset{R_3 \longrightarrow R_3 - (-1)R_2}{\sim} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} = U.$$

Let us consider the 3×3 lower triangular matrix L with all diagonal entries equal to 1 and the subdiagonal entries equal to the respective multipliers used in the above elimination process, namely,

$$L := \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}.$$

It can be easily verified that $LU = A$. Now putting $U\underline{x} = \underline{c}$ and $L\underline{c} = \underline{b} = [5, -2, 9]^T$, we see that the solution of $L\underline{c} = \underline{b}$ is $\underline{c} = [5, -12, 2]^T$ and the solution of $U\underline{x} = \underline{c} = [5, -12, 2]^T$ is $\underline{x} = [1, 1, 2]^T$.

We shall shortly give conditions under which a matrix has an LU-decomposition along with a method of finding such a decomposition. In general, however, a matrix need not have an LU-decomposition, and if it has one, it need not be unique. For example, it is easily seen that

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \quad \text{for any real numbers } a, b, c.$$

On the other hand, it is clear that

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{for any real number } a.$$

Proposition 3. *Suppose an $n \times n$ matrix A can be reduced to its row echelon form U without any row interchanges, that is, by using only the elementary row operations $R_i \longrightarrow R_i - m_{ij}R_j$ for*

$j = 1, \ldots, n-1$ and $i = j+1, \ldots, n$. Define an $n \times n$ matrix $L := [\ell_{ij}]$ as follows:

$$\ell_{ij} := \begin{cases} m_{ij} & \text{if} \quad i > j, \\ 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i < j. \end{cases}$$

Then $A = LU$.

Proof. Let E_{ij} be the elementary matrix corresponding to the elementary row operation $R_i \longrightarrow R_i - m_{ij}R_j$ for $j = 1, \ldots, n-1$ and $i = j+1, \ldots, n$. Consider the product G of these elementary matrices in the same order as the corresponding elementary row operations carried out on A. Thus

$$G := [E_{(n-1)}] \cdots [E_{n2}E_{(n-1)2} \cdots E_{42}E_{32}][E_{n1}E_{(n-1)1} \cdots E_{31}E_{21}].$$

Then $GA = U$. Now let us consider the matrix GL. First, the matrix $E_{21}L$ is obtained by multiplying the first row of L by m_{21} and subtracting it from the second row of L. Hence the matrix $E_{21}L$ has the same entries as the matrix L except that the $(2,1)$th entry m_{21} of L is reduced to 0. Proceeding similarly, the matrix $[E_{n1}E_{(n-1)1} \cdots E_{31}E_{21}]L$ has the same entries as the matrix L except that the entries $m_{21}, \ldots, m_{n1}$ in the first column of L are reduced to 0. Similarly, multiplications by the other elementary matrices appearing in G reduce all the other subdiagonal entries of L to 0, while retaining all diagonal and superdiagonal entries of L in tact. Thus multiplication on the left by G reduces L to the $n \times n$ identity matrix, that is, $GL = I_n$. Hence $GA = U = I_nU = (GL)U = G(LU)$. Since G is invertible, we obtain $A = LU$. □

We have already illustrated the above result by considering a specific example. The numbers m_{ij} appearing in the above result are known as **multipliers**. If for some j and for some $i > j$, there is no need to subtract a multiple of the jth row from the ith row because the relevant entry is already equal to 0, then the corresponding multiplier m_{ij} is equal to 0.

The method of finding an LU-decomposition of a matrix A and then reducing the linear system $A\underline{x} = \underline{b}$ to two triangular systems $U\underline{x} = \underline{c}$ and $L\underline{c} = \underline{b}$ is particularly useful when one wants to solve several linear systems which have the same coefficient matrix A, but the data vector $\underline{b}$ on the right side varies with the system. This is so, because the two factors L and U of A can be stored in the computer once for all, and then for each different data vector $\underline{b}$, the two triangular systems can be solved cheaply, that is, with an operation count of the order of n^2 (as compared to the operation count of the order of n^3 needed for the Forward Elimination Phase of the GEM). This situation typically arises in iterative refinements of an approximate solution of a linear system.

Proposition 4. *Suppose A is an invertible matrix. If A has an LU-decomposition, then it is unique.*

Proof. Let $A = L_1U_1$ and $A = L_2U_2$ be LU-decompositions of A. Since A is invertible, $\det A \neq 0$ and since $\det A = (\det L_1)(\det U_1) = (\det L_2)(\det U_2)$, we see that the determinants of L_1, U_1, L_2 and U_2 are non-zero. In particular, L_2 and U_1 are invertible. Hence $L_2^{-1}L_1 = U_2U_1^{-1}$. This implies that $L_2^{-1}L_1$ is lower triangular as well as upper triangular, so that it is, in fact, a diagonal matrix. Now by appealing to the Gauss-Jordan Method of obtaining the inverse of a matrix, we note that the matrix L_2^{-1} is also lower triangular with all diagonal entries equal to 1. It follows that all the diagonal entries of the matrix $L_2^{-1}L_1$ are equal to 1. Thus $L_2^{-1}L_1 = I_n$, that is, $U_2U_1^{-1} = I_n$. Hence $L_2 = L_1$ and $U_2 = U_1$. □

Proposition 5. *Let $A = (a_{ij})$ be an $n \times n$ matrix and for $k = 1, \dots, n-1$, let $A(k)$ denote the $k \times k$ submatrix of A consisting of the first k rows and k columns of A. If $A(k)$ is invertible for $k = 1, 2, \dots, n-1$, then A has a unique LU-decomposition.*

Proof. Let $A(k)$ be invertible for $k = 1, \dots, n-1$. Since $A(1) = [a_{11}]$ is invertible, $a_{11} \neq 0$. Using elementary row operations with a_{11} as the pivot, we may reduce A to a matrix of the form

$$A' = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a'_{22} & \dots & a'_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a'_{n2} & \dots & a'_{nn} \end{bmatrix}.$$

Now the matrix $A'(2)$ is invertible since it is row equivalent to the invertible matrix $A(2)$. Hence $a'_{22} \neq 0$. Using elementary row operations with a'_{22} as the pivot, we may reduce A' to a matrix of the form

$$A'' = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \dots & a'_{2n} \\ 0 & 0 & a''_{33} & \dots & a''_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & a''_{n3} & \dots & a''_{nn} \end{bmatrix}.$$

Continuing in this manner, we may reduce A to its row echelon form U. Note that the first $n-1$ diagonal entries of U are non-zero. Since no row interchanges are used in this process, Proposition 3 shows that $A = LU$, where L is lower triangular with all diagonal entries equal to 1 and U is upper triangular. This proves the existence of an LU-decomposition of A. To prove its uniqueness, consider another LU-decomposition of A given by $A = L_0U_0$. Since L_0 is lower triangular and invertible, the matrix $L_0^{-1}L$ is lower triangular. Further, since U is upper triangular with the first $n-1$ diagonal entries non-zero and $(L_0^{-1}L)U = U_0$ is also upper triangular, it can be verified that $L_0^{-1}L$ is, in fact, a diagonal matrix. But all its diagonal entries are equal to 1. Hence $L_0^{-1}L = I_n$, the $n \times n$ identity matrix, that is, $L_0 = L$. Consequently, $U_0 = U$ and we are through. $\square$