Beginning Partial Differential Equations

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Beginning Partial Differential Equations

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Contents

Preface

This book is a first course in partial differential equations. The first chapter covers first-order equations, solution of linear and quasi-linear equations, and the role of characteristics in the Cauchy problem. Chapter 2 is devoted to linear second-order equations, classification, the second order Cauchy problem, and the significance of characteristics in existence and uniqueness of solutions, and as carriers of discontinuities. Chapter 3 is a review of Fourier series, integrals, and transforms, and Chapters 4, 5 and 6 develop properties of solutions, and techniques for finding solutions in particular cases, for the wave equation, the heat equation, and Dirichlet and Neumann problems.

Chapters 7 and 8 are new to this edition and are independent of each other. Chapter 7 begins with a classical proof of an existence theorem for the Dirichlet problem. This existence question is then reformulated as a problem of representing a linear functional as an inner product in a Hilbert space, serving as an introduction to the use of function spaces in the study of partial differential equations. The chapter concludes with a brief introduction to distributions and the formulation of another existence theorem.

Chapter 8 is a collection of independent additional topics, including the solution of boundary value problems by eigenfunction expansions, numerical methods, and explicit solutions of Burger's equation, the telegraph equation, and Poisson's equation.

Particularly in working with solutions of wave and heat equations, it is often instructive to use computational software to carry out numerical approximations, to gauge the effects of parameters on solutions, to construct graphs, and to manipulate special functions such as Bessel functions. If such routines are not available, parts of some exercises can be omitted.

Chapter 1

First-Order Equations

1.1 Notation and Terminology

A *partial differential equation* is an equation that contains at least one partial derivative. Examples are

$$\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = xuy^2$$

and

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = f(x,y,z).$$

We often use subscripts to denote partial derivatives. In this notation, $u_x = \partial u/\partial x, u_{xx} = \partial^2 u/\partial x^2, u_{xy} = \partial^2 u/\partial y\, \partial x$, and so on. The partial differential equations listed above can be written, respectively,

$$u_x - xu_y = xuy^2$$

and

$$h_{xx} + h_{yy} + h_{zz} = f(x,y,z). \tag{1.1}$$

A *solution* of a partial differential equation is any function that satisfies the equation. We will often seek solutions satisfying certain conditions and perhaps having the independent variables confined to a specified set of values.

As an example of a solution, the equation

$$4u_x + 3u_y + u = 0 \tag{1.2}$$

has the solution

$$u(x,y) = e^{-x/4}f(3x-4y),$$

in which f can be any differentiable function of a single variable. This can be verified by substituting $u(x, y)$ into the partial differential equation. Chain rule differentiations yield

$$\begin{aligned}u_x &= -\frac{1}{4}e^{-x/4}f(3x-4y) + e^{-x/4}\frac{d}{d(3x-4y)}[f(3x-4y)]\frac{d(3x-4y)}{dx} \\ &= -\frac{1}{4}e^{-x/4}f(3x-4y) + 3e^{-x/4}f'(3x-4y)\end{aligned}$$

and similarly,

$$u_y = -4e^{-x/4}f'(3x-4y).$$

Upon substitution into equation 1.2, we obtain

$$\begin{aligned}4u_x + 3u_y + u = &-e^{-x/4}f(3x-4y) \\ &+ 12e^{-x/4}f'(3x-4y) - 12e^{-x/4}f'(3x-4y) \\ &+ e^{-x/4}f(3x-4y) = 0.\end{aligned}$$

Because of the freedom to choose f, equation 1.2 has infinitely many solutions.

The *order* of a partial differential equation is the order of the highest partial derivative occurring in the equation. Equation 1.2 is of order one and equation 1.1 is of order two.

A partial differential equation is *linear* if it is linear in the unknown function and its partial derivatives. An equation that is not linear is *nonlinear*. For example,

$$x^2u_{xx} - yu_{xy} = u$$

is linear, whereas

$$x^2u_{xx} - yu_{xy} = u^2$$

is nonlinear because of the u^2 term, and

$$(u_{xx})^{1/2} - 4u_{yy} = xu$$

is nonlinear because of the $(u_{xx})^{1/2}$ term.

A partial differential equation is *quasi-linear* if it is linear in its highest-order derivative term(s). The second-order equation

$$u_{xx} + 4yu_{yy} - (u_x)^3 + u_xu_y = \cos(u)$$

is quasi-linear because it is linear in its second derivative (highest-order) terms u_{xx} and u_{yy}. This equation is not linear because of the $\cos(u), u_xu_y$, and $(u_x)^3$ terms. Any linear equation is also quasi-linear.

We now have the vocabulary to begin studying partial differential equations, starting with first order.

Problems for Section 1.1

1. Show that

$$u(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

is a solution of $u_{xx} + u_{yy} + u_{zz} = 0$ for $(x,y,z) \neq (0,0,0)$.

2. Let c be a positive constant. Show that $u(x,t) = f(x+ct) + g(x-ct)$ is a solution of $u_{tt} = c^2 u_{xx}$ for any twice-differentiable functions f and g of one variable.

3. Show that

$$u(x,t) = \frac{1}{2}(\varphi(x+ct) + \varphi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds$$

is a solution of $u_{tt} = c^2 u_{xx}$ for any φ that is twice differentiable and ψ that is differentiable for all real x. c is a positive constant. Show that this solution satisfies the conditions

$$u(x,0) = \varphi(x); u_t(x,0) = \psi(x)$$

for all real x.

4. Show that if p is a continuously differentiable function of one variable, the first-order partial differential equation

$$u_t = p(u)u_x$$

has a solution implicitly defined by

$$u(x,t) = \varphi(x + p(u)t),$$

in which φ can be any continuously differentiable function of one variable. Use this idea to determine (perhaps implicitly) a solution of each of the following equations.

(a) $u_t = ku_x$, with k a nonzero constant

(b) $u_t = uu_x$

(c) $u_t = \cos(u)u_x$

(d) $u_t = e^u u_x$

(e) $u_t = u\sin(u)u_x$

5. Show that

$$u(x,y) = \ln((x-x_0)^2 + (y-y_0)^2)$$

satisfies $u_{xx} + u_{yy} = 0$ for all pairs $(x,\, y)$ of real numbers except (x_0, y_0).

6. Let v and w be solutions of

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + g(x,y)u = 0.$$

Show that $\alpha v + \beta w$ is also a solution for any numbers α and β.

7. In each of the following, classify the equation as linear, quasi-linear and not linear, or not quasi-linear.

   (a) $u^2 u_{xx} + u_y = \cos(u)$

   (b) $x^2 u_x + y^2 u_y + u_{xy} = 2xy$

   (c) $(x - y)u_x^2 + u_{xy} = 1$

   (d) $(x - y)u_x^2 + 2u_y = 4y$

   (e) $x^2 u_{yy} - y u_{xx} = \tan(u)$

   (f) $u_x + u_y^2 - u_{xx} = 4$

   (g) $u_x - u_x u_y - u_y = 0$

   (h) $u u_x + u_{xy} = u^2$

   (i) $u_{xy} - u_x^2 + u_y^2 - \sin(u_x) = 0$

   (j) $u_y / u_x = x^2$

8. Let k be a positive constant. Let

$$u(x,t) = \frac{1}{2\sqrt{\pi k t}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4kt} f(\xi)\, d\xi,$$

   in which f is continuous on the real line. Show that $u_t = k u_{xx}$ for $-\infty < x < \infty, t > 0$. Also determine $u(x,\, t)$ when $f(x) = 1$ for all real x. Hint: Use a change of variables and the standard result that

$$\int_{-\infty}^{\infty} e^{-w^2}\, dw = \sqrt{\pi}.$$

1.2 The Linear First-Order Equation

We will solve the linear first-order partial differential equation in two independent variables:

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y). \tag{1.3}$$

Assume that a, b, c, and f are continuous in some region of the plane, and that $a(x,\, y)$ and $b(x,\, y)$ are not both zero for the same $(x,\, y)$.

The key is to determine a change of variables

$$\xi = \varphi(x,y), \eta = \psi(x,y),$$

which transforms equation 1.3 to the simpler linear equation

$$w_\xi + h(\xi,\eta)w = F(\xi,\eta), \tag{1.4}$$

where $w(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta))$. We will want this transformation to be one-to-one, at least for all $(x,\, y)$ in some set $\mathcal{D}$ of points in the x, y - plane. In this event, on $\mathcal{D}$, we can, at least in theory, solve for x and y as functions of ξ and η. To ensure this, we will require that the Jacobian of the transformation does not vanish in $\mathcal{D}$:

$$J = \begin{vmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{vmatrix} = \varphi_x\psi_y - \varphi_y\psi_x \neq 0$$

for (x, y) in $\mathcal{D}$.

Begin the search for a suitable transformation by computing the chain rule derivatives

$$u_x = w_\xi\xi_x + w_\eta\eta_x, u_y = w_\xi\xi_y + w_\eta\eta_y.$$

Substitute these into equation 1.3 to obtain

$$a(w_\xi\xi_x + w_\eta\eta_x) + b(w_\xi\xi_y + w_\eta\eta_y) + cw = f,$$

which we can write as

$$(a\xi_x + b\xi_y)w_\xi + (a\eta_x + b\eta_y)w_\eta + cw = f. \tag{1.5}$$

This is nearly in the form of equation 1.4 if we choose η so that

$$a\eta_x + b\eta_y = 0$$

for (x, y) in $\mathcal{D}$. If $\eta_y \neq 0$, this requires that

$$\frac{\eta_x}{\eta_y} = -\frac{b}{a}.$$

Suppose for the moment that there is such an η. Putting $\eta(x, y) = c$, with c an arbitrary constant, then

$$d\eta = \eta_x\, dx + \eta_y\, dy = 0$$

implies that

$$\frac{dy}{dx} = -\frac{\eta_x}{\eta_y} = \frac{b}{a}.$$

This means that $\eta = \psi(x, y)$ is an integral of the ordinary differential equation

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}. \tag{1.6}$$

Equation 1.6 is called the *characteristic equation* of the linear equation 1.3. The equation $\eta(x, y) = k =$ constant defines a family of curves in the plane called *characteristic curves*, or *characteristics*, of equation 1.3. We will say more about these in the next section.

Thus far we have found that we can make the coefficient of w_η in the transformed equation 1.5 vanish if we choose $\eta = \psi(x, y)$, with $\psi(x, y) = k$ an equation defining the general solution of the characteristic equation 1.6. With this step alone, equation 1.5 comes close to the transformed equation 1.4 we want to achieve. We can now choose ξ to suit our convenience and the condition that $J \neq 0$. One simple choice is

$$\xi = \varphi(x,y) = x$$

because then

$$J = \begin{vmatrix} 1 & 0 \\ \eta_x & \eta_y \end{vmatrix} = \eta_y$$

and this is nonzero in $\mathcal{D}$ by previous assumption.

Since $\xi_x = 1$ and $\xi_y = 0$, substitution of

$$\xi = x, \eta = \psi(x,y)$$

into equation 1.5 results in

$$a(x,y)w_\xi + c(x,y)w = f(x,y).$$

In this equation, replace each x by ξ and y by $y(\xi,\eta)$ to obtain an equation of the form

$$A(\xi,\eta)w_\xi + C(\xi,\eta)w = p(\xi,\eta).$$

Finally, restricting the variables to a set in which $A(\xi,\eta) \neq 0$, we have

$$w_\xi + \frac{C}{A}w = \frac{p}{A},$$

and this is in the form of equation 1.4 with

$$h(\xi,\eta) = \frac{C(\xi,\eta)}{A(\xi,\eta)} \text{ and } F(\xi,\eta) = \frac{p(\xi,\eta)}{A(\xi,\eta)}.$$

Example 1.1 *The linear equation*

$$x^2u_x + yu_y + xyu = 1$$

has the form of equation 1.3 with $a(x,y) = x^2, b(x,y) = y, c(x,y) = xy$, and $f(x,y) = 1$. We will transform this equation to the simpler equation 1.4. The characteristic equation is

$$\frac{dy}{dx} = \frac{b}{a} = \frac{y}{x^2}.$$

This is a separable first-order ordinary differential equation. Write

$$\frac{1}{y}\,dy = \frac{1}{x^2}\,dx.$$

Integrate and rearrange terms to obtain

$$\ln(y) + \frac{1}{x} = k$$

for $y > 0$ *and* $x \neq 0$ *and* k *an arbitrary constant. This is an integral of the characteristic equation and we choose*

$$\eta = \psi(x, y) = \ln(y) + \frac{1}{x}.$$

Graphs of $\ln(y) + 1/x = k$ *are the characteristics of this partial differential equation. Now, choosing* $\xi = x$, *we have the Jacobian*

$$J = \eta_y = \frac{1}{y} \neq 0,$$

as required. Since $\xi = x$,

$$\eta = \ln(y) + \frac{1}{\xi},$$

so

$$\ln(y) = \eta - \frac{1}{\xi}$$

and

$$y = e^{\eta - 1/\xi}.$$

Now apply the transformation

$$\xi = x, \eta = \ln(y) + \frac{1}{x},$$

to the partial differential equation, with

$$w(\xi, \eta) = u(x, y).$$

Compute

$$u_x = w_\xi \xi_x + w_\eta \eta_x = w_\xi + w_\eta \left(-\frac{1}{x^2}\right) = w_\xi - \frac{1}{\xi^2} w_\eta$$

and

$$u_y = w_\xi \xi_y + w_\eta \eta_y = w_\eta \frac{1}{y} = w_\eta \frac{1}{e^{\eta - 1/\xi}}.$$

The partial differential equation $x^2 u_x + y u_y + xyu = 1$ *transforms to*

$$\xi^2 \left(w_\xi - \frac{1}{\xi^2} w_\eta\right) + e^{\eta - 1/\xi} w_\eta \frac{1}{e^{\eta - 1/\xi}} + \xi e^{\eta - 1/\xi} w = 1$$

or

$$\xi^2 w_\xi + \xi e^{\eta - 1/\xi} w = 1.$$

Then

$$w_\xi + \frac{1}{\xi} e^{\eta - 1/\xi} w = \frac{1}{\xi^2},$$

and this has the form of equation 1.4 in any region of the ξ, η − *plane with* $\xi \neq 0$. ◇

The point to transforming equation 1.3 to the form of equation 1.4 is that we can solve this transformed equation. Think of

$$w_\xi + h(\xi,\eta)w = F(\xi,\eta)$$

as a linear first-order ordinary differential equation in ξ, with η carried along as a parameter. Following the method for ordinary differential equations, multiply this equation by

$$e^{\int h(\xi,\eta)d\xi}$$

to obtain

$$e^{\int h(\xi,\eta)d\xi}w_\xi + h(\xi,\eta)e^{\int h(\xi,\eta)d\xi}w = F(\xi,\eta)e^{\int h(\xi,\eta)d\xi}.$$

Recognize this as

$$\frac{\partial}{\partial \xi}\left(e^{\int h(\xi,\eta)d\xi}w\right) = F(\xi,\eta)e^{\int h(\xi,\eta)d\xi}.$$

Integrate with respect to ξ. Since η is being carried through this process as a parameter, the constant of integration may depend on η. We obtain

$$e^{\int h(\xi,\eta)d\xi}w = \int F(\xi,\eta)e^{\int h(\xi,\eta)d\xi}\,d\xi + g(\eta),$$

in which g is any differentiable function of one variable. Then

$$w(\xi,\eta) = e^{-\int h(\xi,\eta)d\xi}\int F(\xi,\eta)e^{\int h(\xi,\eta)d\xi}\,d\xi + g(\eta)e^{-\int h(\xi,\eta)d\xi}. \tag{1.7}$$

This is the *general solution* of the transformed equation (by general solution we mean one that contains an arbitrary function). Now obtain the general solution of the original equation 1.3 by substituting $\xi = \xi(x,y), \eta = \eta(x,y)$. This general solution will have the form

$$u(x,y) = e^{\alpha(x,y)}[M(x,y) + g(\psi(x,y))], \tag{1.8}$$

in which g is any differentiable function of one variable.

Example 1.2 *We will solve the constant coefficient equation*

$$au_x + bu_y + cu = 0,$$

in which a, b and c are numbers and $a \neq 0$. The characteristic equation is

$$\frac{dy}{dx} = \frac{b}{a}$$

with general solution defined by the equation

$$bx - ay = k,$$

with k any number. Put

$$\xi = x \text{ and } \eta = bx - ay.$$

The characteristics of this differential equation are the straight-line graphs of $bx - ay = k$.

With this transformation, we find by a routine calculation that the partial differential equation transforms to

$$aw_\xi + cw = 0$$

or

$$w_\xi + \frac{c}{a}w = 0.$$

Multiply this equation by $e^{\int (c/a)d\xi}$, which is $e^{c\xi/a}$, to get

$$e^{c\xi/a}w_\xi + \frac{c}{a}we^{c\xi/a} = 0.$$

This is

$$\frac{\partial}{\partial\xi}(e^{c\xi/a}w) = 0.$$

Integrate with respect to ξ to get

$$e^{c\xi/a}w = g(\eta),$$

in which g can be any differentiable function of one variable. Then

$$w(\xi, \eta) = e^{-c\xi/a}g(\eta).$$

Finally, transform this solution back in terms of x and y:

$$u(x, y) = e^{-cx/a}g(bx - ay).$$

This solution is readily verified by substitution into the partial differential equation. $\diamond$

Observe that the solution in this example has the form specified by equation 1.8.

Example 1.3 *The linear equation*

$$u_x + \cos(x)u_y + u = xy$$

has characteristic equation

$$\frac{dy}{dx} = \cos(x).$$

Integrate the characteristic equation to get

$$y - \sin(x) = k,$$

with k any number. This defines the transformation

$$\xi = x, \eta = y - \sin(x).$$

Graphs of $y - \sin(x) = k$ are the characteristics of this partial differential equation.

Now we have

$$y = \eta + \sin(x) = \eta + \sin(\xi)$$

and the partial differential equation transforms to

$$w_\xi + w = \xi[\eta + \sin(\xi)].$$

Multiply this equation by $e^{\int d\xi}$, which is e^ξ, to obtain

$$e^\xi w_\xi + we^\xi = \eta\xi e^\xi + \xi e^\xi \sin(\xi).$$

Write this equation as

$$\frac{\partial}{\partial \xi}(we^\xi) = \eta\xi e^\xi + \xi e^\xi \sin(\xi).$$

Integrate with respect to ξ to obtain

$$\begin{aligned} we^\xi &= \int \eta\xi e^\xi \, d\xi + \int \xi e^\xi \sin(\xi)\, d\xi \\ &= \eta e^\xi(\xi - 1) + \frac{1}{2}\xi e^\xi(\sin(\xi) - \cos(\xi)) + \frac{1}{2}e^\xi \cos(\xi) + g(\eta). \end{aligned}$$

Then

$$w(\xi, \eta) = \eta(\xi - 1) + \frac{1}{2}\xi(\sin(\xi) - \cos(\xi)) + \frac{1}{2}\cos(\xi) + e^{-\xi}g(\eta).$$

Finally,

$$\begin{aligned} u(x, y) = (y - \sin(x))(x - 1) + \frac{1}{2}x(\sin(x) - \cos(x)) \\ + \frac{1}{2}\cos(x) + e^{-x}g(y - \sin(x)), \end{aligned}$$

in which g is any differentiable function of a single variable. ◇

Contrast the idea of the general solution for the linear first-order ordinary differential equation with that for the linear first-order partial differential equation. In the former case, the general solution of

$$y' + d(x)y = p(x)$$

contains an arbitrary constant. Graphs of the solutions obtained by making choices of the constant are curves in the $x, y-$ plane. If we require that $y(x_0) = y_0$, we pick out the unique solution whose graph passes through (x_0, y_0).

But if u is the general solution of the linear first-order partial differential equation 1.3, then $z = u(x,y)$ defines a family of surfaces in 3 - space, each surface corresponding to a choice of the arbitrary function g in equation 1.8. In the next section we investigate the kind of information that should be given in order to pick out one of these surfaces and determine a unique solution.

Problems for Section 1.2

For each of Problems 1 through 12, (a) solve the characteristic equation and sketch graphs of some of the characteristics, (b) define a transformation of the partial differential equation to the form of equation 1.4 and obtain the transformed equation, (c) find the general solution of the transformed equation, (d) find the general solution of the given partial differential equation, and (e) verify the solution by substituting it into the partial differential equation.

1. $3u_x + 5u_y - xyu = 0$
2. $u_x - u_y + yu = 0$
3. $u_x + 4u_y - xu = x$
4. $-2u_x + u_y - yu = 0$
5. $xu_x - yu_y + u = x$
6. $x^2u_x - 2u_y - xu = x^2$
7. $u_x - xu_y = 4$
8. $x^2u_x + xyu_y + xu = x - y$
9. $u_x + u_y - u = y$
10. $u_x - y^2u_y - yu = 0$
11. $u_x + yu_y + xu = 0$
12. $xu_x + yu_y + 2 = 0$
13. Find the general solution of

$$u_x + \alpha(y-1)u_y = \frac{1}{2}\beta f(x)(y-1)u,$$

in which α and β are real numbers and f is continuous on the real line. Use the general solution to find a solution satisfying

$$u(0,y) = y^n,$$

in which n is a nonnegative integer.

1.3 The Significance of Characteristics

In the preceding section we used the characteristic equation of $a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y)$ to define a transformation of this equation to a simpler form that we could solve. In this section we look more closely at the significance of characteristic curves, beginning with an example that will suggest the point we want to make. The characteristic equation of

$$2u_x + 3u_y + 8u = 0$$

is

$$\frac{dy}{dx} = \frac{3}{2}$$

and the characteristics are the straight-line graphs of $3x - 2y = k$. We find that this partial differential equation has the general solution

$$u(x,y) = e^{-4x}g(3x - 2y),$$

in which g can be any differentiable function defined over the real line.

Notice that simply specifying that the solution is to have a given value at a particular point does not determine g uniquely, hence does not determine a unique solution as occurs with ordinary differential equations.

Instead of specifying the solution value at a point, try specifying values of $u(x,\ y)$ along a curve Γ in the plane. To be specific for this discussion, let Γ be the x - axis and give values of $u(x,\ y)$ at points on Γ, say

$$u(x,0) = \sin(x) \text{ for } x \text{ real.}$$

We need

$$u(x,0) = e^{-4x}g(3x) = \sin(x),$$

so

$$g(3x) = e^{4x}\sin(x).$$

Putting $t = 3x$ yields

$$g(t) = e^{4t/3}\sin(t/3).$$

This determines g, and the solution satisfying the condition $u(x,0) = \sin(x)$ on Γ is

$$\begin{aligned} u(x,y) &= e^{-4x}g(3x - 2y) = e^{-4x}e^{4(3x-2y)/3}\sin\left(\frac{1}{3}(3x - 2y)\right) \\ &= e^{-8y/3}\sin\left(x - \frac{2}{3}y\right). \end{aligned}$$

In this example, specifying values of u along Γ uniquely determined the arbitrary function in the general solution, hence determined the unique solution of the partial differential equation having these given values.

Try another choice of Γ, say the line $y = x$. We will try to find a solution having given values along this curve, say

$$u(x, x) = x^4.$$

From the general solution, this requires that

$$u(x, x) = e^{-4x} g(x) = x^4,$$

so

$$g(x) = x^4 e^{4x}.$$

Then

$$u(x, y) = e^{-4x} g(3x - 2y) = e^{8(x-y)}(3x - 2y)^4.$$

This is the unique solution satisfying $u(x, x) = x^4$ along the curve Γ given by $y = x$.

Can we find a unique solution having given values along any (reasonable) curve Γ in the plane? The answer is no, as we can see by taking Γ to be the line $3x - 2y = 1$, and prescribing values $u(x, y)$ is to have along Γ, say

$$u\left(x, \frac{1}{2}(3x - 1)\right) = x^2.$$

This requires that we choose g so that

$$e^{-4x} g\left(3x - 2\frac{1}{2}(3x - 1)\right) = x^2$$

or

$$g(1) = e^{4x} x^2.$$

This is impossible, because $e^{4x}x^2$ is not constant. There is no solution taking the value x^2 at points (x, y) on this Γ.

Why did some choices of Γ give a solution, whereas another choice gave no solution? The difference was that the x - axis and the line $y = x$ are not characteristics of the partial differential equation, whereas the line $3x - 2y = 1$ is a characteristic. This suggests that characteristics have some significance in the context of existence and uniqueness of solutions. To understand this significance, go back to the general solution

$$u(x, y) = e^{\alpha(x,y)}[M(x, y) + g(\psi(x, y))]$$

of the linear first-order partial differential equation 1.3. Suppose that we prescribe $u(x, y) = q(x)$ along a characteristic. Now a characteristic is specified by $\psi(x, y) = k$. If $y = y(x)$ along this characteristic, then

$$q(x) = e^{\alpha(x,y(x))}[M(x, y(x)) + g(k)]$$

or

$$q(x) = e^{\alpha(x,y(x))}[M(x, y(x)) + C], \tag{1.9}$$

in which C is constant. The functions $M(x, y)$ and $\alpha(x,y)$ are determined by the partial differential equation and are not under our control, so equation 1.9 places a constraint on the given function $q(x)$. If $q(x)$ is not of this form for any constant C, there is no solution taking on these prescribed values on Γ. On the other hand, if $q(x)$ is of this form for some C, there are infinitely many such solutions, because we can choose for g any differentiable function such that $g(k) = C$.

Example 1.4 *To illustrate these remarks, begin by finding the general solution of*

$$xu_x + 2x^2u_y - u = x^2e^x. \tag{1.10}$$

The characteristic equation is

$$\frac{dy}{dx} = 2x$$

with general solution defined by $y - x^2 = k$. *The characteristics are parabolas. Let*

$$\xi = x \text{ and } \eta = y - x^2$$

to obtain

$$\xi w_\xi - w = \xi^2 e^\xi,$$

which we write as

$$w_\xi - \frac{1}{\xi}w = \xi e^\xi.$$

Multiply this equation by $e^{\int(-1/\xi)d\xi}$, *which is* $1/\xi$, *to obtain*

$$\frac{1}{\xi}w_\xi - \frac{1}{\xi^2}w = e^\xi$$

or

$$\frac{\partial}{\partial\xi}\left(\frac{1}{\xi}w\right) = e^\xi.$$

Integrate with respect to ξ *to get*

$$\frac{1}{\xi}w = e^\xi + g(\eta),$$

so that

$$w = \xi e^\xi + \xi g(\eta).$$

The general solution of equation 1.10 is

$$u(x,y) = xe^x + xg(y - x^2).$$

Now attempt to find solutions satisfying given conditions along various curves. Suppose that first we seek a solution such that $u(x,y) = \sin(x)$ *on the curve* $y = x^2 + 4$. *Notice that information is being specified along a characteristic. We will need*

$$u(x, x^2 + 4) = xe^x + xg(4) = \sin(x).$$

We must be able to find a constant C such that

$$xe^x + Cx = \sin(x)$$

for all x, and this is impossible. There is no solution satisfying $u(x, y) = \sin(x)$ on the given curve.

Next, suppose we want a solution such that $u(x, y) = xe^x - x$ on the parabola $y = x^2 + 4$. Now we need

$$u(x, x^2 + 4) = xe^x + xg(4) = xe^x - x.$$

This equation requires that $g(4) = -1$. This problem has infinitely many solutions because we can choose g to be any differentiable function of one variable such that $g(4) = -1$. Even though data is specified on a characteristic, the form of the data allows infinitely many solutions.

Finally, suppose we want a solution such that $u(x, y) = \cos(x)$ along the noncharacteristic parabola $y = x^2 + 4x$. Now we need

$$u(x, x^2 + 4x) = xe^x + xg(4x) = \cos(x).$$

This requires that

$$g(4x) = \frac{\cos(x) - xe^x}{x}.$$

Choose

$$g(t) = 4\frac{\cos(t/4) - (t/4)e^{t/4}}{t}$$

for, say, $t > 0$. The solution of the problem (for $x > 0$) is

$$\begin{aligned} u(x, y) &= xe^x + xg(y - x^2) \\ &= xe^x + 4x\left(\frac{\cos\left(\frac{y-x^2}{4}\right) - \frac{1}{4}(y - x^2)e^{(y-x^2)/4}}{y - x^2}\right). \diamond \end{aligned}$$

The problem of finding a solution of equation 1.3 taking on prescribed values on a given curve is called the *Cauchy problem* for this equation, and the information given on the curve is called *Cauchy data*. The examples suggest that we can expect a unique solution of a Cauchy problem if the curve along which data is specified is not characteristic, and either no solution or infinitely many solutions if the curve is characteristic.

Problems for Section 1.3

In each of Problems 1 through 6, determine the characteristic equation, solve it and sketch graphs of some of the characteristics, find the general solution of the partial differential equation, and attempt to find particular solutions satisfying the Cauchy data on the given curves.

1. $3yu_x - 2xu_y = 0$

   (a) $u(x, y) = x^2$ on the line $y = x$

(b) $u(x, y) = 1 - x^2$ on the line $y = -x$

(c) $u(x, y) = 2x$ on the ellipse $3y^2 + 2x^2 = 4$

2. $u_x - 6u_y = y$

(a) $u(x, y) = e^x$ on the line $y = -6x + 2$

(b) $u(x, y) = 1$ on the parabola $y = -x^2$

(c) $u(x, y) = -4x$ on the line $y = -6x$

3. $4u_x + 8u_y - u = 1$

(a) $u(x, y) = \cos(x)$ on the line $y = 3x$

(b) $u(x, y) = x$ on the line $y = 2x$

(c) $u(x, y) = 1 - x$ on the curve $y = x^2$

4. $-4yu_x + u_y - yu = 0$

(a) $u(x, y) = x^3$ on the line $x + 2y = 3$

(b) $u(x, y) = -y$ on $y^2 = x$

(c) $u(x, y) = 2$ on $x + 2y^2 = 1$

5. $yu_x + x^2u_y = xy$

(a) $u(x, y) = 4x$ on the curve $y = \frac{1}{3}x^{3/2}$

(b) $u(x, y) = x^3$ on the curve $3y^2 = 2x^3$

(c) $u(x, y) = \sin(x)$ on the line $y = 0$

6. $y^2u_x + x^2u_y = y^2$

(a) $u(x, y) = x$ on $y = 4x$

(b) $u(x, y) = -2y$ on $y^3 = x^3 - 2$

(c) $u(x, y) = y^2$ on $y = -x$

1.4 The Quasi-Linear Equation

For the general first-order linear equation 1.3, characteristics are certain curves in the x, y–plane, defined so that the Cauchy problem has a unique solution when the Cauchy data is specified along a noncharacteristic.

For the first-order quasi-linear partial differential equation

$$f(x, y, u)u_x + g(x, y, u)u_y = h(x, y, u), \tag{1.11}$$

characteristics are defined as curves in x, y, u - space determined as solutions of

$$\frac{dx}{dt} = f(x, y, u), \frac{dy}{dt} = g(x, y, u), \frac{du}{dt} = h(x, y, u). \tag{1.12}$$

We will show that a solution $u(x, y)$ of equation 1.11 may be interpreted as a surface made up of such characteristics. This idea can be used to obtain solutions containing given noncharacteristic curves, and hence provides a way of solving the Cauchy problem when the partial differential equation is quasi-linear. To understand this process, we need two facts.

Fact 1 Suppose that $u = \varphi(x, y)$ is a solution of equation 1.11 defining a surface $\sum$ and that $P_0 : (x_0, y_0, u_0)$ is a point on $\sum$, so $u_0 = \varphi(x_0, y_0)$. Then the characteristic passing through P_0 lies entirely on $\sum$.

To see why this is true, suppose that the characteristic has parametric equations

$$x = x(t), y = y(t), u = u(t).$$

Because P_0 is on this characteristic, for some t_0,

$$x(t_0) = x_0, y(t_0) = y_0, u(t_0) = u_0.$$

Because this curve is characteristic, we can use equations 1.12 to compute

$$\begin{aligned}\frac{d}{dt}\varphi(x(t), y(t)) &= \varphi_x \frac{dx}{dt} + \varphi_y \frac{dy}{dt}\\ &= \varphi_x f(x, y, u) + \varphi_y g(x, y, u) = h(x, y, u) = \frac{du}{dt}.\end{aligned}$$

Therefore,

$$u(t) = \varphi(x(t), y(t)) + k$$

for some constant k. But then $k = 0$, because we know that

$$u_0 = u(t_0) = \varphi(x(t_0), y(t_0)).$$

Therefore,

$$u(t) = \varphi(x(t), y(t))$$

and the characteristic lies on $\sum$.

Fact 2 If we begin with an arbitrary noncharacteristic curve Γ and construct the family of characteristics passing through points of Γ, the resulting surface $\sum$ is the graph of a solution of the partial differential equation.

To see why this is true, assume that $\sum$ is the graph of $u = \varphi(x, y)$. We want to show that φ is a solution of equation 1.11.

Suppose that Γ is parametrized by

$$x = x(s), y = y(s), z = z(s).$$

At any (x, y, u) on $\sum$,

$$\frac{dx}{ds} = f(x, y, u), \frac{dy}{ds} = g(x, y, u), \frac{du}{ds} = h(x, y, u)$$

because the surface is made up of characteristics. Then

$$\frac{du}{ds} = h(x, y, u) = \varphi_x \frac{dx}{ds} + \varphi_y \frac{dy}{ds} = f\varphi_x + g\varphi_y,$$

so φ is a solution.

These observations suggest the *method of characteristics* for solving the Cauchy problem when the partial differential equation is quasi-linear. Suppose that we want the solution of equation 1.11 assuming prescribed values on a given curve Γ that is not characteristic. Construct the characteristic through each point of Γ. This defines a surface in 3 - space, and this surface is the graph of the solution of this Cauchy problem.

This strategy also suggests why we do not want to specify data along a characteristic C. If we did so, the characteristic through each point of C would be just C itself, and this construction yields just the curve C, not a surface representing a solution of the partial differential equation.

Here are two illustrations of the method of characteristics.

Example 1.5 *We want the solution of*

$$yu_x - xu_y = e^u$$

that passes through the curve Γ *given by* $y = \sin(x), u = 0$. *This means that we want a solution* $u(x,\, y)$ *satisfying*

$$u(x, \sin(x)) = 0.$$

The characteristics of this partial differential equation are specified by

$$\frac{dx}{dt} = y, \frac{dy}{dt} = -x, \frac{du}{dt} = e^u.$$

From the first two of these equations we can write

$$\frac{dy}{dx} = -\frac{x}{y}$$

or

$$y\,dy + x\,dx = 0,$$

with general solution (in terms of t)

$$x = a\cos(t) + b\sin(t) \text{ and } y = b\cos(t) - a\sin(t),$$

with a and b constant. From $du/dt = e^u$ *we obtain*

$$-e^{-u} = t + c.$$

The characteristics therefore have parametric representation

$$x = a\cos(t) + b\sin(t), y = b\cos(t) - a\sin(t), e^{-u} = c - t.$$

Parametrize Γ *as*

$$x = s, y = \sin(s), u = 0.$$

We use s as a parameter on Γ *to distinguish between points on* Γ *and points on characteristics. We want to construct a characteristic through each point of* Γ.

Let P : $(s, \sin(s), 0)$ be a point of Γ and suppose that a characteristic passes through P when $t = 0$ (this is just a scaling of the parameter). Then at $t = 0$,

$$x = a = s, y = b = \sin(s), \text{ and } e^0 = 1 = 0 + c,$$

giving us $a = s$ and $b = \sin(s)$ at P. Therefore, the characteristic intersecting Γ at P has parametric equations

$$x = s\cos(t) + \sin(s)\sin(t), y = \sin(s)\cos(t) - s\sin(t), e^{-u} = 1 - t.$$

Now eliminate t and s from these equations. From the first two equations,

$$s = x\cos(t) - y\sin(t)$$

and

$$\sin(s) = y\cos(t) + x\sin(t).$$

Therefore

$$\sin(x\cos(t) - y\sin(t)) = y\cos(t) + x\sin(t). \tag{1.13}$$

But $e^{-u} = 1 - t$ implies that

$$t = 1 - e^{-u}.$$

Substituting this for t in equation 1.13 gives us

$$\sin(x\cos(1 - e^{-u}) - y\sin(1 - e^{-u})) = y\cos(1 - e^{-u}) + x\sin(1 - e^{-u})).$$

This equation implicitly defines the solution of the Cauchy problem. It is easy to check that $y = \sin(x), u = 0$ satisfies this equation.

Figure 1.1 shows a graph of part of the surface (solution). $\diamond$

Example 1.6 *We will find the solution of the equation*

$$xu_x + yu_y = \sec(u)$$

passing through the curve

$$\Gamma : x = s^2, y = \sin(s), u = 0.$$

The characteristics satisfy

$$\frac{dx}{dt} = x, \frac{dy}{dt} = y, \frac{du}{dt} = \sec(u),$$

which have solutions defined by

$$x = Ae^t, y = Be^t, \sin(u) = t + c.$$

We want to construct the characteristic through each point of Γ. Suppose that a characteristic passes through Γ at P : $(s^2, \sin(s), 0)$ at $t = 0$. Then

$$x = A = s^2, y = B = \sin(s), \sin(0) = 0 = 0 + c,$$

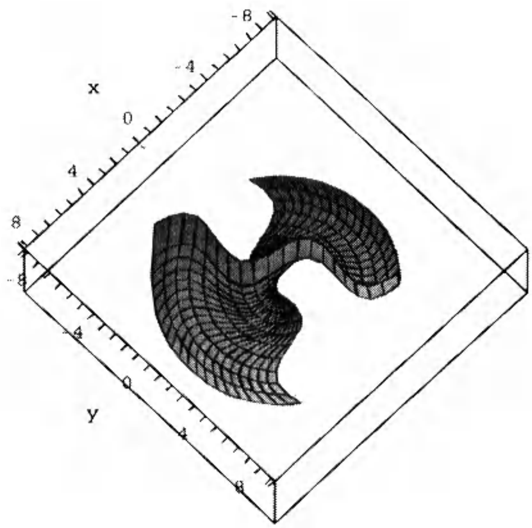

Figure 1.1: Part of the surface of Example 1.5.

so

$$x = s^2, y = \sin(s), \text{ and } c = 0$$

at this point. Then

$$x = s^2 e^t, y = \sin(s)e^t, \text{ and } \sin(u) = t.$$

We want to eliminate s and t from these equations. Since $t = \sin(u)$, *then*

$$y = \sin(s)e^{\sin(u)},$$

so

$$\sin(s) = ye^{-\sin(u)}$$

and

$$s = \arcsin(ye^{-\sin(u)}).$$

Then, using the fact that $x = e^t s^2$, *we have*

$$x = e^{\sin(u)}[\arcsin(ye^{-\sin(u)})]^2.$$

This equation implicitly defines u(x, y) such that $u(s^2, \sin(s)) = 0$. *The solution surface therefore contains the data curve* Γ. *Figure 1.2 shows part of the graph of this surface.* $\diamond$

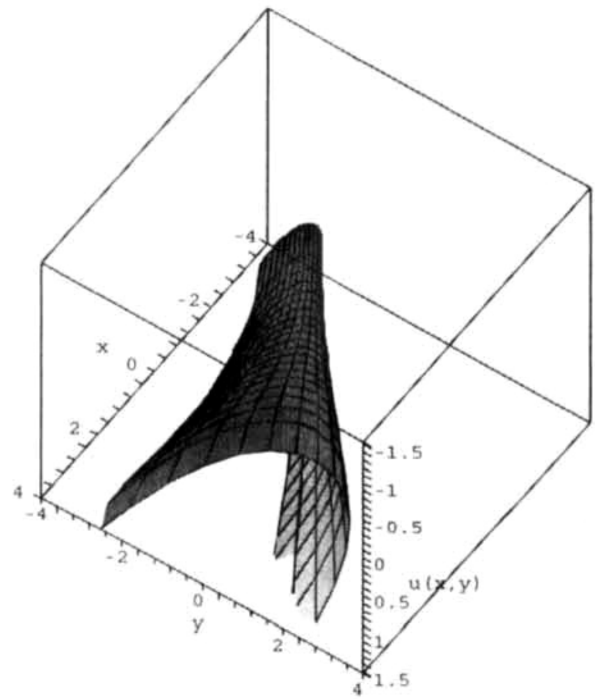

Figure 1.2: Part of the surface of Example 1.6.

Problems for Section 1.4

For each of Problems 1 through 10, use the method of characteristics to find a solution of the partial differential equation that passes through the given curve Γ. The solution may be defined implicitly. Graph part of the solution surface.

1. $xu_x + yu_y = \sec(u); \Gamma$ is the curve defined by $y = x^3, u = 0$.
2. $u_x - xu_y = 4; \Gamma$ is given by $y = 4x, u = 0$.
3. $u_x - y^2u_y = 1; \Gamma$ is given by $y = x^2 + 2; u = 0$.
4. $u_x - y^3u_y = \sec(u); \Gamma$ is given by $y = x^2, u = 0$.
5. $u_x + yu_y = u; \Gamma$ is given by $y = 1 - x, u = 1$.
6. $u_x + y^2u_y = \cos(u); \Gamma$ is given by $x = y^2, u = 0$.
7. $u_x - u_y = u^2; \Gamma$ is given by $y = 2x - 1, u = 4$.
8. $x^3u_x - yu_y = u; \Gamma$ is given by $x = y^2 - 1, u = 1$.
9. $u_x - y^2u_y = u; \Gamma$ is given by $y = 1 - x^2, u = 2$.
10. $xu_x + u_y = e^u; \Gamma$ is given by $y = x - 1, u = 0$.
11. Use the method of characteristics to derive the (implicitly defined) solution

$$u(x, y) = f(x - u(x, y)y)$$

of the problem

$$\begin{aligned} uu_x + u_y &= 0 \\ u(x, 0) &= f(x). \end{aligned}$$

Hint: Think of Γ as the curve $x = s, y = 0, u = f(s)$. Next, use the idea of Problem 4 of Section 1.1 to solve this problem. Compare these solutions.

Chapter 2

Linear Second-Order Equations

2.1 Classification

The general linear second-order partial differential equation, in two independent variables x and y, has the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0, \tag{2.1}$$

in which the coefficients are continuous functions of x and y in some region $\mathcal{D}$ of the plane. The factor of 2 in the coefficient of u_{xy} simplifies an expression we will derive shortly. We assume that $A(x,y), B(x,y)$, and $C(x,y)$ are never zero at the same $(x,\ y)$, so a second derivative term is present at each point. This partial differential equation, although simple in appearance, is rich in theory and important applications, and we explore it in detail in chapters 4 through 6.

We were able to solve the linear first-order partial differential equation by a change of variables. We explore the idea of transforming equation 2.1 into a form more suitable for obtaining solutions, or at least information about solutions. Begin with the general transformation

$$\xi = \xi(x,y), \eta = \eta(x,y).$$

We assume that this is one-to-one, so the Jacobian does not vanish in $\mathcal{D}$:

$$J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0 \text{ for}(x,y) \text{ in } \mathcal{D}.$$

The transformation therefore has an inverse $x = x(\xi,\eta), y = y(\xi,\eta)$. Let $w(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta))$. Use the chain rule to compute

$$\begin{aligned} u_x &= w_\xi \xi_x + w_\eta \eta_x, \\ u_{xx} &= \xi_{xx} w_\xi + \xi_x(w_{\xi\xi}\xi_x + w_{\eta\xi}\eta_x) + \eta_{xx} w_\eta + \eta_x(w_{\xi\eta}\xi_x + w_{\eta\eta}\eta_x) \\ &= w_{\xi\xi}\xi_x^2 + 2w_{\eta\xi}\xi_x\eta_x + w_{\eta\eta}\eta_x^2 + w_\xi \xi_{xx} + w_\eta \eta_{xx}, \end{aligned}$$

and so on, for the other derivatives occurring in equation 2.1. After a routine calculation the transformed equation has the form

$$aw_{\xi\xi} + 2bw_{\xi\eta} + cw_{\eta\eta} + dw_{\xi} + ew_{\eta} + fw + g = 0, \tag{2.2}$$

in which

$$a = A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2, c = A\eta_x^2 + 2B\eta_x\eta_y + C\eta_y^2$$

and

$$b = A\xi_x\eta_x + B(\xi_x\eta_y + \eta_x\xi_y) + C\xi_y\eta_y.$$

The coefficients $d, \cdots, g$ are not difficult to write down, but we will not need them for this discussion.

Equation 2.2 is of exactly the same form as the original equation 2.1. This should not be surprising, since we have not placed any restrictions on the transformation other than that it must be invertible. The idea now is to choose particular transformations to simplify equation 2.2.

A routine calculation shows that

$$b^2 - ac = (B^2 - AC)J^2. \tag{2.3}$$

The quantity $B^2 - AC$ is called the *discriminant* of equation 2.1, and equation 2.3 implies that the original partial differential equation and the transformed equation 2.2 have discriminants of the same sign. The sign of the discriminant is called an *invariant* of the transformation, and the partial differential equation is classified as

hyperbolic in $\mathcal{D}$ if $B^2 - AC > 0$,

parabolic in $\mathcal{D}$ if $B^2 - AC = 0$,

and

elliptic in $\mathcal{D}$ if $B^2 - AC < 0$,

for all (x, y) in $\mathcal{D}$.

Example 2.1 *For*

$$yu_{xx} - 2u_{xu} - xu_{yy} - u_x + \cos(y)u_y - 4 = 0,$$

we have $A(x, y) = y, B(x, y) = -1,$ *and* $C(x, y) = -x.$ *The discriminant is*

$$B^2 - AC = 1 + xy.$$

The equation is hyperbolic for all (x, y) *such that* $xy > -1$, *elliptic "between" the two branches of the hyperbola, that is, when* $xy < -1$, *and parabolic just on the hyperbola* $xy = -1$. $\diamond$

Because of equation 2.3, the classification of a partial differential equation 2.1 remains unchanged through transformations having nonvanishing Jacobian. We will now show that it is possible to transform equation 2.1 to a relatively simple form called its *canonical form*, which varies according to whether the equation is hyperbolic, parabolic, or elliptic. The type of equation determines what properties might be expected of solutions, what kinds of information must be supplied with the equation to have a unique solution, and influences the types of numerical techniques that can be used to approximate solutions.

We develop canonical forms for each classification of equation 2.1.

Problems for Section 2.1

1. Verify equation 2.3.

In each of Problems 2 through 10, classify the equation as hyperbolic, parabolic, or elliptic (in a region of the plane where the coefficients are continuous).

2. $4u_{xx} + u_{xy} - 2u_{yy} - \cos(xy) = 0$
3. $u_{xx} + 4xu_{xy} + 7u_{yy} - u_x + u_y = 0$
4. $yu_{xx} + 4u_{xy} + 4xyu_{yy} - 3u_y + u = 0$
5. $3u_{xx} + 2yu_{yy} + xu_x - u_y + yu = 0$
6. $u_{xy} - 2u_{xx} + (x+y)u_{yy} - xyu = 0$
7. $xu_{xx} - 2u_{xy} + xu_{yy} - x\cos(y) = 0$
8. $yu_{xx} + u_{xy} - x^2u_{yy} - u_x - u = 0$
9. $u_{xx} - yu_{xy} + u_{yy} = 0$
10. $5u_{xx} + 8u_{yy} + yu_x - u_y - 3u = 0$

2.2 The Hyperbolic Canonical Form

Suppose first that $B^2 - AC > 0$ in some region $\mathcal{D}$ of the plane. We will assume that $A(x, y) \neq 0$ for $(x,\ y)$ in $\mathcal{D}$. A similar discussion follows if $C(x, y) \neq 0$. If $A = C = 0$ throughout the region of interest, we will see that the partial differential equation is already in canonical form.

We can make $a = 0$ in equation 2.2 if we choose ξ so that

$$A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2 = 0.$$

Since $A \neq 0$, we can divide this equation by $A\xi_y^2$ to get

$$\left(\frac{\xi_x}{\xi_y}\right)^2 + 2\frac{B}{A}\left(\frac{\xi_x}{\xi_y}\right) + \frac{C}{A} = 0.$$

Then

$$\frac{\xi_x}{\xi_y} = \frac{-B \pm \sqrt{B^2 - AC}}{A}.$$

Along any curve $\xi(x,y) = k$, the slope at any point is given by

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{B \pm \sqrt{B^2 - AC}}{A}. \tag{2.4}$$

We can make $c = 0$ in equation 2.2 if we choose η so that

$$A\eta_x^2 + 2B\eta_x\eta_y + C\eta_y^2 = 0.$$

This is the same form of the equation we had involving ξ_x and ξ_y, and we obtain

$$\frac{\eta_x}{\eta_y} = \frac{-B \pm \sqrt{B^2 - AC}}{A}.$$

Further, the slope along any curve $\eta(x,y) = k$ is

$$\frac{dy}{dx} = -\frac{\eta_x}{\eta_y} = \frac{B \pm \sqrt{B^2 - AC}}{A}. \tag{2.5}$$

We can obtain a transformation with nonzero Jacobian, and making both $a = c = 0$, by choosing different signs before the radical in equations 2.4 and 2.5. Let $\xi(x,y) = k$ be an integral of

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - AC}}{A}, \tag{2.6}$$

and let $\eta(x,y) = K$ be an integral of

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - AC}}{A}. \tag{2.7}$$

The equations 2.6 and 2.7 are the *characteristic equations* for equation 2.1 in the hyperbolic case, and their integrals $\xi(x,y) = k, \eta(x,y) = K$, with k and K arbitrary constants, define two families of curves called *characteristics* for this equation. The transformation

$$\xi = \xi(x,y), \eta = \eta(x,y)$$

has been chosen so that $a = c = 0$ in the transformed equation 2.2, which becomes

$$2bw_{\xi\eta} + dw_\xi + ew_\eta + fw + g = 0.$$

Upon dividing by $2b$, we obtain

$$w_{\xi\eta} + d^*w_\xi + e^*w_\eta + f^*w + g^* = 0. \tag{2.8}$$

This is the *canonical form for the hyperbolic equation.* In general, we write this canonical form for the hyperbolic equation as

$$w_{\xi\eta} + \Phi(\xi, \eta, w, w_\xi, w_\eta) = 0,$$

with the only second derivative term being a mixed second partial derivative with coefficient 1.

Example 2.2 *The equation*

$$u_{xx} + 2\cos(x)u_{xy} - \sin^2(x)u_{yy} - \sin(x)u_y = 0 \tag{2.9}$$

has $A = 1, B = \cos(x)$, *and* $C = -\sin^2(x)$. *The discriminant is*

$$B^2 - AC = \cos^2(x) + \sin^2(x) = 1 > 0.$$

The partial differential equation 2.9 is hyperbolic. From equations 2.6 and 2.7, the characteristic equations are

$$\frac{dy}{dx} = \cos(x) + 1 \;\; and \;\; \frac{dy}{dx} = \cos(x) - 1.$$

Integrate the first of these equations to get $y = \sin(x) + x + k$; *hence let*

$$\xi(x, y) = y - \sin(x) - x.$$

Curves in the family $\xi(x, y) = k$, *for* k *any constant, are characteristics.*

'Solve the other characteristic equation to get $y = \sin(x) - x + K$, *so let*

$$\eta(x, y) = y - \sin(x) + x.$$

Curves in the family $\eta(x, y) = K$, *with* K *any constant, are also characteristics. Some of these characteristics are shown in Figure 2.1.*

Now transform equation 2.9. Let

$$\xi = y - \sin(x) - x, \eta = y - \sin(x) + x,$$

and

$$w(\xi, \eta) = u(x, y).$$

Compute

$$\begin{aligned}
u_x &= w_\xi(-\cos(x) - 1) + w_\eta(-\cos(x) + 1), \\
u_y &= w_\xi + w_\eta, \\
u_{xx} &= \sin(x)w_\xi + \sin(x)w_\eta \\
&\quad - (\cos(x) + 1)[w_{\xi\xi}(-\cos(x) - 1) + w_{\xi\eta}(-\cos(x) + 1)] \\
&\quad + (1 - \cos(x)[w_{\eta\xi}(-\cos(x) - 1) + w_{\eta\eta}(-\cos(x) + 1)], \\
u_{yy} &= w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta},
\end{aligned}$$

and

$$\begin{aligned}
u_{xy} &= w_{\xi\xi}(-\cos(x) - 1) + w_{\xi\eta}(-\cos(x) + 1) \\
&\quad + w_{\eta\xi}(-\cos(x) - 1) + w_{\eta\eta}(-\cos(x) + 1).
\end{aligned}$$

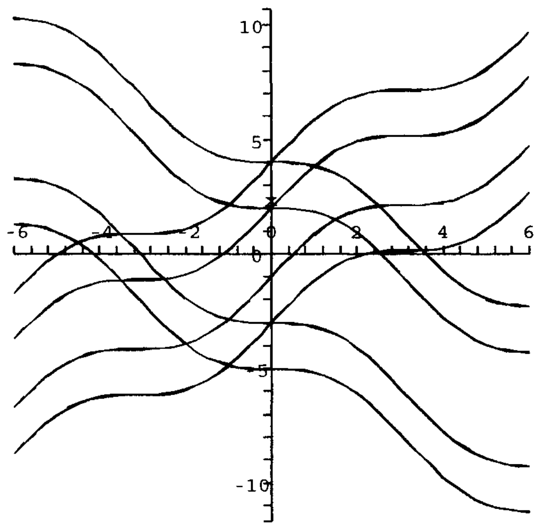

Figure 2.1: Some characteristics in Example 2.1.

Substitute these into equation 2.9 to obtain

$$w_{\xi\eta} = 0.$$

This is the canonical form of equation 2.9. $\diamond$

This example demonstrates one value of the canonical form. Here the canonical form is so simple that we can write solutions immediately. Any function of the form

$$w(\xi, \eta) = f(\xi) + g(\eta)$$

will certainly satisfy $w_{\xi\eta} = 0$ if f and g are twice-differentiable functions of a single variable. Thus, for any such f and g,

$$u(x, y) = f(y - \sin(x) - x) + g(y - \sin(x) + x)$$

is a solution of equation 2.9. For example, if we choose $f(t) = t^2$ and $g(t) = e^t$, we obtain the solution

$$u(x, y) = (y - \sin(x) - x)^2 + e^{y-\sin(x)+x}.$$

In general, however, the canonical form of a hyperbolic equation may contain first derivative and other terms and need not be as simple as this one.

If $C \neq 0$ in some region $\mathcal{D}$, we can again begin with the equation

$$A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2 = 0,$$

but this time divide by $C\xi_x^2$ to obtain

$$\left(\frac{\xi_y}{\xi_x}\right)^2 + 2\frac{B}{C}\left(\frac{\xi_y}{\xi_x}\right) + \frac{A}{C} = 0.$$

Now we solve for ξ_y/ξ_x instead of ξ_x/ξ_y and the discussion proceeds parallel to the preceding one, again resulting in a transformation to the canonical form 2.8.

Problems for Section 2.2

1. Carry out the derivation of the canonical form for the case that $C \neq 0$ throughout $\mathcal{D}$.

2. In each of the following, find the canonical form of the partial differential equation and use this to find a solution in terms of two arbitrary, twice differentiable functions.

   (a) $u_{xx} + 2u_{xy} - 3u_{yy} = 0$
   (b) $u_{xx} + 4u_{xy} + u_{yy} = 0$
   (c) $2u_{xx} + 2u_{xy} - 5u_{yy} = 0$
   (d) $3u_{xx} + 6u_{xy} + 2u_{yy} = 0$

3. Consider the transformation to canonical form in terms of the variables ξ, η in the hyperbolic case. Let $\widehat{\xi} = \xi + \eta$, $\widehat{\eta} = \xi - \eta$, and $\widehat{w}(\widehat{\xi}, \widehat{\eta}) = w(\xi, \eta)$, to obtain

   $$\widehat{w}_{\widehat{\xi}\widehat{\xi}} - \widehat{w}_{\widehat{\eta}\widehat{\eta}} + \Psi(\widehat{\xi}, \widehat{\eta}, \widehat{w}, \widehat{w}_{\widehat{\xi}}, \widehat{w}_{\widehat{\eta}}) = 0.$$

   This is another canonical form commonly used for the hyperbolic second order linear partial differential equation. Obtain this canonical form for the following hyperbolic equations:

   (a) $u_{xx} - 8u_{xy} + 7u_{yy} + u_x = 0$
   (b) $u_{xx} - 10u_{xy} + 16u_{yy} - xu_x = 0$
   (c) $2u_{xx} + 6u_{xy} - 8u_{yy} + yu = 0$
   (d) $u_{xx} + 4u_{xy} - 5u_{yy} + y^2u_y = 0$

4. Show that the hyperbolic canonical form

   $$U_{\xi\xi} - U_{\eta\eta} + \Phi(\xi, \eta, U, U_\xi, U_\eta) = 0$$

   can be obtained from the alternative hyperbolic canonical form

   $$u_{xy} + \Psi(x, y, u, u_x, u_y) = 0$$

   by a rotation of coordinate axes through $\pi/4$ radians.

5. Determine the characteristics of the canonical form $w_{\xi\eta} + \Phi(\xi, \eta, w, w_\xi, w_\eta) = 0$ of the hyperbolic equation.

6. Let u satisfy the hyperbolic equation $u_{xx} - u_{yy} = 0$. Show that u_x, u_y, u_{xx}, u_{yy}, and u_{xy} are also solutions of this equation. Verify this conclusion by direct calculation for the solution
$$u(x, y) = \sin(nx)\cos(ny)$$
of the equation
$$u_{xx} - u_{yy} = 0.$$
Here c is a positive number and n is a positive integer.

2.3 The Parabolic Canonical Form

Suppose that $B^2 - AC = 0$ in $\mathcal{D}$. Now A and C cannot both vanish at any point of $\mathcal{D}$ because then we would also have $B = 0$, and equation 2.1 would be first order. Suppose to be specific that $A(x, y) \neq 0$ throughout $\mathcal{D}$. We can begin as we did in the hyperbolic case, except now equations 2.6 and 2.7 are the same: namely,
$$\frac{dy}{dx} = \frac{B}{A}.$$
This is the characteristic equation for equation 2.1 in the parabolic case. Now the characteristics are graphs of $\xi(x, y) = k$, where this equation implicitly defines an integral of the characteristic equation. We can choose η as *any* continuous function with continuous first and second partial derivatives and such that the Jacobian $J \neq 0$ throughout $\mathcal{D}$. In this way we select a transformation
$$\xi = \xi(x, y), \eta = \eta(x, y).$$
In the transformed equation 2.2 we will have $a = 0$ by the way ξ was chosen. As a bonus, we claim that $b = 0$ automatically. The reason is that
$$b = (A\xi_x + B\xi_y)\eta_x + (B\xi_x + C\xi_y)\eta_y.$$
But $A\xi_x + B\xi_y = 0$ because $\xi_x/\xi_y = -B/A$. Further, $B^2 - AC = 0$ implies that $B/A = C/B$, so $\xi_x/\xi_y = -C/B$ and hence $B\xi_x + C\xi_y = 0$.

With $a = b = 0$, the transformed equation 2.2 is
$$cw_{\eta\eta} + dw_\xi + ew_\eta + fw + g = 0.$$
Upon dividing by c, we obtain the *canonical form of the parabolic equation*:
$$w_{\eta\eta} + d^* w_\xi + e^* w_\eta + f^* w + g^* = 0.$$
We can write this canonical form as
$$w_{\eta\eta} + \Phi(\xi, \eta, w, w_\xi, w_\eta) = 0.$$

Example 2.3 *The equation*

$$9u_{xx} + 12u_{xy} + 4u_{yy} + u_x = 0$$

is parabolic because $B^2 - AC = 6^2 - 4 \cdot 9 = 0$. *The characteristic equation is*

$$\frac{dy}{dx} = \frac{6}{9}$$

with general solution defined by $y - \frac{2}{3}x = k$. *The characteristics are straight lines corresponding to different choices of* k. *Choose*

$$\xi = y - \frac{2}{3}x.$$

We can choose η *as any continuous function with continuous first and second partial derivatives such that the Jacobian of* ξ *and* η *is not zero in the region under consideration. The choice* $\eta = x$ *satisfies these conditions, since*

$$J = \begin{vmatrix} -\frac{2}{3} & 1 \\ 1 & 0 \end{vmatrix} = -1$$

and this is nonzero in the entire plane. One transformation to canonical form is therefore

$$\xi = y - \frac{2}{3}x \text{ and } \eta = x.$$

Let $w(\xi, \eta) = u(x, y)$ *and compute*

$$\begin{aligned} u_x &= -\frac{2}{3}w_\xi + w_\eta, \\ u_y &= w_\xi, \\ u_{xx} &= -\frac{2}{3}\left(-\frac{2}{3}w_{\xi\xi} + w_{\xi\eta}\right) - \frac{2}{3}w_{\eta\xi} + w_{\eta\eta}, \\ u_{yy} &= w_{\xi\xi}, \end{aligned}$$

and

$$u_{xy} = -\frac{2}{3}w_{\xi\xi} + w_{\xi\eta}.$$

Substitute these into the partial differential equation to obtain

$$9w_{\eta\eta} - \frac{2}{3}w_\xi + w_\eta = 0.$$

The canonical form is

$$w_{\eta\eta} - \frac{2}{27}w_\xi + \frac{1}{9}w_\eta = 0. \diamond$$

Example 2.4 *We will find solutions of*

$$u_{xx} + 4u_{xy} + 4u_{yy} = 0. \tag{2.10}$$

Here $A = 1, B = 2$, and $C = 4$, so $B^2 - AC = 0$ and equation 2.10 is parabolic. Solve the characteristic equation

$$\frac{dy}{dx} = \frac{B}{A} = 2$$

to get

$$y - 2x = k.$$

Let $\xi = y - 2x$ and choose η so that $J \neq 0$, say $\eta = x$. We find that

$$u_{xx} = 4w_{\xi\xi} - 4w_{\xi\eta} + w_{\eta\eta},$$
$$u_{yy} = w_{\xi\xi},$$

and

$$u_{xy} = -2w_{\xi\xi} + w_{\xi\eta}.$$

Upon substituting these into equation 2.10, we obtain the canonical form

$$w_{\eta\eta} = 0,$$

which we can write as

$$(w_\eta)_\eta = 0.$$

This implies that w_η is independent of η, say

$$w_\eta = F(\xi).$$

Integrate this equation with respect to η to obtain

$$w(\xi, \eta) = \int F(\xi)\, d\eta = \eta F(\xi) + G(\xi).$$

Therefore,

$$u(x, y) = xF(y - 2x) + G(y - 2x)$$

is a solution for any twice-differentiable functions F and G of a single variable. It is routine to verify by substitution that u(x, y) is indeed a solution of equation 2.10. ◇

Problems for Section 2.3

1. Derive a canonical form of the parabolic equation in the case that $C(x, y) \neq 0$ throughout $\mathcal{D}$.

2. In Example 2.3, the choice $\eta = x$ was arbitrary, requiring only that $J \neq 0$ in $\mathcal{D}$. Determine the canonical form that would result from choosing $\eta = y$. What would result if we chose $\eta = x^2$ and agreed that $\mathcal{D}$ contains no points on the y - axis?

3. For each of the following, find a canonical form of the partial differential equation and use this to find a solution involving two arbitrary functions.

(a) $u_{xx} + 6u_{xy} + 9u_{yy} = 0$

(b) $u_{xx} - 4u_{xy} + 4u_{yy} = 0$

(c) $25u_{xx} + 20u_{xy} + 4u_{yy} = 0$

(d) $9u_{xx} + 12u_{xy} + 4u_{yy} = 0$

4. Determine the characteristics of the canonical form of the parabolic equation.

5. Let u satisfy the parabolic equation $u_y - ku_{xx} = 0$, in which k is a positive number. Show that u_x, u_y, u_{xx}, u_{yy}, and u_{xy} are also solutions. Verify this conclusion by direct calculation for the solution

$$u(x, y) = e^{-\alpha^2\pi^2 y} \cos(\alpha\pi x),$$

in which α can be any nonzero real number.

2.4 The Elliptic Canonical Form

Suppose that $B^2 - AC < 0$ in $\mathcal{D}$. Now $B^2 - AC$ has no real square root, so equations 2.6 and 2.7 used to generate characteristics in the hyperbolic and parabolic cases have no real solutions. The elliptic equation has no characteristics. Nevertheless, we can seek a transformation $\xi = \xi(x, y), \eta = \eta(x, y)$ which simplifies equation 2.2.

We begin by attempting to make the coefficient of $w_{\xi\eta}$ zero in the transformed equation. This will require that

$$\xi_x(A\eta_x + B\eta_y) + \xi_y(B\eta_x + C\eta_y) = 0.$$

Choose any "reasonable" function $\eta(x, y)$. Then along any curve defined by $\xi(x, y) = k$, the last equation tells us that

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{B\eta_x + C\eta_y}{A\eta_x + B\eta_y}.$$

We attempt to solve this equation to determine curves $\xi(x, y) = k$. Assuming that we can do this, we can transform the elliptic partial differential equation to the form

$$aw_{\xi\xi} + cw_{\eta\eta} + \Phi(\xi, \eta, w, w_\xi, w_\eta) = 0. \tag{2.11}$$

Thus far we will have succeeded only in eliminating the mixed derivative term. We do not choose this as the canonical form for the parabolic case because equation 2.11 can be further simplified by a second change of variables. To avoid having to assign new symbols to another transformation, we rewrite equation 2.11 in terms of x and y and imagine that we are starting over with this equation. Thus, we want to simplify

$$Au_{xx} + Cu_{yy} + \Phi(x, y, u, u_x, u_y) = 0. \tag{2.12}$$

We have also written the coefficient functions in uppercase letters to use previous results in which this notation was used.

We will determine a change of variables that transforms equation 2.12 to

$$w_{\xi\xi} + w_{\eta\eta} + \Psi(\xi, \eta, w, w_\xi, w_\eta) = 0.$$

To do this, we must obtain $a = c$ and $b = 0$ in the transformed equation 2.2. Keeping in mind that we already have $B = 0$ in the new starting equation 2.12, we need

$$A\xi_x^2 + C\xi_y^2 = A\eta_x^2 + C\eta_y^2$$

and

$$A\xi_x\eta_x + C\xi_y\eta_y = 0.$$

One way to make headway on the first equation is to try to find ξ and η such that

$$\sqrt{A}\xi_x = \sqrt{C}\eta_y \text{ and } \sqrt{C}\xi_y = -\sqrt{A}\eta_x. \tag{2.13}$$

If we are able to find η, we can put

$$\xi(x,y) = k + \int_{(x_0,y_0)}^{(x,y)} \sqrt{\frac{C}{A}}\eta_y \, dx - \sqrt{\frac{A}{C}}\eta_x \, dy \tag{2.14}$$

in which k is a constant and (x_0, y_0) some conveniently chosen point. Such a function will satisfy equations 2.13 exactly when the line integral defining $\xi(x, y)$ is independent of path, and the condition for this is

$$\left(-\sqrt{\frac{A}{C}}\eta_x\right)_x = \left(\sqrt{\frac{C}{A}}\eta_y\right)_y. \tag{2.15}$$

The strategy is therefore first to attempt to solve equation 2.15 for $\eta(x, y)$ (keeping in mind that A and C are in general functions of x and y), and then use this η and equation 2.14 to obtain ξ.

Assuming success in solving these equations to find a suitable transformation, equation 2.12 transforms to an equation of the form

$$aw_{\xi\xi} + aw_{\eta\eta} + H(\xi, \eta, w, w_\xi, w_\eta) = 0.$$

Finally, upon dividing by a (in a region in which this function is nonzero), we obtain the canonical form

$$w_{\xi\xi} + w_{\eta\eta} + \Psi(\xi, \eta, w, w_\xi, w_\eta) = 0.$$

This is the *canonical form of the elliptic equation.*
Example 2.5 *The equation*

$$u_{xx} + 2u_{xy} + 3u_{yy} + 4u = 0 \tag{2.16}$$

is elliptic, since $B^2 - AC = 1 - (1)(3) = -2 < 0$. *There are no characteristics. We have to start somewhere, so try* $\eta = x + y$ *and consider*

$$\frac{dy}{dx} = \frac{B\eta_x + C\eta_y}{A\eta_x + B\eta_y} = 2.$$

This has general solution defined by $y - 2x = k$, *so let*

$$\xi = y - 2x \text{ and } \eta = x + y.$$

The Jacobian is $J = -3$. *Compute*

$$\begin{aligned} u_x &= -2w_\xi + w_\eta, \\ u_{xx} &= 4w_{\xi\xi} - 4w_{\xi\eta} + w_{\eta\eta}, \\ u_y &= w_\xi + w_\eta, \\ u_{yy} &= w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta}, \end{aligned}$$

and

$$u_{xy} = -2w_{\xi\xi} - w_{\xi\eta} + w_{\eta\eta}.$$

Upon substituting these into the partial differential equation, we obtain

$$3w_{\xi\xi} + 6w_{\eta\eta} + 4w = 0,$$

which we write as

$$w_{\xi\xi} + 2w_{\eta\eta} + \frac{4}{3}w = 0.$$

Thus far we have obtained an intermediate form in which there is no mixed derivative term. To simplify further and obtain the canonical form, write this equation in terms of u, x, *and* y *again to avoid having to use new letters for the second transformation:*

$$u_{xx} + 2u_{yy} + \frac{4}{3}u = 0, \tag{2.17}$$

in which $A = 1$ *and* $C = 2$, *so* $A/C = 1/2$ *and* $C/A = 2$. *Of course, equation 2.17 is again elliptic. Equation 2.15 is*

$$-\frac{1}{\sqrt{2}}\eta_{xx} = \sqrt{2}\eta_{yy}.$$

There are many solutions of this equation. We will choose a simple one, say

$$\eta = x + y.$$

Keep in mind here that we are reusing symbols—this is a new η, *and is only coincidentally the same as that used in the first reduction. Now use equation 2.14 to write*

$$\begin{aligned} \xi(x,y) &= k + \int_{(x_0,y_0)}^{(x,y)} \sqrt{2}(1)\,dx - \frac{1}{\sqrt{2}}(1)\,dy \\ &= k + \sqrt{2}\int_{(x_0,y_0)}^{(x,y)} dx - \frac{1}{2}\,dy. \end{aligned}$$

A potential function for this line integral, which is independent of path, is $\varphi(x,y) = x - \frac{1}{2}y$ *and*

$$\xi(x,y) = k + \sqrt{2}[\varphi(x,y) - \varphi(x_0,y_0)] = \sqrt{2}\left(x - \frac{1}{2}y\right),$$

in which we choose the arbitrary constant k *and the arbitrary point* (x_0, y_0) *so that* $k - \varphi(x_0,y_0) = 0$. *For example, we can choose* $k = 0$ *and* (x_0, y_0) *to be the origin. Finally, we have the transformation*

$$\xi = \sqrt{2}x - \frac{1}{\sqrt{2}}y$$
$$\eta = x + y.$$

It is routine to compute the partial derivatives and substitute into the partial differential equation 2.17 to obtain

$$3w_{\xi\xi} + 3w_{\eta\eta} + \frac{4}{3}w = 0.$$

The canonical form of the partial differential equation 2.16 is

$$w_{\xi\xi} + w_{\eta\eta} + \frac{4}{9}w = 0. \diamond$$

Problems for Section 2.4

In each of Problems 1 through 5, determine the canonical form of the elliptic equation.

1. $u_{xx} + 2u_{xy} + 5u_{yy} + u_y = 0$
2. $3u_{xx} - 2u_{xy} + 2u_{yy} = 0$
3. $u_{xx} + 3u_{yy} - u = 0$
4. $4u_{xx} + u_{xy} + u_{yy} + u_x - xu_y + yu = 0$
5. $2u_{xx} - 2u_{xy} + u_{yy} + \cos(y)u_x = 0$
6. Let u satisfy the elliptic equation $u_{xx} + u_{yy} = 0$. Show that u_x, u_y, u_{xx}, u_{yy} and u_{xy} are also solutions. Verify this conclusion by direct calculation for the solution
$$u(x,y) = \sin(x)\cosh(y).$$

Problems for Sections 2.1 through 2.4

The following problems incorporate ideas from all of the sections 2.1 through 2.4.

In each of Problems 1 through 10, (a) classify the partial differential equation, (b) sketch some of the characteristics (if the equation is not elliptic), and (c) determine the canonical form of the partial differential equation.

1. $u_{xx} - 8u_{xy} + 2u_{yy} + xu_x - yu_y = 0$
2. $4u_{xx} + 2u_{xy} + u_{yy} - u_x + xyu_y + u = 0$
3. $3u_{xx} + 2u_{xy} - u_{yy} + yu_x - u_y = 0$
4. $2u_{xx} - 4u_{xy} + 2u_{yy} - y^2u_x + u_u - xu = 0$
5. $3u_{xx} - 8u_{xy} + 2u_{yy} + (x + y)u_y = 0$
6. $3u_{xx} + 6u_{xy} + 4u_{yy} - u_x - xu_y + xyu = 0$
7. $u_{xx} - 4u_{xy} + 4u_{yy} + u_x + u_y = 0$
8. $6u_{xy} + 5u_{yy} - x^2u_x + yu_y - xu = 0$
9. $2u_{xx} - 10u_{xy} + 8u_{yy} + u_x - u_y = 0$
10. $2u_{xx} - 2u_{xy} - 3u_{yy} + y^2u_x - u = 0$
11. Let a, b, and c be constants and suppose that

$$u_{xx} - u_{yy} + au_x + bu_y + cu = 0.$$

Let $v(x, y) = e^{\alpha x + \beta y}u(x, y)$. Determine constants α, β, and h so that

$$v_{xx} - v_{yy} = -hv.$$

12. Let a, b, and c be constants and suppose that

$$u_y - au_{xx} - bu_x - cu = 0.$$

Let $v(x, y) = e^{\alpha x + \beta y}u(x, y)$. Determine constants α and β so that

$$v_y - hv_{xx} = 0.$$

13. Tricomi's equation is

$$u_{xx} + xu_{yy} = 0.$$

(a) Determine the region of the plane in which Tricomi's equation is, in turn, hyperbolic, parabolic, and then elliptic.

(b) Determine the characteristics of the Tricomi equation in the part of the plane where the equation is hyperbolic.

(c) Can you determine the characteristics of the Tricomi equation in the part of the plane where the equation is parabolic?

14. Consider equation 2.1 in the case that each coefficient is constant. For each of the cases (hyperbolic, parabolic, elliptic), go through the discussion in the text and determine a transformation of equation 2.1 to canonical form. You should find that the transformation to the canonical form in each case is simplified by the assumption of constant coefficients.

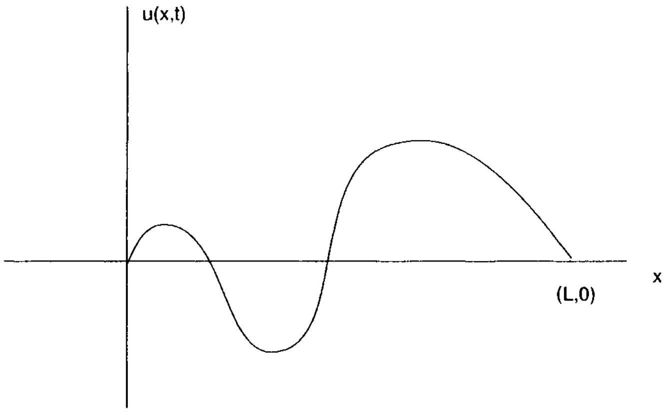

Figure 2.2: Typical shape of the string at time t.

2.5 Some Equations of Mathematical Physics

We derive partial differential equations modeling wave motion, radiation and diffusion, and phenomena derivable from potentials. These will typify the three classifications of linear second-order partial differential equations. Further, having these as "prototypes" affords us insight into properties we might expect solutions to have for each classification.

The Wave Equation

Imagine an elastic string stretched between two pegs, as on a guitar or harp. We want to describe the motion if the string is displaced and released to vibrate in a plane.

Place the string along the x - axis from 0 to L and assume that the plane of motion is the x, y - plane. We seek a function $u(x,\ t)$ such that at any time $t \geq 0$, the graph of the function $y = u(x, t)$ is the shape of the string at that time (Figure 2.2).

To begin with a simple case, neglect damping forces such as air resistance and the weight of the string and assume that the tension $T(x,\ t)$ in the string acts tangentially to the string. Also assume that the mass ρ per unit length is constant. Apply Newton's second law of motion to the segment of string between x and $x + \Delta x$:

net force on this segment due to the tension
= (acceleration of the center of mass of the segment)(mass of the segment).

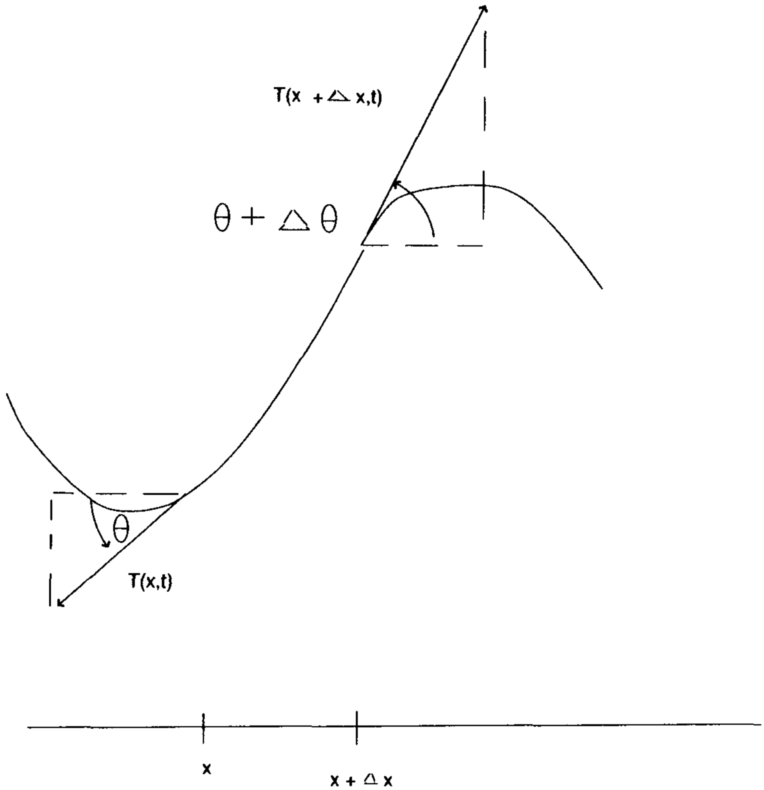

Figure 2.3: Vertical components of the tension at x and $x + \Delta x$.

This is a vector equation. For small Δx, consideration of the vertical component of this equation (note Figure 2.3) gives approximately

$$T(x + \Delta x, t) \sin(\theta + \Delta\theta) - T(x, t) \sin(\theta) = \rho(\Delta x) u_{tt}(\bar{x}, t),$$

in which $\bar{x}$ is the center of mass of the segment. Then

$$\frac{T(x + \Delta x, t) \sin(\theta + \Delta\theta) - T(x, t) \sin(\theta)}{\Delta x} = \rho u_{tt}(\bar{x}, t).$$

$T(x, t) \sin(\theta)$ is the vertical component of the tension. Write this as $v(x,\ t)$. The last equation becomes

$$\frac{v(x + \Delta x, t) - v(x, t)}{\Delta x} = \rho u_{tt}(\bar{x}, t).$$

In the limit as $\Delta x \to 0, \bar{x} \to x$ and this equation yields

$$v_x = \rho u_{tt}.$$

The horizontal component of the tension is $h(x,t) = T(x,t)\cos(\theta)$. Now

$$v(x,t) = h(x,t)\tan(\theta) = h(x,t)u_x$$

so

$$(hu_x)_x = \rho u_{tt}.$$

Since the horizontal component of the tension on the segment is zero,

$$h(x + \Delta x, t) - h(x,t) = 0.$$

Therefore, h is independent of x and $(hu_x)_x = hu_{xx}$, leading to

$$hu_{xx} = \rho u_{tt}$$

or

$$u_{tt} = c^2 u_{xx}, \tag{2.18}$$

where $c^2 = h/\rho$. Equation 2.18 is the *one-dimensional wave equation* (one space dimension), and it is hyperbolic. We associate hyperbolic partial differential equations with wave motion, providing some intuition about how solutions might behave under various conditions.

We would expect the motion of the string to be influenced by whatever is done to the ends of the string. This leads us to seek solutions satisfying conditions of the form

$$u(0,t) = \alpha(t), u(L,t) = \beta(t) \text{ for } t > 0,$$

where $\alpha(t)$ and $\beta(t)$ describe the forces acting on the string's ends. These are called *boundary conditions*. We also specify *initial conditions*

$$u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \text{ for } 0 < x < L.$$

These give, respectively, the initial position from which the string is released and the velocity with which it is released. Under reasonable assumptions on the functions α, β, φ, and ψ, the wave equation, together with these initial and boundary conditions, has a unique solution. This is consistent with the intuition that in the absence of external driving forces acting on the string, the motion is determined completely by the string's initial configuration and velocity and any subsequent action at the ends of the string. The partial differential equation, together with the boundary and initial conditions, is called an *initial-boundary value problem*.

Despite the severe assumptions made in the derivation, the wave equation, together with boundary and initial conditions, has been found to model accurately certain kinds of wave motion, thus forging a link between partial differential equations and a phenomenon of importance in physics, engineering, music, and other areas.

For an ideal string which we imagine covers the entire real line, we have initial conditions but no boundary conditions, since now $-\infty < x < \infty$ (the string has no ends). If the string is thought of as extending over the half-line ($x \geq 0$), we seek solutions satisfying an initial condition and a boundary condition at the left end, $x = 0$. Although we cannot actually produce an infinitely long string, such models are of practical importance in many contexts. For example, an astronomer sending a signal from a radio telescope on Earth may for all practical purposes assume that the signal extends indefinitely from its source.

Derivation of the wave equation can be adapted to include external forces acting on the string. If there is at each point an external force of magnitude $F(x)$ per unit length, acting parallel to the y - axis, then the profile $u(x, t)$ of the string at time t satisfies

$$u_{tt} = c^2 u_{xx} + \frac{1}{\rho} F.$$

In two space dimensions the wave equation is

$$u_{tt} = c^2(u_{xx} + u_{yy}). \tag{2.19}$$

This equation models oscillations of an elastic membrane whose shape at time t is the graph of the surface $z = u(x, y, t)$ in 3 - space. We would seek solutions specifying a given initial position and boundary conditions describing the status of the membrane along its boundary (for example, a drum is usually fastened along a frame, and these boundary points remain motionless).

The Heat Equation

We would like to derive an equation describing the flow of heat energy. Consider a bar of constant density ρ and length L. Assume that the material of the bar conducts heat and that the bar has uniform cross-sectional area A. The lateral surface of the bar is insulated, so there is no heat loss across this surface.

Draw the x - axis along the length of the bar and assume that at a given time, the temperature is the same throughout a given cross - section, although it will in general vary from one cross section to another. We derive an equation for $u(x, t)$, the temperature in the cross section of the bar at x, at time t.

Let c be the specific heat of the material of the bar. This is the amount of heat energy that must be supplied to a unit mass of the material to raise its temperature 1 degree. The segment of bar between x and $x + \Delta x$ has mass $\rho A \, \Delta x$, and it will take approximately $\rho c A u(x, t) \, \Delta x$ units of heat energy to change the temperature of this segment from zero to $u(x, t)$, its temperature at time t. The total heat energy in this segment at time t is

$$E(x, \Delta x, t) = \int_x^{x+\Delta x} \rho c A u(\xi, t) \, d\xi$$

for $t > 0$.

The amount of heat energy within this segment at time t can increase in two ways: heat energy may flow into the segment across its ends (this increase is called the flux of the heat energy), or there may be a source of heat energy within the segment (for example, a chemical reaction or a radioactive material). The rate of change of the temperature within the segment, with respect to time, is therefore

$$\frac{\partial E}{\partial t} = \text{flux} + \text{source} = \int_x^{x+\Delta x} \rho c A \frac{\partial u}{\partial t}(\xi, t)\, d\xi.$$

Assuming for now that there is no energy source within the bar, then

$$\text{flux} = \int_x^{x+\Delta x} \rho c A \frac{\partial u}{\partial t}(\xi, t)\, d\xi. \tag{2.20}$$

Now let $F(x,\ t)$ be the amount of heat energy per unit area flowing across the cross section at x at time t in the direction of increasing x. The rate of flow of heat energy across the section at x into the part of the bar to the right of x is $AF(x,\ t)$, while the rate of flow of heat energy across the section at $x+\Delta x$ into the part of the bar to the right of $x+\Delta x$ is $AF(x+\Delta x, t)$. The rate of flow of heat energy into the segment of bar between x and $x+\Delta x$ at time t is the rate of flow into this segment to the right across the section at x, minus the rate of flow out of this segment to the right across the section at $x+\Delta x$ (Figure 2.4), giving us

$$\text{flux} = AF(x,t) - AF(x+\Delta x, t)$$

or

$$\text{flux} = -A(F(x+\Delta x, t) - F(x,t)). \tag{2.21}$$

Next recall Newton's law of cooling, which states that heat energy flows from the warmer to the cooler segment and that the amount of heat energy is proportional to the temperature gradient. That is,

$$F(x,t) = -K\frac{\partial u}{\partial x}(x,t),$$

with the positive constant of proportionality K called the heat conductivity of the bar. The negative sign in the equation indicates that heat energy flows from the warmer to the cooler segment. Substituting this into equation 2.21 gives us

$$\text{flux} = -A\left(-K\frac{\partial u}{\partial x}(x+\Delta x, t) + K\frac{\partial u}{\partial x}(x,t)\right),$$

which we write as

$$\text{flux} = \int_x^{x+\Delta x} \frac{\partial}{\partial x}\left(KA\frac{\partial u}{\partial x}(\xi, t)\right) d\xi. \tag{2.22}$$

Equating the two expressions for the flux from equations 2.20 and 2.22, we obtain

$$\int_x^{x+\Delta x} \rho c A \frac{\partial u}{\partial t}(\xi, t)\, d\xi = \int_x^{x+\Delta x} \frac{\partial}{\partial x}\left(KA\frac{\partial u}{\partial x}(\xi, t)\right) d\xi$$

or

$$\int_x^{x+\Delta x} \left(\rho c \frac{\partial u}{\partial t}(\xi, t)\, d\xi - \frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}(\xi, t)\right)\right) d\xi = 0.$$

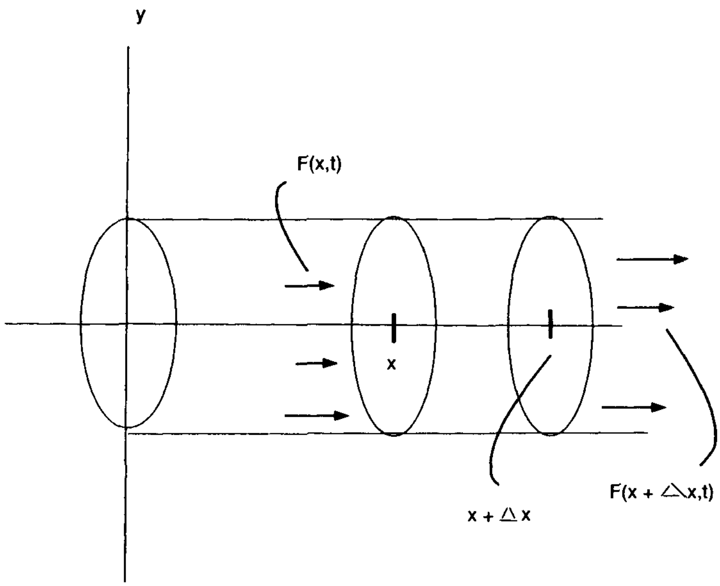

Figure 2.4: Heat flow across sections at x and $x + \Delta x$.

This equation must hold for $0 < x < x + \Delta x < L$. If the integrand were nonzero at any x_0, then, assuming continuity of this integrand, we could choose Δx so small that the integrand is nonzero throughout $[x_0, x_0 + \Delta x]$. In this case the integral of this nonzero continuous function over this interval would be nonzero, a contradiction. We conclude that

$$\rho c \frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$$

for $0 < x < L$. This equation is usually written

$$u_t = k u_{xx}, \tag{2.23}$$

where $k = K/c\rho$ is the *diffusivity* of the material of the bar. Equation 2.23 is the *one-dimensional heat equation*, and it is parabolic. We associate heat flow and diffusion phenomena with parabolic partial differential equations.

As with the wave equation, we usually solve the heat equation subject to additional information needed to determine a unique solution having certain properties. *Boundary conditions*

$$u(0,t) = \alpha(t), u(L,t) = \beta(t) \text{ for } t > 0$$

specify the amount of heat energy introduced at the ends of the bar at time t, and the *initial condition*

$$u(x,0) = f(x) \text{ for } 0 < x < L$$

specifies the temperature of the bar at time zero. We would expect to need this kind of information to determine a unique solution for the temperature function.

As with the wave equation, the heat equation, together with boundary and initial conditions, is called an initial-boundary value problem.

If there is a source of heat energy within the bar, the heat equation takes the form

$$u_t = ku_{xx} + Q(x,t).$$

In two space dimensions, the heat equation (with no sources) is

$$u_t = k(u_{xx} + u_{yy}),$$

and in three space dimensions it is

$$u_t = k(u_{xx} + u_{yy} + u_{zz}).$$

This is often written

$$u_t = k\,\nabla^2 u,$$

where

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$$

is the Laplacian of u (in two dimensions, omit the u_{zz} term).

Laplace's Equation

The *steady-state* case of the heat equation occurs when $u_t = 0$. In this case the heat equation becomes

$$\nabla^2 u = 0.$$

This is *Laplace's equation.* In two dimensions this is the elliptic partial differential equation

$$u_{xx} + u_{yy} = 0,$$

and in three dimensions it is

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

Any function satisfying Laplace's equation in a certain region (of the plane or three space) is said to be *harmonic* in that region.

In the context of heat conduction, Laplace's equation models the steady-state temperature distribution in a medium. This is the limit of the temperature function as $t \to \infty$ and is therefore independent of t. Laplace's equation is important in other areas as well. If $Q(x,\ y,\ z)$ is the electrostatic potential in a region of space that has no electric charges, the negative of the gradient of Q gives the electrostatic intensity at each point (the electrical force exerted on a unit charge at that point). The potential Q satisfies Laplace's equation. In addition, the real and imaginary parts of an analytic complex function are harmonic, so Laplace's equation forms a link between real and complex analysis.

Problems for Section 2.5

1. Show that the function

$$u(x,t) = \sin(n\pi x/L)\cos(n\pi ct/L),$$

with c constant and n any integer, satisfies the one-dimensional wave equation 2.18. L is a positive constant.

2. Show that for any integers m and n, the function defined by

$$u(x,y,t) = k\sin(nx)\cos(my)\cos(\sqrt{n^2+m^2}ct)),$$

with k any constant, satisfies the two-dimensional wave equation 2.19.

3. Obtain the one-dimensional wave equation under the added assumption that the motion of the string is opposed by air resistance, which has a force at each point of magnitude proportional to the square of the velocity at that point.

4. Obtain the one-dimensional wave equation if the motion of the string is being driven by an external force whose magnitude at x and time t is proportional to the product of the magnitude of the velocity at that point and time, and the vertical distance from the x - axis to the string at that time.

5. Show that for any real number α, the function

$$u(x,t) = e^{-\alpha^2\pi^2 t}\cos(\alpha\pi x)$$

satisfies $u_t = u_{xx}$.

6. Show that

$$u(x,t) = t^{-3/2}e^{-x^2/4kt}$$

satisfies $u_t = ku_{xx}$ for $x > 0, t > 0$.

Show that the following functions are harmonic wherever they are defined.

7. xy

8. $x^2 - y^2$

9. $x^3 - 3xy^2$

10. $y^3 - 3x^2y$

11. $x^5 + 5xy^4 - 10x^3y^2$

12. $y^5 + 5x^4y - 10x^2y^3$

13. Show that in polar coordinates, Laplace's equation in two dimensions is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

14. Using the result of Problem 13, show that $r^k \cos(k\theta)$ and $r^k \sin(k\theta)$ are harmonic for k any nonzero integer, with $r > 0$ if k is negative.

2.6 The Second-Order Cauchy Problem

For a first-order ordinary differential equation, one condition for the uniqueness of a solution is the specification of the function value at a given point. The analogue of this for quasi-linear (hence also for linear) first-order partial differential equations is to prescribe function values along a noncharacteristic curve.

For the second-order ordinary differential equation, two pieces of information are needed to determine a unique solution. These could be the value of the solution at a point and the value of its derivative at that point. This leads us to conjecture that conditions for uniqueness of the solution of the second-order linear partial differential equation 2.1 might consist of the specification of values the solution is to have along a given curve, together with some appropriately formulated second condition. In this and the next section we explore this conjecture.

Suppose that $u(x, y) = \varphi(x, y)$ is a solution of equation 2.1, so

$$A\varphi_{xx} + 2B\varphi_{xy} + C\varphi_{yy} + D\varphi_x + E\varphi_y + F\varphi + G = 0. \tag{2.24}$$

Assume that the coefficients $A(x, y), B(x, y), \cdots$ are defined over some region $\mathcal{D}$ of the x, y - plane and that $A(x, y)$, $B(x, y)$, and $C(x, y)$ are not simultaneously zero at any point. In the absence of constraining conditions, this partial differential equation may have many solutions.

Now explore what happens if solution values are specified along some curve Γ. To simplify calculations and illustrate a point, we choose Γ to be the y - axis and specify that φ is to assume values on Γ given by a known function f:

$$\varphi(0, y) = f(y).$$

Assuming that f can be differentiated repeatedly, we can compute partial derivatives of φ at points $(0,\ y)$:

$$\varphi_y(0, y) = f'(y), \varphi_{yy}(0, y) = f''(y), \varphi_{yyy}(0, y) = f'''(y), \cdots .$$

There is, however, insufficient information as yet to determine partial derivatives of φ with respect to x at points $(0,\ y)$. Let us therefore provide such information by specifying that

$$\varphi_x(0, y) = g(y),$$

with g a given function. Assuming that we can repeatedly differentiate g, then

$$\varphi_{xy}(0, y) = g'(y), \varphi_{xyy}(0, y) = g''(y), \cdots .$$

Now we can obtain $\varphi_{xx}(0,y)$ by solving for it in equation 2.24, assuming that $A(0,y) \neq 0$. Further, once we know $\varphi_{xx}(0,y)$, we can obtain y–derivatives of φ_{xx} at $(0, y)$:

$$\varphi_{xxy}(0,y) = \frac{\partial}{\partial y}\varphi_{xx}(0,y), \varphi_{xxyy}(0,y) = \frac{\partial^2}{\partial y^2}\varphi_{xx}(0,y), \cdots .$$

There are still some partial derivatives that we have not computed. But we can obtain these as well at points $(0, y)$. If we differentiate equation 2.24 with respect to x, keeping in mind that the coefficients are functions of x and y, we obtain an equation that we can solve for $\varphi_{xxx}(0,y)$ in terms of quantities calculated previously. We can then differentiate again to get $\varphi_{xxxx}(0,y)$, and so on, to higher-order x - derivatives.

Now here is the reason for obtaining these partial derivatives. For a given y, we can expand $\varphi(x,y)$ as a function of x in a Taylor series about $x = 0$:

$$\varphi(x,y) = \varphi(0,y) + \varphi_x(0,y)x + \frac{1}{2!}\varphi_{xx}(0,y)x^2 + \frac{1}{3!}\varphi_{xxx}(0,y)x^3 + \cdots ,$$

and we can compute the coefficients in this expansion because the information $\varphi(0,y) = f(y)$ and $\varphi_x(0,y) = g(y)$ was chosen to enable us to do this.

We will pause briefly in our discussion to illustrate the ideas up to this point.

Example 2.6 *Suppose that the partial differential equation is*

$$u_{xx} + 4u_{xy} - u_{yy} + yu_x + u = 0,$$

and $u = \varphi(x,y)$ is a solution satisfying

$$\varphi(0,y) = e^{3y} \text{ and } \varphi_x(0,y) = y^4.$$

We will obtain the first few terms of the Taylor expansion of $\varphi(x,y)$ about $x = 0$. This series has the form

$$\varphi(x,y) = \varphi(0,y) + \varphi_x(0,y)x + \frac{1}{2}\varphi_{xx}(0,y)x^2 + \frac{1}{6}\varphi_{xxx}(0,y)x^3 + \frac{1}{24}\varphi_{xxxx}(0,y)x^4 + \cdots .$$

We know $\varphi(0,y)$ and $\varphi_x(0,y)$. To compute $\varphi_{xx}(0,y)$, use the partial differential equation to write

$$\varphi_{xx} = -4\varphi_{xy} + \varphi_{yy} - y\varphi_x - \varphi. \tag{2.25}$$

Then

$$\varphi_{xx}(0,y) = -4\varphi_{xy}(0,y) + \varphi_{yy}(0,y) - y\varphi_x(0,y) - \varphi(0,y).$$

But

$$\varphi_{xy}(0,y) = \frac{\partial}{\partial y}\varphi_x(0,y) = \frac{\partial}{\partial y}y^4 = 4y^3$$

and

$$\varphi_{yy}(0,y) = \frac{\partial^2}{\partial y^2}\varphi(0,y) = \frac{\partial^2}{\partial y^2}e^{3y} = 9e^{3y}.$$

Then

$$\begin{aligned}\varphi_{xx}(0,y) &= -4(4y^3) + 9e^{3y} - y^5 - e^{3y} \\ &= -16y^3 - y^5 + 8e^{3y}.\end{aligned}$$

Next, from equation 2.25,

$$\varphi_{xxx} = -4\varphi_{xxy} + \varphi_{xyy} - y\varphi_{xx} - \varphi_x,$$

in which we interchanged the order in some of the mixed partial derivatives (for example, $\varphi_{xyx} = \varphi_{xxy}$). Now

$$\varphi_{xxy}(0,y) = \frac{\partial}{\partial y}\varphi_{xx}(0,y) = -48y^2 - 5y^4 + 24e^{3y}$$

and

$$\varphi_{xyy}(0,y) = \frac{\partial^2}{\partial y^2}\varphi_x(0,y) = 12y^2.$$

Therefore

$$\begin{aligned}\varphi_{xxx}(0,y) &= -4(-48y^2 - 5y^4 + 24e^{3y}) + 12y^2 \\ &\quad - y(-16y^3 - y^5 + 8e^{3y}) - y^4 \\ &= y^6 + 35y^4 + 204y^2 - 96e^{3y} - 8ye^{3y}.\end{aligned}$$

Thus far we have

$$\begin{aligned}\varphi(x,y) &= e^{3y} + y^4x + \frac{1}{2}(-16y^3 - y^5 + 8e^{3y})x^2 \\ &\quad + (y^6 + 35y^4 + 204y^2 - 96e^{3y} - 8ye^{3y})x^3 + \cdots . \diamond\end{aligned}$$

What is a reasonable generalization of this discussion? Think of the y - axis as a curve in the x, y - plane and observe that φ_x is the derivative of φ in the direction normal to this curve. We therefore conjecture that a reasonable problem (in the sense of having a unique solution) might be to show that equation 2.1 has a unique solution if a curve Γ is specified, along with information about the solution and its derivative in the direction normal to Γ. Such information is called *Cauchy data*. The problem of obtaining a solution of equation 2.1 that satisfies Cauchy data is called the *Cauchy problem* for this partial differential equation.

Thus far we have sought only to motivate the formulation of the Cauchy problem. This does not imply that such a problem always has a solution. Indeed, in the first-order case we saw that it might not if the curve is a characteristic. Next we explore the connection between characteristics and the Cauchy problem in the second-order case.

Problems for Section 2.6

In each of Problems 1 through 5, use the partial differential equation and the Cauchy data on the y - axis to calculate the first four terms of the Taylor expansion about $x = 0$ of the solution $\varphi(x, y)$.

1. $u_{xx} - 4u_{xy} + yu_{yy} + u_x + yu_y = 0; u(0,y) = y^3, u_x(0,y) = 4y$

2. $u_{xx} - x^2u_{xy} + u_x - 3u_y = 0; u(0,y) = 4; u_x(0,y) = y^3$

3. $u_{xx} + xyu_{xy} - xu_{yy} + 8u_x = 0; u(0,y) = -y^2 + y, u_x(0,y) = \cos(y)$

4. $e^{-y}u_{xx} + xu_{xy} + 2u_x - u_y + u = 0; u(0,y) = y^3 - 2y^2, u_x(0,y) = e^y$

5. $u_{xx} - 2u_{xy} + y^2u_{yy} - 3u_x + xu = 0; u(0,y) = \sin(2y), u_x(0,y) = 2y^2$

   In each of Problems 6 through 10, use the partial differential equation and the Cauchy data on the x–axis to calculate the first four terms of the Taylor expansion about $y = 0$ of the solution $\varphi(x, y)$.

6. $u_{xx} - u_{yy} + 2u_x - \sin(y)u = 0; u(x,0) = x^2, u_y(x,0) = xe^{-x}$

7. $u_{xx} - xu_{xy} + u_{yy} - u_x + y^2u = 0; u(x,0) = x^2 - x, u_y(x,0) = \sin(2x)$

8. $u_{xx} - xu_{xy} + u_{yy} - 4xu_y + 2u = 0; u(x,0) = 1 - x^3, u_y(x,0) = x^4$

9. $u_{xx} + xu_{yy} + xu_y - yu = 0; u(x,0) = x\sin(x), u_y(x,0) = x$

10. $u_{xx} - u_{yy} + xyu_x - \cos(x)u = 0; u(x,0) = e^{-x}, u_y(x,0) = x^3$

2.7 Characteristics and the Cauchy Problem

Suppose that we are given a curve Γ in the plane, specified by the equation $\xi(x,y) = 0$. Assume that Γ is smooth and that a unit normal vector $\mathbf{n}(x, y)$ is specified at each point, as in Figure 2.5. This defines a "preferred side" of Γ into which normal vectors drawn from points of Γ are directed.

Let φ be a solution of equation 2.1, which we assume is defined in some region of the plane containing Γ. Suppose φ satisfies Cauchy data that is prescribed on Γ. This means that values $\varphi(x,y)$ and $\varphi_{\mathbf{n}}(x,y)$ are specified at points (x, y) of Γ, with $\varphi_{\mathbf{n}}$ the derivative of φ at (x, y) in the direction of the normal to Γ selected at that point. Our objective is to generalize the discussion of Section 2.6, where Γ was the y–axis, to determine the partial derivatives of φ on Γ. In theory this determines a Taylor expansion of the solution about any point on Γ.

The equation $\xi(x,y) = k$ determines a family of smooth curves in the plane, of which Γ is one. Define another family $\eta(x,y) = k$. This can be done somewhat arbitrarily, but we want these curves to be smooth as well, and no curve from one family should be tangent to a curve from the other family at points of intersection. Such a tangency occurs at a point if and only if

$$\frac{\xi_x}{\xi_y} = \frac{\eta_x}{\eta_y}$$

at this point. We therefore prevent such tangency by requiring that

$$\xi_x\eta_y - \xi_y\eta_x \neq 0.$$

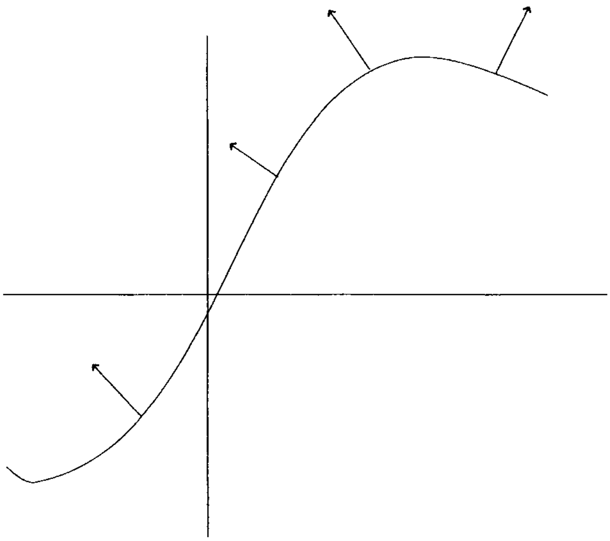

Figure 2.5: Unit normal vectors specifying a preferred side of a curve.

Since $\xi_x\eta_y - \xi_y\eta_x$ is the Jacobian J of the transformation

$$\xi = \xi(x, y), \eta = \eta(x, y),$$

we are really back to the familiar condition that this Jacobian is not zero. This assumption allows us in principle to invert the transformation and write $x = x(\xi, \eta), y = y(\xi, \eta)$.

Applying this transformation to the partial differential equation, we can think of φ as a function of (ξ, η). The transformed equation is equation 2.2.

We will now show that the Cauchy data is sufficient to determine φ_ξ and φ_η along Γ. Imagine that Γ is parametrized in terms of arc length:

$$x = x(s) \text{ and } y = y(s)$$

on Γ. This means that $\xi(x(s), y(s)) \equiv 0$. Let

$$\varphi(x(s), y(s)) = f(s) \tag{2.26}$$

and

$$\varphi_{\mathbf{n}}(x(s), y(s)) = g(s), \tag{2.27}$$

with f and g given as the Cauchy data.

Now

$$\varphi_x = \varphi_\xi\xi_x + \varphi_\eta\eta_x$$

and

$$\varphi_y = \varphi_\xi\xi_y + \varphi_\eta\eta_y.$$

Solve these equations to obtain

$$\varphi_\xi = \frac{\begin{vmatrix} \varphi_x & \eta_x \\ \varphi_y & \eta_y \end{vmatrix}}{\begin{vmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{vmatrix}} = \frac{1}{J}(\varphi_x\eta_y - \varphi_y\eta_x). \tag{2.28}$$

Differentiate equation 2.26 to obtain

$$\varphi_x x'(s) + \varphi_y y'(s) = f'(s). \tag{2.29}$$

The unit tangent to Γ is $x'(s)\mathbf{i} + y'(s)\mathbf{j}$, and the unit normal into the region bounded by Γ is $\mathbf{n} = -y'(s)\mathbf{i} + x'(s)\mathbf{j}$, so, using equation 2.27,

$$\begin{aligned} \varphi_{\mathbf{n}}(x(s), y(s)) &= \nabla\varphi \cdot \mathbf{n} = (\varphi_x\mathbf{i} + \varphi_y\mathbf{j}) \cdot (-y'\mathbf{i} + x'\mathbf{j}) \\ &= -y'\varphi_x + x'\varphi_y = g(s). \end{aligned} \tag{2.30}$$

Solve equations 2.29 and 2.30 to obtain

$$\varphi_x = \frac{\begin{vmatrix} f' & y' \\ g & x' \end{vmatrix}}{\begin{vmatrix} x' & y' \\ -y' & x' \end{vmatrix}} = \frac{x'f' - y'g}{(x')^2 + (y')^2} = x'f' - y'g \tag{2.31}$$

and

$$\varphi_y = \frac{\begin{vmatrix} x' & f' \\ -y' & g \end{vmatrix}}{\begin{vmatrix} x' & y' \\ -y' & x' \end{vmatrix}} = x'g + y'f'. \tag{2.32}$$

Here we have used the fact that the tangent and normal vectors are of length 1, so $(x'(s))^2 + (y'(s))^2 = 1$.

Substitute the results from equations 2.31 and 2.32 into equation 2.28 to obtain

$$\varphi_\xi = \frac{1}{J}[(x'f' - y'g)\eta_y - (x'g + y'f')\eta_x]. \tag{2.33}$$

By similar reasoning, we obtain

$$\varphi_\eta = \frac{1}{J}[(x'g + y'f')\xi_x - (x'f' - y'g)\xi_y]. \tag{2.34}$$

Equations 2.33 and 2.34 mean that the Cauchy data is sufficient to compute φ_ξ and φ_η along Γ in terms of the information provided. Since $\xi = 0$ on Γ, φ_ξ and φ_η are functions only of η along Γ, and we can use the expansions just obtained for φ_ξ and φ_η on Γ to compute $\varphi_{\xi\eta}, \varphi_{\eta\eta}, \varphi_{\xi\eta\eta}, \varphi_{\eta\eta\eta}, \ldots$ along this curve.

However, we must use the transformed partial differential equation 2.2 to solve for $\varphi_{\xi\xi}$. This requires that the coefficient of $\varphi_{\xi\xi}$ in equation 2.2 not vanish. But this coefficient is $A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2$, hence we require that

$$A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \neq 0.$$

Therefore, Γ cannot be a characteristic of the partial differential equation.

Once we have $\varphi_{\xi\xi}$ on Γ as a function of η, we can compute $\varphi_{\xi\xi\eta}, \varphi_{\xi\xi\eta\eta}$, and so on. Continuing in this way, we can compute all partial derivatives of φ on Γ, just as we did in the preceding section when Γ was the y - axis. However, keep in mind the requirement that Γ must not be a characteristic of the partial differential equation.

When we know these partial derivatives, we can expand $\varphi(\xi, \eta)$ in a Taylor series about any point of Γ. If this series converges, it provides a solution of the partial differential equation in a neighborhood of a point of Γ. A classic theorem named for Augustin-Louis Cauchy (1789 - 1857) and Sophie Kowalevski (1850 - 1891) asserts that this series does indeed converge to a unique solution of the Cauchy problem if the coefficients in the partial differential equation, and the functions specifying the solution and its normal derivative along the noncharacteristic curve, are analytic (possess power series expansions).

Example 2.7 *We illustrate these ideas using*

$$u_{xx} + 4u_{xy} - u_{yy} + xu = 0.$$

Let Γ be the line $y = 3x$. Define the family of curves $\xi(x, y) = y - 3x = k$, so Γ is the curve $\xi = 0$.

Specify the following Cauchy data on Γ :

$$u(x, 3x) = \sin(x)$$

and

$$-\frac{3}{\sqrt{10}} u_x(x, 3x) + \frac{1}{\sqrt{10}} u_y(x, 3x) = -7x.$$

Here

$$\mathbf{n} = -\frac{3}{\sqrt{10}}\mathbf{i} + \frac{1}{\sqrt{10}}\mathbf{j}$$

is a unit normal vector to Γ, and

$$u_{\mathbf{n}} = \nabla u \cdot \mathbf{n} = -\frac{3}{\sqrt{10}} u_x + \frac{1}{\sqrt{10}} u_y$$

is the derivative of u in the direction of this normal.

Define $\eta = x + y$, so $\eta(x, y) = c$ defines a second family of curves. This choice of η is somewhat arbitrary, but no curve from one family is tangent to a curve from the other at a point of intersection. Note that

$$J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = -4.$$

We first use this transformation to reduce the partial differential equation to canonical form. Let $u(x, y) = \varphi(\xi, \eta)$ and compute

$$\begin{aligned} u_x &= -3\varphi_\xi + \varphi_\eta, \\ u_{xx} &= 9\varphi_{\xi\xi} - 6\varphi_{\xi\eta} + \varphi_{\eta\eta}, \\ u_y &= \varphi_\xi + \varphi_\eta, \\ u_{yy} &= \varphi_{\xi\xi} + 2\varphi_{\xi\eta} + \varphi_{\eta\eta}, \end{aligned}$$

and

$$u_{xy} = -3\varphi_{\xi\xi} - 2\varphi_{\xi\eta} + \varphi_{\eta\eta}.$$

Substitute these into the partial differential equation to obtain the transformed equation

$$\varphi_{\xi\xi} + 4\varphi_{\xi\eta} - \varphi_{\eta\eta} + \frac{1}{4}(\xi - \eta)\varphi = 0.$$

On Γ, *using arc length as a parameter,*

$$x(s) = \frac{s}{\sqrt{10}} \text{ and } y(s) = \frac{3s}{\sqrt{10}}.$$

Then $(x'(s))^2 + (y'(s))^2 = 1$, *as required to use equations 2.33 and 2.34. The Cauchy data in terms of* s *is*

$$\varphi(x(s), y(s)) = \sin(s/\sqrt{10}) = f(s)$$

and

$$\varphi_{\mathbf{n}}(x(s), y(s)) = -\frac{7s}{\sqrt{10}} = g(s).$$

By equation 2.33,

$$\begin{aligned}
\varphi_\xi(x(s), y(s)) &= -\frac{1}{4}\left[\frac{1}{\sqrt{10}}\frac{1}{\sqrt{10}}\cos\left(\frac{s}{\sqrt{10}}\right) - \frac{3}{\sqrt{10}}\left(\frac{-7s}{\sqrt{10}}\right)\right] \\
&\quad + \frac{1}{4}\left[\frac{1}{\sqrt{10}}\left(\frac{-7s}{\sqrt{10}}\right) + \frac{3}{\sqrt{10}}\frac{1}{\sqrt{10}}\cos\left(\frac{s}{\sqrt{10}}\right)\right] \\
&= \frac{1}{20}\cos\left(\frac{s}{\sqrt{10}}\right) - \frac{7}{10}s,
\end{aligned}$$

and, by equation 2.34,

$$\begin{aligned}
\varphi_\eta(x(s), y(s)) &= -\frac{1}{4}\left[\frac{1}{\sqrt{10}}\left(\frac{-7s}{\sqrt{10}}\right) + \frac{3}{\sqrt{10}}\frac{1}{\sqrt{10}}\cos\left(\frac{s}{\sqrt{10}}\right)\right](-3) \\
&\quad + \frac{1}{4}\left[\frac{1}{\sqrt{10}}\frac{1}{\sqrt{10}}\cos\left(\frac{s}{\sqrt{10}}\right) - \frac{3}{\sqrt{10}}\left(\frac{-7s}{\sqrt{10}}\right)\right] \\
&= \frac{1}{4}\cos\left(\frac{s}{\sqrt{10}}\right).
\end{aligned}$$

These partial derivatives can be written as functions of η *on* Γ, *where* $y = 3x$ *and* $\eta = 4x$; *hence*

$$s = \sqrt{10}x = \frac{\sqrt{10}}{4}\eta.$$

Therefore, on Γ,

$$\varphi_\xi = \frac{1}{20}\cos(\eta/4) - \frac{7}{4\sqrt{10}}\eta$$

and

$$\varphi_\eta = \frac{1}{4}\cos(\eta/4).$$

From these we can compute $\varphi_{\xi\eta}, \varphi_{\xi\eta\eta}, \cdots, \varphi_{\eta\eta}, \varphi_{\eta\eta\eta}, \cdots$ *on* Γ. *In particular, on* Γ,

$$\varphi_{\xi\eta} = \frac{\partial}{\partial\eta}\varphi_\xi = -\frac{1}{80}\sin(\eta/4) - \frac{7}{4\sqrt{10}}$$

and

$$\varphi_{\eta\eta} = -\frac{1}{16}\sin(\eta/4).$$

To compute $\varphi_{\xi\xi}$ *on* Γ, *we must use the transformed partial differential equation to write*

$$\varphi_{\xi\xi} = -4\varphi_{\xi\eta} + \varphi_{\eta\eta} + \frac{1}{4}(\eta - \xi)\varphi.$$

Since $\xi \equiv 0$ *on* Γ, *this equation becomes*

$$\varphi_{\xi\xi} = -4\varphi_{\xi\eta} + \varphi_{\eta\eta} + \frac{1}{4}\eta\varphi.$$

Therefore, on Γ,

$$\begin{aligned}\varphi_{\xi\xi} &= -4\left[-\frac{1}{80}\sin(\eta/4) - \frac{7}{4\sqrt{10}}\right] \\ &\quad - \frac{1}{16}\sin(\eta/4) + \frac{1}{4}\eta\sin(\eta/4) \\ &= -\frac{1}{80}\sin(\eta/4) + \frac{1}{4}\eta\sin(\eta/4) + \frac{7}{\sqrt{10}}.\end{aligned}$$

Using these partial derivatives, we can compute the first few terms of the Taylor series of $\varphi(\xi,\eta)$ *about the origin, which occurs when* $\eta = 0$. *This series is*

$$\begin{aligned}\varphi(\xi,\eta) &= \varphi(0,0) + \varphi_\xi(0,0)\xi + \varphi_\eta(0,0)\eta \\ &\quad + \frac{1}{2}(\varphi_{\xi\xi}(0,0)\xi^2 + 2\varphi_{\xi\eta}(0,0)\xi\eta + \varphi_{\eta\eta}(0,0)\eta^2) + \cdots . \qquad (2.35)\end{aligned}$$

Now

$$\varphi(0,0) = 0, \varphi_\xi(0,0) = \frac{1}{20}, \varphi_\eta(0,0) = \frac{1}{4},$$
$$\varphi_{\xi\eta}(0,0) = -\frac{7}{4\sqrt{10}}, \varphi_{\eta\eta}(0,0) = 0, \text{ and } \varphi_{\xi\xi}(0,0) = \frac{7}{\sqrt{10}},$$

so

$$\varphi(\xi,\eta) = \frac{1}{20}\xi + \frac{1}{4}\eta + \frac{7}{2\sqrt{10}}\xi^2 - \frac{7}{4\sqrt{10}}\xi\eta + \cdots,$$

and we can compute more terms by repeatedly differentiating the partial differential equation and using partial derivatives previously calculated. The resulting series is a solution of the transformed equation in some disk about the origin. ◇

A series solution obtained in this way from the partial differential equation and the Cauchy data can be shown to have a disk of convergence of positive radius if the coefficients in the differential equation are analytic in the region of interest.

Problems for Section 2.7

1. Derive equation 2.34.

2. In the spirit of equations 2.33 and 2.34, derive expressions for $\varphi_{\xi\xi}, \varphi_{\xi\eta}$ and $\varphi_{\eta\eta}$. Hint: Begin with the chain rule derivatives for φ_x and φ_y and obtain $\varphi_{xx}, \varphi_{yy}$, and φ_{xy}. Solve these equations for $\varphi_{\xi\xi}, \varphi_{\eta\eta}$, and $\varphi_{\xi\eta}$. This involves some algebra, and a software routine for carrying out the manipulations is helpful.

3. Carry out the details of the discussion and example for the following problem. Consider the partial differential equation

$$u_{xx} + 4u_{xy} - 3u_{yy} + u_x = 0.$$

Let Γ be the line $y = x$. Check that Γ is not a characteristic of the differential equation. Choose

$$\mathbf{n} = \frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j})$$

as the unit normal to Γ and specify values of u and the normal derivative of u along $\mathbf{n}$ by

$$u(x, x) = x^2,$$
$$-u_x(x, x) + u_y(x, x) = \cos(x).$$

Choosing a suitable $\eta(x, y)$, as allowed in the discussion, compute $\varphi, \varphi_\xi, \varphi_\eta$, $\varphi_{\xi\xi}, \varphi_{\xi\eta}$, and $\varphi_{\eta\eta}$ along Γ. Use these to compute the terms listed in equation 2.35 of the Taylor series for φ about the origin (which is a point of Γ).

4. Carry out the program of Problem 3 with the partial differential equation

$$u_{xx} + 5u_{yy} = 0.$$

Let Γ be the line $y = 2x$ and let the Cauchy data on this line be given by

$$u(x, 2x) = 1 - x^2,$$
$$2u_x(x, 2x) - u_y(x, 2x) = 2 + x.$$

5. Carry out the program of Problem 3 with the partial differential equation

$$u_{xx} - 2u_x - 6u_{yy} = 0.$$

Let Γ be the line $y = 5x$ and let the Cauchy data on this line be given by

$$u(x, 5x) = \sin(x),$$
$$5u_x(x, 5x) - u_y(x, 5x) = x^2.$$

6. Carry out the program of Problem 3 with the partial differential equation

$$u_{xx} - u_{yy} = 0.$$

Let Γ be the line $y = -2x$ and let Cauchy data on Γ be given by

$$u(x, -2x) = 1 + x,$$
$$2u_x(x, -2x) + u_y(x, -2x) = x^2 - x.$$

7. Let φ be a solution of the linear second-order partial differential equation 2.1. Suppose that Cauchy data are prescribed along a curve Γ, which is parametrized in terms of arc length by

$$x = \alpha(s) \text{ and } y = \beta(s).$$

Along this curve we can think of φ and its partial derivatives as functions of s. Show that, along Γ,

$$\varphi_{xx}\alpha' + \varphi_{xy}\beta' = \frac{d}{ds}\varphi_x,$$
$$\varphi_{xy}\alpha' + \varphi_{yy}\beta' = \frac{d}{ds}\varphi_y,$$

and

$$A\varphi_{xx} + 2B\varphi_{xy} + C\varphi_{yy} = -D\varphi_x - E\varphi_y - F\varphi - G.$$

Think of these as three linear algebraic equations to be solved for $\varphi_{xx}, \varphi_{xy}$, and φ_{yy} along Γ, in terms of known quantities along Γ. From algebra, these equations have a unique solution only if the determinant of the coefficients is nonzero. Show that this condition holds if Γ is not characteristic. Further show that if Γ is characteristic, these three equations have a solution only if the terms on the right sides of the equations satisfy a certain condition. Obtain this condition.

2.8 Characteristics as Carriers of Discontinuities

Characteristics are curves along which we may not specify Cauchy data in seeking a unique analytic solution of a second-order linear partial differential equation.

Characteristics may also be regarded as possible carriers of discontinuities of the second derivative. We will discuss what this means.

Suppose that we have made the transformation $\xi = \xi(x, y), \eta = \eta(x, y)$, and obtained the transformed equation 2.2. Suppose also that $\Gamma : \xi(x, y) = \xi_0$ has been chosen, and we want to consider the possibility that $w_{\xi\xi}$ is discontinuous at points of Γ. What does this tell us about Γ?

Let (ξ_0, η) be a point on Γ at which $w_{\xi\xi}$ may be discontinuous, but the other terms appearing in equation 2.2 are continuous. Take the limit of the terms in equation 2.2 as $(\xi, \eta) \to (\xi_0, \eta)$, first from one side of Γ, then from the other. Denote the limit from one side with a superscript $+$, and that from the other side with a superscript $-$. This yields the equations

$$a^+(\xi_0, \eta)w^+_{\xi\xi} + 2b^+(\xi_0, \eta)w^+_{\xi\eta} + c^+(\xi_0, \eta)w^+_{\eta\eta} + d^+(\xi_0, \eta)w^+_{\xi}$$
$$+ \ e^+(\xi_0, \eta)w^+_{\eta} + f^+(\xi_0, \eta)w^+ + g^+(\xi_0, \eta) = 0$$

and

$$a^-(\xi_0, \eta)w^-_{\xi\xi} + 2b^-(\xi_0, \eta)w^-_{\xi\eta} + c^-(\xi_0, \eta)w^-_{\eta\eta} + d^-(\xi_0, \eta)w^-_{\xi}$$
$$+ \ e^-(\xi_0, \eta)w^-_{\eta} + f^-(\xi_0, \eta)w^- + g^-(\xi_0, \eta) = 0.$$

Subtract these equations. Assuming that all terms except possibly $w_{\xi\xi}$ are continuous, most terms cancel because the limits from either side of Γ are equal. We obtain

$$a(\xi_0, \eta)(w^+_{\xi\xi}(\xi_0, \eta) - w^-(\xi_0, \eta)) = 0.$$

This equation can be satisfied in either of two ways: $a(\xi_0, \eta) = 0$ or $w^+_{\xi\xi}(\xi_0, \eta) = w^-_{\xi\xi}(\xi_0, \eta)$. In the latter case the second derivative $w_{\xi\xi}$ is continuous at (ξ_0, η). In the former case, which must hold if $w_{\xi\xi}$ is discontinuous at (ξ_0, η), we have

$$A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2]_{\xi=\xi_0} = 0,$$

and this is exactly the condition for the curve $\xi(x, y) = \xi_0$ to be a characteristic.

We conclude that the characteristics $\xi(x, y) = k$ are curves along which $w_{\xi\xi}$ may be discontinuous. By a similar analysis, characteristics $\eta(x, y) = k$ are curves along which $w_{\eta\eta}$ may have discontinuities.

Geometrically, this means that there may be two solutions of the partial differential equation which branch away from each other at Γ. Imagine Γ lying on a surface $\sum$, which splits along Γ into two parts, $\sum_1$ and $\sum_2$. One solution consists of the surface $\sum$ joined with $\sum_1$ at Γ, and the other solution consists of $\sum$ joined with $\sum_2$ at Γ.

We now have three ways of thinking of characteristics of the second- order linear partial differential equation 2.1:

(1) as curves defining a transformation of the partial differential equation to canonical form

(2) as curves along which Cauchy data do not determine a solution, or perhaps not a unique solution

(3) as curves along which a second derivative of the solution may have a discontinuity.

Problem for Section 2.8

1. Let a, b, and c be real numbers with not both a and b zero. Let Γ be the straight line determined by $ax + by + c = 0$. Define

$$\varphi(x, y) = |ax + by + c|$$

for all $(x,\ y)$ in R^2. Show that φ satisfies Laplace's equation in two dimensions:

$$u_{xx} + u_{yy} = 0.$$

However, show that the normal derivative $\partial\varphi/\partial n$ is discontinuous across this line.

Laplace's equation is elliptic and hence has no characteristic. However, while we have discussed how characteristics can be carriers of discontinuities of second derivative terms, this example suggests that we cannot approach characteristics by thinking of discontinuities of first derivative terms.

Beginning Partial Differential Equations, Second Edition
by Peter V. O'Neil

Chapter 3

Elements of Fourier Analysis

We want to turn now to methods of solution, and properties of solutions, of second-order linear partial differential equations. Fourier series, integrals and transforms will be important in this development, and in this chapter we provide some background in these areas.

3.1 Why Fourier Series?

In his seminal 1807 paper on heat conduction, Joseph Fourier (1768-1830) developed and solved initial-boundary value problems modeling heat flow under a variety of conditions. In particular, he considered the problem:

$$\begin{aligned} u_t &= ku_{xx} \text{ for } 0 < x < \pi, t > 0 \\ u(0,t) &= u(\pi,t) = 0 \\ u(x,0) &= f(x) \text{ for } 0 < x < \pi. \end{aligned}$$

The solution gives the temperature $u(x, t)$ at point x and time t in a bar of material extending from 0 to π and having constant diffusivity k. The ends of the bar are maintained at zero temperature, and the initial temperature (at time $t = 0$) throughout the bar is $f(x)$ in the cross section at x perpendicular to the x - axis.

After some analysis, which we will see in Chapter 5, Fourier was led to look for a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx)e^{-n^2kt},$$

in which the b_n's are constants to be determined. For any choice of these constants, $b_n \sin(nx)e^{-n^2kt}$ satisfies the heat equation and is zero at $x = 0$ and at $x = \pi$. The problem is to choose the b_n's so that $u(x,t)$ satisfies the initial condition $u(x,0) = f(x)$. This requires that

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \text{ for } 0 < x < L.$$

Fourier's attempt to solve a problem in heat conduction led him to the necessity of writing the initial temperature function as an infinite superposition of constant multiples of sines of different frequencies. Because $f(x)$ can be specified somewhat arbitrarily, Fourier's work suggested that almost any function could be expanded in such a series. This astonished many of the mathematicians of Fourier's time and initiated a century of intense investigation of trigonometric series. This chapter is devoted to some of the ideas resulting from this work.

Problems for Section 3.1

1. Solve the boundary value problem for u if $f(x) = \sqrt{3}\sin(2x)$.

2. Solve the initial-boundary value problem for u if $f(x) = 5\sin(x) - 12\sin(4x)$.

3. Prove that $f(x) = x$ cannot be written as a finite sum

$$\sum_{n=1}^{N} b_n \sin(nx)$$

for $0 < x < \pi$, for any choice of real numbers $b_1, \cdots, b_N$.

3.2 The Fourier Series of a Function

Suppose that we are given $f(x)$ for $-\pi \leq x \leq \pi$, and we want to choose numbers $a_0, a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ so that

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)). \tag{3.1}$$

Denoting the constant term as $a_0/2$ is a convention that will simplify some notation.

Proceed informally as follows. Integrate both sides of equation 3.1 and interchange the summation and the integral to obtain

$$\begin{aligned}
&\int_{-\pi}^{\pi} f(x)\,dx \\
&= \frac{1}{2}a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\left(a_n \int_{-\pi}^{\pi} \cos(nx)\,dx + b_n \int_{-\pi}^{\pi} \sin(nx)\,dx\right) \\
&= \pi a_0,
\end{aligned}$$

because all of the integrals in the series are zero. Therefore,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx. \tag{3.2}$$

Now solve for the a_n's for $n = 1, 2, \cdots$ as follows. Multiply equation 3.1 by $\cos(mx)$, with m any positive integer, and integrate the resulting equation, interchanging the summation and the integral. We obtain

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \frac{1}{2}a_0\int_{-\pi}^{\pi}\cos(mx)\,dx$$
$$+\sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos(nx)\cos(mx)\,dx + b_n\int_{-\pi}^{\pi}\sin(nx)\cos(mx)\,dx\right).$$

By a straightforward computation, all the integrals on the right side of this equation are zero except $\int_{-\pi}^{\pi}\cos^2(mx)\,dx$, which occurs when $n = m$ in the summation. This integral equals π. The last equation reduces to

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = a_m\int_{-\pi}^{\pi}\cos^2(mx)\,dx = \pi a_m.$$

Then

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx \text{ for } m = 1, 2, \cdots. \tag{3.3}$$

Finally, multiply equation 3.1 by $\sin(mx)$ and integrate both sides of the resulting equation, interchanging the sum and the integrals to obtain:

$$\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx = \frac{1}{2}a_0\int_{-\pi}^{\pi}\sin(mx)\,dx$$
$$+\sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos(nx)\sin(mx)\,dx + b_n\int_{-\pi}^{\pi}\sin(nx)\sin(mx)\,dx\right).$$

Again, all the integrals on the right are zero except $\int_{-\pi}^{\pi}\sin^2(mx)\,dx$, which occurs in the summation when $n = m$. This integral also equals π. The last equation simplifies to

$$\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx = b_m\int_{-\pi}^{\pi}\sin^2(mx)\,dx = b_m\pi$$

and therefore

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx \text{ for } m = 1, 2, \cdots. \tag{3.4}$$

The reasoning we have just pursued is flawed by the interchange of an integral and an infinite series, which can lead to incorrect results. Nevertheless, this calculation shows how the a_n's and b_n's should be chosen when this interchange is permissible, and suggests that these may be the right choices in general. Although we will encounter some subtleties, this intuition turns out to be correct, at least for functions having certain reasonable properties which we will specify. In anticipation of this, the numbers given by equations 3.2, 3.3, and 3.4 are called the *Fourier coefficients* of f on $[-\pi, \pi]$. With these coefficients, the series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)) \tag{3.5}$$

is defined to be the *Fourier series* of f on $[-\pi, \pi]$.

Notice that we have not claimed that this Fourier series converges to $f(x)$ on $[-\pi, \pi]$. To make use of Fourier series, we will need to know what this series converges to, and what conditions on f will ensure that it converges to $f(x)$. It is not difficult to find very simple examples in which the Fourier series of f does not converge to $f(x)$, at least for some values of x.

Example 3.1 *Let $f(x) = x$ for $-\pi \le x \le \pi$. The Fourier coefficients are*

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = 0,$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx = 0,$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = -2\frac{\cos(n\pi)}{n} = 2\frac{(-1)^{n+1}}{n}.$$

The Fourier series of f on $[-\pi, \pi]$ is

$$\sum_{n=1}^{\infty} 2\frac{(-1)^{n+1}}{n}\sin(nx).$$

This series equals 0 at $x = \pm\pi$, whereas $f(-\pi) = -\pi$ and $f(\pi) = \pi$. At least at $-\pi$ and at π, then, the Fourier series of this function does not converge to the function. At other values of x, it is not obvious what this series converges to. ◇

Problems for Section 3.2

1. Write the Fourier series for

$$f(x) = \begin{cases} -1 & \text{for } -\pi \le x < 0 \\ 1 & \text{for } 0 \le x \le \pi. \end{cases}$$

Does this series converge to $f(x)$ for $-\pi \le x \le \pi$?

2. Write the Fourier series for $f(x) = e^x$ on $[\pi, \pi]$. Are there any points in $[-\pi, \pi]$ at which it is obvious that the Fourier series converges to e^x, or that it does not?

3.3 Convergence of Fourier Series

We will determine the sum of the Fourier series 3.5 for functions satisfying certain conditions. This will involve some preliminaries, culminating in the proof of a convergence theorem.

Periodic Functions

The terms $\sin(nx)$ and $\cos(nx)$ in the Fourier series of f on $[-\pi, \pi]$ are all periodic of period 2π. We will therefore assume that f is periodic of period 2π. This means that $f(x)$ is defined for $-\infty < x < \infty$ and $f(x + 2\pi) = f(x)$ for all x.

Often, we want to expand a function g defined only on $(-\pi, \pi]$ in a Fourier series on $[-\pi, \pi]$. In such a case, we can always think of g as periodic of period 2π by a simple extension strategy. Replicate the graph of g for $-\pi < x \leq \pi$ over successive intervals of length 2π. This produces a function f that is defined over the entire real line and has period 2π, and also

$$g(x) = f(x) \text{ for } -\pi < x \leq \pi.$$

We call such an f the *periodic extension of g to the entire real line*. Figure 3.1 shows a typical function, with part of its periodic extension shown in Figure 3.2.

This extension is simply a device to reconcile the fact that we want to represent $g(x)$ as a Fourier series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$, and that this series is periodic of period 2π. We rarely carry out such an extension explicitly. Indeed, it is usually only the behavior of $g(x)$ on the basic interval $(-\pi, \pi]$ that is of interest, and we often only put values of x in this interval into the Fourier series.

With this periodic extension in the background, we lose no generality in assuming that functions we consider for Fourier series are periodic of period 2π. In this event, any interval of length 2π carries a complete copy of a graph of f. Further, for any number a,

$$\int_{-\pi}^{\pi} f(x)\, dx = \int_{a}^{a+2\pi} f(x)\, dx, \tag{3.6}$$

a fact that is sometimes useful in carrying out calculations.

Dirichlet's Formula

We want to understand the relationship between values of the Fourier series 3.5 and values $f(x)$ of the periodic function f. As with any series, convergence of the Fourier series hinges on the behavior of its partial sums. Denote the Nth partial sum as $S_N(x)$:

$$S_N(x) = \frac{1}{2}a_0 + \sum_{n=1}^{N}(a_n \cos(nx) + b_n \sin(nx)).$$

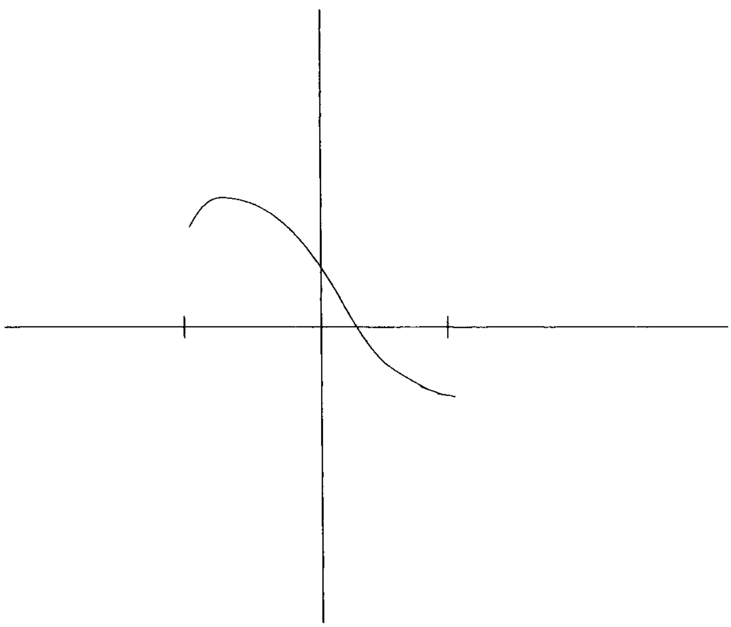

Figure 3.1: a typical function.

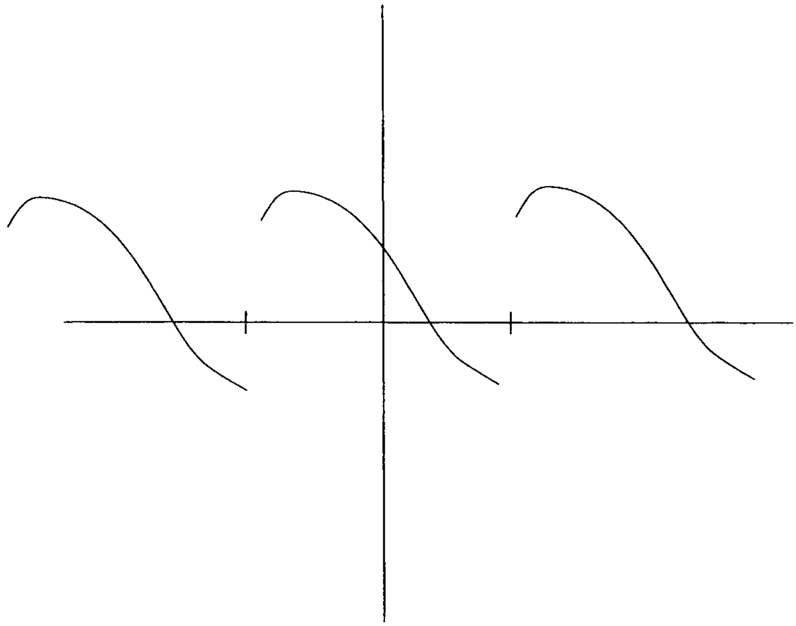

Figure 3.2: Part of the periodic extension of the function of Figure 3.1.

The sum of the Fourier series, at any x, is $\lim_{N\to\infty} S_N(x)$. We will rewrite $S_N(x)$ in a way that will help us determine this limit.

Insert the integral formulas for the Fourier coefficients of f into $S_N(x)$ to obtain

$$\begin{aligned}
S_N(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)\,d\xi \\
&\quad + \sum_{n=1}^{N}\left[\frac{1}{\pi}\int_{-\pi}^{\pi} f(\xi)\cos(n\xi)\,d\xi\cos(nx) + \frac{1}{\pi}\int_{-\pi}^{\pi} f(\xi)\sin(n\xi)\,d\xi\sin(nx)\right] \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\xi)\left[\frac{1}{2} + \sum_{n=1}^{N}[\cos(n\xi)\cos(nx) + \sin(n\xi)\sin(nx)\right]d\xi \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\xi)\left[\frac{1}{2} + \sum_{n=1}^{N}\cos(n(\xi - x))\right]d\xi. \qquad (3.7)
\end{aligned}$$

To simplify the quantity in square brackets in the last integral, let $y = \xi - x$ and let

$$\sigma = \frac{1}{2} + \sum_{n=1}^{N}\cos(ny).$$

Upon multiplying both sides of this equation by $2\sin(y/2)$, we obtain

$$\begin{aligned}
2\sigma\sin(y/2) &= \sin(y/2) + 2\sum_{n=1}^{N}\cos(ny)\sin(y/2) \\
&= \sin(y/2) + \sum_{n=1}^{N}[\sin((n+1/2)y) - \sin((n-1/2)y)] \\
&= \sin(y/2) + [\sin(3y/2) - \sin(y/2)] + [\sin(5y/2) - \sin(3y/2)]\cdots \\
&+ [\sin((N-1/2)y) - \sin((N-3/2)y)] \\
&+ [\sin((N+1/2)y) - \sin((N-1/2)y)].
\end{aligned}$$

This series is telescoping – the first term in each set of square brackets cancels the next set of square brackets. Further, the second term in the first set of brackets is canceled by the $\sin(y/2)$ term. All that remains is the first term in the last set of brackets, and we have

$$2\sigma\sin(y/2) = \sin((N+1/2)y).$$

Then

$$\sigma = \frac{\sin((N+1/2)y)}{2\sin(y/2)} = \frac{\sin((N+1/2)(\xi - x))}{2\sin((\xi - x)/2)},$$

provided that $\sin((\xi - x)/2) \neq 0$. Upon inserting this result into equation 3.7, we obtain

$$S_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\xi)\frac{\sin((N+1/2)(\xi - x))}{2\sin((\xi - x)/2)}\, d\xi. \tag{3.8}$$

Now put $t = \xi - x$ in equation 3.8 to obtain

$$S_N(x) = \frac{1}{\pi}\int_{-\pi-x}^{\pi-x} f(x+t)\frac{\sin((N+1/2)t)}{2\sin(t/2)}\, dt.$$

Since f is periodic of period 2π, this integrand also has period 2π. This means that we can carry out the integral over any interval of length 2π. In particular,

$$S_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\frac{\sin((N+1/2)t)}{2\sin(t/2)}\, dt. \tag{3.9}$$

This is *Dirichlet's formula*, and it is the special form we sought for $S_N(x)$. The function

$$\frac{\sin((N+1/2)t)}{2\sin(t/2)}$$

is called the *Dirichlet kernel*.

Using Dirichlet's formula, we can derive another result that we will use shortly.

Lemma 3.1

$$\frac{1}{\pi}\int_{-\pi}^{0} \frac{\sin((N+1/2)t)}{2\sin(t/2)}\, dt = \frac{1}{\pi}\int_{0}^{\pi} \frac{\sin((N+1/2)t)}{2\sin(t/2)}\, dt = \frac{1}{2}. \diamond$$

Proof Let $f(x) = 1$ in Dirichlet's formula. The Fourier coefficients of this function are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} d\xi = 2$$

and for $n = 1, 2, \cdots$,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(n\xi)\, d\xi = 0 \text{ and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(n\xi)\, d\xi = 0.$$

Therefore $S_N(x) = 1$ and since $f(x+t) = 1$ for all x and t, Dirichlet's formula gives

$$\frac{1}{\pi}\int_{-\pi}^{0} \frac{\sin((N+1/2)t)}{2\sin(t/2)}\, dt + \frac{1}{\pi}\int_{0}^{\pi} \frac{\sin((N+1/2)t)}{2\sin(t/2)}\, dt = 1. \tag{3.10}$$

Let $t = -w$ in the first integral on the left to write

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{0} \frac{\sin((N+1/2)t)}{2\sin(t/2)}\, dt &= \frac{1}{\pi}\int_{\pi}^{0} \frac{\sin((N+1/2)w)}{2\sin(w/2)}(-1)\, dw \\ &= \frac{1}{\pi}\int_{0}^{\pi} \frac{\sin((N+1/2)w)}{2\sin(w/2)}\, dw.\end{aligned}$$

Therefore, the integrals on the left side of equation 3.10 are equal. Since their sum is 1, both integrals equal 1/2. $\diamond$

The Riemann-Lebesgue Lemma

We will now prove a result that will enable us to compute the limit of $S_N(x)$ as $N \to \infty$. Recall that a function g defined on $[a, b]$ is piecewise continuous if it is continuous except perhaps at finitely many points, at each of which it has finite one-sided limits from within the interval. In particular,

$\lim_{x\to a+} g(x)$ and $\lim_{x\to b-} g(x)$ are finite,

and

if $a < t < b$ and g is not continuous at t, then $\lim_{x\to t-} g(x)$ and $\lim_{x\to t+} g(x)$ are finite.

A piecewise continuous function on $[a, b]$ is therefore a function with at most finitely many discontinuities, each a finite jump discontinuity.

Lemma 3.2 (*Riemann-Lebesgue Lemma*) If g is piecewise continuous on $[a, b]$, *then*

$$\lim_{\omega\to\infty} \int_a^b g(t)\sin(\omega t)\, dt = 0. \diamond$$

Proof of the Riemann-Lebesgue Lemma

Suppose first that g is continuous on $[a, b]$ and let $I = \int_a^b g(t)\sin(\omega t)\, dt$. Let $t = \xi + \pi/\omega$, with ω chosen large enough that $b - \pi/\omega \geq a$. Then

$$I = \int_{a-\pi/\omega}^{b-\pi/\omega} g(\xi + \pi/\omega)\sin(\omega\xi + \pi)\, d\xi = -\int_{a-\pi/\omega}^{b-\pi/\omega} g(\xi+\pi/\omega)\sin(\omega\xi)\, d\xi.$$

To maintain t as the variable of integration, replace ξ with t in the last integral:

$$I = -\int_{a-\pi/\omega}^{b-\pi/\omega} g(t+\pi/\omega)\sin(\omega t)\,dt.$$

Now add this expression for I to the definition of I, to write

$$\begin{aligned} 2I &= \int_a^b g(t)\sin(\omega t)\,dt - \int_{a-\pi/\omega}^{b-\pi/\omega} g(t+\pi/\omega)\sin(\omega t)\,dt \\ &= \int_a^{b-\pi/\omega} [g(t) - g(t+\pi/\omega)]\sin(\omega t)\,dt \\ &+ \int_{b-\pi/\omega}^{b} g(t)\sin(\omega t)\,dt - \int_{a-\pi/\omega}^{a} g(t+\pi/\omega)\sin(\omega t)\,dt. \end{aligned} \tag{3.11}$$

Since g is continuous on $[a, b]$, then for some M, $|g(t)| \leq M$ for $a \leq t \leq b$. Then

$$\left|\int_{b-\pi/\omega}^{b} g(t)\sin(\omega t)\,dt\right| \leq M\frac{\pi}{\omega}$$

and

$$\left|\int_{a-\pi/\omega}^{a} g(t+\pi/\omega)\sin(\omega t)\,dt\right| \leq M\frac{\pi}{\omega}.$$

For the other integral in equation 3.11, use the fact that g is uniformly continuous on $[a, b]$. Let $\epsilon > 0$. There is some $\delta > 0$ such that

$$|g(x) - g(y)| < \epsilon/3 \text{ if } |x - y| < \delta.$$

Then

$$|g(t) - g(t+\pi/\omega)| < \epsilon/3 \text{ if } \frac{\pi}{\omega} < \delta.$$

Therefore, for $\omega > \pi/\delta$, and also ω large enough that $b - \pi/\omega \geq a$ and $M\pi/\omega < \epsilon/3$, we have from equation 3.11 and the bounds we have just found that

$$|2I| < M\frac{\pi}{\omega} + M\frac{\pi}{\omega} + \frac{\epsilon}{3} < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

But then

$$|I| < \frac{\epsilon}{2} < \epsilon \text{ if } \omega > \pi/\delta,$$

proving that $\lim_{\omega\to\infty} I = 0$.

This leaves the case that g is continuous except for jump discontinuities at finitely many points $t_1, \cdots, t_k$. Now write I as a sum of integrals from a to t_1, t_1 to $t_2, \cdots$, and finally, from t_n to b. By redefining $g(t)$ at the endpoint of each of these intervals, if necessary, we can write I as a finite sum of integrals, each having the same form as I but with each having a continuous integrand. These integrals all have limit 0 as $\omega \to \infty$ by the case just proved; hence $I \to 0$ as $\omega \to \infty$. This completes the proof of the Riemann - Lebesgue lemma. $\diamond$

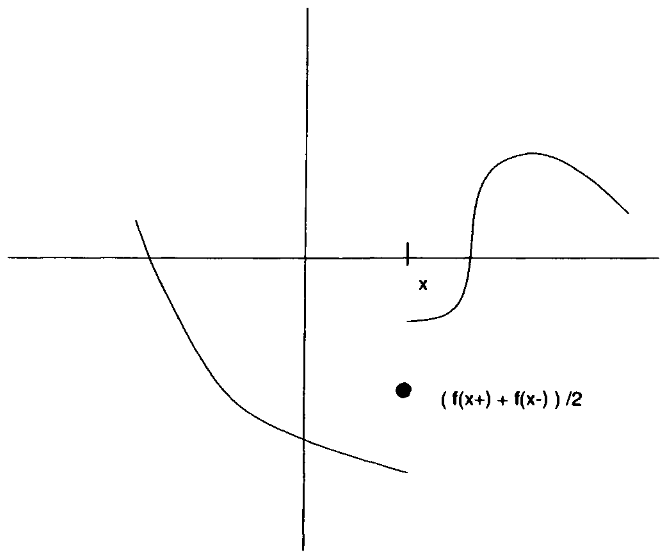

Figure 3.3: Convergence of a Fourier series at a point of discontinuity.

Convergence of the Fourier Series

We can now give what is essentially Dirichlet's proof of a convergence theorem for the Fourier series 3.5. We will use the following notation and terminology.

For any real number x, denote the left limit of f at x by $f(x-)$, and the right limit by $f(x+)$:

$$f(x-) = \lim_{\xi \to x-} f(\xi) \text{ and } f(x+) = \lim_{\xi \to x+} f(\xi).$$

A function is *piecewise smooth* on an interval if the function and its derivative are piecewise continuous on the interval.

Theorem 3.1 (*Convergence of Fourier Series*) *Let f be piecewise smooth on $[-\pi, \pi]$ and periodic of period 2π. Then at each x the Fourier series 3.5 converges to*

$$\frac{1}{2}(f(x+) + f(x-)). \diamond$$

If f is actually continuous at x, then $f(x+) = f(x-) = f(x)$ and the Fourier series of f on $[-\pi, \pi]$ converges to $f(x)$. Thus, over any interval on which f is continuous and satisfies the hypotheses of the theorem, the Fourier series is an exact representation of the function.

If f has a jump discontinuity at x, the Fourier series converges to the average of the left and right limits at x. Geometrically, this number is midway between the ends of the graph at the jump discontinuity (Figure 3.3).

Proof of the Convergence Theorem

Use Dirichlet's formula and Lemma 3.1 to write

$$\begin{aligned}
S_N(x) &= \frac{1}{\pi}\int_{-\pi}^{0} f(x+t)\frac{\sin((N+1/2)t)}{2\sin(t/2)}\,dt \\
&+ \frac{1}{\pi}\int_{0}^{\pi} f(x+t)\frac{\sin((N+1/2)t)}{2\sin(t/2)}\,dt \\
&= \frac{1}{\pi}\int_{-\pi}^{0} f(x+t)\frac{\sin((N+1/2)t)}{2\sin(t/2)}\,dt - \frac{1}{2}f(x-) + \frac{1}{2}f(x-) \\
&+ \frac{1}{\pi}\int_{0}^{\pi} f(x+t)\frac{\sin((N+1/2)t)}{2\sin(t/2)}\,dt - \frac{1}{2}f(x+) + \frac{1}{2}f(x+) \\
&= \frac{1}{\pi}\int_{-\pi}^{0} [f(x+t)-f(x-)]\frac{\sin((N+1/2)t)}{2\sin(t/2)}\,dt + \frac{1}{2}f(x-) \\
&+ \frac{1}{\pi}\int_{0}^{\pi} [f(x+t)-f(x+)]\frac{\sin((N+1/2)t)}{2\sin(t/2)}\,dt + \frac{1}{2}f(x+). \qquad (3.12)
\end{aligned}$$

The proof is complete if we can show that each of the last two integrals in equation 3.12 has limit 0 as $N \to \infty$. In this event, we will have

$$S_N(x) \to \frac{1}{2}(f(x+) + f(x-)),$$

as we want to show. To prove that the last integral has limit 0, let

$$g(t) = \frac{f(x+t) - f(x+)}{2\sin(t/2)} \text{ for } 0 < t \leq \pi.$$

Notice that

$$\begin{aligned}
\lim_{t\to 0+} g(t) &= \lim_{t\to 0+} \frac{f(x+t)-f(x+)}{2\sin(t/2)} \\
&= \lim_{t\to 0+} \frac{f(x+t)-f(x+)}{t}\,\frac{t/2}{\sin(t/2)} \\
&= \lim_{t\to 0+} \frac{f(x+t)-f(x+)}{t}\,\lim_{t\to 0+}\frac{t/2}{\sin(t/2)} = f'(x+)\cdot 1 = f'(x+).
\end{aligned}$$

This limit exists because f' is piecewise continuous on $[-\pi, \pi]$. Define

$$g(0) = f'(x+).$$

Because f is piecewise smooth on $[-\pi, \pi]$ and periodic of period 2π, g is piecewise smooth on $[0, \pi]$. By the Riemann - Lebesgue lemma, with $\omega = N + 1/2$,

$$\begin{aligned}
&\lim_{\omega\to\infty} \int_0^{\pi} g(t)\sin(\omega t)\,dt \\
&= \lim_{N\to\infty} \int_0^{\pi} [f(x+t) - f(x+)]\frac{\sin((N+1/2)(t))}{2\sin(t/2)}\,dt = 0.
\end{aligned}$$

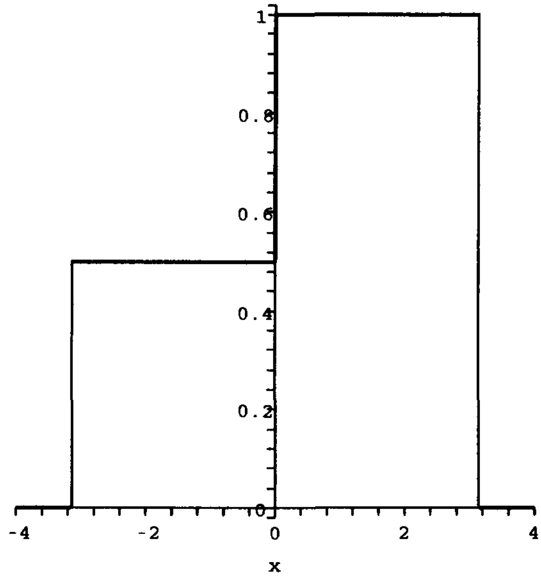

Figure 3.4: $f(x)$ in Example 3.2.

This proves that the last integral in equation 3.12 has limit 0 as $N \to \infty$.

By a similar argument,

$$\lim_{N\to\infty} \int_{-\pi}^{0} [f(x+t) - f(x-)] \frac{\sin((N+1/2)t)}{2\sin(t/2)}\, dt = 0.$$

This proves that

$$\lim_{N\to\infty} S_N(x) = \frac{1}{2}(f(x+) + f(x-)). \diamond$$

Example 3.2 *Let f be periodic of period 2π and let*

$$f(x) = \begin{cases} \frac{1}{2} & \text{for } -\pi < x < 0 \\ 1 & \text{for } 0 \le x \le \pi. \end{cases}$$

A graph of f is shown in Figure 3.4.

A graph of f is shown in Figure 3.4. *The Fourier coefficients are*

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} \frac{1}{2}\, dx + \int_{0}^{\pi} dx\right] = \frac{3}{2};$$

and for $n = 1, 2, \cdots$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} \frac{1}{2}\cos(nx)\, dx + \frac{1}{\pi}\int_{0}^{\pi} \cos(nx)\, dx = 0$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} \frac{1}{2}\sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} \sin(nx)\,dx = \frac{1}{2n\pi}[1 - \cos(n\pi)].$$

The Fourier series of f on $[-\pi, \pi]$ is

$$\frac{3}{4} + \sum_{n=1}^{\infty} \frac{1}{2n\pi}[1 - (-1)^n]\sin(nx),$$

in which we used the fact that $\cos(n\pi) = (-1)^n$ *if n is an integer. Since*

$$1 - (-1)^n = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

we need retain only odd values of n in the Fourier series, which we can write as

$$\frac{3}{4} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)\pi}\sin((2n-1)x).$$

For $0 < x < \pi$, f is continuous and the series converges to $f(x) = 1$. For $-\pi < x < 0$, f is also continuous and the series converges to $f(x) = \frac{1}{2}$. At $x = 0$ the series converges to $\frac{1}{2}(f(0+) + f(0-)) = \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4}$. This is obvious from the series, in which all the sine terms vanish if $x = 0$. At π and at $-\pi$, the series also converges to $\frac{3}{4}$, and this can be verified either by applying the theorem or by again observing that all the sine terms vanish at $\pm\pi$. On $[-\pi, \pi]$, the Fourier series and the function agree except at 0, $-\pi$, and π.

Figures 3.5, 3.6, and 3.7, compare graphs of $f(x)$ with the third, seventh and fifteenth partial sums of this Fourier series, respectively. This provides some insight into how the Fourier series converges. ◇

In this example, the behavior of the partial sums near the point of discontinuity 0 of f is perhaps a little surprising. Near 0, the partial sums have a "jump" which retains approximately the same height no matter how many terms of the partial sum are computed. This is called the *Gibbs phenomenon*, and it is characteristic of the convergence of a Fourier series at a point of discontinuity of the function. The phenomenon is named for the Yale mathematician Josiah Willard Gibbs (1839–1903), who was the first to explain it mathematically, despite the fact that it had been observed by others. The experimental physicist Albert A. Michelson, famed for his experiments on the velocity of light and existence of the aether, had noticed it after he constructed a machine to compute Fourier coefficients and graphed some partial sums.

Change of Scale

This discussion of Fourier series can be recast on an interval $[-L, L]$ by a change of variable in the integrals. Suppose that f is periodic of period $2L$ (so $f(x +$

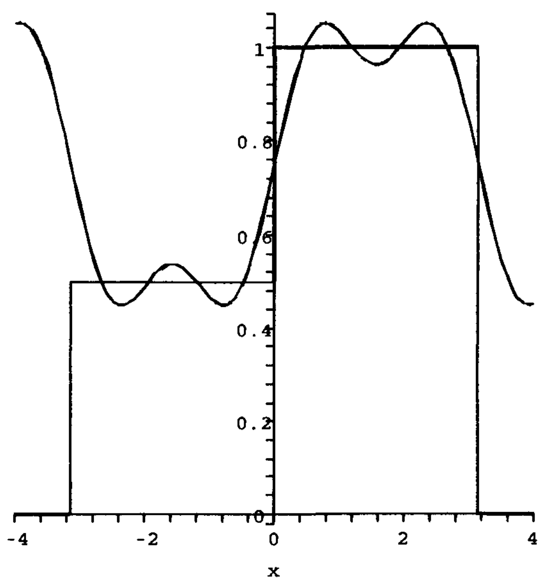

Figure 3.5: $f(x)$ in Example 3.2 and the third partial sum of its Fourier series on $[-\pi, \pi]$.

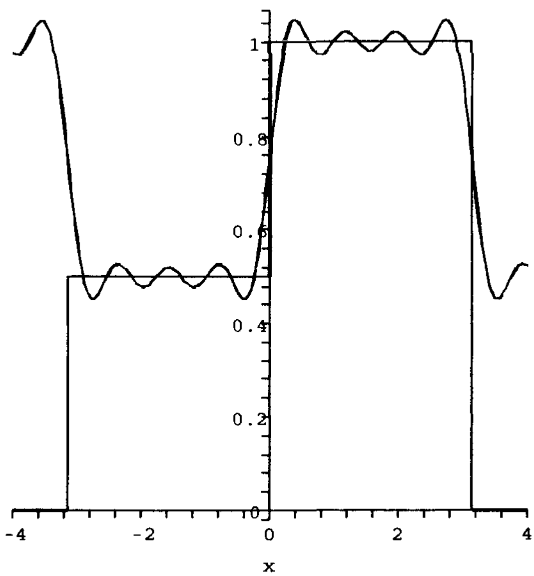

Figure 3.6: $f(x)$ and the seventh partial sum of its Fourier series in Example 3.2.

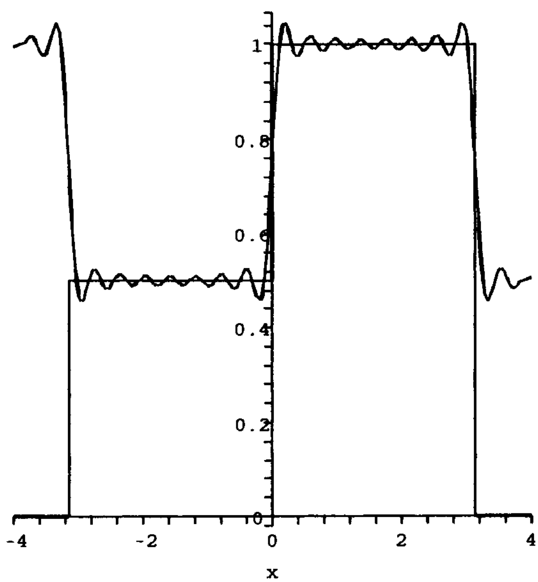

Figure 3.7: $f(x)$ and the fifteenth partial sum of its Fourier series in Example 3.2.

$2L) = f(x))$ and integrable on $[-L, L]$. Since the periodicity condition requires that $f(-L) = f(L)$, we need only define $f(x)$ for $-L < x \leq L$.

The *Fourier coefficients of* f on $[-L, L]$ are

$$a_n = \frac{1}{L}\int_{-L}^{L} f(\xi)\cos\left(\frac{n\pi\xi}{L}\right)\,d\xi \text{ for } n = 0, 1, 2, \ldots$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(\xi)\sin\left(\frac{n\pi\xi}{L}\right)\,d\xi \text{ for } n = 1, 2, \ldots.$$

The *Fourier series* of f on $[-L, L]$ is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right),$$

in which the coefficients are the Fourier coefficients.

The convergence theorem for this series on $[-L, L]$ is an obvious adaptation of the theorem for $[-\pi, \pi]$.

Theorem 3.2 *Let f be piecewise smooth on $[-L, L]$ and periodic of period $2L$. Then at each x the Fourier series of f on $[-L, L]$ converges to*

$$\frac{1}{2}(f(x+) + f(x-)).\diamond$$

Example 3.3 *Let f have period 6 and be defined for $-3 < x \leq 3$ by*

$$f(x) = \begin{cases} 0 & \text{for } -3 < x < 0 \\ x & \text{for } 0 \leq x \leq 3. \end{cases}$$

The Fourier coefficients (with $L = 3$) are computed by routine integrations:

$$a_0 = \frac{1}{3}\int_{-3}^{3} f(x)\,dx = \frac{1}{3}\int_{0}^{3} x\,dx = \frac{3}{2};$$

and for $n = 1, 2, \ldots$,

$$a_n = \frac{1}{3}\int_{0}^{3} x\cos\left(\frac{n\pi x}{3}\right)\,dx = \frac{3}{n^2\pi^2}[(-1)^n - 1]$$

and

$$b_n = \frac{1}{3}\int_{0}^{3} x\sin\left(\frac{n\pi x}{3}\right)\,dx = -\frac{3}{n\pi}(-1)^n.$$

The Fourier series of f on $[-3, 3]$ is

$$\frac{3}{4} + \sum_{n=1}^{\infty}\left[\frac{3}{n^2\pi^2}[(-1)^n - 1]\cos\left(\frac{n\pi x}{3}\right) + \frac{3}{n\pi}(-1)^{n+1}\sin\left(\frac{n\pi x}{3}\right)\right].$$

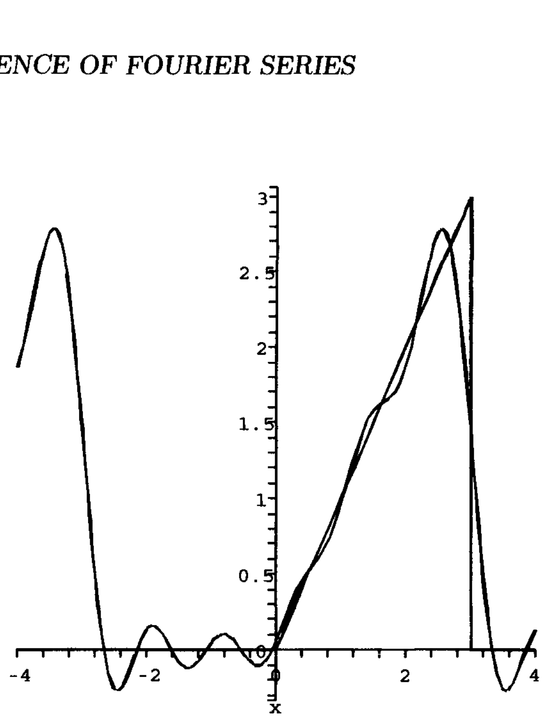

Figure 3.8: $f(x)$in Example 3.3 and the fifth partial sum of its Fourier series on $[-3, 3]$.

This series converges to 0 for $-3 < x \leq 0$, to x for $0 \leq x < 3$, and to $\frac{3}{2}$ for $x = \pm 3$. The last conclusion is obtained by observing that

$$\frac{1}{2}(f(3+) + f(3-)) = \frac{1}{2}(3+0) = \frac{3}{2}$$

and

$$\frac{1}{2}(f(-3+) + f(-3-)) = \frac{1}{2}(0+3) = \frac{3}{2}.$$

The Fourier series in this example represents the function exactly on $(-3, 3)$. This is because f is continuous on $(-3, 3)$, in addition to satisfying the other hypotheses of the theorem. Figure 3.8 compares $f(x)$ to the fifth partial sum of this Fourier series, and Figure 3.9, to the twentieth partial sum. ◇

Problems for Section 3.3

1. Verify equation 3.6.

2. Let f be periodic and differentiable. Prove that f' is also periodic with the same period as f.

3. In each of the following, find the Fourier series of the function on the interval and determine the sum of this series. Graph some partial sums of the Fourier series in comparison with a graph of the function.

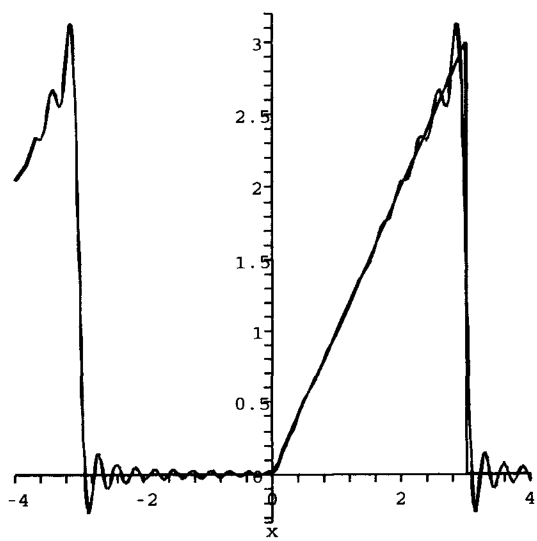

Figure 3.9: $f(x)$ and the twentieth partial sum of its Fourier series in Example 3.3.

(a) $f(x) = -x, -1 < x \leq 1$

(b) $f(x) = \cosh(\pi x), -1 < x \leq 1$

(c) $f(x) = 1 - |x|, -2 < x \leq 2$

(d)
$$f(x) = \begin{cases} -4 & \text{for } -\pi < x < 0 \\ 4 & \text{for } 0 \leq x \leq \pi \end{cases}$$

(e) $f(x) = \sin(2x)$ for $-\pi < x \leq \pi$

(f) $f(x) = x^2 - x + 3$ for $-2 \leq x \leq 2$

(g)
$$f(x) = \begin{cases} -x & \text{for } -5 < x < 0 \\ 1 + x^2 & \text{for } 0 \leq x \leq 5 \end{cases}$$

(h)
$$f(x) = \begin{cases} 1 & \text{for } -\pi < x \leq 0 \\ 2 & \text{for } 0 < x \leq \pi \end{cases}$$

(i) $f(x) = \cos(x/2) - \sin(x)$ for $-\pi < x \leq \pi$

(j)
$$f(x) = \begin{cases} 1 - x & \text{for } -1 < x \leq 0 \\ 0 & \text{for } 0 < x \leq 1 \end{cases}$$

4. In each of the following, determine the sum of the Fourier series of the function on the interval.

(a)
$$f(x) = \begin{cases} 2x & \text{for } -3 < x \leq 0 \\ 0 & \text{for } -2 < x < 1 \\ x^2 & \text{for } 1 \leq x \leq 3 \end{cases}$$

(b)
$$f(x) = \begin{cases} \cos(x) & \text{for } -2 < x < 1/2 \\ \sin(x) & \text{for } 1/2 \leq x \leq 2 \end{cases}$$

(c)
$$f(x) = \begin{cases} -2 & \text{for } -4 < x \leq 2 \\ 1 + x^2 & \text{for } 2 < x < 3 \\ e^{-x} & \text{for } 3 \leq x \leq 4 \end{cases}$$

(d)
$$f(x) = \begin{cases} 1 & \text{for } -2 < x \leq 0 \\ -1 & \text{for } 0 < x < 1/2 \\ x^2 & \text{for } 1/2 \leq x \leq 2 \end{cases}$$

(e)

$$f(x) = \begin{cases} \cos(\pi x) & \text{for } -2 < x < 0 \\ x & \text{for } 0 \le x \le 2 \end{cases}$$

5. Let $f(x) = \frac{1}{2}x^2$ for $-\pi < x \le \pi$. Find the Fourier series of f and sum the series at an appropriately chosen point to determine $\sum_{n=1}^{\infty} 1/n^2$. Use another point to determine $\sum_{n=1}^{\infty}(-1)^n/n$.

6. Prove the following theorem on term-by-term integration of Fourier series. Let f be piecewise continuous on $(-L, L]$ and have Fourier series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right).$$

Then for $-L \le x \le L$,

$$\int_{-L}^{x} f(s)\,ds = \frac{1}{2}a_0(x+L)$$
$$+ \frac{L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[a_n \sin\left(\frac{n\pi x}{L}\right) - b_n\left\{\cos\left(\frac{n\pi x}{L}\right) - \cos(n\pi)\right\}\right].$$

This series on the right is that obtained by integrating the Fourier series for f term by term. This equation is valid even at points at which the Fourier series does not converge to $f(x)$. Hint: Define

$$F(x) = \int_{-L}^{x} f(s)\,ds - \frac{1}{2}a_0 x.$$

Show that F is continuous and F' is piecewise continuous on $[-L, L]$, with $F(-L) = F(L)$. Write the Fourier series of F and relate its coefficients to those of f by integration by parts.

7. Assume that the Fourier series of $f(x)$ on $[-L, L]$ converges to $f(x)$. and can be integrated term by term. Multiply

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

by $f(x)$ and integrate the resulting equation from $-L$ to L to derive *Parseval's equation*,

$$\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{L}\int_{-L}^{L} f(x)^2\,dx.$$

8. Let $f(x) = x^2$ for $[-L, L]$. Write the Fourier series for $f(x)$. Use Parseval's equation (Problem 7) to show that

$$\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^4}{90}.$$

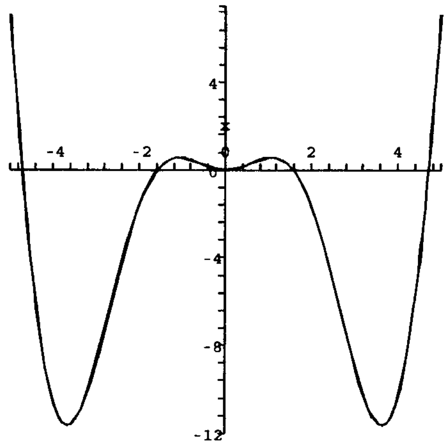

Figure 3.10: an even function.

3.4 Sine and Cosine Expansions

Even and Odd Functions

Suppose that a function g is defined on $[-L, L]$.

We say that g is an *even function* if $g(-x) = g(x)$ for $0 < x \leq L$. The graph of an even function is symmetric about the y - axis. Figure 3.10 shows a typical even function. Examples of even functions are x^n for any even positive integer n, e^{-x^2} and $\cos(n\pi x/L)$ for any integer n.

We say that g is an *odd function* if $g(-x) = -g(x)$ for $0 < x \leq L$. The graph of an odd function is symmetric through the origin. Figure 3.11 shows a typical odd function. Examples of odd functions are x^n for any odd positive integer and $\sin(n\pi x/L)$ for any nonzero integer n. A product of two even functions is even, a product of two odd functions is even, and a product of an odd and an even function is odd.

In calculus texts, it is sometimes pointed out that if g is even on $[-L, L]$, then

$$\int_{-L}^{L} g(x)\,dx = 2\int_{0}^{L} g(x)\,dx; \tag{3.13}$$

and if g is odd on $[-L, L]$, then

$$\int_{-L}^{L} g(x)\,dx = 0. \tag{3.14}$$

These facts are apparent from Figures 3.10 and 3.11 and are straightforward to prove from properties of integrals.

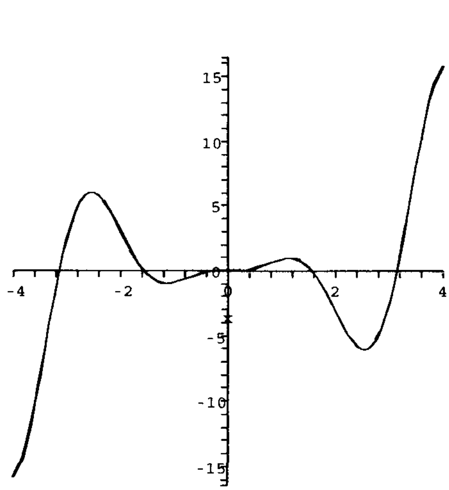

Figure 3.11: an odd function.

We will exploit these properties of even and odd functions to develop what are called *half-range Fourier expansions*, in which we can expand a function defined on $[0, L]$ (and satisfying certain conditions) in two different series, one containing only sine terms and the other containing only cosine terms. The key to these expansions lies in the following observations.

(1) If g is odd on $[-L, L]$, then the Fourier series for g on $[-L, L]$ contains only sine terms. All the coefficients of the cosine terms vanish, since $g(x)\cos(n\pi x/L)$ is odd (product of odd with even), so

$$a_n = \frac{1}{L}\int_{-L}^{L} g(x)\cos\left(\frac{n\pi x}{L}\right)\,dx = 0$$

for $n = 1, 2, \cdots$. Further,

$$a_0 = \frac{1}{L}\int_{-L}^{L} g(x)\,dx = 0.$$

(2) If g is even on $[-L, L]$, the Fourier series for g on $[-L, L]$ contains only cosine terms. All the coefficients of the sine terms now vanish, because $g(x)\sin(n\pi x/L)$ is odd (product of even with odd), so

$$b_n = \frac{1}{L}\int_{-L}^{L} g(x)\sin\left(\frac{n\pi x}{L}\right)\,dx = 0.$$

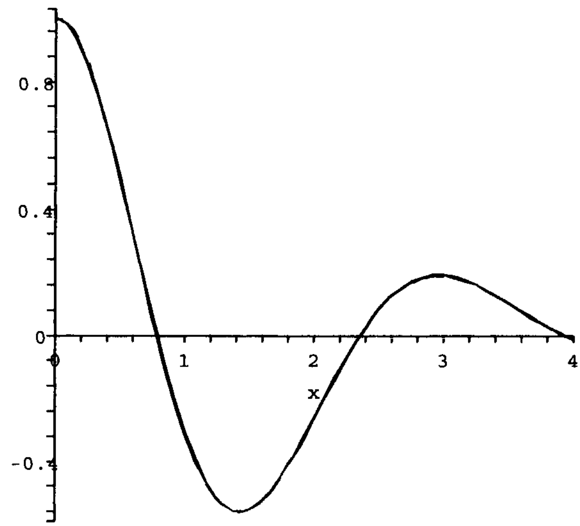

Figure 3.12: a function defined on $[0, L]$.

Fourier Sine Series

Suppose that f is integrable on $[0,\ L]$. We will show how to write a Fourier sine series for f on $[0,\ L]$. Figure 3.12 shows a typical f. Extend f to an odd function h defined on $[-L, L]$ by putting

$$h(x) = \begin{cases} f(x) & \text{for } 0 \leq x < L \\ -f(-x) & \text{for } -L < x < 0. \end{cases}$$

This is shown in Figure 3.13 for the graph of Figure 3.12. Since h is odd, its Fourier series on $[-L, L]$ contains only sine terms. Since $h(x) = f(x)$ for $0 \leq x \leq L$, this gives a Fourier sine representation for $f(x)$ on $[0,\ L]$.

Further, the coefficients in this expansion are

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} h(x)\sin\left(\frac{n\pi x}{L}\right)\, dx \\ &= \frac{2}{L}\int_{0}^{L} h(x)\sin\left(\frac{n\pi x}{L}\right)\, dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx. \end{aligned}$$

The introduction of the odd extension h of f to $[-L, L]$ is an artificial construction enabling us to obtain the sine series of f on $[0,\ L]$, and its coefficients, from a Fourier series on $[-L, L]$. We do not actually make this construction

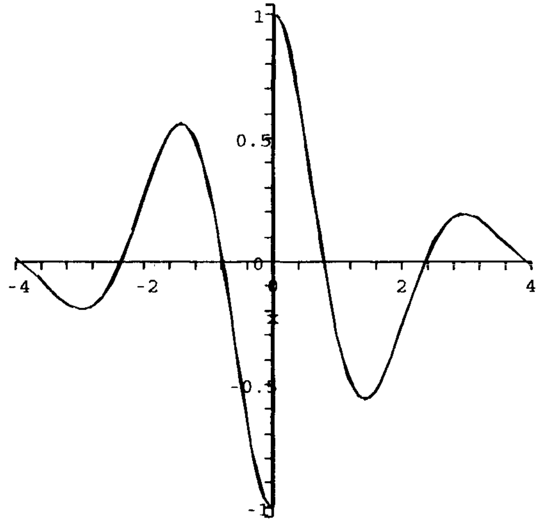

Figure 3.13: Odd extension of the function of Figure 3.12.

each time we want a sine series. Given $f(x)$ for $0 \leq x \leq L$, the *Fourier sine series for f(x) on* $[0, L]$ is

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

Further, applying the convergence theorem for Fourier series to h, we have the following convergence result for sine series: If f is piecewise smooth on $[0, L]$, the sine series for $f(x)$ on $[0, L]$ converges to 0 at $x = 0$ and at $x = L$, and to

$$\frac{1}{2}(f(x+) + f(x-))$$

at each x in $(0, L)$.

Example 3.4 *Let $f(x) = e^x$ for $0 \leq x \leq 2$. The Fourier sine series of e^x on $[0, 2]$ converges to 0 at $x = 0$ and $x = 2$ and to e^x for $0 < x < 2$. The coefficients in the sine expansion are*

$$b_n = \frac{2}{2}\int_0^2 e^{\xi}\sin\left(\frac{n\pi\xi}{2}\right)\,d\xi = \frac{2n\pi}{4 + n^2\pi^2}[1 - (-1)^n e^2].$$

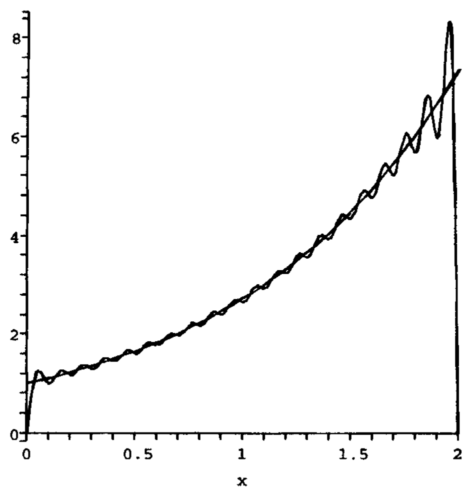

Figure 3.14: The function and the fortieth partial sum of its sine series in Example 3.4.

For $0 < x < 2$, *then,*

$$e^x = \sum_{n=1}^{\infty} \frac{2n\pi}{4 + n^2\pi^2}[1 - (-1)^n e^2] \sin\left(\frac{n\pi x}{2}\right).$$

Figure 3.14 compares a graph of f *with the fortieth partial sum of this sine series.* ◇

Fourier Cosine Series

We can write a Fourier cosine series of a function f defined on $[0, L]$ by first extending f to an even function g defined on $[-L, L]$:

$$g(x) = \begin{cases} f(x) & \text{for } 0 \le x < L \\ f(-x) & \text{for } -L \le x \le 0. \end{cases}$$

Figure 3.15 shows such an extension for the function of Figure 3.12. Then the Fourier series for g on $[-L, L]$ contains only cosine terms and perhaps a constant term. Since $f(x) = g(x)$ for $0 \le x \le L$, this gives a cosine series for $f(x)$ on $[0, L]$. Further, the coefficients are

$$a_n = \frac{1}{L}\int_{-L}^{L} g(\xi)\cos\left(\frac{n\pi\xi}{L}\right) d\xi = \frac{2}{L}\int_0^L f(\xi)\cos\left(\frac{n\pi\xi}{L}\right) d\xi.$$

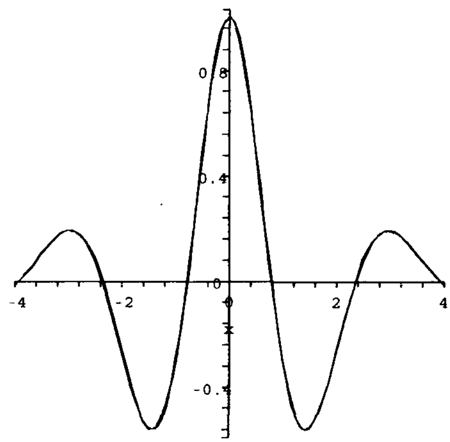

Figure 3.15: Even extension of the function of Figure 3.12.

Again, we need not explicitly introduce the even extension g of f to $[-L, L]$. This is a device enabling us to use a Fourier series on $[-L, L]$ to obtain the coefficients in the Fourier cosine series of f on $[0, L]$.

In sum, the *Fourier cosine series* of f on $[0, L]$ is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

where

$$a_n = \frac{2}{L}\int_0^L f(\xi)\cos\left(\frac{n\pi\xi}{L}\right)\, d\xi$$

for $n = 0, 1, 2, \cdots$.

If we apply the Fourier convergence theorem to g on $[-L, L]$, we obtain a convergence theorem for Fourier cosine series. If f is piecewise smooth on $[0, L]$, its Fourier cosine series converges to

$$\frac{1}{2}(f(x+) + f(x-))$$

for $0 < x < L$. At $x = 0$, the cosine series converges to $f(0+)$, and at $x = L$, to $f(L-)$.

To understand this conclusion about the convergence at 0 and L, apply the Fourier convergence theorem to the even periodic extension g of f. At 0 the cosine series of f (which is the Fourier series of g on $[-L, L]$) converges to

$$\frac{1}{2}(g(0+) + g(0-)).$$

But

$$g(0+) = \lim_{x\to 0+} g(x) = \lim_{x\to 0+} f(x) = f(0+)$$

and

$$g(0-) = \lim_{x\to 0-} g(x) = \lim_{x\to 0+} g(-x) = \lim_{x\to 0+} f(x) = f(0+).$$

Therefore, at 0 the cosine series converges to

$$\frac{1}{2}(f(0+) + f(0+)),$$

which is $f(0+)$.

Similar reasoning proves that the cosine series converges to $f(L-)$ at $x = L$.

Example 3.5 *Let $f(x) = e^x$ for $0 \le x \le 2$. The cosine series on $[0,2]$ converges to e^x for $0 \le x \le 2$. To write this series, evaluate*

$$a_0 = \int_0^2 e^x\,dx = e^2 - 1$$

and for $n = 1, 2, \cdots$,

$$a_n = \int_0^2 e^x \cos\left(\frac{n\pi x}{2}\right)dx = \frac{4}{4+n^2\pi^2}[(-1)^n e^2 - 1].$$

Therefore, for $0 \le x \le 2$,

$$e^x = \frac{1}{2}(e^2-1) + \sum_{n=1}^{\infty} \frac{4}{4+n^2\pi^2}[(-1)^n e^2 - 1]\cos\left(\frac{n\pi x}{2}\right).$$

Figure 3.16 compares a graph of f with the eighth partial sum of this cosine expansion. ◇

Problems for Section 3.4

In each of Problems 1 through 8, write the Fourier sine and the Fourier cosine series of the function. In each case determine the sum of the series and graph some partial sums of the series to compare with the graph of the function.

1. $f(x) = 4$ for $0 \le x \le 3$

2. $$f(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1 \\ -1 & \text{for } 1 < x \le 2 \end{cases}$$

3. $$f(x) = \begin{cases} 0 & \text{for } 0 \le x \le \pi/2 \\ \sin(x) & \text{for } \pi/2 < x \le \pi \end{cases}$$

4. $f(x) = 2x$ for $0 \le x \le 1$

5. $f(x) = x^2$ for $0 \le x \le 2$

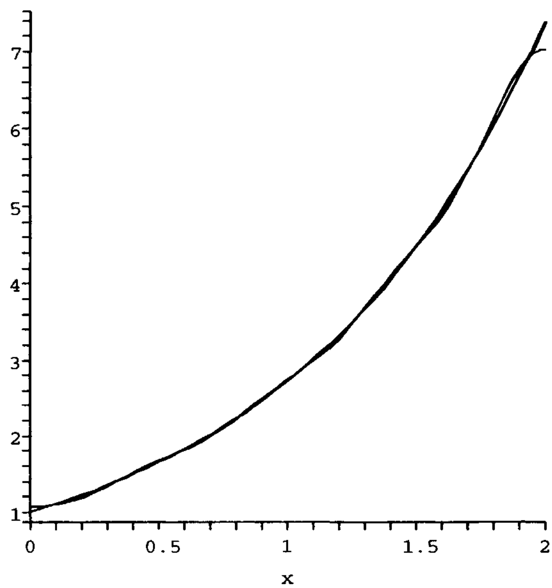

Figure 3.16: f and the eighth partial sum of its cosine expansion in Example 3.5.

6. $f(x) = e^{-x}$ for $0 \le x \le 1$

7. $f(x) = \sin(3x)$ for $0 \le x \le \pi$

8. $$f(x) = \begin{cases} x & \text{for } 0 \le x \le 1 \\ 2 - x & \text{for } 1 \le x \le 2 \end{cases}$$

9. Find the sum of the series $\sum_{n=1}^{\infty} (-1)^n/(4n^2 - 1)$. *Hint*: Expand $\sin(x)$ in a cosine series on $[0, \pi]$ and evaluate this series at an appropriately chosen point.

10. Here is another way of approaching the coefficients in the sine expansion of f on $[0, L]$. Suppose that
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$
Multiply this equation by $\sin(k\pi x/L)$ and integrate the resulting equation from 0 to L, integrating the series term by term. Solve the resulting equation for b_k to obtain the Fourier sine coefficient of f on $[0, L]$.

11. Use the idea of Problem 10 to derive the coefficients in the cosine expansion
$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right).$$
Integrate this equation term by term to obtain a formula for a_0.

3.5 The Fourier Integral

Suppose that f is piecewise smooth on every interval $[L, L]$, and that $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges. In this event we say that f is *absolutely integrable*. If f is not periodic we cannot write $f(x)$ as a Fourier series for all real x. In such a case we may be able to represent f by a Fourier integral, which we will now develop informally.

For any $L > 0$, we can write the Fourier series of f on $[-L, L]$:
$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right).$$
Insert the Fourier coefficients to write this series as
$$\frac{1}{2L}\int_{-L}^{L} f(\xi)\, d\xi + \sum_{n=1}^{\infty}\left[\frac{1}{L}\int_{-L}^{L} f(\xi)\cos\left(\frac{n\pi\xi}{L}\right)\cos\left(\frac{n\pi x}{L}\right) d\xi \right.$$
$$\left. + \frac{1}{L}\int_{-L}^{L} f(\xi)\sin\left(\frac{n\pi\xi}{L}\right)\sin\left(\frac{n\pi x}{L}\right) d\xi\right].$$

We would like to allow $L \to \infty$ to obtain a representation over the entire real line. To determine this limit, let

$$\omega_n = \frac{n\pi}{L} \text{ and } \Delta\omega = \omega_n - \omega_{n-1} = \frac{\pi}{L}.$$

Upon substituting these quantities, the Fourier series of f on $[-L, L]$ becomes

$$\begin{aligned}
&\frac{1}{2\pi}\left(\int_{-L}^{L} f(\xi)\,d\xi\right)\Delta\omega \\
&+\frac{1}{\pi}\sum_{n=1}^{\infty}\left[\left(\int_{-L}^{L} f(\xi)\cos(\omega_n\xi)\,d\xi\right)\cos(\omega_n x)\right. \\
&+\left.\left(\int_{-L}^{L} f(\xi)\sin(\omega_n\xi)\,d\xi\right)\sin(\omega_n x)\right]\Delta\omega.
\end{aligned} \tag{3.15}$$

Now let $L \to \infty$. Then $\Delta\omega \to 0$ and

$$\frac{1}{2\pi}\left(\int_{-L}^{L} f(\xi)\,d\xi\right)\Delta\omega \to 0$$

because $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges to a finite value, by assumption. The other terms in expression 3.15 resemble a Riemann sum for a definite integral. On the basis of this expression, we conjecture that in the limit as $L \to \infty$, expression 3.15 approaches the limit

$$\begin{aligned}
&\frac{1}{\pi}\int_{0}^{\infty}\left[\left(\int_{-\infty}^{\infty} f(\xi)\cos(\omega\xi)\,d\xi\right)\cos(\omega x)\right. \\
&+\left.\left(\int_{-\infty}^{\infty} f(\xi)\sin(\omega\xi)\,d\xi\right)\sin(\omega x)\right]d\omega.
\end{aligned} \tag{3.16}$$

This is the Fourier integral representation of f, and it is possible to prove that under the assumptions made about f, it is equal to

$$\frac{1}{2}(f(x+) + f(x-))$$

for each real x. This conclusion implies that at any point x where the piecewise smooth f is continuous, the Fourier integral 3.16 converges to $f(x)$. We may therefore write this integral representation as

$$\frac{1}{2}(f(x+) + f(x-)) = \int_{0}^{\infty}[A_\omega\cos(\omega x) + B_\omega\sin(\omega x)]\,d\omega,$$

where

$$A_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos(\omega\xi)\,d\xi$$

and

$$B_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\sin(\omega\xi)\,d\xi.$$

These functions of ω are the *Fourier integral coefficients* of f.

By combining terms in equation 3.16 and using a trigonometric identity (similar to the derivation of equation 3.7), this Fourier integral expansion can be written as

$$\frac{1}{2}(f(x+) + f(x-)) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(\xi)\cos(\omega(\xi - x))\,d\xi\,d\omega. \tag{3.17}$$

Example 3.6 *Let*

$$f(x) = \begin{cases} 0 & \text{if } |\, x \,| < 1 \\ 1 & \text{if } -1 \le x \le 1. \end{cases}$$

Certainly, f is piecewise smooth for all x, and absolutely integrable. The Fourier integral coefficients are

$$A_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos(\omega\xi)\,d\xi = \frac{1}{\pi}\int_{-1}^{1}\cos(\omega\xi)\,d\xi = \frac{2\sin(\omega)}{\pi\omega}$$

and

$$B_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\sin(\omega\xi)\,d\xi = 0$$

because f is an even function, so $f(\xi)\sin(\omega\xi)$ is odd. For $-\infty < x < \infty$,

$$\frac{1}{2}(f(x+) + f(x-)) = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin(\omega)}{\omega}\cos(\omega x)\,d\omega.$$

More explicitly,

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin(\omega)}{\omega}d\omega = \begin{cases} 1 & \text{for } -1 < x < 1 \\ 1/2 & \text{for } x = \pm 1 \\ 0 & \text{for } |\, x \,| > 1. \diamond \end{cases}$$

Fourier Cosine and Sine Integrals

Suppose that f is piecewise smooth on $[0, \infty)$ and $\int_0^{\infty} |f(x)|\,dx$ converges. On this half-line, we can expand f in a Fourier sine or cosine integral, whichever we choose, by reasoning as we did for the Fourier sine and cosine series on an interval $[0, L]$ (that is, by making odd or even extensions of f to the entire real line). Without repeating the details, we will summarize the conclusions.

The *Fourier sine integral representation of f* on $[0, \infty)$ is

$$\int_0^{\infty} B_\omega \sin(\omega x)\,d\omega,$$

where

$$B_\omega = \frac{2}{\pi}\int_0^\infty f(\xi)\sin(\omega\xi)\,d\xi.$$

Under the stated conditions on f, this sine integral converges to 0 at $x = 0$ and to

$$\frac{1}{2}(f(x+) + f(x-))$$

for $x > 0$. At any $x > 0$ at which f is continuous, the Fourier sine integral converges to $f(x)$.

The *Fourier cosine integral representation* of f on $[0, \infty)$ is

$$\int_0^\infty A_\omega \cos(\omega x)\,d\omega$$

with

$$A_\omega = \frac{2}{\pi}\int_0^\infty f(\xi)\cos(\omega\xi)\,d\xi.$$

The Fourier cosine integral converges to

$$\frac{1}{2}(f(x+) + f(x-))$$

for $x > 0$ and to $f(0+)$ at $x = 0$. At any $x > 0$ at which f is continuous, the cosine integral also converges to $f(x)$.

Example 3.7 *Let* $f(x) = e^{-kx}$ *for* $x \geq 0$, *with* k *any positive number. Now*

$$\int_0^\infty |f(x)|\,dx = \int_0^\infty e^{-kx}\,dx = \frac{1}{k},$$

so f *is absolutely integrable on* $[0, \infty)$. *Further,* f *is piecewise smooth, in fact, continuous for* $x \geq 0$.

For the sine integral expansion, compute

$$B_\omega = \frac{2}{\pi}\int_0^\infty e^{-k\xi}\sin(\omega\xi)\,d\xi = \frac{2}{\pi}\frac{\omega}{k^2+\omega^2}.$$

For $x > 0$, *the sine integral representation is*

$$e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{\omega}{k^2+\omega^2}\sin(\omega x)\,d\omega. \tag{3.18}$$

For the cosine integral expansion, compute

$$A_\omega = \frac{2}{\pi}\int_0^\infty e^{-k\xi}\cos(\omega\xi)\,d\xi = \frac{2}{\pi}\frac{k}{k^2+\omega^2}.$$

Since $f(0+) = f(0)$, *then for all* $x \geq 0$,

$$e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{1}{k^2+\omega^2}\cos(\omega x)\,d\omega. \tag{3.19}$$

The cosine integral converges to e^{-kx} for $x \geq 0$, while the sine integral converges to e^{-kx} for $x > 0$. The integrals 3.18 and 3.19 are called Laplace's integrals because the coefficient A_ω in the cosine integral is $2/\pi$ times the Laplace transform of $\sin(\omega x)$, *while B_ω in the sine integral is $2/\pi$ times the Laplace transform of* $\cos(\omega x)$. ◇

Problems for Section 3.5

In each of Problems 1 through 6, determine the Fourier integral representation of f, and what this representation converges to.

1.
$$f(x) = \begin{cases} x & \text{for } -\pi \leq x \leq \pi \\ 0 & \text{for } |x| > \pi \end{cases}$$

2.
$$f(x) = \begin{cases} \cos(x) & \text{for } -\alpha \leq x \leq \alpha \\ 0 & \text{for } |x| > \alpha \end{cases}$$
with α a positive constant

3. $f(x) = e^{-|x|}$

4. $f(x) = xe^{-|x|}$

5.
$$f(x) = \begin{cases} |x| & \text{for } -\alpha \leq x \leq \alpha \\ 0 & \text{for } |x| > \alpha \end{cases}$$

6.
$$f(x) = \begin{cases} k & \text{for } 0 < x \leq \alpha \\ -k & \text{for } -\alpha \leq x < 0 \\ 0 & \text{for } |x| > \alpha \end{cases}$$
with k and α positive constants

7. Obtain the integral in equation 3.17 from the Fourier integral representation 3.16.

8. Show that the Fourier integral representation of f can be written
$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\omega(x-\xi)} f(\xi)\, d\xi\, d\omega.$$

In each of Problems 9 through 14, write the Fourier sine integral and the Fourier cosine integral representation of the function. Determine what each integral converges to.

9.
$$f(x) = \begin{cases} \sinh(x) & \text{for } 0 \leq x \leq k \\ 0 & \text{for } x > k \end{cases}$$
with k a positive constant

10.
$$f(x) = \begin{cases} \cos(\pi x) & \text{for } 0 \le x \le 1 \\ 0 & \text{for } x > 1 \end{cases}$$

11. $f(x) = e^{-x}\cos(x)$

12. $f(x) = xe^{-3x}$

13.
$$f(x) = \begin{cases} k & \text{for } 0 \le x \le \alpha \\ 0 & \text{for } x > \alpha \end{cases}$$
with k constant and α a positive constant

14.
$$f(x) = \begin{cases} x^2 & \text{for } 0 \le x \le \alpha \\ 0 & \text{for } x > \alpha \end{cases}$$
with α a positive constant

15. Use the Laplace integrals (see Example 3.7) to compute the Fourier cosine integral of
$$f(x) = \frac{1}{1+x^2}$$
and the Fourier sine integral of
$$g(x) = \frac{x}{1+x^2}.$$

16. Suppose that
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} f'(x) = 0$$
and that $f''(x)$ exists for $x \ge 0$ and $f'(0) = 0$. Show that the integrand in the Fourier cosine integral representation of $f''(x)$ is equal to $-\omega^2$ times the integrand in the Fourier cosine integral representation of $f(x)$.

17. Assume that
$$f(x) = \frac{1}{\pi}\int_0^\infty A(\omega)\cos(\omega x)\,d\omega,$$
in which
$$A(\omega) = \int_{-\infty}^\infty f(t)\cos(\omega t)\,dt.$$

(a) Show that
$$x^2 f(x) = \frac{1}{\pi}\int_0^\infty A^*(\omega)\cos(\omega x)\,d\omega,$$
in which $A^*(\omega) = -A''(\omega)$.

(b) Show that
$$xf(x) = \frac{1}{\pi}\int_0^\infty B^*(\omega)\sin(\omega x)\,d\omega,$$
where $B^*(\omega) = -A'(\omega)$.

3.6 The Fourier Transform

The Fourier transform also plays an important role in partial differential equations. We develop this transform through a complex form of the Fourier integral. Begin with Euler's formula,

$$e^{it} = \cos(t) + i\sin(t).$$

Replacing t with $-t$, we have

$$e^{-it} = \cos(t) - i\sin(t).$$

Solve these equations for $\cos(t)$ to obtain

$$\cos(t) = \frac{1}{2}(e^{it} + e^{-it}).$$

Now suppose that f is piecewise smooth on each interval $[-L, L]$ and $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges. From equation 3.17,

$$\frac{1}{2}(f(x+) + f(x-)) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(\xi)\cos(\omega(\xi - x))\,d\xi\,d\omega$$

for all x. Insert

$$\cos(\omega(\xi - x)) = \frac{1}{2}(e^{i\omega(\xi-x)} + e^{-i\omega(\xi-x)})$$

into this equation to obtain

$$\begin{aligned}
&\frac{1}{2}(f(x+) + f(x-)) \\
&= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(\xi)\frac{1}{2}(e^{i\omega(\xi-x)} + e^{-i\omega(\xi-x)})\,d\xi\,d\omega \\
&= \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^{\infty} f(\xi)e^{i\omega(\xi-x)}\,d\xi\,d\omega + \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^{\infty} f(\xi)e^{-i\omega(\xi-x)}\,d\xi\,d\omega. \quad (3.20)
\end{aligned}$$

In the first integral on the right side of equation 3.20, put $\omega = -w$:

$$\begin{aligned}
&\int_0^\infty \int_{-\infty}^{\infty} f(\xi)e^{i\omega(\xi-x)}\,d\xi\,d\omega \\
&= \int_0^{-\infty} \int_{-\infty}^{\infty} f(\xi)e^{-iw(\xi-x)}\,d\xi(-1)\,dw \\
&= \int_{-\infty}^{0} \int_{-\infty}^{\infty} f(\xi)e^{-iw(\xi-x)}\,d\xi\,dw.
\end{aligned}$$

Substitute this into equation 3.20, but replace the variable of integration w with ω to have the same variable of integration in both integrals. We obtain

$$\begin{aligned}
&\frac{1}{2}(f(x+) + f(x-)) \\
&= \frac{1}{2\pi}\int_{-\infty}^{0}\int_{-\infty}^{\infty} f(\xi)e^{-i\omega(\xi-x)}\,d\xi\,d\omega \\
&+ \frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(\xi)e^{-i\omega(\xi-x)}\,d\xi\,d\omega.
\end{aligned}$$

Finally, combine the integrals with respect to ω, to write

$$\frac{1}{2}(f(x+) + f(x-)) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)e^{-i\omega(\xi-x)}\,d\xi\,d\omega. \tag{3.21}$$

This is the complex Fourier integral representation of f, and it serves as a springboard to the Fourier transform. Write equation 3.21 as

$$\frac{1}{2}(f(x+) + f(x-)) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(\xi)e^{-i\omega\xi}\,d\xi\right)e^{i\omega x}\,d\omega. \tag{3.22}$$

Define

$$\mathcal{F}[f](\omega) = \int_{-\infty}^{\infty} f(\xi)e^{-i\omega\xi}\,d\xi. \tag{3.23}$$

$\mathcal{F}[f]$ is the *Fourier transform* of f. This is a function of ω and is defined for all ω such that this integral converges.

As we might expect with a transform defined by an integral, the Fourier transform is linear. This means that if f and g have Fourier transforms, and α and β are real numbers, then

$$\mathcal{F}[\alpha f + \beta g] = \alpha\mathcal{F}[f] + \beta\mathcal{F}[g].$$

It is often convenient to use the alternative notation of denoting the Fourier transform of a function by a carat over the function:

$$\mathcal{F}[f] = \widehat{f}.$$

In this notation,

$$\mathcal{F}[f](\omega) = \widehat{f}(\omega).$$

Example 3.8 *Let*

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ e^{-\alpha x} & \text{for } x \geq 0. \end{cases}$$

in which α is a positive number. The Fourier transform of f is

$$\begin{aligned}
\widehat{f}(\omega) &= \int_{-\infty}^{\infty} f(\xi)e^{-i\omega\xi}\,d\xi = \int_{0}^{\infty} e^{at}e^{-i\omega\xi}\,d\xi \\
&= \int_{0}^{\infty} e^{-(a+i\omega)\xi}\,d\xi = \lim_{r\to\infty}\int_{0}^{r} e^{-(a+i\omega)\xi}\,d\xi \\
&= \lim_{r\to\infty}\frac{1}{a+i\omega}[1 - e^{-(a+i\omega)r}] = \frac{1}{a+i\omega}.
\end{aligned}$$

In this limit,

$$e^{-(a+i\omega)r} = e^{-ar}[\cos(\omega r) - i\sin(\omega r)] \to 0$$

as $r \to \infty$ because $a > 0$. ◇

Substituting $\mathcal{F}[f](\omega) = \widehat{f}(\omega)$ into equation 3.22 yields

$$\frac{1}{2}(f(x+) + f(x-)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)e^{i\omega x}\, d\omega. \tag{3.24}$$

Equations 3.23 and 3.24 define a *Fourier transform pair*. Given f satisfying certain conditions, we can in theory compute its Fourier transform $\mathcal{F}[f]$ by integration (equation 3.23). Going the other way, equation 3.24 enables us to recover a function from its Fourier transform.

This recovery process has some ambiguity, since there are many functions that have the same Fourier transform. For example, if we change the value of $f(x)$ at finitely many points to obtain a new function, both functions will have the same Fourier transform. However, if we begin with a function f, compute its Fourier transform $\widehat{f}$, and then apply equation 3.24 to $\widehat{f}$, we will recover $f(x)$ at each x where f is continuous. We state the following result without proof.

Theorem 3.3 *Let f and g be real- or complex-valued functions which are continuous on the real line, and suppose that f and g are absolutely integrable. Suppose that $\widehat{f} = \widehat{g}$. Then $f = g$.* ◇

If we use $\mathcal{F}$ for the Fourier transform operation, the inverse transform is denoted $\mathcal{F}^{-1}$. In this notation, $\mathcal{F}[f] = F$ if $\mathcal{F}^{-1}[F] = f$ and the Fourier transform pair can be written

$$\mathcal{F}^{-1}[F](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}\, d\omega = f(x)$$

if

$$\mathcal{F}[f](\omega) = \int_{-\infty}^{\infty} f(\xi)e^{-i\omega\xi}\, d\xi = F(\omega),$$

at least for the case that f is continuous. In Example 3.8,

$$\mathcal{F}[f](\omega) = \frac{1}{a+i\omega} \text{ and } \mathcal{F}^{-1}\left[\frac{1}{a+i\omega}\right](x) = f(x).$$

In practice, both Fourier transforms and inverse Fourier transforms of functions are often computed using a table or a computer routine.

When we use the Fourier transform to solve differential equations, we often exploit the following operational property. It states that under certain conditions, the Fourier transform of the nth derivative of f, is the nth power of $i\omega$, multiplied by the Fourier transform of f. In the theorem, $f^{(k)}(x)$ is the kth derivative of $f(x)$, and $f^{(0)}(x) = f(x)$.

Theorem 3.4 (*Operational Formula for the Fourier Transform*) *Let n be a positive integer. Suppose $f^{(n)}(x)$ is piecewise continuous on the real line and that $\int_{-\infty}^{\infty} |f^{(n-1)}(x)|\, dx$ converges. Assume also that*

$$\lim_{x\to\infty} f^{(k)}(x) = \lim_{x\to-\infty} f^{(k)}(x) = 0$$

for $k = 0, 1, \cdots, n-1$. *Then*

$$\mathcal{F}[f^{(n)}](\omega) = (i\omega)^n \widehat{f}(\omega). \diamond \tag{3.25}$$

We conclude this section with two properties of the transform which are often useful. Other properties are explored in the problems.

Theorem 3.5 (*scaling*) *Let f be continuous on the real line and suppose that $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges. Let k be a positive number. For real x, denote $f_k(x) = kf(kx)$. Then*

$$\widehat{f_k}(\omega) = \widehat{f}(\omega/k). \diamond$$

Theorem 3.6 (*time shifting*) *Let f be continuous on the real line and suppose $\int_{-\infty}^{\infty} |f(x)| dx$ converges. For a given real number t, define $g(x) = f(x-t)$. Then*

$$\widehat{g}(\omega) = e^{-i\omega t}\widehat{f}(\omega). \diamond$$

Proof of Theorem 3.5 This conclusion is proved by a simple change of variable. First,

$$\begin{aligned}\widehat{f_k}(\omega) &= \int_{-\infty}^{\infty} f_k(\xi) e^{-i\omega\xi}\, d\xi \\ &= \int_{-\infty}^{\infty} kf(k\xi) e^{-i\omega\xi}\, d\xi.\end{aligned}$$

Upon letting $z = k\xi$, we obtain

$$\begin{aligned}\widehat{f_k}(\omega) &= \int_{-\infty}^{\infty} kf(z) e^{-i\omega z/k} \frac{1}{k}\, dz \\ &= \int_{-\infty}^{\infty} f(z) e^{-i\omega z/k}\, dz = \widehat{f}(\omega/k). \diamond\end{aligned}$$

Proof of Theorem 3.6 This is also a simple change of variable. Compute

$$\widehat{g}(\omega) = \int_{-\infty}^{\infty} g(\xi) e^{-i\omega\xi}\, d\xi = \int_{-\infty}^{\infty} f(\xi - t) e^{-i\omega\xi}\, d\xi.$$

Let $z = \xi - t$, to obtain

$$\begin{aligned}\widehat{g}(\omega) &= \int_{-\infty}^{\infty} f(z) e^{-i\omega(z+t)}\, dz \\ &= e^{-i\omega t} \int_{-\infty}^{\infty} f(z) e^{-i\omega z}\, dz = e^{-i\omega t}\widehat{f}(\omega). \diamond\end{aligned}$$

Problems for Section 3.6

1. Prove the operational formula for the Fourier transform. Hint: Use integration by parts.

   In Problems 2 through 7, find the complex Fourier integral representation of the function. In these problems k denotes a constant and α a positive constant.

2.
$$f(x) = \begin{cases} 1 & \text{for } -\pi \le x \le \pi \\ 0 & \text{for } \mid x \mid > \pi \end{cases}$$

3.
$$f(x) = \begin{cases} \cos(x) & \text{for } -\alpha \le x \le \alpha \\ 0 & \text{for } \mid x \mid > \alpha \end{cases}$$

4. $f(x) = e^{-|x|}$

5. $f(x) = xe^{-|x|}$

6.
$$f(x) = \begin{cases} \mid x \mid & \text{for } -\alpha \le x \le \alpha \\ 0 & \text{for } \mid x \mid > \alpha \end{cases}$$

7.
$$f(x) = \begin{cases} k & \text{for } 0 < x \le \alpha \\ -k & \text{for } -\alpha \le x < 0 \\ 0 & \text{for } \mid x \mid > \alpha \end{cases}$$

In each of Problems 8 through 13, find the Fourier transform of f. Here k, a, b and α are constants, with $a < b$ and $\alpha > 0$.

8.
$$f(x) = \begin{cases} k & \text{for } -\alpha \le x \le \alpha \\ 0 & \text{for } \mid x \mid > \alpha \end{cases}$$

9.
$$f(x) = \begin{cases} k & \text{for } a \le x \le b \\ 0 & \text{for } x < a \text{ and for } x > b \end{cases}$$

10.
$$f(x) = \begin{cases} \sin(x) & \text{for } -\alpha \le x \le \alpha \\ 0 & \text{for } \mid x \mid > \alpha \end{cases}$$

11. $f(x) = e^{-|x|}$

12. $f(x) = e^{-|x|}\cos(x)$

13.
$$f(x) = \begin{cases} e^{-x} & \text{for } x \ge 0 \\ 0 & \text{for } x < 0 \end{cases}$$

14. The *amplitude spectrum* of f is a graph of the magnitude of its Fourier transform. Generate the amplitude spectrum of each of the functions in Problems 8 through 13.

In each of the following problems, assume that f has a Fourier transform.

15. Prove that
$$\mathcal{F}[e^{ikx}f(x)](\omega) = \widehat{f}(\omega - k)$$
for any real number k. This result is called *frequency shifting.*

16. Let k be any nonzero real number. Prove that
$$\mathcal{F}[f(kx)](\omega) = \frac{1}{|k|}\widehat{f}\left(\frac{\omega}{k}\right).$$

17. Prove that
$$\mathcal{F}[f(-x)](\omega) = \widehat{f}(-\omega).$$
This is called *time reversal.*

18. Prove that
$$\mathcal{F}[\widehat{f}(x)](\omega) = 2\pi f(-\omega).$$
This is called *symmetry* of the Fourier transform.

19. Let k be any real number. Prove that
$$\mathcal{F}[f(x)\cos(kx)](\omega) = \frac{1}{2}\left[\widehat{f}(\omega + k) + \widehat{f}(\omega - k)\right].$$
This result is called *modulation.*

20. Write $\widehat{f}(\omega) = A(\omega) + iB(\omega)$. Show that
$$A(\omega) = \int_{-\infty}^{\infty} f(\xi)\cos(\omega\xi)\,d\xi$$
and
$$B(\omega) = -\int_{-\infty}^{\infty} f(\xi)\sin(\omega\xi)\,d\xi.$$
Relate these conclusions to the Fourier integral coefficients of f.

21. Let a be a positive number. Show that
$$\mathcal{F}[e^{-ax^2}](\omega) = \sqrt{\frac{\pi}{a}}e^{-\omega^2/4a}.$$
Hint: Write the integral defining $\widehat{f}(\omega)$, with $f(t) = e^{-at^2}$. Compute $d\widehat{f}/d\omega$ by differentiating under the integral sign and integrate by parts to show that
$$\frac{d}{d\omega}\widehat{f}(\omega) = -\frac{\omega}{2a}\widehat{f}(\omega).$$
Solve this ordinary differential equation for $\widehat{f}(\omega)$ and evaluate the constant by using the standard integral
$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

22. Let n be a positive integer. Let f be piecewise continuous on $[-L, L]$ for every positive L and suppose that $\int_{-\infty}^{\infty} |t^n f(t)|\, dt$ converges. Prove that

$$\mathcal{F}[t^n f(t)](\omega) = i^n \frac{d^n}{d\omega^n} \widehat{f}(\omega).$$

23. Use the results of Problems 21 and 22 to derive the formula

$$\mathcal{F}[te^{-at^2}](\omega) = -\frac{i\omega}{2a}\sqrt{\frac{\pi}{a}} e^{-\omega^2/4a}.$$

24. Prove that $\widehat{f}(-\omega) = \overline{\widehat{f}(\omega)}$, where the bar indicates complex conjugation.

25. Let f be piecewise continuous on every interval $[-L, L]$. Suppose that $\int_{-\infty}^{\infty} |f(t)|\, dt$ converges. Suppose also that $\widehat{f}(0) = 0$. Prove that

$$\mathcal{F}\left[\int_{-\infty}^{t} f(\tau)\, d\tau\right](\omega) = \frac{1}{i\omega}\widehat{f}(\omega).$$

26. Prove Parseval's theorem. Let f be real-valued, continuous and piecewise smooth on the real line. Assume that $\int_{-\infty}^{\infty} |f(x)|\, dx$ and $\int_{-\infty}^{\infty} f(x)^2\, dx$ converge. Then

$$\int_{-\infty}^{\infty} f^2(x)\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\widehat{f}(\omega)\right|^2 d\omega.$$

<u>Hint</u>: Use equation 3.24 to write $f(x) = (1/2\pi)\int_{-\infty}^{\infty} \widehat{f}(\omega)e^{i\omega x} d\omega$. Multiply both sides of this equation by $f(x)$ and integrate both sides of the resulting equation with respect to x from $-\infty$ to ∞, interchanging the order of integration in the double integral on the right.

27. Prove Plancherel's theorem. Let f and g be real-valued, continuous, and piecewise smooth on the real line, and suppose that f, f^2, g, and g^2 are absolutely integrable. Then

$$\int_{-\infty}^{\infty} f(x)g(x)\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)\overline{\widehat{g}(\omega)}\, d\omega.$$

3.7 Convolution

Many integral transforms (such as the Laplace transform) have a convolution operation, which is designed so that the transform of the convolution of two functions is the product of the transforms of the individual functions.

Let f and g be real- or complex- valued functions defined on the real line. We will say that f *has a convolution with* g if (1) $\int_a^b f(x)\, dx$ and $\int_a^b g(x)\, dx$ exist for every interval $[a, b]$.

(2) For every real number t,

$$\int_{-\infty}^{\infty} |f(t-x)g(x)|\,dx$$

converges.

In this event, the *convolution* $f * g$ of f with g (with respect to the Fourier transform) is the function defined by

$$f * g(t) = \int_{-\infty}^{\infty} f(t-x)g(x)\,dx$$

for every real t. Here $f * g(t)$ denotes the value of the function $f * g$ at t. This is sometimes emphasized by writing $f * g(t)$ as $(f * g)(t)$.

Theorem 3.7 *Suppose that f has a convolution with g. Then*
(1) *g has a convolution with f, and*

$$f * g = g * f.$$

(2) *If f and g both have convolutions with h, and α and β are numbers, then $\alpha f + \beta g$ also has a convolution with h, and*

$$(\alpha f + \beta g) * h = \alpha(f * h) + \beta(g * h). \diamond$$

Conclusion (1) is the commutativity, and (2) is the linearity, of the convolution operation.

Proof of (1) Let $z = t - x$ to write, for any positive r and R, and any real t,

$$\begin{aligned}\int_{-r}^{R} f(t-x)g(x)\,dx &= \int_{t+r}^{t-R} f(z)g(t-z)(-1)\,dz \\ &= \int_{t-R}^{t+r} g(t-z)f(z)\,dz \to \int_{-\infty}^{\infty} g(t-z)f(z)\,dz\end{aligned}$$

as $r \to \infty$ and $R \to \infty$. But in this limit,

$$\int_{-r}^{R} f(t-x)g(x)\,dx \to f * g(t).$$

Then

$$\begin{aligned}f * g(t) &= \int_{-\infty}^{\infty} f(t-x)g(x)\,dx \\ &= \int_{-\infty}^{\infty} g(t-z)f(z)\,dz = g * f(t).\end{aligned}$$

Conclusion (2) follows from elementary properties of integrals, given that the improper integrals involved converge. $\diamond$

It can be shown that if f is bounded and continuous on the real line and $\int_{-\infty}^{\infty} |g(x)|\,dx$ converges, then $f * g$ is bounded and continuous for all real x.

The following theorem is often used in computations involving a convolution.

Theorem 3.8 *Suppose that f and g are bounded and continuous on the real line, and that both $\int_{-\infty}^{\infty} |f(x)|\,dx$ and $\int_{-\infty}^{\infty} |g(x)|\,dx$ converge. Then*

(1)

$$\int_{-\infty}^{\infty} f * g(t)\,dt = \int_{-\infty}^{\infty} f(t)\,dt \int_{-\infty}^{\infty} g(t)\,dt.$$

(2) *For any real ω,*

$$\widehat{f * g}(\omega) = \hat{f}(\omega)\hat{g}(\omega). \diamond$$

Conclusion (2), known as the *convolution theorem*, is a fundamental result we use in solving partial differential equations. This conclusion may also be written

$$\mathcal{F}[f * g] = \mathcal{F}[f]\mathcal{F}[g].$$

We will not provide a complete proof of (1), but here is an outline of the argument. Write

$$\begin{aligned}
\int_{-\infty}^{\infty} f * g(t)\,dt &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t-x)g(x)\,dx\right) dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t-x)g(x)\,dt\right) dx \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t-x)\,dt\right) g(x)\,dx.
\end{aligned}$$

Now, for any real x,

$$\int_{-\infty}^{\infty} f(t-x)\,dt = \int_{-\infty}^{\infty} f(t)\,dt.$$

Then

$$\begin{aligned}
\int_{-\infty}^{\infty} f * g(t)\,dt &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t)\,dt\right) g(x)\,dx \\
&= \int_{-\infty}^{\infty} f(t)\,dt \int_{-\infty}^{\infty} g(x)\,dx \\
&= \int_{-\infty}^{\infty} f(t)\,dt \int_{-\infty}^{\infty} g(t)\,dt.
\end{aligned}$$

This argument may seem plausible enough, but a careful reader should have the feeling that something tricky happened somewhere in the middle. This intuition is correct. The crux of the argument, a change in the order of integration, is not valid without justification. We will not pursue the proof of a result on change of order of integration. However, the hypotheses of the theorem are sufficient to justify this interchange.

Using (1) of the theorem, we can prove the convolution theorem.
Proof of (2) Let

$$F(t) = e^{-i\omega t} f(t), G(t) = e^{-i\omega t} g(t)$$

for real t and ω. Then

$$\begin{aligned}
\widehat{f * g}(\omega) &= \int_{-\infty}^{\infty} f * g(t) e^{-i\omega t}\, dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t-x) g(x)\, dx \right) e^{-i\omega t}\, dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-i\omega t} f(t-x) g(x)\, dx \right) dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-i\omega(t-x)} f(t-x) e^{-i\omega x} g(x)\, dx \right) dt.
\end{aligned}$$

Upon recognizing that the integral within the large parentheses is the convolution of F with G, we now have, by application of (1) of the theorem,

$$\begin{aligned}
\widehat{f * g}(\omega) &= \int_{-\infty}^{\infty} F * G(t)\, dt \\
&= \int_{-\infty}^{\infty} F(t)\, dt \int_{-\infty}^{\infty} G(t)\, dt \\
&= \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt \int_{-\infty}^{\infty} g(t) e^{-i\omega t}\, dt \\
&= \widehat{f}(\omega) \widehat{g}(\omega). \diamond
\end{aligned}$$

Problems for Section 3.7

1. Let $f(t) = e^{-t}$ and
$$g(t) = \begin{cases} t & \text{for } | t | \leq 1 \\ 0 & \text{for } | t | > 1. \end{cases}$$
Compute $f * g$.

2. Let k be a positive number. Define $f(t) = e^{-|t|}$ and $g(t) = e^{-k|t|}$. Determine $f * g$.

3. Let k be a positive number. Define
$$f(t) = \begin{cases} t \text{ for } |t| \leq k \\ 0 \text{ for } |t| > k. \end{cases}$$
Determine $f * f$.

4. Suppose that f and g are bounded and continuous on the real line and that $\int_{-\infty}^{\infty} |f(t)|\, dt$ and $\int_{-\infty}^{\infty} |g(t)|\, dt$ converge. Prove that
$$\int_{-\infty}^{\infty} |f * g(t)|\, dt \leq \int_{-\infty}^{\infty} |f(t)|\, dt \int_{-\infty}^{\infty} |g(t)|\, dt.$$

5. Let f have a convolution with g. Define the support of a function to be the closure of the set of points at which the function does not vanish:

$$S(F) = \overline{\{x|F(x) \neq 0\}}.$$

Prove that

$$S(f * g) \subset \{x + y | x \in S(f) \text{ and } y \in S(g)\}.$$

6. Suppose that $f * g$, $f' * g$, and $f * g'$ all exist. Prove that

$$(f * g)' = f' * g + f * g'.$$

7. (Convolution and the Dirac Delta Function) The *Heaviside function* H is defined by

$$H(t) = \begin{cases} t & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

In terms of H, the Dirac delta function is defined by

$$\delta(t) = \lim_{\epsilon \to 0+} \frac{1}{2\epsilon}[H(t + \epsilon) - H(t - \epsilon)].$$

Strictly speaking, this is not a function, but is an object called a distribution (see Section 7.3). It is instructive to attempt a graph of $\delta(t)$, noting that for $\epsilon > 0$,

$$H(t_\epsilon) - H(t - \epsilon) = \begin{cases} 1 & \text{for } -\epsilon \leq t < \epsilon \\ 0 & \text{for } t \geq \epsilon \text{ and for } t < -\epsilon. \end{cases}$$

Assuming that the limit can be interchanged with the operation of taking the Fourier transform, define

$$\widehat{\delta}(\omega) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \left[\widehat{H}(t + \epsilon) - \widehat{H}(t - \epsilon)\right].$$

(a) Show that

$$\mathcal{F}[H(t + \epsilon) - H(t - \epsilon)](\omega) = 2\frac{\sin(\epsilon\omega)}{\omega}.$$

(b) Conclude from (a) and the definition of $\widehat{\delta}(\omega)$ that

$$\widehat{\delta}(\omega) \equiv 1.$$

(c) Show that, if f has a convolution with δ, then

$$f * \delta = \delta * f = f.$$

This means that the Dirac delta function behaves like a group identity element under the convolution operation.

3.8 Fourier Sine and Cosine Transforms

We have seen how the Fourier integral suggests the Fourier transform. In a similar way, Fourier sine and cosine integrals suggest the Fourier sine and cosine transforms.

Suppose that f is piecewise smooth and $|f|$ is integrable on $[0, \infty)$. The *Fourier sine transform* of f is denoted $\mathcal{F}_S[f]$ and is defined by

$$\mathcal{F}_S[f](\omega) = \int_0^\infty f(x)\sin(\omega x)\,dx. \tag{3.26}$$

This is a function of ω. The Fourier sine transform of f is also denoted $\widehat{f}_S$.

To see how the definition arises from the Fourier sine integral, recall that the sine integral representation of f is

$$\frac{1}{2}(f(x+) + f(x-)) = \int_0^\infty B_\omega \sin(\omega x)\,d\omega,$$

where

$$B_\omega = \frac{2}{\pi}\int_0^\infty f(\xi)\sin(\omega\xi)\,d\xi.$$

Now recognize that

$$B_\omega = \frac{2}{\pi}\widehat{f}_S(\omega),$$

so

$$\frac{1}{2}(f(x+) + f(x-)) = \frac{2}{\pi}\int_0^\infty \widehat{f}_S(\omega)\sin(\omega x)\,d\omega. \tag{3.27}$$

Equation 3.27 defines the *inverse Fourier sine transform*, and it recovers f from the sine transform of f in the sense of returning $f(x)$ where f is continuous, and the average of left and right limits where f has a jump discontinuity.

Equations 3.26 and 3.27 form a *Fourier sine transform pair*. The first equation gives the Fourier sine transform of f, and the second enables us to recover f from its sine transform.

In a similar way, we can define the *Fourier cosine transform* by

$$\mathcal{F}_C[f](\omega) = \int_0^\infty f(x)\cos(\omega x)\,dx. \tag{3.28}$$

The cosine transform of $f(x)$ is also denoted $\widehat{f}_C(\omega)$.

This definition arises from the Fourier cosine integral. The Fourier cosine integral representation of f is

$$\frac{1}{2}(f(x+) + f(x-)) = \int_0^\infty A_\omega \cos(\omega x)\,d\omega,$$

in which

$$A_\omega = \frac{2}{\pi}\int_0^\infty f(x)\cos(\omega x)\,dx.$$

Since

$$A_\omega = \frac{2}{\pi}\widehat{f}_C(\omega),$$

then

$$\frac{1}{2}(f(x+) + f(x-)) = \frac{2}{\pi}\int_0^\infty \widehat{f}_C(\omega)\cos(\omega x)\,d\omega. \tag{3.29}$$

The right side of equation 3.29 is the *inverse Fourier cosine transform* of f. It enables us to recover f from its transform, in the sense of producing $f(x))$ at each x at which f is continuous.

Equations 3.28 and 3.29 form a *transform pair for the Fourier cosine transform.*

The operational rules used in applying these transforms to partial differential equations are given by the following theorem.

Theorem 3.9 (*Operational Rules for the Sine and Cosine Transforms*) *Let f and f' be continuous on $[0,\infty)$. Assume that $f(x) \to 0$ and $f'(x) \to 0$ as $x \to \infty$. Suppose that f'' is piecewise continuous on $[0,\infty)$. Then*

(1)

$$\mathcal{F}_S[f''](\omega) = -\omega^2\widehat{f}_S(\omega) + \omega f(0).$$

(2)

$$\mathcal{F}_C[f''](\omega) = -\omega^2\widehat{f}_C(\omega) - f'(0). \diamond$$

Both conclusions are proved by two integrations by parts.

Problems for Section 3.8

In each of Problems 1 through 5, calculate the Fourier sine transform and the Fourier cosine transform of the function. In these problems k and α denote constants, with $\alpha > 0$.

1. $f(x) = e^{-x}$

2. $f(x) = xe^{-ax}$, with a any positive number

3. $$f(x) = \begin{cases} \cos(x) & \text{for } 0 \le x \le \alpha \\ 0 & \text{for } x > \alpha \end{cases}$$

4. $$f(x) = \begin{cases} k & \text{for } 0 \le x \le \alpha \\ -k & \text{for } \alpha < x \le 2\alpha \\ 0 & \text{for } x > \alpha \end{cases}$$

5. $f(x) = e^{-x}\cos(x)$

6. Prove the operational rule for the Fourier sine transform.

7. Prove the operational rule for the Fourier cosine transform.

8. Show that under appropriate conditions on f,

$$\mathcal{F}_S[f^{(4)}(x)](\omega) = \omega^4 \widehat{f}_S(\omega) - \omega^3 f(0) + \omega f''(0),$$

in which $f^{(4)}$ is the fourth derivative of f. The phrase "appropriate conditions" means conditions sufficient to apply the operational rule for the Fourier sine transform to $(f'')''$.

9. Prove that under appropriate conditions on f,

$$\mathcal{F}_C[f^{(4)}(x)](\omega) = \omega^4 \widehat{f}_C(\omega) + \omega^2 f'(0) - f^{(3)}(0).$$

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by Peter V. O'Neil

Chapter 4

The Wave Equation

This chapter is devoted to solutions and properties of solutions of the constant coefficient, hyperbolic, second-order linear partial differential equation, focusing on the wave equation $u_{tt} = c^2 u_{xx} + F(x,t)$ as a prototype.

4.1 d'Alembert's Solution of the Cauchy Problem

Begin with the homogeneous wave equation

$$u_{tt} = c^2 u_{xx}$$

with independent variables x and t and with c a positive constant. We often think of t as time and x as a space dimension in interpreting solutions as waves.

The characteristic equations 2.6 and 2.7 for $u_{tt} = c^2 u_{xx}$ are

$$\frac{dt}{dx} = \frac{1}{c} \text{ and } \frac{dt}{dx} = -\frac{1}{c}$$

with general solutions defined implicitly by $x - ct = k_1$ and $x + ct = k_2$. The straight line graphs of these equations are the characteristics of the wave equation. We will use these to derive the canonical form. Let

$$\xi = x - ct, \eta = x + ct$$

and

$$U(\xi, \eta) = u(x,t).$$

By the chain rule,

$$\begin{aligned}
u_x &= U_\xi \xi_x + U_\eta \eta_x = U_\xi + U_\eta, \\
u_{xx} &= (U_{\xi\xi} + U_{\xi\eta}) + (U_{\eta\xi} + U_{\eta\eta}) = U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}, \\
u_t &= U_\xi(-c) + U_\eta(c),
\end{aligned}$$

and

$$\begin{aligned} u_{tt} &= -c(U_{\xi\xi}(-c) + U_{\xi\eta}(c)) + c(U_{\eta\xi}(-c) + U_{\eta\eta}(c)) \\ &= c^2 U_{\xi\xi} - 2c^2 U_{\xi\eta} + c^2 U_{\eta\eta}. \end{aligned}$$

Then

$$u_{tt} - c^2 u_{xx} = 0 = 4c^2 U_{\xi\eta}$$

and the canonical form is

$$U_{\xi\eta} = 0. \tag{4.1}$$

To solve this equation, and hence also the wave equation, we will follow a line of reasoning developed by the French mathematician Jean Le Rond d'Alembert in the 1740s. Begin by writing the canonical form 4.1 as

$$(U_\eta)_\xi = 0.$$

This implies that U_η is independent of ξ and hence is a function of η only, say

$$U_\eta = w(\eta).$$

Integration of this equation with respect to η yields

$$U(\xi, \eta) = \int w(\eta)\, d\eta + F(\xi),$$

in which the "constant" of the integration with respect to η may be a function of ξ. Since $\int w(\eta)\, d\eta$ is just another function of η, say $G(\eta)$, U must have the form

$$U(\xi, \eta) = F(\xi) + G(\eta),$$

in which F and G can be any functions of a single variable having two continuous derivatives.

We conclude that any function of the form

$$u(x, t) = F(x - ct) + G(x + ct) \tag{4.2}$$

with F and G twice continuously differentiable is a solution of the homogeneous wave equation. Conversely, any solution of this equation has this form. Equation 4.2 is *d'Alembert's solution* of the wave equation.

We will use d'Alembert's solution to solve the Cauchy problem on the real line for the wave equation

$$\begin{gathered} u_{tt} = c^2 u_{xx} \text{ for } -\infty < x < \infty, t > 0 \\ u(x, 0) = \varphi(x), u_t(x, 0) = \psi(x) \text{ for } -\infty < x < \infty. \end{gathered}$$

Interpreting t as time, these conditions are the *initial conditions*, giving the displacement and velocity of the string at time zero. Intuitively, this information and the wave equation governing the motion should be enough to determine the shape of the wave at any later time. We will show that indeed this Cauchy problem has a unique solution which depends continuously on the initial data φ and ψ. Usually, it is assumed that φ and ψ are continuous, which are reasonable expectations for initial position and velocity functions.

We know that any solution of the wave equation has the form

$$u(x,t) = F(x-ct) + G(x+ct).$$

The idea is to choose F and G to produce a solution satisfying the initial conditions. Now

$$u(x,0) = F(x) + G(x) = \varphi(x) \tag{4.3}$$

and

$$u_t(x,0) = -cF'(x) + cG'(x) = \psi(x). \tag{4.4}$$

Integrate equation 4.4 and rearrange terms to obtain

$$-F(x) + G(x) = \frac{1}{c}\int_0^x \psi(s)\,ds - F(0) + G(0).$$

Add this equation to equation 4.3 to obtain

$$2G(x) = \varphi(x) + \frac{1}{c}\int_0^x \psi(s)\,ds - F(0) + G(0).$$

Therefore,

$$G(x) = \frac{1}{2}\varphi(x) + \frac{1}{2c}\int_0^x \psi(s)\,ds - \frac{1}{2}F(0) + \frac{1}{2}G(0). \tag{4.5}$$

But then, from equation 4.3,

$$\begin{aligned} F(x) &= \varphi(x) - G(x) \\ &= \frac{1}{2}\varphi(x) - \frac{1}{2c}\int_0^x \psi(s)\,ds + \frac{1}{2}F(0) - \frac{1}{2}G(0). \end{aligned} \tag{4.6}$$

Finally, use equations 4.5 and 4.6 to write the solution 4.2 as

$$\begin{aligned} u(x,t) &= F(x-ct) + G(x+ct) \\ &= \frac{1}{2}\varphi(x-ct) - \frac{1}{2c}\int_0^{x-ct} \psi(s)\,ds + \frac{1}{2}(F(0) - G(0)) \\ &+ \frac{1}{2}\varphi(x+ct) + \frac{1}{2c}\int_0^{x+ct} \psi(s)\,ds - \frac{1}{2}(F(0) - G(0)). \end{aligned}$$

After cancellations,

$$u(x,t) = \frac{1}{2}(\varphi(x-ct) + \varphi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds. \tag{4.7}$$

This is *d'Alembert's formula* for the solution of the Cauchy problem for the wave equation on the real line.

Example 4.1 *Solve the Cauchy problem*

$$u_{tt} = 9u_{xx} \ for - \infty < x < \infty, t > 0$$
$$u(x,0) = \cos(x), u_t(x,0) = \sin(2x) \ for - \infty < x < \infty.$$

With $\varphi(x) = \cos(x), \psi(x) = \sin(2x)$, *and* $c = 3$, *the solution is*

$$\begin{aligned} u(x,t) &= \frac{1}{2}(\cos(x-3t) + \cos(x+3t)) \\ &+ \frac{1}{6}\int_{x-3t}^{x+3t} \sin(2s)\,ds \\ &= \frac{1}{2}(\cos(x-3t) + \cos(x+3t)) \\ &- \frac{1}{12}(\cos(2(x+3t)) - \cos(2(x-3t))). \end{aligned}$$

This solution can be written

$$u(x,t) = \cos(x)\cos(3t) + \frac{1}{6}\sin(2x)\sin(6t). \diamond$$

The d'Alembert formula shows that a solution of the Cauchy problem for the wave equation exists and is unique. We use this formula to show that the solution depends continuously on the initial data. This means that small changes in the initial data (initial position and/or velocity) cause correspondingly small changes in the solution.

Theorem 4.1 (*continuous dependence on initial data*) *Let* u_1 *be the solution of* $u_{tt} = c^2 u_{xx}$ *on* $-\infty < x < \infty, t > 0$, *satisfying*

$$u(x,0) = \varphi_1(x) \text{ and } \frac{\partial u}{\partial t}(x,0) = \psi_1(x).$$

Let u_2 *be the solution satisfying*

$$u(x,0) = \varphi_2(x) \text{ and } \frac{\partial u}{\partial t}(x,0) = \psi_2(x).$$

Let $\epsilon > 0$ *and* $T > 0$. *Then there exists a positive number* δ *such that if*

$$|\varphi_1(x) - \varphi_2(x)| < \delta \text{ and } |\psi_1(x) - \psi_2(x)| < \delta \tag{4.8}$$

for $-\infty < x < \infty$ *and* $0 \le t \le T$, *then*

$$|u_1(x,t) - u_2(x,t)| < \epsilon$$

for $-\infty < x < \infty$ *and* $0 \le t \le T. \diamond$

Proof We know that

$$u_1(x,t) = \frac{1}{2}(\varphi_1(x-ct) + \varphi_1(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_1(s)\,ds$$

and

$$u_2(x,t) = \frac{1}{2}(\varphi_2(x-ct) + \varphi_2(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_2(s)\, ds.$$

Suppose now that we are given positive numbers ϵ and T as in the statement of the theorem. Then

$$\begin{aligned}
&|u_1(x,t) - u_2(x,t)| \\
&\le \frac{1}{2}|\varphi_1(x-ct) - \varphi_2(x-ct)| + \frac{1}{2}|\varphi_1(x+ct) - \varphi_2(x+ct)| \\
&+ \frac{1}{2c}\int_{x-ct}^{x+ct} |\psi_1(s) - \psi_2(s)|\, ds \\
&< \frac{1}{2}\delta + \frac{1}{2}\delta + \frac{1}{2c}(2cT)\delta = (1+T)\delta.
\end{aligned}$$

We will therefore have

$$|u_1(x,t) - u_2(x,t)| < \epsilon$$

for all x and $0 \le t \le T$ if we choose δ as any positive number such that

$$(1+T)\delta < \epsilon.$$

Thus we may choose δ as any number satisfying $0 < \delta < \epsilon/(1+T)$. $\diamond$

We say that the Cauchy problem for the wave equation on the real line is *well posed* because (1) a solution exists, (2) the solution is determined uniquely by the Cauchy data (initial conditions), and (3) the solution depends continuously on the initial conditions. A problem is *ill posed* if one of these criteria fails to be met. We will see that the Cauchy problem for the heat equation is ill posed.

Derivation of d'Alembert's Formula Using the Fourier Transform

We will show how the Fourier transform can be used to derive d'Alembert's formula. We want to solve

$$\begin{aligned}
u_{tt} &= c^2 u_{xx} \\
u(x,0) &= \varphi(x), u_t(x,0) = \psi(x)
\end{aligned}$$

for $-\infty < x < \infty, t > 0$. Apply the Fourier transform to the wave equation, thinking of x as the variable in which the transform is carried out and carrying t along as a parameter. Formally,

$$\mathcal{F}[u_{tt}] = c^2\mathcal{F}[u_{xx}].$$

Now,

$$\begin{aligned}
\mathcal{F}[u_{tt}](\omega) &= \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial t^2}(x,t)e^{-i\omega x}\, dx \\
&= \frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}\, dx = \frac{\partial^2 \widehat{u}}{\partial t^2}(\omega,t).
\end{aligned}$$

The operations $\partial^2/\partial t^2$ and $\int_{-\infty}^{\infty} \cdots dx$ commute (that is, the partial differentiation with respect to t can be taken outside of the integration with respect to x) because x and t are independent and u is assumed continuous with continuous first partial derivatives.

Computing $\mathcal{F}[u_{xx}](\omega)$ is another matter, since now the differentiation is with respect to x, the variable with respect to which the transform is taken. Now we must apply the operational formula for the Fourier transform with $n = 2$, obtaining

$$\mathcal{F}[u_{xx}](\omega) = (i\omega)^2 \widehat{u}(\omega, t) = -\omega^2 \widehat{u}(\omega, t).$$

In applying this operational formula, we seek solutions satisfying the hypotheses of the formula. Specifically, we assume that

$$\int_{-\infty}^{\infty} |u_x(x,t)|\, dx$$

converges for $t \geq 0$, and that $u(x,t) \to 0$ and $u_x(x,t) \to 0$ as $x \to \pm\infty$, for $t \geq 0$.

We now have

$$\widehat{u}_{tt}(\omega, t) = -c^2\omega^2 \widehat{u}(\omega, t)$$

or

$$\widehat{u}_{tt} + c^2\omega^2 \widehat{u} = 0.$$

Think of this as an ordinary differential equation in t for $\widehat{u}(\omega, t)$, with ω carried along as a parameter. The general solution for $\widehat{u}(\omega, t)$ is

$$\widehat{u}(\omega, t) = a(\omega)\cos(\omega ct) + b(\omega)\sin(\omega ct).$$

The coefficients in this solution may depend on ω. To solve for these coefficients, first apply the Fourier transform to the initial condition $u(x, 0) = \varphi(x)$ to get

$$\widehat{u}(\omega, 0) = \widehat{\varphi}(\omega) = a(\omega).$$

Next apply the transform to the initial condition $u_t(x, 0) = \psi(x)$:

$$\widehat{u}_t(\omega, 0) = \widehat{\psi}(\omega).$$

Here we used the fact that $\partial/\partial t$ passes through the transform. But $\widehat{u}_t(\omega, 0) = c\omega b(\omega)$, so

$$b(\omega) = \frac{1}{c\omega}\widehat{\psi}(\omega).$$

Therefore,

$$\widehat{u}(\omega, t) = \widehat{\varphi}(\omega)\cos(c\omega t) + \frac{1}{c\omega}\widehat{\psi}(\omega)\sin(c\omega t).$$

This gives the transform of the solution in terms of quantities that are known, or computable from given information. From this expression we must extract $u(x, t)$. We will exploit the facts that

$$\cos(T) = \frac{1}{2}(e^{iT} + e^{-iT}) \text{ and } \sin(T) = \frac{1}{2i}(e^{iT} - e^{-iT}).$$

We will also find it convenient to define

$$\Psi(x) = \int_{-\infty}^{x} \psi(s)\,ds.$$

In this way $\Psi'(x) = \psi(x)$ and we can use the operational formula for the Fourier transform to write

$$\widehat{\psi}(\omega) = \widehat{\Psi}'(\omega) = i\omega\widehat{\Psi}(\omega).$$

This eliminates the $1/\omega$ factor in one term of $\widehat{u}(\omega, t)$ and enables us to write

$$\begin{aligned}
\widehat{u}(\omega, t) &= \widehat{\varphi}(\omega)\cos(c\omega t) + \frac{1}{c}i\widehat{\Psi}(\omega)\sin(c\omega t) \\
&= \widehat{\varphi}(\omega)\frac{1}{2}(e^{ic\omega t} + e^{-ic\omega t}) + \frac{1}{2c}\widehat{\Psi}(\omega)(e^{ic\omega t} - e^{-ic\omega t}) \\
&= \frac{1}{2}(\widehat{\varphi}(\omega)e^{ic\omega t} + \widehat{\varphi}(\omega)e^{-ic\omega t}) + \frac{1}{2c}\left(\widehat{\Psi}(\omega)e^{ic\omega t} - \widehat{\Psi}(\omega)e^{-ic\omega t}\right).
\end{aligned}$$

Now recognize that $\widehat{\varphi}(\omega)e^{ic\omega t}$ is the Fourier transform of the translated function $\varphi(x + ct)$, by the time-shifting property of the Fourier transform. With similar recognition of the other terms in the last line of the last equation, we have

$$\widehat{u}(\omega, t) = \mathcal{F}\left[\frac{1}{2}(\varphi(x + ct) + \varphi(x - ct)) + \frac{1}{2c}(\Psi(x + ct) - \Psi(x - ct))\right](\omega, t).$$

By the uniqueness of the inverse Fourier transform of a continuous function,

$$u(x, t) = \frac{1}{2}(\varphi(x + ct) + \varphi(x - ct)) + \frac{1}{2c}(\Psi(x + ct) - \Psi(x - ct)).$$

Finally,

$$\begin{aligned}
\Psi(x + ct) - \Psi(x - ct) &= \int_{-\infty}^{x+ct} \psi(s)\,ds - \int_{-\infty}^{x-ct} \psi(s)\,dx \\
&= \int_{x-ct}^{x+ct} \psi(s)\,ds
\end{aligned}$$

and this is d'Alembert's solution.

Problems for Section 4.1

In each of Problems 1 through 9, solve

$$\begin{gathered}
u_{tt} = c^2 u_{xx} \text{ for } -\infty < x < \infty, t > 0 \\
u(x, 0) = \varphi(x), u_t(x, 0) = \psi(x) \text{ for } -\infty < x < \infty
\end{gathered}$$

for the given φ, ψ, and c.

1. $\varphi(x) = \cos(3x), \psi(x) = x, c = 7$

2. $\varphi(x) = x^2, \psi(x) = \sin(2x), c = 4$
3. $\varphi(x) = e^{-|x|}, \psi(x) = \sin^2(x), c = 3$
4. $\varphi(x) = \cosh(x), \psi(x) = 2x, c = 2$
5. $\varphi(x) = \cos(x) - \sin(x), \psi(x) = \sin(x), c = 2$
6. $\varphi(x) = 2 + x, \psi(x) = e^x, c = 1$
7. $\varphi(x) = \cos(x), \psi(x) = xe^{-x}, c = 4$
8. $\varphi(x) = \sin(3x), \psi(x) = \cos(3x), c = 1$
9. $\varphi(x) = x^3, \psi(x) = x\cos(x), c = 3$
10. Let w be a solution of the problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \text{ for } -\infty < x < \infty, t > 0 \\ u(x,0) &= \varphi(x), u_t(x,0) = 0 \text{ for } -\infty < x < \infty. \end{aligned}$$

Let v be a solution of

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \text{ for } -\infty < x < \infty, t > 0 \\ u(x,0) &= 0, u_t(x,0) = \psi(x) \text{ for } -\infty < x < \infty. \end{aligned}$$

Prove that $w + v$ is a solution of

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \text{ for } -\infty < x < \infty, t > 0 \\ u(x,0) &= \varphi(x), u_t(x,0) = \psi(x) \text{ for } -\infty < x < \infty. \end{aligned}$$

11. Write the solution of the Cauchy problem for the wave equation for $c = 1, \varphi(x) = \sin(x)$, and $\psi(x) = 0$. Then write the solution with $\varphi(x) = \sin(x)+\epsilon, \psi(x) = 0$, with ϵ any positive number. Show that these solutions differ in magnitude by ϵ for all x and $t \geq 0$.

12. Write the solution of the Cauchy problem for $c = 1, \varphi(x) = \cos(x)$, and $\psi(x) = x$. Then write the solution with $\varphi(x) = \cos(x)+\epsilon$ and $\psi(x) = x+\epsilon$, with ϵ any positive number. Show that these solutions differ in magnitude by no more than $\epsilon(1 + T)$ for all x, if $0 \leq t \leq T$.

13. Let $u(x, t)$ be a solution of the problem

$$\begin{aligned} u_{xx} &= u_{tt} + f(x,t) \text{ for } -\infty < x < \infty, t > 0 \\ u(x,0) &= u_t(x,0) = 0. \end{aligned}$$

Let $w(x, t, T)$ be a solution of the problem

$$\begin{aligned} w_{xx} - w_{tt} &= 0 \text{ for } -\infty < x < \infty, t > T \\ w(x,T,T) &= 0, w_t(x,T,T) = -f(x,T). \end{aligned}$$

Prove that

$$u(x,t) = \int_0^t w(x,t,T)\,dT.$$

This conclusion is known as *Duhamel's principle* for the wave equation.

14. Verify that

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)\,d\xi\,d\tau$$

is a solution of the problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + f(x,t) \text{ for } -\infty < x < \infty, t > 0 \\ u_t(x,0) &= 0 \text{ for } -\infty < x < \infty \\ u(0,t) &= 0 \text{ for } t > 0. \end{aligned}$$

15. Solve $u_{tt} = c^2 u_{xx}$ on the half-line $x > 0$, subject to the initial conditions $u(x,0) = u_t(x,0) = 0$ and the boundary condition $u(0,t) = h(t)$. Hint: Substitute the initial and boundary conditions into the general expression $u(x,t) = f(x+ct)+g(x-ct)$, and determine f and g to solve this problem. Obtain $u(x,t) = 0$ for $x - ct > 0$ and $u(x,t) = h(t - x/c)$ for $x - ct < 0$.

4.2 d'Alembert's Solution as a Sum of Waves

Write d'Alembert's formula for the solution of the Cauchy problem for the wave equation as

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(\varphi(x-ct) - \frac{1}{c}\int_0^{x-ct} \psi(s)\,ds\right) \\ &\quad + \frac{1}{2}\left(\varphi(x+ct) + \frac{1}{c}\int_0^{x+ct} \psi(s)\,ds\right) \\ &= F(x-ct) + B(x+ct), \end{aligned} \tag{4.9}$$

in which

$$F(x) = \frac{1}{2}\varphi(x) - \frac{1}{2c}\int_0^x \psi(s)\,ds \tag{4.10}$$

and

$$B(x) = \frac{1}{2}\varphi(x) + \frac{1}{2c}\int_0^x \psi(s)\,ds. \tag{4.11}$$

The graph of $F(x - ct)$ is the graph of $F(x)$ translated ct units to the right. We may therefore think of $F(x - ct)$ as a *forward wave*, or right wave, propagating the graph of F to the right with velocity c. The graph of $B(x + ct)$ is the graph of $B(x)$ translated ct units to the left, so $B(x + ct)$ is a *backward wave*, or left wave, propagating the graph of B to the left with velocity c. The resulting motion is a superposition of these forward and backward waves. The

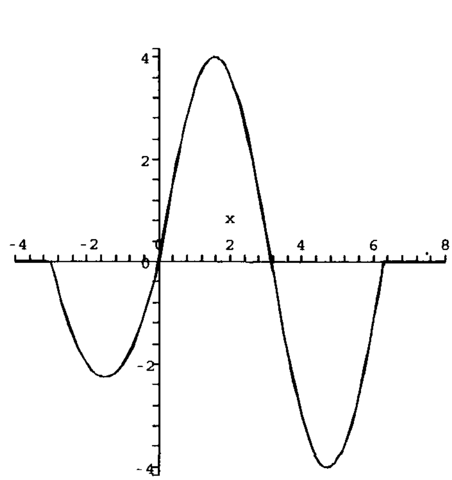

Figure 4.1: initial position

fact that these waves are moving with constant velocity means that the solution waves are propagated with finite speed. This is a characteristic property of solutions of the hyperbolic wave equation.

Figures 4.1 through 4.8 display these superpositions of forward and backward waves for the case $c = 1$, zero initial velocity, and initial position function given by

$$\varphi(x) = \begin{cases} x^2 + 3x & \text{for } -3 \leq x \leq 0 \\ 4\sin(x) & \text{for } 0 \leq x \leq 2\pi \\ 0 & \text{for } x < -3 \text{ and } x > 2\pi. \end{cases}$$

Figure 4.1 shows the wave at $t = 0$. Figure 4.2 shows the profile at time $t = 1/2$. This is the superposition of the initial position moved $ct = 1/2$ unit to the right and $1/2$ unit to the left. Figure 4.3 shows the wave at time $t = 1$ and is the superposition of the original profile moved 1 unit to the right and 1 unit to the left. Figures 4.4 through 4.8 show the waves at times $t = 3/2, 2, 4, 5$, and 6, respectively. Because the initial position is nonzero only on an interval of finite length, eventually the forward and backward waves separate completely, as has happened in this example by time $t = 5$. After this the wave profile consists of two copies of the original wave, moving away from each other at velocity $2c$.

Similarly, Figures 4.9 through 4.14 show profiles of the string having $c = 1$, zero initial velocity, and initial position function

$$\varphi(x) = \begin{cases} \cos(x) & \text{for } -\pi/2 \leq x \leq \pi/2 \\ 0 & \text{for } \mid x \mid \geq \pi/2. \end{cases}$$

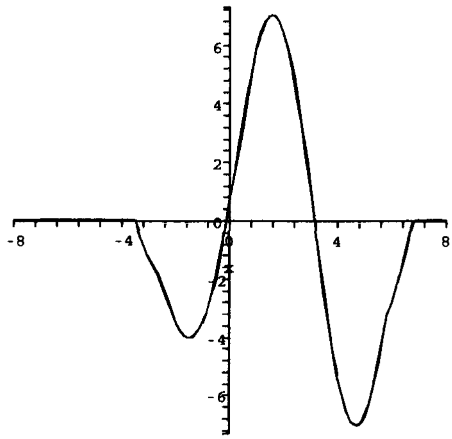

Figure 4.2: string profile at $t = 1/2$

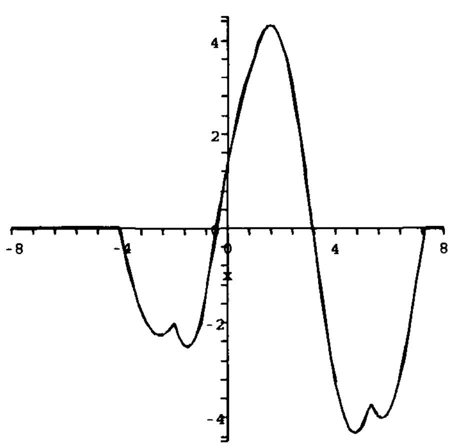

Figure 4.3: $t = 1$

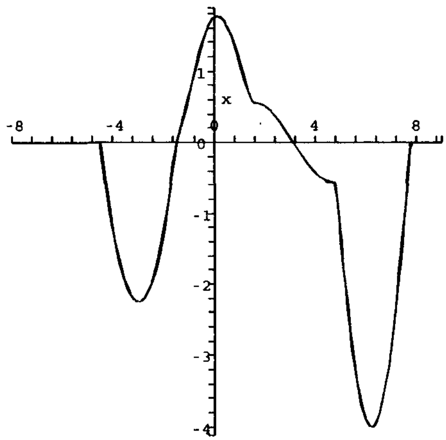

Figure 4.4: $t = 3/2$

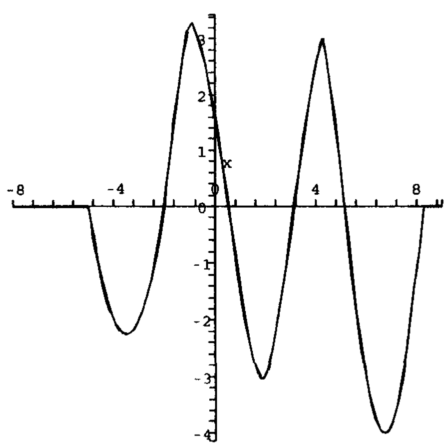

Figure 4.5: $t = 2$

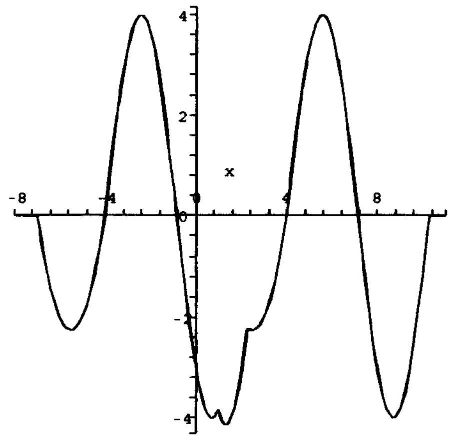

Figure 4.6: $t = 4$

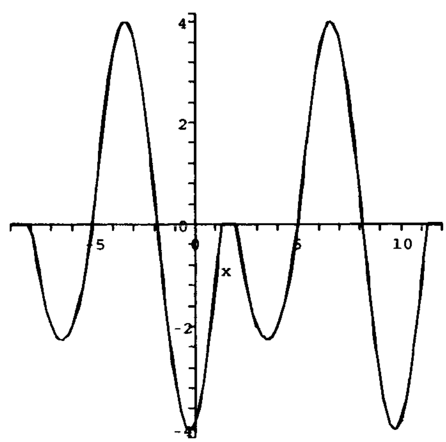

Figure 4.7: $t = 5$

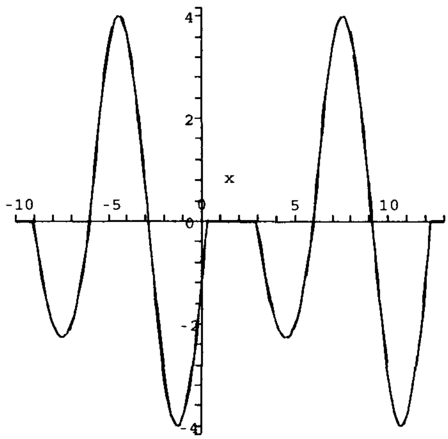

Figure 4.8: $t = 6$

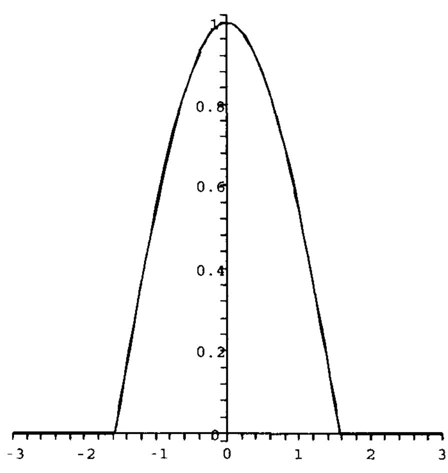

Figure 4.9: Initial position $\varphi(x) = \cos(x)$ for $-\pi/2 \leq x \leq \pi/2$.

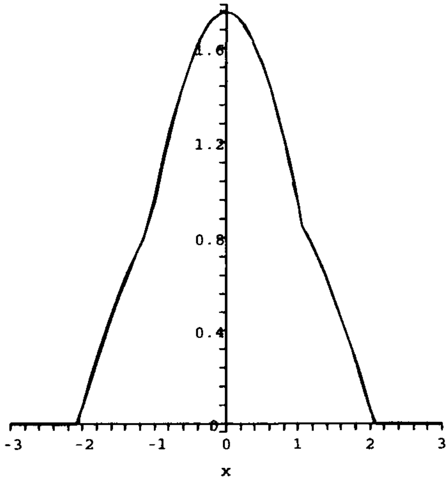

Figure 4.10: $t = 1/2$

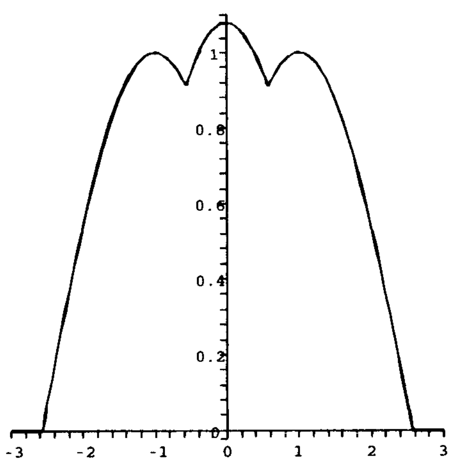

Figure 4.11: $t = 1$

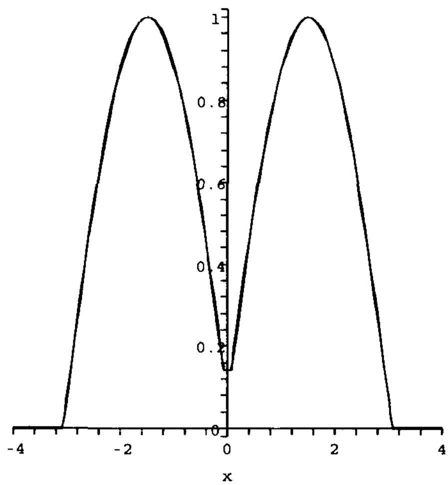

Figure 4.12: $t = 3/2$

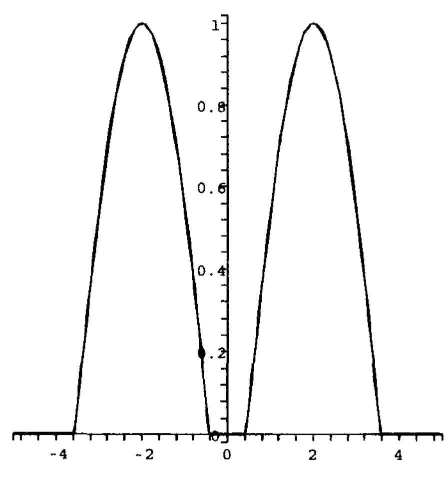

Figure 4.13: $t = 2$

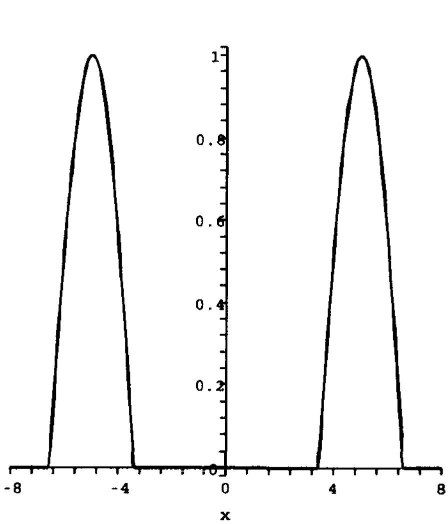

Figure 4.14: $t = 5$

Notice that in this example the forward and backward waves have separated by time $t = 2$ (Figure 4.13), and by time $t = 5$ they have moved a fair distance apart. At time increases, these waves will continue to move away from each other to the left and right.

Problems for Section 4.2

In Problems 1 through 6, write the solution of the Cauchy problem

$$u_{tt} = u_{xx} \text{ for } -\infty < x < \infty, t > 0$$
$$u(x,0) = \varphi(x), u_t(x,0) = 0 \text{ for } -\infty < x < \infty$$

as the sum of a forward wave and a backward wave. In the spirit of Figures 4.1 through 4.14, graph the initial position and then graph the solution at different times, showing the solution as a superposition of forward and backward waves along the real line.

1. $\varphi(x) = e^{-x^2}$

2.

$$\varphi(x) = \begin{cases} 1 - |x| & \text{for } -1 \le x \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

3.

$$\varphi(x) = \begin{cases} x\cos(x) & \text{for } -\pi/2 \le x \le \pi/2 \\ 0 & \text{for } |x| > \pi/2 \end{cases}$$

4.

$$\varphi(x) = \begin{cases} 1 - x^2 & \text{for } |x| \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

5.

$$\varphi(x) = \begin{cases} x^3 - x^2 - 2 & \text{for } -1 \le x \le 2 \\ 0 & \text{for } x < -1 \text{ or } x > 2 \end{cases}$$

6.

$$\varphi(x) = \begin{cases} x^3 - x^2 - 4x + 4 & \text{for } -2 \le x \le 2 \\ 0 & \text{for } |x| > 2 \end{cases}$$

4.3 The Characteristic Triangle

We will revisit the Cauchy problem for the wave equation on the real line. We want to know what information is needed to determine $u(x_0, t_0)$ at a point x_0 and time t_0.

A point $(x,\ t)$ lies in the x, t - plane, with the first coordinate specifying a location on the line, and the second, the time at which we are looking at that location. For example, if we move up the line $x = x_0$ along points (x_0, t), we remain at the location $x = x_0$ on the real line, but as the height above the horizontal axis increases we are looking at this point of the wave at later times. A graph in this x, t - plane is not to be confused with a graph of $y = u(x, t)$ at a particular time. This graph is drawn in the x, y - plane and gives a picture of the entire wave at that time. If we think of the problem as modeling a vibrating string, these graphs are snapshots of the string at selected times.

At (x_0, t_0), the Cauchy problem for the wave equation has the solution

$$u(x_0, t_0) = \frac{1}{2}(\varphi(x_0 - ct_0) + \varphi(x_0 + ct_0)) + \frac{1}{2c}\int_{x_0 - ct_0}^{x_0 + ct_0} \psi(s)\, ds,$$

where φ gives the initial position and ψ the initial velocity. The terms $\varphi(x_0 - ct_0)$ and $\varphi(x_0 + ct_0)$ are the initial position function evaluated at two numbers, $x_0 - ct_0$ and $x_0 + ct_0$, and depend only on these two numbers. The integral term in d'Alembert's formula depends on values of the initial velocity function at times over the entire interval $[x_0 - ct_0, x_0 + ct_0]$.

The lines $x - ct$ = constant and $x + ct$ = constant are the characteristics of the wave equation. These are straight lines in the x, t - plane having, respectively, slope $1/c$ and $-1/c$. The two unique characteristics passing through (x_0, t_0) are the lines

$$x - ct = x_0 - ct_0 \text{ and } x + ct = x_0 + ct_0$$

shown in Figure 4.15, and they intersect the x - axis at $(x_0 - ct_0, 0)$ and $(x_0 + ct_0, 0)$, respectively. These two points, together with (x_0, t_0), form the vertices of a solid triangle called the *characteristic triangle*, and $u(x_0, t_0)$ depends on the initial position function φ evaluated at the base vertices of this triangle and on the integral of the initial velocity over the base side of this triangle. Because of this, the interval $[x_0 - ct_0, x_0 + ct_0]$ is called the *domain of dependence* of the point (x_0, t_0).

Now consider again equation 4.9, in which we wrote the d'Alembert solution as the sum of a forward wave F and a backward wave B:

$$u(x,t) = F(x - ct) + B(x + ct),$$

with F and B defined by equations 4.10 and 4.11. In particular,

$$u(x_0, t_0) = F(x_0 - ct_0) + B(x_0 + ct_0).$$

Along the characteristic $x - ct = x_0 - ct_0$, $F(x - ct)$ has the constant value $F(x_0 - ct_0)$. This means that a disturbance initially at $(x_0 - ct_0, 0)$ is carried forward *with constant velocity* c *along this characteristic.* Along the characteristic $x + ct = x_0 + ct_0$, $B(x + ct)$ is the constant $B(x_0 + ct_0)$. A disturbance initially at $(x_0 + ct_0, 0)$ is carried backward with velocity c unchanged along this characteristic. *A solution of the wave equation propagates disturbances with constant velocity* c *along its characteristics.*

Influence of a Space Interval on the Wave Motion

We can also investigate the influence on the ensuing wave motion of an arbitrary interval on the horizontal axis. Suppose that we have points $(a, 0)$ and $(b, 0)$. There are two characteristics through each point. These divide the half-plane $t > 0$ into six regions, labeled I through VI in Figure 4.16. Region I is the characteristic triangle having base vertices $(a, 0)$ and $(b, 0)$. We examine in turn a typical point in each region and how the wave (solution) at this point is influenced by what is happening on $[a, b]$.

If (x, t) is a point in region II, its domain of dependence (Figure 4.17) does not intersect the characteristic triangle I. The same is true for a point in region III. A point x on the string at a time t, with (x, t) in one of these regions, is not influenced in any way by a disturbance at time $t = 0$ in $[a, b]$.

As Figure 4.18 suggests, a backward wave originating at a point in $[a, b]$ can reach a point in region IV, and a forward wave can reach a point in region V. A point x on the string, at a time t such that (x, t) is in region IV, is reached by a backward wave originating at time zero at a point in $[a, b]$. If (x, t) is in region V, this point on the string is reached by a forward wave originating at time zero at a point in $[a, b]$.

If (x, t) is in region VI, enough time has passed (t is sufficiently large) that the forward and backward waves in the d'Alembert solution have already passed and x is not further influenced by a displacement in $[a, b]$ at time zero, but remains at rest after a displacement of $(1/2c)\int_a^b \psi(s)\, ds$.

Finally, region I is the characteristic triangle and the solution at a point in this region is influenced by both forward and backward waves originating at a point on $[a, b]$.

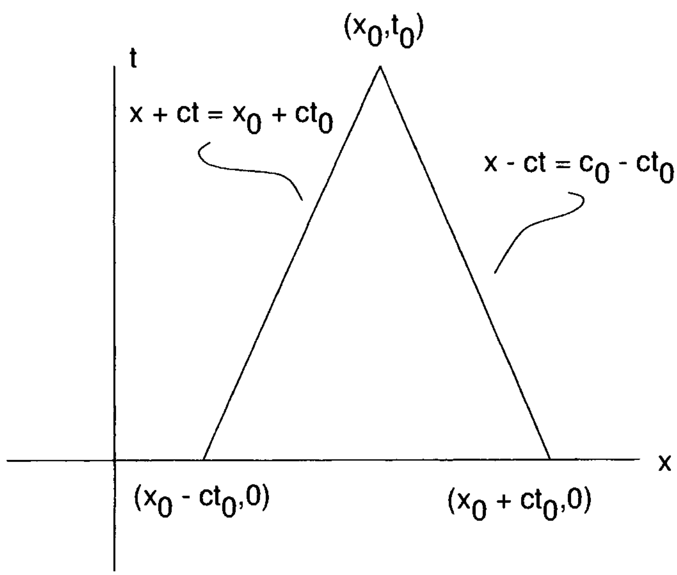

Figure 4.15: Characteristic triangle at (x_0, t_0).

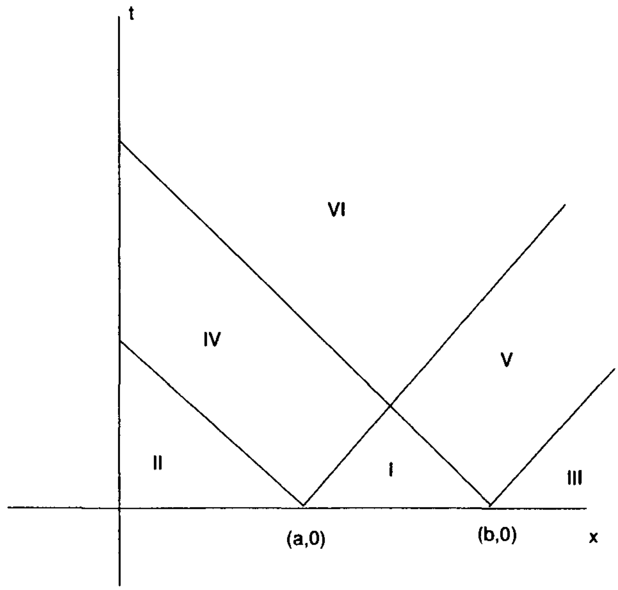

Figure 4.16: Characteristics partitioning the quarter-plane $x > 0, t > 0$.

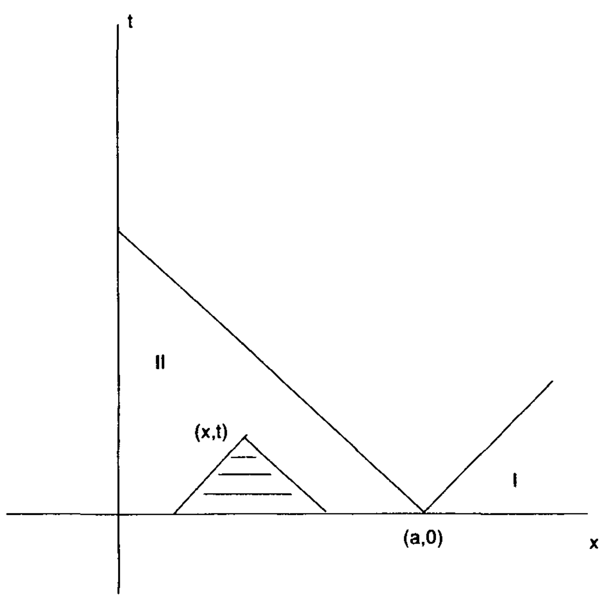

Figure 4.17: Domain of dependence of a point in region II.

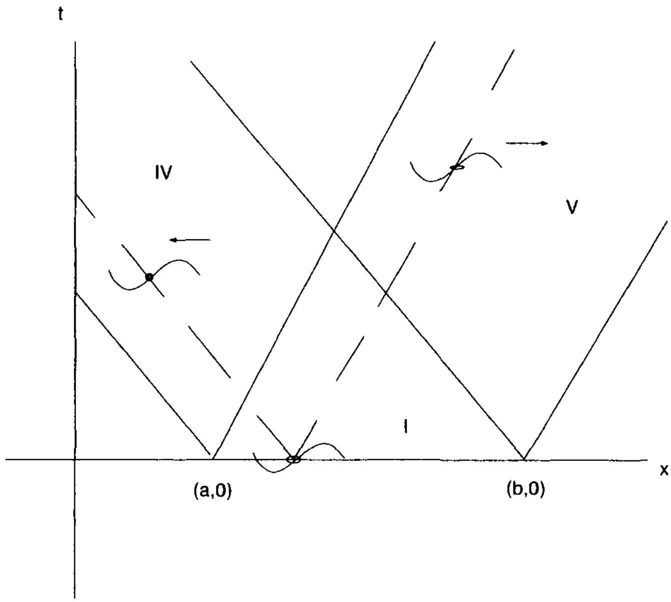

Figure 4.18: Forward and backward waves originating in (a, b).

Problems for Section 4.3

1. Suppose that the initial displacement and velocity functions vanish outside an interval of finite length. Specifically, let a be a positive number and suppose that $\varphi(x) = 0$ and $\psi(x) = 0$ for $|x| > a$. Prove that the solution of the Cauchy problem

$$u_{tt} = c^2 u_{xx} \text{ for } -\infty < x < \infty, t > 0$$
$$u(x, 0) = \varphi(x), u_t(x, 0) = \psi(x) \text{ for } -\infty < x < \infty$$

vanishes outside $[-a - ct, a + ct]$.

2. Suppose that there is some interval $[a, b]$ on the real line outside of which φ and ψ are identically zero. Let u be a solution of the Cauchy problem

$$u_t = u_{xx} \text{ for } -\infty < x < \infty, t > 0$$
$$u(x, 0) = \varphi(x), u_t(x, 0) = \psi(x),$$

and suppose that u is continuous with continuous first and second partial derivatives. Show that

$$\int_{-\infty}^{\infty} [u_x^2 + u_t^2]_{t=T}\, dx = \int_{-\infty}^{\infty} [u_x^2 + u_t^2]_{t=0}\, dx$$

for every $T \geq 0$. Hint: First establish the identity

$$2u_t(u_{xx} - u_{tt}) = (2u_t u_x)_x - (u_x^2 + u_t^2)_t.$$

Form the double integral of this identity over the rectangular region $-R - 2T \leq x \leq R + 2T, 0 \leq t \leq T$ and apply Green's theorem. The integral

$$\int_\alpha^\beta [u_x^2 + u_t^2]_{t=T}\, dx$$

is called the *energy* of $u(x, t)$ in the interval $[\alpha, \beta]$ at time T. The conclusion above states that the energy of the solution over the entire line is constant over time (conservation of energy).

4.4 The Wave Equation on a Half-Line

We will solve the problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \text{ for } x > 0, t > 0 \\ u(x,0) &= \varphi(x), u_t(x,0) = \psi(x) \text{ for } x \geq 0 \\ u(0,t) &= 0 \text{ for } t \geq 0. \end{aligned}$$

This problem models vibrations along a string stretched from zero to infinity. Because $x \geq 0$, this string has a left end, and the condition $u(0,t) = 0$ for all time means that this end is fixed. The condition $u(0,t) = 0$ is a *boundary condition* and this problem is called an *initial-boundary value problem.*

We will show how to use the d'Alembert formula for the initial value problem on the entire line, to write a solution of this problem on the half-line. Define odd extensions of φ and ψ to the entire line:

$$\Phi(x) = \begin{cases} \varphi(x) & \text{for } x \geq 0 \\ -\varphi(-x) & \text{for } x < 0. \end{cases}$$

and

$$\Psi(x) = \begin{cases} \psi(x) & \text{for } x \geq 0 \\ -\psi(-x) & \text{for } x < 0. \end{cases}$$

On the half-line $x \geq 0, \Phi$ agrees with φ and Ψ with ψ. Further, for all $x \neq 0$,

$$\Phi(-x) = -\Phi(x)$$

and

$$\Psi(-x) = -\Psi(x).$$

Now consider the following Cauchy problem for the entire line:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \text{ for } -\infty < x < \infty \\ u(x,0) &= \Phi(x), u_t(x,0) = \Psi(x) \text{ for } -\infty < x < \infty \end{aligned}$$

The solution of this problem is

$$u(x,t) = \frac{1}{2}(\Phi(x-ct) + \Phi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \Psi(s)\,ds. \tag{4.12}$$

u satisfies the wave equation for $-\infty < x < \infty$ and $t > 0$, hence also for $x > 0$ and $t > 0$. Further, if $x \geq 0$, then

$$u(x,0) = \frac{1}{2}(\Phi(x) + \Phi(x)) = \Phi(x) = \varphi(x)$$

and

$$\begin{aligned} u_t(x,0) &= \frac{1}{2}(-c\Phi'(x) + c\Phi'(x)) + \frac{1}{2c}(c\Psi(x) - (-c)\Psi(x)) \\ &= \Psi(x) = \psi(x). \end{aligned}$$

Therefore, u satisfies the initial conditions of the problem on the half-line. Finally, we claim that $u(0,t) = 0$. Compute

$$u(0,t) = \frac{1}{2}(\Phi(-ct) + \Phi(ct)) + \frac{1}{2c}\int_{-ct}^{ct} \Psi(s)\,ds.$$

But

$$\Phi(-ct) + \Phi(ct) = -\varphi(ct) + \varphi(ct) = 0$$

because Φ is the odd extension of φ; and

$$\int_{-ct}^{ct} \Psi(s)\,ds = 0$$

because Ψ is also an odd function.

We conclude that the solution 4.12 of the extended initial value problem on the entire line is also the solution of the initial-boundary value problem on the half-line.

This method of solution is sometimes called the *method of images*. In effect, we cast an image of the problem across the y - axis to create a problem on the whole real line, which we have solved previously. If the image is cast appropriately (in this case, using odd functions), the solution on the line gives the solution on the half-line.

Example 4.2 *Solve*

$$\begin{aligned} u_{tt} &= u_{xx} \text{ for } x > 0, t > 0 \\ u(x,0) &= 1 - e^{-x}, u_t(x,0) = \cos(x) \text{ for } x \geq 0 \\ u(0,t) &= 0 \text{ for } t \geq 0. \end{aligned}$$

Here $\varphi(x) = 1 - e^{-x}$ *and* $\psi(x) = \cos(x)$. *Form the odd extensions of these functions to the entire line:*

$$\Phi(x) = \begin{cases} 1 - e^{-x} & \text{for } x \geq 0 \\ -1 + e^{x} & \text{for } x < 0. \end{cases}$$

and

$$\Psi(x) = \begin{cases} \cos(x) & \text{for } x \geq 0 \\ -\cos(x) & \text{for } x < 0. \end{cases}$$

Then $\Phi(-x) = -\Phi(x)$ *and* $\Psi(-x) = -\Psi(x)$.

The solution of the initial-boundary value problem on the half-line is

$$u(x,t) = \frac{1}{2}(\Phi(x-t) + \Phi(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} \Psi(s)\, ds$$

for $x \geq 0,\ t \geq 0$.

It is natural to write this solution in terms of the original initial position and velocity functions φ *and* ψ, *and we can do this because* Φ *and* Ψ *are defined in terms of these functions.*

If $x - t \geq 0$, *then*

$$\begin{aligned} u(x,t) &= \frac{1}{2}(\Phi(x-t) + \Phi(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} \Psi(s)\, ds \\ &= \frac{1}{2}(1 - e^{-(x-t)} + 1 - e^{-(x+t)}) + \frac{1}{2}\int_{x-t}^{x+t} \cos(s)\, ds. \end{aligned}$$

This yields

$$u(x,t) = 1 - e^{-x}\cosh(t) + \cos(x)\sin(t) \text{ for } x - t \geq 0.$$

Now suppose that $x - t < 0$. *Keep in mind that* $x + t > 0$ *if* $x > 0$ *and* $t > 0$. *Now*

$$\begin{aligned} u(x,t) &= \frac{1}{2}(\Phi(x-t) + \Phi(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} \Psi(s)\, ds \\ &= \frac{1}{2}(-1 + e^{x-t} + 1 - e^{-(x+t)}) + \frac{1}{2}\int_{x-t}^{0} -\cos(s)\, ds + \frac{1}{2}\int_{0}^{x+t} \cos(s)\, ds \\ &= \frac{1}{2}(e^x e^{-t} - e^{-x}e^{-t}) - \left[\frac{1}{2}\sin(s)\right]_{x-t}^{0} + \left[\frac{1}{2}\sin(s)\right]_{0}^{x+t} \\ &= e^{-t}\sinh(x) + \sin(x)\cos(t). \end{aligned}$$

In summary, the solution is

$$u(x,t) = \begin{cases} 1 - e^{-x}\cosh(t) + \cos(x)\sin(t) & \text{for } x \geq t \geq 0 \\ e^{-t}\sinh(x) + \sin(x)\cos(t) & \text{for } 0 < x < t \diamond. \end{cases}$$

Problems for Section 4.4

In each of Problems 1 through 10, write the solution of the problem for the given initial position and velocity functions:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \text{ for } x > 0, t > 0 \\ u(x,0) &= \varphi(x), u_t(x,0) = \psi(x) \text{ for } x \geq 0 \\ u(0,t) &= 0 \text{ for } t \geq 0. \end{aligned}$$

1. $\varphi(x) = x^2, \psi(x) = \sin(x), c = 1$

2. $\varphi(x) = x^2, \psi(x) = x, c = 4$

3. $\varphi(x) = 1 - e^x, \psi(x) = x^2, c = 2$

4. $\varphi(x) = 1 - \cos(x), \psi(x) = e^{-x}, c = 4$

5. $\varphi(x) = x\sin(x), \psi(x) = x^2, c = 2$

6. $\varphi(x) = \sinh^2(x), \psi(x) = x, c = 7$

7. $\varphi(x) = x^3, \psi(x) = e^{-x}, c = 3$

8. $\varphi(x) = x^2 - x, \psi(x) = x - 1, c = 3$

9. $\varphi(x) = \cosh(x) - 1, \psi(x) = \sin(x), c = 5$

10. $\varphi(x) = x^3 + x, \psi(x) = \cos(2x), c = 6$

4.5 A Half-Line with Moving End

We will solve the problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \text{ for } x > 0, t > 0 \\ u(x,0) &= \varphi(x), u_t(x,0) = \psi(x) \text{ for } x \geq 0 \\ u(0,t) &= f(t) \text{ for } t \geq 0. \end{aligned}$$

This is an initial-boundary value problem which differs from that of the preceding section if $f(t)$ is not identically zero. We say that this boundary condition is *nonhomogeneous.* The problem models vibrations in a string stretched from zero to infinity, with given initial displacement and velocity and with its behavior at the left end governed by the function f. Imagine a person standing at the end of the string and moving it according to the function f.

We can solve this problem using d'Alembert's formula for the initial value problem on the entire line, but now we must take into account the behavior of the left end of the string.

Suppose first that $x_0 \geq ct_0$. Now $[x_0 - ct_0, x_0 + ct_0]$ lies entirely on the nonnegative part of the x - axis, and not enough time has passed for the initial displacement $f(t)$ at time zero to reach any point in this interval. In this case, $u(x_0, t_0)$ is not influenced by f and the d'Alembert formula holds. By equation 4.9,

$$u(x_0, t_0) = F(x_0 - ct_0) + B(x_0 + ct_0) \text{ for } x_0 \geq ct_0. \tag{4.13}$$

F and B are the forward and backward waves defined by equations 4.10 and 4.11.

Now suppose that $x_0 < ct_0$. As shown in Figure 4.19, the characteristic triangle through (x_0, t_0) intersects the t - axis and has part of its base on the negative x - axis. Now d'Alembert's formula does not apply because $\varphi(x)$ and

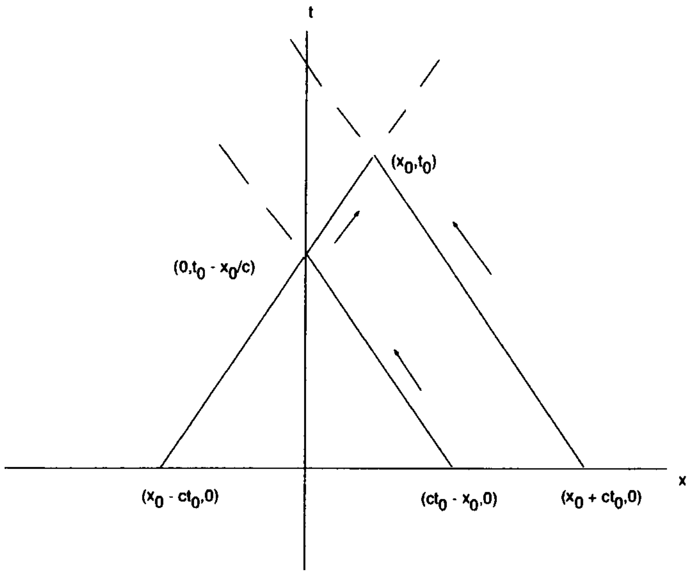

Figure 4.19: Reflection of a backward wave by the t - axis.

$\psi(x)$ are not defined for $x_0 - ct_0 < x < 0$; hence $F(x - ct_0)$ is not defined. However, putting $x = 0$ into equation 4.13 formally yields

$$u(0, t_0) = f(t_0) = F(-ct_0) + B(ct_0).$$

This suggests that we can extend F to this negative value by defining

$$F(-ct_0) = f(t_0) - B(ct_0).$$

Both function values on the right are well defined. Further, since t_0 can be any positive number, we can think of ct_0 as any positive number and use this equation as a model to define

$$F(-x) = f\left(\frac{x}{c}\right) - B(x) \tag{4.14}$$

for any positive number x. This extends F to negative values. For simplicity we are using the same symbol F for the extended function. Now put, for $x_0 - ct_0 <$ 0,

$$\begin{aligned} F(x_0 - ct_0) &= F(-(ct_0 - x_0)) \\ &= f\left(\frac{ct_0 - x_0}{c}\right) - B(ct_0 - x_0) \end{aligned}$$

or

$$F(x_0 - ct_0) = f\left(t_0 - \frac{x_0}{c}\right) - B(ct_0 - x_0).$$

Substituting this into d'Alembert's solution 4.9, we have

$$u(x_0, t_0) = f\left(t_0 - \frac{x_0}{c}\right) - B(ct_0 - x_0) + B(x_0 + ct_0) \text{ for } x_0 - ct_0 < 0.$$

In view of the definition of the backward wave B, this equation can be written

$$\begin{aligned} u(x_0, t_0) = {} & f\left(t_0 - \frac{x_0}{c}\right) + \frac{1}{2}(\varphi(x_0 + ct_0) - \varphi(ct_0 - x_0)) \\ & + \frac{1}{2c}\int_{ct_0 - x_0}^{x_0 + ct_0} \psi(s)\, ds \text{ for } x_0 < ct_0. \end{aligned} \tag{4.15}$$

Using equations 4.13 and 4.15, we can now compute $u(x_0, t_0)$ for $x_0 - ct_0 \geq 0$ and for $x_0 - ct_0 < 0$ in terms of the initial and boundary data of the problem. The value of the solution in the case $x_0 - ct_0 < 0$ depends on the initial position at $x_0 + ct_0$ and at $ct_0 - x_0$ and on the initial velocity on $[ct_0 - x_0, x_0 + ct_0]$. This is the domain of dependence of the solution when $x_0 < ct_0$.

Refer again to Figure 4.19. The characteristic through (x_0, t_0) and $(x_0 - ct_0, 0)$ intersects the t - axis at $(0, t_0 - x_0/c)$ when $x_0 < ct_0$. The reflection of $(x_0 - ct_0, 0)$ across the t - axis is the point $(ct_0 - x_0, 0)$, and the line through this point and $(0, t_0 - x_0/c)$ is again a characteristic. We may think of the solution 4.15 for $x_0 < ct_0$ as the result of a backward wave from $(x_0 + ct_0, 0)$, together with a backward wave from $(ct_0 - x_0, 0)$, reflected by the t - axis at $(0, t_0 - x_0/c)$.

We have used the zero subscript to discuss the solution at a particular point and maintain x and t as variables. However, we now drop the subscript and write the solution at any $(x,\ t)$ with $x \geq 0, t \geq 0$:

$$\begin{aligned} u(x,t) = {} & \frac{1}{2}(\varphi(x - ct) + \varphi(x + ct)) \\ & + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds \text{ for } x \geq ct \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} u(x,t) = {} & f\left(t - \frac{x}{c}\right) + \frac{1}{2}(\varphi(x + ct) - \varphi(ct - x)). \\ & + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(s)\, ds \text{ for } x < ct. \end{aligned} \tag{4.17}$$

From equation 4.17, $u(0, t) = f(t)$ for $t \geq 0$, so this solution satisfies the boundary condition.

Example 4.3 *Solve*

$$\begin{aligned} u_{tt} &= 4u_{xx} \text{ for } x > 0, t > 0 \\ u(x, 0) &= \sin(3x), u_t(x, 0) = x \text{ for } x \geq 0 \\ u(0, t) &= 1 - e^{-t} \text{ for } t \geq 0. \end{aligned}$$

In the context of the preceding discussion, $\varphi(x) = \sin(3x), \psi(x) = x, f(t) = 1 - e^{-t}$ *and* $c = 2$. *We can write the solution using equations 4.16 and 4.17:*

$$\begin{aligned} u(x,t) &= \frac{1}{2}(\sin(3(x-2t)) + \sin(3(x+2t))) \\ &+ \frac{1}{4}\int_{x-2t}^{x+2t} s\,ds \text{ for } x - 2t \geq 0 \\ u(x,t) &= 1 - e^{-t+x/2} + \frac{1}{2}(\sin(3(x+2t)) - \sin(3(2t-x)) \\ &+ \frac{1}{4}\int_{2t-x}^{x+2t} s\,ds \text{ for } x - 2t < 0. \end{aligned}$$

After some manipulation this solution is

$$u(x,t) = \begin{cases} \sin(3x)\cos(6t) + xt & \text{for } x - 2t \geq 0 \\ \sin(3x)\cos(6t) + xt + 1 - e^{-t}e^{x/2} & \text{for } x - 2t < 0. \end{cases}$$

It is routine to show that this function satisfies the initial and boundary conditions. ◇

Problems for Section 4.5

In each of Problems 1 through 10, write the solution of $u_{tt} = c^2 u_{xx}$ for $x > 0, t > 0$ with the given initial and boundary information.

1. $\varphi(x) = x, \psi(x) = e^{-x}, f(t) = t^2, c = 1$
2. $\varphi(x) = x^2, \psi(x) = 2x, f(t) = t, c = 3$
3. $\varphi(x) = \sin(x), \psi(x) = x, f(t) = 1 - e^t, c = 7$
4. $\varphi(x) = 1 - \cos(x), \psi(x) = \cos(2x), f(t) = \sin(t), c = 5$
5. $\varphi(x) = \cos(x), \psi(x) = x^2, f(t) = t + \cos(t), c = 3$
6. $\varphi(x) = \sin^2(x), \psi(x) = xe^x, f(t) = t^2, c = 1$
7. $\varphi(x) = e^{-x}, \psi(x) = \sin(x), f(t) = 1 - t, c = 3$
8. $\varphi(x) = x^3, \psi(x) = x, f(t) = \sin(3t), c = 4$
9. $\varphi(x) = x + x^2, \psi(x) = 1, f(t) = 2t, c = 2$
10. $\varphi(x) = 1 - x, \psi(x) = \cos(2x), f(t) = e^{-t}, c = 5$

4.6 A Nonhomogeneous Problem on the Real Line

Consider the nonhomogeneous problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + P(x,t) \text{ for } -\infty < x < \infty, t > 0 \\ u(x,0) &= \varphi(x), u_t(x,0) = \psi(x) \text{ for } -\infty < x < \infty. \end{aligned} \tag{4.18}$$

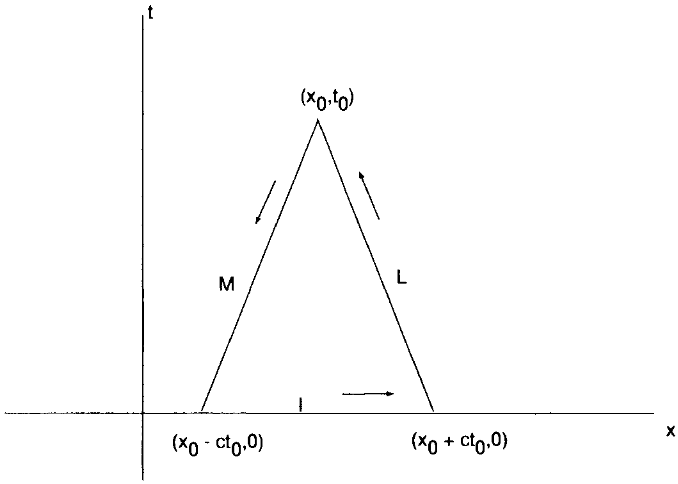

Figure 4.20: Characteristic triangle Δ.

In thinking of the wave equation as modeling vibrations along a string, $P(x,t)$ plays the role of an external driving or damping force. We exploit the characteristic triangle to write the solution of this initial value problem.

Suppose that we want the solution at (x_0, t_0). Let Δ denote the characteristic triangle having vertices $(x_0, t_0), (x_0 - ct_0, 0)$, and $(x_0 + ct_0, 0)$, as in Figure 4.20. Δ includes the sides L, M, and I of the triangle. Calculate the integral of $-P$ over this triangle:

$$-\iint_\Delta P(x,t)\,dA = \iint_\Delta \left(c^2 u_{xx} - u_{tt}\right)\,dA$$
$$= \iint_\Delta \left(\frac{\partial}{\partial x}(c^2 u_x) - \frac{\partial}{\partial t}(u_t)\right)\,dA.$$

Apply Green's theorem to the last integral, converting the double integral to a line integral around the boundary C of Δ. This piecewise smooth curve consists of three line segments and is oriented counterclockwise. We obtain

$$-\iint_\Delta P(x,t)\,dA = \oint_C u_t\,dx + c^2 u_x\,dt.$$

Evaluate the line integral over each of the line segments comprising C.

On I, $t = 0$ (so $dt = 0$) and x varies from $x_0 - ct_0$ to $x_0 + ct_0$. Then

$$\int_I u_t\,dx + c^2 u_x\,dt = \int_{x_0-ct_0}^{x_0+ct_0} u_t(x,0)\,dx = \int_{x_0-ct_0}^{x_0+ct_0} \psi(s)\,ds.$$

On L, $x - ct = x_0 - ct_0$, so $dx = c\,dt$ and

$$\begin{aligned}\int_L u_t\,dx + c^2 u_x\,dt &= \int_L u_t c\,dt + c^2 u_x \frac{1}{c}\,dx \\ &= c\int_L u_t\,dt + u_x\,dx = c\int_L du \\ &= c(u(x_0 - ct_0, 0) - u(x_0, t_0))\end{aligned}$$

because in the counterclockwise orientation L extends from (x_0, t_0) to $(x_0 - ct_0, 0)$.

On M, $x + ct = x_0 + ct_0$, so $dx = -c\,dt$ and

$$\begin{aligned}\int_M u_t\,dx + c^2 u_x\,dt &= \int_M u_t(-c)\,dt + c^2 u_x\left(-\frac{1}{c}\right)dx \\ &= -c\int_M du = -c(u(x_0, t_0) - u(x_0 + ct_0, 0)).\end{aligned}$$

Upon summing these line integrals, we obtain

$$\begin{aligned}-\iint_\Delta P(x,t)\,dA = &\int_{x_0-ct_0}^{x_0+ct_0} \psi(s)\,ds \\ &+ cu(x_0 - ct_0, 0) - 2cu(x_0, t_0) + cu(x_0 + ct_0, 0).\end{aligned}$$

Now $u(x_0 - ct_0, 0) = \varphi(x_0 - ct_0)$ and $u(x_0 + ct_0, 0) = \varphi(x_0 + ct_0)$. We can therefore solve for $u(x_0, t_0)$ in the last equation to obtain

$$\begin{aligned}u(x_0, t_0) = &\frac{1}{2}(\varphi(x_0 + ct_0) + \varphi(x_0 - ct_0)) + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} \psi(s)\,ds \\ &+ \frac{1}{2c}\iint_\Delta P(x,t)\,dA.\end{aligned}$$

We have used x_0 and t_0 in this derivation to be able to use $(x,\ t)$ to denote a point of Δ. However, once we have this formula it is convenient to drop the subscript notation and write the solution $u(x,t)$ at an arbitrary point (x,t) as

$$\begin{aligned}u(x,t) = &\frac{1}{2}(\varphi(x + ct) + \varphi(x - ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds \\ &+ \frac{1}{2c}\iint_\Delta P(X,T)\,dX\,dT. \qquad (4.19)\end{aligned}$$

In this notation, Δ is the characteristic triangle having vertices (x,t), $(x - ct, 0)$ and $(x + ct, 0)$, and (X,T) is a variable point of this triangle. The double integral is over this triangle, with X and T as variables of integration.

The solution of this nonhomogeneous problem is the d'Alembert formula for the homogeneous Cauchy problem ($P(x,t)$ identically zero), plus a constant times the integral of the forcing term over the characteristic triangle.

Example 4.4 *Solve*

$$u_{tt} = 4u_{xx} + x\cos(t) \text{ for } -\infty < x < \infty, t > 0$$
$$u(x,0) = e^{-x}, u_t(x,0) = \sin(x) \text{ for } -\infty < x < \infty.$$

Figure 4.21 shows the characteristic triangle Δ having (x,t) as a vertex. Evaluate

$$\iint_\Delta X\cos(T)\,dX\,dT = \int_0^t \left(\int_{x-2t+2T}^{x+2t-2T} X\,dX\right)\cos(T)\,dT$$
$$= \int_0^t 4x(t-T)\cos(T)\,dT = 4x(1-\cos(t)).$$

The solution is

$$u(x,t) = \frac{1}{2}(e^{-x-2t} + e^{-x+2t}) + \frac{1}{4}\int_{x-2t}^{x+2t} \sin(s)\,ds + x(1-\cos(t))$$
$$= e^{-x}\cosh(t) - \frac{1}{4}(\cos(x+2t) - \cos(x-2t)) + x(1-\cos(t)).$$

Some simplification yields

$$u(x,t) = e^{-x}\cosh(t) + \frac{1}{2}\sin(x)\sin(2t) + x(1-\cos(t)). \diamond$$

Problems for Section 4.6

In each of Problems 1 through 10, solve problem 4.18 with the given initial position, initial velocity, and $P(x, t)$.

1. $\varphi(x) = x, \psi(x) = e^{-x}, P(x,t) = x+t, c = 4$
2. $\varphi(x) = \sin(x), \psi(x) = 2x, P(x,t) = 2xt, c = 2$
3. $\varphi(x) = x^2 - x, \psi(x) = \cos(2x), P(x,t) = t\cos(x), c = 8$
4. $\varphi(x) = x^2, \psi(x) = xe^{-x}, P(x,t) = x\sin(t), c = 4$
5. $\varphi(x) = \cosh(x), \psi(x) = 1, P(x,t) = 3xt^2, c = 3$
6. $\varphi(x) = 1+x, \psi(x) = \sin(x), P(x,t) = x - \cos(t), c = 7$
7. $\varphi(x) = \cos(2x), \psi(x) = 1 - \cos(x), P(x,t) = t^2, c = 2$
8. $\varphi(x) = x^3, \psi(x) = \sin^2(x), P(x,t) = e^{-x}\cos(t), c = 1$
9. $\varphi(x) = x\sin(x), \psi(x) = e^{-x}, P(x,t) = xt, c = 2$
10. $\varphi(x) = 1 - x^2, \psi(x) = x\sin(x), P(x,t) = t\sin(x), c = 6$

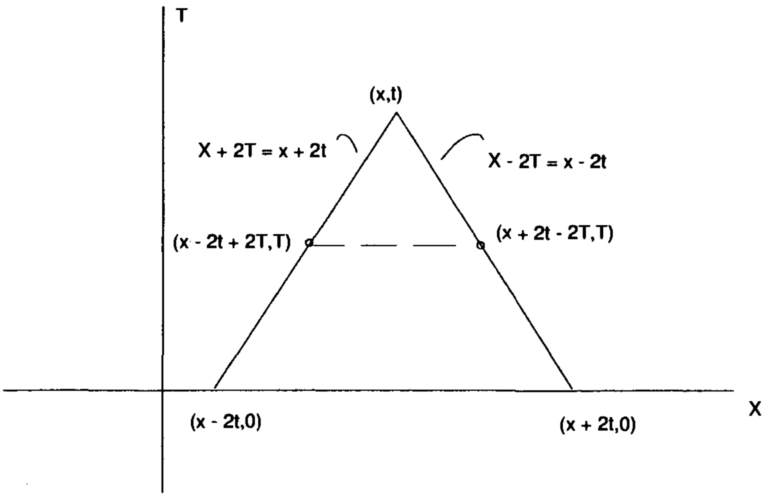

Figure 4.21: Determining the limits of integration over Δ in Example 4.4.

4.7 A General Problem on a Closed Interval

We now shift our emphasis from problems on the line or half-line to problems on a closed interval. In this section we exploit characteristics to solve the initial-boundary value problem:

$$\begin{gathered} u_{tt} = c^2 u_{xx} \text{ for } 0 < x < L, t > 0 \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \text{ for } 0 < x < L \\ u(0,t) = a(t), u(L,t) = b(t) \text{ for } t > 0. \end{gathered}$$

This problem models the motion of a string with given initial position φ and initial velocity ψ, having free ends whose motion is dictated by control functions a at the left end and b at the right end.

The characteristics of the wave equation are the lines

$$x - ct = \text{constant and } x + ct = \text{constant}.$$

The solution is based on segments of characteristics and the way they partition the strip $0 \leq x \leq L, t \geq 0$ into triangles and quadrilaterals, which are labeled I, II, III, $\cdots$ in Figure 4.22.

Begin with the characteristic $x - ct = 0$ of slope $1/c$ through the origin. Think of a beam of light emanating from the origin and shining along this characteristic and think of the t - axis and the line $x = L$ as reflective walls. The beam will strike the wall $x = L$ at $(L, L/c)$ and reflect back upward to

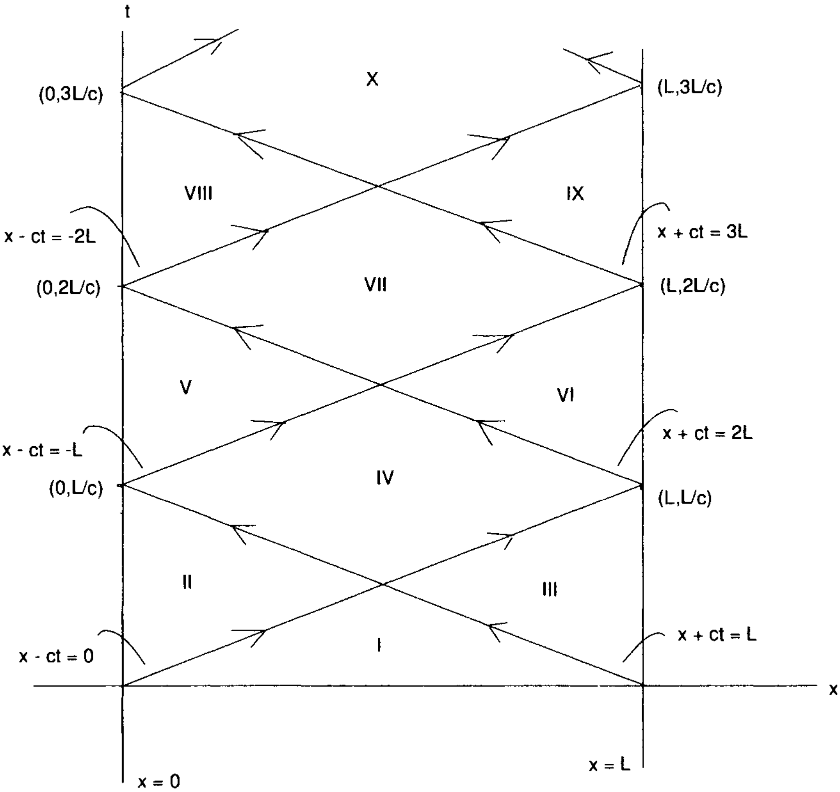

Figure 4.22: Characteristics partitioning the strip $0 \leq x \leq L, t \geq 0$.

the left along the characteristic $x + ct = 2L$, where it will strike the t - axis at $(0, 2L/c)$. From here the beam will reflect upward to the right along the characteristic $x - ct = -2L$ and strike the wall $x = L$ at $(L, 3L/c)$. Here the beam will reflect upward to the left along the characteristic $x+ct = 4L$ to strike the t - axis at $(0, 4L/c)$, and the reflection process continues indefinitely.

Similarly, a beam originating at $(L, 0)$ and directed along the characteristic $x + ct = L$ will strike the t - axis at $(0, L/c)$, then reflect upward to the right along the characteristic $x - ct = -L$ to strike the line $x = L$ at $(L, 2L/c)$. From here it will reflect back along the characteristic $x + ct = 3L$ to the t - axis, and continue on.

We know $u(x, t)$ on the sides of the strip from the initial and boundary conditions. On the bottom side,

$$u(x, 0) = \varphi(x).$$

On the left side,

$$u(0, t) = a(t)$$

and on the right side,

$$u(L, t) = b(t).$$

Now we want to find $u(x,\ t)$ at interior points of the strip.

The key lies in the following observation. Figure 4.23 shows a *characteristic quadrilateral* formed from segments of four characteristics. P_1 and P_2 are opposite vertices, as are Q_1 and Q_2. We claim that if u is any solution of the wave equation $u_{tt} = c^2 u_{xx}$ (regardless of initial and boundary conditions), then

$$u(P_1) + u(P_2) = u(Q_1) + u(Q_2). \tag{4.20}$$

This can be proved using the fact that any solution of the one-dimensional wave equation has the form

$$u(x, t) = F(x - ct) + G(x + ct).$$

The ramification of equation 4.20 is that if we know $u(x, t)$ at three vertices of a characteristic quadrilateral, we can determine its value at the fourth vertex.

Now begin working through the regions shown in Figure 4.22. If $(x,\ t)$ is in region I, then $u(x, t)$ is given by d'Alembert's formula, equation 4.7.

Next suppose that $P : (x, t)$ is in region II. Form a characteristic quadrilateral as in Figure 4.24, having one vertex on the line $x = 0$ and two vertices on the piece of the characteristic from the origin bounding region I. From equation 4.20,

$$u(P) = u(A) + u(B) - u(C).$$

But $u(A)$ is known because A is on the left boundary of the strip. And $u(B)$ and $u(C)$ are known because these are on the boundary of region I, where we know $u(x, t)$. We can therefore solve for $u(x, t)$ at any point in region II.

If $P : (x, t)$ is in region III, the construction shown in Figure 4.25 enables us to use equation 4.20 to again evaluate $u(x, t)$ in terms of known values of u at three vertices of a characteristic quadrilateral.

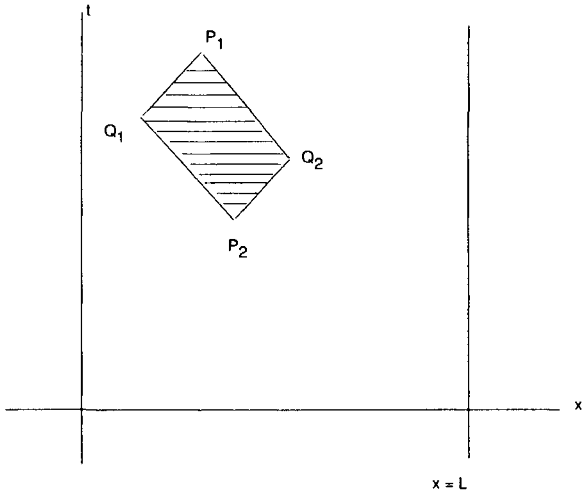

Figure 4.23: typica. characteristic quadrilateral

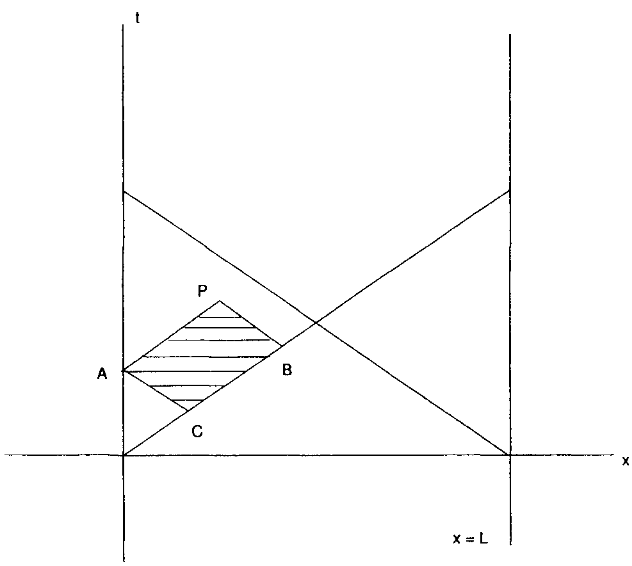

Figure 4.24: characteristic quadrilateral in region II

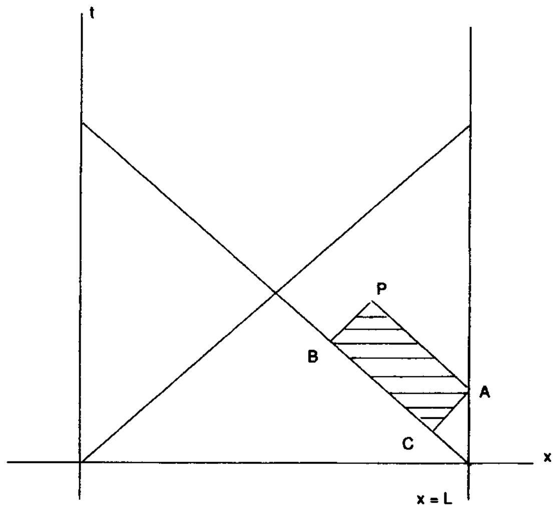

Figure 4.25: characteristic quadrilateral in region III

Now suppose that $P : (x, t)$ is in region IV. Construct a characteristic quadrilateral as shown in Figure 4.26 and again use equation 4.20 to solve for $u(x, t)$ in terms of previously determined values of u at the other three vertices of the quadrilateral.

Once we know $u(x, t)$ in region IV, we can continue these constructions to solve for $u(x, t)$ at points of regions V and VI, then move on to region VII, then to VIII, IX; and so on, working our way up Figure 4.22.

Example 4.5 *Solve the problem*

$$\begin{aligned} u_{tt} &= 9u_{xx} \text{ for } 0 < x < 4, t > 0 \\ u(x,0) &= \varphi(x) = x\sin(\pi x) \text{ for } 0 < x < 4 \\ u_t(x,0) &= \psi(x) = x(x-4) \text{ for } 0 < x < 4 \\ u(0,t) &= u(4,t) = 9\pi t^2 \text{ for } t > 0. \end{aligned}$$

We will construct $u(x, t)$ at various points (x, t). Since (x, t) is a point at which we will find the solution, we will use X and T as coordinates in the plane in computing equations of lines and coordinates of points.

Characteristics through the origin and (4, 0) divide the strip $0 \leq X \leq 4, T \geq 0$ into four regions, labeled I, II, III, and IV, as in part of Figure 4.22.

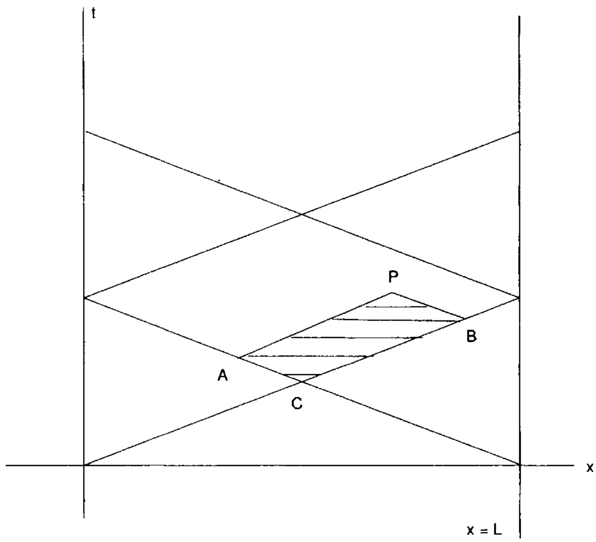

Figure 4.26: characteristic quadrilateral in region IV

First let $P:(x,t)$ be in region I. Here d'Alembert's formula applies and

$$\begin{aligned} u(x,t) &= \frac{1}{2}(\varphi(x+3t)+\varphi(x-3t)) + \frac{1}{6}\int_{x-3t}^{x+3t}\psi(s)\,ds \\ &= \frac{1}{2}(x+3t)\sin(\pi(x+3t)) + \frac{1}{2}(x-3t)\sin(\pi(x-3t)) \\ &+ x^2t + 3t^3 - 4xt. \end{aligned} \tag{4.21}$$

Next suppose that $P:(x,t)$ is in region II. Figure 4.27 shows the characteristic quadrilateral having P as one vertex, together with the relevant lines and points, which are found by routine geometry. From equation 4.20,

$$u(x,t) = u\left(0, t-\frac{x}{3}\right) + u\left(\frac{3t+x}{2}, \frac{3t+x}{6}\right) - u\left(\frac{3t-x}{2}, \frac{3t-x}{6}\right).$$

We can compute $u(0,t-x/3)$ from the boundary condition at $x=0$, and we can compute

$$u\left(\frac{3t+x}{2}, \frac{3t+x}{6}\right) \text{ and } u\left(\frac{3t-x}{2}, \frac{3t-x}{6}\right)$$

from equation 4.21, which gives the solution in region I. This is a lengthy but

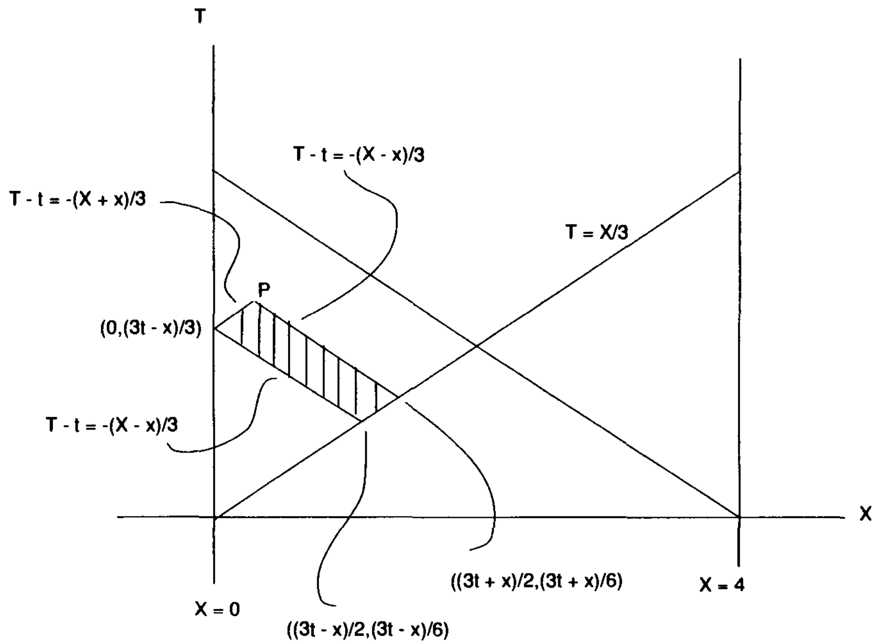

Figure 4.27: characteristic quadrilateral in region II in Example 4.5

routine calculation, and we obtain

$$u(x,t) = 9\pi\left(\frac{x-3t}{3}\right)^2 + \frac{1}{2}(x+3t)\sin(\pi(x+3t)$$
$$-\frac{1}{2}(x-3t)\sin(\pi(x-3t)) + 3xt^2 + \frac{1}{9}x^3 - 4xt.$$

This gives $u(x,t)$ for (x,t) in region II.

Now suppose that $P:(x,t)$ is in region III. Figure 4.28 shows the characteristic quadrilateral and the relevant points and lines for this case. Again applying equation 4.20, we have

$$u(x,t) = u\left(\frac{4-3t+x}{2}, \frac{4+3t-x}{6}\right)$$
$$+\ u\left(4, \frac{-4+3t+x}{3}\right) - u\left(\frac{12-x-3t}{2}, \frac{x+3t-4}{6}\right).$$

We know $u(4,(-4+3t+x)/3)$ from the given boundary condition at $x=4$, and we know

$$u\left(\frac{4+x-3t}{2}, \frac{4+3t-x}{6}\right) \text{ and } u\left(\frac{12-x-3t}{2}, \frac{x+3t-4}{6}\right)$$

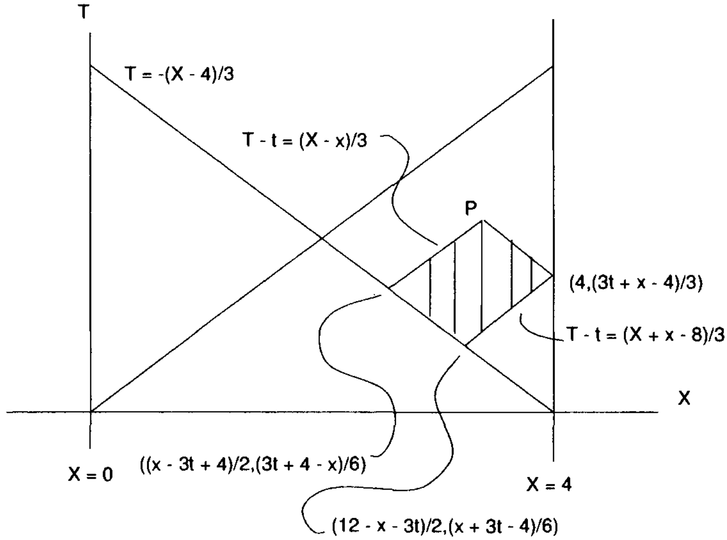

Figure 4.28: characteristic quadrilateral in region III in Example 4.5

from the solution in region I. As in region II, a routine calculation yields

$$\begin{aligned} u(x,t) &= \frac{1}{2}(x-3t)\sin(\pi(x-3t)) - \frac{1}{2}(8-x-3t)\sin(\pi(8-x-3t)) \\ &+ 9\pi\left(\frac{-4+x+3t}{3}\right)^2 + \frac{64}{9} + 12t^2 \\ &+ 4xt + \frac{4}{3}x^2 - 3xt^2 - \frac{1}{9}x^3 - 16t - \frac{16}{3}x. \end{aligned}$$

This is the solution at points in region III.

We can now work our way up the strip, using equation 4.20 to calculate $u(x,t)$ in region IV of Figure 4.22, then in regions V, VI, and VII, and so on. ◇

The calculations involved in obtaining solution values in this way can be quite involved and should be done using a computational software package.

In the discussion thus far we have glossed over a significant point, compatibility of the initial and boundary conditions at $(0,0)$ and at $(L,0)$. To illustrate, consider $u(0,0)$. On the one hand, this is $u(x,0)$ at $x=0$, hence $u(0,0)=\varphi(0)$. But $u(0,0$ is also $u(0,t)$ at $t=0$, so $u(0,0)=a(0)$. This makes sense only if

$$\varphi(0)=a(0).$$

In general, to have a solution u that is continuous with continuous first and second partial derivatives on $0 \le x \le L, t \ge 0$, the following *compatibility* *conditions* must be satisfied by the initial and boundary data:

$$a(0) = \varphi(0), a'(0) = \psi(0), a''(0) = c^2\varphi''(0) \tag{4.22}$$

at the left end, and

$$b(0) = \varphi(L), b'(0) = \psi(L), b''(0) = c^2\varphi''(L) \tag{4.23}$$

at the right end of the interval. It is easy to check that these conditions are satisfied in the example.

Problems for Section 4.7

1. Derive equation 4.20. Hint: Recall equation 4.2.

2. Derive the compatibility conditions 4.22 and 4.23.

   In each of Problems 3 through 7, obtain $u(x,t)$ for (x,t) in regions I, II and III. Verify that the initial and boundary conditions are compatible in each problem.

3.
$$\begin{aligned} u_{tt} &= u_{xx} \text{ for } 0 < x < 2, t > 0 \\ u(x,0) &= x(2-x), u_t(x,0) = 0 \text{ for } 0 < x < 2 \\ u(0,t) &= -t^2 = u(2,t) \text{ for } t > 0 \end{aligned}$$

4.
$$\begin{aligned} u_{tt} &= 4u_{xx} \text{ for } 0 < x < 3, t > 0 \\ u(x,0) &= \sin(\pi x), u_t(x,0) = 0 \text{ for } 0 < x < 3 \\ u(0,t) &= u(3,t) = t^3 \text{ for } t > 0 \end{aligned}$$

5.
$$\begin{aligned} u_{tt} &= 9u_{xx} \text{ for } 0 < x < 2, t > 0 \\ u(x,0) &= x^2(2-x), u_t(x,0) = 0 \text{ for } 0 < x < 2 \\ u(0,t) &= 18t^2, u(2,t) = -36t^2 \text{ for } t > 0 \end{aligned}$$

6.
$$\begin{aligned} u_{tt} &= 16u_{xx} \text{ for } 0 < x < 1, t > 0 \\ u(x,0) &= x\sin(\pi x), u_t(x,0) = 4x \text{ for } 0 < x < 1 \\ u(0,t) &= 16\pi t^2, u(1,t) = -16\pi t^2 + 4t \text{ for } t > 0 \end{aligned}$$

7.
$$\begin{aligned} u_{tt} &= u_{xx} \text{ for } 0 < x < 2, t > 0 \\ u(x,0) &= x(2-x)^2, u_t(x,0) = x^2 \text{ for } 0 < x < 2 \\ u(0,t) &= -4t^2, u(2,t) = 2t^2 + 4t \text{ for } t > 0 \end{aligned}$$

8. Obtain $u(x,t)$ for (x,t) in regions I, II, III and IV for the problem:

$$\begin{aligned} u_{tt} &= 4u_{xx} \text{ for } 0 < x < 1, t > 0 \\ u(x,0) &= x\cos(\pi x) + x, u_t(x,0) = x^2 \text{ for } 0 < x < 1 \\ u(0,t) &= 0, u(1,t) = 2\pi t^2 + t \text{ for } t > 0 \end{aligned}$$

9. In Example 4.5, obtain $u(x,t)$ for (x,t) in region IV.

4.8 Fourier Series Solutions on a Closed Interval

Thus far we have developed methods of solution which exploit the characteristics of the wave equation. We will now move to the use of Fourier techniques, beginning with a problem on a bounded interval:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \text{ for } 0 < x < L, t > 0 \\ u(x,0) &= \varphi(x), u_t(x,0) = \psi(x) \text{ for } 0 \le x \le L \\ u(0,t) &= u(L,t) = 0 \text{ for } t \ge 0. \end{aligned}$$

To use the *Fourier method*, or *separation of variables*, attempt a solution of the form $u(x,t) = X(x)T(t)$, the product of a function of x and a function of t. Substitute this into the wave equation to obtain

$$XT'' = c^2 X'' T$$

or

$$\frac{T''}{c^2 T} = \frac{X''}{X}.$$

In this equation the left side is a function just of t and the right side, just of x. We have separated the variables. Because x and t are independent, we can put any positive value of t we like into the left side, and the right side must equal the resulting number for all x in (0, L). Therefore, X''/X is constant for $0 < x < L$, so T''/c^2T equals the same constant for $t > 0$. This means that for some constant λ,

$$\frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda.$$

λ is called the *separation constant*. The negative sign on λ is common practice, and we would reach the same conclusion if we used λ in place of $-\lambda$.

We now have

$$X'' + \lambda X = 0 \text{ and } T'' + \lambda c^2 T = 0.$$

Since $u(0,t) = X(0)T(t) = 0$ for all t, we infer that $X(0) = 0$. Similarly, $u(L,t) = X(L)T(t) = 0$ implies that $X(L) = 0$. Actually, we could have $T(t) = 0$ for all t if $u(x,t) = 0$, which is the correct solution if the initial displacement and velocity are identically zero. A string lying horizontally, initially at rest, given no displacement, and not acted upon by any driving force, does not move.

The problem for X is therefore

$$X'' + \lambda X = 0; X(0) = X(L) = 0. \tag{4.24}$$

We want values of λ (called *eigenvalues*) for which there are nontrivial solutions for X (called *eigenfunctions*) satisfying this ordinary differential equation and the boundary conditions.

Look at the possibilities for λ.

If $\lambda = 0$, then $X'' = 0$ and $X(x) = ax + b$ for some constants a and b. But $X(0) = b = 0$, and $X(L) = aL = 0$ implies that $a = 0$ also. There is no nontrivial solution for X in problem 4.24 in the case $\lambda = 0$, so 0 is not an eigenvalue of this problem.

If $\lambda < 0$, say $\lambda = -k^2$, then $X'' - k^2 X = 0$, so

$$X = ae^{kx} + be^{-kx}.$$

Now $X(0) = a + b = 0$ implies that $a = -b$, so

$$X = a(e^{kx} - e^{-kx}).$$

If $a \neq 0$, then $X(L) = a(e^{kL} - e^{-kL}) = 0$ can be satisfied only if $e^{kL} - e^{-kL} = 0$, or $e^{2kL} = 1$. But then $2kL = 0$, which is impossible if k and L are nonzero. This means that $a = 0$ and we have only the trivial solution in this case also. Problem 4.24 has no negative eigenvalue.

If $\lambda > 0$, say $\lambda = k^2$ with $k > 0$, then $X'' + k^2 X = 0$ and

$$X = a\cos(kx) + b\sin(kx).$$

Now $X(0) = a = 0$, so $X = b\sin(kx)$. But then

$$X(L) = b\sin(kL) = 0.$$

This equation is satisfied if $b = 0$, but then $X \equiv 0$ and we will have obtained no values of λ for which there are nontrivial solutions of problem 4.24. There is another possibility—we can require that kL be an integer multiple of π,

$$kL = n\pi \text{ for } n = 1, 2, 3, \cdots.$$

Then $k = n\pi/L$, with n any positive integer, and we obtain eigenvalues

$$\lambda_n = k^2 = n^2\pi^2/L^2 \text{ for } n = 1, 2, \cdots.$$

For each positive integer n, λ_n is an eigenvalue of this problem, with corresponding eigenfunction

$$X_n(x) = \sin(n\pi x/L)$$

or any constant nonzero multiple of this function.

Now we know the admissible values of λ and corresponding solutions for X satisfying problem 4.24. With these values of λ, the problem for T has the form

$$T'' + \frac{n^2\pi^2c^2}{L^2}T = 0$$

with general solution

$$T_n(t) = a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)$$

for $n = 1, 2, \cdots$.

For each positive integer n, we now have infinitely many functions

$$\begin{aligned} u_n(x,t) &= X_n(x)T_n(t) \\ &= \left[a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right) \end{aligned} \tag{4.25}$$

which satisfy the wave equation and the boundary conditions $u(0,t) = u(L,t) = 0$. The remaining issue is to find a solution satisfying the initial conditions.

It was Fourier's brilliant insight to attempt a solution $u(x, t)$ as a superposition of all of the functions $u_n(x,t)$:

$$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right). \tag{4.26}$$

We must choose the a_n's and the b_n's so that this u satisfies the initial conditions. First, we need

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = \varphi(x).$$

This is the Fourier sine expansion on $[0, L]$ of the initial position function, so choose

$$a_n = \frac{2}{L}\int_0^L \varphi(\xi) \sin\left(\frac{n\pi\xi}{L}\right) d\xi.$$

To solve for the b_n's, formally differentiate the series term by term to obtain

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{n\pi c}{L}\left[-a_n \sin\left(\frac{n\pi ct}{L}\right) + b_n \cos\left(\frac{n\pi ct}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right).$$

The initial velocity condition requires that

$$u_t(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \sin\left(\frac{n\pi x}{L}\right) = \psi(x). \tag{4.27}$$

This is the Fourier sine expansion of ψ on $[0, L]$, so choose

$$\frac{n\pi c}{L} b_n = \frac{2}{L}\int_0^L \psi(\xi) \sin\left(\frac{n\pi\xi}{L}\right) d\xi$$

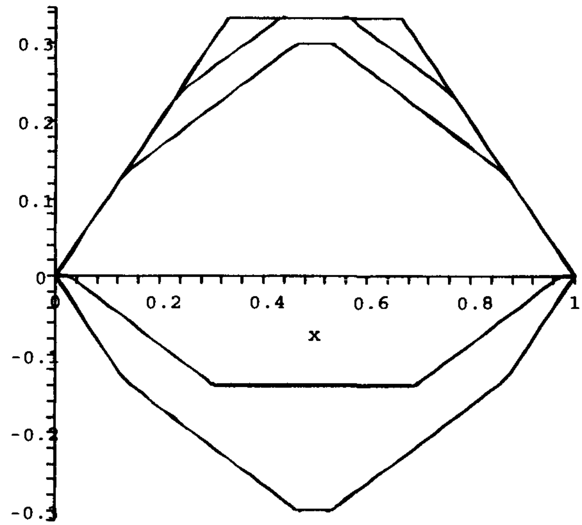

Figure 4.29: String profiles in Example 4.6 at times $t = 0, 0.05, 0.1, 1.4$, and 1.68.

or

$$b_n = \frac{2}{n\pi c}\int_0^L \psi(\xi)\sin\left(\frac{n\pi\xi}{L}\right)\,d\xi.$$

With these choices of the a_n's and b_n's, equation 4.26 is the solution of the initial-boundary value problem, assuming that φ and ψ have Fourier sine representations on $[0, L]$.

Example 4.6 *Suppose that a string of length* 1 *unit is lifted to an initial position given by* $\varphi(x)$ *and released from rest. The problem for the wave function is*

$$\begin{aligned}
u_{tt} &= 4u_{xx} \text{ for } 0 < x < 1, t > 0\\
u(x,0) = \varphi(x) &= \begin{cases} x & \text{for } 0 \le x \le 1/3 \\ 1/3 & \text{for } 1/3 \le x \le 2/3 \\ 1-x & \text{for } 2/3 \le x \le 1 \end{cases}\\
u_t(x,0) &= \psi(x) = 0 \text{ for } 0 \le x \le 1\\
u(0,t) &= u(1,t) = 0 \text{ for } t \ge 0.
\end{aligned}$$

With $\psi(x) \equiv 0$, each $b_n = 0$ in the solution 4.26. Compute

$$\begin{aligned} a_n &= 2\int_0^1 \varphi(\xi)\sin(n\pi\xi)\,d\xi \\ &= 2\left(\int_0^{1/3} \xi\sin(n\pi\xi)\,d\xi + \int_{1/3}^{2/3} \frac{1}{3}\sin(n\pi\xi)\,d\xi + \int_{2/3}^{1}(1-\xi)\sin(n\pi\xi)\,d\xi\right) \\ &= \frac{2}{n^2\pi^2}(\sin(n\pi/3)+\sin(2n\pi/3)). \end{aligned}$$

The solution is

$$u(x,t) = \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(\sin(n\pi/3)+\sin(2n\pi/3))\sin(n\pi x)\cos(2n\pi t).$$

Figure 4.29 shows the position of the string at times $t = 0$ (the initial position) and then times $t = 0.05, 0.1, 1.4$, and 1.68. These positions appear, respectively, from top to bottom in the diagram. ◇

Example 4.7 *Now suppose that the string has no initial displacement but is given an initial velocity*

$$\begin{aligned} u_{tt} &= 36u_{xx} \text{ for } 0 < x < 1, t > 0 \\ u(x,0) &= \varphi(x) = 0 \text{ for } 0 \le x \le 1 \\ u_t(x,0) &= \psi(x) = 3(1-x) \text{ for } 0 < x < 1 \\ u(0,t) &= u(1,t) = 0, \text{ for } t \ge 0. \end{aligned}$$

With $\varphi(x) \equiv 0$, each $a_n = 0$ in the solution 4.26, while

$$b_n = \frac{1}{n\pi}\int_0^1 (1-\xi)\sin(n\pi\xi)\,d\xi = \frac{1}{n^2\pi^2}.$$

The solution is

$$u(x,t) = \sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\sin(n\pi x)\sin(6n\pi t).$$

Figure 4.30 shows the shape of the string at times $t = 0.041, 0.053, 0.067, 0.074, 0.081, 0.092, 1.13, 1.18$, and 1.21. For $t = 0.041$ through $t = 0.092$, the peaks of the profiles move upward from left to right in the graph, reaching a high point at about $u = 0.04$. At $t = 1.13$ the peak has continued to move farther to the right, but this high point is now slightly below $u = 0.03$, so the wave is now moving downward. At $t = 1.18$ the wave has moved below the horizontal axis and peaks slightly below $u = -0.01$, while at $t = 1.21$ it has moved lower still, peaking farther left and just below $u = -0.03$. ◇

Example 4.8 *This example has an initial displacement and a nonzero initial velocity*

$$\begin{aligned} u_{tt} &= 9u_{xx} \text{ for } 0 < x < \pi, t > 0 \\ u(x,0) &= x^2(\pi - x). u_t(x,0) = \sin(x) \text{ for } 0 \le x \le \pi \\ u(0,t) &= u(\pi,t) = 0 \text{ for } t \ge 0. \end{aligned}$$

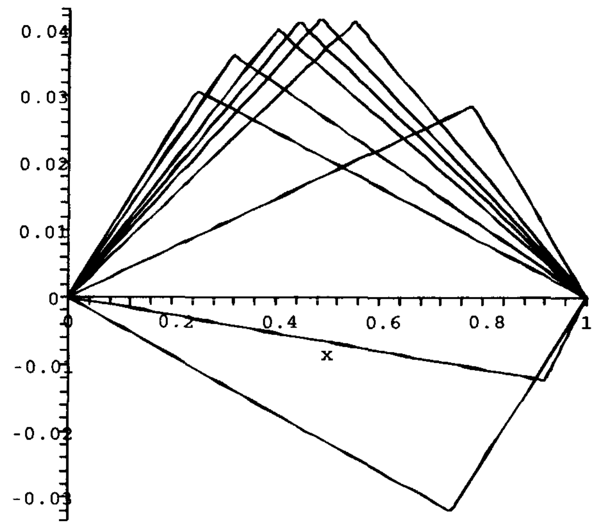

Figure 4.30: String profiles over increasing times in Example 4.7.

The coefficients in the solution 4.26 are

$$a_n = \frac{2}{\pi}\int_0^{\pi} \xi^2(\pi - \xi)\sin(n\xi)\,d\xi = -\frac{4}{n^3}[1 + 2(-1)^n]$$

and

$$b_n = \frac{2}{3n\pi}\int_0^{\pi} \sin(\xi)\sin(n\xi)\,d\xi = \begin{cases} 0 & \text{if } n = 2, 3, \cdots \\ 1/3 & \text{if } n = 1. \end{cases}$$

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} -\frac{4}{n^3}[1 + 2(-1)^n]\sin(nx)\cos(3nt) + \frac{1}{3}\sin(x)\sin(3t).$$

Figure 4.31 shows graphs of the position of the string at times $t = 0, 0.13, 0.27, 0.41, 0.57, 0.74,$ *and* 0.91. *The peaks of the waves move downward to the left as t increases over these times. For the last two times the wave has moved entirely below the horizontal axis.* ◇

Comparison of the Fourier and d'Alembert Solutions

The solution of this initial-boundary value problem on $[0, L]$ can be obtained from the d'Alembert solution on the entire real line. We will do this and compare the result with the solution by separation of variables.

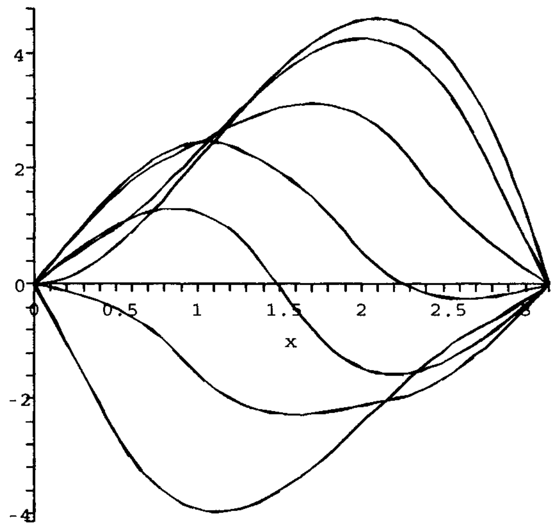

Figure 4.31: Successive positions of the string in Example 4.8.

For the problem on $[0,\ L]$, the initial position and velocity functions are defined only for $0 \leq x \leq L$. Define odd extensions of these functions to $[-L, L]$, and from these define periodic extensions φ_p of φ and ψ_p of ψ to the entire real line, each having period $2L$. Now we can write the d'Alembert solution

$$u(x,t) = \frac{1}{2}(\varphi_p(x-ct) + \varphi_p(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_p(s)\,ds$$

for the problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \text{ for } -\infty < x < \infty, t > 0 \\ u(x,0) &= \varphi_p(x), u_t(x,0) = \psi_p(x) \text{ for } -\infty \leq x \leq \infty \\ u(0,t) &= u(L,t) = 0 \text{ for } t \geq 0. \end{aligned}$$

When x is restricted to $[0,\ L]$, this expression for $u(x,\ t)$ also provides the solution for

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \text{ for } 0 < x < L, t > 0 \\ u(x,0) &= \varphi(x), u_t(x,0) = \psi(x) \text{ for } 0 \leq x \leq L \\ u(0,t) &= u(L,t) = 0 \text{ for } t \geq 0. \end{aligned}$$

We will now show how this d'Alembert solution follows from the solution by Fourier's method. First write the Fourier solution as

$$u(x,t) =$$
$$\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)$$
$$= \sum_{n=1}^{\infty} \frac{1}{2}a_n \left[\sin\left(\frac{n\pi(x-ct)}{L}\right) + \sin\left(\frac{n\pi(x+ct)}{L}\right)\right]$$
$$+ \sum_{n=1}^{\infty} \frac{1}{2}b_n \left[\cos\left(\frac{n\pi(x-ct)}{L}\right) - \cos\left(\frac{n\pi(x+ct)}{L}\right)\right]. \tag{4.28}$$

Now $\varphi(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/L)$ for $0 \le x \le L$, and this sine series has period $2L$, so

$$\varphi_p(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

for all x, and

$$\sum_{n=1}^{\infty} \frac{1}{2}a_n \left[\sin\left(\frac{n\pi(x-ct)}{L}\right) + \sin\left(\frac{n\pi(x+ct)}{L}\right)\right]$$
$$= \frac{1}{2}(\varphi_p(x-ct) + \varphi_p(x+ct)).$$

For the series of cosine terms on the right side of equation 4.28, write

$$\sum_{n=1}^{\infty} \frac{1}{2}b_n \left[\cos\left(\frac{n\pi(x-ct)}{L}\right) - \cos\left(\frac{n\pi(x+ct)}{L}\right)\right]$$
$$= \sum_{n=1}^{\infty} \frac{1}{2}\frac{n\pi}{L}b_n \int_{x-ct}^{x+ct} \sin\left(\frac{n\pi s}{L}\right) ds$$
$$= \frac{1}{2c}\int_{x-ct}^{x+ct} \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \sin\left(\frac{n\pi s}{L}\right) ds$$
$$= \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_p(s)\, ds.$$

The solution by separation of variables is therefore d'Alembert's solution in different form. This was to be expected, given the uniqueness of the solution.

Problems for Section 4.8

In each of Problems 1 through 10, write the solution of the initial-boundary value problem on $[0, L]$ for the given information, using the Fourier method. Generate graphs of the wave profile at different times, as in the examples.

1. $\varphi(x) = 1 - \cos(x), \psi(x) = 0, c = 3, L = 2\pi$

2. $\varphi(x) = 0, \psi(x) = x, c = 2, L = 1$

3. $\varphi(x) = x(1 - x), \psi(x) = 0, c = 6, L = 1$

4. $\varphi(x) = \sin(x), \psi(x) = 0, c = 4, L = \pi$

5. $\varphi(x) = 0, \psi(x) = e^{-x}, c = 3, L = 2$

6. $\varphi(x) = 1 - \cos(x), \psi(x) = 0, c = 4, L = 2\pi$

7. $\varphi(x) = \sin(x), \psi(x) = x, c = 3, L = \pi$

8. $\varphi(x) = x^2(2 - x), \psi(x) = x^2, c = 5, L = 2$

9. $\varphi(x) = \sin^2(x), \psi(x) = 1, c = 1, L = \pi$

10. $\varphi(x) = x(1 - x^2), \psi(x) = \cos(x), c = 3, L = 1$

11. In interpreting solutions of the wave equation as describing the motion of a vibrating string, it is interesting to study the effect c has on the motion. Take the solution found in Example 4.6 and adjust it for c equal to 0.2, 0.8, 1.5, 3, and 6. For each time used in the example, graph the solution for each of these values of c on the same x, u - axes. For each of these times, this displays a measure of the effect of c on the wave profile. What effect does increasing c appear to have on the motion at specific times?

12. Repeat the program of Problem 11 for the solution in Example 4.7.

13. Repeat the program of Problem 11 for the solution in Example 4.8.

14. Use separation of variables to solve the telegraph equation

$$u_{tt} + Au_t + Bu = c^2 u_{xx} \text{ for } 0 < x < L, t > 0$$

in which A, B, and c are positive constants. The boundary conditions are

$$u(0, t) = u(L, t) = 0 \text{ for } t > 0$$

and the initial conditions are

$$u(x, 0) = \varphi(x) \text{ and } u_t(x, 0) = 0 \text{ for } 0 < x < L.$$

Assume that $A^2L^2 < 4(BL^2 + c^2\pi^2)$. Hint: Proceed as in the derivation of equation 4.26, except now the differential equation for $T(t)$ is more complicated.

15. Continue with the telegraph equation

$$u_{tt} + Au_t + Bu = u_{xx} \text{ for } 0 < x < L, t > 0.$$

with $c = 1$. Suppose that

$$u(0, t) = u(L, t) = 0 \text{ for } t \geq 0$$
$$u(x, 0) = \varphi(x), u_t(x, 0) = \psi(x) \text{ for } 0 \leq x \leq L.$$

(a) Prove that for any $T > 0$,

$$\int_0^L (u_x^2 + u_t^2 + bu^2)_{t=T}\, dx \le \int_0^L (u_x^2 + u_t^2 + bu^2)_{t=0}\, dx.$$

Hint: First show that

$$(2u_t u_x)_x = (u_x^2 + u_t^2 + bu^2)_t + 2au_t^2.$$

(b) Use the integral inequality from (a) to show that the initial boundary value problem for the telegraph equation can have only one solution.

16. The current $i(x,t)$ and voltage $v(x,t)$ at time t and distance x from one end of a transmission line satisfy the system

$$\begin{aligned} -v_x &= Ri + Li_t \\ -i_x &= Sv + Kv_t, \end{aligned}$$

where R is the resistance, L the inductance, S the leakage conductance, and K the capacitance to ground, all per unit length and all assumed constant. By differentiating appropriately and eliminating i, show that v satisfies the telegraph equation (Problem 14). Similarly, show that i also satisfies the telegraph equation.

17. Let $\theta(x,t)$ be the angular displacement at time t of the cross-section at x of a homogeneous cylindrical shell about the x - axis. It can be shown that

$$\theta_{tt} = a^2\theta_{xx} \text{ for } 0 < x < L, t > 0.$$

Solve this equation subject to the conditions

$$\theta(x,0) = \varphi(x), \theta_t(x,0) = 0 \text{ for } 0 < x < L$$

if the ends of the shell are fixed elastically, which means that

$$\theta_x(0,t) - \alpha\theta(0,t) = 0, \theta_x(L,t) + \alpha\theta(L,t) = 0$$

for $t > 0$ and some positive constant α.

4.9 A Nonhomogeneous Problem on a Closed Interval

Problems involving a nonhomogeneous wave equation can sometimes be solved by separation of variables after a change of variables. As an illustration of this technique, we solve the initial-boundary value problem

$$\begin{aligned} u_{tt} &= u_{xx} + Ax \text{ for } 0 < x < L, t > 0 \\ u(x,0) &= u_t(x,0) = 0 \text{ for } 0 \le x \le L \\ u(0,t) &= u(L,t) = 0 \text{ for } t \ge 0. \end{aligned}$$

This problem models the motion of a string pegged at its ends, with zero initial velocity and horizontal initial position, but with a forcing term acting parallel to the u - axis in the plane of motion and proportional to the distance from the left end of the string. A is a positive constant.

Let $u(x,t) = X(x)T(t)$, to get

$$XT'' = X''T + Ax.$$

We cannot isolate all terms involving x on one side of an equation and all terms involving t on the other. Separation of variables fails here. In such a case it is sometimes possible to transform the problem into one to which separation of variables applies. Let

$$u(x,t) = U(x,t) + f(x).$$

The idea is to choose f so that U satisfies an initial-boundary value problem that we can solve. Substituting u into the wave equation yields

$$U_{tt} = U_{xx} + f''(x) + Ax$$

and this is just the homogeneous wave equation $U_{tt} = U_{xx}$ for U if we choose f so that

$$f''(x) + Ax = 0.$$

Now look at the boundary conditions. First

$$U(0,t) = u(0,t) - f(0) = -f(0)$$

and

$$U(L,t) = u(L,t) - f(L) = -f(L)$$

for $t \geq 0$. These conditions will reduce to $U(0,t) = 0$ and $U(L,t) = 0$ if we choose f so that

$$f(0) = f(L) = 0.$$

We therefore want to choose f so that

$$f''(x) + Ax = 0 \text{ and } f(0) = f(L) = 0.$$

First integrate $f''(x) = -Ax$ twice to obtain

$$f(x) = -\frac{A}{6}x^3 + Bx + C.$$

Then

$$f(0) = C = 0$$

and

$$f(L) = -\frac{A}{6}L^3 + BL = 0.$$

Then $B = (A/6)L^2$, so

$$f(x) = -\frac{1}{6}Ax^3 + \frac{1}{6}AL^2x = \frac{A}{6}x(L^2 - x^2).$$

Finally, look at the initial conditions in the problem for U. We need

$$U(x,0) = u(x,0) - f(x) = -f(x)$$

and

$$U_t(x,0) = u_t(x,0) = 0.$$

In summary, the initial-boundary value problem for U is

$$\begin{aligned}
U_{tt} &= U_{xx} \text{ for } 0 < x < L, t > 0 \\
U(x,0) &= -\frac{A}{6}x(L^2 - x^2) \text{ for } 0 \le x \le L \\
U_t(x,0) &= 0 \text{ for } 0 \le x \le L \\
U(0,t) &= U(L,t) = 0 \text{ for } t \ge 0.
\end{aligned}$$

From equation 4.26 the solution of this problem is

$$U(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi t}{L}\right),$$

in which

$$a_n = \frac{2}{L}\int_0^L -\frac{A}{6}\xi(L^2 - \xi^2)\sin\left(\frac{n\pi\xi}{L}\right)d\xi = \frac{2AL^3(-1)^n}{n^3\pi^3}.$$

Therefore,

$$u(x,t) = \frac{2AL^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi t}{L}\right) + \frac{A}{6}x(L^2 - x^2).$$

Figures 4.32 through 4.35 show graphs of this solution for $t = 0.0009, 0.0019$, 0.0032, and 0.0047, with $A = 1$ and $L = \pi$.

Example 4.9 *We will solve the initial-boundary value problem*

$$\begin{aligned}
u_{tt} &= c^2 u_{xx} + K \text{ for } 0 < x < \pi, t > 0 \\
u(0,t) &= u(\pi,t) = 0 \text{ for } t > 0 \\
u(x,0) &= \varphi(x) = x(\pi - x), u_t(x,0) = 0 \text{ for } 0 \le x \le \pi,
\end{aligned}$$

in which K is a positive constant.

Let $u(x,t) = U(x,t) + f(x)$. Again, the idea is to choose f so that the problem for U is solvable by standard techniques. Substitute $u(x,t) = U(x,t) + f(x)$ into the partial differential equation, to get

$$U_{tt} = c^2(U_{xx} + f''(x)) + K,$$

which is the homogeneous wave equation if we choose f so that $c^2 f''(x) + K = 0$, or

$$f''(x) = -\frac{K}{c^2}.$$

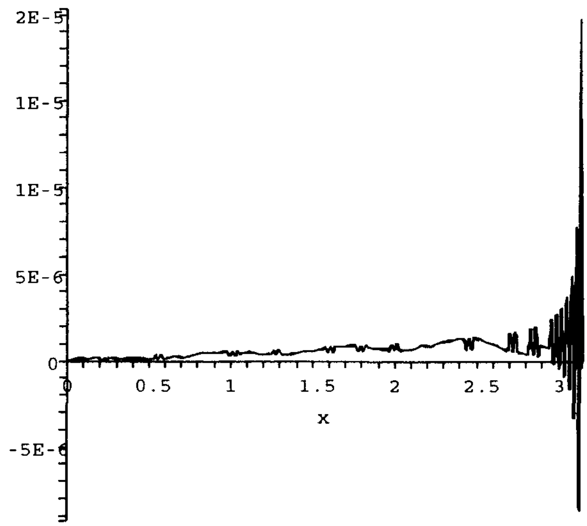

Figure 4.32: String profile at time $t = 0.0009$.

Next,

$$u(0,t) = U(0,t) + f(0) = U(0,t) = 0$$

if $f(0) = 0$, *and*

$$u(\pi,t) = U(\pi,t) + f(\pi) = U(\pi,t) = 0$$

if $f(\pi) = 0$. *Thus choose f to be the solution of*

$$f''(x) = -\frac{K}{c^2} \text{ and } f(0) = f(\pi) = 0.$$

By integrating twice and solving for the constants of integration, we obtain

$$f(x) = -\frac{K}{2c^2}x^2 + \frac{K\pi}{2c^2}x.$$

Finally,

$$u_t(x,0) = U_t(x,0) = 0 \; for \; t \geq 0$$

and

$$u(x,0) = U(x,0) + f(x) = \varphi(x)$$

if

$$U(x,0) = \varphi(x) - f(x) \; for \; 0 \leq x \leq L.$$

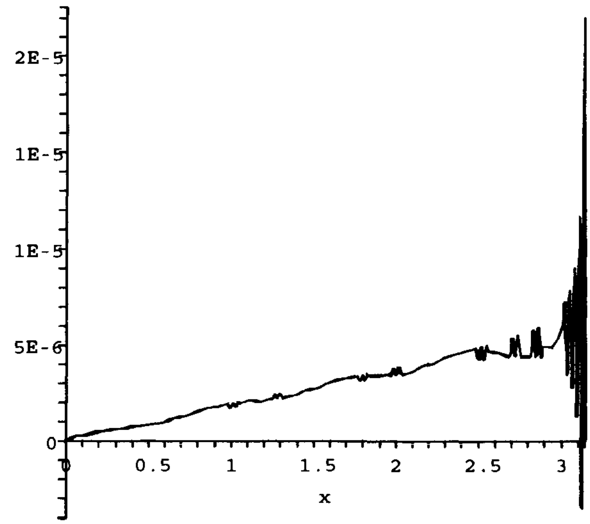

Figure 4.33: Profile at time $t = 0019$.

The problem for U is

$$U_t = c^2 U_{xx} \text{ for } 0 < x < L, t > 0$$
$$U(x,0) = x(\pi - x) + \frac{K}{2c^2}x(x-\pi), U_t(x,0) = 0 \text{ for } 0 \le x \le \pi$$
$$U(0,t) = U(\pi,t) = 0 \text{ for } t \ge 0.$$

The solution for $U(x, t)$ is

$$U(x,t) = \sum_{n=1}^{\infty} a_n \sin(nx)\cos(nct)$$

in which

$$a_n = \frac{2}{\pi}\int_0^{\pi}\left(\xi(\pi-\xi) + \frac{K}{2c^2}\xi(\xi-\pi)\right)\sin(n\xi)\,d\xi$$
$$= \frac{(2K-4c^2)((-1)^n - 1)}{n^3c^2\pi}.$$

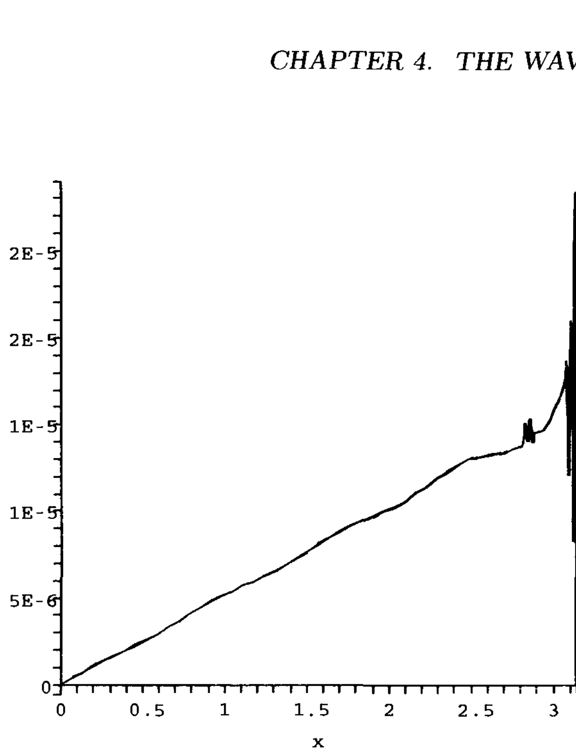

Figure 4.34: Profile at time $t = 0032$.

Then

$$\begin{aligned} u(x,t) &= U(x,t) + f(x) \\ &= \sum_{n=1}^{\infty} \left(\frac{(2K - 4c^2)((-1)^n - 1)}{n^3 c^2 \pi} \right) \sin(nx)\cos(nct) + \frac{Kx}{2c^2}(\pi - x). \end{aligned}$$

We may think of this initial-boundary value problem as modeling the motion of a string initially placed in the configuration of the parabola $y = x(\pi - x)$, for $0 \leq x \leq \pi$, released from rest, and having a constant vertical driving force acting on it. If the driving force has magnitude k per unit length and ρ is the constant density of the string, then $K = k/\rho$. Figure 4.36 shows string profiles at times $t = 0.86, 0.92, 0.98, 1.08, 1.13$, and 1.25, with $c = 2$ and $K = 3$. The string moves downward from its initial parabolic position for the times considered. ◇

Problems for Section 4.9

1. In Example 4.9, write the solution for the following values of K: 0.4, 2.5, 5.6, 10.2, and 15.1. Let $c = 2$. For each value of K, plot the graph of the solutions for the same value of t on the same u, t - axes, using the time values in Figure 4.36. This will suggest, for each time, the effect of the magnitude of the forcing term K on the resulting motion.

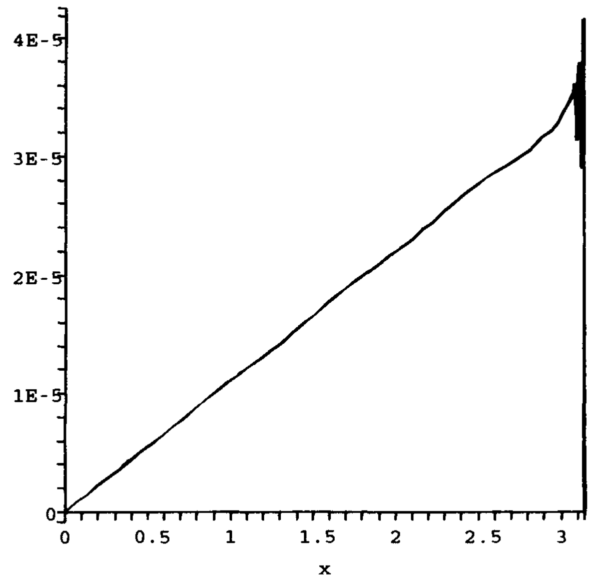

Figure 4.35: $t = 0.0047$.

2. Solve

$$\begin{aligned} u_{tt} &= 9u_{xx} + Ax^2 \text{ for } 0 < x < 1, t > 0 \\ u(x,0) &= u_t(x,0) = 0 \text{ for } 0 \leq x \leq 1 \\ u(0,t) &= u(1,t) = 0 \text{ for } t \geq 0, \end{aligned}$$

in which A is a positive constant. Generate sets of graphs of the wave profile at different times, first with $A = 1.5$, then with $A = 5.7$, and finally, with $A = 15.3$. This provides some feeling for the influence of the forcing term Ax^2.

3. Solve

$$\begin{aligned} u_{tt} &= 4u_{xx} + \cos(x) \text{ for } 0 < x < \pi, t > 0 \\ u(x,0) &= u_t(x,0) = 0 \text{ for } 0 \leq x \leq \pi \\ u(0,t) &= u(\pi,t) = 0 \text{ for } t \geq 0. \end{aligned}$$

Generate graphs of the solution at different times.

4. Write the solution of Problem 3 adjusted to have $\cos(\alpha x)$ in place of $\cos(x)$. Choose a value of t and graph the solutions corresponding to $\alpha = 0.5$, 1.2, 1.6, 2.1, and 2.8 on the same set of axes. Repeat this experiment

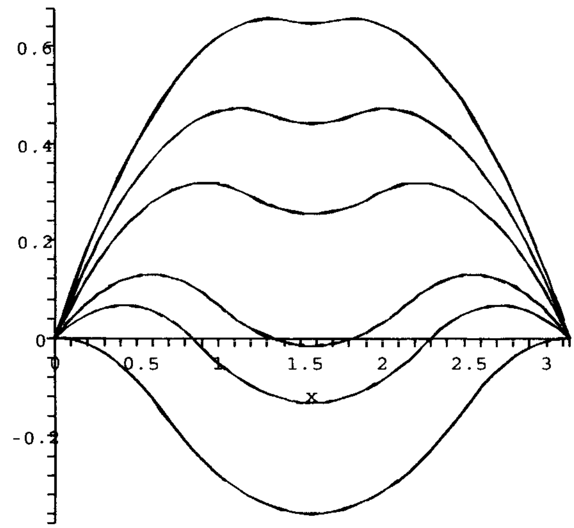

Figure 4.36: String profiles at successive times in Example 4.9.

for different times. What is the effect on the motion of increasing the frequency factor α in the forcing term?

5. Solve

$$\begin{aligned} u_{tt} &= u_{xx} - 3e^{-x} \text{ for } 0 < x < 2, t > 0 \\ u(x,0) &= u_t(x,0) = 0 \text{ for } 0 \leq x \leq 2 \\ u(0,t) &= u(2,t) = 0 \text{ for } t \geq 0. \end{aligned}$$

Draw graphs of the wave profile at various times.

6. Solve

$$\begin{aligned} u_{tt} &= 16u_{xx} - e^{-x} \text{ for } 0 < x < 3, t > 0 \\ u(x,0) &= u_t(x,0) = 0 \text{ for } 0 \leq x \leq 3 \\ u(0,t) &= u(3,t) = 0 \text{ for } t \geq 0. \end{aligned}$$

Draw graphs of the solution at various times.

7. Solve

$$\begin{aligned} u_{tt} &= 9u_{xx} + 4x \text{ for } 0 < x < 1, t > 0 \\ u(0,t) &= u(1,t) = 0 \text{ for } t > 0 \\ u(x,0) &= 0, u_t(x,0) = 1 \text{ for } 0 < x < 1. \end{aligned}$$

8. Solve

$$
\begin{aligned}
u_{tt} &= 4u_{xx} \text{ for } 0 < x < 9, t > 0 \\
u(0,t) &= 0, u(9,t) = 1 \text{ for } t > 0 \\
u(x,0) &= 0, u_t(x,0) = x \text{ for } 0 < x < 9.
\end{aligned}
$$

Draw graphs showing the shape of the wave profile at various times.

9. Try to adapt the method of this section to solve

$$
\begin{aligned}
u_{tt} &= 4u_{xx} + t^2 \text{ for } 0 < x < 1, t > 0 \\
u(x,0) &= u_t(x,0) = 0 \text{ for } 0 \le x \le 1 \\
u(0,t) &= u(1,t) = 0 \text{ for } t \ge 0.
\end{aligned}
$$

10. Consider the general wave equation

$$u_{tt} = c^2 u_{xx} + g(x,t) \text{ for } 0 < x < L, t > 0.$$

Suppose that u is a solution which is continuous with continuous first and second partial derivatives. Prove that

$$\frac{d}{dt}\int_a^b \frac{1}{2}(u_t^2 + c^2 u_x^2)\,dx = (c^2 u_t u_x)_a^b + \int_a^b g u_t\,dx$$

for $0 < a \le x \le b < L$ any $t > 0$. Hint: Multiply the wave equation by u_t, integrate the resulting equation from a to b, and use integration by parts on one of the resulting integrals. It will also be useful to note that

$$u_t u_{tt} = \frac{1}{2}\frac{\partial}{\partial t} u_t^2,$$

with a similar relationship for $u_x u_{xt}$.

11. Assume that the problem

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx} + f(x,t) \text{ for } 0 < x < L, t > 0 \\
u(0,t) &= A(t), u(L,t) = B(t) \text{ for } t \ge 0 \\
u(x,0) &= \varphi(x), u_t(x,0) = \psi(x) \text{ for } 0 \le x \le L
\end{aligned}
$$

has a solution, assuming that f, A, B, φ, and ψ are continuous and that the conditions are compatible at the endpoints. Show that this solution is unique. Hint: Suppose that there are two solutions u and v and let $w = u - v$. Determine the initial-boundary value problem satisfied by w and let

$$E(t) = \frac{1}{2}\int_0^L (w_t^2 + c^2 w_x^2)\,dx$$

for $t \ge 0$. Calculate $E'(t)$, using an integration by parts on the term involving w_{xt}, and show that $E'(t) = 0$, hence $E(t)$ is constant. Finally, show that $E(t)$ is identically zero and conclude that $w_t^2 + c^2 w_x^2$ is identically zero. Use this to show that $w(x,t) \equiv 0$.

4.10 The Cauchy Problem by Fourier Integral

We have solved the Cauchy problem

$$u_{tt} = c^2 u_{xx} \text{ for } -\infty < x < \infty, t > 0$$
$$u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \text{ for } -\infty < x < \infty. \tag{4.29}$$

with d'Alembert's formula. We will illustrate the Fourier method on this unbounded interval, considering the special case that the initial velocity $\psi(x)$ is identically zero.

Let $u(x,t) = X(x)T(t)$ in the wave equation to obtain

$$XT'' = c^2 X'' T$$

or, as with the bounded interval,

$$X'' + \lambda X = 0, T'' + \lambda c^2 T = 0$$

for some separation constant λ. Consider cases on λ, noting that the ordinary differential equation for X has no boundary conditions to place limitations on solutions. In this absence, we impose the requirement that solutions must be bounded.

If $\lambda = 0$, then $X = ax + b$. There are no boundary conditions to restrict a or b. For a bounded solution, let $a = 0$, so $X =$ constant. This means that $\lambda = 0$ is an eigenvalue with constant eigenfunctions.

If $\lambda < 0$, write $\lambda = -\omega^2$ with $\omega > 0$. Then $X = ae^{\omega x} + be^{-\omega x}$. But $e^{\omega x} \to \infty$ as $x \to \infty$, so we must choose $a = 0$ to have a bounded solution. And $e^{-\omega x} \to \infty$ as $x \to -\infty$, so we must choose $b = 0$ also. The case $\lambda < 0$ allows no nontrivial bounded solutions for X. This problem has no negative eigenvalue.

If $\lambda > 0$, write $\lambda = \omega^2$ with $\omega > 0$. In this case, $X = a\cos(\omega x) + b\sin(\omega x)$. This is a bounded solution for X for any choices of a and b. Every positive number $\lambda = \omega^2$ is an eigenvalue, with eigenfunctions

$$a\cos(\omega x) + b\sin(\omega x).$$

Since this eigenfunction is constant if $\omega = 0$, we can combine the cases $\lambda = 0$ and $\lambda > 0$ to write

$$X_\omega(x) = a_\omega \cos(\omega x) + b_\omega \sin(\omega x)$$

for $\lambda = \omega^2 \geq 0$.

Now turn to the equation for T. In view of what we have found about λ, this is

$$T'' + c^2\omega^2 T = 0$$

with $\omega \geq 0$. If $\omega = 0$, this has solution $T =$ constant. If $\omega > 0$, then

$$T = d\cos(\omega ct) + k\sin(\omega ct).$$

Because the initial velocity is zero,

$$u_t(x,0) = X(x)T'(0) = 0,$$

implying that $T'(0) = 0$. This forces us to choose $k = 0$.

For $\omega \geq 0$, we now have functions of the form

$$u_\omega(x,t) = [a_\omega \cos(\omega x) + b_\omega \sin(\omega x)] \cos(\omega ct)$$

which satisfy the wave equation and the condition that the initial velocity is zero. To satisfy the condition that the initial position is given by $\varphi(x)$, we must generally attempt a superposition, which in this case is over all $\omega \geq 0$. Such a superposition is accomplished by integrating $u_\omega(x,t)$ over $\omega \geq 0$, with $\int_0^\infty \cdots d\omega$ replacing the infinite series $\sum_{n=1}^\infty$ superposition used in the case of a bounded interval. There we had an eigenvalue $\lambda_n = n^2\pi^2/L^2$ corresponding to each positive integer (*discrete spectrum*), while in the present circumstance the eigenvalues cover the right half-line $\omega \geq 0$ (*continuous spectrum*). The totality of eigenvalues is called the *spectrum* of the partial differential equation.

We are therefore led to attempt a solution in the form of an integral superposition

$$\begin{aligned} u(x,t) &= \int_0^\infty u_\omega(x,t)\, d\omega \\ &= \int_0^\infty [a_\omega \cos(\omega x) + b_\omega \sin(\omega x)] \cos(\omega ct)\, d\omega. \end{aligned} \tag{4.30}$$

We require that

$$u(x,0) = \int_0^\infty [a_\omega \cos(\omega x) + b_\omega \sin(\omega x)]\, dx = \varphi(x).$$

This is the Fourier integral representation of $\varphi(x)$ on the real line, so choose

$$a_\omega = \frac{1}{\pi}\int_{-\infty}^\infty \varphi(s)\cos(\omega s)\, ds \text{ and } b_\omega = \frac{1}{\pi}\int_{-\infty}^\infty \varphi(s)\sin(\omega s)\, ds.$$

When this integral representation converges to $\varphi(x)$, equation 4.30 is the solution of this Cauchy problem.

Example 4.10 *Let*

$$\varphi(x) = \begin{cases} \cos(x) & \text{for } -\pi/2 \leq x \leq \pi/2 \\ 0 & \text{for } |x| > \pi/2. \end{cases}$$

Then

$$a_\omega = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos(s)\cos(\omega s)\, ds = \begin{cases} \frac{2}{\pi}\frac{\cos(\pi\omega/2)}{1-\omega^2} & \text{for } \omega \neq 1 \\ \frac{1}{2} & \text{for } \omega = 1. \end{cases}$$

Notice that a_ω is actually a continuous function of ω, since

$$\lim_{\omega \to 1} a_\omega = \frac{1}{2} = a_1.$$

Next,

$$b_\omega = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos(s)\sin(\omega s)\,ds = 0.$$

The solution for this initial displacement, and zero initial velocity, is

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\cos(\pi\omega/2)}{1-\omega^2}\cos(\omega x)\cos(\omega ct)\,d\omega. \diamond$$

Solution by Fourier Transform

We will illustrate the use of the Fourier transform for the Cauchy problem just solved (again with $\psi = 0$). Since $-\infty < x < \infty$, we can attempt a Fourier transform $\mathcal{F}$ in x, carrying t through the transform as a parameter. Let $\mathcal{F}[u(x,t)](\omega) = \widehat{u}(\omega,t)$.

Apply $\mathcal{F}$ to the wave equation:

$$\mathcal{F}[u_{tt}] = c^2\mathcal{F}[u_{xx}].$$

Now

$$\begin{aligned}\mathcal{F}[u_{tt}(x,t)](\omega) &= \int_{-\infty}^{\infty} u_{tt}(x,t)e^{-i\omega x}\,dx \\ &= \frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}\,dx = \widehat{u}_{tt}(\omega,t).\end{aligned}$$

The operation $\partial/\partial t$ of differentiation with respect to time passes through the transform because the transform is with respect to x, and x and t are independent.

For the transform of u_{xx}, use the operational formula for the Fourier transform (equation 3.25):

$$\mathcal{F}[u_{xx}(x,t)](\omega) = (i\omega)^2\widehat{u}(\omega,t) = -\omega^2\widehat{u}(\omega,t).$$

Therefore, application of the transform in the space variable to the wave equation yields

$$\widehat{u}_{tt}(\omega,t) = -c^2\omega^2\widehat{u}(\omega,t).$$

Think of this as an ordinary differential equation in t, with ω carried along as a parameter. The general solution is

$$\widehat{u}(\omega,t) = \alpha_\omega\cos(\omega ct) + \beta_\omega\sin(\omega ct).$$

Now,

$$\alpha_\omega = \widehat{u}(\omega,0) = \mathcal{F}[u(x,0)](\omega) = \mathcal{F}[\varphi(x)](\omega) = \widehat{\varphi}(\omega),$$

the Fourier transform of the initial position function. And

$$\omega c\beta_\omega = \widehat{u}_t(\omega,0) = \mathcal{F}[u_t(x,0)](\omega) = 0$$

because we are taking the initial velocity to be zero. Therefore, $\beta_\omega = 0$ and

$$\widehat{u}(\omega, t) = \widehat{\varphi}(\omega)\cos(\omega ct).$$

This is the Fourier transform of the solution of the Cauchy problem. Apply the inverse Fourier transform (equation 3.24) to write the solution

$$u(x,t) = \mathcal{F}^{-1}[\widehat{u}(\omega,t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{\varphi}(\omega)\cos(\omega ct)e^{i\omega x}\,d\omega. \tag{4.31}$$

We will show that this solution equals the solution 4.30 obtained using the Fourier integral. For this discussion only, let $u_{tr}(x,t)$ denote the solution 4.31 obtained by transform, and $u_{fi}(x,t)$ the solution 4.30 by Fourier integral. Begin by inserting the formula for $\widehat{\varphi}(\omega)$ into the solution by transform 4.31:

$$\begin{aligned}
u_{tr}(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\varphi(s)e^{-i\omega s}\,ds\right)\cos(\omega ct)e^{i\omega x}\,d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\omega(s-x)}\cos(\omega ct)\,d\omega\varphi(s)\,ds \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\cos(\omega(s-x)) - i\sin(\omega(s-x)))\cos(\omega ct)\,d\omega\varphi(s)\,ds \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos(\omega(s-x))\cos(\omega ct)\varphi(s)\,d\omega\,ds,
\end{aligned}$$

the coefficient of i being omitted because $u_{tr}(x,t)$ is real. The integrand in the last integral is an even function in ω, being unchanged by a replacement of ω by $-\omega$. Therefore,

$$u_{tr}(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty}\cos(\omega(s-x))\cos(\omega ct)\varphi(s)\,d\omega\,ds.$$

Now go back to the solution 4.30 by Fourier integral and insert the formulas for the coefficients to obtain

$$\begin{aligned}
&u_{fi}(x,t) = \\
&\quad\frac{1}{\pi}\int_0^{\infty}\left[\left(\int_{-\infty}^{\infty}\varphi(s)\cos(\omega s)\,ds\right)\cos(\omega x)\right. \\
&\quad\left.+\left(\int_{-\infty}^{\infty}\varphi(s)\sin(\omega s)\,ds\right)\sin(\omega x)\right]\cos(\omega ct)\,d\omega \\
&\quad= \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\int_0^{\infty}[\cos(\omega s)\cos(\omega x) + \sin(\omega s)\sin(\omega x)]\cos(\omega ct)\,d\omega\right)\varphi(s)\,ds \\
&\quad= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^{\infty}\cos(\omega(s-x))\cos(\omega ct)\varphi(s)\,d\omega\,ds = u_{tr}(x,t).
\end{aligned}$$

Problems for Section 4.10

In each of Problems 1 through 6, solve the boundary value problem 4.29 for the given initial position, assuming zero initial velocity, using the Fourier integral approach through separation of variables.

1. $\varphi(x) = e^{-|x|}$

2. $$\varphi(x) = \begin{cases} 1 - x^2 & \text{for } -1 \leq x \leq 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

3. $$\varphi(x) = \begin{cases} \sin(x) & \text{for } -\pi \leq x \leq \pi \\ 0 & \text{for } |x| > \pi \end{cases}$$

4. $$\varphi(x) = \begin{cases} 4 - |x| & \text{for } -4 \leq x \leq 4 \\ 0 & \text{for } |x| > 4 \end{cases}$$

5. $$\varphi(x) = \begin{cases} e^{-x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

6. $$\varphi(x) = \begin{cases} \sin^2(x) & \text{for } -4\pi \leq x \leq 4\pi \\ 0 & \text{for } |x| > 4\pi \end{cases}$$

7. Use separation of variables to write an integral solution of
$$u_{tt} = c^2 u_{xx} \text{ for } -\infty < x < \infty, t > 0$$
$$u(x,0) = 0, u_t(x,0) = \psi(x) \text{ for } -\infty < x < \infty.$$

8. Solve the boundary value problem of Problem 7 using the Fourier transform. Show that the solution obtained by transform agrees with that obtained by the separation of variables.

In each of Problems 9 through 12, use the Fourier method to solve the boundary value problem 4.29 with zero initial displacement and the given initial velocity function.

9. $\psi(x) = e^{-|x|}$

10. $$\psi(x) = \begin{cases} k & \text{for } -\alpha \leq x \leq \alpha \\ 0 & \text{for } |x| > \alpha, \end{cases}$$
in which k and α are positive constants.

11. $$\psi(x) = \begin{cases} \cos(x) & \text{for } -\pi/2 \leq x \leq \pi/2 \\ 0 & \text{for } |x| > \pi/2 \end{cases}$$

12. $$\psi(x) = \begin{cases} x & \text{for } -5 \leq x \leq 5 \\ 0 & \text{for } |x| > 5 \end{cases}$$

4.11 A Wave Equation in Two Space Dimensions

We will solve the initial-boundary value problem

$$\begin{aligned} u_{tt} &= c^2(u_{xx} + u_{yy}) \text{ for } 0 < x < a, 0 < y < b \\ u(x,0,t) &= u(x,b,t) = 0 \text{ for } 0 \leq x \leq a, t \geq 0 \\ u(0,y,t) &= u(a,y,t) = 0 \text{ for } 0 \leq y \leq b, t \geq 0 \\ u(x,y,0) &= \varphi(x,y), u_t(x,y,0) = 0 \text{ for } 0 \leq x \leq a, 0 \leq y \leq b. \end{aligned}$$

This problem models vibrations of a membrane fastened on a rectangular frame, given an initial displacement $\varphi(x,y)$ at (x, y) but no initial velocity.

To attempt a separation of variables with this problem, put $u(x,y,t) = X(x)Y(y)T(t)$ into the wave equation to obtain

$$XYT'' = c^2(X''YT + XY''T)$$

or

$$\frac{T''}{c^2T} - \frac{Y''}{Y} = \frac{X''}{X}.$$

The left side of this equation can be fixed by choosing specific values of y and t. Therefore the right side is constant for $0 < x < a$. For some constant λ,

$$\frac{T''}{c^2T} - \frac{Y''}{Y} = \frac{X''}{X} = -\lambda.$$

Now

$$\frac{T''}{c^2T} + \lambda = \frac{Y''}{Y} \text{ and } X'' + \lambda X = 0.$$

In the equation for T and Y, the left side depends only on t and the right side only on y, and these variables are independent. Therefore, for some constant μ,

$$\frac{T''}{c^2T} + \lambda = \frac{Y''}{Y} = -\mu.$$

Now we have

$$Y'' + \mu Y = 0 \text{ and } T'' + (\lambda + \mu)c^2T = 0.$$

With two space variables we need two separation constants because it is impossible to isolate three variables independently on opposite sides of one equation.

Now consider the boundary conditions. First,

$$u(0,y,t) = X(0)Y(y)T(t) = 0$$

implies that $X(0) = 0$. Similarly, $u(a,y,t) = 0$ implies that $X(a) = 0$. The problem for X is

$$X'' + \lambda X = 0; X(0) = X(a) = 0.$$

We solved this problem in Section 4.8, finding eigenvalues and eigenfunctions

$$\lambda_n = \frac{n^2\pi^2}{a^2} \text{ and } X_n(x) = \sin\left(\frac{n\pi x}{a}\right) \text{ for } n = 1, 2, \cdots.$$

Since $u(x,0,t) = u(x,b,t) = 0$, then $Y(0) = Y(b) = 0$ and the problem for Y is

$$Y'' + \mu Y = 0; Y(0) = Y(b) = 0.$$

This problem has eigenvalues and eigenfunctions

$$\mu_m = \frac{m^2\pi^2}{b^2} \text{ and } Y_m(y) = \sin\left(\frac{m\pi y}{b}\right) \text{ for } m = 1, 2, \cdots.$$

Notice that n and m independently take on all positive integer values.

Finally, since $u_t(x,y,0) = X(x)Y(y)T'(0) = 0$, then $T'(0) = 0$. The problem for T is

$$T'' + \left(\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}\right)c^2T = 0 \text{ and} T'(0) = 0.$$

This has as a solution any constant multiple of

$$T_{nm}(y) = \cos(\alpha_{nm}ct),$$

in which

$$\alpha_{nm} = \sqrt{\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}}.$$

For each positive integer n and m we now have a function

$$u_{nm}(x,y,t) = b_{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\cos(\alpha_{nm}ct)$$

that satisfies the wave equation, the boundary conditions, and the initial condition $u_t(x,y,0) = 0$. There remains to satisfy the initial position condition, and this generally requires a superposition of these functions. This is a double sum, since n and m independently take on all positive integer values. Let

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\cos(\alpha_{nm}ct). \tag{4.32}$$

We need to choose the coefficients to satisfy

$$u(x,y,0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) = \varphi(x,y). \tag{4.33}$$

This is a *double Fourier sine expansion* of φ on the rectangle. We will reason informally to obtain the coefficients as follows. Fix y in [0, b]. For this $y, \varphi(x,y)$ is a function of x on [0, a]. Think of

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} b_{nm}\sin\left(\frac{m\pi y}{b}\right)\right)\sin\left(\frac{n\pi x}{a}\right) = \varphi(x,y) \tag{4.34}$$

as the Fourier sine expansion of this function of x for the selected y. This means that the coefficient (the summation in large parentheses in equation 4.34) is the Fourier sine coefficient in this expansion. Therefore, for each $n = 1, 2, \cdots$,

$$\sum_{m=1}^{\infty} b_{nm} \sin\left(\frac{m\pi y}{b}\right) = \frac{2}{a}\int_0^a \varphi(\xi, y) \sin\left(\frac{n\pi\xi}{a}\right) d\xi = h_n(y). \tag{4.35}$$

The integral in equation 4.35 defines a function $h_n(y)$ for $0 \le y \le b$. We can think of

$$h_n(y) = \sum_{m=1}^{\infty} b_{nm} \sin\left(\frac{m\pi y}{b}\right)$$

as the Fourier sine expansion of h_n on $[0, b]$, for $n = 1, 2, 3, \cdots$. Therefore, b_{nm} is the mth Fourier sine coefficient in this expansion:

$$\begin{aligned}
b_{nm} &= \frac{2}{b}\int_0^b h_n(\zeta) \sin\left(\frac{m\pi\zeta}{b}\right) d\zeta \\
&= \frac{2}{b}\int_0^b \int_0^a \left(\frac{2}{a}\varphi(\xi, \zeta) \sin\left(\frac{n\pi\xi}{a}\right) d\xi\right) \sin\left(\frac{m\pi\zeta}{b}\right) d\zeta \\
&= \frac{4}{ab}\int_0^b \int_0^a \varphi(\xi, \zeta) \sin\left(\frac{n\pi\xi}{a}\right) \sin\left(\frac{m\pi\zeta}{b}\right) d\xi\, d\zeta.
\end{aligned}$$

By tracking back through this derivation of b_{nm} and using the theorem on convergence of Fourier sine series, it is possible to derive sufficient conditions on φ for the double sine series 4.34 to converge to $\varphi(x, y)$ for $0 \le x \le a, 0 \le y \le b$. When this occurs, equation 4.32 defines the solution of the initial-boundary value problem.

Example 4.11 *We solve this problem for the case that $a = b = \pi$ and*

$$\varphi(x, y) = xy(\pi - x)(\pi - y).$$

Compute

$$\begin{aligned}
b_{nm} &= \frac{4}{\pi^2}\int_0^\pi \int_0^\pi \xi(\pi - \xi) \sin(n\xi) \zeta(\pi - \zeta) \sin(m\zeta)\, d\xi\, d\zeta \\
&= \frac{16}{n^3 m^3 \pi^2}[(-1)^n - 1][(-1)^m - 1].
\end{aligned}$$

The solution is

$$u(x, y, t) = \frac{16}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{[(-1)^n - 1][(-1)^m - 1]}{n^3 m^3} \sin(nx) \sin(my) \cos(\alpha_{nm} ct)$$

in which

$$\alpha_{nm} = \sqrt{n^2 + m^2}. \diamond$$

Problems for Section 4.11

1. Solve

$$\begin{aligned}
u_t &= 9(u_{xx} + u_{yy}) \text{ for } 0 < x < 3, 0 < y < 6, t > 0 \\
u(x,0,t) &= u(x,6,t) = 0 \text{ for } 0 \le x \le 3, t \ge 0 \\
u(0,y,t) &= u(3,y,t) = 0 \text{ for } 0 \le y \le 6, t \ge 0 \\
u(x,y,0) &= \sin\left(\frac{\pi x}{3}\right) y(6-y), u_t(x,y,0) = 0.
\end{aligned}$$

2. Solve

$$\begin{aligned}
u_{tt} &= 4(u_{xx} + u_{yy}) \text{ for } 0 < x < \pi, 0 < y < 2\pi, t > 0 \\
u(x,0,t) &= u(x,2\pi,t) = 0 \text{ for } 0 \le x \le \pi, t \ge 0 \\
u(0,y,t) &= u(\pi,y,t) = 0 \text{ for } 0 \le y \le 2\pi, t \ge 0 \\
u(x,y,0) &= x^2(\pi - x)y^2(2\pi - y) \text{ for } 0 \le x \le \pi, 0 \le y \le 2\pi, \\
u_t(x,y,0) &= 0 \text{ for } 0 \le x \le \pi, 0 \le y \le 2\pi.
\end{aligned}$$

3. Solve

$$\begin{aligned}
u_{tt} &= 16(u_{xx} + u_{yy}) \text{ for } 0 < x < 1, 0 < y < 1, t > 0 \\
u(x,0,t) &= u(x,1,t) = 0 \text{ for } 0 \le x \le 1, t \ge 0 \\
u(0,y,t) &= u(1,y,t) = 0 \text{ for } 0 \le y \le 1, t \ge 0 \\
u(x,y,0) &= 0, u_t(x,y,0) = \sin(\pi x)\sin(\pi y) \text{ for } 0 \le x \le 1, 0 \le y \le 1.
\end{aligned}$$

4. Solve

$$\begin{aligned}
u_{tt} &= 9(u_{xx} + u_{yy}) \text{ for } 0 < x < \pi, 0 < y < \pi, t > 0 \\
u(x,0,t) &= u(x,\pi,t) = 0 \text{ for } 0 \le x \le \pi, t \ge 0 \\
u(0,y,t) &= u(\pi,y,t) = 0 \text{ for } 0 \le y \le \pi, t \ge 0 \\
u(x,y,0) &= \sin(x)\sin(y), u_t(x,y,0) = xy \text{ for } 0 \le x \le \pi, 0 \le y \le \pi.
\end{aligned}$$

5. Solve

$$\begin{aligned}
u_{tt} &= u_{xx} + u_{yy} \text{ for } 0 \le x \le 1, 0 \le y \le 1 \\
u(x,0,t) &= u(x,1,t) = 0 \text{ for } 0 \le x \le 1, t \ge 0 \\
u(0,y,t) &= u(1,y,t) = 0 \text{ for } 0 \le y \le 1, t \ge 0 \\
u(x,y,0) &= x(x-1)^2 y(y-1) \text{ for } 0 \le y \le 1 \\
u_t(x,y,0) &= xy^2 \text{ for } 0 \le x \le 1, 0 \le y \le 1.
\end{aligned}$$

6. Solve

$$\begin{aligned}
u_{tt} &= u_{xx} + u_{yy} \text{ for } 0 < x < 1, 0 < y < \pi, t > 0 \\
u(x, 0, t) &= u(x, \pi, t) = 0 \text{ for } 0 \le x \le 1, t \ge 0 \\
u(0, y, t) &= u(1, y, t) = 0 \text{ for } 0 \le y \le \pi, t \ge 0 \\
u(x, y, 0) &= x\cos(\pi x/2) \text{ for } 0 \le x \le 1, 0 \le y \le \pi \\
u_t(x, y, 0) &= x + y \text{ for } 0 \le x \le 1, 0 \le y \le \pi.
\end{aligned}$$

4.12 The Kirchhoff - Poisson Solution

We will derive an integral solution of the Cauchy problem for the wave equation in three space dimensions. Let R^3 denote three-dimensional space consisting of all points (x, y, z). The Cauchy problem is

$$\begin{gathered}
u_{tt} = u_{xx} + u_{yy} + u_{zz} \\
u(x, y, z, 0) = \varphi(x, y, z), u_t(x, y, z, 0) = \psi(x, y, z)
\end{gathered}$$

for (x, y, z) in R^3 and $t \ge 0$. We refer to this problem as CP. We have let $c = 1$ in this wave equation as a convenience.

Let VCP denote the special Cauchy problem

$$\begin{gathered}
u_{tt} = u_{xx} + u_{yy} + u_{zz} \\
u(x, y, z, 0) = 0, u_t(x, y, z, 0) = \psi(x, y, z)
\end{gathered}$$

for (x, y, z) in R^3.

We will show that if we can solve the apparently simpler VCP for arbitrary ψ, we can solve CP as well. Two observations are needed.

Observation 1 If w is the solution of CP having initial position φ and initial velocity zero, and v is the solution of CP having initial position zero and initial velocity ψ, then $w + v$ is the solution of CP having initial position φ and initial velocity ψ.

To verify this, let $h = w + v$. It is routine to check that h satisfies the wave equation. Next,

$$h(x, y, z, 0) = w(x, y, z, 0) + v(x, y, z, 0) = \varphi(x, y, z)$$

and

$$h_t(x, y, z, 0) = w_t(x, y, z, 0) + v_t(x, y, z, 0) = \psi(x, y, z).$$

We can therefore solve CP by solving two simpler problems: one with zero initial velocity, the other with zero initial position.

Observation 2 If v is the solution of VCP with initial velocity $\eta(x, y, z)$, then v_t is the solution of CP with initial position $\eta(x, y, z)$ and zero initial velocity.

To verify this, let $u = v_t$. We must first show that u satisfies the wave equation. Compute

$$\begin{aligned} u_{tt} &= (v_t)_{tt} = (v_{tt})_t = (v_{xx} + v_{yy} + v_{zz})_t \\ &= (v_t)_{xx} + (v_t)_{yy} + (v_t)_{zz} = (u_{xx} + u_{yy} + u_{zz}). \end{aligned}$$

Next,

$$u(x, y, z, 0) = v_t(x, y, z, 0) = \eta(x, y, z).$$

Finally,

$$\begin{aligned} u_t(x, y, z, 0) &= v_{tt}(x, y, z, 0) \\ &= [v_{xx} + v_{yy} + v_{zz}]_{t=0} = 0 \end{aligned}$$

because in problem VCP, $v(x, y, z, 0) = 0$ for all $(x,\ y,\ z)$.

Using these observations, we can solve CP if we can solve VCP. Let u_ψ be the solution of VCP having initial velocity ψ, and let u_φ, be the solution of VCP having initial velocity φ. By observation 2, $\partial u_\varphi/\partial t$ is the solution of CP with initial position φ and zero initial velocity. By observation 1, the solution of CP is

$$u = \frac{\partial u_\varphi}{\partial t} + u_\psi. \tag{4.36}$$

Now here is the point. We claim that there is an integral formula for the solution of VCP. Hence, using equation 4.36, we will be able to write a formula for the solution of CP in terms of integrals involving the initial position and velocity functions.

Here is the integral formula for the solution of VCP.

Theorem 4.2 (*Integral Solution of VCP*) *Let ψ be continuous with continuous first and second partial derivatives for all $(x,\ y,\ z)$. Then for $t > 0$ and (x, y, z) in R^3, the solution of VCP is*

$$u(x, y, z, t) = \frac{1}{4\pi t} \iint_{S(x,y,z;t)} \psi(\xi, \eta, \tau)\, d\sigma_t, \tag{4.37}$$

in which $S(x,\ y,\ z;\ t)$ is the sphere of radius t centered at $(x,\ y,\ z)$. $\diamond$

$S(x,\ y,\ z;\ t)$ consists of all (ξ, η, ζ) satisfying

$$(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2 = t^2.$$

The solution of VCP, at any point $(x,\ y,\ z)$ and time $t > 0$, is therefore $1/4\pi t$ times the surface integral of the initial velocity function over the sphere of radius t about the point. In this surface integral, $d\sigma_t$ is the differential element of surface area on $S(x,\ y,\ z;\ t)$. This solution is known as *Kirchhoff's formula.*

Proof We must first show that u as defined by equation 4.37 satisfies the wave equation. It will be convenient to think of points in R^3 as vectors and to denote vectors by boldface letters. Write $\mathbf{x} = (x, y, z)$ as an arbitrary point in 3 - space, so $u(x,\ y,\ z,\ t)$ can be written $u(\mathbf{x},\ t)$. We will denote the sphere $S(x, y, z; t)$ as S_t. Let U be the sphere of radius 1 about the origin and let $d\sigma$

be the differential element of surface area on U. Then $d\sigma_t = t^2\, d\sigma$. Let $\mathbf{n}_t$ be the unit outer normal vector on S_t, and $\mathbf{n}$ the unit outer normal on U.

In vector notation, equation 4.37 is

$$u(\mathbf{x}, t) = \frac{1}{4\pi t} \iint_{S_t} \psi(\boldsymbol{\xi})\, d\sigma_t \tag{4.38}$$

with $\boldsymbol{\xi} = (\xi, \eta, \tau)$ as the variable of integration in this surface integral over S_t. Transform this surface integral over S_t into one over U, in which the variable of integration is $\boldsymbol{\zeta}$, by putting

$$\boldsymbol{\xi} = \mathbf{x} + t\boldsymbol{\zeta}.$$

We obtain

$$u(\mathbf{x}, t) = \frac{t}{4\pi} \iint_{U} \psi(\mathbf{x} + t\boldsymbol{\zeta})\, d\sigma, \tag{4.39}$$

with $\boldsymbol{\zeta}$ the variable of integration over the unit sphere U. From equation 4.39,

$$\begin{aligned} u_{xx} + u_{yy} + u_{zz} &= \frac{t}{4\pi} \iint_{U} [\psi_{xx} + \psi_{yy} + \psi_{zz}]_{\mathbf{x}+t\boldsymbol{\zeta}}\, d\sigma \\ &= \frac{1}{4\pi t} \iint_{S_t} (\psi_{xx}(\boldsymbol{\xi}) + \psi_{yy}(\boldsymbol{\xi}) + \psi_{zz}(\boldsymbol{\xi}))\, d\sigma_t. \end{aligned} \tag{4.40}$$

Again using equation 4.39, we obtain

$$\begin{aligned} u_t(\mathbf{x}, t) &= \frac{1}{4\pi} \iint_{U} \psi(\mathbf{x} + t\boldsymbol{\zeta})\, d\sigma + \frac{t}{4\pi} \iint_{U} \nabla\psi(\mathbf{x} + t\boldsymbol{\zeta}) \cdot \mathbf{n}\, d\sigma \\ &= \frac{1}{t} u(\mathbf{x}, t) + \frac{1}{4\pi t} \iint_{S_t} \nabla\psi(\boldsymbol{\xi}) \cdot \mathbf{n}_t\, d\sigma_t, \end{aligned} \tag{4.41}$$

in which ∇ is the gradient operator. Apply the divergence theorem of Gauss to the last integral:

$$\iint_{S_t} \nabla\psi(\boldsymbol{\xi}) \cdot \mathbf{n}_t\, d\sigma_t = \iiint_{B_t} \operatorname{div}(\nabla\psi)\, dV,$$

where B_t is the solid ball consisting of all points on or interior to S_t. It is routine to verify that

$$\operatorname{div}(\nabla\psi) = \psi_{xx} + \psi_{yy} + \psi_{zz}.$$

Therefore, equation 4.41 becomes

$$u_t(\mathbf{x}, t) = \frac{1}{t} u(\mathbf{x}, t) + \frac{1}{4\pi t} \iiint_{B_t} (\psi_{xx} + \psi_{yy} + \psi_{zz})\, dV.$$

Denoting this triple integral by I, the last equation is

$$u_t(\mathbf{x}, t) = \frac{1}{t} u(\mathbf{x}, t) + \frac{1}{4\pi t} I. \tag{4.42}$$

Then

$$\begin{aligned}
u_{tt}(\mathbf{x},t) &= -\frac{1}{t^2}u(\mathbf{x},t) + \frac{1}{t}u_t(\mathbf{x},t) + \frac{1}{4\pi t}I_t - \frac{1}{4\pi t^2}I \\
&= \frac{1}{t}\left(-\frac{1}{t}u(\mathbf{x},t) + u_t(\mathbf{x},t) - \frac{1}{4\pi t}I\right) + \frac{1}{4\pi t}I_t \\
&= \frac{1}{4\pi t}I_t,
\end{aligned} \tag{4.43}$$

with the term in large parentheses vanishing in view of equation 4.42. Now,

$$I_t = \iint_{S_t} (\psi_{xx}(\boldsymbol{\xi}) + \psi_{yy}(\boldsymbol{\xi}) + \psi_{zz}(\boldsymbol{\xi}))\, d\sigma_t. \tag{4.44}$$

By equations 4.43, 4.44, and 4.40,

$$\begin{aligned}
u_{tt} &= \frac{1}{4\pi t}\iint_{S_t} (\psi_{xx}(\boldsymbol{\xi}) + \psi_{yy}(\boldsymbol{\xi}) + \psi_{zz}(\boldsymbol{\xi}))\, d\sigma_t \\
&= u_{xx} + u_{yy} + u_{zz}.
\end{aligned}$$

Therefore, the function u as defined by equation 4.37 satisfies the wave equation.

There remains to show that u satisfies the initial conditions. First,

$$u(\mathbf{x},0) = 0$$

by putting $t = 0$ into equation 4.39. Finally, from the first line of equation 4.41,

$$\begin{aligned}
u_t(\mathbf{x},0) &= \frac{1}{4\pi}\iint_U \psi(\mathbf{x})\, d\sigma \\
&= \psi(\mathbf{x})\frac{1}{4\pi}\iint_U d\sigma = \psi(\mathbf{x})
\end{aligned}$$

since

$$\iint_U d\sigma = \text{area of } U = 4\pi.$$

This completes the proof of the theorem. ◇

Corollary 4.1 *The solution of CP is*

$$u(\mathbf{x},t) = \frac{1}{4\pi}\frac{\partial}{\partial t}\left[\frac{1}{t}\iint_{S_t}\varphi(\boldsymbol{\xi})\, d\sigma_t\right] + \frac{1}{4\pi t}\iint_{S_t}\psi(\boldsymbol{\xi})\, d\sigma_t. \diamond \tag{4.45}$$

The conclusion follows immediately from the theorem and equation 4.36. This integral expression is known as *Poisson's formula* for the solution of the Cauchy problem for the wave equation in three dimensions.

Poisson's formula has an important consequence for wave motion in 3-space. For a given point (x, y, z) and time $t, u(x,y,z,t)$ depends only on t and data given on the sphere S_t of radius t about the point. This was noticed by the Dutch natural philosopher Christian Huygens (1629–1695), a contemporary of

Newton and Leibniz. S_t is the *domain of influence* of the initial conditions at time t.

To see the physical effect of this observation, imagine an initial disturbance that vanishes outside a solid ball B_R of radius R about the origin, and picture yourself standing at (x, y, z) outside this ball. Let ρ be the distance from the origin to (x, y, z). S_t intersects B_R if and only if

$$\rho - R \leq t \leq \rho + R.$$

This means that you feel the disturbance first at time $\rho - R$, but after time $\rho + R$ the effect is gone, because the initial disturbance vanishes outside B_R and the domain of influence no longer intersects any nonzero data. This effect is known as *Huygens' principle*.

If we think of the solution as a sound wave, this means that at first you hear nothing at (x, y, z). Then at time $\rho - R$ you hear the sound, continuing to experience it until time $\rho + R$, at which time the sound vanishes. This is what happens when someone standing some distance away blows a whistle for a short period of time. At first you hear nothing, then the whistle sound, then nothing again.

Problems for Section 4.12

1. Suppose that in Problem CP the initial position function is identically zero, and the initial velocity is constant. Show that the solution is independent of $\mathbf{x}$. Does this make sense from a physical point of view, thinking of the solution as a sound wave?

2. State a version of Duhamel's principle (Problem 13 of Section 4.1) for the three-dimensional wave problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2} = f(x, y, z, t)$$

for $(x, y, z) \in R^3, t > 0$,

$$u(x, y, z, 0) = u_t(x, y, z, 0) = 0.$$

Call this initial value problem, problem K. Use the requested version of Duhamel's principle and Kirchhoff's formula 4.37 for the solution of problem VCP to show that the solution of problem K is given by

$$u(\mathbf{x}, t) = -\frac{1}{4\pi} \iiint\limits_{\overline{B}(\mathbf{x},t)} \frac{1}{r} f(\mathbf{y}, t - r)\, d\mathbf{y},$$

in which $r = |\mathbf{x} - \mathbf{y}|$. Assume that f is continuous for all $(\mathbf{x}, t)$ with $\mathbf{x}$ in $R^3, t \geq 0$.

3. Write an integral formula for the solution of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2} = f(\mathbf{x}, t) \text{ for } \mathbf{x} \text{ in } R^3, t > 0$$

$$u(\mathbf{x}, 0) = \varphi(\mathbf{x}), u_t(\mathbf{x}, 0) = \psi(\mathbf{x}).$$

4.13 Hadamard's Method of Descent

We discuss a method for solving the Cauchy problem for the wave equation in two space dimensions, using Poisson's formula for the solution of the corresponding problem in 3-space. The method is due to the French mathematician Jacques Hadamard (1865–1963) and is called the *method of descent* because it descends from three to two space dimensions.

The Cauchy problem in two space variables is

$$u_{tt} = u_{xx} + u_{yy}$$
$$u(x, y, 0) = \varphi(x, y), u_t(x, y, 0) = \psi(x, y)$$

for all $(x,\ y)$ in the plane and for $t \geq 0$. Again we have let $c = 1$.

The idea is to think of the plane R^2 as consisting of points $(x,\ y,\ 0)$ in R^3, and of this Cauchy problem in the plane as a Cauchy problem in R^3 in which the initial data and partial differential equation are independent of z.

We may use Poisson's formula 4.45 to write the solution of this Cauchy problem in R^3 as

$$u(x, y, t) = \frac{1}{4\pi}\frac{\partial}{\partial t}\left[\frac{1}{t}\iint_{S_t} \varphi(\xi, \eta)\, d\sigma_t\right] + \frac{1}{4\pi t}\iint_{S_t} \psi(\xi, \eta)\, d\sigma_t.$$

Since φ and ψ are functions of just x and y, we can project these surface integrals to double integrals over a region in the plane. Recall that $\mathbf{n}_t$ denotes the unit outer normal vector to S_t. Consider a typical surface element on S_t having area $d\sigma$. This surface element projects onto a $d\xi$ by $d\eta$ rectangle centered at $(x,\ y,\ 0)$ in the ξ, η - plane. The area of this rectangle is related to the area of the surface element by

$$d\xi\, d\eta = \mathbf{n}_t \cdot \mathbf{k}\, d\sigma = \frac{1}{t}\sqrt{t^2 - (\xi - x)^2 - (\eta - y)^2}\, d\sigma.$$

Here we have used the fact that the sphere circle of radius t about $(x,\ y,\ 0)$ has the equation

$$(\xi - x)^2 + (\eta - y)^2 + \zeta^2 = t^2,$$

with (ξ, η, ζ) an arbitrary point on S_t. Since the sphere has both an upper and lower hemisphere, we introduce a factor of 2 and write

$$\iint_{S_t} \varphi(\xi, \eta)\, d\sigma_t = 2t\iint_{D_t} \frac{\varphi(\xi, \eta)}{\sqrt{t^2 - (\xi - x)^2 - (\eta - y)^2}}\, d\xi\, d\eta$$

in which D_t is the disk of radius t about $(x,\ y)$ in the plane, and consists of all (ξ, η) with

$$(\xi - x)^2 + (\eta - y)^2 \leq t^2.$$

A similar formula holds for the integral of ψ over S_t. The solution of the Cauchy problem in the plane is therefore

$$u(x,y,t) = \frac{1}{2\pi}\frac{\partial}{\partial t}\left[\iint_{D_t} \frac{\varphi(\xi,\eta)}{\sqrt{t^2-(\xi-x)-(\eta-y)^2}}\,d\xi\,d\eta\right] + \frac{1}{2\pi}\iint_{D_t} \frac{\psi(\xi,\eta)}{\sqrt{t^2-(\xi-x)^2-(\eta-y)^2}}\,d\xi\,d\eta. \tag{4.46}$$

There is a significant difference between the solution 4.45 in 3 - space, and the solution 4.46 in the plane. The domain of influence of the initial data at a point in 3-space at time t is the surface of a sphere of radius t about the point, and Huygens' principle holds. In the plane, however, the domain of influence at (x,y) at time t is the *solid disk* of radius t about the point, not just its boundary circle. A disturbance (say, in a small disk D about the origin) will eventually be felt at $(x,\ y)$ and will then be felt at all later times because once D_t intersects D at some time, it will do so at all later times. Huygens' principle does not hold in the plane. A wave disturbance in a plane will eventually be felt at any point, and then at that point at all later times, although with decreasing magnitude.

It is an interesting exercise to descend one more dimension, going from equation 4.46 to the solution of the Cauchy problem on the real line. It should be no surprise that this will result in d'Alembert's formula.

Problems for Section 4.13

1. Use the method of descent to derive d'Alembert's formula for the one-dimensional Cauchy problem from Poisson's formula for two space dimensions (equation 4.46).

2. Using d'Alembert's formula, explain why Huygens' principle fails to hold in one space dimension. Give a physical model for this (consider waves along a vibrating string).

Beginning Partial Differential Equations, Second Edition
by Peter V. O'Neil

Chapter 5

The Heat Equation

The wave equation is hyperbolic. We turn next to the heat equation, which is parabolic, beginning with the Cauchy problem for this equation.

5.1 The Cauchy Problem and Initial Conditions

We will show that the Cauchy problem

$$u_t = u_{xx} \text{ for } -\infty < x < \infty, t > 0$$
$$u(0,t) = \varphi(t), u_x(0,t) = \psi(t) \text{ for } t \geq 0$$

for the heat equation on the real line is not well posed.

The characteristics of $u_t = u_{xx}$ are the lines $t =$ constant. The Cauchy problem specifies data about the function and its normal derivative on a line that is not characteristic. For this problem to be well posed, it is required that it have a unique solution and that the solution depend continuously on the data. We show by an example that the last criterion can fail to be met.

For each positive integer n, define a Cauchy problem by setting $\psi_n(t) \equiv 0$ and $\varphi_n(t) = (2/n)\sin(2n^2t)$. It is routine to check that this problem has the solution

$$u_n(x,t) = \frac{1}{n}(e^{nx}\sin(2n^2t + nx) + e^{-nx}\sin(2n^2t - nx)).$$

Now,

$$\frac{2}{n}\sin(2n^2t) \to 0 \text{ as } n \to \infty.$$

We can therefore make the initial data as small in magnitude as we like by choosing n large enough. However, as n increases, we can, for certain values of x and t, make the solution as large in magnitude as we like.

The solution therefore does not depend continuously on the initial data.

Initial and Boundary Conditions

For the time being we restrict ourselves to the one-dimensional heat equation $u_t = ku_{xx}$, which models the temperature distribution in a wire or bar of material, with $u(x, t)$ the temperature at time t in the cross section at x. Generally, t will range over the nonnegative real numbers. The space variable x may vary over the entire real line (infinite medium) or the half-line $x \geq 0$ (semi-infinite medium), or may be restricted to some finite interval, usually $[0, L]$.

In the latter case there are certain kinds of boundary conditions that are commonly encountered. Conditions of the form

$$u(0,t) = g(t), u(L,t) = h(t) \text{ for } t \geq 0$$

specify the temperature at the ends of the bar for all time.

Alternatively, we might have conditions of the form

$$u_x(0,t) = g(t), u_x(L,t) = h(t) \text{ for } t \geq 0.$$

These boundary conditions give information about radiation of energy from the ends of the bar at times $t \geq 0$. When $h(t) = g(t) = 0$ for $t \geq 0$, these boundary conditions are called *insulation conditions*, and they state that there is no transfer of heat energy across the ends of the bar into the surrounding medium.

We can also specify *free radiation*, or *convection*, boundary conditions, in which the bar loses heat by radiation from its ends into the surrounding medium. If we assume that the medium is kept at a constant temperature of zero, these conditions take the form

$$u_x(0,t) + Au(0,t) = 0, u_x(L,t) + Au(L,t) = 0 \text{ for } t \geq 0,$$

in which A is constant.

An *initial-boundary value problem* for the heat equation on a finite interval consists of the heat equation, together with an initial condition and boundary conditions at the ends of the bar.

For example, the initial-boundary value problem

$$\begin{aligned} u_t &= ku_{xx} \text{ for } 0 < x < L, t > 0 \\ u(x,0) &= f(x) \text{ for } 0 \leq x \leq L \\ u(0,t) &= u(L,t) = 0 \text{ for } t \geq 0 \end{aligned}$$

models heat conduction in a bar of length L, with initial temperature in the cross section at x given by $f(x)$, and with ends kept at temperature zero for all times $t \geq 0$. With reasonable assumptions on f, this problem will be seen to be well posed—it has a unique solution that depends continuously on the initial data. This is consistent with the intuition that the heat equation governing the energy distribution, together with the initial temperature throughout the bar, and the temperature at the ends of the bar at all later times, should uniquely determine the temperature throughout the bar at any time. Our intuition also tells us that small changes in the initial temperature should cause small changes in the resulting temperature distribution.

As another example of an initial-boundary value problem, we could have

$$\begin{aligned} u_t &= ku_{xx} \text{ for } 0 < x < L, t > 0 \\ u(x,0) &= f(x) \text{ for } 0 \leq x \leq L \\ u(0,t) &= T_1, u_x(L,t) = -A[u(L,t) - T_2] \text{ for } t \geq 0. \end{aligned}$$

This problem models the temperature distribution in a bar of length L, with initial temperature in the cross section at x equal to $f(x)$, the temperature at the left end maintained at the constant value T_1, and the right end of the bar radiating heat energy into a medium that is kept at constant temperature T_2.

In the case of a semi-infinite medium, $0 \leq x < \infty$, we will still require an initial condition

$$u(x,0) = f(x) \text{ for } x \geq 0.$$

Now, however, there is only one endpoint of the bar, and we specify a boundary condition only at $x = 0$. This might give the temperature at the left end at all times $t \geq 0$, or be in the form of an insulation condition or a free radiation condition. The absence of a boundary condition at a right end of the bar is compensated for by seeking a bounded solution. This is consistent with the physical interpretation of $u(x, t)$ as a temperature distribution, which we expect to be a bounded function. However, aside from physical interpretations, initial-boundary value problems can have unbounded solutions.

For an infinite medium $-\infty < x < \infty$ there are no boundary conditions, only an initial condition

$$u(x, 0) = f(x) \text{ for all } x.$$

Again, we will seek bounded solutions for this initial value problem.

Problems for Section 5.1

1. Verify the assertion that $u_n(x, t)$ can be made as large in magnitude as we like as n increases, for certain values of x and t.

2. Formulate an initial-boundary value problem modeling heat conduction in a homogeneous bar of length L and uniform cross section if the left end is kept at temperature zero and the right end is insulated. The initial temperature distribution in the cross section at x is $f(x)$.

3. Formulate an initial-boundary value problem modeling heat conduction in a homogeneous bar of length L and uniform cross section if the left end is kept at temperature $\alpha(t)$, the right end at constant temperature K, and the initial temperature in the cross section at x is $f(x)$.

4. Suppose that u is a solution of $u_t = ku_{xx}$ for $0 < x < L, t > 0$. Prove that

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\int_0^L u^2(x,t)\,dx\right) = k[u(x,t)u_x(x,t)]_{x=0}^{x=L} - k\int_0^L u_x^2(x,t)\,dx.$$

Hint: Multiply the heat equation by u and integrate.

Use this result to show that the problem

$$\begin{aligned} u_t &= ku_{xx} \text{ for } 0 < x < L, t > 0 \\ u(0,t) &= u(L,t) = 0 \text{ for } t \geq 0 \\ u(x,0) &= f(x) \text{ for } 0 \leq x \leq L \end{aligned}$$

can have only one solution.

5. Prove the following version of Duhamel's principle for the heat equation (see Problem 13 of Section 4.1 for a version related to the wave equation). Let u satisfy

$$\begin{aligned} u_t &= u_{xx} + f(x,t) \text{ for } 0 < x < L, t > 0 \\ u(0,t) &= u(L,t) = 0 \text{ for } t \geq 0 \\ u(x,0) &= 0 \text{ for } 0 \leq x \leq L. \end{aligned}$$

For $T > 0$, let $v(x, t, T)$ satisfy

$$\begin{aligned} v_t &= v_{xx} \text{ for } 0 < x < L, t > T \\ v(x,T,T) &= f(x,T) \text{ for } 0 \leq x \leq L \\ v(0,t,T) &= v(L,t,T) = 0 \text{ for } t \geq T. \end{aligned}$$

Prove that

$$u(x,t) = \int_0^t v(x,t,T)\,dT.$$

5.2 The Weak Maximum Principle

We will derive a property enjoyed by any continuous solution of the heat equation in one space dimension, independent of initial and boundary conditions.

Any real-valued function that is continuous on the compact rectangle $0 \leq x \leq L$ and $0 \leq t \leq T$ in the x, t - plane must achieve a maximum on this rectangle. We claim that if the function is a solution of $u_t = ku_{xx}$, this maximum is achieved at a point on the lower horizontal side or a vertical side of the rectangle.

Theorem 5.1 (*Weak Maximum Principle*) *Let v be a solution of $u_t = ku_{xx}$ that is continuous on the closed rectangle R consisting of all (x, t) with $0 \leq x \leq$*

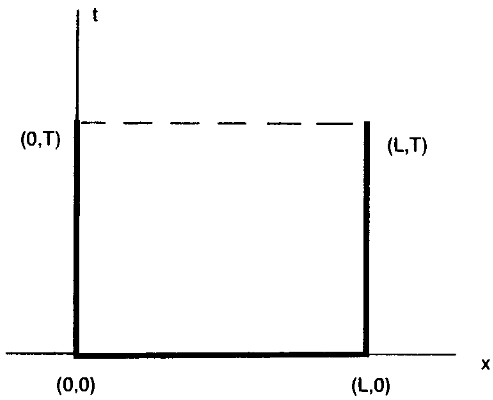

Figure 5.1: Weak maximum principle: $u(x,t)$ has a maximum on a lower or vertical side of R.

$L, 0 \le t \le T$. *Then* $v(x,\ t)$ *assumes its maximum value on* R *at a point on the base* $t = 0$ *or on a vertical side* $x = 0$ *or* $x = L$. $\diamond$

There is a physically based rationale for this conclusion. A solution of the heat equation on $[0,\ L]$ may be interpreted as a temperature distribution in a bar of length L. In the absence of new sources of heat energy, we would not expect the temperature in the bar at any later time to exceed either the initial temperature ($t = 0$) throughout the bar or the temperature at the ends of the bar (at $x = 0$ or $x = L$).

An analytical argument can be constructed along the following lines.

Proof Let C consist of points $(x,\ 0)$ with $0 \le x \le L$, and points $(0,\ t)$ and (L, t) with $0 \le t \le T$. C consists of the lower and vertical sides of the rectangle in Figure 5.1.

We know that $v(x,\ t)$ achieves a maximum value M on the closed rectangle R. $v(x,\ t)$ must also achieve a maximum value M_C on the closed set C. We will prove the theorem by showing that $M = M_C$.

Clearly, $M_C \le M$, since every point of C is also in R. Suppose that $M - M_C = \epsilon > 0$. Choose any point (x_0, t_0) in R such that $v(x_0, t_0) = M$. Since

$\epsilon > 0$, (x_0, t_0) is not in C, so $0 < x_0 < L$ and $0 < t_0 \leq T$. Define

$$w(x,t) = v(x,t) + \frac{\epsilon}{4L^2}(x - x_0)^2.$$

Consider $w(x, t)$ at points in C. First, for $0 \leq x \leq L$,

$$w(x,0) = v(x,0) + \frac{\epsilon}{4L^2}(x - x_0)^2 \leq M - \epsilon + \frac{\epsilon}{4L^2}L^2 = M - \frac{3\epsilon}{4} < M.$$

In similar fashion,

$$w(0,t) < M \text{ and } w(L,t) < M \text{ for } 0 \leq t \leq T.$$

But $w(x_0, t_0) = v(x_0, t_0) = M$. Therefore the maximum of $w(x, t)$ on R is at least M and is achieved at a point (x_1, t_1) of R not on C. Because $0 < x_1 < L$ and $0 < t_1 \leq T$, then

$$w_t(x_1, t_1) \geq 0 \text{ and } w_{xx}(x_1, t_1) \leq 0.$$

Then

$$w_t(x_1, t_1) - kw_{xx}(x_1, t_1) \geq 0.$$

But

$$\begin{aligned} w_t(x_1, t_1) - kw_{xx}(x_1, t_1) &= v_t(x_1, t_1) - kv_{xx}(x_1, t_1) - k\frac{\epsilon}{2L^2} \\ &= -\frac{k\epsilon}{2L^2} < 0. \end{aligned}$$

This is a contradiction, and we conclude that $M = M_C$, completing the proof. ◇

We have proved that a continuous solution achieves its maximum at a point on the lower or vertical sides of R. We have *not* proved that this solution cannot also achieve its maximum at another point of R. Indeed, this occurs with a constant solution. For this reason the theorem is referred to as the *weak maximum principle*. By considering $-v$, we can use the theorem to show that a continuous solution also achieves its minimum at a point on the lower or vertical sides of R.

We will derive two important consequences of the weak maximum principle for the general initial-boundary value problem

$$\begin{aligned} u_t &= ku_{xx} + F(x,t) \text{ for } 0 < x < L, t > 0 \\ u(x,0) &= f(x) \text{ for } 0 \leq x \leq L \\ u(0,t) &= g(t), u(L,t) = h(t) \text{ for } t \geq 0. \end{aligned} \tag{5.1}$$

It is assumed that f, g, and h are continuous and that F is continuous for $0 \leq x \leq L$ and $t \geq 0$. Our first consequence is a uniqueness result for this problem.

Theorem 5.2 (*Uniqueness*) *The initial-boundary value problem 5.1 can have only one continuous solution.* ◇

Proof Suppose that u_1 and u_2 are continuous solutions. Let $w = u_1 - u_2$. It is routine to check that

$$\begin{aligned} w_t - kw_{xx} &= 0 \text{ for } 0 < x < L, t > 0 \\ w(x,0) &= 0 \text{ for } 0 \leq x \leq L \\ w(0,t) &= w(L,t) = 0 \text{ for } t \geq 0. \end{aligned}$$

Now let $T > 0$ and let R be the rectangle defined by $0 \leq x \leq L, 0 \leq t \leq T$. The initial and boundary conditions mean that $w(x,t) = 0$ on the lower and vertical sides of R. Since w achieves its maximum on R on these sides, then $w(x,t) \leq 0$ for $0 \leq x \leq L$ and $t \geq 0$.

But $w(x, t)$ also achieves its minimum on R on these sides, so $0 \leq w(x,t)$ for $0 \leq x \leq L$ and $t \geq 0$.

We conclude that $w(x,t) = 0$ for $0 \leq x \leq L$ and $0 \leq t \leq T$, so $u_1 = u_2$ on R. Since T is any positive number, then $u_1(x,t) = u_2(x,t)$ for $0 \leq x \leq L$ and $t \geq 0$. ◇

We can also use the weak maximum principle to show that solutions of problem 5.1 depend continuously on the initial and boundary data. The problem is therefore well posed.

Theorem 5.3 (*Continuous Dependence on Data*) *For $j = 1, 2$, let u_j be the continuous solution of*

$$\begin{aligned} u_t &= ku_{xx} + F(x,t) \text{ for } 0 < x < L, t > 0 \\ u(x,0) &= f_j(x) \text{ for } 0 \leq x \leq L \\ u(0,t) &= g_j(t), u(L,t) = h_j(t) \text{ for } t \geq 0. \end{aligned}$$

Let ϵ be a positive number such that

$$|f_1(x) - f_2(x)| \leq \epsilon \text{ for } 0 \leq x \leq L$$

and

$$|g_1(t) - g_2(t)| \leq \epsilon \text{ and } |h_1(t) - h_2(t)| \leq \epsilon$$

for $t \geq 0$. Then

$$|u_1(x,t) - u_2(x,t)| \leq \epsilon$$

for $0 \leq x \leq L$ and $t \geq 0$. ◇

Proof Let $w = u_1 - u_2$. Then

$$\begin{aligned} w_t - kw_{xx} &= 0 \text{ for } 0 < x < L, t > 0 \\ w(x,0) &= f_1(x) - f_2(x) \text{ for } 0 \leq x \leq L \\ w(0,t) &= g_1(t) - g_2(t), w(L,t) = h_1(t) - h_2(t) \text{ for } t \geq 0. \end{aligned}$$

Now let T be any positive number and consider values of $w(x, t)$ on the lower and vertical sides of the rectangle $R : 0 \leq x \leq L, 0 \leq t \leq T$. On the lower side of R,

$$|w(x,0)| = |f_1(x) - f_2(x)| \leq \epsilon.$$

On the left vertical side,

$$|w(0,t)| = |g_1(t) - g_2(t)| \le \epsilon,$$

and on the right vertical side,

$$|w(L,t)| = |h_1(t) - h_2(t)| \le \epsilon.$$

By the weak maximum principle, $|w(x,t)| \le \epsilon$ for $0 \le x \le L$ and $0 \le t \le T$. Since T can be any positive number, this proves the theorem. $\diamond$

Problems for Section 5.2

1. Fill in the details of this alternative proof of the weak maximum principle. Let

$$M = \max_{(x,t)\in R} v(x,y)$$

There is at least one point (x_0, t_0) in R at which $v(x_0,t_0) = M$. If $M_C < M$, this point is not on C. Define $w(x,t) = v(x,t) - \epsilon(t - t_0)$, with ϵ any positive number. Then w is continuous on R and $w(x_0,t_0) = v(x_0,t_0) = M > M_C$. Choose ϵ sufficiently small that $w(x_0,t_0)$ is greater than the maximum value achieved by $w(x,\ t)$ on C. Then the maximum of $w(x,\ t)$ on R is achieved at a point (x_1,t_1) not on C. Now reason as in the proof given in the text.

2. Prove the following theorem. Suppose that

$$\begin{aligned} u_t &= k u_{xx} \text{ for } -\infty < x < \infty, t > 0 \\ u(x,0) &= f(x) \text{ for } -\infty < x < \infty. \end{aligned}$$

Suppose that f is continuous on the entire real line. Let $u(x,t) \to 0$ uniformly in t as $x \to \pm\infty$. Then

$$|u(x,t)| \le \max |f(x)|.$$

Hint: Apply the weak maximum principle on $-a \le x \le a, t \ge 0$, and then let $a \to \infty$.

5.3 Solutions on Bounded Intervals

In this section we solve initial-boundary value problems involving the one-dimensional heat equation on a bounded interval [0, L].

Ends of the Bar Maintained at Temperature Zero

We will solve the initial-boundary value problem

$$\begin{aligned} u_t &= k u_{xx} \text{ for } 0 < x < L, t > 0 \\ u(0,t) &= u(L,t) = 0 \text{ for } t \ge 0 \\ u(x,0) &= f(x) \text{ for } 0 \le x \le L. \end{aligned} \tag{5.2}$$

The solution $u(x, t)$ models the temperature distribution in a homogeneous bar of length L and uniform cross section, with given initial temperature $f(x)$ at the cross section at x and whose ends are kept at temperature zero.

Let $u(x,t) = X(x)T(x)$ and substitute into the heat equation to obtain

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda,$$

in which λ is the separation constant. Then

$$X'' + \lambda X = 0 \text{ and } T' + \lambda kT = 0.$$

Now $u(0,t) = X(0)T(t) = 0$ and $u(L,t) = X(L)T(t) = 0$ imply that $X(0) = X(L) = 0$. The problem for X is therefore

$$X'' + \lambda X = 0; X(0) = X(L) = 0.$$

This is exactly the problem for X obtained in separating variables in the wave equation, and we know that the eigenvalues are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \text{ for } n = 1, 2, \cdots$$

with corresponding eigenfunctions constant multiples of

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$

The problem for T now has the form

$$T' + \frac{kn^2\pi^2}{L^2}T = 0,$$

with solutions constant multiples of $e^{-kn^2\pi^2t/L^2}$. For each positive integer n, we now have a function

$$u_n(x,t) = b_n \sin(n\pi x/L)e^{-kn^2\pi^2t/L^2},$$

which satisfies the heat equation and the boundary conditions. To satisfy the initial condition $u(x,0) = f(x)$, generally we must use an infinite superposition

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/L)e^{-kn^2\pi^2t/L^2}.$$

We need to choose the b_n's so that

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/L).$$

This is the Fourier sine expansion of the initial temperature function f. Hence choose

$$b_n = \frac{2}{L}\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi.$$

This yields the solution

$$u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi\right)\sin\left(\frac{n\pi x}{L}\right)e^{-n^2\pi^2kt/L^2}. \qquad (5.3)$$

Example 5.1 *Solve*

$$u_t = ku_{xx} \text{ for } 0 < x < L, t > 0$$
$$u(0,t) = u(L,t) = 0 \text{ for } t > 0$$

$$u(x,0) = f(x) = \begin{cases} x & \text{for } 0 \le x \le L/2 \\ L - x & \text{for } L/2 \le x \le L. \end{cases}$$

Compute

$$\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi = \frac{2L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right).$$

The solution of this initial-boundary value problem is

$$u(x,t) = \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{L}\right)e^{-n^2\pi^2kt/L^2}.$$

Figure 5.2 shows the evolution of this temperature distribution with time, using $k = 1$ and $L = \pi$. As we might expect in the absence of a source, the temperature profile decreases as time increases.

It is interesting to use different values of k and compare the effects on the temperature function at selected times. We leave this experiment to the student. ◇

Example 5.2 *Suppose that the initial temperature is given on different segments of a homogeneous metal bar of length* 1 *by*

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x < 1/3 \\ 100 & \text{for } 1/3 \le x \le 2/3 \\ 0 & \text{for } 2/3 < x \le 1. \end{cases}$$

Suppose that $k = 1$. The temperature function is given by equation 5.3. Compute

$$\int_0^1 f(\xi)\sin(n\pi\xi)\,d\xi$$
$$= \int_0^{1/3}\sin(n\pi\xi)\,d\xi + \int_{1/3}^{2/3}100\sin(n\pi\xi)\,d\xi$$
$$= \frac{99\cos(n\pi/3) + 1 - 100\cos(2n\pi/3)}{n\pi}.$$

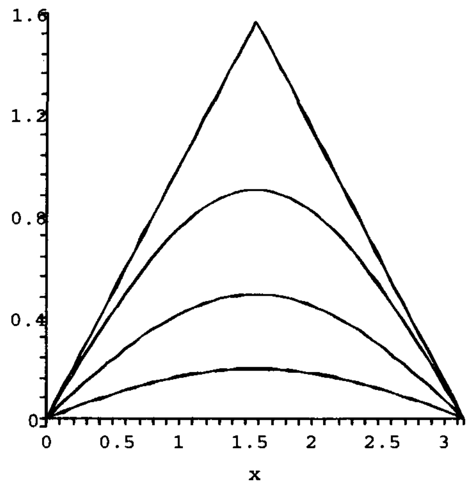

Figure 5.2: Decreasing temperature profiles at times $t = 0, 0.351, 0.951$, and 1.87 in Example 5.1.

The temperature distribution is

$$u(x,t) = 2\sum_{n=1}^{\infty}\left(\frac{99\cos(n\pi/3) + 1 - 100\cos(2n\pi/3)}{n\pi}\right)\sin(n\pi x)e^{-n^2\pi^2 t}.$$

Figure 5.3 shows the temperature function at times $t = 0.0003, 0.003,$ *and* 0.035. *For these times, the temperature function begins very close to the initial temperature. As t increases, the temperature decreases and is also more evenly distributed throughout the medium.* ◇

Temperature in a Bar with Insulated Ends

Model the temperature distribution in a bar with insulated ends by

$$\begin{aligned} u_t &= ku_{xx} \text{ for } 0 < x < L, t > 0 \\ u_x(0,t) &= u_x(L,t) = 0 \text{ for } t \geq 0 \\ u(x,0) &= f(x) \text{ for } 0 \leq x \leq L. \end{aligned}$$

We can also solve this problem by separation of variables. Let $u(x,t) = X(x)T(t)$. Proceeding as we have before, we obtain

$$X'' + \lambda X = 0 \text{ and } T' + \lambda kT = 0.$$

Now $u_x(0,t) = X'(0)T(t) = 0$ implies that $X'(0) = 0$ and $u_x(L,t) = 0$ implies that $X'(L) = 0$. The boundary value problem for X is therefore

$$X'' + \lambda X = 0; X'(0) = X'(L) = 0.$$

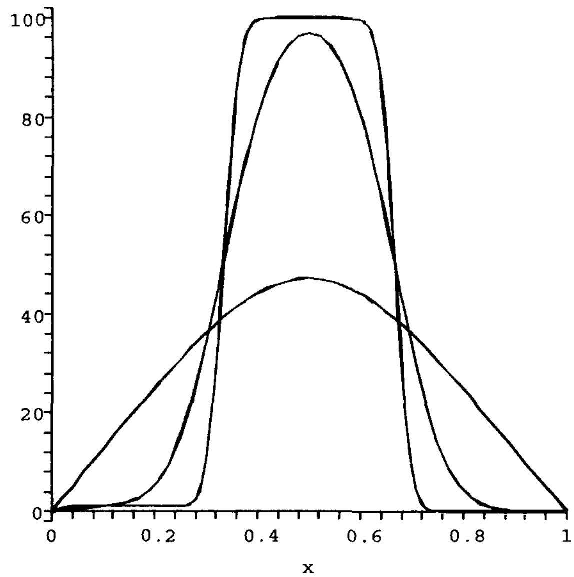

Figure 5.3: Decreasing temperature profiles as time increases in Example 5.2.

Examine cases on λ to determine the eigenvalues and eigenfunctions of this problem. If $\lambda = 0$, then $X'' = 0$, so $X(x) = ax + b$. Now $X'(0) = X'(L) = a = 0$ and b can be any constant. Therefore, 0 is an eigenvalue of this problem, with constant eigenfunctions.

If $\lambda < 0$, write $\lambda = -\alpha^2$ with $\alpha > 0$. Now $X'' - \alpha^2 X = 0$ has the general solution

$$X(x) = ae^{\alpha x} + be^{-\alpha x}.$$

The condition

$$X'(0) = \alpha a - \alpha b = 0$$

implies that $a = b$, so $X(x) = a(e^{\alpha x} + e^{-\alpha x})$. But then

$$X'(L) = ka(e^{\alpha L} - e^{-\alpha L}) = 0.$$

Because $e^{\alpha L} - e^{-\alpha L} > 0$, this forces $a = 0$. The solution is therefore trivial, so this problem has no negative eigenvalue.

If $\lambda > 0$, write $\lambda = \alpha^2$ for $\alpha > 0$. Now $X'' + \alpha^2 X = 0$ has the general solution

$$X(x) = a\cos(\alpha x) + b\sin(\alpha x).$$

Since

$$X'(0) = b\alpha = 0,$$

we conclude that $b = 0$. Then $X(x) = a\cos(\alpha x)$. Now

$$X'(L) = -a\alpha \sin(\alpha L) = 0$$

implies that αL must be an integer multiple of π, so

$$\alpha = \frac{n\pi}{L} \text{ for } n = 1, 2, \cdots$$

and

$$\lambda_n = \frac{n^2\pi^2}{L^2}.$$

These numbers are eigenvalues of the problem. Corresponding to each such eigenvalue is an eigenfunction

$$X_n(x) = \cos\left(\frac{n\pi x}{L}\right).$$

In summary, the eigenvalues and corresponding eigenfunctions for the present problem are

$$\lambda_0 = 0 \text{ and } X_0(x) = 1$$

and

$$\lambda_n = \frac{n^2\pi^2}{L^2} \text{ and } X_n(x) = \cos\left(\frac{n\pi x}{L}\right) \text{ for } n = 1, 2, \cdots.$$

Now recall that the differential equation for T is

$$T' + \lambda k T = 0.$$

Corresponding to $\lambda = 0$, this equation has the solution

$$T_0(t) = \text{constant}.$$

Corresponding to $\lambda = n^2\pi^2/L^2$ for $n = 1, 2, \cdots, T_n(t)$ can be any constant multiple of

$$e^{-n^2\pi^2 kt/L^2}.$$

We therefore have functions

$$u_0(x,t) = \text{constant}$$

and

$$u_n(x,t) = a_n \cos\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 kt/L^2} \text{ for } n = 1, 2, \cdots$$

which satisfy the heat equation and the insulation conditions for any choices of the constant coefficients. To satisfy the initial condition, we attempt a superposition

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t) \\ &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 kt/L^2}. \end{aligned} \tag{5.4}$$

The constant term is denoted $a_0/2$ in anticipation of a Fourier cosine expansion. We need

$$u(x,0) = f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right).$$

This is the Fourier cosine expansion of f on $[0,\ L]$, so choose

$$a_n = \frac{2}{L}\int_0^L f(\xi)\cos\left(\frac{n\pi\xi}{L}\right)\,d\xi \text{ for } n = 0, 1, 2, \cdots.$$

This gives the solution of the heat equation with insulation boundary conditions.
Example 5.3 *Suppose that we have a homogeneous bar with insulated ends of length π and with material having $k = 4$. Suppose that the initial temperature is given by*

$$f(x) = \begin{cases} 0 & \text{for } 0 \le x < \pi/2 \\ 50 & \text{for } \pi/2 \le x \le \pi. \end{cases}$$

Compute the numbers

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(\xi)\cos(n\xi)\,d\xi.$$

We get

$$a_0 = \frac{2}{\pi}\int_{\pi/2}^{\pi} 50 d\xi = 50$$

and for $n = 1, 2, \cdots$,

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_{\pi/2}^{\pi} 50\cos(n\xi)\,d\xi \\ &= -\frac{100}{n\pi}\sin(n\pi/2). \end{aligned}$$

The solution is

$$u(x,t) = 25 - \sum_{n=1}^{\infty}\left(\frac{100}{n\pi}\sin(n\pi/2)\right)\cos(nx)e^{-4n^2t}.$$

Figure 5.4 shows the temperature distribution at times $t = 0, 0.003, 0.025, 0.097$, and 0.38. At time $t = 0.003$ the temperature function is very close to the initial distribution $f(x)$. However, as t increases the temperature graph is seen to be leveling out toward the steady-state value of 25, which is the limit of $u(x,t)$ as $t \to \infty$. $\diamond$

Ends of the Bar at Different Temperatures

We will solve the initial-boundary value problem

$$\begin{aligned} u_t &= ku_{xx} \text{ for } 0 < x < L, t > 0 \\ u(0,t) &= T_1, u(L,t) = T_2 \text{ for } t \ge 0 \\ u(x,0) &= f(x) \text{ for } 0 \le x \le L \end{aligned}$$

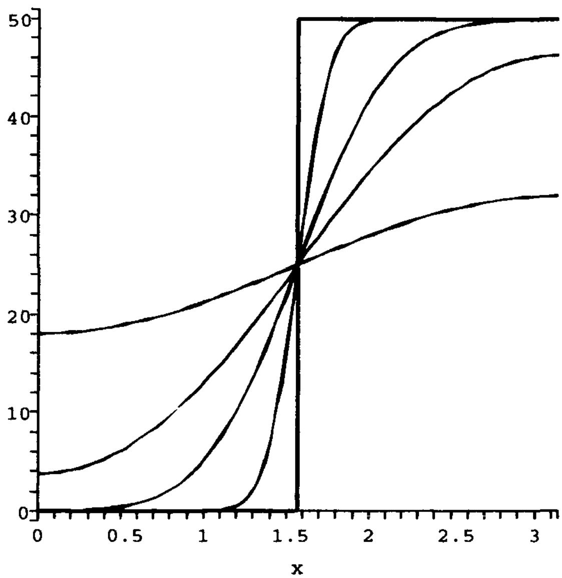

Figure 5.4: Temperature profiles in Example 5.3.

in which T_1 and T_2 are distinct positive numbers. This problem models temperature distribution in a bar in which the ends are kept at different constant temperatures.

If we attempt a separation of variables $u(x,t) = X(x)T(t)$, there is a rude surprise. We require that

$$u(0,t) = T_1 = X(0)T(t).$$

When $T_1 = 0$, this implies that $X(0) = 0$. However, if $T_1 \neq 0$, then $T(t) = T_1/X(0)$ for $t \geq 0$, hence

$$u(x,t) = \frac{T_1}{X(0)}X(x),$$

independent of t. This is an unrealistic conclusion. Similarly, if we apply the boundary condition at $x = L$, we have to conclude that

$$u(x,t) = \frac{T_2}{X(L)}X(x).$$

We are led to this state of affairs by attempting to apply separation of variables when the boundary conditions are nonhomogeneous, specifying a nonzero temperature at an end.

We can overcome this difficulty by perturbing the temperature function to define a new initial-boundary value problem having homogeneous boundary conditions, which we know how to solve. To do this, begin by putting

$$u(x,t) = U(x,t) + \psi(x).$$

Substitute u into the heat equation to obtain

$$U_t = k(U_{xx} + \psi''(x)),$$

and this is a standard heat equation for U if $\psi''(x) = 0$. Thus choose $\psi(x) = Cx + D$. Now consider the boundary conditions. First,

$$u(0,t) = T_1 = U(0,t) + \psi(0)$$

becomes the homogeneous condition $U(0,t) = 0$ if

$$\psi(0) = T_1.$$

This leads us to choose $D = T_1$. So far, $\psi(x) = Cx + T_1$.

Next, $u(L,t) = U(L,t) + \psi(L) = T_2$ becomes $U(L,t) = 0$ if

$$\psi(L) = CL + T_1 = T_2.$$

This suggests that we set

$$C = \frac{1}{L}(T_2 - T_1).$$

Now

$$\psi(x) = \frac{1}{L}(T_2 - T_1)x + T_1.$$

With this choice of ψ, the initial-boundary value problem for U is

$$\begin{aligned} U_t &= kU_{xx} \text{ for } 0 < x < L, t > 0 \\ U(0,t) &= U(L,t) = 0 \\ U(x,0) &= u(x,0) - \psi(x) \\ &= f(x) - \frac{1}{L}(T_2 - T_1)x - T_1 \text{ for } 0 \leq x \leq L. \end{aligned}$$

We have solved this initial-boundary value problem. By equation 5.3,

$$U(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 kt/L^2}$$

with

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L \left(f(\xi) - \frac{1}{L}(T_2 - T_1)\xi - T_1\right) \sin\left(\frac{n\pi\xi}{L}\right) d\xi \\ &= \frac{2}{L}\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right) d\xi + 2\frac{(-1)^n T_2 - T_1}{n\pi}. \end{aligned}$$

The solution of the original initial-boundary value problem is

$$u(x,t) = U(x,t) + \frac{1}{L}(T_2 - T_1)x + T_1.$$

We may interpret this solution as a decomposition of the temperature distribution into a transient part $U(x, t)$, which decays to zero as $t \to \infty$, and a steady-state part $\psi(x)$, which is independent of time. Indeed,

$$\psi(x) = \lim_{t\to\infty} u(x,t).$$

Such decompositions are commonly seen in physical systems. For example, the current in a circuit can be written as the sum of a transient part that decays to zero as $t \to \infty$, and a steady-state part, which is the limit of the solution as $t \to \infty$. Solutions of mixing problems may also have a transient term and a steady-state term.

Example 5.4 *We will solve the initial-boundary value problem*

$$u_t = 7u_{xx} \text{ for } 0 < x < 5, t > 0$$
$$u(0,t) = 1, u(5,t) = 4 \text{ for } t > 0$$

$$u(x,0) = f(x) = \begin{cases} 3 - x & \text{for } 0 < x < 3 \\ 10(x-3) & \text{for } 3 < x < 5. \end{cases}$$

Following the discussion, let

$$u(x,t) = U(x,t) + \frac{3}{5}x + 1,$$

where U is the solution of the initial-boundary value problem

$$U_t = 7U_{xx} \text{ for } 0 < x < 5, t > 0$$
$$U(0,t) = U(5,t) = 0 \text{ for } t > 0$$

$$U(x,0) = u(x,0) - \frac{3}{5}x - 1 = \begin{cases} 2 - \frac{8}{5}x & \text{for } 0 < x \le 3 \\ -31 + \frac{47}{5}x & \text{for } 3 \le x < 5. \end{cases}$$

This problem has the solution

$$U(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/5) e^{-7n^2\pi^2 t/25},$$

in which

$$\begin{aligned} b_n &= \frac{2}{5}\int_0^5 U(\xi,0)\sin(n\pi\xi/5)\,d\xi \\ &= \frac{2}{5}\left[\int_0^3 \left(2 - \frac{8}{5}\xi\right)\sin(n\pi\xi/5)\,d\xi + \int_3^5 \left(-31 + \frac{47}{5}\xi\right)\sin(n\pi\xi/5)\,d\xi\right] \\ &= -110\frac{\sin(3n\pi/5)}{n^2\pi^2} + \frac{4}{n\pi} - \frac{32}{n\pi}(-1)^n. \end{aligned}$$

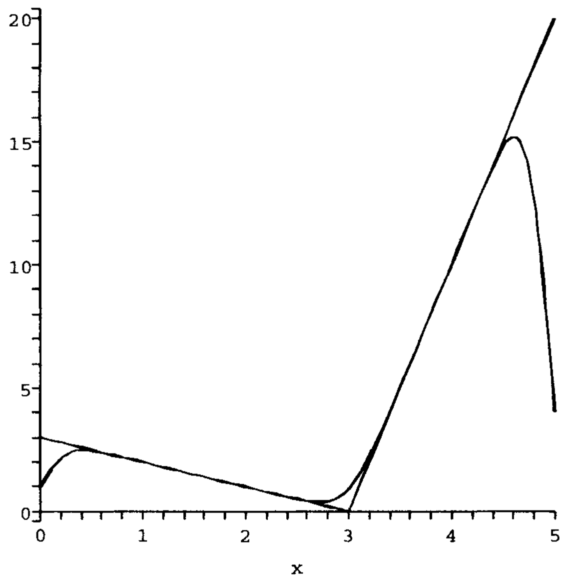

Figure 5.5: Temperature function at times 0 and 0.003 in Example 5.4.

Then

$$u(x,t) = \frac{3}{5}x + 1$$
$$+ \sum_{n=1}^{\infty}\left(-110\frac{\sin(3n\pi/5)}{n^2\pi^2} + \frac{4}{n\pi} - \frac{32}{n\pi}(-1)^n\right)\sin\left(\frac{n\pi x}{5}\right)e^{-7n^2\pi^2 t/25}$$

Figure 5.5 shows the temperature function at times $t = 0$ *and* $t = 0.003$. *Figure 5.6 shows the solution at times* $t = 0.26, 0.87,$ *and* 0.32. *These show the temperature decreasing toward the the graph of the straight line* $u = \frac{3}{5}x + 1$, *which is the steady state-limit of* $u(x,t)$. $\diamond$

Diffusion of Charges in a Transistor

The heat equation is so named because it models heat conduction in various settings. It can, however, model other diffusion phenomena. Let $h(x,\ t)$ be the concentration of positive charge carriers at time t and position x in a transistor occupying the interval $[0,\ L]$. It has been shown that h is modeled by the

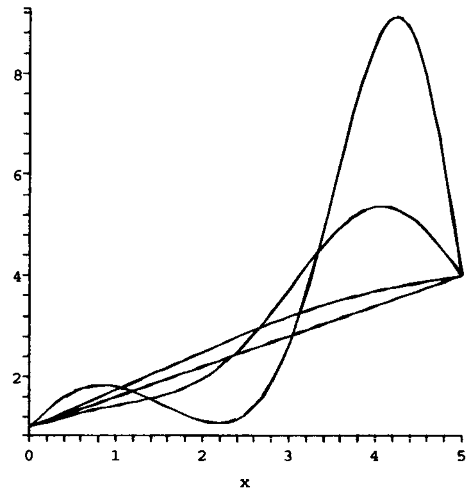

Figure 5.6: Temperature functions approaching the steady state limit $u = \frac{3}{5}x+1$ in Example 5.4.

initial-boundary value problem:

$$h_t = k\left(h_{xx} - \frac{a}{L}h_x\right) \text{ for } 0 < x < L, t > 0$$
$$h(0,t) = h(L,t) = 0 \text{ for } t \geq 0$$
$$h(x,0) = \frac{KL}{ka}(1 - e^{-a(1-x/L)}) \text{ for } 0 \leq x \leq L,$$

in which a and K are positive constants.

If we put $h(x,t) = X(x)T(t)$ into the partial differential equation, we get

$$\frac{T'}{kT} = \frac{X''}{X} - \frac{a}{L}\frac{X'}{X} = -\lambda,$$

and the differential equation for X is

$$X'' - \frac{a}{L}X' + \lambda X = 0,$$

with $X(0) = X(L) = 0$. Solving this and taking cases on λ will cost some labor. Here is another approach which is easier. Set

$$h(x,t) = e^{\alpha x+\beta t}u(x,t)$$

and attempt to choose α and β so that the resulting problem for u is one that we have already solved. Substitute $h(x, t)$ into the partial differential equation to get

$$\begin{aligned}
&\beta e^{\alpha x+\beta t}u + e^{\alpha x+\beta t}u_t \\
&= k(\alpha^2 e^{\alpha x+\beta t}u + 2\alpha e^{\alpha x+\beta t}u_x + e^{\alpha x+\beta t}u_{xx}) \\
&- k\frac{a}{L}(\alpha e^{\alpha x+\beta t}u + e^{\alpha x+\beta t}u_x).
\end{aligned}$$

This equation can be written

$$u_t = ku_{xx} + u_x\left(2\alpha k - \frac{ka}{L}\right) + u\left(k\alpha^2 - \frac{\alpha ka}{L} - \beta\right).$$

This is the standard heat equation for u if we choose α and β so that the terms in parentheses vanish. Thus choose

$$\alpha = \frac{a}{2L} \text{ and } \beta = -\frac{ka^2}{4L^2}.$$

Next,

$$h(0,t) = e^{\beta t}u(0,t) = 0$$

implies that $u(0,t) = 0$, and $h(L,t) = 0$ implies that $u(L,t) = 0$. Further, $u(x,t) = e^{-\alpha x-\beta t}h(x,t)$. The initial-boundary value problem for u is

$$\begin{aligned}
u_t &= ku_{xx} \text{ for } 0 < x < L, t > 0 \\
u(0,t) &= u(L,t) = 0 \text{ for } t \geq 0 \\
u(x,0) &= \frac{KL}{ka}e^{-ax/2L}(1 - e^{-a(1-x/L)}) \text{ for } 0 \leq x \leq L.
\end{aligned}$$

By equation 5.3 the solution of this problem for u is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 kt/L^2},$$

in which

$$\begin{aligned}
b_n &= \frac{2K}{ka}\int_0^L e^{-a\xi/2L}(1 - e^{-a(1-\xi/L)})\sin\left(\frac{n\pi\xi}{L}\right)d\xi \\
&= \frac{8KL}{ka}\frac{n\pi(1-e^{-a})}{4n^2\pi^2 + a^2}.
\end{aligned}$$

Therefore

$$u(x,t) = \frac{8KL}{ka}(1 - e^{-a})\sum_{n=1}^{\infty}\frac{n\pi}{4n^2\pi^2 + a^2}\sin\left(\frac{n\pi x}{L}\right)e^{-n^2\pi^2 kt/L^2}$$

and

$$h(x,t) = e^{ax/2L}e^{-ka^2t/4L^2}u(x,t).$$

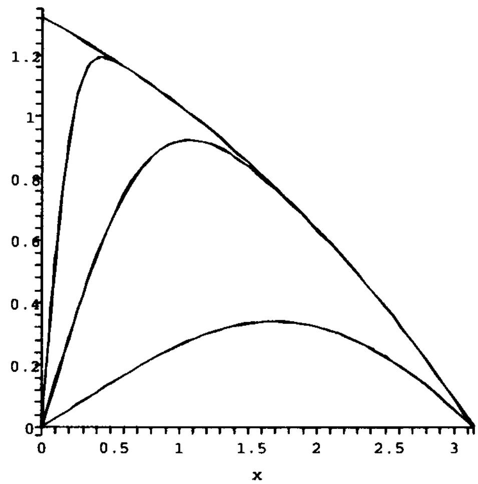

Figure 5.7: $h(x,t)$ decreasing as t increases over $0, 0.005, 0.052$, and 0.37.

Figure 5.7 shows the diffusion function decreasing over times $t = 0.005, 0.052$, and 0.37, with $L = \pi$, $K = 2$, $k = 3$, and $a = 1$.

We have just used a transformation of the form $h(x,t) = e^{\alpha x + \beta t}u(x,t)$ to exchange a potentially difficult problem for one that we had already solved. For some problems, a different transformation might be needed, and there is no guarantee that a useful transformation will be found. Nevertheless, this is an important strategy to keep in mind.

Problems for Section 5.3

In each of Problems 1 through 7, solve $u_t = ku_{xx}$ with the given boundary and initial conditions. Graph the solution at different times for the case $k = 1$ and (for Problem 7) $L = \pi$.

1. $u(0,t) = u(1,t) = 0$ for $t \geq 0; u(x,0) = \sin(\pi x)$ for $0 \leq x \leq 1$

2. $u(0,t) = u(3,t) = T$ for $t \geq 0; u(x,0) = T$ for $0 \leq x \leq 3$

3. $u_x(0,t) = u_x(4,t) = 0$ for $t \geq 0; u(x,0) = x^2$ for $0 \leq x \leq 4$

4. $u(0,t) = u(2,t) = 0$ for $t \geq 0; u(x,0) = \sin(\pi x)$ for $0 \leq x \leq 2$

5. $u_x(0,t) = u_x(6,t) = 0$ for $t \geq 0; u(x,0) = e^{-x}$ for $0 \leq x \leq 6$

6. $u(0,t) = 3, u(5,t) = \sqrt{7}$ for $t \geq 0; u(x,0) = x^2$ for $0 \leq x \leq 5$

7. $u_x(0,t) = u_x(L,t) = 0$ for $t \geq 0; u(x,0) = L - x^2$ for $0 \leq x \leq L$

8. Given the partial differential equation

$$w_t = kw_{xx} + hw,$$

with h constant, determine a choice of the constant α so that if

$$w(x,t) = e^{\alpha t}u(x,t),$$

u satisfies the canonical one-dimensional heat equation.

9. Let a and b be real numbers and $k > 0$. For the partial differential equation

$$w_t = k(w_{xx} + aw_x + bw),$$

determine constants α and β so that if

$$w(x,t) = e^{\alpha x + \beta t}u(x,t),$$

u satisfies the canonical heat equation.

10. Let u be a solution of the heat equation

$$u_t = ku_{xx} \text{ for } 0 < x < L, t > 0.$$

Let $0 < a < b < L$. Prove that for any positive t,

$$\frac{d}{dt}\int_a^b \frac{1}{2}u(x,t)^2\,dx = kuu_x]_a^b - \int_a^b ku_x(x,t)^2\,dx.$$

This is the analogue for the heat equation of the energy integral defined previously for the wave equation. Hint: Multiply the heat equation by u and integrate both sides from a to b, applying integration by parts where appropriate.

11. The problem

$$\begin{aligned} u_t &= ku_{xx} \text{ for } 0 < x < L, t > 0 \\ u(0,t) &= u_x(L,t) = 0 \text{ for } t \geq 0 \\ u(x,0) &= f(x) \end{aligned}$$

models heat conduction in a bar of length L with the left end kept at temperature zero but with an insulation condition on the right end. Using separation of variables $u(x,t) = X(x)T(t)$, show that the problem to solve for X is

$$X'' + \lambda X = 0; X(0) = X'(L) = 0.$$

By considering cases on λ, show that this problem has eigenvalues

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4L^2}$$

for $n = 1, 2, \cdots$, and corresponding eigenfunctions

$$X_n(x) = \sin\left(\frac{(2n-1)\pi x}{2L}\right).$$

For each λ_n, show that corresponding solutions for T are constant multiples of

$$T_n(t) = e^{-(2n-1)^2\pi^2 kt/4L^2}.$$

Show that for $n = 1, 2, \cdots$, the functions

$$u_n(x,t) = b_n \sin\left(\frac{(2n-1)\pi x}{2L}\right) e^{-(2n-1)^2\pi^2 kt/4L^2}$$

are solutions of the heat equation satisfying both boundary conditions. To satisfy the initial condition, attempt a superposition

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{(2n-1)\pi x}{2L}\right) e^{-(2n-1)^2\pi^2 kt/4L^2}.$$

We require that

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{(2n-1)\pi x}{2L}\right).$$

Explain why this is not a Fourier sine expansion of f on $[0, L]$. However, it is an expansion of f in a series of eigenfunctions of a boundary value problem for X. Show that

$$\int_0^L \sin\left(\frac{(2n-1)\pi x}{2L}\right) \sin\left(\frac{(2m-1)\pi x}{2L}\right) dx = 0$$

if n and m are distinct positive integers, and derive a formula for the b_n's by reasoning informally as was done in motivating the choice of the Fourier coefficients in Section 3.2.

12. Solve

$$\begin{aligned} u_t &= ku_{xx} \text{ for } 0 < x < L, t > 0 \\ u_x(0,t) &= u(L,t) = 0 \text{ for } t \geq 0 \\ u(x,0) &= f(x) \text{ for } 0 \leq x \leq L. \end{aligned}$$

13. Solve

$$\begin{aligned} u_t &= ku_{xx} - hu_x \text{ for } 0 < x < L, t > 0 \\ u(0,t) &= u(L,t) = 0 \text{ for } t \geq 0 \\ u(x,0) &= f(x) \text{ for } 0 \leq x \leq L, \end{aligned}$$

in which h is a positive number.

14. Solve

$$\begin{aligned} u_t &= 4u_{xx} - 2u_x \text{ for } 0 < x < L, t > 0 \\ u(0,t) &= u(L,t) = 0 \text{ for } t > 0 \\ u(x,0) &= x^2(L-x) \text{ for } 0 \le x \le L. \end{aligned}$$

Graph the solution for some values of the time, with $L = \pi$.

15. Solve

$$\begin{aligned} u_t &= u_{xx} - 6u_x \text{ for } 0 < x < \pi, t > 0 \\ u(0,t) &= u(\pi,t) = 0 \text{ for } t > 0 \\ u(x,0) &= \sin(x) \text{ for } 0 \le x \le \pi. \end{aligned}$$

Graph the solution for selected values of t.

16. Solve

$$\begin{aligned} u_t &= ku_{xx} - hu \text{ for } 0 < x < L, t > 0 \\ u_x(0,t) &= u_x(L,t) = 0 \text{ for } t \ge 0 \\ u(x,0) &= f(x) \text{ for } 0 \le x \le L, \end{aligned}$$

in which h is a positive constant. Hint: Note Problem 8.

17. Solve

$$\begin{aligned} u_t &= 4u_{xx} - 8u \text{ for } 0 < x < 2\pi, t > 0 \\ u_x(0,t) &= u_x(2\pi,t) = 0 \text{ for } t > 0 \\ u(x,0) &= x(2\pi - x) \text{ for } 0 \le x \le 2\pi. \end{aligned}$$

Generate some graphs of the solution for selected values of t.

18. Solve

$$\begin{aligned} u_t &= 7u_{xx} \text{ for } 0 < x < 5, t > 0 \\ u(0,t) &= 1, u(5,t) = 4 \text{ for } t > 0 \\ u(x,0) &= e^{-x} \text{ for } 0 < x < 5. \end{aligned}$$

Graph the solution for selected values of t.

19. A thin, homogeneous bar of length L has insulated ends and initial temperature B, a constant. Find the temperature distribution in the bar.

20. A thin, homogeneous bar of length L has initial temperature equal to a constant B, and the right end ($x = L$) is insulated while the left end is kept at temperature zero. Find the temperature distribution in the bar.

21. A thin, homogeneous bar of thermal diffusivity 9, length 2 cm and insulated sides has its left end maintained at temperature zero while the right end is insulated. The bar has an initial temperature given by $f(x) = x^2$. Determine the temperature distribution in the bar. Calculate $\lim_{t\to\infty} u(x,t)$.

22. Suppose that we have a long, thin, homogeneous bar of length L, with sides poorly insulated. Heat radiates freely from the bar along its length. Assuming a positive transfer coefficient A and a constant temperature T in the surrounding medium, the equation for the temperature distribution is

$$u_t = ku_{xx} - A(u - T).$$

Assume insulated ends and an initial temperature of $f(x)$. Solve for $u(x,t)$. Hint: Let $w = u - T$ and recall Problem 8.

23. Solve

$$\begin{aligned} u_t &= ku_{xx} \text{ for } -L < x < L, t > 0 \\ u(-L,t) &= u(L,t), u_x(-L,t) = u_x(L,t) \text{ for } t \geq 0 \\ u(x,0) &= f(x) \text{ for } 0 \leq x \leq L. \end{aligned}$$

This initial-boundary value problem models heat conduction in an insulated wire of length $2L$ which is thought of as bent into the form of a ring, with the ends joined. The boundary conditions identify the points $(-L,t)$ and (L, t) in the sense that the temperature and flux at these points are the same. This problem is known as *Fourier's ring*.

24. Solve

$$\begin{aligned} u_t &= k(u_{xx} - 3u_x + 2u) \text{ for } 0 \leq x \leq L, t > 0 \\ u(0,t) &= u(L,t) = 0 \text{ for } t > 0 \\ u(x,0) &= f(x) \text{ for } 0 \leq x \leq L. \end{aligned}$$

25. Solve the following special case of the *Stefan problem*:

$$\begin{aligned} u_t &= u_{xx} \text{ for } 0 < x < f(t), t > 0 \\ u(0,t) &= A \text{ for } t \geq 0 \\ u(f(t),t) &= 0 \text{ for } t > 0 \\ u_x(f(t),t) &= -f'(t) \text{ for } t > 0, \end{aligned}$$

in which both u and f are to be determined and A is a given constant. Hint: Assume a solution

$$u(x,t) = B + C \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right) \text{ for } 0 < x < f(t).$$

Here erf is the *error function*, which is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} \, d\xi.$$

Show that this choice of u satisfies the heat equation. Now determine B, C, and f to satisfy the other conditions of the problem.

5.4 The Heat Equation on the Real Line

We sometimes use a Fourier series to write the solution of an initial-boundary value problem on a bounded domain. On an unbounded domain we may turn to Fourier integrals and transforms, or other transforms such as the Laplace transform. In this section we solve an initial value problem involving the heat equation on the entire real line.

The problem

$$\begin{aligned} u_t &= ku_{xx} \text{ for } -\infty < x < \infty, t > 0 \\ u(x,0) &= f(x) \text{ for all } x \end{aligned} \tag{5.5}$$

models heat flow in a homogeneous bar occupying the x - axis. The bar has no endpoints and therefore there are no boundary conditions. We compensate for this absence of information by seeking a bounded solution.

Substitute $u(x,t) = X(x)T(t)$ into the heat equation to obtain, as in the bounded interval case,

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda$$

or

$$X'' + \lambda X = 0 \text{ and } T' + \lambda kT = 0.$$

There are no boundary conditions on X.

In Section 4.10 we found that the eigenvalues and bounded eigenfunctions for this problem for X are

$$\lambda = \omega^2 \geq 0 \text{ and } X_\omega(x) = a_\omega \cos(\omega x) + b_\omega \sin(\omega x).$$

Now solve for T. Corresponding to $\lambda = 0$, the equation for T is $T' = 0$, so $T =$ constant. If $\lambda = \omega^2 > 0$, then $T' + \omega^2 kT = 0$ with general solution $T = ce^{-\omega^2 kt}$. We now have

$$u_\omega(x,t) = [a_\omega \cos(\omega x) + b_\omega \sin(\omega x)]e^{-\omega^2 kt}$$

as a bounded solution of the heat equation for $-\infty < x < \infty$, with a_ω and b_ω real numbers which are as yet arbitrary.

To satisfy the initial condition, attempt a superposition

$$u(x,t) = \int_0^\infty [a_\omega \cos(\omega x) + b_\omega \sin(\omega x)]e^{-\omega^2 kt}\, d\omega. \tag{5.6}$$

We must choose the coefficients a_ω and b_ω so that

$$u(x,0) = f(x) = \int_0^\infty [a_\omega \cos(\omega x) + b_\omega \sin(\omega x)]\, d\omega.$$

Since this is the Fourier integral expansion of f, choose

$$a_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos(\omega\xi)\,d\xi$$

and

$$b_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\sin(\omega\xi)\,d\xi.$$

Example 5.5 *Suppose that the initial temperature is given by $f(x) = e^{-|x|}$. Then*

$$a_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-|\xi|}\cos(\omega\xi)\,d\xi = \frac{2}{\pi}\frac{1}{1+\omega^2}$$

and

$$b_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-|\xi|}\sin(\omega\xi)\,d\xi = 0$$

(the last integral is zero by inspection because the integrand is an odd function). With this initial temperature function, the solution is

$$u(x,t) = \frac{2}{\pi}\int_0^{\infty} \frac{1}{1+\omega^2}\cos(\omega x)e^{-\omega^2 kt}\,d\omega. \diamond$$

Solution by Fourier Transform

We solve problem 5.5 using the Fourier transform as an illustration of this technique. Since $-\infty < x < \infty$, we may attempt a Fourier transform of $u(x,\,t)$ in x, treating t as a parameter carried along in the process. Denote the Fourier transform of $u(x,\,t)$ in the x - variable by $\widehat{u}(\omega,t)$. That is, $\mathcal{F}[u(x,t)](\omega) = \widehat{u}(\omega,t)$.

Apply $\mathcal{F}$ to the heat equation:

$$\mathcal{F}[u_t] = \mathcal{F}[ku_{xx}]$$

or, since k is constant,

$$\mathcal{F}[u_t] = k\mathcal{F}[u_{xx}].$$

Since the transform is with respect to x, $\mathcal{F}$, and the operation $\partial/\partial t$ of differentiating with respect to time commute,

$$\begin{aligned}\mathcal{F}[u_t](\omega) &= \int_{-\infty}^{\infty} \frac{\partial u(x,t)}{\partial t}e^{-i\omega x}\,dx \\ &= \frac{\partial}{\partial t}\int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}\,dx = \frac{\partial}{\partial t}\widehat{u}(\omega,t).\end{aligned}$$

For the transform of u_{xx}, apply the operational rule (equation 3.25) with $n = 2$:

$$\mathcal{F}[u_{xx}](\omega) = (i\omega)^2\widehat{u}(\omega,t) = -\omega^2\widehat{u}(\omega,t).$$

The transform of the heat equation is

$$\widehat{u}_t = -k\omega^2\widehat{u}$$

or

$$\widehat{u}_t + k\omega^2\widehat{u} = 0.$$

This can be thought of as an ordinary differential equation in t, with ω as a parameter. The solution is

$$\widehat{u}(\omega, t) = A_\omega e^{-\omega^2 kt}, \tag{5.7}$$

in which the coefficient A_ω may depend on ω. To determine A_ω, transform the initial condition $u(x, 0) = f(x)$ and use equation 5.7 at $\omega = 0$ to obtain

$$\mathcal{F}[u(x,0)](\omega) = \widehat{u}(\omega, 0) = A_\omega = \mathcal{F}[f](\omega) = \widehat{f}(\omega).$$

Now we have

$$\widehat{u}(\omega, t) = \widehat{f}(\omega)e^{-\omega^2 kt}.$$

This is the Fourier transform of the solution. Apply the inverse Fourier transform (equation 3.24) to obtain

$$u(x,t) = \mathcal{F}^{-1}[\widehat{f}(\omega)e^{-\omega^2 kt}] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)e^{-\omega^2 kt}e^{i\omega x}\, d\omega.$$

To write this solution in terms of f (instead of $\widehat{f}$), insert into this integral the definition of $\widehat{f}(\omega)$ from equation 3.23:

$$\begin{aligned}
u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(\xi)e^{-i\omega\xi}\, d\xi\right) e^{-\omega^2 kt}e^{i\omega x}\, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)e^{-i\omega(\xi - x)}e^{-\omega^2 kt}\, d\xi\, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)[\cos(\omega(\xi - x)) - i\sin(\omega(\xi - x))]e^{-\omega^2 kt}\, d\xi\, d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)\cos(\omega(\xi - x))e^{-\omega^2 kt}\, d\xi\, d\omega \\
&\quad - \frac{i}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)\sin(\omega(\xi - x))e^{-\omega^2 kt}\, d\xi\, d\omega.
\end{aligned}$$

Because $u(x, t)$ is real valued,

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)\cos(\omega(\xi - x))e^{-\omega^2 kt}\, d\xi\, d\omega. \tag{5.8}$$

A Single Integral Expression for the Solution

Equation 5.8 is the solution of the initial-boundary value problem 5.5 obtained using the Fourier transform. It is possible to write this solution in a neater form by exploiting the integral

$$\int_{-\infty}^{\infty} e^{-\zeta^2} \cos\left(\frac{\alpha\zeta}{\beta}\right) d\zeta = \sqrt{\pi} e^{-\alpha^2/4\beta^2}. \tag{5.9}$$

Problem 4 suggests a derivation of equation 5.9. To use this integral to simplify the solution, let

$$\zeta = \omega\sqrt{kt}, \alpha = x - \xi, \text{ and } \beta = \sqrt{kt}$$

in the integral with respect to ω in equation 5.8. Now,

$$d\zeta = \sqrt{kt}\, d\omega \text{ and } \frac{\alpha\zeta}{\beta} = \omega(x - \xi),$$

so

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-\omega^2 kt} \cos(\omega(\xi - x))\, d\omega &= \int_{-\infty}^{\infty} e^{-\zeta^2} \cos\left(\frac{\alpha\zeta}{\beta}\right) \frac{1}{\sqrt{kt}}\, d\zeta \\ &= \frac{1}{\sqrt{kt}} \sqrt{\pi} e^{-(x-\xi)^2/4kt}.\end{aligned}$$

Therefore, equation 5.8 can be written

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{\frac{\pi}{kt}} e^{-(x-\xi)^2/4kt} f(\xi)\, d\xi,$$

and the solution of problem 5.5 is

$$u(x,t) = \frac{1}{2\sqrt{\pi kt}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4kt} f(\xi)\, d\xi. \tag{5.10}$$

If we had just started with this function, it might not be obvious that $u(x, t)$ is the solution of problem 5.5. The fact that $u(x, t)$ satisfies the heat equation can be shown by differentiating with respect to x and t under the integral sign, operations that can be justified by uniform convergence arguments. To show that $u(x, 0) = f(x)$, make the change of variable

$$\zeta = \frac{\xi - x}{\sqrt{4kt}}$$

in equation 5.10 to obtain

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + \sqrt{4kt}\zeta) e^{-\zeta^2}\, d\zeta.$$

Because f is bounded, this integral converges uniformly in x and t, and the limit and integral can be interchanged to write

$$\lim_{t\to 0+} u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(\lim_{t\to 0+} f(x + \sqrt{4kt}\zeta)\right) e^{-\zeta^2}\, d\zeta.$$

Now

$$\lim_{t\to 0+} f(x+\sqrt{4kt}\zeta) = f(x)$$

by the continuity of f, and $f(x)$ passes through the integration with respect to ζ to yield

$$\lim_{t\to 0+} u(x,t) = u(x,0) = f(x)\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\zeta^2}\,d\zeta = f(x),$$

in which we have used the standard integral

$$\int_{-\infty}^{\infty} e^{-\zeta^2}\,d\zeta = \sqrt{\pi}.$$

A Solution by Convolution

Consider again the initial value problem 5.5. We derived the solution 5.6 using the Fourier integral, and the solution 5.8 using the Fourier transform. These solutions are different expressions for the same function. We also derived the more compact expression 5.10 for this solution, involving only a single integral.

It is also possible to proceed directly from the boundary value problem 5.5 to the solution 5.10 by using the Fourier transform and the convolution theorem. By applying the Fourier transform in the space variable to the heat equation (as we did with the wave equation in Section 4.10), we obtain

$$\widehat{u}(\omega,t) = \widehat{f}(\omega)e^{-k\omega^2 t}.$$

We would like to write $\widehat{f}(\omega)e^{-k\omega^2 t}$ as the Fourier transform of some continuous function, because then $u(x,\ t)$ must be this function. To do this, it is enough to write just $e^{-k\omega^2 t}$ as the transform of some function, because then $\widehat{u}(x,t)$ will be the product of two transformed functions, and this is the transform of the convolution of the individual functions.

To carry out this strategy we use the following two results:

(1) If $m > 0$ and we set $g_m(x) = mg(mx)$, then $\widehat{g_m}(\omega) = \widehat{g}(\omega/m)$.

This is the scaling lemma for the Fourier transform.

(2) If $g(x) = e^{-x^2/2}$, then $\widehat{g}(\omega) = \sqrt{2\pi}e^{-\omega^2/2} = \sqrt{2\pi}g(\omega)$.

This can be found in many standard tables of Fourier transforms of functions.

Now proceed as follows. Let $g(x) = e^{-x^2/2}$. With $m = 1/\sqrt{2kt}$,

$$\begin{aligned} e^{-k\omega^2 t} &= g(\sqrt{2kt}\omega) = \frac{1}{\sqrt{2\pi}}\widehat{g}(\sqrt{2kt}\omega) = \frac{1}{\sqrt{2\pi}}\widehat{g}(\omega/m) \\ &= \frac{1}{\sqrt{2\pi}}\widehat{g_m}(\omega) = \frac{1}{\sqrt{2\pi}}\widehat{g_{1/\sqrt{2kt}}}(\omega). \end{aligned}$$

Then

$$\begin{aligned}\widehat{u}(\omega,t) &= \widehat{f}(\omega)\frac{1}{\sqrt{2\pi}}\widehat{g_{1/\sqrt{2kt}}}(\omega)\\ &= \frac{1}{\sqrt{2\pi}}\widehat{g_{1/\sqrt{2kt}}}\widehat{f}(\omega).\end{aligned}$$

We have written the factors in this order to obtain the particular form of the solution given in equation 5.10. By the convolution theorem for the Fourier transform,

$$\begin{aligned}u(x,t) &= \frac{1}{\sqrt{2\pi}}\left(g_{1/\sqrt{2kt}} * f\right)(x)\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g_{1/\sqrt{2kt}}(x-\xi)f(\xi)\,d\xi\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2kt}}g\left((x-\xi)/\sqrt{2kt}\right)f(\xi)\,d\xi\\ &= \frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^{\infty} e^{-(x-\xi)^2/4kt}f(\xi)\,d\xi,\end{aligned}$$

and this is equation 5.10.

Problems for Section 5.4

1. Suppose that the function f in the initial value problem 5.5 is an odd function. Show that the solution of the problem is odd (in the space variable). Show that if f is an even function, the solution is even in the space variable.

2. Substitute the Fourier integral coefficients into the solution 5.6 obtained by Fourier integral to verify that the solutions obtained by Fourier integral and Fourier transform agree. Hint: This is like the argument carried out in Section 4.10 for solutions of the wave equation on the real line.

3. Let a be a positive number and suppose that $f(x) = 0$ for $|x| > a$. Suppose also that $f(x) > 0$ for $-a < x < a$. Let u be the solution of

$$\begin{aligned}u_t &= ku_{xx} \text{ for } -\infty < x < \infty, t > 0\\ u(x,0) &= f(x) \text{ for } -\infty < x < \infty.\end{aligned}$$

   Prove that $u(x,t) > 0$ for $-\infty < x < \infty$ and for all $t > 0$. This conclusion can be interpreted to mean that the parabolic heat equation propagates disturbances with infinite velocity, in contrast to what we saw with the hyperbolic wave equation.

4. Derive equation 5.9 as follows. Let

$$F(x) = \int_0^{\infty} e^{-\zeta^2}\cos(x\zeta)\,d\zeta.$$

(a) Show that this integral converges uniformly for all x, as does the integral obtained by interchanging d/dx and $\int_0^\infty \cdots d\zeta$.

(b) Compute $F'(x)$ by interchanging d/dx and $\int_0^\infty \cdots d\zeta$. By integrating by parts, show that

$$F'(x) = -\frac{x}{2}F(x).$$

(c) Solve this differential equation to obtain $F(x)$ to within a constant.

(d) Evaluate this constant by using the standard integral

$$\int_0^\infty e^{-\zeta^2}\, d\zeta = \frac{\sqrt{\pi}}{2}.$$

For each of Problems 5 through 9, obtain the solution of problem 5.5 by separation of variables (Fourier integral), then by Fourier transform, and finally by convolution. Is it obvious that each method yields the same solution?

5. $f(x) = e^{-4|x|}$

6.
$$f(x) = \begin{cases} \sin(x) & \text{for } |x| \le \pi \\ 0 & \text{for } |x| > \pi \end{cases}$$

7.
$$f(x) = \begin{cases} x & \text{for } 0 \le x \le 4 \\ 0 & \text{for } x < 0 \text{ and for } x > 4 \end{cases}$$

8.
$$f(x) = \begin{cases} e^{-x} & \text{for } -1 \le x \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

9.
$$f(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1 \\ -1 & \text{for } -1 \le x < 0 \\ 0 & \text{for } |x| > 1 \end{cases}$$

10. Let
$$f(x) = \begin{cases} 1 & \text{for } -1 \le x \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$
in the solution 5.10. Expand the integrand in a Maclaurin series and integrate term by term to obtain a series solution for $u(x, t)$.

11. Carry out the program of Problem 10, with
$$f(x) = \begin{cases} x & \text{for } -2 \le x \le 2 \\ 0 & \text{for } |x| > 2 \end{cases}$$

12. Use separation of variables and then the Fourier transform to solve

$$u_t = ku_{xx} \text{ for } -\infty < x < \infty, t > 0$$
$$u_x(x,0) = \begin{cases} 1 \text{ for } -h \le x \le h \\ 0 \text{ for } |x| > h. \end{cases}$$

Show that the same solution is obtained by both methods.

13. Use separation of variables and then a Fourier transform to solve

$$u_t = ku_{xx} \text{ for } -\infty < x < \infty, t > 0$$
$$u_x(x,0) = \begin{cases} \cos(\pi x) \text{ for } -1 \le x \le 1 \\ 0 \text{ for } |x| > 1. \end{cases}$$

Show that the same solution is obtained by both methods.

14. Consider the problem

$$u_t = u_{xx} \text{ for } -\infty < x < \infty, t > 0$$
$$u(x,0) = 0.$$

Define

$$s(x,t) = \begin{cases} xt^{-3/2}e^{-x^2/4t} \text{ for } -\infty < x < \infty, t > 0 \\ 0 \text{ for } t = 0. \end{cases}$$

Show that s satisfies the heat equation for $-\infty < x < \infty$ and $t > 0$, and that

$$\lim_{t \to 0+} s(x,t) = 0.$$

Hence conclude that s is continuous on $-\infty < x < \infty, t \ge 0$ and is a solution of the initial value problem. But $u(x,t) \equiv 0$ is also a solution. Does this show that this problem does not have a unique solution?

15. For the problem

$$u_t = ku_{xx} \text{ for } -\infty < x < \infty, t > 0$$
$$u(x,0) = f(x) \text{ for } -\infty < x < \infty,$$

determine a function $G(x,\ t)$ such that the solution can be written

$$u(x,t) = \int_{-\infty}^{\infty} G(x-\xi,t)f(\xi)\,d\xi.$$

G is called the *Green's function* for this initial-boundary value problem. Physically, it may be thought of as the temperature at x, at time t, due to a heat source concentrated at ξ at time zero. Show that

$$\int_{-\infty}^{\infty} G(x-\xi,t)\,d\xi = 1$$

for all x.

16. Derive a Duhamel-type principle for the problem

$$\begin{aligned} u_t &= u_{xx} + f(x,t) \text{ for } -\infty < x < \infty, t > 0 \\ u(x,0) &= 0. \end{aligned}$$

Using this principle, derive the solution

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi(t-\tau)}} e^{-(x-\xi)^2/4(t-\tau)} f(\xi,\tau)\, d\xi\, d\tau.$$

5.5 The Heat Equation on the Half-Line

We continue the theme begun in the last section by looking at problems involving the heat equation on the half-line $x \geq 0$. Consider the initial-boundary value problem

$$\begin{aligned} u_t &= ku_{xx} \text{ for } x \geq 0 \\ u(0,t) &= 0 \text{ for } t \geq 0 \\ u(x,0) &= f(x). \end{aligned} \tag{5.11}$$

This problem is posed on $[0, \infty)$ and so has a boundary condition at the left end. We seek a bounded solution.

Let $u(x,t) = X(x)T(t)$ and proceed as with the Fourier integral solution on the entire real line to obtain

$$X'' + \lambda X = 0 \text{ and } T' + \lambda kT = 0.$$

Now $u(0,t) = X(0)T(t) = 0$ implies that

$$X(0) = 0.$$

Examine the cases on λ. If $\lambda = 0$, then $X = ax + b$. For a bounded solution we need $a = 0$. But then $X(0) = b = 0$, so this case has only the trivial solution for X and 0 is not an eigenvalue.

If $\lambda = -\omega^2$ with $\omega > 0$, then $X = ae^{\omega x} + be^{-\omega x}$. This is unbounded for $x \geq 0$ unless $a = 0$. Then $X = be^{-\omega x}$. But then $X(0) = b = 0$, so this case also yields only the trivial solution.

If $\lambda = \omega^2$ with $\omega > 0$, then $X = a\cos(\omega x) + b\sin(\omega x)$. This is bounded for $x \geq 0$. However, $X(0) = a = 0$, so we retain only the sine term.

With $\lambda = \omega^2$ the equation for T is $T' + \omega^2 kT = 0$ with solutions $T = ce^{-\omega^2 kt}$. We are therefore led to the functions

$$u_\omega(x,t) = b_\omega \sin(\omega x)e^{-\omega^2 kt}$$

satisfying the heat equation and the boundary condition. To satisfy the initial condition form, a superposition

$$u(x,t) = \int_0^\infty b_\omega \sin(\omega x)e^{-\omega^2 kt}\, d\omega.$$

We require that

$$u(x,0) = f(x) = \int_0^\infty b_\omega \sin(\omega x)\, d\omega.$$

Choose b_ω as the coefficient in the Fourier sine integral expansion of f:

$$b_\omega = \frac{2}{\pi}\int_0^\infty f(\xi)\sin(\omega\xi)\, d\xi.$$

Solution by Fourier Sine Transform

We illustrate a transform technique to solve this problem on $[0, \infty)$. Since $x \geq 0$, we can try using a Fourier sine or cosine transform in x. The operational formula for the cosine transform requires that we know $u_x(0,t)$, and we do not. The operational formula for the sine transform uses $u(0, t)$, which we are given. We therefore choose the sine transform in x and apply it to the heat equation. As with the Fourier transform in the preceding section, $\partial/\partial t$ passes through $\mathcal{F}_S$ and we use the operational formula to compute $\mathcal{F}_S[u_{xx}]$. Letting $\mathcal{F}_S[U(x,t)](\omega) = \widehat{u}_S(\omega, t)$, we obtain

$$\frac{\partial}{\partial t}\widehat{u}_S(\omega,t) = k[-\omega^2\widehat{u}_S(\omega,t) + \omega u(0,t)].$$

Since $u(0,t) = 0$,

$$\frac{\partial}{\partial t}\widehat{u}_S(\omega,t) + \omega^2 k\widehat{u}_S(\omega,t) = 0.$$

Think of this as a first-order differential equation in t. The general solution is

$$\widehat{u}_S(\omega,t) = b_\omega e^{-\omega^2 kt}.$$

Now

$$\widehat{u}_S(\omega,0) = b_\omega = \mathcal{F}_S[u(x,0)](\omega) = \mathcal{F}_S[f(x)](\omega) = \widehat{f}_S(\omega).$$

Therefore,

$$\widehat{u}_S(\omega,t) = \widehat{f}_S(\omega)e^{-\omega^2 kt}.$$

This is the sine transform of the solution. Apply the inverse sine transform (equation 3.27) to obtain

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \widehat{f}_S(\omega)e^{-\omega^2 kt}\sin(\omega x)\, d\omega.$$

Upon inserting the definition of $\widehat{f}_S(\omega)$, we obtain the solution

$$u(x,t) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(\xi)\sin(\omega\xi)\sin(\omega x)e^{-\omega^2 kt}\, d\xi\, d\omega.$$

This solution is often written in a different way. Use the identity

$$\sin(\omega\xi)\sin(\omega x) = \frac{1}{2}[\cos(\omega(x-\xi)) - \cos(\omega(x+\xi))]$$

and interchange the order of integration in the solution, to write

$$u(x,t) = \frac{1}{\pi}\int_0^\infty \left(\int_0^\infty e^{-\omega^2 kt}\cos(\omega(x-\xi))\,d\omega\right) f(\xi)\,d\xi$$
$$-\frac{1}{\pi}\int_0^\infty \left(\int_0^\infty e^{-\omega^2 kt}\cos(\omega(x+\xi))\,d\omega\right) f(\xi)\,d\xi.$$

Now use the fact that

$$\frac{1}{\pi}\int_0^\infty e^{-\omega^2 kt}\cos(\alpha\omega)\,d\omega = \frac{1}{2\sqrt{\pi kt}}e^{-\alpha^2/4kt},$$

which can be obtained by a change of variables in the integral 5.9. This enables us to write the solution as

$$u(x,t) = \frac{1}{2\sqrt{\pi kt}}\int_0^\infty \left[e^{-(x-\xi)^2/4kt} - e^{-(x+\xi)^2/4kt}\right] f(\xi)\,d\xi. \tag{5.12}$$

Note the similarity in form between this solution on the half-line and the solution 5.10 on the entire line.

A Problem with a Nonhomogeneous Boundary Condition

We will solve the initial-boundary value problem

$$u_t = ku_{xx} \text{ for } x > 0, t > 0$$
$$u(x,0) = A \text{ for } x > 0$$
$$u(0,t) = \begin{cases} B \text{ for } 0 < t < t_0 \\ 0 \text{ for } t > t_0. \end{cases}$$

in which A, B, and t_0 are positive constants. This models a homogeneous bar extending from 0 to infinity. At time zero the temperature throughout the bar for $x > 0$ is a constant A. The left end is kept at temperature B from time zero until time t_0, after which it is at temperature zero.

The boundary condition is written more conveniently using the Heaviside function, which is defined by

$$H(t) = \begin{cases} 1 \text{ for } t \geq 0 \\ 0 \text{ for } t < 0. \end{cases}$$

In terms of H,

$$u(0,t) = B[1 - H(t-t_0)].$$

We will solve this problem by using a Laplace transform in t. Let $\mathcal{L}$ denote the Laplace transform, defined by

$$\mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}\,dt.$$

Let $\mathcal{L}[u(x,t)](s) = U(x,s)$. The variable of the transformed function is s, and x is carried along as a parameter.

Since the transform is in the t variable, $\partial/\partial x$ and $\mathcal{L}$ commute, so

$$\mathcal{L}[u_{xx}(x,t)](s) = \frac{\partial^2}{\partial x^2}U(x,s).$$

For the transform of u_t, integrate by parts to obtain

$$\begin{aligned}\mathcal{L}[u_t(x,t)](s) &= \int_0^\infty u_t(x,t)e^{-st}\,dt \\ &= u(x,t)e^{-st}]_0^\infty - \int_0^\infty u(x,t)(-s)e^{-st}\,dt \\ &= -u(x,0) + s\int_0^\infty u(x,t)e^{-st}\,dt \\ &= -A + sU(x,s).\end{aligned}$$

Therefore,

$$\mathcal{L}[u_t] = \mathcal{L}[ku_{xx}]$$

becomes

$$-A + sU(x,s) = k\frac{\partial^2 U(x,s)}{\partial x^2}$$

or

$$\frac{\partial^2 U}{\partial x^2} - \frac{s}{k}U = -\frac{A}{k}.$$

Think of this as a constant-coefficient second-order differential equation for U in terms of the variable x. Its general solution is

$$U(x,s) = a(s)e^{\sqrt{s/k}x} + b(s)e^{-\sqrt{s/k}x} + \frac{A}{s},$$

in which the coefficients may be functions of s. We impose the condition that $a(s) = 0$ because $e^{\sqrt{s/k}x} \to \infty$ as $x \to \infty$ and we want a bounded solution. Then

$$U(x,s) = b(s)e^{-\sqrt{s/k}x} + \frac{A}{s}.$$

To solve for $b(s)$, use the condition that $u(0,t) = B[1 - H(t - t_0)]$. Then

$$\begin{aligned}\mathcal{L}[u(0,t)](s) = U(0,s) &= b(s) + \frac{A}{s} \\ &= \mathcal{L}[B[1 - H(t - t_0)]](s) \\ &= \frac{B}{s} - \frac{B}{s}e^{-t_0 s},\end{aligned}$$

in which the transform of $1 - H(t - t_0)$ can be computed by integration, by consulting a table, or with computer software. We conclude that

$$b(s) = \frac{B - A}{s} - \frac{B}{s}e^{-t_0 s}$$

and therefore,

$$U(x,s) = \left[\frac{B-A}{s} - \frac{B}{s}e^{-t_0 s}\right] e^{-\sqrt{s/k}x} + \frac{A}{s}.$$

The solution is $u(x,t) = \mathcal{L}^{-1}[U(x,s)]$ and this inverse can be determined by consulting a table or using software. There is also a complex integral formula for the inverse Laplace transform of a function. We obtain

$$\begin{aligned} u(x,t) = &\left[A \operatorname{erf}\left(\frac{x}{2\sqrt{kt}}\right) + B \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right)\right](1 - H(t-t_0)) \\ &+ A \operatorname{erf}\left(\frac{x}{2\sqrt{kt}}\right) H(t-t_0) \\ &+ \left[B \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right) - B \operatorname{erfc}\left(\frac{x}{2\sqrt{k(t-t_0)}}\right)\right] H(t-t_0). \end{aligned}$$

Here erf is the error function defined in Problem 25 of Section 5.3, and erfc is the complementary error function, defined by

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\xi^2}\, d\xi.$$

Problems for Section 5.5

In each of Problems 1 through 5, solve the initial-boundary value problem 5.11 by separation of variables (Fourier integral) and also by Fourier sine transform.

1. $f(x) = e^{-\alpha x}$, with α a positive number

2. $f(x) = xe^{-\alpha x}, \alpha$ positive

3. $$f(x) = \begin{cases} 1 & \text{for } 0 \le x \le h \\ 0 & \text{for } x > h \end{cases}$$

4. $$f(x) = \begin{cases} x & \text{for } 0 \le x \le 2 \\ 0 & \text{for } x > 2 \end{cases}$$

5. $$f(x) = \begin{cases} 1 & \text{for } 0 \le x \le h \\ -1 & \text{for } h < x \le 2h \\ 0 & \text{for } x > 2h \end{cases}$$

6. Solve

$$\begin{aligned} u_t &= ku_{xx} \text{ for } x > 0, t > 0 \\ u(x,0) &= f(x) \text{ for } x > 0 \\ u_x(0,t) &= 0 \text{ for } t > 0 \end{aligned}$$

7. Solve Problem 6 with

$$f(x) = \begin{cases} e^{-x} \text{ for } 0 \le x \le 1 \\ 0 \text{ for } x > 1. \end{cases}$$

8. Use separation of variables to solve

$$u_t = ku_{xx} \text{ for } x > 0, t > 0$$
$$u_x(x,0) = \begin{cases} A \text{ for } 0 \le x \le h \\ 0 \text{ for } x > h. \end{cases}$$

Solve the problem again using the Fourier sine transform. Show that the same solution is obtained by both methods.

9. Carry out the program of Problem 8 for

$$u_t = ku_{xx} \text{ for } x > 0, t > 0$$
$$u_x(x,0) = \begin{cases} x \text{ for } 0 \le x \le 2 \\ 0 \text{ for } x > 2. \end{cases}$$

10. Carry out the program of Problem 8 for

$$u(x,0) = \begin{cases} 1 - x \text{ for } 0 \le x \le 1 \\ 0 \text{ for } x > 1. \end{cases}$$

11. Solve

$$u_t = u_{xx} - u \text{ for } x > 0, t > 0$$
$$u_x(0,t) = f(t) \text{ for } t > 0$$
$$u(x,0) = 0 \text{ for } , x > 0$$

using the Fourier sine or cosine transform, as appropriate.

12. Solve

$$u_t = u_{xx} - tu \text{ for } x > 0, t > 0$$
$$u(x,0) = xe^{-x} \text{ for } x > 0$$
$$u(0,t) = 0 \text{ for } t > 0.$$

13. (a) Solve the problem

$$u_t = ku_{xx} \text{ for } x > 0, t > 0$$
$$u(x,0) = 0 \text{ for } x > 0$$
$$u(0,t) = B \text{ for } t > 0$$

in which B is constant. Hint: Transform the problem into one solved previously by setting $v = u - B$.

Show that the solution can be written in the form

$$u(x,t) = B\left(1 - \operatorname{erf}\left(x/\sqrt{4kt}\right)\right)$$

for $x > 0, t > 0$.

(b) Solve the problem

$$\begin{aligned} u_t &= ku_{xx} \text{ for } x > 0, t > 0 \\ u(x,0) &= 0 \text{ for } x > 0 \\ u(0,t) &= B(t) \text{ for } t > 0. \end{aligned}$$

Show that the solution of this problem can be written in the form

$$u(x,t) = \int_0^t \frac{\partial}{\partial t} W(x, \tau, t - \tau)\, d\tau,$$

in which $W(x, \tau, t)$ is the solution of the problem

$$\begin{aligned} W_t &= kW_{xx} \text{ for } x > 0, t > 0 \\ W(x,0) &= 0 \text{ for } x > 0 \\ W(0,t) &= B(\tau) \text{ for } t > 0. \end{aligned}$$

Note that, in the last condition, $W(0,\ t)$ is constant for each choice of τ.

(c) Use the result of part (a) to solve the problem for W in part (b) and write the solution for $u(x,\ t)$ in the problem for u in part (b). This solution can be simplified by an appropriate change of variables.

5.6 The Debate Over the Age of the Earth

The solution 5.12 for the heat equation in a semi-infinite medium played an important role in one of the more significant and interesting scientific controversies ever to take place. The entire story is told in Joe D. Burchfield's book *Lord Kelvin and The Age of the Earth*, published by Macmillan in 1975. We will recall some of the background and then use the solution 5.12 to derive Lord Kelvin's estimate for the age of the Earth.

In the eighteenth century a common belief, among natural philosophers and nonscientists alike, was that the Earth and everything on it (plants, animals, humankind) had been created at about the same time. This meant that the age of the Earth was approximately the age of the human race, and this was estimated to be several thousand years. Some scholars estimated the age of the Earth by forming a list of families in the Bible and guessing at the average age of each generation. One biblical chronology, constructed in the third century A.D. by Julius Africanus, made the underlying assumption that history is comprised of a cosmic week, with each week 1,000 years. Estimating that Christ was born in the sixth day of this cosmic week, Africanus arrived at a creation date of about 5500 B.C. for the Earth. This would make the Earth about 7500 years old today.

In the nineteenth century, scientists began to suspect that certain geological processes, such as the formation of mountains and canyons, must have taken a very long time. Estimates of the age of the Earth were extended to tens of thousands of years. At the extreme, some geologists held that the Earth's history extended indefinitely back, as there did not appear to be any evidence of a beginning (or of a coming end).

Then, in the period 1830–1833, Sir Charles Lyell's monumental three-volume work *Principles of Geology* was published. In it Lyell argued that processes such as erosion and sedimentation in rivers had to occur over extended periods of time. Unlike his predecessors and colleagues, Lyell thought in terms of many millions of years, perhaps hundreds of millions, a bold step beyond anything previously imagined. This view became known as the *uniformitarian theory*, because it disputed the belief that early violent geological activity had caused the present-day Earth to form over a relatively short time and maintained that the Earth reached its present state over an extended period of gradual, nearly uniform change.

Charles Darwin, who in 1859 published his classic *On the Origin of Species*, was very much a supporter of Lyell's view because evolution required longer periods of time than the several thousand years the eighteenth-century scientists had been willing to allow.

In the 1850s, Lord Kelvin (Sir William Thomson, 1824–1907) entered the arena with a completely different perspective, that of a physicist. Kelvin was instrumental in the development of the branch of physics known as thermodynamics, and was sensitive to arguments involving generation and conservation of energy. He began by considering the sun as the primary source of energy for the Earth. Since nothing was known of nuclear processes at this time, it was at first believed that the sun must be a gigantic chemical reaction. But this notion was soon discarded for want of a sufficiently large source of fuel. Kelvin therefore turned to the idea that the sun generated heat energy converted from gravitational potential energy as the sun contracts. This led him to an initial estimate that the sun, and therefore the Earth, could not be more than 100 million years old.

Soon Kelvin expanded on this argument. He had observed from data taken from mines extending deep into the Earth that the temperature of the Earth increases with depth. He reasoned that this must be caused by a continual loss of heat energy from the interior of the planet by conduction into the upper crust. However, it can also be observed that this upper crust does not increase in temperature on a year-to-year basis. Hence Kelvin reasoned that there must be a continual loss of heat energy from the Earth. Further, Kelvin, and many geologists, believed the Earth to be a solid ball that had solidified at a uniform temperature. Thus, if one could make an estimate of this initial temperature and assuming that there is a continual loss of heat energy, it should be possible, by determining the rate of heat loss and current temperatures in the crust, to estimate the age of the Earth.

Kelvin was prepared mathematically for this task. Unlike geologists, who at this time did not approach their science by any kind of mathematical modeling, Kelvin was quite expert at constructing models and solving the resulting equations. He began by assuming that the Earth is an approximately homogeneous sphere of radius R. If $\psi(\mathbf{x}, t)$ is the temperature at a point $\mathbf{x}$ (a vector denoting a point in 3-space, with origin at the center of the sphere) and time t, then ψ satisfies the three-dimensional heat equation

$$\frac{\partial \psi}{\partial t}(\mathbf{x}, t) = k\nabla^2\psi(\mathbf{x}, t) \text{ for } |\mathbf{x}| < R, t > 0.$$

Kelvin had solved this equation while engaged in studies of heat conduction. However, he now became convinced that very near the surface of the Earth, the temperature has remained constant and is on average approximately its initial value. This led him to neglect the curvature of the Earth and model heat conduction through the Earth as being along the half-line from its center. Let $u(x, t)$ be the temperature of the earth at distance x from the center and time t, for $x > 0, t > 0$. The one-dimensional problem to be solved for u is

$$\begin{aligned} u_t(x,t) &= ku_{xx}(x,t) \text{ for } x > 0, t > 0 \\ u(x,0) &= c \text{ for } x > 0 \\ u(0,t) &= 0 \text{ for } t > 0. \end{aligned}$$

Kelvin solved this problem, obtaining the integral expression given in equation 5.12. We repeat this expression here for ease of reference, noting that in the present context $f(x) = c$:

$$u(x,t) = \frac{c}{2\sqrt{\pi kt}} \int_0^\infty \left[e^{-(x-\xi)^2/4kt} - e^{-(x+\xi)^2/4kt}\right] d\xi.$$

Kelvin had no way of measuring the Earth's temperature at large depths. However, by making measurements as he descended into mine tunnels, he could estimate the rate of increase of temperature as one moved from the surface toward the center of the Earth. This suggested that he compute the rate of change of temperature with depth. From the solution, compute

$$\frac{\partial u}{\partial x} = \frac{c}{2\sqrt{\pi kt}} \int_0^\infty \left[-\frac{x-\xi}{2kt}e^{-(x-\xi)^2/4kt} + \frac{x+\xi}{2kt}e^{-(x+\xi)^2/4kt}\right] d\xi.$$

Then

$$\begin{aligned} \frac{\partial u}{\partial x}(0,t) &= \frac{c}{2\sqrt{\pi kt}} \int_0^\infty \frac{\xi}{kt} e^{-\xi^2/4kt}\, d\xi \\ &= \frac{c}{\sqrt{\pi kt}} \left[-e^{-\xi^2/4kt}\right]_0^\infty = \frac{c}{\sqrt{\pi kt}}. \end{aligned}$$

Solve this equation for t to obtain

$$t = \frac{1}{\pi k}\left(\frac{c}{u_x(0,t)}\right)^2.$$

As mentioned, Kelvin had used measurements taken in mine shafts to estimate $u_x(0, t)$. He also took measurements of the temperature of rocks and molten lava to estimate c and k. These led him to estimate the age of the Earth at between 100 million and 400 million years, with the variation to take into account possible errors in the values taken for c, k, and $u_x(0, t)$.

There were several reasons why Kelvin's estimates differed so much from the numbers we accept today. Kelvin had no way of understanding the nuclear processes generating energy from the sun, or radioactive minerals as heat sources within the Earth. More significantly, his concept of heat conduction within the Earth itself was flawed and did not take into account convection in the partly fluid interior of the planet. Nevertheless, the effect of Kelvin's analysis was profound. Many geologists disagreed with his conclusions, but they were at a loss to contradict his mathematics, which they did not understand. The mathematical arguments did indeed appear formidable, and Kelvin's deserved high status in the scientific community tended to lend further credence to his position. Despite his primary involvement with physics and thermodynamics, Kelvin retained an interest in geology and the age of the Earth until his death early in the twentieth century. Partly as a result of his influence, the science of geology became more attuned to the careful collection and analysis of data based on geological samples, and also, to a lesser extent, on mathematical modeling.

There is an amusing footnote to the story of Kelvin and the age of the Earth. In 1904, Ernest Rutherford's studies of radium led to his discovery of heat generation by radioactive processes. One day Rutherford was scheduled to give a lecture on his discovery, and he expected Kelvin to be present. Rutherford was worried about Kelvin's reaction to this new factor accounting for heat energy within the Earth. During the lecture Kelvin appeared to doze off, but he suddenly came awake when Rutherford approached the crucial point. Rutherford mentioned Kelvin's estimate of the age of the Earth, but diplomatically emphasized that no one could have foreseen this newly found source of heat. Kelvin seemed satisfied, and all was well.

5.7 The Nonhomogeneous Heat Equation

We will consider the initial-boundary value problem

$$\begin{aligned} u_t &= ku_{xx} + F(x,t) \text{ for } 0 < x < L, t > 0 \\ u(0,t) &= u(L,t) = 0 \text{ for } t \geq 0 \\ u(x,0) &= f(x) \text{ for } 0 \leq x \leq L. \end{aligned} \tag{5.13}$$

This is a nonhomogeneous problem because of the term $F(x,\ t)$ in the heat equation. This could represent a source of heat energy in the medium.

In the case $F \equiv 0$ we obtained a solution (equation 5.3) of the form

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 kt/L^2},$$

in which the b_n 's are the coefficients in the Fourier sine expansion of f on $[0, L]$. For the nonhomogeneous problem, we will take a cue from that case and attempt a solution

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi x}{L}\right). \tag{5.14}$$

The problem is to determine each $T_n(t)$.

First observe that for a given t, equation 5.14 can be thought of as the Fourier sine expansion of $u(x, t)$, considered a function of x, with $T_n(t)$ the *nth* Fourier coefficient in this expansion. Of course we do not yet know $u(x, t)$, but on the basis of equation 5.14, we can formally write

$$T_n(t) = \frac{2}{L}\int_0^L u(\xi,t)\sin\left(\frac{n\pi\xi}{L}\right)\,d\xi. \tag{5.15}$$

Now assume that for each $t \geq 0$, $F(x, t)$, as a function of x, can also be expanded in a Fourier sine series:

$$F(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin\left(\frac{n\pi x}{L}\right)\,dx, \tag{5.16}$$

where

$$B_n(t) = \frac{2}{L}\int_0^L F(\xi,t)\sin\left(\frac{n\pi\xi}{L}\right)\,d\xi \tag{5.17}$$

is the coefficient in this expansion, and may depend on t.

Differentiate equation 5.15 to obtain

$$T_n'(t) = \frac{2}{L}\int_0^L u_t(\xi,t)\sin\left(\frac{n\pi\xi}{L}\right)\,d\xi. \tag{5.18}$$

Since $u_t = ku_{xx} + F(x, t)$, equation 5.18 becomes

$$T_n'(t) = \frac{2k}{L}\int_0^L u_{xx}(\xi,t)\sin\left(\frac{n\pi\xi}{L}\right)\,d\xi + \frac{2}{L}\int_0^L F(\xi,t)\sin\left(\frac{n\pi\xi}{L}\right)\,d\xi.$$

In view of equation 5.17,

$$T_n'(t) = \frac{2k}{L}\int_0^L u_{xx}(\xi,t)\sin\left(\frac{n\pi\xi}{L}\right)\,d\xi + B_n(t). \tag{5.19}$$

Apply integration by parts twice to the integral in equation 5.19, using the boundary conditions and, at the last step, equation 5.15:

$$\int_0^L u_{xx}(\xi,t)\sin\left(\frac{n\pi\xi}{L}\right)d\xi$$
$$= \sin\left(\frac{n\pi\xi}{L}\right)u_x(\xi,t)\Big]_0^L - \int_0^L u_x(\xi,t)\frac{n\pi}{L}\cos\left(\frac{n\pi\xi}{L}\right)d\xi$$
$$= -\int_0^L \frac{n\pi}{L}u_x(\xi,t)\cos\left(\frac{n\pi\xi}{L}\right)d\xi$$
$$= -\frac{n\pi}{L}u(\xi,t)\cos\left(\frac{n\pi\xi}{L}\right)\Big]_0^L + \frac{n\pi}{L}\int_0^L u(\xi,t)\left(-\frac{n\pi}{L}\right)\sin\left(\frac{n\pi\xi}{L}\right)d\xi$$
$$= -\frac{n^2\pi^2}{L^2}\int_0^L u(\xi,t)\sin\left(\frac{n\pi\xi}{L}\right)d\xi$$
$$= -\frac{n^2\pi^2}{L^2}\frac{L}{2}T_n(t) = -\frac{n^2\pi^2}{2L}T_n(t).$$

Substituting this into equation 5.19 yields

$$T_n'(t) = -k\frac{n^2\pi^2}{L^2}T_n(t) + B_n(t).$$

For $n = 1, 2, \cdots$, we now have an ordinary differential equation for $T_n(t)$:

$$T_n' + k\frac{n^2\pi^2}{L^2}T_n = B_n(t). \tag{5.20}$$

Next, use equation 5.15 to obtain the condition,

$$T_n(0) = \frac{2}{L}\int_0^L u(\xi,0)\sin\left(\frac{n\pi\xi}{L}\right)d\xi$$
$$= \frac{2}{L}\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi = b_n,$$

the *nth* coefficient in the Fourier sine expansion of f on $[0, L]$.

Solve the ordinary differential equation 5.20 subject to the condition $T_n(0) = b_n$ to obtain the unique solution

$$T_n(t) = \int_0^t e^{-kn^2\pi^2(t-\tau)/L^2}B_n(\tau)\,d\tau + b_n e^{-kn^2\pi^2t/L^2}.$$

Finally, substitute this into equation 5.14 to obtain

$$u(x,t) = \sum_{n=1}^{\infty}\left(\int_0^t e^{-kn^2\pi^2(t-\tau)/L^2}B_n(\tau)\,d\tau\right)\sin\left(\frac{n\pi x}{L}\right)$$
$$+\sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)e^{-kn^2\pi^2t/L^2}, \tag{5.21}$$

in which $B_n(\tau)$ is determined by equation 5.17, and

$$b_n = \frac{2}{L}\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi$$

for $n = 1, 2, \cdots$. We leave it for the student to verify that $u(x, t)$ as defined by equation 5.21 is the solution of problem 5.13.

Example 5.6 *Solve the initial-boundary value problem*

$$\begin{aligned} u_t &= 4u_{xx} + t^2\cos(x/2) \text{ for } 0 < x < \pi, t > 0 \\ u(0,t) &= u(\pi,t) = 0 \text{ for } t \geq 0 \\ u(x,0) &= f(x) = \begin{cases} 0 \text{ for } 0 \leq x \leq \pi/2 \\ 25 \text{ for } \pi/2 < x < \pi. \end{cases} \end{aligned}$$

In the notation of the discussion, $F(x,t) = t^2\cos(x/2)$ *and* $L = \pi$. *First compute*

$$B_n(t) = \frac{2}{\pi}\int_0^\pi t^2\cos(\xi/2)\sin(n\xi)\,d\xi = \frac{8}{\pi}\frac{2n}{4n^2-1}t^2$$

from equation 5.17. Now we can evaluate the integral occurring in the first summation of equation 5.21:

$$\int_0^t \frac{8}{\pi}\frac{2n}{4n^2-1}\tau^2 e^{-4n^2(t-\tau)}\,d\tau = \frac{1}{2}\frac{-4n^2t + 8n^4t^2 + 1 - e^{-4n^2t}}{n^5\pi(4n^2-1)}.$$

Finally, compute

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi f(\xi)\sin(n\xi)\,d\xi \\ &= \frac{2}{\pi}\int_{\pi/2}^\pi 25\sin(n\xi)\,d\xi = \frac{50}{n\pi}(\cos(n\pi/2) - (-1)^n). \end{aligned}$$

The solution is

$$\begin{aligned} u(x,t) = &\sum_{n=1}^{\infty}\left(\frac{1}{2}\frac{-4n^2t + 8n^4t^2 + 1 - e^{-4n^2t}}{n^5\pi(4n^2-1)}\right)\sin(nx) \\ &+ \sum_{n=1}^{\infty}\frac{50}{n\pi}(\cos(n\pi/2) - (-1)^n)\sin(nx)e^{-4n^2t}. \end{aligned}$$

The second summation in $u(x, t)$ *is the solution of the initial-boundary value problem if the source term* $t^2\cos(x/2)$ *is omitted. If we denote this solution as* $u_h(x,t)$ *(the subscript* h *is for the fact that the heat equation without* $F(x, t)$ *is homogeneous), then*

$$u_h(x,t) = \sum_{n=1}^{\infty}\frac{50}{n\pi}(\cos(n\pi/2) - (-1)^n)\sin(nx)e^{-4n^2t}$$

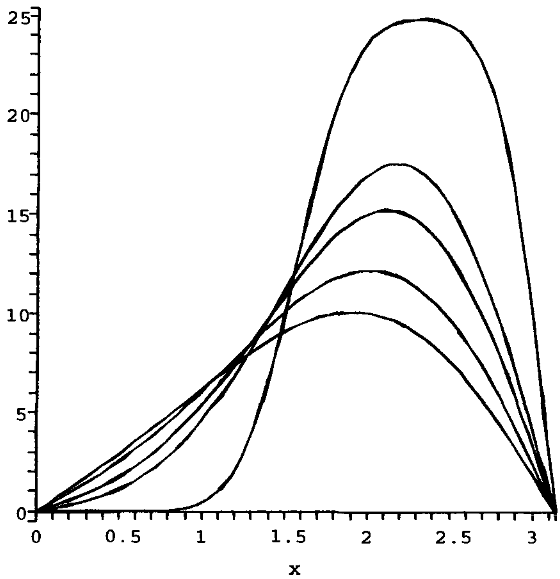

Figure 5.8: Decaying temperature profile in Example 5.6 in the absence of a source term.

and

$$u(x,t) = u_h(x,t) + \sum_{n=1}^{\infty} \left(\frac{1}{2} \frac{-4n^2t + 8n^4t^2 + 1 - e^{-4n^2t}}{n^5\pi(4n^2-1)} \right) \sin(nx).$$

This way of writing the solution clarifies which terms in the solution arise from the $t^2\cos(x/2)$ *term in the partial differential equation.*

Figure 5.8 shows the evolution of the solution $u_h(x,t)$ *for the problem with the source term omitted, at times* $t = 0.009, 0.067, 0.098, 0.05,$ *and* 0.13. *With no source, the temperature profile simply decreases with time. To gauge the effect of the source term at one time, Figure 5.9 compares the solution* $u_h(x, 1.2)$ *with* $u(x, 1.2)$, *at which time solution without the source term has decayed considerably below the solution with the source.* ◇

Problems for Section 5.7

In each of Problems 1 through 5, solve the problem 5.13. In each case, graph the solution as a function of x for different values of t.

1. $F(x,t) = t, f(x) = x(L-x)$
2. $F(x,t) = x\sin(t), f(x) = 1$
3. $F(x,t) = \cos(x), f(x) = x^2(L-x)$

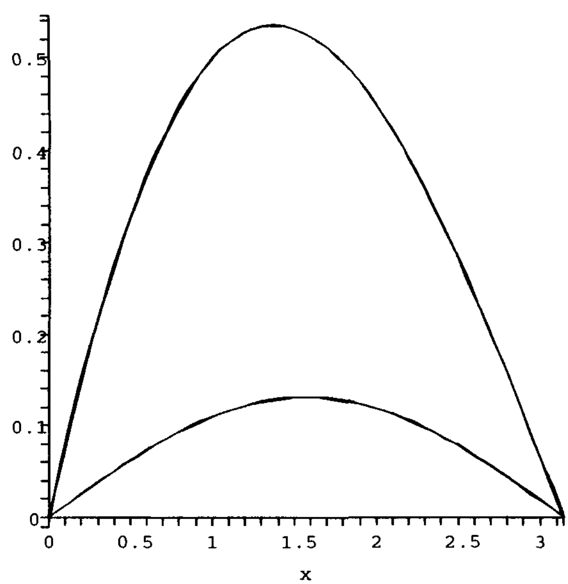

Figure 5.9: Comparison of $u_h(x, 1.2)$ (lower graph) and $u(x, 1.2)$ (upper graph) in Example 5.6.

4.
$$F(x,t) = \begin{cases} K & \text{for } 0 \le x \le L/2 \\ 0 & \text{for } L/2 < x \le L \end{cases}$$

and $f(x) = \sin(\pi x/L)$

5. $F(x,t) = xt, f(x) = K$

6. Adapt the method of solution of the problem 5.13 to obtain a solution of the problem

$$\begin{aligned} u_t &= ku_{xx} + F(x,t) \text{ for } 0 < x < L, t > 0 \\ u_x(0,t) &= u_x(L,t) = 0 \text{ for } t > 0 \\ u(x,0) &= f(x) \text{ for } 0 < x < L. \end{aligned}$$

Hint: Begin with

$$u(x,t) = \frac{1}{2}T_0(t) + \sum_{n=1}^{\infty} T_n(t)\cos(n\pi x/L),$$

where

$$T_n(t) = \frac{2}{L}\int_0^L u(\xi,t)\cos(n\pi\xi/L)\,d\xi.$$

Write

$$F(x,t) = \frac{1}{2}A_0(t) + \sum_{n=1}^{\infty} A_n(t)\cos(n\pi x/L)$$

with

$$A_n(t) = \frac{2}{L}\int_0^L F(\xi,t)\cos(n\pi\xi/L)\,d\xi.$$

Now follow the reasoning used in this section in solving problem 5.13.

7. Solve the initial-boundary value problem of Problem 6 for the case $F(x,t) = xt, f(x) = 1$.

8. Solve

$$\begin{aligned} u_t &= ku_{xx} \text{ for } 0 < x < L, t > 0 \\ u(0,t) &= \alpha(t), u_x(L,t) = \beta(t) \text{ for } t > 0 \\ u(x,0) &= 0 \text{ for } 0 < x < L. \end{aligned}$$

Hint: Attempt a solution

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin\left(\frac{(2n-1)\pi x}{2L}\right)$$

(note the hint to Problem 11 of Section 5.3). Let

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2.$$

Show that

$$T_n'(t) + k\lambda_n T_n(t) = b_n(t),$$

where

$$b_n(t) = \frac{2k}{L}[\sqrt{\lambda_n}\alpha(t) + (-1)^{n+1}\beta(t)].$$

Solve for $T_n(t)$, subject to $T_n(0) = 0$.

9. Solve the initial-boundary value problem of Problem 8 with $\alpha(t) = 1, \beta(t) = t$.

10. Solve

$$\begin{aligned} u_t &= ku_{xx} + F(x,t) \text{ for } 0 < x < L, t > 0 \\ u(x,0) &= 0 \text{ for } 0 < x < L \\ u_x(0,t) &= \alpha(t), u_x(L,t) = 0 \text{ for } t > 0. \end{aligned}$$

5.8 The Heat Equation in Two Space Variables

We solve an initial-boundary value problem for the temperature distribution in a thin flat homogeneous (constant density) rectangular plate having initial temperature $f(x, y)$ at (x, y) if the sides are maintained at temperature zero. This setting is modeled by the problem

$$\begin{aligned} u_t &= k(u_{xx} + u_{yy}) \text{ for } 0 < x < a, 0 < y < b, t > 0 \\ u(x,0,t) &= u(x,b,t) = 0 \text{ for } 0 \le x \le a, t \ge 0 \\ u(0,y,t) &= u(a,y,t) = 0 \text{ for } 0 \le y \le b, t \ge 0 \\ u(x,y,0) &= f(x,y) \text{ for } 0 \le x \le a, 0 \le y \le b. \end{aligned}$$

Let $u(x,y,t) = X(x)Y(y)T(t)$ to attempt a separation of variables. The heat equation yields

$$XYT' = k(X''YT + XY''T)$$

or

$$\frac{T'}{kT} - \frac{Y''}{Y} = \frac{X''}{X}.$$

Both sides must equal the same constant, which we will call $-\lambda$. Then

$$\frac{X''}{X} = -\lambda \text{ and } \frac{T'}{kT} - \frac{Y''}{Y} = -\lambda.$$

Write the last equation as

$$\frac{T'}{kT} + \lambda = \frac{Y''}{Y}.$$

Both sides of this equation must also be constant (since t and y are independent). Call this constant $-\mu$. Then

$$\frac{T'}{kT} + \lambda = \frac{Y''}{Y} = -\mu.$$

We now have separated the variables, obtaining

$$X'' + \lambda X = 0, Y'' + \mu Y = 0, \text{ and } T' + (\lambda + \mu)kT = 0.$$

Now take into account the boundary conditions. Since

$$u(0, y, t) = X(0)Y(y)T(t) = 0$$

we conclude that $X(0) = 0$. Using the other boundary conditions, we obtain

$$X(a) = Y(0) = Y(b) = 0.$$

The problems for X and Y are therefore

$$X'' + \lambda X = 0; X(0) = X(a) = 0$$

and

$$Y'' + \mu Y = 0; Y(0) = Y(b) = 0.$$

Except for notation, these problems for X and Y are the same and have been solved before. The eigenvalues and eigenfunctions are

$$\lambda_n = \frac{n^2\pi^2}{a^2} \text{ and } X_n(x) = \sin\left(\frac{n\pi x}{a}\right)$$

for $n = 1, 2, \cdots$, and

$$\mu_m = \frac{m^2\pi^2}{b^2} \text{ and } Y_m(y) = \sin\left(\frac{m\pi y}{b}\right)$$

for $m = 1, 2, \cdots$. These indices are independent—the choice of n does not influence the choice of m.

The equation for T is

$$T' + k\left(\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}\right)T = 0.$$

Solutions are constant multiples of

$$e^{-k\alpha_{nm}t},$$

in which

$$\alpha_{nm} = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}$$

for $n, m = 1, 2, \cdots$.

We can now form functions

$$u_{nm}(x,y,t) = c_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) e^{-k\alpha_{nm}t}.$$

These functions satisfy the heat equation and the boundary conditions for n and m any positive integers and for any numbers c_{nm}. To satisfy the initial condition, use the superposition

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) e^{-k\alpha_{nm}t}.$$

The initial condition requires that

$$u(x,y,0) = f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right).$$

This is the double Fourier sine expansion of $f(x, y)$ on the rectangle $0 \le x \le a, 0 \le y \le b$. In Section 4.11 we found the coefficients of such an expansion:

$$c_{nm} = \frac{4}{ab}\int_0^b \int_0^a f(\xi,\eta) \sin\left(\frac{n\pi\xi}{a}\right) \sin\left(\frac{m\pi\eta}{b}\right) d\xi\, d\eta$$

for $n = 1, 2, \cdots$ and $m = 1, 2, \cdots$.

For example, suppose that

$$f(x,y) = x(a-x)y(b-y).$$

Then

$$\begin{aligned} c_{nm} &= \frac{4}{ab}\int_0^b \int_0^a \xi(a-\xi)\eta(b-\eta) \sin\left(\frac{n\pi\xi}{a}\right) \sin\left(\frac{m\pi\eta}{b}\right) d\xi\, d\eta \\ &= \frac{16a^2b^2}{n^3m^3\pi^6}[1-(-1)^n][1-(-1)^m]. \end{aligned}$$

The solution in this case is

$$u(x,y,t) =$$
$$\frac{16a^2b^2}{\pi^6}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1-(-1)^n}{n^3}\frac{1-(-1)^m}{m^3} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) e^{-k\alpha_{nm}t}.$$

Problems for Section 5.8

1. Solve

$$\begin{aligned} u_t &= u_{xx} + u_{yy} \text{ for } 0 < x < \pi, 0 < y < \pi, t > 0 \\ u(x,0,t) &= u(x,\pi,t) = 0 \text{ for } 0 \le x \le \pi, t \ge 0 \\ u(0,y,t) &= u(\pi,y,t) = 0 \text{ for } 0 \le y \le \pi, t \ge 0 \\ u(x,y,0) &= \sin(x)y(y-\pi) \text{ for } 0 \le x \le \pi, 0 \le y \le \pi. \end{aligned}$$

2. Solve

$$\begin{aligned} u_t &= u_{xx} + u_{yy} \text{ for } 0 < x < 1, 0 < y < 1, t > 0 \\ u_x(0, y, t) &= u_x(1, y, t) = 0 \text{ for } 0 \le y \le 1, t \ge 0 \\ u_y(x, 0, t) &= u_y(x, 1, t) = 0 \text{ for } 0 \le x \le 1, t \ge 0 \\ u(x, y, 0) &= K \text{ for } 0 \le x \le 1, 0 \le y \le 1. \end{aligned}$$

3. Solve

$$\begin{aligned} u_t &= k(u_{xx} + u_{yy}) \text{ for } 0 < x < \pi, 0 < y < \pi, t > 0 \\ u(x, 0, t) &= u(x, \pi, t) = 0 \text{ for } 0 \le x \le \pi, t \ge 0 \\ u_x(0, y, t) &= u_x(\pi, y, t) = 0 \text{ for } 0 \le y \le \pi, t \ge 0 \\ u(x, y, 0) &= x\cos(x/2)y(\pi - y) \text{ for } 0 \le x \le \pi, 0 \le y \le \pi. \end{aligned}$$

4. Let C be a simple closed curve in the x, y - plane, bounding a region D. Formulate and prove a weak maximum principle for the two-dimensional heat equation
$$u_t = k(u_{xx} + u_{yy})$$
on the three-dimensional region consisting of all points (x, y, z) with $0 \le t \le T$ and (x, y) in D.

5. Let D be a region of the x, y - plane bounded by a simple closed curve C. Consider the problem

$$\begin{aligned} u_t &= k(u_{xx} + u_{yy}) \text{ for } (x, y) \text{ in } D, t > 0. \\ u(x, y, 0) &= \varphi(x, y) \text{ for } (x, y) \text{ in } D \\ u(x, y, t) &= \psi(x, y, t) \text{ for } (x, y) \text{ on } C, t > 0. \end{aligned}$$

Let φ and ψ be continuous. Assuming that this problem has a solution, prove that this solution is unique. Hint: Use the result of Problem 4.

Beginning Partial Differential Equations, Second Edition
by Peter V. O'Neil

Chapter 6

Dirichlet and Neumann Problems

6.1 The Setting of the Problems

Having considered the wave equation, which is hyperbolic, and the heat equation, which is parabolic, we now take up Dirichlet and Neumann problems, which are elliptic. We begin with some notation and terminology.

The *Laplacian* $\nabla^2 u$ of a function $u(x_1, x_2, \cdots, x_n)$ is

$$\nabla^2 u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

In the case of two independent variables, we often write x and y instead of x_1 and x_2, and in 3 - space we usually write x, y, and z.

The partial differential equation

$$\nabla^2 u = 0$$

is *Laplace's equation*. A continuous function satisfying Laplace's equation, and having continuous first and second partial derivatives, is called a *harmonic function*. Usually, we discuss harmonic functions for the variables restricted to a specified set of values.

The Laplacian appears in the heat equation, which can be written

$$u_t = k\,\nabla^2 u.$$

The steady-state case of the heat equation occurs when $u_t = 0$, and then the heat equation becomes Laplace's equation $\nabla^2 u = 0$.

We need some notation and definitions to analyze solutions of Laplace's equation. Let R^n denote the usual n - dimensional space of ordered n - tuples $(x_1, \cdots, x_n)$, in which each x_j is a real number. R^2 is interpreted as the plane.

We may also think of R^n as consisting of vectors with n components. We denote vectors in bold type.

The distance between $\mathbf{x} = (x_1, \cdots, x_n)$ and $\mathbf{y} = (y_1, \cdots, y_n)$ in R^n is denoted $|\mathbf{x} - \mathbf{y}|$ and is defined by the usual metric

$$|\mathbf{x} - \mathbf{y}| = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}.$$

This distance function is *symmetric*, which means that

$$|\mathbf{x} - \mathbf{y}| = |\mathbf{y} - \mathbf{x}|.$$

It also satisfies the *triangle inequality:*

$$|\mathbf{x} - \mathbf{y}| \leq |\mathbf{x} - \mathbf{z}| + |\mathbf{z} - \mathbf{y}|.$$

We will use the standard notation $\mathbf{x} \in A$ if $\mathbf{x}$ is an element of a set A.

Given a point $\mathbf{x}_0 \in R^n$ and a positive number r, the *open ball of radius r about* $\mathbf{x}_0$ consists of all points in R^n at distance less than r from $\mathbf{x}_0$. This set is denoted $B(\mathbf{x}_0, r)$. Thus $\mathbf{y} \in B(\mathbf{x}_0, r)$ if and only if $|\mathbf{y} - \mathbf{x}_0| < r$. In the plane an open ball consists of all points enclosed by a circle (but not on the circle itself), and in 3 - space an open ball consists of all points enclosed by a sphere (but not on the surface of the sphere).

A *neighborhood* of $\mathbf{x}_0$ is an open ball $B(\mathbf{x}_0, r)$ centered at $\mathbf{x}_0$, in which r can be any positive number.

If S is a set of points in R^n, a point $\mathbf{x}$ in R^n is a *boundary point* of S if every neighborhood of $\mathbf{x}$ contains at least one point in S and one point not in S. We call $\mathbf{x}$ an *interior point* of S if there is some neighborhood of $\mathbf{x}$ which contains only points of S. A point cannot be both a boundary point and an interior point of S. An interior point of S necessarily belongs to S (because it is the center of some neighborhood containing only points of S), while a boundary point of S may or may not belong to S.

Every point belonging to S must be either a boundary point or an interior point. However, there may be boundary points of S that do not belong to S.

S is an *open set* if all of its points are interior points. A set is *closed* if it contains all of its boundary points. A set may be neither open nor closed. Indeed, if a set contains a boundary point, it cannot be open, but it need not be closed either, because there may be other boundary points the set does not contain.

Example 6.1 *Let S consist of all points (x, y) on the line $x + y = 1$ in the plane. Then S consists of all points $(x, 1 - x)$. S is shown in Figure 6.1 S has no interior point. Given any point P of S, there is no circle about P containing only points of this line..*

Every point of S is a boundary point. Every circle about P contains P in S, and also points off the line, hence not in S. Every point of S is a boundary point, and S has no other boundary points. $\diamond$

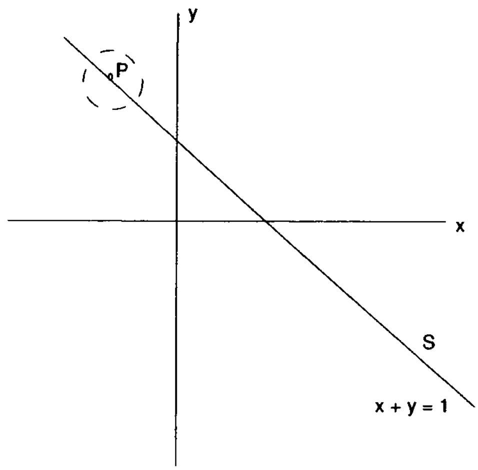

Figure 6.1: S in Example 6.1 has no interior point.

Example 6.2 *Let S consist of points (x, y) in the plane with $x \geq 0$ and $y > 0$. S is sketched in Figure 6.2. The positive y - axis is drawn as a solid line to indicate that points on this half-line are in S, while the nonnegative x - axis is drawn as a dashed line to indicate that these points are not in S.*

The points (x, y) with $x > 0$ and $y > 0$ are interior points of S. About any such point P we can draw an open ball $B(P, r)$ containing only points with positive coordinates, hence lying entirely in S.

The boundary points of S are the points $(0, y)$ with $y \geq 0$ and points $(x, 0)$ with $x \geq 0$. The boundary points $(0, y)$ with $y > 0$ belong to S, while the boundary points $(x, 0)$ with $x \geq 0$ do not belong to S.

S is not closed because S does not contain all of its boundary points. S is not open because S contains some of its boundary points. $\diamond$

Example 6.3 *Any open ball $B(\mathbf{x}_0, r)$ in R^n is an open set. About any point within $B(\mathbf{x}_0, r)$ we can place a ball of sufficiently small radius that it contains only points of $B(\mathbf{x}_0, r)$.* $\diamond$

Given a set S of points in R^n, the *boundary* of S consists of all the boundary points of S, and is denoted ∂S.

Example 6.4 *Let S consist of all points (x, y) in R^2 with $-1 < x < 1$. S may be visualized as the strip shown in Figure 6.3. S is open, every point being an interior point. The boundary points are points $(1, y)$ and points $(-1, y)$, lying on the vertical sides of the strip. Thus ∂S consists of all points (α, y) with $\alpha = \pm 1$.* $\diamond$

The boundary of a ball $B(\mathbf{x}_0, r)$ consists of all points $\mathbf{x}$ at distance r from

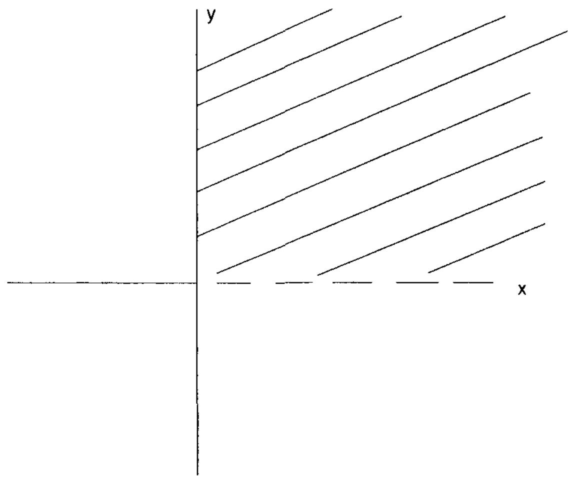

Figure 6.2: The set of points (x, y) with $x \geq 0$ and $y > 0$.

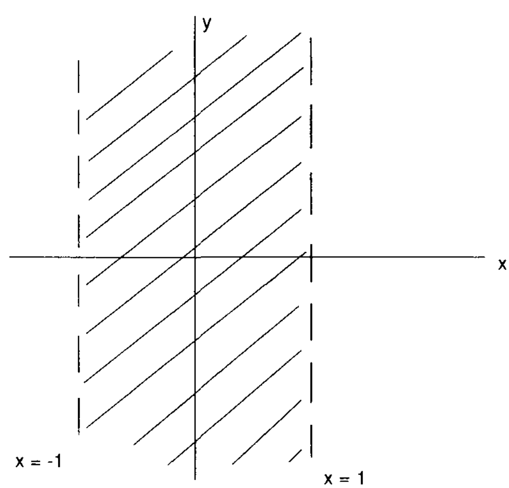

Figure 6.3: The open strip $-1 < x < 1$ in R^2 in Example 6.4.

$\mathbf{x}_0$:

$$|\mathbf{x} - \mathbf{x}_0| = r.$$

These boundary points of $B(\mathbf{x}_0, r)$ form the *n - sphere* $S(\mathbf{x}_0, r)$ of radius r about $\mathbf{x}_0$. In the plane this consists of all points on a circle about some point; in 3 - space, the boundary consists of all points on a sphere about a point.

The *closure* of a set S consists of all points of S together with all the boundary points of S and is denoted $\overline{S}$. If S is closed, then S already contains all of its boundary points, so $S = \overline{S}$. Conversely, $\overline{S}$ is always a closed set (containing all of its boundary points). Hence $S = \overline{S}$ if and only if S is closed. The closure of the strip S in Example 6.4 is the set $\overline{S}$ consisting of all points (x, y) with $-1 \leq x \leq 1$. The closure of $B(\mathbf{x}_0, r)$ consists of exactly those $\mathbf{x}$ in R^n satisfying $|\mathbf{x} - \mathbf{x}_0| \leq r$.

A set S of points in R^n is *connected* if for any two points $\mathbf{x}$ and $\mathbf{y}$ of S there is a polygonal path consisting of a finite number of straight-line segments, lying entirely in S, and having $\mathbf{x}$ and $\mathbf{y}$ as endpoints. (We are using the word connected here where a topologist might use the term *polygonally connected.*) For example, the set of points in the plane consisting of all (x, y) with $x < 0$ and $y < 0$, together with all (x, y) with $y > 1 - x$, is not connected. This set is shown in Figure 6.4. It is impossible to connect a point in the set and lying above the line $x + y = 1$ with a point in the lower-left quarter plane without going outside this set. An open, connected set of points in R^n is called a *domain.* For example, the right half-plane in R^2, consisting of (x, y) with $x > 0$, is a domain. Any open ball is a domain.

S is a *bounded set* if there is some number M such that $|\mathbf{x}| \leq M$ for all $\mathbf{x}$ in S. A set is bounded if it can be fit entirely within some open ball about the origin. If a set is not bounded, we say that it is *unbounded.*

Boundedness says nothing about whether the set is open, closed, not open, or not closed. Note also the difference between the concepts "boundary of a set" and "a set is bounded." A set may be unbounded (contain points arbitrarily far from the origin) and still have boundary points. For example, the upper half plane in R^2 consists of all points (x, y) with $y > 0$. This set is unbounded. But this set has a boundary consisting of all points $(x, 0)$ on the horizontal axis in the plane.

A closed, bounded subset of R^n is called a *compact set.*

Now return to Laplace's equation and harmonic functions. Given a domain Ω, a *Dirichlet problem* for Ω consists of finding a function that is harmonic in Ω and assumes given values on the boundary of Ω. That is, given Ω and a function f that is defined on $\partial\Omega$, we seek a function u such that

$$\nabla^2 u(\mathbf{x}) = 0 \text{ for } \mathbf{x} \in \Omega$$

and

$$u(\mathbf{x}) = f(\mathbf{x}) \text{ for } \mathbf{x} \in \partial\Omega.$$

In the plane the boundary of a domain will often be a piecewise smooth curve having a continuous tangent except possibly at finitely many points. In

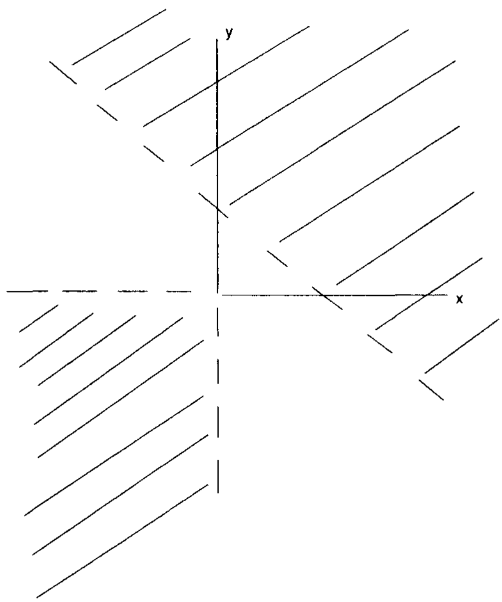

Figure 6.4: A disconnected set in the plane.

R^3 we often consider domains whose boundaries are piecewise smooth surfaces, composed of finitely many smooth pieces, each of which has a continuous normal vector. For example, a sphere in R^3 is smooth, whereas a cube is piecewise smooth, consisting of six smooth faces.

A *Neumann problem* for Ω consists of finding a function that is harmonic in Ω and whose normal derivative assumes given values on the boundary. The normal derivative of a function at a boundary point is the directional derivative of the function in the direction of the outer normal to the boundary at that point. If $\mathbf{n}(\mathbf{x})$ is the unit outer, or exterior, normal to $\partial\Omega$, the normal derivative of $u(\mathbf{x})$ is denoted

$$\frac{\partial u}{\partial n}$$

and is the dot product of the gradient of u, with $\mathbf{n}$:

$$\frac{\partial u(\mathbf{x})}{\partial n} = \nabla u(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}).$$

Figure 6.5 shows unit outer normals at the boundary of a typical domain in the plane.

A Neumann problem therefore consists of determining a function u satisfying

$$\nabla^2 u = 0 \text{ in } \Omega$$

$$\frac{\partial u}{\partial n} = f \text{ on } \partial\Omega$$

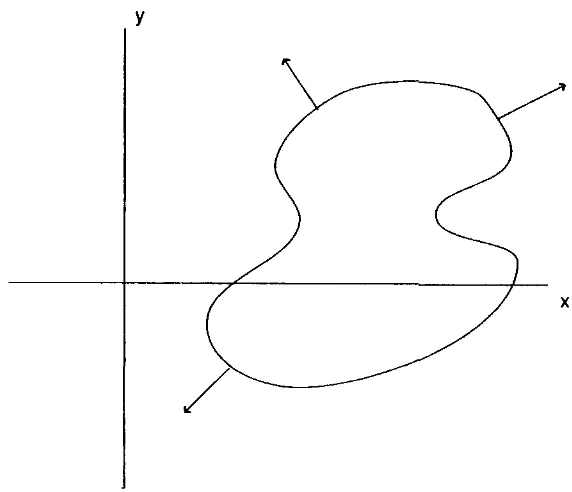

Figure 6.5: Unit outer normals to a domain in R^2.

in which f is a given function.

Both the Dirichlet and Neumann problems are boundary value problems. Initial conditions are not relevant for Dirichlet and Neumann problems.

Sometimes the Dirichlet problem is called the *first boundary value problem* for a domain, and the Neumann problem, the *second boundary value problem.* There is also a *third*, or *mixed, boundary value problem*, in which we seek a function that is harmonic in a domain and assumes values on the boundary that are given as a combination of Dirichlet- and Neumann- type boundary data. This problem, also known as *Robin's problem*, takes the form of seeking u satisfying

$$\nabla^2 u = 0 \text{ in } \Omega$$
$$a(\mathbf{x})u(\mathbf{x}) + b(\mathbf{x})\frac{\partial u(\mathbf{x})}{\partial n} = f(\mathbf{x}) \text{ for } \mathbf{x} \in \partial\Omega,$$

in which a, b, and f are given functions.

We do not consider Cauchy data for Laplace's equation because this Cauchy problem is not well posed. The following example illustrates this and is due to the French mathematician Jacques Hadamard. For each positive integer n, the problem

$$\begin{aligned} \nabla^2 u(x,y) &= 0 \text{ for all real } x \text{ and } y \\ u(x,0) &= 0 \text{ for all } x \\ u_y(x,0) &= \frac{1}{n}\sin(nx) \text{ for all } x \end{aligned}$$

has the unique solution

$$u(x,y) = \frac{1}{2n^2}(e^{ny} - e^{-ny})\sin(nx).$$

As n is chosen larger, the data given for u_y along the horizontal axis tends to zero. However, the solution $u(x, y)$ assumes values arbitrarily large in magnitude for certain (x, y): for example, for points $((2n+1)\pi/2, n)$. Problem 8 provides another example of this type of behavior.

Problems for Section 6.1

1. Let S consist of all (x, y) in R^2 with $0 \le x < 1, 0 < y \le 1$. Determine all interior points of S, all boundary points, the closure of S, and whether S is open, closed, or neither open nor closed. Is S connected?

2. Let S consist of all (x, y) in R^2 with $x < 0$ and $1 \le y \le 4$. Determine all interior points of S, all boundary points, the closure of S, and whether S is open, closed, or neither open nor closed. Is S connected?

3. Let S consist of all points (x, y) in R^2 with $1 < x^2 + y^2 < 4$. Determine if S is a domain.

4. Let S consist of all points (x, y) in R^2 with $-1 < x < 6$. Determine if S is a domain.

5. Let S consist of all points in R^2 with rational coordinates. Determine all interior points of S, all boundary points, the closure of S, and whether S is open, closed, or neither open nor closed. Is S connected?

6. Consider the same questions as in Problem 5, applied to the set T consisting of all points in R^2 with irrational coordinates.

7. Does the set of all points on a plane in R^3 constitute a domain?

8. Show that $u_n(x,y) = e^{-\sqrt{n}}\sin(nx)e^{ny}$ is a solution of

$$\begin{aligned} \nabla^2 u(x,y) &= 0 \text{ for all } x \text{ and } t > 0 \\ u(x,0) &= 0, u_y(x,0) = ne^{-\sqrt{n}}\sin(nx) \end{aligned}$$

for each positive integer n. Prove that for any positive y_0,

$$u_n(x, y_0) \to \infty \text{ as } n \to \infty.$$

6.2 Some Harmonic Functions

It will be useful to have some harmonic functions at our disposal. It is obvious that constant multiples of harmonic functions are harmonic, and finite sums of harmonic functions are harmonic.

Harmonic Functions in the Plane

Rectangular Coordinates

In rectangular coordinates in R^2, one way of generating harmonic functions is to recall that the real and imaginary parts of an analytic function of a complex variable are both harmonic. This means that if we take any analytic complex function and separate the real and imaginary parts, we obtain a pair of harmonic functions.

To illustrate, consider $f(z) = z^2$, where $z = x+iy$ and x and y are real. Then

$$f(z) = (x+iy)^2 = x^2 - y^2 + 2ixy = u(x,y) + iv(x,y),$$

where

$$u(x,y) = x^2 - y^2 \text{ and } v(x,y) = 2xy.$$

It is routine to check that $\nabla^2 u = 0$ and $\nabla^2 v = 0$ for all $(x,\, y)$. These functions are harmonic in the entire plane R^2.

As another example, write $\cos(z)$ as

$$\begin{aligned}\cos(z) &= \frac{1}{2}(e^{iz} + e^{-iz}) = \frac{1}{2}(e^{i(x+iy)} + e^{-i(x+iy)})\\ &= \frac{1}{2}(e^{-y}e^{ix} + e^{y}e^{-ix})\\ &= \frac{1}{2}[e^{-y}(\cos(x) + i\sin(x)) + e^{y}(\cos(x) - i\sin(x))]\\ &= \cos(x)\frac{1}{2}(e^{y} + e^{-y}) - i\frac{1}{2}\sin(x)(e^{y} - e^{-y}).\end{aligned}$$

This yields the harmonic functions $\cos(x)(e^y + e^{-y})$ and $\sin(x)(e^y - e^{-y})$.

Polar Coordinates

In rectangular coordinates in the plane, Laplace's equation is

$$u_{xx} + u_{yy} = 0.$$

We will derive the polar coordinate form of this equation. Let

$$x = r\cos(\theta), y = r\sin(\theta)$$

and

$$U(r,\theta) = u(r\cos(\theta), r\sin(\theta)).$$

Calculate

$$\begin{aligned}
U_r &= u_x \cos(\theta) + u_y \sin(\theta), \\
U_{rr} &= u_{xx} \cos^2(\theta) + 2u_{xy} \cos(\theta)\sin(\theta) + u_{yy} \sin^2(\theta), \\
U_\theta &= -u_x r \sin(\theta) + u_y r \cos(\theta), \\
U_{\theta\theta} &= -u_x r \cos(\theta) - u_y r \sin(\theta) \\
&\quad - r\sin(\theta)[u_{xx}(-r\sin(\theta)) + u_{xy} r \cos(\theta)] \\
&\quad + r\cos(\theta)[u_{yx}(-r\sin(\theta)) + u_{yy}(r\cos(\theta)].
\end{aligned}$$

Then

$$U_{\theta\theta} = -rU_r + r^2[u_{xx}\sin^2(\theta) - 2u_{xy}\cos(\theta)\sin(\theta) + u_{yy}\sin^2(\theta)].$$

Now observe that

$$U_{rr} + \frac{1}{r^2}U_{\theta\theta} = u_{xx} + u_{yy} - \frac{1}{r}U_r.$$

Therefore, $u_{xx} + u_{yy} = 0$ implies that

$$U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} = 0. \tag{6.1}$$

This is Laplace's equation in polar coordinates. Using equation 6.1, we can generate additional harmonic functions in the plane as follows.

To begin with a simple case, there are important harmonic functions which depend only on r, the distance from the origin. Such θ - independent functions can be found as follows. If $U_\theta = 0$, equation 6.1 becomes just $U_{rr} + (1/r)U_r = 0$, or $rU_{rr} + U_r = 0$. This can be written

$$(rU_r)_r = 0;$$

hence $rU_r = c$, a constant. But then $U_r = c/r$ with solutions

$$U(r) = c\ln(r) + k.$$

This function is harmonic for $r > 0$ and any choices of the constants c and k.

We can find harmonic functions of both r and θ by separating variables in equation 6.1. With $U(r, \theta) = R(r)\Theta(\theta)$, we get

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

or

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta}.$$

Since the right side depends only on r, and the left, only on θ, and r and θ are independent, both sides must equal the same constant, which we will call λ. Then

$$r^2R'' + rR' - \lambda R = 0 \text{ and } \Theta'' + \lambda\Theta = 0.$$

Now (r, π) and $(r, -\pi)$ are polar coordinates of the same point, so we require that

$$\Theta(\pi) = \Theta(-\pi). \tag{6.2}$$

This is a *periodicity condition.*

It is routine to solve for Θ by considering possibilities for λ. If $\lambda = 0$, then $\Theta = a\theta + b$ and we must choose $a = 0$ to satisfy condition 6.2, obtaining $\Theta(\theta) = \text{constant}$. If $\lambda = -k^2$ with $k \neq 0$, then

$$\Theta = ae^{k\theta} + be^{-k\theta}$$

and this function cannot satisfy condition 6.2 with nonzero real a or b for real θ. If $\lambda = k^2$, then

$$\Theta(\theta) = a\cos(k\theta) + b\sin(k\theta)$$

and this function will satisfy equation 6.2 exactly when k is chosen to be an integer.

Now consider the equation for R. This is an Euler differential equation and we attempt solutions $R = r^\alpha$. Substitute this into the differential equation for R, with $\lambda = k^2$, to obtain

$$\alpha(\alpha - 1) + \alpha - k^2 = 0,$$

so $\alpha = \pm k$ and R must have the general form $R(r) = cr^k + dr^{-k}$.

This means that $U(r, \theta)$ has the form

$$\begin{aligned} U(r,\theta) &= R(r)\Theta(\theta) \\ &= (cr^k + dr^{-k})(a\cos(k\theta) + b\sin(k\theta)), \end{aligned}$$

in which k is any integer. This includes the case of the constant function obtained when $k = 0$.

In summary, we have obtained the harmonic functions

$$1 \text{ and } \ln(r)$$

which are independent of θ, with $\ln(r)$ defined for $r > 0$, as well as θ - dependent functions

$$r^k\cos(k\theta) \text{ and } r^k\sin(k\theta)$$

defined for $r \geq 0$ and all θ, and

$$r^{-k}\cos(k\theta) \text{ and } r^{-k}\sin(k\theta),$$

defined for $r > 0$ and all θ, with $k = 1, 2, \cdots$.

These harmonic functions will be used to solve Dirichlet and Neumann problems.

Harmonic Functions in R^3

Laplace's equation in spherical coordinates (ρ, θ, φ) is

$$u_{\rho\rho} + \frac{2}{\rho} u_\rho + \frac{1}{\rho^2 \sin^2(\varphi)} u_{\theta\theta} + \frac{1}{\rho^2} u_{\varphi\varphi} + \frac{\cot(\varphi)}{\rho^2} u_\varphi = 0,$$

in which θ is the polar angle and φ the azimuthal angle.

The function

$$u(\rho) = \frac{1}{\rho} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

is harmonic in R^3 with the origin removed and is independent of θ and φ. It is possible to use separation of variables to find other harmonic functions in spherical coordinates, but we will not pursue this project.

Problems for Section 6.2

In Problems 1 through 4, produce harmonic functions u and v by writing the given complex function in the form $f(z) = u(x, y) + iv(x, y)$.

1. $f(z) = z^3 - 2z$

2. $f(z) = \sin(z)$ Hint: Use Euler's formula and the fact that

$$\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

3. $f(z) = z\cos(z)$

4. $f(z) = \sin^2(z)$

5. Let u be harmonic in a domain Ω in R^2. Let a and b be real numbers and let Ω^* consist of all translations $(x + a, y + b)$ of points (x, y) in Ω. Define

$$v(x, y) = u(x - a, y - b)$$

for (x, y) in Ω^*. Prove that v is harmonic in Ω^*. This means that translations of harmonic functions are harmonic functions.

6. Let θ be any real number and let

$$x^* = \cos(\theta)x + \sin(\theta)y$$
$$y^* = -\sin(\theta)x + \cos(\theta)y,$$

a rotation about the origin in the plane. Suppose that u is harmonic in Ω, and let Ω^* be obtained by applying the rotation to all points of Ω. Let $w(x^*, y^*) = u(x, y)$. Show that w is harmonic in Ω^*. This means that rotations take harmonic functions to harmonic functions.

7. Derive the spherical coordinate form of Laplace's equation.

8. Let u be harmonic on the open ball $B(\mathbf{0}, a)$ of radius a about the origin in R^3. Define an inversion ι in this ball to be the mapping that sends a point $\mathbf{x} \neq \mathbf{0}$ in this ball to the point $\iota(\mathbf{x})$ exterior to the ball, with the property that $\iota(\mathbf{x})$ is on the line from the origin through $\mathbf{x}$, and the product of the distance from the origin to $\mathbf{x}$ and from the origin to $\iota(\mathbf{x})$ is a^2:

$$|\mathbf{x}||\iota(\mathbf{x})| = a^2.$$

(a) Show that

$$\iota(\mathbf{x}) = \frac{a^2}{|\mathbf{x}|^2}\mathbf{x}$$

for $\mathbf{x} \in B(\mathbf{0}, a)$ and $\mathbf{x} \neq \mathbf{0}$.

(b) Prove that every point outside the closure of $B(\mathbf{0}, a)$ is the inversion of exactly one point in $B(\mathbf{0}, a)$. Conversely, show that every point except the origin in $B(\mathbf{0}, a)$ is the inversion of exactly one point outside the closure of this ball.

(c) Let $w(\mathbf{x}) = u(\iota(\mathbf{x}))$ for $\mathbf{x}$ outside $\overline{B(\mathbf{0}, a)}$. Prove that w is harmonic in $\overline{B(\mathbf{0}, a)}$ with the origin deleted and that $w(\mathbf{x}) = u(\mathbf{x})$ for $\mathbf{x}$ in $S(\mathbf{0}, a)$.

6.3 Representation Theorems

In this section we develop integral representations of functions that we will use to derive properties of harmonic functions and to solve Dirichlet and Neumann problems. We begin with a general representation theorem for (not necessarily harmonic) functions defined on a domain in R^3 and then discuss its analogue for functions defined on a domain in the plane.

If Ω is a bounded domain in R^n, let $C^2(\overline{\Omega})$ denote the set of functions that are continuous, with continuous first and second partial derivatives throughout $\overline{\Omega}$.

A surface in R^3 is called a *closed surface* if it bounds a volume. For example, a sphere is a closed surface, as is a cube, whereas a hemisphere is not. A hemisphere capped by a disk of the same radius is a closed surface.

A Representation Theorem in R^3

We will use the following result named for the nineteenth century Englishman George Green.

Lemma 6.1 (*Green's Second Identity in* R^3) *Let* Ω *be a bounded domain in* R^3 *and assume that* $\partial\Omega$ *is a piecewise smooth closed surface. Let* u, $v \in C^2(\overline{\Omega})$. *Then*

$$\iiint_{\overline{\Omega}} (u\nabla^2 v - v\nabla^2 u)\, dV = \iint_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) d\sigma. \diamond$$

In this lemma, u and v are not assumed to be harmonic.

Proof Let $\mathbf{n}$ be the unit outer normal to $\partial\Omega$. Since this surface is piecewise smooth, $\mathbf{n}$ is piecewise continuous. Apply Gauss's divergence theorem to the following surface integrals:

$$\begin{aligned}&\iint_{\partial\Omega} u\frac{\partial v}{\partial n}\,d\sigma - \iint_{\partial\Omega} v\frac{\partial u}{\partial n}\,d\sigma\\ &= \iint_{\partial\Omega} u\nabla v\cdot\mathbf{n}\,d\sigma - \iint_{\partial\Omega} v\nabla u\cdot\mathbf{n}\,d\sigma\\ &= \iiint_{\overline{\Omega}} \operatorname{div}(u\nabla v)\,dV - \iiint_{\overline{\Omega}} \operatorname{div}(v\nabla u)\,dV.\end{aligned}$$

Now it is a routine calculation to show that

$$\operatorname{div}(u\nabla v) - \operatorname{div}(v\nabla u) = u\nabla^2 v - v\nabla^2 u,$$

completing the proof. ◇

Using this lemma, we can derive an integral representation of $C^2(\overline{\Omega})$ functions, which we use to write an integral representation for harmonic functions (equation 6.5 below).

Theorem 6.1 (*Representation Theorem for C^2) Functions in R^3 Let Ω be a bounded domain in R^3 with piecewise smooth closed boundary surface $\partial\Omega$. Let $u \in C^2(\overline{\Omega})$. Then, at any $\mathbf{x}$ in Ω,*

$$\begin{aligned}u(\mathbf{x}) = &\frac{1}{4\pi}\iint_{\partial\Omega}\left[\frac{1}{|\mathbf{y}-\mathbf{x}|}\frac{\partial u(\mathbf{y})}{\partial n} - u(\mathbf{y})\frac{\partial}{\partial n}\frac{1}{|\mathbf{y}-\mathbf{x}|}\right]d\sigma_{\mathbf{y}}\\ &-\frac{1}{4\pi}\iiint_{\overline{\Omega}}\frac{\nabla^2 u(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|}dV_{\mathbf{y}}. \diamond \end{aligned}\tag{6.3}$$

The notations $d\sigma_{\mathbf{y}}$ and $dV_{\mathbf{y}}$ are simply reminders that the variable of integration in these surface and volume integrals is $\mathbf{y}$, while $\mathbf{x}$ is any point of Ω. As with Green's second identity, no assumption is made here that u is harmonic. The theorem gives the value of u at an arbitrary point of Ω in terms of a surface integral, which uses only information about u on the boundary of Ω, and an integral involving the Laplacian of u, over the closure of the entire domain (domain together with its boundary surface). This suggests a connection with harmonic functions that we explore after proving the theorem.

Proof Let $v(\mathbf{y}) = 1/|\mathbf{y}-\mathbf{x}|$ for $\mathbf{y}$ in Ω and $\mathbf{y} \neq \mathbf{x}$. We would like to apply Green's second identity to u and v. This cannot be done over Ω because v is not defined at $\mathbf{x}$. However, since Ω is open, $\mathbf{x}$ is an interior point and there is an open ball $B(\mathbf{x},\epsilon)$ about $\mathbf{x}$ containing only points of Ω. Denote this ball as B. By choosing the radius of B small enough, we can actually make $\overline{B} \subset \Omega$. Let Ω_ϵ be the set formed by removing all points of $\overline{B}$ from Ω. Note that we remove not only points of B, but points on the sphere bounding B. It is routine to check that Ω_ϵ is also a domain (open and connected).

Although u is not assumed to be harmonic, v is harmonic for $\mathbf{y} \neq \mathbf{x}$. By Green's second identity, since $\nabla^2 v = 0$ in Ω_ϵ,

$$-\iiint\limits_{\overline{\Omega_\epsilon}} \frac{\nabla^2 u(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|} dV_{\mathbf{y}}$$
$$= \iint_{\partial\Omega_\epsilon} \left(u(\mathbf{y}) \frac{\partial}{\partial n} \left(\frac{1}{|\mathbf{y}-\mathbf{x}|} \right) - \frac{1}{|\mathbf{y}-\mathbf{x}|} \frac{\partial u(\mathbf{y})}{\partial n} \right) d\sigma_{\mathbf{y}}.$$

Now $\partial\Omega_\epsilon$ consists of two disjoint pieces: the boundary $\partial\Omega$ of Ω, and $S(\mathbf{x},\epsilon)$, the sphere bounding B. Denote this sphere as S. The surface integral over $\partial\Omega_\epsilon$ is the sum of the surface integrals over $\partial\Omega$ and S, and the last equation can be written

$$-\iiint\limits_{\overline{\Omega_\epsilon}} \frac{\nabla^2 u(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|} dV_{\mathbf{y}}$$
$$= \iint_{\partial\Omega} \left(u(\mathbf{y}) \frac{\partial}{\partial n} \left(\frac{1}{|\mathbf{y}-\mathbf{x}|} \right) - \frac{1}{|\mathbf{y}-\mathbf{x}|} \frac{\partial u(\mathbf{y})}{\partial n} \right) d\sigma_{\mathbf{y}}$$
$$+ \iint_{S} \left(u(\mathbf{y}) \frac{\partial}{\partial n} \left(\frac{1}{|\mathbf{y}-\mathbf{x}|} \right) - \frac{1}{|\mathbf{y}-\mathbf{x}|} \frac{\partial u(\mathbf{y})}{\partial n} \right) d\sigma_{\mathbf{y}}. \tag{6.4}$$

We want to determine what happens to the terms in this equation as $\epsilon \to 0$.

First, $u \in C^2(\overline{\Omega})$ is sufficient for convergence of the improper integral

$$\iiint\limits_{\overline{\Omega}} \frac{\nabla^2 u(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|} dV_{\mathbf{y}}.$$

Therefore, as $\epsilon \to 0$,

$$\iiint\limits_{\overline{\Omega_\epsilon}} \frac{\nabla^2 (\mathbf{y})}{|\mathbf{y}-\mathbf{x}|} dV_{\mathbf{y}} \to \iiint\limits_{\overline{\Omega}} \frac{\nabla^2 u(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|} dV_{\mathbf{y}}.$$

Next, the surface integral over $\partial\Omega$ on the right side of equation 6.4 is independent of ϵ and so remains unchanged as $\epsilon \to 0$.

Finally, we will show that the surface integral over S in equation 6.4 has limit $4\pi u(\mathbf{x})$ as $\epsilon \to 0$. For $\mathbf{y}$ on S,

$$\frac{1}{|\mathbf{y}-\mathbf{x}|} = \frac{1}{\epsilon}.$$

To compute the normal derivative of v on S, recall that the line from any point of a sphere to its center is normal to the sphere. This means that for any $\mathbf{y}$ on S, the vector $\mathbf{y}-\mathbf{x}$ is normal to S and points out of the ball B. We want the normal vector oriented away from Ω_ϵ, so we must choose a vector along

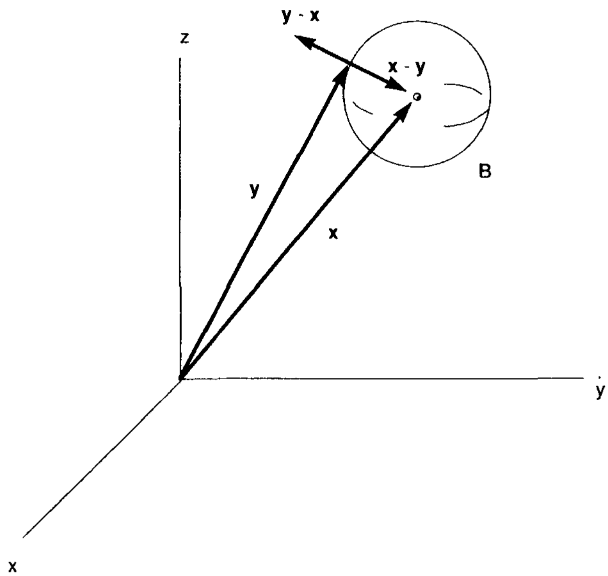

Figure 6.6: Exterior normal $\mathbf{y}-\mathbf{x}$ to B at $\mathbf{y}$.

$-(\mathbf{y}-\mathbf{x})$, toward $\mathbf{x}$, and into the ball (hence exterior to Ω_ϵ). These vectors are illustrated in Figure 6.6. To obtain a unit vector with this orientation, let

$$\mathbf{y} = y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k} \text{ and } \mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$$

and choose

$$\begin{aligned}\mathbf{n}(\mathbf{y}) &= -\frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}\\ &= -\frac{(y_1-x_1)\mathbf{i}+(y_2-x_2)\mathbf{j}+(y_3-x_3)\mathbf{k}}{\sqrt{(y_1-x_1)^2+(y_2-x_2)^2+(y_3-x_3)^2}}.\end{aligned}$$

Now compute

$$\begin{aligned}\frac{\partial}{\partial n}\left(\frac{1}{|\mathbf{y}-\mathbf{x}|}\right) &= \nabla\left(\frac{1}{|\mathbf{y}-\mathbf{x}|}\right)\cdot\mathbf{n}(\mathbf{y})\\ &= -\frac{(y_1-x_1)\mathbf{i}+(y_2-x_2)\mathbf{j}+(y_3-x_3)\mathbf{k}}{((y_1-x_1)^2+(y_2-x_2)^2+(y_3-x_3)^2)^{3/2}}\cdot\mathbf{n}(\mathbf{y})\\ &= \frac{(y_1-x_1)^2+(y_2-x_2)^2+(y_3-x_3)^2}{((y_1-x_1)^2+(y_2-x_2)^2+(y_3-x_3)^2)^2}\\ &= \frac{1}{(y_1-x_1)^2+(y_2-x_2)^2+(y_3-x_3)^2} = \frac{1}{\epsilon^2}.\end{aligned}$$

Then

$$\begin{aligned}
&\iint_S \left(u(\mathbf{y}) \frac{\partial}{\partial n} \left(\frac{1}{|\mathbf{y}-\mathbf{x}|} \right) - \frac{1}{|\mathbf{y}-\mathbf{x}|} \frac{\partial u(\mathbf{y})}{\partial n} \right) d\sigma_{\mathbf{y}} \\
&= \iint_S \left(\frac{1}{\epsilon^2} u(\mathbf{y}) - \frac{1}{\epsilon} \frac{\partial u(\mathbf{y})}{\partial n} \right) d\sigma_{\mathbf{y}} \\
&= \frac{1}{\epsilon^2} \iint_S u(\mathbf{x})\, d\sigma_{\mathbf{y}} + \iint_S \left(\frac{1}{\epsilon^2} [u(\mathbf{y}) - u(\mathbf{x})] - \frac{1}{\epsilon} \frac{\partial u(\mathbf{y})}{\partial n} \right) d\sigma_{\mathbf{y}}.
\end{aligned}$$

Because the integration is with respect to $\mathbf{y}$, the constant $u(\mathbf{x})$ factors through the integral and

$$\frac{1}{\epsilon^2} \iint_S u(\mathbf{x})\, d\sigma_{\mathbf{y}} = \frac{1}{\epsilon^2} u(\mathbf{x})(\text{area of } S) = 4\pi u(\mathbf{x}).$$

Finally,

$$\begin{aligned}
&\left| \iint_S \left(\frac{1}{\epsilon^2} [u(\mathbf{y}) - u(\mathbf{x})] - \frac{1}{\epsilon} \frac{\partial u(\mathbf{y})}{\partial n} \right) d\sigma_{\mathbf{y}} \right| \\
&\le \frac{1}{\epsilon^2} (4\pi\epsilon^2) \max_{\mathbf{y}\in S} |u(\mathbf{y}) - u(\mathbf{x})| + \frac{1}{\epsilon} (4\pi\epsilon^2) \max_{\mathbf{y}\in S} \left| \frac{\partial u(\mathbf{y})}{\partial n} \right| \to 0
\end{aligned}$$

as $\epsilon \to 0$ because $u(\mathbf{y}) \to u(\mathbf{x})$ as $\mathbf{y} \to \mathbf{x}$ and, for the last term, because

$$\left| \frac{\partial u(\mathbf{y})}{\partial n} \right|$$

is bounded for $\mathbf{y}$ in S.

Therefore, in the limit as $\epsilon \to 0$, equation 6.4 becomes

$$\begin{aligned}
&- \iiint_{\overline{\Omega}} \frac{\nabla^2 u(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|} dV_{\mathbf{y}} \\
&= \iint_{\partial\Omega} \left(u(\mathbf{y}) \frac{\partial}{\partial n} \left(\frac{1}{|\mathbf{y}-\mathbf{x}|} \right) - \frac{1}{|\mathbf{y}-\mathbf{x}|} \frac{\partial u(\mathbf{y})}{\partial n} \right) d\sigma_{\mathbf{y}} + 4\pi u(\mathbf{x}).
\end{aligned}$$

This is equivalent to the conclusion of the theorem. ◇

This representation theorem does not assume that u is harmonic. If, however, $\nabla^2 u = 0$ in Ω, the triple integral in equation 6.3 vanishes and

$$u(\mathbf{x}) = \frac{1}{4\pi} \iint_{\partial\Omega} \left(\frac{1}{|\mathbf{y}-\mathbf{x}|} \frac{\partial u(\mathbf{y})}{\partial n} - u(\mathbf{y}) \frac{\partial}{\partial n} \frac{1}{|\mathbf{y}-\mathbf{x}|} \right) d\sigma_{\mathbf{y}} \tag{6.5}$$

for $\mathbf{x} \in \Omega$. This is the integral representation we need for a harmonic $C^2(\overline{\Omega})$ function in a bounded domain Ω in R^3.

The importance of this result for the Dirichlet problem is that it relates the value of a harmonic function u at an interior point $\mathbf{x}$ of Ω to an integral over the boundary, hence to a quantity that depends only on values of u on the boundary. This is exactly what we want to have, because in a Dirichlet problem we are given values the function is to assume on the boundary.

A Representation Theorem in R^2

The discussion in R^3 can be adapted to functions defined on a domain in the plane. In this adaptation, surface integrals are replaced by line integrals, triple integrals by double integrals, and v is chosen as a logarithm function. We outline the details, beginning with the R^2 version of Green's second identity.

Lemma 6.2 (*Green's Second Identity in* R^2) *Let* Ω *be a bounded domain in* R^2 *and suppose that* $\partial\Omega$ *is a piecewise smooth closed curve. Let* $u, v \in C^2(\overline{\Omega})$. *Then*

$$\iint_{\overline{\Omega}} (u\nabla^2 v - v\nabla^2 u)\, dA = \oint_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) ds. \diamond$$

The proof is similar to that for the R^3 version, with Green's theorem playing the role of Gauss's divergence theorem.

Now apply this lemma with

$$v(\mathbf{y}) = \ln\left(\frac{1}{|\mathbf{y}-\mathbf{x}|}\right).$$

This function is harmonic in the domain formed by removing $\mathbf{x}$ from Ω. By an argument similar to that just done in R^3, we obtain the following.

Theorem 6.2 (*Representation Theorem for* C^2) *Functions in* R^2 *Let* Ω *be a bounded domain in* R^2 *whose boundary* $\partial\Omega$ *is a piecewise smooth closed curve. Let* $u \in C^2(\overline{\Omega})$. *Then, for any* $\mathbf{x}$ *in* Ω,

$$u(\mathbf{x}) = \frac{1}{2\pi}\oint_{\partial\Omega} \left(\ln\left(\frac{1}{|\mathbf{y}-\mathbf{x}|}\right)\frac{\partial u(\mathbf{y})}{\partial n} - u(\mathbf{y})\frac{\partial}{\partial n}\ln\left(\frac{1}{|\mathbf{y}-\mathbf{x}|}\right)\right) ds$$
$$- \frac{1}{2\pi}\iint_{\overline{\Omega}} \nabla^2 u(\mathbf{y}) \ln\left(\frac{1}{|\mathbf{y}-\mathbf{x}|}\right) dA. \diamond$$

In R^3 the integrals have a factor $1/4\pi$ which arises from the fact that the area of a sphere of radius ϵ is $4\pi\epsilon^2$. In R^2 we enclose $\mathbf{x}$ by a circle of radius ϵ and length $2\pi\epsilon$ and in computing the limit of the integrals obtain a factor $1/2\pi$.

As in the three-dimensional case, the representation theorem does not assume that u is harmonic. If, however, $\nabla^2 u = 0$ in Ω, the double integral term vanishes and the theorem gives the value of $u(\mathbf{x})$ at any interior point, in terms of information about $u(\mathbf{y})$ for $\mathbf{y}$ on $\partial\Omega$:

$$u(\mathbf{x}) = \frac{1}{2\pi}\oint_{\partial\Omega} \left(\ln\left(\frac{1}{|\mathbf{y}-\mathbf{x}|}\right)\frac{\partial u(\mathbf{y})}{\partial n} - u(\mathbf{y})\frac{\partial}{\partial n}\ln\left(\frac{1}{|\mathbf{y}-\mathbf{x}|}\right)\right) ds \qquad (6.6)$$

for $\mathbf{x} \in \Omega$. This is the two-dimensional analogue of the integral representation 6.5 in R^3.

These representation theorems have versions in R^n using an n - dimensional analogue of Green's second identity. In this case the integrals have a factor of the reciprocal of the area of the unit sphere in R^n.

It is interesting that for $n \geq 3$, the integral representations all have the same general appearance, allowing for adjustment in the dimension of the integral.

Only in the plane is there an intrinsic difference in the integrand itself, seen in the appearance of the logarithm term $\ln(1/|\mathbf{y}-\mathbf{x}|)$, which replaces the simpler term $1/|\mathbf{y}-\mathbf{x}|$ that occurs in higher dimensions. This is just one of many examples that one can find in mathematics in which the behavior of some object of interest is significantly different in the plane than in higher dimensions.

Problems for Section 6.3

1. Prove Green's second identity for a domain in the plane.

2. Write out the details of a proof of the representation theorem for R^2 (at the same level of rigor as the proof given in R^3).

6.4 Two Properties of Harmonic Functions

We will begin to reap some benefits of the considerable background that we have developed. We start by using representation theorems to show that if u is harmonic in a bounded domain Ω, then at any $\mathbf{x}$ in $\Omega, u(\mathbf{x})$ is the average of function values on any sphere centered at $\mathbf{x}$ and lying entirely in Ω. This is the *mean value property for harmonic functions*, and it is valid in R^n. We prove it for $n = 3$.

Theorem 6.3 (*Mean Value Property for Harmonic Functions in* R^3) *Let u be harmonic in a bounded domain Ω in R^3 and let $\mathbf{x} \in \Omega$. Then*

$$u(\mathbf{x}) = \frac{1}{4\pi\epsilon^2}\iint_{S(\mathbf{x},\epsilon)} u(\mathbf{y})\,d\sigma_{\mathbf{y}},$$

provided that ϵ is sufficiently small that all points in $\overline{B(\mathbf{x},\epsilon)}$ are in Ω. $\diamond$

Proof Since u is harmonic, the representation theorem, applied to the ball of radius ϵ about $\mathbf{x}$, enables us to write

$$u(\mathbf{x}) = \frac{1}{4\pi}\iint_S \left(\frac{1}{|\mathbf{y}-\mathbf{x}|}\frac{\partial u(\mathbf{y})}{\partial n} - u(\mathbf{y})\frac{\partial}{\partial n}\frac{1}{|\mathbf{y}-\mathbf{x}|}\right) d\sigma_{\mathbf{y}},$$

in which $S = S(\mathbf{x},\epsilon)$. Repeating a calculation done in proving the representation theorem, we obtain

$$\frac{1}{|\mathbf{y}-\mathbf{x}|} = \frac{1}{\epsilon} \text{ and } \frac{\partial}{\partial n}\frac{1}{|\mathbf{y}-\mathbf{x}|} = -\frac{1}{\epsilon^2}$$

for $\mathbf{y}$ on S. This normal derivative is negative because the exterior normal points out of the ball away from $\mathbf{x}$, opposite the orientation in the representation theorem. Now

$$u(\mathbf{x}) = \frac{1}{4\pi}\iint_S \frac{1}{\epsilon}\frac{\partial u(\mathbf{y})}{\partial n}\,d\sigma_{\mathbf{y}} + \frac{1}{4\pi}\iint_S \frac{1}{\epsilon^2}u(\mathbf{y})\,d\sigma_{\mathbf{y}}.$$

The first integral on the right is zero by Gauss's divergence theorem:

$$\iint_S \frac{\partial u(\mathbf{y})}{\partial n}\, d\sigma_{\mathbf{y}} = \iint_S \nabla u(\mathbf{y}) \cdot \mathbf{n}\, d\sigma_{\mathbf{y}}$$
$$= \iiint_B \operatorname{div}(\nabla u) dV_{\mathbf{y}} = \iiint_B \nabla^2 u dV_{\mathbf{y}} = 0.$$

Therefore

$$u(\mathbf{x}) = \frac{1}{4\pi\epsilon^2} \iint_S u(\mathbf{y})\, d\sigma_{\mathbf{y}}. \diamond$$

Since $4\pi\epsilon^2$ is the area of S, this is the average of $u(x, y)$ on S.

Here is the mean value property for dimension 2.

Theorem 6.4 (*Mean Value Property for Harmonic Functions in* R^2) *Let* u *be harmonic in a bounded domain* Ω *in* R^2 *and let* $\mathbf{x}$ *be in* Ω. *Then*

$$u(\mathbf{x}) = \frac{1}{2\pi\epsilon} \oint_C u(\mathbf{y})\, ds_y,$$

in which C *is a circle of radius* ϵ *about* $\mathbf{x}$ *and* ϵ *is sufficiently small that this circle and all points interior to it are in* Ω. $\diamond$

Note the appearance of $2\pi\epsilon$, which is the length of C. The theorem is proved by using the representation theorem in R^2.

Another important consequence of the representation theorems is known as the maximum principle. Suppose that u is harmonic in a bounded domain Ω and continuous on $\overline{\Omega}$. Since $\overline{\Omega}$ is a compact set, $u(\mathbf{x})$ must achieve a maximum at points of $\overline{\Omega}$. We prove that this maximum must occur on the boundary of Ω, and cannot occur at an interior point (except in the trivial case that u is a constant function). This is the *maximum principle for harmonic functions.*

The proof makes use of the following fact from topology. Suppose that Γ is a polygonal path in a domain Ω, consisting of a finite number of line segments of finite length. Then there is a number ρ such that every point P of Γ is at distance $\geq \rho$ from each point of the boundary of Ω (Figure 6.7). That is, a polygonal path in a domain cannot come arbitrarily close to the boundary of Ω, but instead, must have a positive minimum distance from $\partial\Omega$.

Theorem 6.5 (*Maximum Principle for Harmonic Functions*) *Let* u *be harmonic and nonconstant in a bounded domain* Ω *in* R^n *and continuous on* $\overline{\Omega}$. *Then* u *achieves its maximum and minimum values on* $\overline{\Omega}$ *only at points of* $\partial\Omega$. $\diamond$

Proof Let $M = \max_{\mathbf{x}\in\overline{\Omega}} u(\mathbf{x})$. We want to show that this maximum can be achieved only at one or more points on the boundary of Ω.

Suppose instead that $\mathbf{x}_0$ is in Ω and $u(\mathbf{x}_0) = M$. Let $B = B(\mathbf{x}_0, \delta)$, with δ chosen so that $\overline{B}$ is entirely within Ω. This can be done because $\mathbf{x}_0$ is an interior point of Ω. We will prove that $u(\mathbf{x}) = M$ for all $\mathbf{x}$ in $\overline{B}$. To prove this, choose any ϵ with $0 < \epsilon < \delta$ and consider the open ball B_ϵ of radius ϵ about $\mathbf{x}_0$. If $u(\mathbf{x}) < M$ for any $\mathbf{x}$ on $S(\mathbf{x}_0, \epsilon)$, then by continuity of u we would have $u(\mathbf{x}) < M$ at all points on some set of positive area about $\mathbf{x}$ on $S(\mathbf{x}_0, \epsilon)$. Then the average of u over $S(\mathbf{x}_0, \epsilon)$ would be less than M. But by the mean value

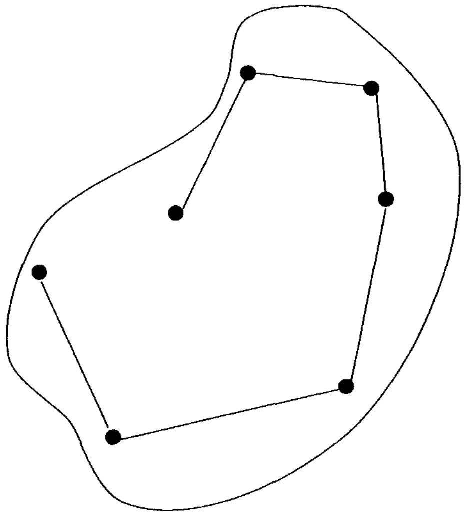

Figure 6.7: A polygonal path in a domain remains at some positive distance from the boundary.

property, $u(\mathbf{x}_0) = M$ is the average of u over $S(\mathbf{x}_0, \epsilon)$. This is a contradiction. We conclude that $u(\mathbf{x}) = M$ for every point on $S(\mathbf{x}_0, \epsilon)$. Since ϵ is any number with $0 < \epsilon < \delta$, we must have $u(\mathbf{x}) = M$ at every point of B. By continuity, $u(\mathbf{x}) = M$ at every point of $\overline{B}$.

We now claim that $u(\mathbf{x}) = M$ at every point of Ω. For, suppose that $\mathbf{y}$ is any point of Ω with $\mathbf{y} \neq \mathbf{x}_0$. We will show that $u(\mathbf{y}) = M$. Because Ω is connected, there is a polygonal path Γ from $\mathbf{x}_0$ to $\mathbf{y}$ in Ω and consisting of a finite number of straight-line segments. Suppose that every point of Γ has distance at least ρ from any point of $\partial\Omega$. Since ρ is a positive number and Γ has finite length, we can move along Γ from $\mathbf{x}_0$ to $\mathbf{y}$, choosing a finite number of intermediary points $\mathbf{x}_1, \cdots, \mathbf{x}_n$ along Γ, and positive numbers $\epsilon_0, \epsilon_1, \cdots, \epsilon_n$ such that each $\overline{B}(\mathbf{x}_j, \epsilon_j)$ contains only points of Ω, and

$$\mathbf{x}_1 \text{ is in } B(\mathbf{x}_0, \epsilon_0),$$
$$\mathbf{x}_j \text{ is in } B(\mathbf{x}_{j-1}, \epsilon_{j-1}) \text{ for } j = 2, \cdots, n$$

and

$$\mathbf{y} \text{ is in } B(\mathbf{x}_n, \epsilon_n).$$

This idea is illustrated in Figure 6.8, where $B(\mathbf{x}_j, \epsilon_j)$ is denoted B_j for $j = 0, 1, \cdots, n$.

Now u has the constant value M in $B(\mathbf{x}_0, \epsilon_0)$, and $\mathbf{x}_1$ is in this ball, so $u(\mathbf{x}_1) = M$. But then u has the constant value M on $B(\mathbf{x}_1, \epsilon_1)$. Since $\mathbf{x}_2$ is in

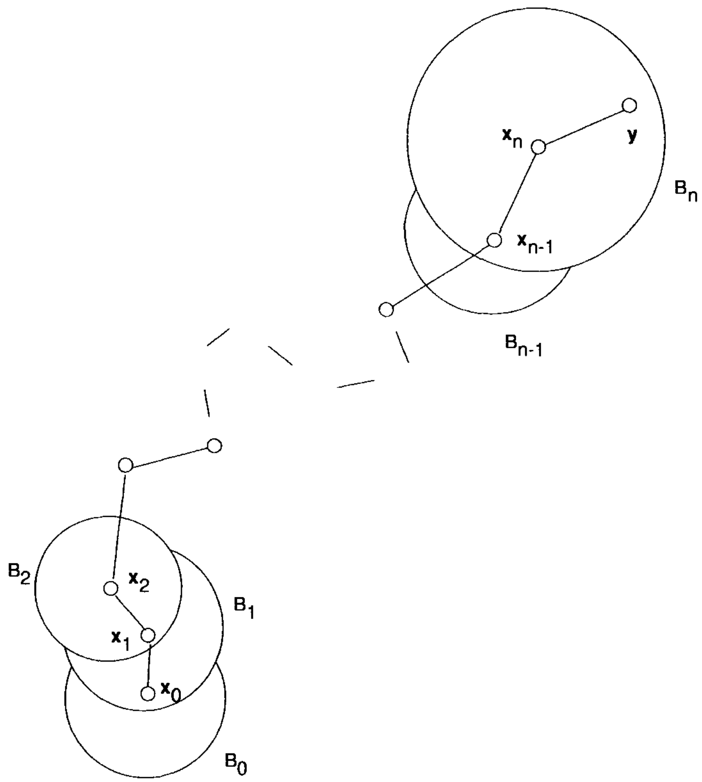

Figure 6.8: Chain of open balls connecting $\mathbf{y}$ to $\mathbf{x}_0$ along a polygonal path.

this ball, $u(\mathbf{x}_2) = M$. Proceeding along Γ, we obtain

$$u(\mathbf{x}_2) = \cdots = u(\mathbf{x}_n) = M,$$

and at the last step, $u(\mathbf{y}) = M$. Therefore, u is constant on Ω, hence by continuity, on $\overline{\Omega}$. This contradiction proves that u cannot assume a maximum value at a point of Ω, and must therefore assume it at a point on $\partial\Omega$. $\diamond$

By applying this argument to $-u$, we find that u can achieve its minimum only at a boundary point.

Problems for Section 6.4

1. Write a complete proof of the mean value property for harmonic functions in the plane.

2. Let Ω be a domain in R^2 bounded by the piecewise smooth closed curve $\partial\Omega$. Let $u, v \in C^2(\overline{\Omega})$. Prove that

$$\oint_{\partial\Omega} u\frac{\partial v}{\partial n}\,ds = \iint_{\overline{\Omega}} (u\nabla^2 v + \nabla u \cdot \nabla v)\,dA.$$

   This is Green's first identity (in the plane).

3. Use Green's first identity to prove Green's second identity for a bounded domain in the plane.

4. Let Ω be a bounded domain in R^2 having a piecewise smooth curve as boundary. Let $u \in C^2(\overline{\Omega})$ and suppose that

$$\nabla^2 u = 0 \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega.$$

   Use Green's first identity to show that $u \equiv 0$ on $\overline{\Omega}$.

5. Fill in the details of this alternative proof of the mean value theorem for functions harmonic in a bounded domain in the plane. Let (x, y) be in Ω and consider an open ball B of radius ϵ about (x, y) and lying entirely in Ω. If (ξ, η) is in B, we can write

$$\xi = x + r\cos(\theta) \text{ and } \eta = y + r\sin(\theta)$$

   with $0 \le r < \epsilon$ and $0 \le \theta \le 2\pi$. Apply Green's first identity (Problem 2) to show that

$$\int_0^{2\pi} u_r(x + r\cos(\theta), y + r\sin(\theta))r\,d\theta = \iint_B r\nabla^2 u\,dr\,d\theta$$

   and conclude that the integral on the left is zero. Therefore,

$$\int_0^{2\pi} \frac{\partial}{\partial r} u(x + r\cos(\theta), y + r\sin(\theta))\,d\theta = 0.$$

Conclude that

$$\int_0^{2\pi} u(x + r\cos(\theta), y + r\sin(\theta))\, d\theta$$

is independent of r, hence is equal to the particular value of this integral at $r = 0$. Hence conclude that

$$\int_0^{2\pi} u(x, y)\, d\theta = \int_0^{2\pi} u(x + r\cos(\theta), y + r\sin(\theta))\, d\theta.$$

From this, show that

$$u(x, y) = \frac{1}{2\pi}\int_0^{2\pi} u(x + r\cos(\theta), y + r\sin(\theta))\, d\theta.$$

Finally, explain how this shows that $u(x,\ y)$ is the average of function values on a circle of radius ϵ about $(x,\ y)$.

6. We will outline another proof of the maximum principle for functions in the plane. Fill in the details. Let u be harmonic in a bounded domain in R^2 having piecewise continuous closed boundary $\partial\Omega$. We want to show that the maximum of $u(x,\ y)$ on $\overline{\Omega}$ is achieved only on $\partial\Omega$.

   First prove that if v is continuous on $\overline{\Omega}$ and $v_{xx} + v_{yy} > 0$ on Ω, then $v(x,\ y)$ achieves its maximum only on $\partial\Omega$. (Suppose that v achieved its maximum at an interior point (x_0, y_0) of Ω. Show that $v_{xx}(x_0, y_0) \le 0$ and $v_{yy}(x_0, y_0) \le 0$, obtaining a contradiction). Now let

$$M_{\partial\Omega} = \max_{(x,y)\in\partial\Omega} u(x, y)$$

   and let

$$v(x, y) = u(x, y) + \epsilon(x^2 + y^2)$$

   for any positive ϵ. Show that v must achieve its maximum M_v on $\overline{\Omega}$ at a point of $\partial\Omega$. Then

$$v(x, y) \le M_v \le M_{\partial\Omega} + \epsilon D,$$

   where D is the largest value of $x^2 + y^2$ for $(x,\ y)$ in $\overline{\Omega}$. Thus show that

$$u(x, y) \le M_{\partial\Omega} + \epsilon D.$$

   Finally, use the fact that ϵ can be as close to zero as we like.

7. Prove the normal derivative lemma. Let Ω be a domain in R^2 and let u be continuous on $\overline{\Omega}$ and harmonic in Ω. Let the maximum of u on $\overline{\Omega}$ be achieved at a boundary point $\mathbf{x}_0$, and suppose that $\partial\Omega$ has a tangent at $\mathbf{x}_0$. Suppose also that the normal derivative

$$\left.\frac{\partial u}{\partial n}\right]_{\mathbf{x}=\mathbf{x}_0}$$

exists and that u is not the constant function. Then

$$\left.\frac{\partial u}{\partial n}\right]_{\mathbf{x}=\mathbf{x}_0} > 0.$$

Hint: Recall that

$$\left.\frac{\partial u}{\partial n}\right]_{\mathbf{x}=\mathbf{x}_0} = \lim_{h\to 0+} \frac{1}{h}(u(\mathbf{x}_0) - u(\mathbf{x}_0 - h\mathbf{n})),$$

where $\mathbf{n}$ is the unit outer normal to $\partial\Omega$ at $\mathbf{x}_0$. It is enough to prove the lemma when $\Omega = B(\mathbf{x}_0, r)$. Define

$$v(\mathbf{x}) = \ln(r/|\mathbf{x}|)$$

for $\mathbf{x} \neq \mathbf{0}$. Then $w(\mathbf{x}) = u(\mathbf{x}) + hv(\mathbf{x})$ is harmonic for $r/2 < |\mathbf{x}| < r$ and continuous on the closure of this domain. Show that $\max\limits_{|\mathbf{x}|=r/2} u(\mathbf{x}) < u(\mathbf{x}_0)$. Hence show that $w(\mathbf{x}) < u(\mathbf{x}_0)$ for h sufficiently small. Conclude that w achieves a maximum at $\mathbf{x} = \mathbf{x}_0$; hence conclude that

$$\left.\frac{\partial w}{\partial n}\right]_{\mathbf{x}=\mathbf{x}_0} \geq 0.$$

From this complete the proof.

The lemma is valid for domains in R^n, with the concept of tangent line at $\mathbf{x}_0$ appropriately generalized (tangent plane in R^3, and so on). For $n > 2$ adapt the proof outlined above by letting

$$v(\mathbf{x}) = |\mathbf{x}|^{2-n} - r^{2-n}.$$

6.5 Is the Dirichlet Problem Well Posed?

A Dirichlet problem for a bounded domain need not have a solution. We will sketch a physically motivated example in R^3 due to the French mathematician Henri Lebesgue.

First define a surface as follows. Imagine a sphere of radius 1 about the origin in R^3, made of thin, highly elastic rubber. Place the point of a thin needle at (0, 1, 0) and push the point inward toward the origin, without puncturing the sphere. This deforms part of the sphere into a thin spike pointing into the interior of the sphere toward the origin. The resulting surface is shown in Figure 6.9, with a darkened strip we will use shortly. Let Ω be the domain bounded by this surface (inside the sphere, outside the spike). The deformed surface is $\partial\Omega$.

Next define a function f over $\partial\Omega$. Let $f(x, y, z) = 0$ for $(x,\ y,\ z)$ on the spike part of the surface, up to the shaded strip. At the strip, proceeding from the spike outward over the shaded part of the sphere, make $f(x,\ y,\ z)$ grow at

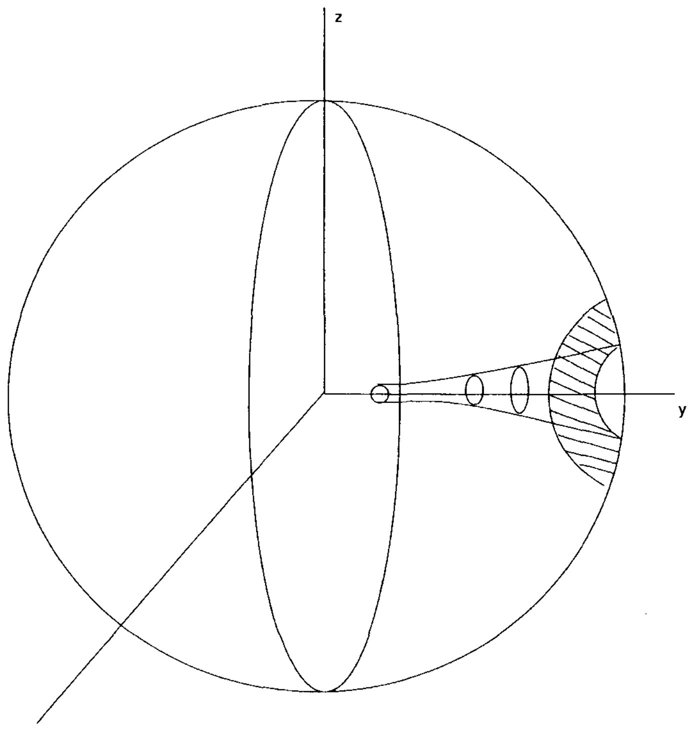

Figure 6.9: Lebesgue's example of a Dirichlet problem with no solution.

a rapid rate until it reaches a value T at the outer edge of the shaded part, beyond which it maintains the constant value T over the rest of the spherical part of the surface. T can be chosen arbitrarily large. We can also make the spike thin enough, and the shaded part of sufficiently small diameter, that the shaded area and spike have an arbitrarily small surface area.

We will argue that the Dirichlet problem

$$\nabla^2 u = 0 \text{ in } \Omega,$$
$$u = f \text{ on } \partial\Omega$$

has no solution. For suppose that u is a solution. Think of $u(x, y, z)$ as the steady-state temperature distribution throughout Ω. But the temperature on the surface $\partial\Omega$ is a constant T except over the spike and the darkened layer, which can be constructed to have as small a surface area as we like. We would therefore expect the steady-state temperature over Ω to be nearly the constant T, say equal to $T - \epsilon$ for some small positive ϵ:

$$u(x, y, z) = T - \epsilon \text{ for } (x, y, z) \in \Omega.$$

But now u cannot be continuous on $\overline{\Omega}$, since $u(x, y, z) = 0$ at points on the spike and $u(x, y, z) = T - \epsilon$ at points inside the sphere arbitrarily close to the spike.

This example, which can be done rigorously, suggests that an existence theorem for the Dirichlet problem must place some conditions on the domain, or perhaps its boundary, as well as on the boundary data function. In Chapter 7 we prove a theorem giving conditions on both f and Ω sufficient to ensure a continuous solution.

However, using the maximum principle, it is not difficult to show that a Dirichlet problem on a bounded domain can have at most one continuous solution.

Theorem 6.6 (*Uniqueness of the Solution of a Dirichlet Problem*) *Let Ω be a bounded domain in R^n and let f be continuous on $\partial\Omega$. Then the Dirichlet problem*

$$\nabla^2 u = 0 \text{ in } \Omega$$
$$u = f \text{ on } \partial\Omega$$

has at most one solution that is continuous on $\overline{\Omega}$. ◇

Proof Suppose w and v are solutions that are continuous on $\overline{\Omega}$. Let $h = w - v$. Then h is harmonic in Ω and $h = 0$ on $\partial\Omega$. The maximum and minimum values of h on $\overline{\Omega}$ are achieved on $\partial\Omega$ and are therefore zero. But then

$$|h(\mathbf{x})| = |w(\mathbf{x}) - v(\mathbf{x})| = 0$$

for all $\mathbf{x}$ in Ω; hence $u = v$. ◇

Further, a continuous solution of a Dirichlet problem on a bounded domain depends continuously on the data. This means that small perturbations in the boundary data cause correspondingly small perturbations in the solution.

Theorem 6.7 (*Continuous Dependence on Boundary Data*) *Let Ω be a bounded domain in R^n. Let f and g be continuous on $\partial\Omega$. Let w be the continuous solution of*

$$\nabla^2 u = 0 \text{ in } \Omega \text{ and } u = f \text{ on } \partial\Omega.$$

Let v be the continuous solution of

$$\nabla^2 u = 0 \text{ in } \Omega \text{ and } u = g \text{ on } \partial\Omega.$$

Suppose that ϵ is a positive number and

$$|f(\mathbf{x}) - g(\mathbf{x})| < \epsilon \text{ for } \mathbf{x} \in \partial\Omega.$$

Then

$$|w(\mathbf{x}) - v(\mathbf{x})| < \epsilon \text{ for } \mathbf{x} \in \overline{\Omega}. \diamond$$

We now have some information about solutions of Dirichlet problems. For the remainder of this chapter we develop techniques for writing solutions of Dirichlet problems and then Neumann problems.

Problem for Section 6.5

1. Prove Theorem 6.7. Hint: Let $h = w - v$ and apply the maximum principle.

6.6 Dirichlet Problem for a Rectangle

We will solve the Dirichlet problem

$$\begin{aligned}
\nabla^2 u(x,y) &= 0 \text{ for } 0 < x < a, 0 < y < b \\
u(x,0) &= 0 \text{ for } 0 \le x \le a \\
u(0,y) &= u(a,y) = 0 \text{ for } 0 \le y \le b \\
u(x,b) &= f(x) \text{ for } 0 \le x \le a.
\end{aligned}$$

This problem models the steady-state temperature distribution in a thin flat plate, with temperatures on the lower and vertical sides kept at zero and temperature $f(x)$ along the top side.

We can solve this problem by separation of variables. Let $u(x,y) = X(x)Y(y)$ in Laplace's equation, to obtain

$$X''Y + XY'' = 0$$

or

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda,$$

in which both X''/X and Y''/Y must be constant because x and y are independent. Then

$$X'' + \lambda X = 0 \text{ and } Y'' - \lambda Y = 0.$$

From the boundary conditions,

$$u(x,0) = X(x)Y(0) = 0$$

so $Y(0) = 0$. Similarly, $X(0) = X(a) = 0$.

The problem for X is the familiar boundary value problem

$$X'' + \lambda X = 0; X(0) = X(a) = 0$$

with eigenvalues

$$\lambda = \frac{n^2\pi^2}{a^2} \text{ for } n = 1, 2, \cdots$$

and eigenfunctions constant multiples of

$$\sin\left(\frac{n\pi x}{a}\right).$$

For any eigenvalue, the corresponding equation for Y is

$$Y'' - \frac{n^2\pi^2}{a^2}Y = 0$$

with general solution

$$Y = ce^{n\pi y/a} + de^{-n\pi y/a}.$$

But $Y(0) = 0$, so $d = -c$ and Y is of the form

$$Y = 2c \sinh\left(\frac{n\pi y}{a}\right).$$

For $n = 1, 2, \cdots$, we now have functions

$$u_n(x,y) = b_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right),$$

which are harmonic in the rectangle and satisfy the homogeneous boundary conditions on the lower and vertical sides. To find a solution satisfying the condition on the side $y = b$, try a superposition

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right). \tag{6.7}$$

We must choose the b_n's so that

$$u(x,b) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi b}{a}\right) = f(x).$$

This is a Fourier sine expansion of f on $[0, a]$. Choose the coefficient of $\sin(n\pi x/a)$ to be the Fourier sine coefficient:

$$b_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a f(\xi)\sin\left(\frac{n\pi\xi}{a}\right) d\xi.$$

Then

$$b_n = \frac{2}{a\ \sinh(n\pi b/a)} \int_0^a f(\xi) \sin\left(\frac{n\pi\xi}{a}\right) d\xi.$$

With this choice of the b_n's, equation 6.7 gives the solution of this Dirichlet problem for a rectangle (with nonzero data prescribed on the top side only).

For example, suppose that $f(x) = x^2(a - x)$. Compute

$$\int_0^a \xi^2(a-\xi)\sin\left(\frac{n\pi\xi}{a}\right) d\xi = -\frac{a^4}{n^3\pi^3}[1 + 2(-1)^n].$$

In this case the solution is

$$u(x, y) = -2\frac{a^3}{\pi^3}\sum_{n=1}^{\infty} \frac{1 + 2(-1)^n}{n^3 \sinh(n\pi b/a)} \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right).$$

In this example the boundary conditions were homogeneous (identically zero) on the vertical and bottom sides of the rectangle, and a given function on the top side. This problem was solved in a straightforward fashion by separation of variables. We may have nonzero boundary conditions specified on each side of the rectangle. In this event, write four Dirichlet problems, in each of which there is nonzero data on only one side. The solution of the original problem is the sum of the solutions of these four problems.

Problems for Section 6.6

In each of Problems 1 through 5, solve the Dirichlet problem for the rectangle.

1.

$$u(0, y) = u(1, y) = 0 \text{ for } 0 \le y \le \pi$$
$$u(x, 0) = \sin(\pi x), u(x, \pi) = 0 \text{ for } 0 \le x \le 1$$

2.

$$u(0, y) = y(2 - y), u(3, y) = 0 \text{ for } 0 \le y \le 2$$
$$u(x, 0) = u(x, 2) = 0 \text{ for } 0 \le x \le 3$$

3.

$$u(0, y) = u(1, y) = 0 \text{ for } 0 \le y \le 4$$
$$u(x, 0) = 0, u(x, 4) = x\cos(\pi x/2) \text{ for } 0 \le x \le 1$$

4.

$$u(0, y) = \sin(y), u(\pi, y) = 0 \text{ for } 0 \le y \le \pi$$
$$u(x, 0) = x(\pi - x), u(x, \pi) = 0 \text{ for } 0 \le x \le \pi$$

5.

$$u(0, y) = 0, u(2, y) = \sin(y) \text{ for } 0 \le y \le \pi$$
$$u(x, 0) = 0, u(x, \pi) = x\sin(\pi x) \text{ for } 0 \le x \le 2$$

Apply separation of variables to solve each of the following mixed boundary value problems.

6.

$$\nabla^2 u = 0 \text{ for } 0 < x < a, 0 < y < b$$
$$u(x, 0) = u_y(x, b) = 0 \text{ for } 0 \le x \le a$$
$$u(0, y) = 0, u(a, y) = g(y) \text{ for } 0 \le y \le b$$

7.

$$\nabla^2 u = 0 \text{ for } 0 < x < a, 0 < y < b$$
$$u(x, 0) = 0, u(x, b) = f(x) \text{ for } 0 \le x \le a$$
$$u(0, y) = u_x(a, y) = 0 \text{ for } 0 \le y \le b$$

6.7 Dirichlet Problem for a Disk

We will solve the Dirichlet problem for a disk of radius ρ about the origin in the plane. Now it is convenient to use polar coordinates, in terms of which the problem is

$$\nabla^2 u(r, \theta) = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0,$$
$$u(\rho, \theta) = f(\theta) \text{ for } -\pi \le \theta \le \pi.$$

We know from Section 6.2 that the functions

$$1, r^n\cos(n\theta), r^n\sin(n\theta)$$

are harmonic on the disk, for $n = 1, 2, \cdots$. We attempt a superposition of these functions as a solution:

$$u(r, \theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)). \tag{6.8}$$

The harmonic functions $r^{-n}\cos(n\theta)$ and $r^{-n}\sin(n\theta)$, also found in Section 6.2, are not used in this superposition because these are undefined at $r = 0$, the center of the disk.

To satisfy the boundary condition, we must choose the coefficients so that

$$u(\rho, \theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \rho^n \cos(n\theta) + b_n \rho^n \sin(n\theta)) = f(\theta).$$

This is a Fourier expansion of f on $[-\pi, \pi]$, leading us to choose

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\xi)\, d\xi$$
$$a_n \rho^n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\xi) \cos(n\xi)\, d\xi$$

and

$$b_n \rho^n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\xi) \sin(n\xi)\, d\xi.$$

Then

$$a_n = \frac{1}{\rho^n \pi} \int_{-\pi}^{\pi} f(\xi) \cos(n\xi)\, d\xi \text{ for } n = 0, 1, 2, \cdots$$

and

$$b_n = \frac{1}{\rho^n \pi} \int_{-\pi}^{\pi} f(\xi) \sin(n\xi)\, d\xi \text{ for } n = 1, 2, \cdots .$$

For example, suppose that we want to solve

$$\nabla^2 u = 0 \text{ for } 0 \le r < 3, -\pi \le \theta \le \pi$$
$$u(3, \theta) = |\cos(\theta/2)| \text{ for } -\pi \le \theta \le \pi.$$

The solution is given by equation 6.8 and we need only compute the coefficients:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |\cos(\xi/2)|\, d\xi = \frac{4}{\pi},$$
$$a_n = \frac{1}{3^n \pi} \int_{-\pi}^{\pi} |\cos(\xi/2)| \cos(n\xi)\, d\xi = \frac{4(-1)^{n+1}}{3^n \pi (4n^2 - 1)},$$
$$b_n = \frac{1}{3^n \pi} \int_{-\pi}^{\pi} |\cos(\xi/2)| \sin(n\xi)\, d\xi = 0.$$

The solution is

$$u(r, \theta) = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n (4n^2 - 1)} r^n \cos(n\theta)$$

for $0 \le r \le 3$ and $-\pi \le \theta \le \pi$.

Problems for Section 6.7

In each of Problems 1 through 5, solve the Dirichlet problem.

1. Solve the Dirichlet problem for the disk $r < \rho$ if $u(\rho, \theta) = \cos^2(\theta)$.

2. Solve the Dirichlet problem for the disk $r < \rho$ if $u(\rho, \theta) = \sin^3(\theta) + \cos^2(\theta)$.

3. Solve the Dirichlet problem for the disk $r < \rho$ if $u(\rho, \theta) = \sin(\theta)$.

4. Solve the Dirichlet problem

$$\nabla^2 u(x,y) = 0 \text{ for } x^2 + y^2 < 9$$
$$u(x,y) = x^2 \text{ for } x^2 + y^2 = 9.$$

Hint: Convert the problem to polar coordinates.

5. Solve the Dirichlet problem

$$\nabla^2 u(x,y) = 0 \text{ for } x^2 + y^2 < 16$$
$$u(x,y) = x^2y^2 \text{ for } x^2 + y^2 = 16.$$

6. Consider the Dirichlet problem for the disk $r < \rho$, with $u(\rho,\theta) = f(\theta)$. Suppose that f is periodic of period 2π and is an odd function of θ on $[-\pi,\pi]$. Show that the solution $u(r,\theta)$ is also an odd function of θ. If f is an even function of θ, is the solution even in θ?

7. Solve the Dirichlet problem for an annulus (the domain between two concentric circles). The problem is

$$\nabla^2 u(r,\theta) = 0 \text{ for } \rho_1 < r < \rho_2, 0 \le \theta \le 2\pi$$
$$u(\rho_1,\theta) = g(\theta), u(\rho_2,\theta) = f(\theta) \text{ for } 0 \le \theta \le 2\pi.$$

Hint: Attempt a series similar to that of equation 6.8, except now include terms $\ln(r), r^{-n}\sin(n\theta)$, and $r^{-n}\cos(n\theta)$ in the expansion, since the origin is not in the annulus. Use the boundary conditions on both of the boundary circles to find the coefficients.

In each of Problems 8 through 13, data is given on the bounding circles of an annulus. Use the solution of Problem 7 to write a series solution of the Dirichlet problem for the annulus.

8. $u(1,\theta) = 1, u(2,\theta) = 2$

9. $u(1,\theta) = 1, u(2,\theta) = \cos(\theta)$

10. $u(1,\theta) = \sin(\theta), u(2,\theta) = \cos(\theta)$

11. $u(1,\theta) = 1, u(2,\theta) = \cos^2(\theta)$

12. $u(2,\theta) = \sin(2\theta), u(4,\theta) = \sin(4\theta)$

13. $u(2,\theta) = \sin^2(\theta), u(4,\theta) = \cos^2(\theta)$

6.8 Poisson's Integral Representation for a Disk

We have just obtained an infinite series solution for the Dirichlet problem for a disk about the origin in the plane. We will write an integral formula for this solution. Take the radius to be $\rho = 1$ initially.

Begin with the solution 6.8 with the integral formulas for the coefficients inserted:

$$
\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)\,d\xi \\
&\quad + \sum_{n=1}^{\infty} \frac{1}{\pi} r^n \left[\int_{-\pi}^{\pi} f(\xi)\cos(n\xi)\,d\xi\cos(nx) + \int_{-\pi}^{\pi} f(\xi)\sin(n\xi)\,d\xi\sin(nx)\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[1 + 2\sum_{n=1}^{\infty} r^n(\cos(n\xi)\cos(nx) + \sin(n\xi)\sin(nx))\right] f(\xi)\,d\xi \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[1 + 2\sum_{n=1}^{\infty} r^n\cos(n(\theta-\xi))\right] f(\xi)\,d\xi.
\end{aligned}
$$

The quantity

$$P(r;\zeta) = \frac{1}{2\pi}\left[1 + 2\sum_{n=1}^{\infty} r^n\cos(n\zeta)\right]$$

is called the *Poisson kernel*, and in terms of $P(r;\zeta)$ the solution is

$$u(r,\theta) = \int_{-\pi}^{\pi} P(r;\theta-\xi)f(\xi)\,d\xi \tag{6.9}$$

for $0 \le r < 1$ and $-\pi \le \theta \le \pi$. This is another example of a general theme in which a kernel function is used to write a quantity of interest as the integral of the product of the kernel function with a given function. Previously, we wrote the partial sum of a Fourier series of f as the integral of f times the Dirichlet kernel. We will see this idea again in Section 6.16 when we write a solution of a Dirichlet problem as the integral of f times the normal derivative of a function called the Green's function for the problem.

Thus far we have just manipulated the solution 6.8 and given part of it a name. We will now write the Poisson kernel in closed form (that is, without a summation), a tactic we also pursued with the Dirichlet kernel. Recall Euler's formula,

$$e^{i\zeta} = \cos(\zeta) + i\sin(\zeta).$$

Let $z = re^{i\zeta}$. Then

$$z^n = r^n e^{in\zeta} = r^n\cos(n\zeta) + ir^n\sin(n\zeta)$$

and $r^n\cos(\zeta)$ may be thought of as the real part of z^n:

$$r^n\cos(n\zeta) = \mathrm{Re}(z^n).$$

Now

$$1 + 2\sum_{n=1}^{\infty} r^n \cos(n\zeta) = \mathrm{Re}\left(1 + 2\sum_{n=1}^{\infty} z^n\right).$$

From the geometric series, we have, for $|z| = r < 1$,

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z},$$

yielding

$$\begin{aligned}
&1 + 2\sum_{n=1}^{\infty} r^n \cos(n\zeta)\\
&= \mathrm{Re}\left(1 + 2\frac{z}{1-z}\right) = \mathrm{Re}\left(\frac{1+z}{1-z}\right)\\
&= \mathrm{Re}\left(\frac{1+re^{i\zeta}}{1-re^{i\zeta}}\right).
\end{aligned}$$

One way to compute the real part of the last quantity is to multiply numerator and denominator by $1 - re^{-i\zeta}$ (the complex conjugate of the denominator):

$$\begin{aligned}
\frac{1+re^{i\zeta}}{1-re^{i\zeta}} &= \left(\frac{1+re^{i\zeta}}{1-re^{i\zeta}}\right)\left(\frac{1-re^{-i\zeta}}{1-re^{-i\zeta}}\right)\\
&= \frac{1-r^2+r(e^{i\zeta}-e^{-i\zeta})}{1+r^2-r(e^{i\zeta}+e^{-i\zeta})}\\
&= \frac{1-r^2+2ir\sin(\zeta)}{1+r^2-2r\cos(\zeta)}.
\end{aligned}$$

The real part of this is $(1-r^2)/(1+r^2-2r\cos(\zeta))$, so

$$\begin{aligned}
1 + 2\sum_{n=1}^{\infty} r^n \cos(n\zeta) &= \mathrm{Re}\left(\frac{1+re^{i\zeta}}{1-re^{i\zeta}}\right).\\
&= \frac{1-r^2}{1+r^2-2r\cos(\zeta)}.
\end{aligned}$$

This gives the closed form of the Poisson kernel:

$$P(r;\zeta) = \frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos(\zeta)}.$$

Upon substituting this into equation 6.9, we obtain

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-\xi)+r^2} f(\xi)\, d\xi. \qquad (6.10)$$

This is *Poisson's formula* for the solution of the Dirichlet problem for the unit disk, and it is valid for $0 \le r < 1$ and $-\pi \le \theta \le \pi$.

By a change of variables we can write the integral solution for a disk of any positive radius ρ:

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\rho^2 - r^2}{\rho^2 - 2\rho r\cos(\theta-\xi) + r^2} f(\xi)\, d\xi \tag{6.11}$$

for $0 \le r < \rho$ and $-\pi \le \theta \le \pi$.

Assuming that f has period 2π, this integral can be evaluated over any interval of length 2π and we sometimes see the equivalent expression

$$u(r,\theta) = \frac{1}{2\pi}\int_{0}^{2\pi} \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r\cos(\theta-\xi)} f(\xi)\, d\xi,$$

which is also called Poisson's formula.

One consequence of equation 6.11 is that

$$\begin{aligned} u(0,0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)\, d\xi = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\rho,\xi)\, d\xi \\ &= \frac{1}{2\pi\rho}\int_{-\pi}^{\pi} u(\rho,\xi)\, ds, \end{aligned} \tag{6.12}$$

since the differential element of arc length on the circle of radius ρ about the origin is $ds = \rho\, d\xi$. This equation states that the value of u at the center of the disk is the average of its values on the bounding circle, consistent with the mean value property for harmonic functions.

We will use Poisson's integral solution 6.11 to derive Harnack's inequality, and from this another important property of harmonic functions. Since

$$-1 \le \cos(\theta - \xi) \le 1,$$

we can replace the cosine term in equation 6.11 by -1 and then by 1 to obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\rho^2 - r^2}{\rho^2 + r^2 + 2\rho r} f(\xi)\, d\xi \le u(r,\theta) \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r} f(\xi)\, d\xi.$$

Then

$$\frac{(\rho-r)(\rho+r)}{(\rho+r)^2}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)\, d\xi \le u(r,\theta) \le \frac{(\rho-r)(\rho+r)}{(\rho-r)^2}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)\, d\xi,$$

which, in view of equation 6.12, we can write as

$$\frac{\rho - r}{\rho + r}u(0,0) \le u(r,\theta) \le \frac{\rho + r}{\rho - r}u(0,0) \tag{6.13}$$

for $0 \le \rho < r$ and $-\pi \le \theta \le \pi$. This is *Harnack's inequality*.

Suppose now that u is harmonic and nonnegative in the entire plane. Then Harnack's inequality must hold for every positive ρ and upon taking the limit as $\rho \to \infty$, we obtain

$$u(0,0) \le u(r,\theta) \le u(0,0),$$

hence $u(r,\theta) = u(0,0)$. We conclude that a nonnegative function that is harmonic in R^2 must be constant. This is a theorem of Liouville and also follows from Liouville's theorem in complex analysis (a bounded entire function must be constant).

Problems for Section 6.8

1. Derive equation 6.11 from equation 6.10.

2. By differentiating under the integral sign, show that the function defined by equation 6.11 is a solution of Laplace's equation in polar coordinates for the disk $r < \rho$.

3. Show that for $0 \le r < \rho$ and $-\pi \le \theta \le \pi$,
$$\int_{-\pi}^{\pi} \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2r\rho\cos(\theta - \xi)}\, d\xi = 2\pi.$$

4. Solve the Dirichlet problem for the exterior of a disk:
$$\begin{aligned} \nabla^2 u(r,\theta) &= 0 \text{ for } r > \rho, -\pi \le \theta \le \pi \\ u(\rho,\theta) &= f(\theta). \end{aligned}$$
Hint: Seek a solution that is bounded as $r \to \infty$. Obtain the Poisson integral formula
$$u(r,\theta) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2r\rho\cos(\theta - \xi)} f(\xi)\, d\xi.$$

5. Suppose that u is harmonic in R^2 but is not nonnegative. Does it follow that u must be constant?

6. Prove Liouville's theorem. Let u be harmonic in the plane R^2, and suppose that u is not a constant function. Then u can have neither an upper bound nor a lower bound. (This result holds in R^n.)

   Sometimes an opportunistic use of a formula can yield additional results. Problems 7 through 10 are examples of this.

7. Choose $u(r,\theta) = r^n \sin(n\theta)$ in equation 6.11 and evaluate $u(\rho/2, \pi/2)$ to derive the integral
$$\int_0^{2\pi} \frac{\sin(n\xi)}{5 - 4\sin(\xi)}\, d\xi = \frac{\pi}{3 \cdot 2^{n-1}} \sin(n\pi/2),$$
in which n is any positive integer.

8. Continuing Problem 7, what integral formula do we obtain by evaluating $u(\rho/2, \pi)$?

9. Derive integral formulas by choosing $u(r,\theta) = r^n\cos(n\theta)$ in equation 6.11 and evaluating $u(\rho/2, \pi/2)$ and $u(\rho/2, \pi)$.

10. Derive an integral formula by using $u(r,\theta) \equiv 1$ in Poisson's formula.

6.9 Dirichlet Problem for the Upper Half-Plane

Dirichlet problems for unbounded domains can sometimes be solved using Fourier integrals and transforms. We illustrate this idea with the problem for the upper half plane, whose boundary is the horizontal axis:

$$\begin{aligned} \nabla^2 u(x,y) &= 0 \text{ for } -\infty < x < \infty, y > 0 \\ u(x,0) &= f(x) \text{ for } -\infty < x < \infty. \end{aligned} \tag{6.14}$$

Here Ω is not a bounded domain, and we impose the condition that the solution must be a bounded function.

Solution by Fourier Integral

We can solve this problem by separation of variables. Let $u(x,y) = X(x)Y(y)$ and substitute into the differential equation to obtain

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

or

$$X'' + \lambda X = 0 \text{ and } Y'' - \lambda Y = 0.$$

Take cases on λ.

If $\lambda = 0$, then $X'' = 0$, so $X = ax + b$. We must choose $a = 0$ to have a bounded solution, so $X = b$, a constant solution.

If $\lambda = -\omega^2$ with $\omega > 0$, then $X = ae^{\omega x} + be^{-\omega x}$. Now $e^{\omega x} \to \infty$ as $x \to \infty$, so we must choose $a = 0$; and $e^{-\omega x} \to \infty$ as $x \to -\infty$, so we must also choose $b = 0$. This case yields no nontrivial bounded solution for X.

If $\lambda = \omega^2$ with $\omega > 0$, then $X = a\cos(\omega x) + b\sin(\omega x)$, a bounded solution. This includes the constant solution in the case $\lambda = 0$.

Now consider the y - dependence. With $\lambda = \omega^2$ we have $Y'' - \omega^2 Y = 0$, so $Y = ae^{\omega y} + be^{-\omega y}$. Since $e^{\omega y} \to \infty$ as $y \to \infty$, choose $a = 0$. Since $y > 0, be^{-\omega y}$ is a bounded solution for Y.

For each $\omega \geq 0$ we now have a function

$$u_\omega(x,y) = [a_\omega \cos(\omega x) + b_\omega \sin(\omega x)]e^{-\omega y}$$

that is harmonic on the half-plane $y > 0$. To satisfy the boundary condition we must generally superimpose these solutions over all $\omega \geq 0$, and this is done by integrating. Let

$$u(x,y) = \int_0^\infty [a_\omega \cos(\omega x) + b_\omega \sin(\omega x)]e^{-\omega y}\, d\omega. \tag{6.15}$$

The boundary condition requires that

$$u(x,0) = \int_0^\infty [a_\omega \cos(\omega x) + b_\omega \sin(\omega x)]\, d\omega = f(x).$$

This is the Fourier integral expansion of f; hence choose

$$a_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos(\omega\xi)\,d\xi$$

and

$$b_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\sin(\omega\xi)\,d\xi.$$

Here we are assuming that f has a Fourier integral representation on the entire real line.

The solution 6.15 can be written in a more compact form by inserting these coefficients, changing the order of integration, and performing some routine manipulations:

$$\begin{aligned} &u(x,y) = \\ &\frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} [\cos(\omega\xi)\cos(\omega x) + \sin(\omega\xi)\sin(\omega x)] f(\xi) e^{-\omega y}\,d\xi\,d\omega \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\int_0^\infty \cos(\omega(\xi - x))e^{-\omega y}\,d\omega\right) f(\xi)\,d\xi. \end{aligned}$$

The inner integral is

$$\begin{aligned} &\int_0^\infty e^{-\omega y}\cos(\omega(\xi - x))\,d\omega \\ &= \frac{e^{-\omega y}}{y^2 + (\xi - x)^2}(-y\cos(\omega(\xi - x)) + (\xi - x)\sin(\omega(\xi - x)))\Big]_0^\infty \\ &= \frac{y}{y^2 + (\xi - x)^2}. \end{aligned}$$

This yields a relatively simple expression for the solution of the Dirichlet problem for the upper half-plane:

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)}{y^2 + (\xi - x)^2}\,d\xi. \tag{6.16}$$

Solution by Fourier Transform

We solve the problem for the upper half-plane again, this time by Fourier transform to illustrate the technique. Apply the Fourier transform in x to Laplace's equation, letting $\mathcal{F}[u(x,y)](\omega) = \widehat{u}(\omega, y)$. Since y is independent of x,

$$\begin{aligned} \mathcal{F}[u_{yy}(x,y)](\omega) &= \int_{-\infty}^{\infty} \frac{\partial^2 u(x,y)}{\partial y^2} e^{-i\omega x}\,dx \\ &= \frac{\partial^2}{\partial y^2}\left(\int_{-\infty}^{\infty} u(x,y)e^{-i\omega x}\,dx\right) = \frac{\partial^2}{\partial y^2}\widehat{u}(\omega, y). \end{aligned}$$

Next, by the operational formula 3.25 for the Fourier transform,

$$\mathcal{F}[u_{xx}(x,y)](\omega) = -\omega^2\widehat{u}(\omega,y).$$

The result of applying the transform to Laplace's equation is

$$\frac{\partial^2\widehat{u}}{\partial y^2} - \omega^2\widehat{u} = 0.$$

This has the general solution

$$\widehat{u}(\omega,y) = a(\omega)e^{\omega y} + b(\omega)e^{-\omega y},$$

in which the coefficients may be functions of ω. Now $y > 0$ in the upper half plane. Since $e^{\omega y} \to \infty$ as $y \to \infty$ if $\omega > 0$, we must have $a(\omega) = 0$ if $\omega > 0$ in order to have a bounded solution. But $e^{-\omega y} \to \infty$ as $y \to \infty$ if $\omega < 0$, so we must have $b(\omega) = 0$ if $\omega < 0$. Thus,

$$\widehat{u}(\omega,y) = \begin{cases} b(\omega)e^{-\omega y} \text{ for } \omega \geq 0 \\ a(\omega)e^{\omega y} \text{ for } \omega \leq 0. \end{cases}$$

We can consolidate this notation by writing

$$\widehat{u}(\omega,y) = c(\omega)e^{-|\omega|y}.$$

To solve for $c(\omega)$, recall that $u(x,0) = f(x)$. Therefore,

$$\widehat{u}(\omega,0) = \widehat{f}(\omega) = c(\omega),$$

so

$$\widehat{u}(\omega,y) = \widehat{f}(\omega)e^{-|\omega|y}.$$

This is the Fourier transform of the solution. Apply the inverse Fourier transform

$$\begin{aligned} u(x,y) = \mathcal{F}^{-1}[\widehat{u}(\omega,y)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{u}(\omega,y)e^{i\omega x}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)e^{-|\omega|y}e^{i\omega x}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(\xi)e^{-i\omega\xi}\,d\xi\right)e^{-|\omega|y}e^{i\omega x}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-|\omega|y}e^{-i\omega(\xi-x)}\,d\omega\right)f(\xi)\,d\xi. \end{aligned}$$

Now

$$e^{-i\omega(\xi-x)} = \cos(\omega(\xi-x)) - i\sin(\omega(\xi-x))$$

and a straightforward integration gives

$$\int_{-\infty}^{\infty} e^{-|\omega|y}e^{-i\omega(\xi-x)}\,d\omega = \frac{2y}{y^2+(\xi-x)^2}.$$

The solution by Fourier transform is

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)}{y^2 + (\xi - x)^2}\, d\xi,$$

in agreement with the solution obtained by separation of variables.

Problems for Section 6.9

In each of Problems 1 through 4, obtain the solution of the Dirichlet problem for the upper half-plane, using the given $f(x)$. In Problems 1 through 3, carry out the integrations in the solution 6.16.

1. $f(x) = 0$ for $x < 0$ and for $x > 1$, and $f(x) = x$ for $0 \leq x \leq 1$

2. $$f(x) = \begin{cases} 0 & \text{for } |x| > 2 \\ x^2 & \text{for } -2 \leq x \leq 2 \end{cases}$$

3. $$f(x) = \begin{cases} 0 & \text{for } |x| > c \\ k & \text{for } -c \leq x \leq c \end{cases}$$

   with k constant and c a positive constant

4. $f(x) = e^{-|x|}$

5. Use of the Fourier integral to solve the Dirichlet problem for the upper half-plane imposes conditions on the boundary data function f, since we must be able to write a Fourier integral representation for f. This method therefore fails for certain choices of f. In particular, consider the Dirichlet problem

$$\begin{aligned} \nabla^2 u &= 0 \text{ for } -\infty < x < \infty, y > 0 \\ u(x,0) &= k \text{ for } -\infty < x < \infty, \end{aligned}$$

   in which k is a nonzero constant. Find the bounded solution of this problem and produce infinitely many solutions that are unbounded. What does equation 6.16 give for $f(x) = k$, constant?

6.10 Dirichlet Problem for the Right Quarter-Plane

Sometimes it is possible to exploit the solution of a Dirichlet problem on one domain to solve a Dirichlet problem on another domain. We illustrate this idea by solving the Dirichlet problem for the right quarter-plane:

$$\begin{aligned} \nabla^2 u &= 0 \text{ for } x > 0, y > 0 \\ u(x,0) &= f(x) \text{ for } x \geq 0 \\ u(0,y) &= 0 \text{ for } y \geq 0. \end{aligned} \tag{6.17}$$

The boundary consists of the nonnegative x - and y - axes, and we are prescribing nonzero boundary data only on the horizontal boundary.

This problem can be solved by separation of variables, as we did for the upper half-plane. We can also obtain the solution by using the Fourier sine transform. Here is another approach. We know an integral solution of the Dirichlet problem for the upper half-plane. If we fold the upper half-plane along the vertical axis, we obtain the right quarter-plane. This suggests that we explore the possibility of using the solution 6.16 to produce the solution for the right quarter-plane. To do this we manufacture a Dirichlet problem for the upper half-plane in such a way that the solution $u(x, y)$ satisfies the given problem for the right quarter-plane when $x > 0$ and $y > 0$.

Define

$$g(x) = \begin{cases} f(x) & \text{for } x \geq 0 \\ \text{any} & \text{for } x < 0, \end{cases}$$

where by "any" we mean, for the moment, give $g(x)$ any continuous definition for negative x.

We know that the Dirichlet problem

$$\nabla^2 u = 0 \text{ for } -\infty < x < \infty, y > 0$$
$$u(x, 0) = g(x) \text{ for } -\infty < x < \infty$$

for the upper half-plane has the solution

$$u_{hp}(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{g(\xi)}{y^2 + (\xi - x)^2} \, d\xi.$$

Write this as

$$u_{hp}(x, y) = \frac{y}{\pi} \left[\int_{-\infty}^{0} \frac{g(\xi)}{y^2 + (\xi - x)^2} \, d\xi + \int_{0}^{\infty} \frac{g(\xi)}{y^2 + (\xi - x)^2} \, d\xi \right].$$

If we put $w = -\xi$ in the first integral in this equation for $u_{hp}(x, y)$, we obtain

$$\int_{-\infty}^{0} \frac{g(\xi)}{y^2 + (\xi - x)^2} \, d\xi = \int_{\infty}^{0} \frac{g(-w)}{y^2 + (w + x)^2} (-1) \, dw.$$

Using ξ again for the variable of integration, this integral is

$$\int_{0}^{\infty} \frac{g(-\xi)}{y^2 + (\xi + x)^2} \, d\xi.$$

Therefore

$$\begin{aligned} u_{hp}(x, y) &= \frac{y}{\pi} \int_{0}^{\infty} \left(\frac{g(-\xi)}{y^2 + (\xi + x)^2} + \frac{g(\xi)}{y^2 + (\xi - x)^2} \right) d\xi \\ &= \frac{y}{\pi} \int_{0}^{\infty} \left(\frac{g(-\xi)}{y^2 + (\xi + x)^2} + \frac{f(\xi)}{y^2 + (\xi - x)^2} \right) d\xi. \end{aligned}$$

Now we will fill in the "any" in the definition of g. The function $u_{hp}(x,y)$ will vanish on the positive y - axis, where $x = 0$ if $g(-\xi) = -f(\xi)$ for $\xi \geq 0$. This occurs if we set

$$g(x) = -f(-x) \text{ for } x < 0.$$

That is, make g the odd extension of f to the entire real line. Now

$$u_{hp}(x,y) = \frac{y}{\pi}\int_0^\infty \left(\frac{1}{y^2+(\xi-x)^2} - \frac{1}{y^2+(\xi+x)^2}\right) f(\xi)\,d\xi \tag{6.18}$$

is the solution of the Dirichlet problem 6.14 for the upper half-plane. But u_{hp} is also harmonic on the right-quarter plane, vanishes when $x = 0$, and equals $f(x)$ for $x \geq 0$ if $y = 0$. Therefore, equation 6.18 also gives the solution of the problem 6.17 for the right quarter-plane.

As an example, consider the problem

$$\begin{aligned} \nabla^2 u &= 0 \text{ for } x > 0, y > 0 \\ u(0,y) &= 0 \text{ for } y \geq 0 \\ u(x,0) &= xe^{-x} \text{ for } x \geq 0. \end{aligned}$$

The solution is

$$u(x,y) = \frac{y}{\pi}\int_0^\infty \left(\frac{1}{y^2+(\xi-x)^2} - \frac{1}{y^2+(\xi+x)^2}\right) \xi e^{-\xi}\,d\xi.$$

As another example, look at the problem 6.17 when $f(x) \equiv 1$. Now the solution 6.18 is

$$u(x,y) = \frac{y}{\pi}\int_0^\infty \frac{1}{y^2+(\xi-x)^2}\,d\xi - \frac{y}{\pi}\int_0^\infty \frac{1}{y^2+(\xi+x)^2}\,d\xi.$$

In this simple case these integrals can be evaluated in closed form. First,

$$\begin{aligned} \int_0^\infty \frac{1}{y^2+(\xi-x)^2}\,d\xi &= \int_0^\infty \frac{1}{y^2\left[1+\left(\frac{\xi-x}{y}\right)^2\right]}\,d\xi \\ &= \frac{1}{y}\left(\frac{\pi}{2} - \arctan\left(-\frac{x}{y}\right)\right) = \frac{\pi}{2y} + \frac{1}{y}\arctan\left(\frac{x}{y}\right). \end{aligned}$$

Similarly,

$$\int_0^\infty \frac{1}{y^2+(\xi+x)^2}\,d\xi = \frac{\pi}{2y} - \frac{1}{y}\arctan\left(\frac{x}{y}\right).$$

Therefore,

$$\begin{aligned} u(x,y) &= \frac{y}{\pi}\left[\frac{\pi}{2y} + \frac{1}{y}\arctan\left(\frac{x}{y}\right) - \frac{\pi}{2y} + \frac{1}{y}\arctan\left(\frac{x}{y}\right)\right] \\ &= \frac{2}{\pi}\arctan\left(\frac{x}{y}\right). \end{aligned}$$

It is routine to check that this function is harmonic in the right quarter-plane, that $u(0, y) = 0$, and that for $x > 0$,

$$\lim_{y \to 0+} \frac{2}{\pi} \arctan\left(\frac{x}{y}\right) = \frac{2}{\pi}\frac{\pi}{2} = 1.$$

Problems for Section 6.10

1. Find a bounded solution of the Dirichlet problem for the left quarter-plane:

$$\begin{aligned} \nabla^2 u &= 0 \text{ for } x < 0, y > 0 \\ u(0, y) &= 0 \text{ for } y > 0 \\ u(x, 0) &= f(x) \text{ for } x < 0. \end{aligned}$$

2. Find a bounded solution of the Dirichlet problem for the quarter-plane $x < 0, y < 0$:

$$\begin{aligned} \nabla^2 u &= 0 \text{ for } x < 0, y < 0 \\ u(0, y) &= 0 \text{ for } y < 0 \\ u(x, 0) &= f(x) \text{ for } x < 0. \end{aligned}$$

3. Find a bounded solution of the Dirichlet problem

$$\begin{aligned} \nabla^2 u &= 0 \text{ for } x > 0, y > 0 \\ u(0, y) &= k \text{ for } y > 0 \\ u(x, 0) &= 0 \text{ for } x < 0, \end{aligned}$$

with k constant.

4. Find a bounded solution of

$$\begin{aligned} \nabla^2 u &= 0 \text{ for } x > 0, y > 0 \\ u(0, y) &= 0 \text{ for } y > 0 \\ u(x, 0) &= x \text{ for } x < 0. \end{aligned}$$

6.11 Dirichlet Problem for a Rectangular Box

We will illustrate a Dirichlet problem in three independent variables. Consider

$$\begin{aligned} \nabla^2 u(x, y, z) &= 0 \text{ for } 0 < x < a, 0 < y < b, 0 < z < c \\ u(x, y, 0) &= u(x, y, c) = 0 \\ u(0, y, z) &= u(a, y, z) = 0 \\ u(x, 0, z) &= 0, u(x, b. z) = f(x, z). \end{aligned}$$

Homogeneous boundary data are prescribed on five faces of the rectangular box, and boundary data f is given on the side $y = b$. If nonhomogeneous data were given on more than one side, we would independently consider the Dirichlet problem for only one nonzero side at a time and add these solutions.

Let $u(x, y, z) = X(x)Y(y)Z(z)$, substitute into Laplace's equation and divide by XYZ to obtain

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

or

$$\frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} = -\lambda$$

for some constant λ because x, y, and z are independent. Then

$$X'' + \lambda X = 0.$$

Further, from the conditions $u(0, y, z) = u(a, y, z) = 0$, we obtain the boundary conditions

$$X(0) = X(a) = 0.$$

This problem for X has eigenvalues $\lambda_n = n^2\pi^2/a^2$ and eigenfunctions $X_n(x) = \sin(n\pi x/a)$ for $n = 1, 2, \cdots$.

Now write

$$\frac{Z''}{Z} = \lambda - \frac{Y''}{Y} = -\mu,$$

with μ constant, because y and z are independent. Now

$$Z'' + \mu Z = 0,$$

and the conditions $u(x, y, 0) = u(x, y, c) = 0$ imply that

$$Z(0) = Z(c) = 0.$$

This problem for Z has solutions $\mu_m = m^2\pi^2/c^2$ for $m = 1, 2, \cdots$ and $Z_m(z) = \sin(m\pi z/c)$ for $m = 1, 2, \cdots$.

The equation for Y is

$$Y'' - \left(\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{c^2}\right) Y = 0$$

for $n, m = 1, 2, \cdots$. This has general solution of the form

$$Y = pe^{\beta_{nm}y} + qe^{-\beta_{nm}y}$$

in which

$$\beta_{nm} = \sqrt{\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{c^2}}.$$

Since $u(x, 0, z) = 0$, then $Y(0) = 0$ and $p = -q$. Thus Y must be a constant multiple of $\sinh(\beta_{nm}y)$.

For each positive integer n and m we now have harmonic functions

$$u_{nm}(x,y,z) = b_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi z}{c}\right) \sinh(\beta_{nm} y).$$

To satisfy the initial condition, we generally need a superposition

$$u(x,y,z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi z}{c}\right) \sinh(\beta_{nm} y).$$

We must choose the coefficients so that

$$\begin{aligned} u(x,b,z) &= f(x,z) \\ &= \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} b_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi z}{c}\right) \sinh(\beta_{nm} b). \end{aligned}$$

This is the double Fourier sine expansion of f on $0 \leq x \leq a, 0 \leq z \leq c$. We have encountered such an expansion before. The coefficients are

$$b_{nm} = \frac{4}{ac \sinh(\beta_{nm} b)} \int_0^a \int_0^c f(\xi,\zeta) \sin\left(\frac{n\pi\xi}{a}\right) \sin\left(\frac{m\pi\zeta}{c}\right) d\zeta \, d\xi.$$

Problems for Section 6.11

1. Solve

$$\begin{aligned} \nabla^2 u(x,y,z) &= 0 \text{ for } 0 < x < 1, 0 < y < 1, 0 < z < 1 \\ u(x,y,0) &= u(x,y,1) = 0 \\ u(0,y,z) &= u(1,y,z) = 0 \\ u(x,0,z) &= 0, u(x,1,z) = x\cos(\pi x/2) z(1-z). \end{aligned}$$

2. Solve

$$\begin{aligned} \nabla^2 u(x,y,z) &= 0 \text{ for } 0 < x < 1, 0 < y < 1, 0 < z < \pi \\ u(x,y,0) &= 0, u(x,y,\pi) = \sin(\pi x)\sin(\pi y) \\ u(x,0,z) &= u(x,1,z) = 0 \\ u(0,y,z) &= u(1,y,z) = 0. \end{aligned}$$

3. Solve

$$\begin{aligned} \nabla^2 u(x,y,z) &= 0 \text{ for } 0 < x < \pi, 0 < y < \pi, 0 < z < \pi \\ u(x,y,0) &= \sin(3x)\sin(y), u(x,y,\pi) = x(\pi - x)y(\pi - y) \\ u(x,0,z) &= u(x,\pi,z) = 0 \\ u(0,y,z) &= u(\pi,y,z) = 0. \end{aligned}$$

6.12 The Neumann Problem

In this and the following three sections we discuss some aspects of the Neumann problem. Following this, we conclude the chapter with the Green's function for Dirichlet problems, and conformal mapping techniques.

The Neumann problem for a domain Ω is to find a function that is harmonic on Ω and whose normal derivative takes on given values on the boundary:

$$\begin{aligned} \nabla^2 u &= 0 \text{ in } \Omega \\ \frac{\partial u}{\partial n} &= f \text{ on } \partial\Omega. \end{aligned} \tag{6.19}$$

Certainly, if u is a solution of a Neumann problem, so is u plus any constant, since the boundary data is given for the normal derivative, not for the function itself, and the constant will vanish in the differentiation. We will show that continuous solutions of the Neumann problem are unique to within this additive constant.

Theorem 6.8 *Let Ω be a bounded domain in R^n having a closed, piecewise smooth boundary $\partial\Omega$. Let f be continuous on $\partial\Omega$. Let w and v be continuous solutions of the Neumann problem 6.19. Then w and v differ by a constant on $\overline{\Omega}$.* ◇

We prove the theorem for a domain in R^2. The proof makes use of Green's first identity.

Lemma 6.3 (*Green's First Identity in R^2*) *Let Ω be a bounded domain in R^2 having a closed, piecewise smooth curve as boundary. Let g and h be in $C^2(\overline{\Omega})$. Then*

$$\oint_{\partial\Omega} g\frac{\partial h}{\partial n}\,ds = \iint_{\overline{\Omega}} (g\nabla^2 h + \nabla g \cdot \nabla h)\,dA. \diamond$$

Proof First use Green's theorem to write

$$\oint_{\partial\Omega} g\frac{\partial h}{\partial n}\,ds = \iint_{\overline{\Omega}} \operatorname{div}(g\nabla h)\,dA.$$

Now verify that

$$\operatorname{div}(g\nabla h) = g\nabla^2 h - \nabla g \cdot \nabla h, \tag{6.20}$$

completing the proof of the lemma. ◇

If $g = h$ in the lemma, we obtain

$$\oint_{\partial\Omega} g\frac{\partial g}{\partial n}\,ds = \iint_{\overline{\Omega}} (g\nabla^2 g + |\nabla g|^2)\,dA. \tag{6.21}$$

Now we can prove the theorem. Let $u = v - w$. Then

$$\nabla^2 u = 0 \text{ in } \Omega$$

and

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} - \frac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega.$$

Apply equation 6.21 with $g = u$ to conclude that

$$\iint_{\overline{\Omega}} |\nabla u|^2 \, dA = 0.$$

Then

$$u_x^2 + u_y^2 = 0$$

on Ω, hence

$$u_x = u_y = 0$$

on Ω. But then $u(x, y) =$ constant on Ω. By continuity $u(x, y) =$ constant on $\overline{\Omega}$.

A similar argument can be used to prove the theorem for a bounded domain in R^3. In this case Gauss's divergence theorem is used to prove the three-dimensional version of Green's first identity, a surface integral replaces the line integral, and a volume integral replaces the double integral in the plane over the domain Ω.

We do not attempt to derive sufficient conditions for the existence of a solution of the Neumann problem. However, it is easy to derive an important necessary condition. Let u be a solution of a Neumann problem on Ω in R^2. By equation 6.21 with $g = 1$ and $h = u$,

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} \, ds = 0.$$

Since $\partial u/\partial n = f$ on $\partial\Omega$,

$$\int_{\partial\Omega} f \, ds = 0.$$

We have proved that a necessary condition for there to exist a solution of the Neumann problem 6.19 in R^2 is that the integral of the data function over the boundary must be zero.

For a domain in R^3, the analogous condition is that the surface integral $\int\int_{\partial\Omega} f \, d\sigma$ of the data function over the boundary surface must be zero.

This condition has a physical interpretation. If we think of Laplace's equation $\nabla^2 u = 0$ as the steady-state heat equation, this integral condition means that in the steady-state case the net flow of heat energy across the boundary of the domain must be zero. This is a conservation of energy condition, in the absence of sources or sinks of heat energy in the domain.

Problems for Section 6.12

1. Derive equation 6.20.

2. Prove Theorem 6.8 for a domain in R^3.

3. Prove that $\int\int_{\partial\Omega} f \, d\sigma = 0$ is necessary for the existence of a solution of the Neumann problem in the case that Ω is a bounded domain in R^3.

4. Let Ω be a bounded domain in the plane, and suppose that $\partial\Omega$ is a piecewise smooth closed curve. Let q be continuous on $\overline{\Omega}$ and f on $\partial\Omega$. Prove that the problem

$$\begin{aligned} \nabla^2 u &= q \text{ in } \Omega \\ \frac{\partial u}{\partial n} &= f \text{ on } \partial\Omega \end{aligned}$$

can have a solution only if

$$\iint_{\overline{\Omega}} q(x,y)\,dA = \oint_{\partial\Omega} f\,ds.$$

5. Consider the boundary value problem

$$\begin{aligned} \nabla^2 u &= 0 \text{ on } \Omega \\ \frac{\partial u}{\partial n} + hu &= f \text{ on } \partial\Omega. \end{aligned}$$

Assume that Ω is a bounded domain in the plane, with piecewise smooth closed boundary curve $\partial\Omega$. Let h and f be continuous on $\partial\Omega$ and assume that h is not identically zero and that $h(\mathbf{x}) \geq 0$ for $\mathbf{x}$ in $\partial\Omega$. Prove that this problem can have only one solution.

6. Prove the conclusion of Problem 5 for the case that Ω is a bounded domain in R^3, with piecewise smooth closed boundary surface $\partial\Omega$.

7. Let f be continuous on the domain Ω in R^2, and let g be continuous on the piecewise smooth curve bounding Ω. Let k be a real number. Prove that the problem

$$\begin{aligned} \nabla^2 u + ku &= f \text{ on } \Omega \\ \frac{\partial u}{\partial n} &= g \text{ on } \partial\Omega \end{aligned}$$

can have at most one solution if $k < 0$.

8. Let Ω be a domain in R^2, whose boundary consists of two piecewise smooth curves C and K. Let f be continuous on Ω, and let g be continuous on C. Let k be a real number. Prove that the problem

$$\begin{aligned} \nabla^2 u &= f \text{ on } \Omega \\ u &= g \text{ on } C \\ \frac{\partial u}{\partial n} + ku &= 0 \text{ on } K \end{aligned}$$

can have at most one solution.

6.13 Neumann Problem for a Rectangle

We will solve a Neumann problem for a rectangle:

$$\begin{aligned}\nabla^2 u(x,y) &= 0 \text{ for } 0 < x < a, 0 < y < b\\ u_y(x,0) &= u_y(x,b) = 0 \text{ for } 0 \le x \le a\\ u_x(0,y) &= 0 \text{ for } 0 \le y \le b\\ u_x(a,y) &= g(y) \text{ for } 0 \le y \le b.\end{aligned}$$

This is a Neumann problem because the partial derivatives specified are normal to the respective sides of the rectangle.

A necessary condition for the existence of a solution is that

$$\oint_{\partial\Omega} \frac{\partial u}{\partial n}\, ds = 0;$$

in which $\partial\Omega$ is the piecewise smooth curve bounding the rectangle. For the given boundary conditions this condition reduces to

$$\int_0^b g(y)\, dy = 0.$$

We assume that g satisfies this condition. We will soon see why there can be no solution if this condition is not met.

To use separation of variables, let $u(x,y) = X(x)Y(y)$ and obtain in the usual way

$$X'' + \lambda X = 0 \text{ and } Y'' - \lambda Y = 0,$$

in which λ is the separation constant. Now

$$u_y(x,0) = X(x)Y'(0) = 0$$

implies that $Y'(0) = 0$, and $u_y(x,b) = 0$ implies that $Y'(b) = 0$. The problem for Y is

$$Y'' - \lambda Y = 0; Y'(0) = Y'(b) = 0.$$

This problem has eigenvalues and eigenfunctions

$$\lambda_n = -\frac{n^2\pi^2}{b^2} \text{ and } Y_n = \cos\left(\frac{n\pi y}{b}\right)$$

for $n = 0, 1, 2, \cdots$.

Now the problem for X is

$$X'' - \frac{n^2\pi^2}{b^2} X = 0.$$

If $n = 0, X = cx + d$. But $u_x(0,y) = 0$ implies that $X'(0) = 0$, so $c = 0$ and $X_0 =$ constant for $n = 0$.

If n is a positive integer, the differential equation for X has general solution

$$X = ce^{n\pi x/b} + de^{-n\pi x/b}.$$

$X'(0) = 0$ implies that

$$\frac{n\pi}{b}c - \frac{n\pi}{b}d = 0,$$

so $c = d$ and

$$X_n = \cosh\left(\frac{n\pi x}{b}\right).$$

For each nonnegative integer n, we therefore have functions

$$u_0(x, y) = \text{constant}$$

and for $n = 1, 2, \cdots$,

$$u_n(x, y) = a_n \cosh\left(\frac{n\pi x}{b}\right)\cos\left(\frac{n\pi y}{b}\right).$$

To satisfy the boundary condition $u_x(a, y) = g(y)$, use a superposition of these functions, which we will write as

$$u(x, y) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cosh\left(\frac{n\pi x}{b}\right)\cos\left(\frac{n\pi y}{b}\right).$$

We need

$$u_x(a, y) = \sum_{n=1}^{\infty} \alpha_n \frac{n\pi}{b}\sinh\left(\frac{n\pi a}{b}\right)\cos\left(\frac{n\pi y}{b}\right) = g(y). \tag{6.22}$$

This is a Fourier cosine expansion of g in which the *nth* coefficient is given by

$$\alpha_n \frac{n\pi}{b}\sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b g(\xi)\cos\left(\frac{n\pi\xi}{b}\right)d\xi$$

or

$$\alpha_n = \frac{2}{n\pi\sinh(n\pi a/b)}\int_0^b g(\xi)\cos\left(\frac{n\pi\xi}{b}\right)d\xi.$$

The constant term in the Fourier cosine expansion of g on $[0, b]$ is $(1/b)\int_0^b g(\xi)$ The expansion 6.22 which is needed for a solution requires that this term be zero. But this is exactly the condition

$$\oint \frac{\partial u}{\partial n} ds = 0,$$

which is necessary for this Neumann problem to have a solution. The solution is

$$u(x, y) = \alpha_0 + \sum_{n=1}^{\infty}\left(\frac{2}{n\pi\sinh(n\pi a/b)}\int_0^b g(\xi)\cos\left(\frac{n\pi\xi}{b}\right)d\xi\right)\cosh\left(\frac{n\pi x}{b}\right)\cos\left(\frac{n\pi y}{b}\right),$$

in which α_0 is an arbitrary constant. Recall that for a Neumann problem, we can only expect to obtain a solution to within an additive constant.

Problems for Section 6.13

1. Solve

$$\begin{aligned} \nabla^2 u(x,y) &= 0 \text{ for } 0 < x < 1, 0 < y < 1 \\ u_x(0,y) = u_x(1,y) &= 0 \text{ for } 0 \le y \le 1 \\ u_y(x,0) &= 4\cos(\pi x), u_y(x,1) = 0 \text{ for } 0 \le x \le 1. \end{aligned}$$

2. Solve

$$\begin{aligned} \nabla^2 u &= 0 \text{ for } 0 < x < 1, 0 < y < \pi \\ u_x(0,y) &= y - \frac{\pi}{2}, u_x(1,y) = \cos(y) \text{ for } 0 \le y \le \pi \\ u_y(x,0) &= u_y(x,\pi) = 0 \text{ for } 0 \le x \le 1. \end{aligned}$$

3. Solve

$$\begin{aligned} \nabla^2 u &= 0 \text{ for } 0 < x < \pi, 0 < y < \pi \\ u_x(0,y) &= u_x(\pi,y) = 0 \text{ for } 0 \le y \le \pi \\ u_y(x,0) &= \cos(3x), u_y(x,\pi) = 6x - 3\pi \text{ for } 0 \le x \le \pi. \end{aligned}$$

4. Use separation of variables to solve the mixed boundary value problem

$$\begin{aligned} \nabla^2 u(x,y) &= 0 \text{ for } 0 < x < \pi, 0 < y < \pi \\ u(x,0) &= f(x), u(x,\pi) = 0 \text{ for } 0 \le x \le \pi \\ u_x(0,y) &= u_x(\pi,y) = 0 \text{ for } 0 \le y \le \pi. \end{aligned}$$

   Does this problem have a unique solution?

5. Attempt separation of variables to solve

$$\begin{aligned} \nabla^2 u(x,y) &= 0 \text{ for } 0 < x < 1, 0 < y < 1 \\ u(x,0) &= u(x,1) = 0 \text{ for } 0 \le x \le 1 \\ u_x(0,y) &= 3y^2 - 2y, u_y(1,y) = 0 \text{ for } 0 \le y \le 1. \end{aligned}$$

6.14 Neumann Problem for a Disk

We will solve the Neumann problem for a disk centered about the origin in the plane. In polar coordinates the problem is

$$\begin{aligned} \nabla^2 u(r,\theta) &= 0 \text{ for } 0 \le r < \rho, -\pi \le \theta \le \pi \\ \frac{\partial u}{\partial n}(\rho,\theta) &= f(\theta) \text{ for } -\pi \le \theta \le \pi. \end{aligned}$$

At any point on a circle about the origin, the line from the origin through the point is along the normal to the circle at that point, so the normal derivative in polar coordinates is just the radial derivative $\partial/\partial r$. The boundary condition can be written

$$\frac{\partial u}{\partial r}(\rho,\theta) = f(\theta) \text{ for } -\pi \leq \theta \leq \pi.$$

A necessary condition for a solution to exist is that

$$\int_{-\pi}^{\pi} f(\theta)\, d\theta = 0$$

and we assume that f satisfies this condition.

Attempt a solution

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta).$$

We need

$$\frac{\partial u}{\partial r}(\rho,\theta) = \sum_{n=1}^{\infty} a_n n\rho^{n-1}\cos(n\theta) + b_n n\rho^{n-1}\sin(n\theta) = f(\theta).$$

This is a Fourier expansion of f on $[-\pi,\pi]$ and the coefficients are

$$a_n = \frac{1}{\pi n\rho^{n-1}}\int_{-\pi}^{\pi} f(\xi)\cos(n\xi)\, d\xi$$

and

$$b_n = \frac{1}{\pi n\rho^{n-1}}\int_{-\pi}^{\pi} f(\xi)\sin(n\xi)\, d\xi$$

for $n = 1, 2, \cdots$.

Upon inserting these coefficients, the solution is

$$\begin{aligned} u(r,\theta) &= \frac{1}{2}a_0 \\ &+ \frac{\rho}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{\rho}\right)^n \int_{-\pi}^{\pi}[\cos(n\xi)\cos(n\theta) + \sin(n\xi)\sin(n\theta)]f(\xi)\, d\xi. \end{aligned} \tag{6.23}$$

with a_0 an arbitrary constant. The factor $1/2$ with a_0 is customary and is convenient in the ensuing calculation.

It is possible to sum this series to obtain an integral expression for the solution which is analogous to Poisson's formula for the Dirichlet problem for a disk. Interchange the summation and the integral and use a trigonometric identity, as we have done before, to write equation 6.23 as

$$u(r,\theta) = \frac{1}{2}a_0 + \frac{\rho}{\pi}\int_{-\pi}^{\pi}\left(\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{\rho}\right)^n \cos(n(\theta-\xi))\right) f(\xi)\, d\xi. \tag{6.24}$$

The interchange of $\sum_{n=1}^{\infty}$ and $\int_{-\pi}^{\pi}$ is justified by the uniform convergence of this series as a function of ξ for $0 \leq r < \rho$. To sum the series in large parentheses in equation 6.24, let $z = Re^{i\zeta}$. Then

$$z^n = R^n \cos(n\zeta) + iR^n \sin(n\zeta)$$

and

$$\sum_{n=1}^{\infty} R^n \cos(n\zeta) = \sum_{n=1}^{\infty} \mathrm{Re}(z^n) = \mathrm{Re}\left[\sum_{n=1}^{\infty} z^n\right].$$

If $0 \leq R < 1, |z| < 1$ and

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}.$$

Now compute

$$\frac{1}{2} + \sum_{n=1}^{\infty} z^n = \frac{1}{2} + \frac{z}{1-z} = \frac{1+z}{2(1-z)}.$$

To find the real part of this expression, first multiply numerator and denominator by $1 - \overline{z}$:

$$\begin{aligned}\frac{1}{2} + \sum_{n=1}^{\infty} z^n &= \frac{1}{2}\frac{1+z}{1-z}\frac{1-\overline{z}}{1-\overline{z}} \\ &= \frac{1}{2}\frac{1-\overline{z}+z-z\overline{z}}{1-z-\overline{z}+z\overline{z}}.\end{aligned}$$

But

$$\begin{aligned}z + \overline{z} &= R\cos(\zeta) + iR\sin(\zeta) + [R\cos(\zeta) - iR\sin(\zeta)] = 2R\cos(\zeta), \\ z - \overline{z} &= -2iR\sin(\zeta)\end{aligned}$$

and

$$z\overline{z} = R^2.$$

Therefore,

$$\frac{1}{2} + \sum_{n=1}^{\infty} z^n = \frac{1}{2}\left(\frac{1 - 2iR\sin(\zeta) - R^2}{1 - 2R\cos(\zeta) + R^2}\right).$$

Then

$$\begin{aligned}\frac{1}{2} + \sum_{n=1}^{\infty} R^n \cos(n\zeta) &= \mathrm{Re}\left[\frac{1}{2} + \sum_{n=1}^{\infty} z^n\right] \\ &= \frac{1}{2}\left(\frac{1-R^2}{1+R^2-2R\cos(\zeta)}\right).\end{aligned}$$

so

$$\sum_{n=1}^{\infty} R^n \cos(n\zeta) = \frac{1}{2}\left(\frac{1-R^2}{1+R^2-2R\cos(\zeta)}\right) - \frac{1}{2}$$
$$= \frac{R\cos(\zeta) - R^2}{1+R^2-2R\cos(\zeta)}.$$

Upon dividing by R we obtain

$$\sum_{n=1}^{\infty} R^{n-1}\cos(n\zeta) = \frac{\cos(\zeta) - R}{1+R^2-2R\cos(\zeta)}.$$

This series also converges uniformly as a function of R, so

$$\int_0^R \sum_{n=1}^{\infty} t^{n-1}\cos(n\zeta)\,dt = \sum_{n=1}^{\infty}\int_0^R t^{n-1}\cos(n\zeta)\,dt$$
$$= \sum_{n=1}^{\infty}\frac{1}{n}R^n\cos(n\zeta)$$
$$= \int_0^R \frac{\cos(\zeta)-t}{1+t^2-2t\cos(\zeta)}\,dt = -\frac{1}{2}\ln(1+R^2-2R\cos(\zeta)).$$

Now put $R = r/\rho$ and $\zeta = \theta - \xi$, to obtain

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{\rho}\right)^n \cos(n(\theta-\xi)) = -\frac{1}{2}\ln\left(1+\frac{r^2}{\rho^2} - 2\frac{r}{\rho}\cos(\theta-\xi)\right).$$

Finally, we can write the solution 6.24 as

$$u(r,\theta) = \frac{1}{2}a_0 - \frac{\rho}{2\pi}\int_{-\pi}^{\pi}\ln\left(1+\frac{r^2}{\rho^2} - 2\frac{r}{\rho}\cos(\theta-\xi)\right) f(\xi)\,d\xi,$$

in which a_0 can be any real number. This is the solution of the Neumann problem for the disk of radius ρ about the origin in R^2.

Problems for Section 6.14

1. Use the expression 6.23 to solve

$$\nabla^2 u(r,\theta) = 0 \text{ for } 0 \le r < \rho, -\pi \le \theta \le \pi$$
$$\frac{\partial u}{\partial r}(\rho,\theta) = \sin(3\theta) \text{ for } -\pi \le \theta \le \pi.$$

2. Solve

$$\nabla^2 u(r,\theta) = 0 \text{ for } 0 \le r < \rho, -\pi \le \theta \le \pi$$
$$\frac{\partial u}{\partial r}(\rho,\theta) = \cos(2\theta) \text{ for } -\pi \le \theta \le \pi.$$

3. Solve

$$\nabla^2 u(x,y) = 0 \text{ for } 0 \le x^2+y^2 < 9$$
$$\frac{\partial u}{\partial n} = 4xy \text{ for } x^2+y^2 = 9.$$

Hint: Convert the problem to polar coordinates.

4. Solve

$$\nabla^2 u(x,y) = 0 \text{ for } x^2+y^2 < 1$$
$$\frac{\partial u}{\partial n} = x \text{ for } x^2+y^2 = 1.$$

5. Solve

$$\nabla^2 u(x,y) = 0 \text{ for } x^2+y^2 < 1$$
$$\frac{\partial u}{\partial n} = xy^2 \text{ for } x^2+y^2 = 1.$$

6.15 Neumann Problem for the Upper Half-Plane

As an illustration of a Neumann problem in an unbounded domain, consider the following problem for the upper half-plane:

$$\nabla^2 u(x,y) = 0 \text{ for } -\infty < x < \infty, y > 0$$
$$u_y(x,0) = f(x) \text{ for } -\infty < x < \infty.$$

u_y is the normal derivative to the horizontal axis.

Although $\partial\Omega$ in this problem is not a closed curve, it is still possible to show that a necessary condition for a solution to exist is that the integral of the data function over the boundary is zero. Thus we assume that

$$\int_{-\infty}^{\infty} f(x)\,dx = 0.$$

There is an elegant device for reducing this problem to one that we have already solved. Let $v = u_y$. Then

$$\begin{aligned}\nabla^2 v &= v_{xx} + v_{yy} = (u_y)_{xx} + (u_y)_{yy} \\ &= (u_{xx})_y + (u_{yy})_y = (\nabla^2 u)_y = 0 \text{ for } -\infty < x < \infty, y > 0\end{aligned}$$

and

$$v(x,0) = u_y(x,0) = f(x) \text{ for } -\infty < x < \infty.$$

We conclude that v is the solution of a Dirichlet problem for the upper half-plane, if u is a solution of the Neumann problem. But we know the solution of this Dirichlet problem. By equation 6.16 it is

$$v(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)}{y^2+(\xi-x)^2}\,d\xi.$$

Since $v = u_y$, we can recover u from this solution for v by integrating with respect to y:

$$\begin{aligned} u(x,y) &= \frac{1}{\pi}\iint_{-\infty}^{\infty} \frac{y}{y^2+(\xi-x)^2} f(\xi)\,d\xi\,dy \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\int \frac{y}{y^2+(\xi-x)^2}\,dy\right) f(\xi)\,d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \ln(y^2+(\xi-x)^2) f(\xi)\,d\xi + c, \end{aligned} \tag{6.25}$$

in which c is an arbitrary constant.

Problems for Section 6.15

1. Solve the Neumann problem for the lower half-plane:

$$\begin{aligned} \nabla^2 u(x,y) &= 0 \text{ for } -\infty < x <, y < 0 \\ u_y(x,0) &= f(x) \text{ for } -\infty < x < \infty. \end{aligned}$$

Assume that f is bounded and continuous. Is any other condition on f required for this problem to have a solution?

2. Solve the Neumann problem for the right quarter-plane:

$$\begin{aligned} \nabla^2 u(x,y) &= 0 \text{ for } x > 0, y > 0 \\ u_x(0,y) &= 0 \text{ for } y \geq 0 \\ u_y(x,0) &= f(x) \text{ for } x \geq 0. \end{aligned}$$

Assume that f is bounded and continuous. Are any other conditions on f required for this problem to have a solution?

3. Solve the Neumann problem for the right half-plane:

$$\begin{aligned} \nabla^2 u(x,y) &= 0 \text{ for } x > 0, -\infty < y < \infty \\ u_y(0,y) &= g(y) \text{ for } -\infty < y < \infty. \end{aligned}$$

Assume that g is bounded and continuous. Is any other condition on g required?

4. Solve the Neumann problem for the left half-plane:

$$\begin{aligned} \nabla^2 u(x,y) &= 0 \text{ for } x < 0, -\infty < y < \infty \\ u_y(0,y) &= g(y) \text{ for } -\infty < y < \infty. \end{aligned}$$

5. Solve the problem

$$\begin{aligned} \nabla^2 u(x,y) &= 0 \text{ for } x > 0, y > 0 \\ u(0,y) &= 0 \text{ for } y \geq 0 \\ u_y(x,0) &= f(x) \text{ for } x \geq 0. \end{aligned}$$

Hint: The solution can be obtained using equation 6.25.

6.16 Green's Function for a Dirichlet Problem

In this section we develop an integral representation for the solution of a Dirichlet problem, based on a function called the *Green's function*. We will do this for problems in R^3. This discussion can be adapted to problems in the plane or dimensions higher than 3.

Let Ω be a bounded domain in R^3 whose boundary $\partial\Omega$ is a piecewise smooth closed surface. If $u \in C^2(\overline{\Omega})$ and $\nabla^2 u = 0$ in Ω, we know that at any $\mathbf{x} \in \Omega$,

$$u(\mathbf{x}) = \frac{1}{4\pi} \iint_{\partial\Omega} \left(\frac{1}{|\mathbf{y}-\mathbf{x}|} \frac{\partial u(\mathbf{y})}{\partial n} - u(\mathbf{y}) \frac{\partial}{\partial n} \frac{1}{|\mathbf{y}-\mathbf{x}|} \right) d\sigma_{\mathbf{y}}.$$

Now refer to Green's second identity in R^3. If v is any function that is also harmonic in Ω, the volume integral in Green's second identity is zero and we obtain

$$\iint_{\partial\Omega} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) d\sigma_{\mathbf{y}} = 0.$$

Since this integral is zero, we can add it to the integral representation of u and obtain, after rearranging terms,

$$\begin{aligned} u(\mathbf{x}) = & \iint_{\partial\Omega} \left(\frac{1}{4\pi} \frac{1}{|\mathbf{y}-\mathbf{x}|} + v(\mathbf{y}) \right) \frac{\partial u(\mathbf{y})}{\partial n} \, d\sigma_{\mathbf{y}} \\ & - \iint_{\partial\Omega} u(\mathbf{y}) \frac{\partial}{\partial n} \left(v(\mathbf{y}) + \frac{1}{4\pi} \frac{1}{|\mathbf{y}-\mathbf{x}|} \right) d\sigma_{\mathbf{y}}. \end{aligned} \qquad (6.26)$$

Equation 6.26 represents $u(\mathbf{x})$ as a sum of two surface integrals. The first involves the normal derivative of u on the boundary of Ω, and no information is given about this quantity. We can cause this integral to be zero if we choose v so that the term in large parentheses in this integral is zero. This will leave the second surface integral in equation 6.26, which involves values of u on the boundary (these are given to us), and the normal derivative of a function that is known once we choose v.

Thus, choose v so that

$$\begin{aligned} \nabla^2 v(\mathbf{y}) &= 0 \text{ in } \Omega \\ v(\mathbf{y}) &= -\frac{1}{4\pi} \frac{1}{|\mathbf{y}-\mathbf{x}|} \text{ for } \mathbf{y} \text{ in } \partial\Omega. \end{aligned}$$

For each $\mathbf{x}$ in Ω, this problem for v is a Dirichlet problem on Ω. Therefore, v depends both on $\mathbf{y}$ (the name given to the variable in the problem for v) and on $\mathbf{x}$ (the point at which we are seeking to represent u). For this reason we denote v as $v(\mathbf{x}, \mathbf{y})$.

Now define

$$G(\mathbf{x}, \mathbf{y}) = v(\mathbf{x}, \mathbf{y}) + \frac{1}{4\pi} \frac{1}{|\mathbf{y}-\mathbf{x}|}. \qquad (6.27)$$

G is the *Green's function* for the Dirichlet problem for u. By equation 6.26 and the way G has been defined, the solution of the Dirichlet problem for u is

$$u(\mathbf{x}) = -\iint_{\partial\Omega} f(\mathbf{y})\frac{\partial}{\partial n}G(\mathbf{x},\mathbf{y})\,d\sigma_{\mathbf{y}}. \tag{6.28}$$

One can show that G is symmetric:

$$G(\mathbf{x},\mathbf{y}) = G(\mathbf{y},\mathbf{x}).$$

As a function of $\mathbf{y}$, G is harmonic in $\Omega_{\mathbf{x}} = \Omega - \{\mathbf{x}\}$ and vanishes on $\partial\Omega$. Finally, as $\mathbf{y} \to \mathbf{x}, G(\mathbf{x},\mathbf{y}) \to \infty$ in the same way that $1/|\mathbf{y}-\mathbf{x}|$ does.

Green's Function for a Sphere

Equation 6.28 reduces the problem of solving a Dirichlet problem on a bounded domain in R^3 to one of finding a Green's function for this domain. This is a formidable task in itself for many domains. Sometimes, however, a domain has a particular property, often some kind of symmetry, that enables us to construct a Green's function explicitly. We illustrate this for a ball about the origin.

Let $\Omega = B(\mathbf{0}, a)$, the open ball of radius a about the origin in R^3. The first step in constructing the Green's function for Ω is to solve, for each $\mathbf{x}$ in Ω, the problem

$$\begin{aligned} \nabla^2 v(\mathbf{y}) &= 0 \text{ for } \mathbf{y} \text{ in } \Omega \\ v(\mathbf{y}) &= -\frac{1}{4\pi}\frac{1}{|\mathbf{y}-\mathbf{x}|} \text{ for } \mathbf{y} \text{ in } S(\mathbf{0}, a). \end{aligned}$$

An ideal candidate for v might appear to be $-1/4\pi|\mathbf{y}-\mathbf{x}|$ itself, since this is a harmonic function and equals itself on $\partial\Omega$. The problem is that this function is not defined at $\mathbf{x}$ in Ω. We therefore perform an inversion in the sphere. An inversion ι maps a point $\mathbf{y}$ inside the ball $B(\mathbf{0},\, a)$ to a point ι $(\mathbf{y})$ outside and on the line from the origin through $\mathbf{y}$ and having the property that

$$|\mathbf{y}||\iota(\mathbf{y})| = a^2.$$

This mapping is illustrated in Figure 6.10 and is defined by

$$\iota(\mathbf{y}) = \frac{a^2}{|\mathbf{y}|^2}\mathbf{y}$$

(see Problem 8, Section 6.2). This function maps the origin maps to ∞. Points on the sphere $S(\mathbf{0},\, a)$ map to themselves.

Now take the function

$$-\frac{1}{4\pi}\frac{1}{|\mathbf{y}-\mathbf{x}|}$$

and replace $\mathbf{y}$ with its image under the inversion to form the function

$$-\frac{1}{4\pi}\frac{1}{\left|\frac{a^2}{|\mathbf{y}|^2}\mathbf{y}-\mathbf{x}\right|}.$$

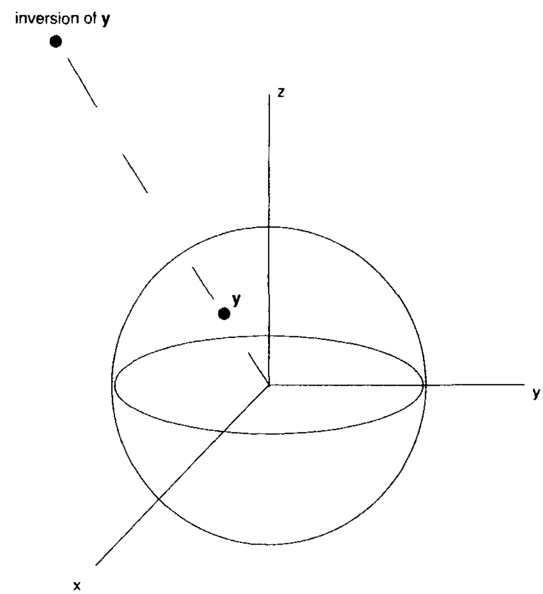

Figure 6.10: Inversion mapping.

This can be written

$$-\frac{1}{4\pi}\frac{|\mathbf{y}|}{a}\frac{1}{\left|\frac{a}{|\mathbf{y}|}\mathbf{y}-\frac{|\mathbf{y}|}{a}\mathbf{x}\right|}$$

and it easy to check that this function is harmonic in Ω as a function of $\mathbf{y}$ (keep in mind that $\iota(\mathbf{y})$ is outside the ball if $\mathbf{y}$ is inside). Further, if we omit the factor $|\mathbf{y}|/a$, the resulting function

$$-\frac{1}{4\pi}\frac{1}{\left|\frac{a}{|\mathbf{y}|}\mathbf{y}-\frac{|\mathbf{y}|}{a}\mathbf{x}\right|}$$

is also harmonic in Ω. This function is equal to $-1/4\pi|\mathbf{y}-\mathbf{x}|$ if $\mathbf{y}$ is on the sphere, because then $|\mathbf{y}|=a$. Therefore, choose

$$v(\mathbf{x},\mathbf{y})=-\frac{1}{4\pi}\frac{1}{\left|\frac{a}{|\mathbf{y}|}\mathbf{y}-\frac{|\mathbf{y}|}{a}\mathbf{x}\right|}.$$

The Green's function for the ball is

$$G(\mathbf{x},\mathbf{y})=\frac{1}{4\pi}\frac{1}{|\mathbf{y}-\mathbf{x}|}-\frac{1}{4\pi}\frac{1}{\left|\frac{a}{|\mathbf{y}|}\mathbf{y}-\frac{|\mathbf{y}|}{a}\mathbf{x}\right|},$$

and the solution of the Dirichlet problem for $B(\mathbf{0},\ a)$ is

$$u(\mathbf{x}) = -\iint_{S(\mathbf{0},a)} f(\mathbf{y})\frac{\partial}{\partial n}G(\mathbf{x},\mathbf{y})\,d\sigma_y.$$

This defines a function that is harmonic on Ω and equals f on the sphere.

It is possible to show that

$$\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n} = -\frac{1}{4\pi a}\frac{a^2 - r^2}{(a^2 + r^2 - 2ar\cos(\Theta))^{3/2}}, \tag{6.29}$$

in which $\Theta(\mathbf{x},\mathbf{y})$ is the angle between $\mathbf{x}$ and $\mathbf{y}$, and $r = |\mathbf{x}|$. This gives the solution of the Dirichlet problem for $B(\mathbf{0},\ a)$ in R^3 as

$$u(\mathbf{x}) = \frac{1}{4\pi a}\iint_{S(\mathbf{0},a)} \frac{a^2 - r^2}{(a^2 + r^2 - 2ar\cos(\Theta))^{3/2}} f(\mathbf{y})\,d\sigma_y, \tag{6.30}$$

and this is the three-dimensional analogue of Poisson's integral formula 6.11 for a disk in R^2. One can also derive an analogous integral formula for the solution of the Dirichlet problem for an open ball in R^n. This proves the existence of a solution of the Dirichlet problem for an open ball in R^n, assuming that the function given on the boundary is continuous.

The Method of Electrostatic Images

Sometimes a physical interpretation of a mathematical concept provides insight into the solution of equations or the determination of important functions. This occurs with Green's functions for certain domains in R^3.

The Green's function $G(\mathbf{x},\ \mathbf{y})$ for a domain Ω in R^3 can be interpreted as the electrostatic potential resulting from a unit charge at $\mathbf{x}$ in Ω, with the bounding surface $\partial\Omega$ a grounded conducting surface (the electrostatic potential is zero on such a surface). Now, recall that

$$G(\mathbf{x},\mathbf{y}) = v(\mathbf{x},\mathbf{y}) + \frac{1}{4\pi}\frac{1}{|\mathbf{y}-\mathbf{x}|}.$$

The second term on the right side of this expression is the potential due to a unit charge at $\mathbf{x}$. The term $v(\mathbf{x},\ \mathbf{y})$ is the potential due to the induced charge distribution on $\partial\Omega$. We can therefore find $v(\mathbf{x},\ \mathbf{y})$, and hence the Green's function for Ω, if we can find the induced charge distribution on $\partial\Omega$.

As any student in a classical electricity and magnetism course can testify, this is usually not an easy task. However, there is a clever way of approaching this difficulty, which sometimes meets with success. We can also think of $v(\mathbf{x},\ \mathbf{y})$ as the potential due to "imaginary" charges placed at appropriate points outside Ω. These imaginary charges are called *electrostatic images* of the unit charges in Ω, and the idea is to locate them outside Ω so that the potential $v(\mathbf{x},\ \mathbf{y})$ caused by these charges satisfies

$$v(\mathbf{x},\mathbf{y}) = -\frac{1}{4\pi}\frac{1}{|\mathbf{y}-\mathbf{x}|}$$

for $\mathbf{y}$ in $\partial\Omega$. The idea of determining $v(\mathbf{x},\ \mathbf{y})$ in this way is called the *method of electrostatic images.*

It would be nice if we could now solve all Dirichlet problems, but this is not realistic. However, in cases where Ω has some property that enables us to place the electrostatic images, we can write the Green's function and use equation 6.28 to obtain an integral formula for the solution of the Dirichlet problem for Ω.

Example 6.5 *Let Ω be the half-space consisting of all points above the x, y - plane. Thus Ω consists of all points $(x,\ y,\ z)$ with $z > 0$. We will determine the Green's function for Ω.*

Here $\partial\Omega$ is the x, y - plane, consisting of points $(x,\ y,\ 0)$. Corresponding to a unit charge at $\mathbf{x} = (x, y, z)$ in Ω, place an imaginary unit charge at $\mathbf{x}^* = (x, y, -z)$ placed symmetrically with respect to $\mathbf{x}$ across the x, y - plane. These two charges cancel each other, and the potential due to their joint effect is zero on $\partial\Omega$. Therefore, the Green's function for Ω is

$$G(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi}\frac{1}{|\mathbf{y}-\mathbf{x}|} - \frac{1}{4\pi}\frac{1}{|\mathbf{y}-\mathbf{x}^*|}.$$

Since

$$|\mathbf{y}-\mathbf{x}| = |\mathbf{y}-\mathbf{x}^*|$$

if $\mathbf{y}$ is in the x, y - plane, then indeed $G(\mathbf{x},\mathbf{y}) = 0$ for $\mathbf{y}$ in $\partial\Omega$, as required. ◇

Problems for Section 6.16

1. Adapt the discussion of this section to write the analogue of equation 6.28 for the Dirichlet problem for a bounded domain in the plane. In doing this it is necessary to write the R^2 analogue of the expression 6.27. Hint: Use a logarithm in adapting equation 6.27 to R^2.

2. Poisson's equation for a domain Ω in R^3 has the form

$$\nabla^2 u(\mathbf{x}) = F(\mathbf{x}) \text{ on } \Omega.$$

This is Laplace's equation if $F(\mathbf{x})$ is identically zero. Consider the problem of solving Poisson's equation subject to the boundary condition

$$u(\mathbf{x}) = f(\mathbf{x}) \text{ for } \mathbf{x} \text{ in } \partial\Omega.$$

Begin with the representation theorem and adapt the line of reasoning of this section to derive the solution

$$u(\mathbf{x}) = -\iiint_{\Omega} F(\mathbf{y})G(\mathbf{x},\mathbf{y})dV_y - \iint_{\partial\Omega} f(\mathbf{y})\frac{\partial}{\partial n}G(\mathbf{x},\mathbf{y})\,d\sigma_y,$$

in which $G(\mathbf{x},\ \mathbf{y})$ is the Green's function for the Dirichlet problem

$$\nabla^2 u = 0 \text{ in } \Omega;\ u = f \text{ on } \partial\Omega.$$

Assume that F and f are continuous and that Ω is a bounded domain.

3. Use the representation theorem for R^2 to write an integral formula for the solution of Poisson's equation satisfying given boundary data, for a bounded domain in the plane.

Problems 4 through 12 relate to Green's function for the sphere.

4. Derive equation 6.29.

5. Prove that for $0 \leq r < 1$,

$$\iint_{S(\mathbf{0},1)} \frac{1-r^2}{(1+r^2-2r\cos(\Theta))^{3/2}}\, d\sigma_y = 4\pi.$$

6. Show that the solution of the Dirichlet problem for the disk $B(\mathbf{0}, a)$ can be written

$$u(\mathbf{x}) = -\oint_{\partial\Omega} f(\mathbf{y}) \frac{\partial}{\partial n} G(\mathbf{x},\mathbf{y})\, ds_y,$$

where

$$G(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\left[\ln\left(\frac{1}{|\mathbf{y}-\mathbf{x}|}\right) - \ln\left(\frac{1}{\left|\frac{a}{|\mathbf{y}|}\mathbf{y} - \frac{|\mathbf{y}|}{a}\mathbf{x}\right|}\right)\right]$$

is the Green's function for a disk in R^2.

7. Use the Green's function for a disk in R^2 (Problem 6) to give another derivation of Poisson's integral formula for the solution of the Dirichlet problem for a disk.

8. Use the integral expression in equation 6.30 to prove a three-dimensional version of Harnack's inequality.

9. Use the result of Problem 8 to prove a version of Liouville's theorem that is valid in R^3.

10. Use the results of Problems 2 and 6 to write an integral formula for a solution of the problem:

$$\nabla^2 u(x) = F(x) \text{ in } \Omega$$
$$u(x) = f(x) \text{ on } \partial\Omega,$$

where Ω is the disk of radius ρ about the origin in R^2. Convert this integral solution to polar coordinates.

11. Write an integral expression for the solution of the problem

$$\nabla^2 u(r,\theta) = \cos(\theta) \text{ for } 0 \leq r < \rho, 0 \leq \theta \leq 2\pi$$
$$u(\rho,\theta) = 1 \text{ for } 0 \leq \theta \leq 2\pi.$$

12. Write an integral expression for the solution of

$$\nabla^2 u(r,\theta) = 1 \text{ for } 0 \le r < \rho, 0 \le \theta \le 2\pi$$
$$u(\rho,\theta) = 1 \text{ for } 0 \le \theta \le 2\pi.$$

Problems 13 through 20 have to do with the method of electrostatic images.

13. Use the method of electrostatic images to derive the Green's function for the unit ball in R^3, obtained previously using the inversion mapping. Explain how the use of the inversion mapping and the method of electrostatic images for the sphere are really different ways of using the same idea.

14. Let Ω consist of all (x, y, z) with $y > 0$ and $z > 0$. Determine the Green's function for Ω. Hint: Corresponding to a positive unit charge at (x, y, z) in Ω, place a negative unit charge at $(x, -y, z)$, then balance this with a positive unit charge at $(x, -y, -z)$, and finally, balance the entire system with a negative unit charge at $(x, y, -z)$.

15. Find the Green's function for the domain bounded by two parallel planes $z = 0$ and $z = 1$. Hint: Place an imaginary charge so that each plane $z = 0, \pm 1, \pm 2, \cdots$ has zero potential.

16. Consider the Dirichlet problem for the upper half-space:

$$\nabla^2 u(x,y,z) = 0 \text{ for } z > 0 \text{ and } u(x,y,0) = f(x,y).$$

Use the Green's function for the upper half-space in Example 6.5, together with equation 6.28, to derive the solution

$$u(x,y,z) = \frac{z}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(\xi,\eta)}{[(\xi-x)^2+(\eta-y)^2+z^2]^{3/2}}\,d\xi\,d\eta.$$

17. The motivating notion of an electrostatic charge would seem to restrict its use to domains in R^3. However, once we understand the use of imaginary charges (really an exploitation of symmetry) to construct Green's functions for certain domains in 3 - space, we can jettison the physical motivation and use the method for certain domains in the plane. Now we must replace the potential

$$\frac{1}{4\pi}\frac{1}{|\mathbf{y}-\mathbf{x}|}$$

in R^3 due to a unit charge at $\mathbf{x}$ with the potential

$$\frac{1}{2\pi}\ln\left(\frac{1}{|\mathbf{y}-\mathbf{x}|}\right),$$

which we have found to be appropriate for problems in the plane. Use this idea and the method of electrostatic images to find the Green's function for the unit disk. Use this Green's function and the two-dimensional analogue of equation 6.28 (see Problem 1) to derive an integral solution of the Dirichlet problem for the unit disk. How does this expression compare with the Poisson integral formula?

18. Find the Green's function for the upper half-plane $y > 0$ and use this and the two-dimensional analogue of equation 6.28 to write an integral solution for the Dirichlet problem for the upper half-plane. Compare this expression with a previously derived integral solution of this problem (Equation 6.16).

19. Find the Green's function for the domain

$$\Omega = \{(x, y)|0 < y < 1\}.$$

This is a horizontal strip in the x, y - plane. Hint: Recall finding the Green's function for the domain bounded by two parallel planes in R^3, Problem 15.

20. Find the Green's function for the right quarter-plane $x > 0, y > 0$. Use this to write an integral formula for the solution of the Dirichlet problem for the right quarter-plane. Compare this solution with the solution (equation 6.18) obtained previously.

6.17 Conformal Mapping Techniques

Complex functions, thought of as mappings, can sometimes be used to solve Dirichlet problems. We will develop this idea, assuming some familiarity with analytic functions of a complex variable.

6.17.1 Conformal Mappings

If f is a complex function, we often write z as the independent variable and $w = f(z)$ as the dependent variable. In thinking of f from a geometric point of view, it is convenient to make two copies of the complex plane, one called the z - plane, one the w - plane. As z varies over a given set of points in the z - plane, we can observe how $f(z)$ varies over a corresponding set in the w - plane. In this way, f can be thought of as associating points or sets of points in the z - plane with their images in the w - plane.

If $f(z)$ is defined at least for all z in some set D of complex numbers in the z - plane, and K is a set of complex numbers in the w - plane, we say that f *maps D into K*, and write $f : D \to K$ if $f(z) \in K$ for each $z \in D$. Then each point z of D has its image $f(z)$ in K. This mapping is *onto* if every point of K is the image of some point in D under the mapping. This occurs if for each $w \in K$, there is some $z \in D$ with $w = f(z)$. The mapping is *one-to-one* if f maps distinct points to distinct points. This means that $f(z_1) = f(z_2)$ can

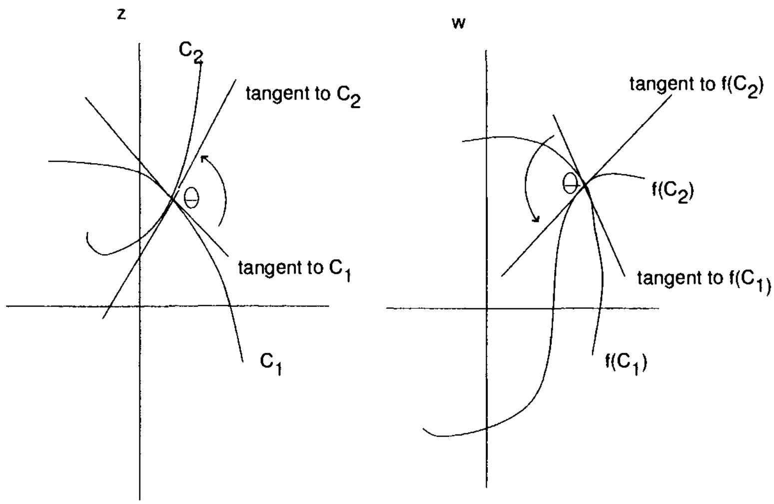

Figure 6.11: Angle-preserving mapping.

occur only if $z_1 = z_2$. The concepts of onto and one-to-one are independent of each other.

A mapping f: $D \to K$ is *conformal* if it preserves angles and orientation. This occurs when f has the following two properties:

(1) *f preserves angles.* This means that for any z in D, any two smooth curves in D intersecting at an angle θ at z have images in K that intersect at $f(z)$ at the same angle. This idea is illustrated in Figure 6.11.

(2) f preserves *orientation.* This means that a counterclockwise rotation in D is mapped by f to a counterclockwise rotation in K. This is illustrated in Figure 6.12, where the counterclockwise sense of orientation from L_1 to L_2 is mapped to a similarly counterclockwise orientation from the images of these lines. By contrast, Figure 6.13 illustrates a mapping that is not orientation preserving.

Many familiar complex functions, thought of as mappings, are conformal.

Theorem 6.9 *Let f: $D \to K$ be an analytic function mapping a domain D onto a domain K. Suppose that $f'(z) \neq 0$ for every $z \in D$. Then f is a conformal mapping.* $\diamond$

Thus, for analytic mappings between domains, nonvanishing of the derivative is sufficient to guarantee that the mapping preserves both angles and orientation.

It is routine to show that a composition of conformal mappings is again conformal. This is apparent by following the stages of the composition mapping

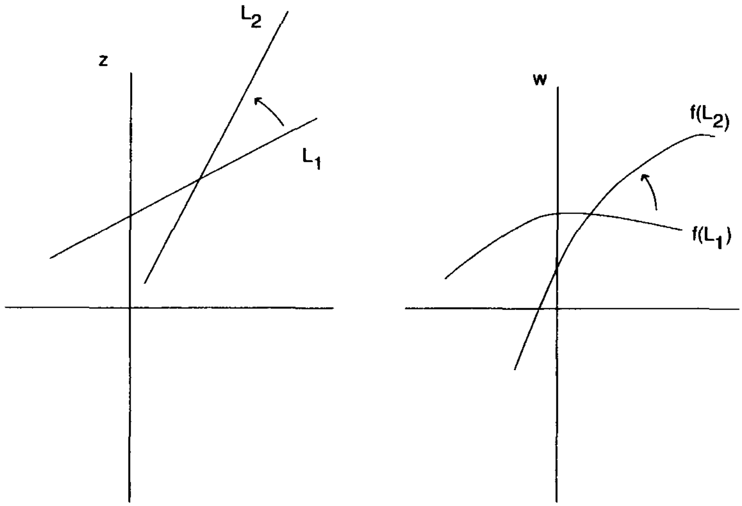

Figure 6.12: Orientation-preserving mapping.

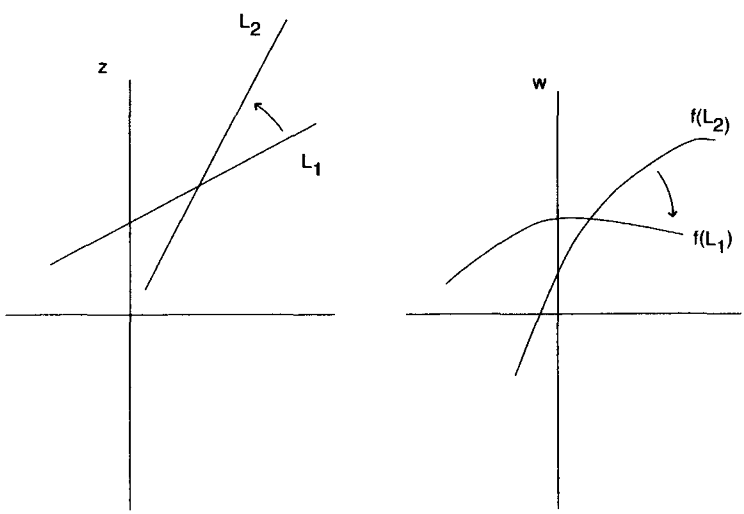

Figure 6.13: Orientation-reversing mapping.

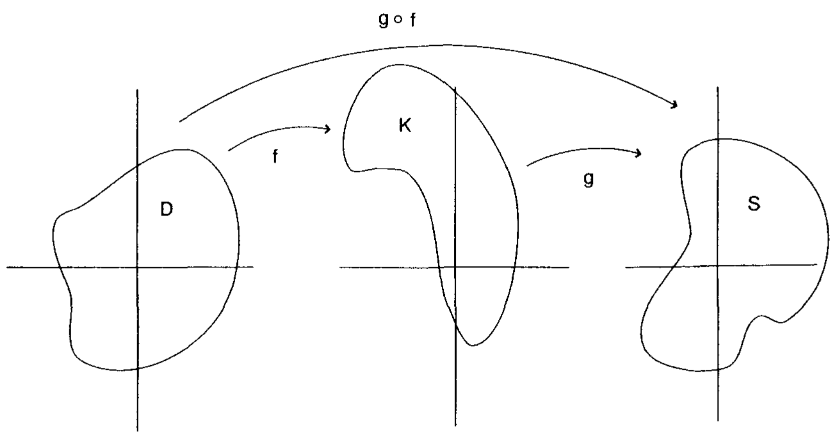

Figure 6.14: A composition of conformal mappings is conformal.

$f \circ g$ as z is mapped to $g(z)$ and then to $f(g(z))$. This is shown in Figure 6.14.

In using conformal mappings to solve Dirichlet problems, we need to construct conformal mappings between given domains. This can be a daunting task. However, a classical theorem due to Riemann at least implies the existence of such a mapping under very broad conditions.

Theorem 6.10 (*Riemann Mapping Theorem*) *Let K be a domain in the w - plane and assume that K is not the entire complex plane. Then there exists a one-to-one conformal mapping of the unit disk $|z| < 1$ onto K.* $\diamond$

This theorem guarantees the existence of a conformal mapping of the unit disk onto a given domain (that is not the plane). However, usually we want to map a domain D one-to-one onto a domain K. To use the theorem to conclude that there is such a mapping, put a copy of the unit disk in a third plane between the z - and w - planes, as in Figure 6.15. By the Riemann mapping theorem, there is a one-to-one conformal mapping F of this unit disk onto K, and there is also a one-to-one conformal mapping G in the other direction of this unit disk onto D. Then G^{-1} is a one-to-one conformal mapping of D onto the unit disk, and the composition $F \circ G^{-1}$ is a one-to-one conformal mapping of D onto K.

Problems for Section 6.17.1

1. Let $f(z) = z^2$. Prove that f maps the z - plane onto the w - plane and is not one-to-one. Show that the image of a vertical line $x = a$ and of a horizontal line $y = b$ are both parabolas in the w-plane. How do these

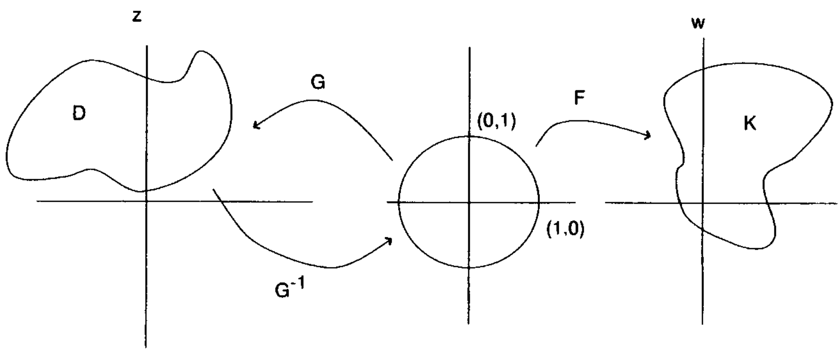

Figure 6.15: Constructing a conformal mapping $F \circ G^{-1} : D \to K$.

parabolas differ? Hint: Write $z = x + iy$ and $w = u + iv$, so

$$f(z) = (x + iy)^2 = x^2 - y^2 + 2ixy = u + iv.$$

Thus, in general, $u = x^2 - y^2$ and $v = 2xy$. Now for the case $x = a$, determine a relationship between u and v, yielding the image of the vertical line in the w - plane. Similarly, determine a relationship between u and v when $y = b$ to obtain the image of a horizontal line.

2. Let $f(z) = e^z$. Show that a vertical line $x = a$ maps to a circle of radius e^a about the origin in the w - plane, and a horizontal line $y = b$ maps to a half-line from the origin, making an angle of b radians with the positive real axis in the w - plane.

3. Let $f(z) = \cos(z)$. Show that a vertical line $x = a$ maps to a branch of the hyperbola
$$\frac{u^2}{\cos^2(a)} - \frac{v^2}{\sin^2(a)} = 1$$
provided that $\cos(a)$ and $\sin(a)$ are nonzero. Show that the image of a horizontal line $y = b$ is the ellipse
$$\frac{u^2}{\cosh^2(b)} + \frac{v^2}{\sinh^2(b)} = 1$$
provided that $b \neq 0$.

4. Show that the mapping

$$w = f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

maps the circle $|z| = r$ onto an ellipse with foci ± 1.

5. Show that the mapping $f(z) = \overline{z}$ is not conformal. Here, if $z = x + iy$, then $\overline{z} = x - iy$ is the conjugate of z.

6.17.2 Bilinear Transformations

We have just observed that there always exists a one-to-one conformal mapping of a given domain onto another domain that is not the entire plane. Now suppose that we are given two such domains and we want to produce a conformal mapping between them. There is no formula for doing this, but there are some techniques that are helpful. In particular, special conformal mappings called linear fractional transformations can sometimes be used.

A *bilinear* or *linear fractional transformation* is a function of the form

$$T(z) = \frac{az + b}{cz + d},$$

with a, b, c, and d complex numbers and $ad - bc \neq 0$. This condition assures that the mapping is one-to-one and has an inverse mapping. $T(z)$ is defined for all z except $-d/c$. Further,

$$T'(z) = \frac{ad - bc}{(cz + d)^2}$$

and this is nonzero for $z \neq -d/c$, hence T is a conformal mapping defined on the complex plane with the number $-d/c$ deleted.

There are three special bilinear transformations which are fundamental in a sense we will explain shortly.

Translation A mapping $T(z) = z + b$ is called a *translation.* If $b = \alpha + i\beta$, the effect of applying this mapping to z is to shift z horizontally by α and vertically by β. For example, $T(z) = z + 2 - i$ takes each complex number and moves it 2 units to the right and 1 unit down.

Rotation/Magnification A mapping

$$T(z) = az$$

is called a *rotation/magnification* if a is a nonzero complex number. If we write a and z in polar form as $a = Ae^{ib}$ and $z = re^{i\theta}$, then

$$T(z) = az = Are^{i(\theta+b)}.$$

Thus the image point $T(z)$ has magnitude Ar and argument $\theta + b$. The effect of T on z is to stretch (if $A > 1$) or shrink (if $0 < A < 1$) the distance from

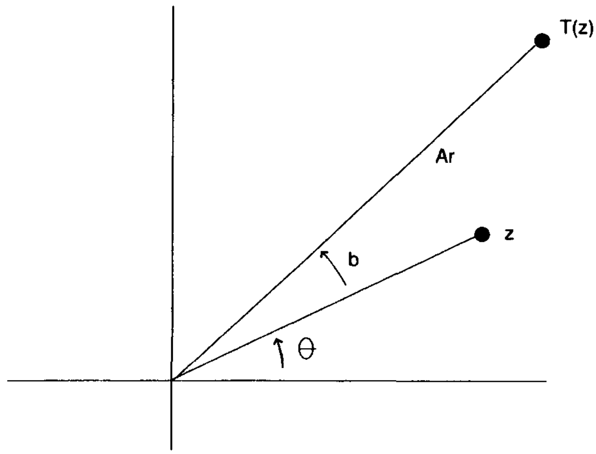

Figure 6.16: Magnification by $A > 1$ and counterclockwise rotation ($b > 0$).

the origin to z, and to rotate the line from the origin to z through an angle of b radians (counterclockwise if b is positive, clockwise if b is negative). This is shown in Figure 6.16 for the case $A > 1$ and $b > 0$, and in Figure 6.17 for $0 < A < 1$ and $b < 0$.

Inversion The mapping $T(z) = 1/z$ is an *inversion*. We have already seen inversions in disks of arbitrary radius, and this is the special case that the radius is 1. This inversion maps complex numbers interior to the unit circle (except the origin) to numbers in the exterior, and vice versa (Figure 6.18). If z_1 is on the unit circle, then $T(z_1) = 1/z_1 = \overline{z_1}$ is on the unit circle at the other intersection of the circle with the vertical line through z_1.

Any bilinear transformation can be achieved as a sequence of transformations of these three kinds. To see this, begin with

$$T(z) = \frac{az+b}{cz+d}$$

and consider two cases. If $c = 0$, then

$$T(z) = \frac{a}{d}z + \frac{b}{d}$$

and we can think of this as the end result of the sequence of mappings

$$z \xrightarrow{\text{rot/mag}} \frac{a}{d}z \xrightarrow{\text{translation}} \frac{a}{d}z + \frac{b}{d} = T(z).$$

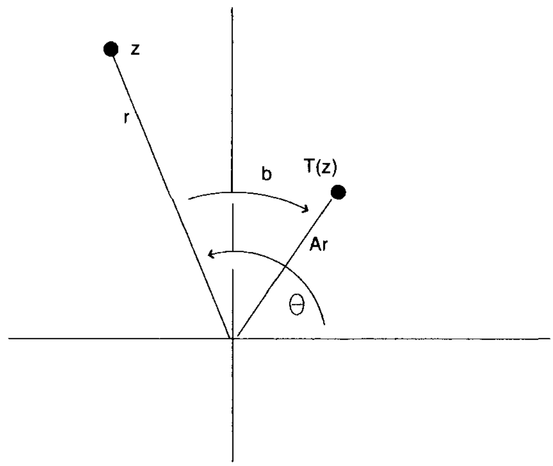

Figure 6.17: Magnification by $A < 1$ and clockwise rotation ($b < 0$).)

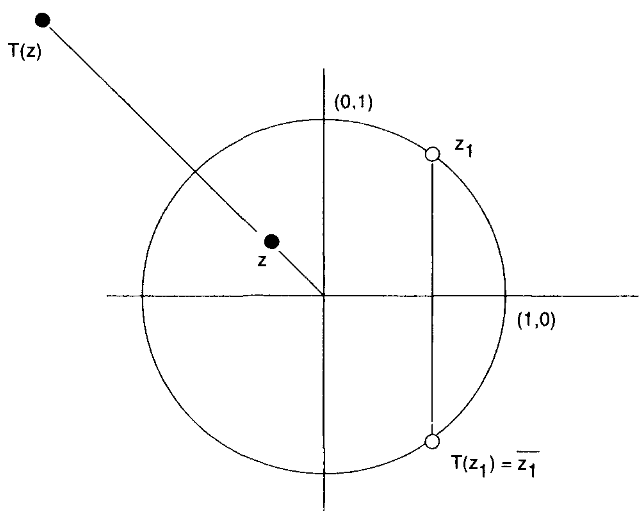

Figure 6.18: Inversion map.

If $c \neq 0$, then write

$$\begin{aligned} z &\xrightarrow{\text{rot/mag}} cz \xrightarrow{\text{translation}} cz+d \xrightarrow{\text{inversion}} \frac{1}{cz+d} \\ &\xrightarrow{\text{rot/mag}} \frac{bc-ad}{c}\frac{1}{cz+d} \xrightarrow{\text{translation}} \frac{bc-ad}{c}\frac{1}{cz+d}+\frac{a}{c} \\ &= \frac{az+b}{cz+d} = T(z). \end{aligned}$$

This means that any bilinear transformation can be "factored" into simpler components. This is sometimes useful in analyzing a particular transformation or in constructing a transformation to certain specifications. This way of decomposing bilinear transformations is also important in understanding properties of these mappings.

Theorem 6.11 *Let T be a bilinear transformation. Then T maps a circle to a circle or straight line, and a straight line to a circle or straight line.* $\diamond$

It is obvious that a translation maps a circle to a circle and a line to a line. Similarly, a rotation/magnification maps a circle to a circle and a line to a line. We need to determine the effect of an inversion on a circle or line. Take an arbitrary circle or line in the plane having equation

$$A(x^2+y^2)+Bx+Cy+R=0.$$

This is a straight line if $A = 0$ and a circle otherwise, except for degenerate cases (such as $B = C = R = 0$). If $z = x + iy$, this equation can be written

$$A|z|^2+\frac{B}{2}(z+\overline{z})+\frac{C}{2i}(z-\overline{z})+R=0.$$

Now let $w = 1/z$ to see the effect of an inversion on this equation. The image of the locus of this equation in the w - plane is

$$A\frac{1}{|w|^2}+\frac{B}{2}\left(\frac{1}{w}+\frac{1}{\overline{w}}\right)+\frac{C}{2i}\left(\frac{1}{w}-\frac{1}{\overline{w}}\right)+R=0.$$

Upon multiplying by $w\overline{w}$, which is the same as $|w|^2$, this equation becomes

$$R|w|^2+\frac{B}{2}(w+\overline{w})-\frac{C}{2i}(w-\overline{w})+A=0,$$

and this is a circle if $R \neq 0$ and a line if $R = 0$ and B and C are not both zero.

With this as background, we now claim that we can always produce a bilinear transformation mapping three given points to three given points.

Theorem 6.12 *Let z_1, z_2, z_3 be distinct complex numbers, and w_1, w_2, w_3 distinct complex numbers. Then there is a bilinear transformation T such that $T(z_j) = w_j$ for $j = 1, 2, 3$.* $\diamond$

To obtain such a transformation explicitly, think of $w = T(z)$ and solve for w in the equation

$$\begin{aligned} &(w_1-w)(w_3-w_2)(z_1-z_2)(z_3-z) \\ &= (z_1-z)(z_3-z_2)(w_1-w_2)(w_3-w). \end{aligned} \qquad (6.31)$$

For example, suppose that we want to map $3 \to i, 1-i \to 4$, and $2-i \to 6+2i$. Let

$$z_1 = 3, z_2 = 1-i, z_3 = 2-i,$$
$$w_1 = i, w_2 = 4, w_3 = 6+2i.$$

Make these substitutions into equation 6.31 to get

$$(i-w)(2+2i)(2+i)(2-i-z) = (3-z)(1)(i-4)(6+2i-w).$$

Upon solving for w, we obtain

$$w = T(z) = \frac{(20+4i)z - (16i+68)}{(6+5i)z - (22+7i)}.$$

It is routine to check that this mapping sends the three given points to their respective targets.

Problems for Section 6.17.2

1. Show that the inversion $T(z) = 1/z$ maps the vertical line $x = a \neq 0$ to the circle
$$\left(u - \frac{1}{2a}\right)^2 + v^2 = \frac{1}{4a^2}.$$

2. Show that the inverse of a bilinear mapping
$$T(z) = \frac{az+b}{cz+d}$$
is also a bilinear mapping.

In each of Problems 3 through 8, find a bilinear mapping that sends the given points to the images indicated.

3. $1 \to 1, 2 \to -i, 3 \to 1+i$

4. $i \to i, 1 \to -i, 2 \to 0$

5. $1 \to 1+i, 2i \to 3-i, 4 \to 4$

6. $6+i \to 2-i, i \to 3i, 5 \to -i$

7. $1 \to 6-4i, 1+i \to 2, 3+4i \to -2$

8. $2 \to -3i, 1 \to 1-i, 2+i \to 0$

In each of Problems 9 through 14, write the bilinear mapping as a sequence of mappings, each of which is a translation, rotation/magnification or inversion.

9.
$$T(z) = \frac{iz-4}{z}$$

10.
$$T(z) = \frac{z-1}{z+3+i}$$

11.
$$T(z) = \frac{z-4}{2z+i}$$

12.
$$T(z) = i(z+6) - 2 + i$$

13.
$$T(z) = \frac{(-2+3i)z}{z+4}$$

14.
$$T(z) = \frac{6i}{z+8}$$

6.17.3 Construction of Conformal Mappings Between Domains

In attempting to construct a conformal mapping between given domains, the following observation is very useful: *A conformal mapping of a domain D onto a domain K will map the boundary of D to the boundary of K.*

We use this fact as follows. Suppose that D is bounded by a piecewise smooth curve C_D (not necessarily closed) which separates the z - plane into two domains, D and D^*. These are called *complementary domains*. Similarly, suppose that K is bounded by a piecewise smooth curve C_K in the w - plane, separating this plane into complementary domains K and K^* (Figure 6.19). Try to find a conformal mapping f that sends points of C_D to points of C_K. This may be easier than trying to find a mapping of the entire domain directly. Such an f will then send D to either K or to K^*. To determine which, choose any convenient $z_0 \in D$. If $f(z_0)$ is in K, then $f : D \to K$ and f is a suitable conformal mapping. If $f(z_0)$ is in K^*, then $f : D \to K^*$ and f is not the mapping we want. However, it is often possible to produce the conformal mapping we want from f, sometimes by using an inversion.

To sum, first try to produce a mapping between boundaries, then use this to define a conformal mapping between the given domains. We consider some examples.

Example 6.6 *Map the disk $|z| < 1$ conformally onto the disk $|w| < 3$.*

Clearly, all we need to do here is expand the unit disk. Use the magnification $f(z) = 3z$ (Figure 6.20). ◇

Notice that this mapping carries the boundary $|z| = 1$ of D onto the boundary $|w| = 3$ of K. Here C_D is the unit circle $|z| = 1$, and this separates the z - plane into the complementary domains $D : |z| < 1$ and $\mathcal{D} : |z| > 1$. The complementary domains in the w - plane are $K : |w| < 3$ and $\mathcal{K} : |w| > 3$, both having the circle $C_K : |w| = 3$ as boundary. Now pick any point in D, say $z = 0$. Since $f(0) = 0$ is in $|w| < 3$, then f maps $|z| < 1$ onto $|w| < 3$. f also

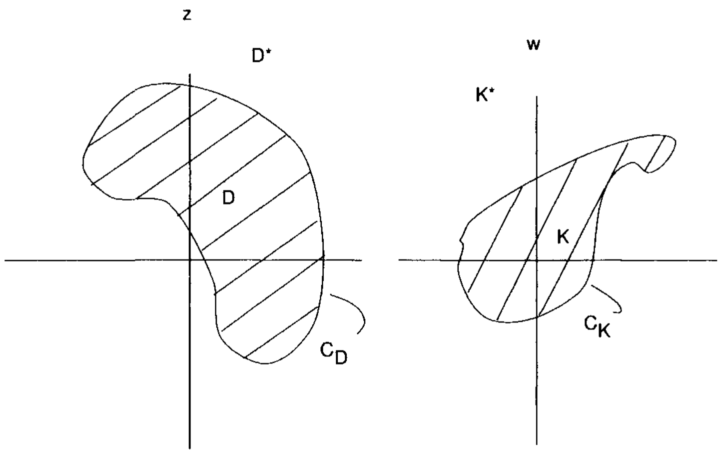

Figure 6.19: Complementary domains in the $z-$ and $w-$ planes.

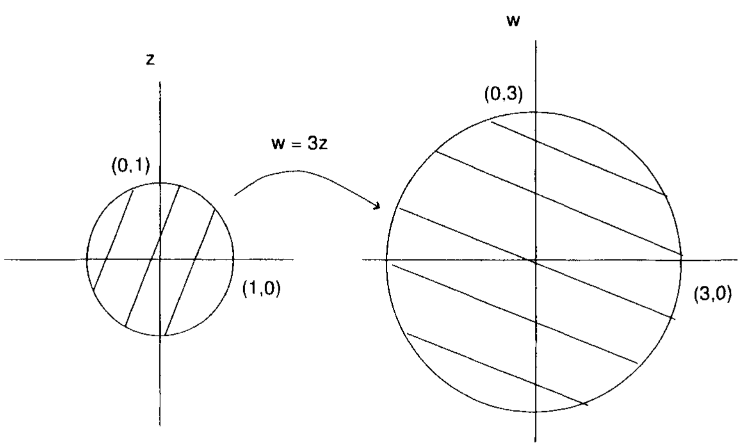

Figure 6.20: Mapping of $\mid z \mid< 1$ onto $midw \mid< 3$.

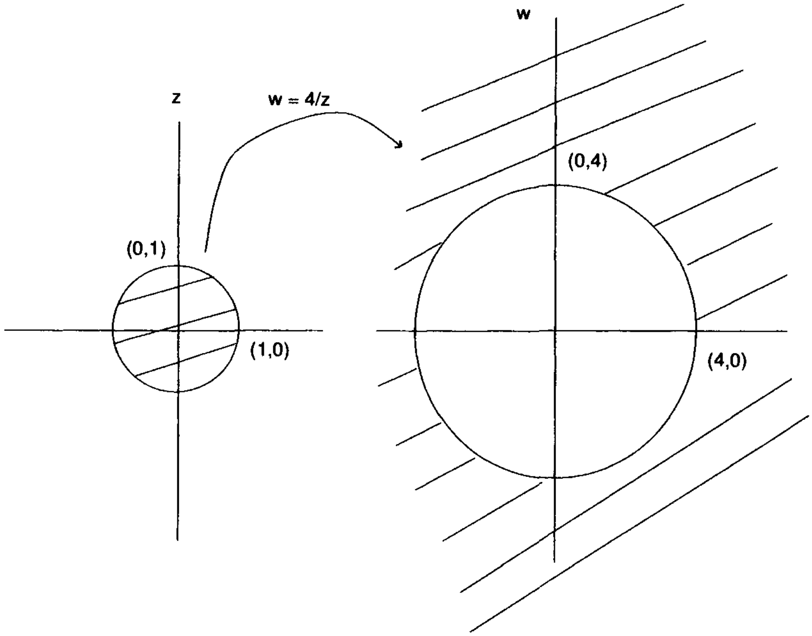

Figure 6.21: Mapping of $| z | < 1$ onto $| w | > 4$.

maps $|z| = 1$ onto $|w| = 3$. Although this conclusion is obvious in this simple example, it serves to illustrate the discussion of the preceding paragraph.

Example 6.7 *Map $|z| < 1$ conformally onto $|w| > 4$.*

We know that an inversion maps the interior of the unit disk to its exterior, so all we have to do is combine an inversion with a magnification by 4 to expand the radius to 4 (Figure 6.21). This suggests the mapping

$$w = f(z) = \frac{4}{z}. \diamond$$

To illustrate the preceding remarks again, notice that the boundary circle $|z| = 1$ maps to the circle $|w| = 4$. This circle bounds two complementary domains in the w - plane: the interior of this circle and its exterior. Since $f(1/2) = 8$ is exterior to $|w| = 4$, then f maps $|z| < 1$ to the exterior of $|w| = 4$, as we want.

Example 6.8 *Map the unit disk $|z| < 1$ onto the disk $|w - i| < 3$ of radius 3 centered at i.*

We want to expand the unit disk by a factor of 3 and move it up 1 unit to have center i. Construct the mapping in two steps, as suggested in Figure 6.22. Place an intermediate ζ - plane between the z - and w - planes and first map

$$\zeta = 3z.$$

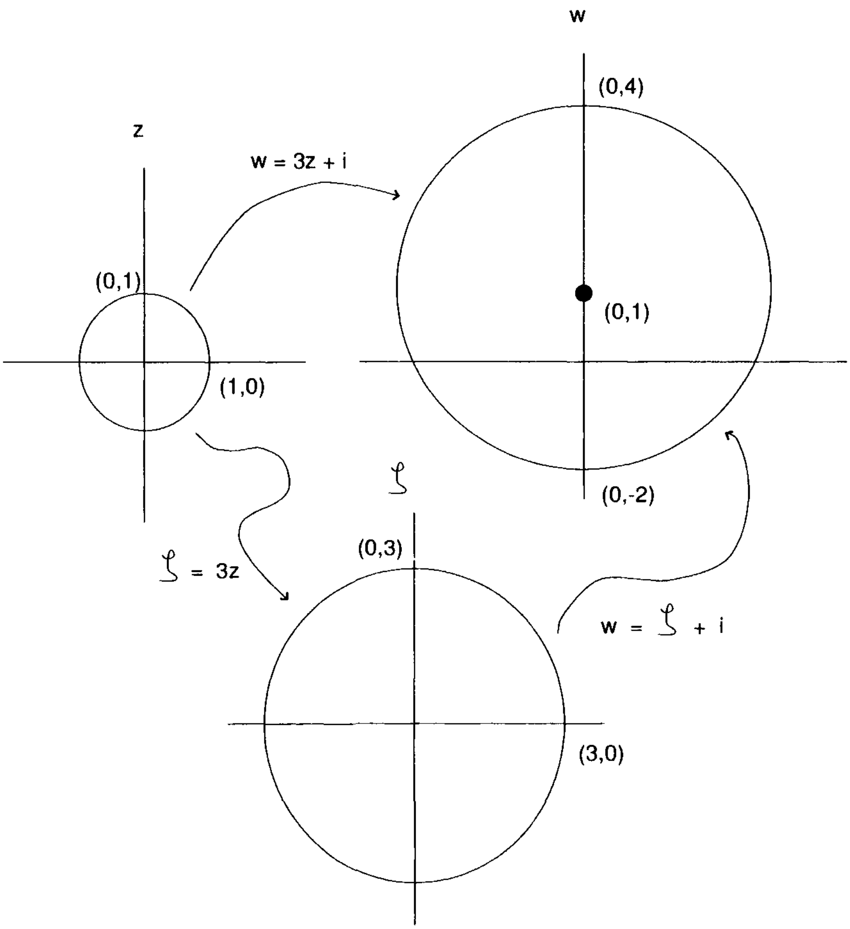

Figure 6.22: Mapping of $| z |< 1$ onto $| w - i |< 3$.

This expands the unit disk to the disk of radius 3 *about the origin in the* ζ *- plane. Now translate this circle 1 unit vertically to center it at* i *in the* w *- plane by putting*

$$w = \zeta + i.$$

The mapping we want is the composition of these mappings:

$$w = \zeta + i = 3z + i.$$

This takes $|z| = 1$ *to* $|w - i| = 3$, *since*

$$|w - i| = |3z| = 3|z| = 3$$

if $|z| = 1.\diamond$

Example 6.9 *Map the right half-plane* $\text{Re}(z) > 0$ *conformally onto the unit disk* $|w| < 1$.

The domains are shown in Figure 6.23. The boundary of the half-plane is the imaginary axis $\text{Re}(z) = 0$. *We will map this line onto the circle* $|w| = 1$, *the boundary of the target domain. One way to do this is to pick three points on* $\text{Re}(z) = 0$ *and map them to three points on* $|w| = 1$. *There is, however, a subtlety. To maintain positive orientation (counterclockwise on closed curves), choose three points in succession down the imaginary axis, so a person walking along these points sees the half-plane* $\text{Re}(z) > 0$ *on the left, and map these to points chosen counterclockwise (positive orientation) around* $|w| = 1$ *(a person walking around this circle counterclockwise sees the domain* $|w| < 1$ *on the left). To save on calculation, give a little thought to picking "simple" points.*

With this in mind, choose, say, $i, 0$, *and* $-i$ *moving down the vertical axis, and* $1, i$, *and* -1 *in the counterclockwise order around* $|w| = 1$. *Of course, infinitely many other choices can be made. Thus we will seek a bilinear transformation mapping*

$$i \to 1, 0 \to i, -i \to -1.$$

Solve for w *in the equation*

$$(1 - w)(-1 - i)(i)(-i - z) = (i - z)(-i)(1 - i)(-1 - w)$$

to obtain

$$w = T(z) = -i\left(\frac{z - 1}{z + 1}\right).$$

This mapping must take the right half-plane to either $|w| < 1$ *or to its complementary domain* $|w| > 1$. *To see which, choose a point in the right half-plane, say,* $z = 1$. *Now*

$$T(1) = 0\ .$$

is in $|w| < 1$, *so* T *is a conformal mapping sending* $\text{Re}(z) > 0$ *onto* $|w| < 1.\diamond$

Suppose in the last example that we had wanted to map $\text{Re}(z) > 0$ to the exterior of the unit circle, $|w| > 1$. Using the mapping just found onto the interior of the unit circle, we can now perform an inversion to obtain

$$f(z) = \frac{1}{T(z)} = i\left(\frac{z + 1}{z - 1}\right).$$

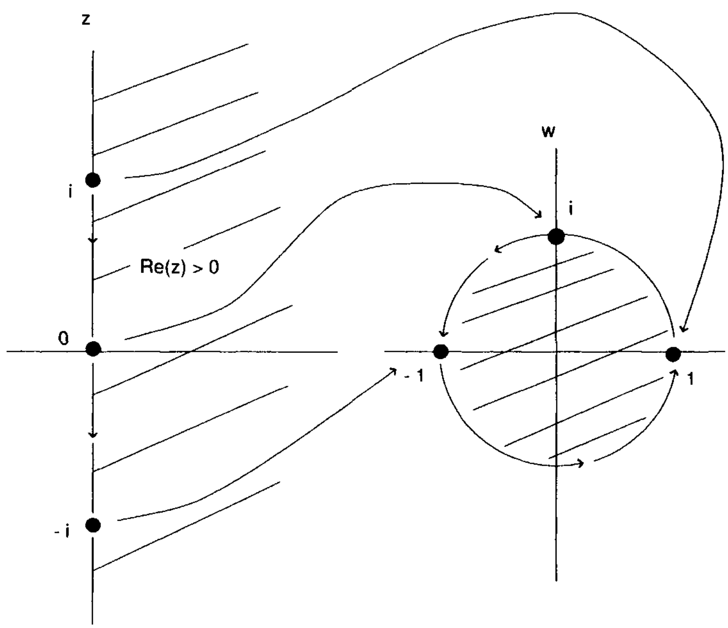

Figure 6.23: Constructing a conformal mapping of $Re(z) > 0$ onto $|\, w \,| < 1$.

For example, $1 + i$ is in the right half-plane, and its image under this mapping is

$$f(1+i) = i\left(\frac{2+i}{i}\right) = 2+i,$$

which lies in $|w| > 1$, as expected.

Example 6.10 *Map the right half-plane* $\text{Re}(z) > 0$ *conformally onto the disk* $|w - i| < 3$.

We can do this using a composition of mappings already formed. Put an intermediate ζ *- plane between the* z *- and* w *- planes. Map* $\text{Re}(z) > 0$ *onto* $|\zeta| < 1$ *by using the last example:*

$$\zeta = -i\left(\frac{z-1}{z+1}\right).$$

Next map $|\zeta| < 1$ *onto* $|w - i| < 3$ *by*

$$w = 3\zeta + i.$$

Finally, form the composition

$$\begin{aligned} w &= 3\zeta + i \\ &= -3i\left(\frac{z-1}{z+1}\right) + i = 2i\frac{2-z}{z+1}. \end{aligned}$$

The sequence of mappings is displayed in Figure 6.24. ◇

We cannot always achieve the mapping we want by a bilinear transformation. In particular, these always map {circles, lines} to {circles, lines}, and therefore have intrinsic limitations. For domains bounded by polygons, the Schwarz-Christoffel transformation, which is defined by an integral, is often used. However, at this point we return to the use of mappings to solve Dirichlet problems.

Problems for Section 6.17.3

1. Consider the mapping $f(z) = \sin(z)$. Show that the half-line $x = -\pi/2, y \geq 0$ maps to the interval $(-\infty, -1]$ on the real axis in the w - plane; that the half-line $x = \pi/2, y \geq 0$ maps to the interval $[1, \infty)$; and that the interval $[-\pi/2, \pi/2]$ on the real axis in the z - plane maps to $[-1, 1]$. Let S be the strip consisting of all z with $-\pi/2 < \text{Re}(z) < \pi/2$ and $\text{Im}(z) > 0$ in the z - plane. The boundary of this strip consists of the three line segments just mapped. Thus the boundary of S maps to the real axis in the w - plane. Using this fact and the preceding discussion of boundaries and mappings of domains, explain why f maps S onto the upper half-plane $\text{Im}(w) > 0$ in the w - plane.

2. Determine the image of the rectangle $0 \leq x \leq \pi, 0 \leq y \leq \pi$ under the mapping $w = e^z$.

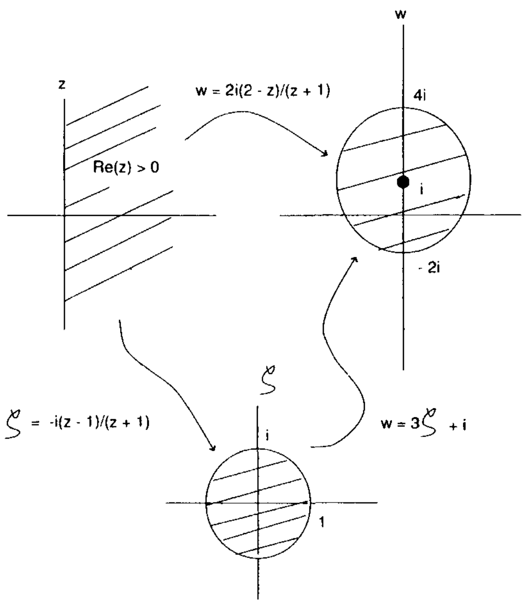

Figure 6.24: Conformal mapping of the right half-plane onto $| w - i |< 3$.

3. Determine the image of the rectangle $\pi/2 \leq x \leq \pi, 1 \leq y \leq 3$ under the mapping $w = \cos(z)$.

4. Find a conformal mapping of $|z| < 3$ onto $|w - 1 + i| > 6$. Hint: Pick three points in counterclockwise order on $|z| = 3$, and map these by a bilinear transformation to three points chosen in clockwise order on $|w - 1 + i| = 6$. There are infinitely many ways to do this.

5. Find a conformal mapping of $|z + 2i| < 1$ onto $|w - 3| > 2$.

6. Find a conformal mapping of $\text{Re}(z) > 1$ onto $\text{Im}(w) > -1$. Hint: Choose three points (in the right order) along the boundary of $\text{Re}(z) > 1$ and map them to three chosen points (in the right order) along the boundary of $\text{Im}(w) > -1$.

7. Find a conformal mapping of $\text{Re}(z) < -4$ onto $|w + 1 - 2i| > 3$.

8. Find a conformal mapping of $|z - 1 + 3i| > 1$ onto $\text{Re}(w) < -5$.

6.17.4 An Integral Solution of the Dirichlet Problem for a Disk

We have Poisson's integral formula for the solution of the Dirichlet problem for a disk. We will use complex analysis to derive a different integral representation of this solution (from which Poisson's formula follows easily). This alternative representation is particularly suited to using conformal mappings to write solutions of Dirichlet problems.

Begin with a function u that is harmonic in the disk $|u| < 1 + \epsilon$, slightly larger than the unit disk. We will derive an integral representation for $u(x, y)$.

We have already observed that the real and imaginary parts of any analytic complex function are harmonic. It is also true that given a harmonic function u, there is a harmonic function v (the *harmonic conjugate* of u) such that $f = u+iv$ is an analytic complex function on $|z| < 1 + \epsilon$. Further, by adding a constant if necessary, we may choose v so that $v(0,0) = 0$. It is possible to write a formula for v, but we want v only as leverage to employ complex function methods. Only u appears in the final result.

Expand f in a Maclaurin series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then

$$\begin{aligned} u(x,y) = \mathrm{Re}[f(x+iy)] &= \frac{1}{2}\left(f(z) + \overline{f(z)}\right) \\ &= \frac{1}{2}\sum_{n=0}^{\infty}(a_n z^n + \overline{a_n}\overline{z^n}) = a_0 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n z^n + \overline{a_n}\overline{z^n}). \end{aligned}$$

Now let ζ be on the unit circle γ. Then $|\zeta|^2 = \zeta\overline{\zeta} = 1$, so $\overline{\zeta} = 1/\zeta$ and

$$u(\zeta) = a_0 + \frac{1}{2}\sum_{n=0}^{\infty}(a_n\zeta^n + \overline{a_n}\zeta^{-n}).$$

Choose any integer m, multiply this equation by $\zeta^m/2\pi i$, and integrate over γ. The series and the integral can be interchanged within the open disk of convergence, and we obtain

$$\begin{aligned} \frac{1}{2\pi i}\oint_\gamma u(\zeta)\zeta^m\,d\zeta &= \frac{a_0}{2\pi i}\oint_\gamma \zeta^m\,d\zeta \\ &+ \frac{1}{2}\frac{1}{2\pi i}\sum_{n=1}^{\infty}\left(a_n\oint_\gamma \zeta^{n+m}\,d\zeta + \overline{a_n}\oint_\gamma \zeta^{-n+m}\,d\zeta\right). \end{aligned} \tag{6.32}$$

But

$$\oint_\gamma \zeta^k\,d\zeta = \begin{cases} 0 \text{ if } k \neq -1 \\ 2\pi i \text{ if } k = -1. \end{cases}$$

Thus, if $m = -1$ in equation 6.32, we get

$$\frac{1}{2\pi i}\oint_\gamma u(\zeta)\frac{1}{\zeta}\,d\zeta = a_0,$$

and if $m = -n - 1$ with $n = 1, 2, \cdots$, we get

$$\frac{1}{2\pi i} \oint_\gamma u(\zeta)\zeta^{-n-1}\, d\zeta = \frac{1}{2} a_n.$$

Substitute these coefficients into the Maclaurin series for f, keeping in mind that ζ is the variable of integration on γ, to obtain

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n z^n \\ &= \frac{1}{2\pi i} \oint_\gamma u(\zeta)\zeta^{-1}\, d\zeta + \sum_{n=1}^{\infty} \frac{1}{\pi i} \oint_\gamma u(\zeta)\zeta^{-n-1}\, d\zeta z^n \\ &= \frac{1}{2\pi i} \oint_\gamma \left[1 + 2\sum_{n=1}^{\infty} \left(\frac{z}{\zeta}\right)^n\right] \frac{u(\zeta)}{\zeta}\, d\zeta. \end{aligned}$$

Since $|z| < 1$ and $|\zeta| = 1, |z/\zeta| < 1$ and the geometric series in the integrand converges to the familiar result

$$\sum_{n=1}^{\infty} \left(\frac{z}{\zeta}\right)^n = \frac{z/\zeta}{1 - z/\zeta} = \frac{z}{\zeta - z}.$$

Therefore,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_\gamma \left(1 + \frac{z}{\zeta - z}\right) \frac{u(\zeta)}{\zeta}\, d\zeta \\ &= \frac{1}{2\pi i} \oint_\gamma u(\zeta) \left(\frac{\zeta + z}{\zeta - z}\right) \frac{1}{\zeta}\, d\zeta. \end{aligned}$$

If values of u are prescribed on the boundary circle γ, say $u(\zeta) = g(\zeta)$ for a given function g, then for $|z| < 1$,

$$f(z) = \frac{1}{2\pi i} \oint_\gamma g(\zeta) \left(\frac{\zeta + z}{\zeta - z}\right) \frac{1}{\zeta}\, d\zeta. \tag{6.33}$$

This integral formula determines $f(z)$ at points in the open unit disk, given values of $\text{Re}[f(z)]$ on the boundary unit circle. The solution of the Dirichlet problem for the unit disk, which asks for a harmonic function taking values given by g on the boundary unit circle, is retrieved from this formula as

$$u(x, y) = \text{Re}[f(x + iy)].$$

Problem for Section 6.17.4

1. Derive Poisson's integral formula by putting $z = re^{i\theta}$ and $\zeta = e^{i\varphi}$ into equation 6.33 and extracting the real part of the resulting expression.

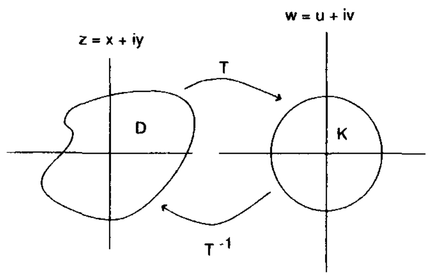

Figure 6.25: Constructing a conformal mapping to solve a Dirichlet problem.

6.17.5 Solution of Dirichlet Problems by Conformal Mapping

We are now poised to solve Dirichlet problems by using conformal mappings. Equation 6.33 was derived with this in mind. The idea is to map a Dirichlet problem for a domain D in the z - plane to a Dirichlet problem for the unit disk in the w - plane. We can solve the problem for the unit disk, then map this solution back to the z - plane for the original problem on D. This process appears as a change of variables in the integral solution for the disk.

Now for the details. Suppose that we know a differentiable, one-to-one conformal mapping $T : D \to K$, with K the unit disk $|w| < 1$, as in Figure 6.25. Assume that T maps the boundary curve C of D onto the unit circle γ bounding K and that T^{-1} is also a differentiable conformal mapping. To help follow the notation, we use ζ to denote an arbitrary point of γ, ξ for an arbitrary point of C, and $(\widetilde{x}, \widetilde{y})$ for an arbitrary point of the w - plane.

Suppose that we want to solve a Dirichlet problem for D:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \text{ for } (x,y) \in D \\ u(x,y) &= g(x,y) \text{ for } (x,y) \in C. \end{aligned}$$

If $w = T(z)$, then $z = T^{-1}(w)$. Define

$$\widetilde{g}(w) = g(T^{-1}(w)) = g(z)$$

for z on C and consider the Dirichlet problem on the unit disk $|w| < 1$:

$$\begin{aligned} \widetilde{u}_{xx} + \widetilde{u}_{yy} &= 0 \text{ for } |\widetilde{x} + i\widetilde{y}| < 1 \\ \widetilde{u}(\widetilde{x}, \widetilde{y}) &= \widetilde{g}(\widetilde{x}, \widetilde{y}) \text{ for } |\widetilde{x} + i\widetilde{y}| = 1. \end{aligned}$$

From equation 6.33 the solution of this problem for the disk in the w - plane is the real part of

$$\widetilde{f}(w) = \frac{1}{2\pi i} \oint_{\gamma} \widetilde{g}(\zeta) \left(\frac{\zeta + w}{\zeta - w} \right) \frac{1}{\zeta} \, d\zeta.$$

Define

$$f(z) = \widetilde{f}(T(z)).$$

Then

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \widetilde{g}(\zeta) \left(\frac{\zeta + T(z)}{\zeta - T(z)} \right) \frac{1}{\zeta} \, d\zeta \tag{6.34}$$

and

$$u(x, y) = \text{Re}[f(x + iy)]$$

is the solution of the original Dirichlet problem for the domain D in the x, y - plane.

It is convenient to put the integral expression 6.34 entirely in terms of the boundary curve of D and variables in the original z - plane. Recalling that T maps C onto γ, set $\zeta = T(\xi)$ for ξ on C to change variables in the integral in equation 6.34. This yields

$$f(z) = \frac{1}{2\pi i} \int_C \widetilde{g}(T(\xi)) \left(\frac{T(\xi) + T(z)}{T(\xi) - T(z)} \right) \frac{1}{T(\xi)} T'(\xi) \, d\xi.$$

But by definition of $\widetilde{g}, \widetilde{g}(T(\xi)) = g(T^{-1}(T(\xi))) = g(\xi)$, so

$$f(z) = \frac{1}{2\pi i} \int_C g(\xi) \left(\frac{T(\xi) + T(z)}{T(\xi) - T(z)} \right) \frac{T'(\xi)}{T(\xi)} \, d\xi, \tag{6.35}$$

and the solution of the Dirichlet problem for D is

$$u(x, y) = \text{Re}[f(x + iy)].$$

We emphasize that the integral in this solution is over C, which need not be a closed curve. This is the reason for the notation $\int_C$ rather than $\oint_C$.

The Dirichlet Problem for the Right Half-Plane

We illustrate this discussion by solving the Dirichlet problem for the right half-plane:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \text{ for } x > 0, -\infty < y < \infty \\ u(0, y) &= g(0, y) \text{ for } -\infty < y < \infty, \end{aligned}$$

with g a given function. We could write $g(y)$ instead of $g(0, y)$, since the first coordinate plays no role in this function defined on the vertical boundary of the right half-plane. However, we denote points on this boundary as $(0, y)$, and hence have adopted this notation.

We know a conformal mapping of the right half-plane onto the unit disk:

$$w = T(z) = -i \left(\frac{z - 1}{z + 1} \right).$$

Other conformal mappings between these domains could also be used. Compute

$$T'(z) = \frac{-2i}{(z+1)^2}.$$

From equation 6.35, the solution of the Dirichlet problem is the real part of

$$f(z) = \frac{1}{2\pi i}\int_C g(\xi)\left(\frac{-i\frac{\xi-1}{\xi+1} - i\frac{z-1}{z+1}}{-i\frac{\xi-1}{\xi+1} + i\frac{z-1}{z+1}}\right)\frac{\xi+1}{-i(\xi-1)}\frac{-2i}{(\xi+1)^2}\,d\xi,$$

in which C is the boundary of the right half-plane (the imaginary axis). Although this integrand may appear formidable, routine algebra yields the more tractable expression

$$f(z) = \frac{1}{\pi i}\int_C g(\xi)\left(\frac{\xi z - 1}{\xi - z}\right)\frac{1}{\xi^2 - 1}\,d\xi.$$

On C, the vertical axis, $\xi = it$, where t varies from ∞ to $-\infty$. Remember that ξ must move down this axis to maintain positive orientation. We obtain

$$\begin{aligned} f(z) &= \frac{1}{\pi i}\int_{\infty}^{-\infty} g(it)\left(\frac{itz-1}{it-z}\right)\frac{-1}{1+t^2}\,dt \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} g(it)\left(\frac{itz-1}{it-z}\right)\frac{1}{1+t^2}\,dt. \end{aligned}$$

We must extract the real part of this integral. Since t is real, $it = (0,t)$ and $g(it) = g(0,t)$ is real. Further, $1/(1+t^2)$ is real. Therefore, the real part of the integral will depend on the real part of the term in large parentheses in the integrand. With $z = x + iy$, calculate

$$\begin{aligned} \frac{itz-1}{it-z} &= \frac{itx - ty - 1}{i(t-y) - x} \\ &= \left(\frac{itx - ty - 1}{i(t-y) - x}\right)\left(\frac{-i(t-y) - x}{-i(t-y) - x}\right) \\ &= \frac{tx(t-y) - itx^2 + ity(t-y) + txy + i(t-y) + x}{x^2 + (t-y)^2}. \end{aligned}$$

The real part of this expression is

$$\frac{x(1+t^2)}{x^2 + (t-y)^2}.$$

Putting everything together, we finally have the solution of the Dirichlet problem for the right half-plane:

$$\begin{aligned} u(x,y) &= \mathrm{Re}[f(x+iy)] \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} g(it)\frac{x(1+t^2)}{x^2 + (t-y)^2}\frac{1}{1+t^2}\,dt \\ &= \frac{x}{\pi}\int_{-\infty}^{\infty} g(0,t)\frac{1}{x^2 + (t-y)^2}\,dt. \end{aligned}$$

This is similar in form to the solution 6.16 for the upper half-plane.

As a specific example, consider the Dirichlet problem for the right half-plane:

$$u_{xx} + u_{yy} = 0 \text{ for } x > 0, -\infty < y < \infty$$
$$u(0, y) = e^{-|y|} \text{ for } -\infty < y < \infty.$$

The solution is

$$u(x, y) = \frac{x}{\pi} \int_{-\infty}^{\infty} \frac{1}{x^2 + (t-y)^2} e^{-|t|}\, dt.$$

Problems for Section 6.17.5

1. Use the conformal mapping method to write an integral solution for the Dirichlet problem for the upper half-plane. Compare this expression with equation 6.16.

2. Use the conformal mapping method to solve the Dirichlet problem for the right quarter-plane $x > 0, y > 0$.

3. Use a conformal mapping to write the solution of the Dirichlet problem for the disk $|z - z_0| < R$.

4. Use a conformal mapping to write the solution of the Dirichlet problem for the infinite strip S consisting of all (x, y) with $-\infty < x < \infty, -\pi/2 < y < \pi/2$. <u>Hint</u>: A conformal mapping of S onto the unit disk is given by
$$w = -i\left(\frac{e^z - 1}{e^z + 1}\right).$$

5. Use a conformal mapping to write an integral solution for the Dirichlet problem for the right half-plane $\text{Re}(z) > 1$. <u>Hint</u>: A translation maps this domain onto $\text{Re}(z) > 0$.

6. Use a conformal mapping to write an integral solution for the Dirichlet problem for the lower half-plane $\text{Im}(z) < 0$.

7. Use a conformal mapping to write an integral solution for the Dirichlet problem for the exterior of a disk $|z - z_0| > R$.

Beginning Partial Differential Equations, Second Edition
by Peter V. O'Neil

Chapter 7

Existence Theorems

In Chapter 6 we developed techniques for solving Dirichlet problems for certain sets and boundary functions. Lebesgue's example demonstrates that there are Dirichlet problems with no solution. We therefore seek some conditions on the domain and the boundary function which will guarantee that the corresponding Dirichlet problem has a solution. In this treatment we restrict ourselves to problems in the plane, although these results extend to higher dimensions.

Section 7.1 provides an existence theorem along classical lines. In Section 7.2 we develop a Hilbert space perspective of the Dirichlet problem, as an introduction to the modern function space approach to existence questions for partial differential equations. Finally, in Section 7.3, we introduce distributions and state a general existence theorem in the language of these generalized functions.

7.1 A Classical Existence Theorem

The method of proof we develop in this section is called the méthode de balayage, after the French mathematician Henri Poincaré, who conceived it. Because it was later refined by Perron, it is known today as the Poincaré-Perron method. The idea is to construct a set of (not necessarily harmonic) functions having the correct values on the boundary of the domain. From these a function that is harmonic on the domain, and takes on the appropriate boundary values, is manufactured, provided that at each point of the boundary, a condition we will formulate is satisfied.

As preparation for these constructions, we need two results about sequences of harmonic functions. The first states that uniform convergence of a sequence of harmonic functions on the boundary of a domain ensures that the sequence converges uniformly on the entire domain.

Theorem 7.1 (*Harnack's First Theorem*) *Let D be a bounded domain in the plane. Suppose that $u_n(x,y)$ is a sequence of functions harmonic on D and continuous on $\overline{D}$. Suppose that $u_n(x,y)$ converges uniformly on ∂D. Then $u_n(x,y)$ converges uniformly on $\overline{D}$ to a function that is harmonic on D.* $\diamond$

Proof Let $\epsilon > 0$. Because of the uniform convergence on ∂D, there is a positive number N such that

$$| u_n(x,y) - u_m(x,y) | < \epsilon$$

for $n, m \geq N$ and $(x,y) \in \partial D$. By the maximum principle applied to each u_n, we also have

$$| u_n(x,y) - u_m(x,y) | < \epsilon$$

for $n, m \geq N$ and $(x,y) \in D$. Therefore, $u_n(x,y)$ converges uniformly on $\overline{D}$. Suppose that this sequence converges to $U(x,y)$. It remains to show that U is harmonic on D. To do this, let $P_0 \in D$ and let C be a circle of radius R centered at P_0, with C and the disk it encloses contained in D. In polar coordinates centered at P_0, write $P_0 = \rho e^{i\theta}$. Because each u_n is harmonic, then

$$u_n(P_0) = \frac{1}{2\pi}\int_0^{2\pi} u_n(Re^{i\xi})\frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\xi-\theta)}\,d\xi.$$

By the uniform convergence of u_n, we can take the limit, as $n \to \infty$, through the integral to obtain

$$\begin{aligned}\lim_{n\to\infty} u_n(P_0) &= U(\rho e^{i\theta})\\ &= \frac{1}{2\pi}\int_0^{2\pi} U(Re^{i\xi})\frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\xi-\theta)}\,d\xi.\end{aligned}$$

This proves that U is harmonic on the disk bounded by C; hence U is harmonic on D. $\diamond$

Using the first Harnack theorem, we can prove the second.

Theorem 7.2(*Harnack's Second Theorem*) *Let u_n be a sequence of nonnegative functions that are harmonic on D. Suppose that this sequence converges at some point (a,b) in D. Then u_n converges on D to a harmonic function, and this convergence is uniform on each compact subset of D.* $\diamond$

Proof First show that u_n converges uniformly on every disk within D and centered at (a,b). Let D_1 be such a disk, having radius R and centered at (a,b), and assume that the closure of the disk D_2 of radius $R+\epsilon$ centered at (a,b) is also contained in D. By Harnack's inequality,

$$\frac{R+\epsilon-\rho}{R+\epsilon+\rho}u_n(a,b) \leq u_n(\rho e^{i\theta}) \leq \frac{R+\epsilon+\rho}{R+\epsilon-\rho}u_n(a,b)$$

for any point in D_2 having polar coordinates $\rho e^{i\theta}$ centered at (a,b). Therefore, convergence of u_n at (a,b) implies convergence on $\overline{D}_2$. By Harnack's first theorem, the limit function is also harmonic on D_2.

We now want to prove the convergence of u_n at an arbitrary point P of D. The strategy here is similar to that used to prove the maximum principle. Connect (a,b) to P by a polygonal path L in D. For some positive number δ, each point of L is at a distance at least δ from ∂D. Construct circles of radius $\delta/2$ and centers proceeding along L from (a,b) to P. Starting from the first circle (about (a,b)), u_n converges uniformly in this circle, hence in the next circle, and so on, until in a finite number of steps the last circle (about P) is reached. This proves convergence of u_n at P. Further, again by Harnack's first theorem, the limit function is harmonic about P.

Finally, it must be shown that convergence of u_n is uniform on each compact subset of D. Let F be such a subset. By compactness of F, it is possible to cover F by a finite union of open disks $O_1, \cdots, O_m$, with each $\overline{O}_j \subset D$. But u_n converges at the center of each of these disks, hence, by what has already been proved, uniformly on each $\overline{O}_j$, and therefore also uniformly on F. This completes the proof of Harnack's second theorem. $\diamond$

We can now develop the machinery to carry out the Poincaré-Perron approach to an existence theorem. For the remainder of this section, D is a bounded domain in the plane and f is continuous on ∂D. The argument exploits the fact that a function is harmonic on an open set D exactly when it is harmonic on every open disk within D. This will enable us to make use of the fact that we know an integral solution of a Dirichlet problem on a disk.

Now suppose that K is any open disk (that is, points enclosed by a circle but not including boundary points) within D, and v is continuous on $\overline{D}$. Since the Dirichlet problem for a circle has a solution (Poisson integral formula), there is a function $f_{K,v}$ which is harmonic on K, and $f_{K,v}(Q) = v(Q)$ for Q on the boundary circle of K. Extend $f_{K,v}$ to all of $\overline{D}$ by setting $f_{K,v}(P) = v(P)$ for $P \in \overline{D} - K$. This definition is illustrated in Figure 7.1. In this way we associate with each function continuous on $\overline{D}$, and each open disk K within D, a unique continuous function that is harmonic on K and agrees with v within the closure of D, but outside K (that is, on $\overline{D} - K$).

We write $v \leq w$ on D if $v(P) \leq w(P)$ for $P \in D$. In this notation it is sometimes useful to observe that if $v \leq w$ on D, then $f_{K,v} \leq f_{K,w}$ for each open disk $K \subset D$. This fact is easily seen using the Poisson integral formula. Suppose that K has radius R. If $P = \rho e^{i\theta}$ is in K, using polar coordinates with origin at the center of K, then

$$f_{K,v}(P) = \frac{1}{2\pi} \int_0^{2\pi} v(\xi) \frac{R^2 - \rho^2}{R^2 + \rho^2 - 2R\rho\cos(\theta - \xi)}\, d\xi$$

and similarly,

$$f_{K,w}(P) = \frac{1}{2\pi} \int_0^{2\pi} w(\xi) \frac{R^2 - \rho^2}{R^2 + \rho^2 - 2R\rho\cos(\theta - \xi)}\, d\xi.$$

Now, if $v(\xi) \leq w(\xi)$, the integrand in the expression for $f_{K,v}(P)$ is less than or equal to the integrand in the expression for $f_{K,w}(P)$; hence $f_{K,v} \leq f_{K,w}$ on K. Indeed,

$$f_{K,v}(P) = v(P) \leq w(P) = f_{K,w}(P)$$

for P on the boundary circle of K, so in fact, $f_{K,v} \leq f_{K,w}$ on $\overline{K}$, not just on the open disk K.

If v is continuous on $\overline{D}$, we say that v is *superharmonic* (in D) if $f_{K,v} \leq v$ for every open disk $K \subset D$. If this inequality is reversed so that $v \leq f_{K,v}$ for each

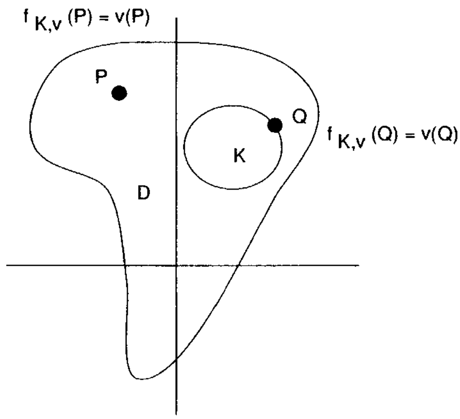

Figure 7.1: Definition of $f_{K,v}$.

open disk $K \subset D$, v is called *subharmonic*. We develop a sequence of properties of superharmonic and subharmonic functions, leading to an existence theorem.

(1) Every harmonic function on D is superharmonic on D and also subharmonic on D.

This follows immediately from the observation that v is harmonic on D exactly when $f_{K,v} = v$ for each open disk K within D.

(2) A finite sum of superharmonic functions is superharmonic, and a finite sum of subharmonic functions is subharmonic.

This is a consequence of the fact that

$$f_{K,v_1+v_2} = f_{K,v_1} + f_{K,v_2}.$$

(3) If v is superharmonic and u is harmonic on D, then $v + u$ and $v - u$ are superharmonic.

To prove this, first note by (1) that u is superharmonic, so by (2), $v + u$ is superharmonic. Similarly, $-u$ is harmonic, so the sum $v - u$ is superharmonic.

(4) If v is superharmonic and w is subharmonic, $v - w$ is superharmonic.

Let K be a disk within D. Because v is superharmonic, $f_{K,v} \leq v$ on K and because w is subharmonic, $w \leq f_{K,w}$ on K. Then $-f_{K,w} \leq -w$ on K, so

$$f_{K,v-w} = f_{K,v} - f_{K,w} \leq v - w$$

on K, showing that (4) is true.

(5) If v is superharmonic on $\overline{D}$, v assumes its minimum value over $\overline{D}$ on ∂D.

This result is reminiscent of the maximum principle for harmonic functions. To prove it, begin with the fact that the continuous function v assumes a minimum value on the compact set $\overline{D}$. We will assume that v is not constant. It is enough to show that this minimum value cannot be assumed at a point of the open set D, allowing us to conclude that in this nonconstant case it must be assumed on the boundary. Suppose then that v has its minimum value at $Q \in D$. For some P in D, $v(Q) < v(P)$. Draw an open disk K within D and about Q, but not containing P. Then

$$v(Q) < v(P) = f_{K,v}(P),$$

and this contradicts the superhomonicity of v. This proves (5).

So far the ideas of superharmonic and subharmonic functions have not been related to f. This is done through the concepts of upper and lower functions. Define v to be an *upper function* for f on D if v is superharmonic and $f \leq v$ on ∂D. Thus an upper function for f is a superharmonic function v that dominates f on the boundary of D. (Recall here that f is defined only on ∂D.) Notice that there must exist upper functions for f, since any constant function $v(P) = c$ that bounds f on $\overline{D}$ is such a function. In similar fashion, v is a *lower function* for f if v is subharmonic, and $v \leq f$ on ∂D.

The strategy now is to define a function u on $\overline{D}$ by setting, for $P \in \overline{D}$, $u(P)$ equal to the greatest lower bound of the numbers $v(P)$, taken over all upper functions v for f on D. Notice that this set of function values at P is bounded below (by $f(P)$), so this greatest lower bound always exists as a real number. We will show that u is harmonic on D and that with a certain condition which we will state, $u(P) = f(P)$ on the boundary of D. This will provide the existence criterion we seek and complete the proof of an existence theorem.

(6) If v is an upper function and w a lower function for f, then

$$w(P) \leq u(P)$$

for all $P \in \overline{D}$. That is, every upper function dominates every lower function on $\overline{D}$.

This conclusion follows directly from statements (4) and (5). Because v is superharmonic and w is subharmonic, then by (4), $v - w$ is superharmonic, and so by (5), $v - w$ assumes its minimum value on ∂D. But $w(P) \leq f(P) \leq v(P)$ for P on the boundary of D. Then $v(P) - w(P) \geq 0$ on the boundary, hence $v(P) - w(P) \geq 0$ on $\overline{D}$. This proves statement (6).

(7) If v is an upper function for f, and D_1 is an open disk within D, then $f_{D_1,v}$ is an upper function for f on D.

Denote $w = f_{K,v}$. We need to show that w is superharmonic and dominates f on the boundary of D.

First, if $P \in \partial D$, then $P \in \overline{D} - D_1$, so

$$f(P) \leq v(P) = f_{D_1,v} = w(P).$$

This shows that w dominates f on the boundary of D.

It remains to be shown that w is superharmonic. Let K be any open disk within D. We must show that $f_{K,w}(P) \leq w(P)$ on K. The proof of this lies in the consideration of cases on respective positions of K and D_1 within D. The four cases are:

Case 1: $K \subset D_1$.

Case 2: $K \subset \overline{D} - D_1$.

Case 3: $D_1 \subset K$.

Case 4: If none of cases 1 through 3 apply, the boundaries of K and D_1 intersect.

For case 1, suppose that $P \in K$ and K has radius R. Using polar coordinates from the center of K, write $P = \rho e^{i\theta}$. Because $f_{K,w} = w$ on the boundary of K,

$$f_{K,w}(P) = \frac{1}{2\pi} \int_0^{2\pi} w(Re^{i\xi}) \frac{R^2 - \rho^2}{R^2 + \rho^2 - 2R\rho\cos(\theta - \xi)}\, d\xi.$$

But because $D_1 \subset K$, this integral is also the Poisson integral representation of $w(P)$. Thus $f_{K,w} = w$ on K in this case.

We omit the details for cases 2 and 3, and move on to case 4. Suppose in particular that K and D_1 have a nonempty intersection, but neither is a subset of the other. First observe that $w \leq v$ on ∂K because v is superharmonic. This means that $f_{K,w} \leq f_{K,v}$ on K. But then, again because v is superharmonic, $f_{K,v} \leq v$. Now $w = v$ outside K, so $f_{K,w} \leq w$ outside D_1. (This applies to case 3 as well.) But w and $f_{K,w}$ are harmonic in $D_1 \cap K$, so $f_{K,w} \leq w$ on $\partial(D_1 \cap K)$, and hence also on $K \cap D_1$.

Next we claim that the minimum of any finite set of upper functions is also an upper function.

(8) Let $v_1, \cdots, v_n$ be upper functions for f on D. Let

$$v(P) = \min(v_1(P), \cdots, v_n(P))$$

for $P \in \overline{D}$. Then v is an upper function for f.

To prove (8), first suppose that $P \in \partial D$. Then $f(P) \leq v_j(P)$ for each j, so $f(P) \leq v(P)$. Thus v dominates f on the boundary of D.

To show that v is superharmonic, suppose that K is a disk within D. If $P \in K$, then $v(P) = v_j(P)$ for some j, so

$$v(P) = v_j(P) \geq f_{K,v_j}(P) = f_{K,v}(P).$$

We are now ready to produce a candidate for a solution of this Dirichlet problem on D.

(9) For $P \in D$, define $u(P)$ to be the greatest lower bound of the numbers $v(P)$ taken over all upper functions v for f on D. Then u is harmonic on D.

To prove this, it is enough to prove that u is harmonic on every open disk in D. Let K be such a disk, centered at P_0, and let ϵ be any positive number. There is an upper function v_1 that is harmonic on K and satisfies

$$v_1(P_0) \leq u(P_0 + \epsilon.$$

In fact, the definition of u guarantees the existence of an upper function v_1 satisfying this inequality, and if v_1 is not harmonic on K, we can replace v_1 by f_{K,v_1}, which is also an upper function satisfying this inequality, and is harmonic on K.

Now take w_1 to be an upper function satisfying

$$w_1(P_0) \leq u(P_0) + \frac{\epsilon}{2}.$$

Define $v_2 = f_{K,\min(v_1,w_1)}$. By (7) and (8), v_2 is an upper function for f on D and is harmonic on K.

Continuing in this way, we obtain a decreasing sequence of upper functions $v_1, v_2, \cdots$ which are harmonic on K and bounded below (because every upper function is at least as large as every lower function). By Harnack's second theorem, there is a function v which is the uniform limit of $v_1, v_2, \cdots$ on K. Therefore, v is also harmonic on K.

To show that u is harmonic on D, we will now show that $v = u$ on K, keeping in mind that K is an arbitrarily chosen disk within D.

Suppose that the equality $v = u$ on K is false. Then there must be an upper function w and a point $Q \in K$ such that

$$w(Q) < v(Q).$$

Let the distance between P_0 and Q be r and consider the disk K_Q of radius r centered at P_0. Then Q is on the boundary of K_Q. Define

$$w_n = f_{K_Q,\min(w,v_n)}.$$

Then each w_n is an upper function for f on D. Now we know that v_n converges uniformly to v on $\overline{K_Q}$. Therefore, w_n also converges uniformly on $\overline{K_Q}$. By choosing n sufficiently large, we can make $w_n(P_0)$ as close as we like to $f_{K_Q,min(w,v_n)}(P_0)$, and

$$f_{K_Q,\min(w,v_n)}(P_0) < v(P_0) = u(P_0).$$

But this contradicts the definition of $u(P_0)$ as the greatest lower bound of all the upper functions for f on D, evaluated at P_0. This completes the proof that u is harmonic on D.

If all we wanted was a function harmonic on D, we would not need all of the machinery we have developed. The issue is to produce a harmonic function agreeing with f on the boundary of D. We will now show that $u = f$ on ∂D for the function u produced in (9) if a certain condition is satisfied at each point of ∂D.

To this end, we introduce the notion of a *barrier function* at a boundary point Q as a function that is superharmonic and positive at all points of $\overline{D}$ except at Q, where it has the value zero. More carefully, if $Q \in \partial D$, a function ω_Q is called a *barrier function* (for D at Q) if the following conditions hold:

(i) ω_Q is continuous on $\overline{D}$.

(ii) ω_Q is superharmonic.

(iii) $\omega_Q(Q) = 0$.

(iv) $\omega_Q(P) > 0$ for all $P \in \overline{D}$ different from Q.

We will now prove that u and f agree at each boundary point of D at which a barrier function exists.

(10) Let $Q \in \partial D$. Suppose that a barrier function ω_Q for D exists. Then $u(Q) = f(Q)$.

To prove this, begin by letting ϵ be any positive number. Because f is continuous on ∂D, there is some open disk D_Q about Q such that

$$f(Q) - \epsilon \leq f(P) \leq f(Q) + \epsilon$$

for $P \in \partial D_Q \cap \partial D$. Now, by assumption, there is a barrier function ω_Q. Define a new function F_U on $\overline{D}$ by

$$F_U(P) = f(Q) + \epsilon + C\omega_Q(P).$$

Here C is, for the moment, any positive number.

We claim first that F_U is superharmonic on D. The reason for this is that any constant function is superharmonic and ω_Q is superharmonic, so (with C positive) $C\omega_Q$ is superharmonic and a sum of superharmonic functions is superharmonic. This holds for any positive number C.

Next, notice that by choosing C sufficiently large, $f(P) \leq F_U(P)$ for all $P \in \partial D$. Therefore, F_U is an upper function for f on D.

Next, define

$$F_L(P) = f(Q) - \epsilon - C\omega_Q(P).$$

By an argument like that used for F_U, we can show that for sufficiently large C, F_L is a lower function for f on D.

For the rest of this proof, suppose that C has been chosen sufficiently large that F_U is an upper function and F_L is a lower function for f on D, for the same C. Then, because $\omega_Q(Q) = 0$, we have $F_L(Q) = f(Q) - \epsilon$ and $F_U(Q) = f(Q) + \epsilon$, so

$$f(Q) - \epsilon = F_L(Q) \leq u(Q) \leq F_U(Q) = f(Q) + \epsilon.$$

Since ϵ can be chosen arbitrarily small, this implies that $u(Q) = f(Q)$, completing the proof.

We will summarize this discussion in an existence theorem.

Theorem 7.3 *Suppose that D is a bounded domain in the plane, f is continuous on ∂D, and there is a barrier function ω_Q for each $Q \in \partial D$. Then there is a continuous function u that is harmonic on D and agrees with f on ∂D.* $\diamond$

We have proved this result for the Dirichlet problem in the plane because it is easier to visualize circles and disks in the plane than their higher-dimensional analogues. However, the definitions and arguments pass over almost verbatim to higher dimensions, providing an existence theorem for the Dirichlet problem for a domain in R^n.

As an application of Theorem 7.3 we will prove that if D is bounded by a simple closed curve, the Dirichlet problem for D, with continuous data function on ∂D, has a solution.

To prove this, we need only produce a barrier function at each point of ∂D, and the assumption that ∂D is a simple closed curve enables us to do this. To begin, suppose that D is entirely within the right quarter-plane $x > 0, y > 0$. Let $Q = (x_0, y_0) \in \partial D$, and define

$$\varphi(x,y) = \ln\left(\frac{\sqrt{x^2+y^2}}{2\delta}\right)$$

and

$$\psi(x,y) = \arctan\left(\frac{y}{x}\right),$$

with δ the diameter of the bounded domain D. This diameter is defined to be the greatest lower bound of the distances between pairs of points of D, and is a real number because D is bounded. Next define

$$\omega_Q(x,y) = -\frac{\varphi(x-x_0, y-y_0)}{(\varphi(x-x_0,y-y_0))^2 + (\psi(x-x_0,y-y_0))^2}.$$

By definition of δ, we have

$$(x-x_0)^2 + (y-y_0)^2 \leq \delta^2$$

for each $(x,y) \in \overline{D}$. The introduction of δ into the definition of $\varphi(x,y)$ guarantees that $\varphi(x,y) < 0$, and therefore $\omega_Q(x,y) > 0$ for (x,y) in $\overline{D}$ and $(x,y) \neq (x_0,y_0)$.

Now $\omega_Q(x,y)$ is not defined at (x_0,y_0). However, it is easy to verify that as $(x,y) \to (x_0,y_0)$, $\omega_Q(x,y) \to 0$, so we can define $\omega_Q(x_0,y_0) = 0$.

Finally, it is routine to check that ω_Q is a barrier function at Q for each point on ∂D. Therefore, the Dirichlet problem for this domain has a solution if the function specified on the boundary is continuous.

If D is not entirely within the upper right quarter-plane, we can translate D to this quadrant and use the fact that translates of harmonic functions are harmonic. The translated problem has a solution, by the argument just given. This solution can then be translated back to a solution for the original domain.

Problems for Section 7.1

1. Prove that assertion that v is harmonic on D if and only if $f_{K,v} = v$ for every open disk $K \subset D$.

2. Prove the assertion made in (2) that
$$f_{K,v_1+v_2} = f_{K,v_1} + f_{K,v_2}.$$

3. Write out the details of the proof of (7) in cases 2 and 3.

4. Suppose that the Dirichlet problem for a domain D in the plane has a solution. Let D^* be a translation of D. Prove that the Dirichlet problem for D^* also has a solution.

5. (Substantial Project) The proof of the existence theorem focused on superharmonic functions and upper functions. Develop a separate proof of an existence theorem using subharmonic functions and lower functions, with the solution $u(P)$ at P defined as the least upper bound of the numbers $v(P)$ taken over all lower functions of f on D. A derivation can follow the treatment using upper functions but reversing inequalities where appropriate to base the solution on lower functions.

7.2 A Hilbert Space Approach

In this section we outline another approach to the existence of a solution to the Dirichlet problem, through the theory of linear functionals defined on an algebraic and topological structure called a Hilbert space. Although the discussion focuses on the Dirichlet problem, the ideas are widely applicable to the study of partial differential equations in general. Our purpose is to show how this approach leads naturally to certain kinds of function spaces and generalizations of functions called distributions, which we discuss in the next section.

To begin this program, we assume familiarity with the idea of a vector space over the real number field. The standard example of such a space is R^n, the vector space of n - tuples $(x_1, x_2, \cdots, x_n)$ with each x_j a real number. Another example is $C([a, b])$, the vector space of real-valued continuous functions defined on $[a, b]$. R^n has dimension n, while $C([a, b])$ is infinite dimensional.

Some vector spaces have defined on them an *inner product.* This operation associates with each pair of objects x and y in the space, a real number denoted $< x, y >$, called the inner product of x with y. (If the vector space is over the complex field, the inner product can be complex valued, with appropriate alterations in the definition.) The inner product operation is assumed to satisfy the following conditions.

(1) The inner product is commutative:
$$< x, y > = < y, x > .$$

(2) The inner product is linear. If α and β are scalars (real numbers) and x, y, and z are vectors, then

$$< \alpha x + \beta y, z > = \alpha < x, z > + \beta < y, z > .$$

(3) For any nonzero vector x, $< x, x >$ is positive.

(4) If O denotes the zero vector, $< O, O > = 0$.

When such an operation is defined, we call the vector space an *inner product space*.

In R^n, the usual dot product is an inner product. In the plane, for vectors $x = (a_1, a_2)$ and $y = (b_1, b_2)$, this inner product is

$$< x, y > = a_1 b_1 + a_2 b_2.$$

This standard inner product on R^2 admits various geometric interpretations to accompany its algebraic properties. One of these is orthogonality (perpendicularity) of vectors. In the plane, two vectors x and y are orthogonal exactly when their inner product is zero. We extend this terminology to arbitrary inner product spaces, referring to x and y as orthogonal when $< x, y > = 0$.

In $C([a, b])$ we can define an inner product by

$$< f, g > = \int_a^b f(x)g(x)dx.$$

Properties (1) through (4) of the inner product are easily verified. In this space, two functions f and g are orthogonal when

$$< f, g > = \int_a^b f(x)g(x)\, dx = 0.$$

We have seen the usefulness of this type of orthogonality in the context of Fourier series, where $\sin(n\pi x/L)$ and $\cos(m\pi x/L)$ are orthogonal on $[-L, L]$. It is this orthogonality that enables us to determine the coefficients in the Fourier expansion of a function on $[-L, L]$.

A *normed linear space* is a vector space on which a norm has been defined. The *norm* of a vector x is a real number $\| x \|$ with the following properties:

(1) $\| x \| \geq 0$ for all vectors x in the space.

(2) $\| x \| = 0$ if and only if x is the zero vector.

(3) $\| cx \| = | c | \| x \|$ for any vector x in the space and any scalar (real number) c.

(4) $\| x + y \| \leq \| x \| + \| y \|$ for any vectors x and y in the space.

Property (4) of the norm is called the triangle inequality. In a normed linear space, the norm can be used to define a metric, or distance function, by setting the distance between vectors x and y equal to $\| x - y \|$. We may also think of

$\| x \|$ as the distance between the zero vector and x, which is what we would normally think of as the length of x.

Of course, many norms might be defined on a given vector space. In the plane R^2, the usual norm is given by

$$\| (a,b) \| = \sqrt{a^2 + b^2},$$

so the distance between (a, b) and (c, d) is

$$\| (a,b) - (c,d) \| = \sqrt{(a-c)^2 + (b-d)^2},$$

the usual distance between points in the plane; and

$$\| (a,b) \| = \sqrt{a^2 + b^2}$$

is the length of a vector (a, b) in the plane if this vector is given the usual interpretation as the arrow from the origin to the point (a, b).

We can always define a norm on an inner product space by setting

$$\| x \| = \sqrt{< x, x >}.$$

In $C([a, b])$, with the integral inner product defined previously, the norm of a function f is

$$\| f \| = \sqrt{< f, f >} = \left(\int_a^b (f(x))^2 \, dx \right)^{1/2};$$

and the distance between f and g is

$$\| f - g \| = \left(\int_a^b (f(x) - g(x))^2 \, dx \right)^{1/2},$$

a number that can be interpreted as the square root of the area bounded by the horizontal axis and the graph of $(f - g)^2$ on the interval $[a, b]$.

Now suppose that we have a vector space V with an inner product and a norm defined by this inner product. With the distance function defined by the norm, V is a metric space and so has a topological as well as an algebraic structure. The open ball of radius r centered at x_0 is the set of all vectors $x \in V$ at distance less than r from x_0, that is, all x satisfying $\| x - x_0 \| < r$. In form, this looks just like the definition of a disk in the plane except that V is now a normed linear space and the vectors are elements of V. A set $B \subset V$ is open in this norm topology if about every element of B there is an open ball contained entirely in B. Other concepts, such as interior and boundary points of a set, and closed set, are defined similarly from the corresponding notions in the plane, with the norm metric on V replacing the usual metric in the plane.

This norm topology gives rise to a concept of convergence of sequences of points of V. Suppose that $v_n \in V$ for $n = 1, 2, \cdots$. We say that this sequence *converges* to $v \in V$ (in the norm topology) if the distance between v_n and v has limit zero as n increases to ∞, that is, if

$$\lim_{n\to\infty} \| v_n - v \| = 0.$$

Following the lead from real sequences, a sequence v_n in V is called a *Cauchy sequence* if

$$\lim_{m,n\to\infty} \| v_n - v_m \| = 0.$$

It is easy to check that every convergent sequence in V is Cauchy. Conversely, if every Cauchy sequence of elements of V converges in V, we say that V is *complete*. A complete inner product space (with respect to the norm metric generated by the inner product) is called a *Hilbert space*.

R^n is a finite-dimensional Hilbert space, because any Cauchy sequence of $n-$ vectors converges to an n - vector.

The space l^2 of all infinite real sequences x such that $\sum_{n=1}^{\infty} x_n^2$ converges is a Hilbert space, with inner product

$$< x, y > = \sum_{n=1}^{\infty} x_n y_n$$

and norm

$$\| x \| = \sqrt{< x, x >} = \left(\sum_{n=1}^{\infty} x_n^2 \right)^{1/2}.$$

Another important Hilbert space is $L^2(D)$, the space of real-valued functions defined on a compact set $D \subset R^n$ and such that $\int_D f^2$ is finite. Here $\int_D f^2$ denotes the n - fold integral of f^2 over D. $L^2(D)$ is a Hilbert space with the inner product $< f, g > = \int_D fg$ and norm

$$\| f \| = \left(\int_D f^2 \right)^{1/2}.$$

By contrast, the inner product space $C([a, b])$ is not a Hilbert space, because it is not complete. If f_n is a Cauchy sequence in $C([a, b])$, then

$$\lim_{m,n\to\infty} \left(\int_a^b (f_n(x) - f_m(x))^2 \, dx \right)^{1/2} = 0$$

and this does not imply the existence of a continuous function g on $[a, b]$ such that

$$\lim_{n\to\infty} \left(\int_a^b (f_n(x) - g(x))^2 \, dx \right)^{1/2} = 0.$$

If V is an inner product space that is not complete, then V can be embedded as a dense subset of a Hilbert space H. This means that there is a Hilbert space such that $V \subset H$, and if $h \in H$ and $\epsilon > 0$, there is an element of V within distance ϵ of h. We call such H the *completion* of V. Perhaps the simplest example is the real number system as a completion of the rational numbers. The set Q of rational numbers is incomplete because a Cauchy sequence of rationals may not converge to a rational number, hence may have no limit in Q. However, any Cauchy sequence of reals does have a real limit, and if it is not rational, then it is an irrational number. A sequence of rationals may converge to an irrational limit. Further, given any real number x and any positive number ϵ, there is a rational number within distance ϵ of x, so Q is dense in the space of real numbers.

A *linear functional* on an inner product space V is a real-valued function φ satisfying the linearity condition

$$\varphi(\alpha x + \beta y) = \alpha\varphi(x) + \beta\varphi(y)$$

for $x, y \in V$ and scalars (real numbers) α and β.

A linear functional φ is *bounded* if for some positive number M,

$$| \varphi(x) | \leq M \| x \|$$

for all x in V.

Bounded linear functionals on an inner product space V are intimately tied to the inner product on V. If $y_0 \in V$, then

$$\varphi(x) = < x, y_0 >$$

defines a bounded linear functional on V. Perhaps surprisingly, in a Hilbert space, this works the other way. In the early twentieth century the Hungarian mathematician Frederich Riesz observed that if φ is any bounded linear functional on a Hilbert space H, there is a unique $y_0 \in H$ such that $\varphi(x) = < x, y_0 >$. Every bounded linear functional on a Hilbert space can be written as an inner product with some fixed vector in the space. This is the *Riesz representation theorem*, and it has become a fundamental tool of functional analysis.

We now sketch how these ideas about Hilbert spaces and bounded linear functionals bear on the question of existence of solutions of Dirichlet problems. Although this discussion can be carried out for sets in R^n, we focus on domains in the plane, R^2. This discussion will also serve as motivation for a brief introduction to distributions in the next section.

Suppose then that D is a bounded domain in the plane. In using Green's theorem as we will shortly, we assume that D has a piecewise smooth boundary. Given a function f defined on ∂D, we want to show that there is a function u that is harmonic on D and equal to f on ∂D. To begin, we recast this problem as one of determining a particular bounded linear functional on an appropriately chosen Hilbert space.

Consider the modified Dirichlet problem, in which we assume that f is defined on all of $\overline{D}$, not just on ∂D. Let $v = u - f$ and denote $w = \nabla^2 f$, which is a known function because f is given. The Dirichlet problem now becomes

$$\nabla^2 v = -w$$

with $v = 0$ on ∂D. This suggests that we work in the space $C_0^1(\overline{D})$, consisting of all functions that are continuous, with continuous first partial derivatives, on $\overline{D}$ but which vanish on ∂D. On this space, define the inner product

$$< f, g > = \iint_{\overline{D}} \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} \right) dA.$$

In particular,

$$< f, f > = \| f \|^2 = \iint_{\overline{D}} | \nabla f |^2 \, dA$$

is called the *Dirichlet integral* of f.

Now suppose that $v \in C_0^1(\overline{D})$ and $\nabla^2 v = 0$. If $u \in C_0^1(\overline{D})$, Green's theorem gives us the following:

$$\begin{aligned}
&\iint_{\overline{D}} \left[\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial y} \right) \right] dA \\
&= \iint_{\overline{D}} \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) dA + \iint_{\overline{D}} u \, \nabla^2 \, v dA \\
&= < u, v > + \int\int_{\overline{D}} u \, \nabla^2 \, v \, dA \\
&= < u, v > - \iint_{\overline{D}} uw \, dA \\
&= \int_{\partial D} \left(u\frac{\partial v}{\partial x} \right) dx + \left(u\frac{\partial v}{\partial y} \right) dy = 0.
\end{aligned}$$

Here we used the fact that $v = 0$ on ∂D. We conclude that

$$< u, v > = \iint_{\overline{D}} uw \, dA.$$

Keeping in mind that w is known in terms of the given f, the integral on the right defines a linear functional over $C_0^1(\overline{D})$, and this linear functional, acting on u, is represented by the inner product of u with some v. We can now reformulate this modified Dirichlet problem as: Find a v in $C_0^1(\overline{D})$ such that

$$\varphi(u) = \iint_{\overline{D}} uw \, dA = < u, v > .$$

The Riesz representation theorem will ensure that such a v exists if we can overcome two obstacles. The Riesz theorem applies to bounded linear functionals, and we have not shown that φ is bounded. Further, the theorem is stated for a Hilbert space, and $C_0^1(\overline{D})$ is not (it is an inner product space but is not complete with respect to the norm defined by the inner product).

The second difficulty is dealt with by embedding $C_0^1(\overline{D})$ as a dense subspace of a Hilbert space $H_0^1(\overline{D})$. This embedding extracts a cost, introducing new objects. In the case of the (incomplete) rationals embedded in the (complete) real number system, many of the reals are not in the original set of rationals.

Similarly, when we form $H_0^1(\overline{D})$, we obtain new objects which are not functions in the traditional sense but are generalizations of functions called distributions. We will say more about distributions in the next section.

Here is an argument to show that φ is bounded. We need a positive number M such that

$$\mid \varphi(u) \mid \leq M \parallel u \parallel \tag{7.1}$$

for all $u \in H_0^1(\overline{D})$.

Begin with the general Cauchy inequality

$$\left(\iint_{\overline{D}} uw\, dA\right)^2 \leq \left(\iint_{\overline{D}} u^2\, dA\right)\left(\iint_{\overline{D}} w^2\, dA\right).$$

Since w is a known function, the integral of w^2 is a known constant independent of $u \in H_0^1(\overline{D})$. It will therefore be sufficient to find a number K such that

$$\iint_{\overline{D}} u^2\, dA \leq K \parallel u \parallel$$

for $u \in H_0^1(\overline{D})$.

We will first show this for $u \in C_0^1(\overline{D})$. Since D is a bounded domain, we can enclose it in a closed square S defined by $-k \leq x \leq k, -k \leq y \leq k$. Extend u to S by defining $u(x,y) = 0$ at points in $K - \overline{D}$. Then, at any point (x,y) of the square,

$$\begin{aligned}
u^2(x,y) &= \left(\int_{-k}^{x} \frac{\partial u}{\partial x}(\xi,y)d\xi\right)^2 \\
&\leq (x+k)\int_{-k}^{x}\left(\frac{\partial u}{\partial x}(\xi,y)\right)2d\xi \leq 2k\int_{-k}^{k}\left(\frac{\partial u}{\partial x}(\xi,y)\right)^2 d\xi.
\end{aligned}$$

Therefore,

$$\int_{-k}^{k} u^2(x,y)\, dx \leq 4k^2 \int_{-k}^{k}\left(\frac{\partial u}{\partial x}\right)^2 (\xi,y)\, d\xi.$$

Integrate this inequality with respect to y from $-k$ to k to get

$$\int_{-k}^{k}\int_{-k}^{k} u^2(\xi,\eta)d\xi d\eta \leq 4k^2 \int_{-k}^{k}\int_{-k}^{k}\left(\frac{\partial u}{\partial x}\right)^2 (\xi,\eta)d\xi d\eta.$$

Then

$$\iint_K u^2\, dA \leq 4k^2 \iint_K \left(\frac{\partial u}{\partial x}\right)^2 dA.$$

This implies that

$$\begin{aligned}
\iint_{\overline{D}} u^2\, dA &\leq 4k^2 \iint_{\overline{D}}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dA \\
&= 4k^2 \parallel u \parallel^2 .
\end{aligned}$$

This gives the bound we want if $u \in C_0^1(\overline{D})$. We must now establish this bound on $\mid \varphi(u) \mid$ when $u \in H_0^1(\overline{D})$, the completion of $C_0^1(\overline{D})$, but not in $C_0^1(\overline{D})$ itself. In this case, u is the limit of a Cauchy sequence v_n of elements of $C_0^1(\overline{D})$. Because this sequence is Cauchy,

$$\lim_{m,n\to\infty} \| v_n - v_m \| = 0.$$

Because of the bound 7.1, which has been proved for functions in $C_0^1(\overline{D})$, this implies that

$$\lim_{m,n\to\infty} \iint_{\overline{D}} (v_n - v_m)^2 \, dA = 0.$$

Then $\varphi(v_n)$ is a Cauchy sequence of real numbers, and therefore converges, and we can define $\varphi(u)$ for $u \in H_0^1(\overline{D})$ by $\varphi(u) = \lim_{n\to\infty} \varphi(v_n)$. But this limit also extends through the norm, $\lim_{n\to\infty} \| v_n \| = \| u \|$. Therefore, the inequality 7.1 extends to elements of $H_0^1(\overline{D})$.

Now φ is a bounded linear functional on the Hilbert space $H_0^1(\overline{D})$. By Riesz's theorem, there exists $v \in H_0^1(\overline{D})$ such that $\varphi(u) = < u, v >$, and this is enough to assert the existence of a solution of the modified Dirichlet problem.

There are two issues hanging over this conclusion. First, the problem solved is the modified problem, not the Dirichlet problem itself. Second, the solution is an element of $H_0^1(\overline{D})$, and may not be a function in the conventional sense. It is a generalized function, or distribution. With a fair amount of additional analysis, which we will not pursue, one can show that this distribution is actually a function that is continuous with continuous first and second partial derivatives on $\overline{D}$, and satisfies the boundary condition (with some conditions on ∂D), hence yields a solution of the Dirichlet problem in the classical sense.

Problems for Section 7.2

1. Prove the Schwarz inequality for an inner product space: If x and y are vectors, then

$$|< x, y >| \leq \| x \| \| y \| .$$

Hint: Begin by expanding the right side of the inequality $0 \leq < x - \alpha y, x - \alpha y >$ and then choose $\alpha = < x, x > / < x, y >$, assuming that $< x, y > \neq 0$.

2. Prove that orthogonal vectors in a Hilbert space satisfy the Pythagorean relationship

$$\| x + y \|^2 + \| x - y \|^2 = 2 \left(\| x \|^2 + \| y \|^2 \right) .$$

3. Prove that for vectors in a Hilbert space,

$$\| x + y \| = \| x \| + \| y \|$$

if and only if $x = \alpha y$ for some nonnegative number α.

4. A subset S of a Hilbert space H is said to be convex if for each x and y in S, $\alpha x + (1-\alpha)y$ is also in S. This means that the straight-line segment in H between any two vectors in S is contained in S. Prove that if S is a closed, convex subset of H, then S contains a unique element of smallest norm. Hint: Let d be the greatest lower bound of $\| x \|$, taken over all $x \in S$. Show that there is a sequence x_n of elements of S such that $\lim_{n\to\infty} \| x_n \| = d$. Use the convexity of S to argue that $\frac{1}{2}(x_n + x_m) \in S$ for each positive integer n and m, and use the result of Problem 2 to show that $\lim_{n,m\to\infty} \| x_n - x_m \| = 0$. Thus show that x_n converges to some $x_0 \in S$, and that $\| x_0 \| = d$. Finally, show that x_0 is unique among elements of S having norm d.

5. Let M be a closed subset of a Hilbert space H, and let $x \in H$. Show that there is a unique representation $x = y + z$, with $y \in M$ and z orthogonal to M (that is, $< z, s > = 0$ for every $s \in S$). Hint: If $x \in M$, choose $y = x, z = O$, where O is the zero vector of H. Thus suppose that x is not in M. Let S consist of all $x - y$ for $y \in M$. Show that S is closed and convex and use the result of Problem 4 to produce a unique element x_0 of smallest norm in S. Then $x_0 = x - x_1$ for some $x_1 \in M$. Show that $< x_0, y > = 0$ for all $y \in M$, and then write $x = x_1 + x_0$.

6. Prove the Riesz representation theorem. Let φ be a bounded linear functional on a Hilbert space H. Then for some unique $y \in H$, $\varphi(x) = < x, y >$ for all $x \in H$. Hint: First prove the theorem if $\varphi(x) = 0$ for all $x \in H$. Now suppose that this is not the case. Let M consist of all x such that $\varphi(x) = 0$. (M is the *kernel* of φ.) Show that this is a closed subspace of H. Use the result of Problem 5 to produce a nonzero $z \in H$ such that z is orthogonal to every element of M. By multiplying by a real number if necessary, we may suppose that $\varphi(z) = 1$. If $x \in H$, show that $w = x - \varphi(x)z \in M$. Thus we may write $x = w + \varphi(x)z$, a sum of a vector in M and a vector orthogonal to M. Finally, let $y = z/ \| z \|^2$. Show that $\varphi(x) = < x, y >$ and that y is unique.

7.3 Distributions and an Existence Theorem

In the mid-nineteenth century, a major initiative led to the development of a theory of *distributions*, or *generalized functions*. This movement was led by the French school of abstract analysis, in particular Laurent Schwartz and Claude Chevalley. Part of the motivation lay in existence questions in partial differential equations. Attempts to use powerful and relatively new tools from functional analysis led to the realization that the traditional concept of function was too limited. Functions often do not behave well. They may have discontinuities, or fail to be differentiable at various points, or have n derivatives but not $n+1$ derivatives. As we will see, distributions are always continuous and differentiable. The derivative of a distribution is a distribution, so there is a second derivative, and so on. Further, although functions are not distributions, every function is associated with a distribution. This section is devoted to some fundamental facts about distributions, culminating in an example of an existence theorem formulated in terms of distributions.

Distributions are defined by their action (to be defined shortly) on a set of test functions. These test functions are chosen to suit particular problems, but have certain characteristics in common. For us, a test function will be a real-valued function, defined on a subset of R^n, having partial derivatives of all orders and having compact support. This means that the function vanishes outside some compact set.

To simplify the notation of the discussion, we focus on the case $n = 1$ and develop some ideas about distributions based on test functions defined over the real line but having compact support. We then suggest how the discussion can be generalized to test functions defined on subsets of R^n. Thus suppose that we have some set $\mathbf{F}$ of functions, called *test functions*, each defined on the set R of real numbers. Each test function must be infinitely differentiable (we say that the function is C^∞) and have compact support. In this discussion we denote functions in $\mathbf{F}$ by lowercase Greek letters. The derivative of order k of φ will be denoted $\varphi^{(k)}$.

We need a very strong kind of convergence for sequences in $\mathbf{F}$, which we now define. A sequence φ_n of functions in $\mathbf{F}$ *converges to zero* (the zero function) if the following two conditions are met:

(1) For each positive integer k, the sequence of kth derivatives $\varphi^{(k)}$ converges uniformly to zero.

(2) There is an interval $[a, b]$ such that each φ_n vanishes outside $[a, b]$. This means that there is an interval $[a, b]$ containing the support of each function in the sequence. When this holds, we say that the sequence φ_n has uniformly bounded supports.

We are now in a position to define the concept of a distribution (with respect to this set $\mathbf{F}$ of test functions).

Suppose that T is a mapping of $\mathbf{F}$ into R. If $\varphi \in \mathbf{F}$, we denote the real number to which T maps φ as $< T, \varphi >$. We call T a *distribution* if T satisfies the following:

(1) If $a, b \in R$ and $\varphi, \psi \in \mathbf{F}$, then

$$< T, a\varphi + b\psi > = a < T, \varphi > + b < T, \psi > .$$

(2) If φ_n is a sequence of functions in $\mathbf{F}$ converging to zero in the sense just defined, the sequence of numbers $< T, \varphi_n >$ converges to zero.

The notation $< T, \varphi >$ for the action of T on φ makes computations with distributions analogous to computations with an inner product. Taking a cue from linear transformations between vector spaces, property (1) of the definition is called the *linearity* of a distribution. Property (2) is called *continuity*.

Example 7.1 *We will show that it is natural to define the Dirac delta function as a distribution. Begin with the Heaviside step function* h *defined by*

$$\begin{cases} h(x) = 0 & \text{if } x \leq 0 \\ h(x) = 1 & \text{if } x > 0. \end{cases}$$

This function is named after Oliver Heaviside (1850-1925), an English electrical engineer who helped popularize use of the Laplace transform in solving circuit problems. If ϵ *is any positive number, define*

$$\delta_\epsilon(x) = \frac{1}{\epsilon}(h(x) - h(x-\epsilon)).$$

This is a pulse of magnitude $1/\epsilon$ *and duration* ϵ*. The Dirac function, named for the Nobel laureate physicist P.A.M. Dirac, is defined by*

$$\delta(x) = \lim_{\epsilon \to 0+} \delta_\epsilon(x).$$

This symbol is used to represent an impulse of infinite magnitude applied over an interval of infinitely short duration. Think of striking your thumb with a hammer. Although this force is not infinite, it is a very large force applied to a small area over a very short period of time. From the thumb's perspective, this force probably appears infinite in magnitude. It is traditional to refer to δ *as a function, but it is not in the usual sense of that term. We will show how to define it as a distribution. The motivation for the definition is to look at*

$$\int_{-\infty}^{\infty} \delta(x)\varphi(x)\,dx,$$

for any test function φ*. First compute*

$$\begin{aligned} &\int_{-\infty}^{\infty} \delta_\epsilon(x)\varphi(x)\,dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\epsilon}(h(x) - h(x-\epsilon))\varphi(x)\,dx \\ &= \frac{1}{\epsilon}\int_0^{\epsilon} \varphi(x)\,dx \\ &= \frac{1}{\epsilon}(\epsilon)\varphi(x_\epsilon) = \varphi(x_\epsilon) \end{aligned}$$

for some number x_ϵ *between* 0 *and* ϵ*. As* $\epsilon \to 0$*,* $x_\epsilon \to 0$ *and* $\delta_\epsilon(x) \to \delta(x)$*, yielding*

$$\int_{-\infty}^{\infty} \delta(x)\varphi(x)\,dx = \varphi(0).$$

This suggests that we define δ *as a distribution by specifying its action on any test function as*

$$< \delta, \varphi > = \varphi(0).$$

This defines a mapping from the set of test functions to the real line, hence a distribution. When we define the derivative of a distribution, we will return to this example and show that the delta distribution is the derivative of the Heaviside distribution. ◇

If f is a piecewise continuous function defined on R, we can define a mapping T_f from $\mathbf{F}$ to R by

$$< T_f, \varphi > = \int_{-\infty}^{\infty} f(x)\varphi(x)\, dx$$

for $\varphi \in \mathbf{F}$. It is routine to check that T_f is a distribution. In this way, each piecewise continuous function is associated in a natural way with a distribution.

Certain operations can be defined on distributions by defining their actions on test functions. In particular, if T and P are distributions, define $T + P$ by

$$< T + P, \varphi > = < T, \varphi > + < P, \varphi > .$$

If a is a real number, define aT by

$$< aT, \varphi > = a < T, \varphi > .$$

Define a derivative T' of T by

$$< T', \varphi > = - < T, \varphi' > .$$

Because every test function is differentiable, it follows that every distribution is differentiable.

Finally, if g is a C^∞ function, define gT by

$$< gT, \varphi > = < T, g\varphi > .$$

It is routine to check that each of these defines a distribution.

The derivative of a distribution satisfies rules that are familiar from derivatives of functions. In particular:

(1) $(S + T)' = S' + T'$.

(2) $(aT)' = aT'$ for any real number a.

(3) $(gT)' = g'T + gT'$ if g is a C^∞ function.

Conclusion (3) reminds us of a product rule for differentiation, except here the product is of a C^∞ function multiplied by a distribution.

With a mild abuse of notation, we can also define distributions $T(ax)$ and $T(x + a)$ by, respectively,

$$< T(ax), \varphi > = \frac{1}{|a|} < T, \varphi(x/a) >$$

for a a nonzero real number, and

$$< T(x + a), \varphi > = < T, \varphi(x - a) > .$$

These distributions interact with the derivative operation for distributions as follows:

$$(T(ax))' = aT'(ax)$$

for nonzero, real a, and

$$(T(x+a))' = T'(x+a).$$

There is a sense in which we can write

$$T'(x) = \lim_{h \to 0} \frac{T(x+h) - T(x)}{h},$$

suggesting that at least in form, the derivative of a distribution resembles the derivative of a function. To see why such an equation holds, recall that

$$< T', \varphi > = - < T, \varphi' >$$

for any test function φ. Further,

$$< (T(x+h) - T(x))/h, \varphi > = < T, (\varphi(x-h) - \varphi(x))/h > .$$

As $h \to 0$, $(\varphi(x-h) - \varphi(x))/h \to -\varphi'(x)$. The limit formula for $T'(x)$ now follows from the continuity condition in the definition of distribution.

Example 7.2 *The Heaviside step function is a legitimate function but may be thought of as a distribution in the sense defined previously. We will differentiate h as a distribution. If φ is a test function, then*

$$\begin{aligned} < h', \varphi > &= - < h, \varphi' > = - \int_0^\infty \varphi'(x)\,dx \\ &= \varphi(0) \; = < \delta, \varphi > . \end{aligned}$$

This means that $h' = \delta$ in the sense of distributions. As a function, of course, h is discontinuous and nondifferentiable at the origin. $\diamond$

Thus far we have restricted the discussion to distributions defined using test functions that are real-valued functions of a single real variable. This discussion generalizes to the use of test functions that are real-valued functions of n real variables. In doing this, we introduce some standard terminology. Let α_j be a positive integer for $j = 1, \cdots, n$ and let

$$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n).$$

Then the differential operator D^α is defined by

$$D^\alpha = \frac{\partial^{\alpha_1 + \alpha_2 + \cdots + \alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

It is also convenient to denote

$$\mid \alpha \mid = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

Now test functions are functions on R^n to R, having compact support, and also continuous derivatives of all orders, where the derivatives of φ are now partial derivatives $D^\alpha(\varphi)$, for all possible choices of α. With this agreement, the formal definitions of distributions, and derivatives of distributions, follow.

Using the D^α notation, we mention the idea of a Sobolev space. If $D \subset R^n$ and n is a nonnegative integer, the Sobolev space $H^n(D)$ consists of functions u in $L^2(D)$ (that is, their squares are integrable over D), with each $D^\alpha \in L^2(D)$ if $|\alpha| \leq n$. Sobolev spaces are important in current work on existence questions and are named for the twentieth-century Russian mathematician S.L. Sobolev, who was among the first to use function spaces in the study of partial differential equations.

We will conclude with an example of the use of distributions in an existence theorem.

Theorem 7.4 *Let D be a bounded subset of R^n. Let $f \in L^2(\overline{D})$. Then there is a unique distribution u satisfying $\nabla^2 u = f$ on D and $u = 0$ on ∂D.*

For this theorem, the fairly strong condition that f is an L^2 function is a counterpoint to the absence of any conditions on D other than that it is a bounded set.

Problems for Section 7.3

1. Suppose that T_n is a distribution for each positive integer n. We say that T_n converges to a distribution L, and write $T_n \to L$, if

$$< T_n, \varphi > \to < L, \varphi >$$

   for every test function φ. Now suppose that $T_n \to L$ and $S_n \to P$. Prove that $S_n + T_n \to S + T$ and, for any number a, $aT_n \to aL$.

2. Suppose that $T_n \to L$. Show that $T_n' \to L'$.

3. Suppose that f and g are piecewise continuous, and T is a distribution. Show that $T_{f+g} = T_f + T_g$ and, for any number a, $T_{af} = aT_f$.

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by Peter V. O'Neil

Chapter 8

Additional Topics

This chapter offers an introduction to two additional general topics (solutions by eigenfunction expansions and numerical approximations of solutions) as well as an analysis of three important partial differential equations. These are Burger's equation (a diffusion equation), the telegraph equation (a wave equation), and Poisson's equation (of which Laplace's equation is a special case). These topics are independent of each other and can be read in any order.

8.1 Solutions by Eigenfunction Expansions

When we separated variables in the one-dimensional wave and heat equations on an interval $[0, L]$, we encountered the problem

$$X'' + \lambda X = 0; X(0) = X(L) = 0, \tag{8.1}$$

in which λ is the separation variable.

This problem for X was easily solved, yielding numbers (eigenvalues) $\lambda_n = (n\pi/L)^2$ and the corresponding eigenfunctions $X_n(x) = \sin(n\pi x/L)$ for positive integer values of n. These eigenfunctions were crucial to solving problems involving wave motion and heat conduction, by expanding a given function f (an initial position, velocity, or temperature) in an eigenfunction series

$$f(x) = \sum_{n=1}^{\infty} c_n X_n(x),$$

which in this case is a Fourier sine series.

Separation of variables in a partial differential equation may lead to a more complicated problem than 8.1 for the eigenvalues and eigenfunctions. This occurs in Examples 8.1 and 8.2, which we will use to motivate some of the ideas we now develop. Ultimately, these ideas will enable us to solve the problems posed in these examples.

Example 8.1 *Suppose that we have a solid sphere of radius κ, and we know the temperature on the surface at all times. We would like to solve for the steady-state temperature distribution throughout the sphere.*

Center the sphere at the origin and use spherical coordinates(ρ, θ, φ). If we assume symmetry about the z - axis, then the heat equation is independent of the polar angle θ and is given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial^2 u}{\partial \varphi^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} + \frac{\cot(\varphi)}{\rho^2}\frac{\partial u}{\partial \varphi}.$$

The steady-state case is that $\partial u/\partial t = 0$, and the heat equation becomes

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial^2 u}{\partial \varphi^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} + \frac{\cot(\varphi)}{\rho^2}\frac{\partial u}{\partial \varphi} = 0.$$

The boundary condition is $u(\kappa, \varphi) = f(\varphi)$ for a given function f.

Separate variables by putting $u(\rho, \varphi) = R(\rho)\Phi(\varphi)$. A routine calculation yields the equation for Φ:

$$\frac{1}{\sin(\varphi)}[\Phi' \sin(\varphi)]' + \lambda\Phi = 0, \tag{8.2}$$

in which λ is the separation constant. This equation determines the eigenvalues λ and corresponding eigenfunctions $\Phi(\varphi)$ for this problem. Unlike the problem encountered previously for X, however, equation 8.2 is not easily solved. Further, if we have to expand f in a series of these eigenfunctions, we need some extension of the theory of the Fourier trigonometric series used with the standard heat and wave equations, in order to know the coefficients in the expansion, and convergence properties of the resulting series. $\diamond$

Example 8.2 *Suppose that a membrane, such as a drumhead, is fastened onto a circular frame and struck, causing it to vibrate. Suppose that the membrane is initially in the x, y - plane centered at the origin. If $z(x, y, t)$ is the vertical displacement at time t of the point on the membrane originally at $(x, y, 0)$, then z satisfies the three-dimensional wave equation. In cylindrical coordinates, this is*

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}\right).$$

If the motion of the membrane is symmetric about the origin, z is independent of θ and this equation becomes

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r}\right).$$

To separate the variables, substitute $z(r, t) = R(r)T(t)$ into the partial differential equation. We find that the equation for R is

$$R'' + \frac{1}{r}R' + \frac{\lambda}{c^2}R = 0, \tag{8.3}$$

with λ the separation constant. Values of λ for which equation equation 8.3 has nontrivial solutions are the eigenvalues of this problem. For each eigenvalue λ a corresponding solution for $R(r)$ is an eigenfunction. It is not immediately apparent what the eigenvalues and eigenfunctions are in this problem. Again, once we find the eigenfunctions, we may have to expand a given function f in a series of these eigenfunctions in order to write the solution for the displacement function. A theory of eigenfunction expansions is needed to enable us to do this. ◇

General Eigenfunction Expansions

Before solving the problems of Examples 8.1 and 8.2, we develop a general context in which we can expand a given function (for our purposes, usually a data function in an initial-boundary value problem) in a series of eigenfunctions arising from separation of variables in the partial differential equation.

Such a separation of variables often leads (perhaps after some manipulation) to a differential equation of the form

$$(ry')' + (q + \lambda p)y = 0. \tag{8.4}$$

This is the case in both of the examples just discussed. Equation 8.4 is called the *Sturm - Liouville differential equation*. It is assumed that r, p, q, and r' are continuous on the interval $[a, b]$ of interest, and $p(x) > 0$ and $r(x) > 0$ on (a, b).

There are different kinds of Sturm - Liouville problems, distinguished by the coefficient functions and boundary conditions. A *regular Sturm - Liouville problem* consists of equation 8.4, together with boundary conditions of the form

$$A_1 y(a) + A_2 y'(a) = 0, B_1 y(b) + B_2 y'(b) = 0.$$

Here A_1, A_2, B_1, and B_2 are given numbers, with A_1 and A_2 not both zero and B_1 and B_2 not both zero. The problem 8.1 is a regular Sturm - Liouville problem, with X written in place of y.

A *singular Sturm - Liouville problem* consists of the Sturm - Liouville differential equation and one of the following sets of conditions:

(1) $r(a) = 0$ and there is no boundary condition at a.

(2) $r(b) = 0$ and there is no boundary condition at b.

(3) $r(a) = r(b) = 0$ and there are no boundary conditions at a or b. In this case we require that solutions be bounded on $[a, b]$.

In a regular or singular Sturm - Liouville problem a value of λ for which there are nontrivial solutions is called an *eigenvalue*. For each such λ, nontrivial solutions are called *eigenfunctions corresponding to this eigenvalue*.

For example, the differential equation

$$((1 - x^2)y')' + \lambda y = 0 \tag{8.5}$$

is called *Legendre's equation*, and we will encounter it when we solve for the steady-state temperature distribution of the sphere in Example 8.1. Legendre's equation by itself constitutes a singular Sturm - Liouville problem on $[-1,1]$. Here $r(x) = 1 - x^2$ and $r(-1) = r(1) = 0$, so there are no boundary conditions at -1 or 1.

As another example,

$$(xy')' + \left(\lambda x - \frac{\nu}{x}\right) y = 0 \tag{8.6}$$

is called *Bessel's equation of order* ν. This is a second-order differential equation. The phrase "order ν" refers to the parameter ν appearing in the equation. Bessel's equation of various orders arise in many contexts, including the analysis of the skin effect for alternating current in a coaxial cable and elasticity of beams. Solutions of Bessel's equation of order ν are called Bessel functions of order ν. There is an extensive literature on Bessel functions, and certain Bessel functions have been incorporated into computational software routines to facilitate their use in engineering and science applications.

In general, it can be proved that a Sturm - Liouville problem has an infinite sequence of eigenvalues which can be ordered

$$\lambda_1 < \lambda_2 < \cdots$$

so that $\lambda_n \to \infty$ as $n \to \infty$. This means that the eigenvalues cannot accumulate at any finite value. We see this behavior in the eigenvalues $(n\pi/L)^2$ of problem 8.1.

In the case of the Sturm - Liouville problem 8.1 on $[0, L]$, the integral over this interval of a product of eigenfunctions corresponding to distinct eigenfunctions is zero:

$$\begin{aligned} &\int_0^L X_n(x) X_m(x)\, dx \\ &= \int_0^L \sin(n\pi x/L) \sin(m\pi x/L)\, dx = 0 \end{aligned}$$

if $n \neq m$. Thinking of this integral as a dot product of eigenfunctions (like the inner product defined on $C([a,b])$ in Section 7.2), we say that these eigenfunctions are *orthogonal* on $[0, L]$.

A similar property holds for eigenfunctions of any regular or singular Sturm - Liouville problem. If y_n is an eigenfunction corresponding to the eigenvalue λ_n, and y_m is an eigenfunction corresponding to the eigenvalue λ_m, we claim that

$$\int_a^b p(x) y_n(x) y_m(x)\, dx = 0$$

if $n \neq m$. Here the dot (or inner) product is of the form

$$< f, g > = \int_a^b p(x) f(x) g(x)\, dx.$$

with $p(x)$ the nonnegative function appearing in the Sturm - Liouville differential equation

$$(ry')' + (q + \lambda p)y = 0.$$

This type of inner product is called a *weighted inner product*, with *weight function* $p(x)$. In the language of inner product spaces, two functions are *orthogonal with respect to the weight function* $p(x)$ if their weighted inner product is zero. Our claim, then, is that eigenfunctions corresponding to distinct eigenvalues of a Sturm - Liouville problem are orthogonal, with weight function p. (Often we simply refer to the eigenfunctions as orthogonal, with the weight function p understood.)

To verify this weighted orthogonality, start with the Sturm - Liouville equation 8.4. We have

$$(ry_n')' + (q + \lambda_n p)y_n = 0$$

and

$$(ry_m')' + (q + \lambda_m p)y_m = 0.$$

Multiply the first of these equations by y_m and the second by y_n and subtract to get

$$(ry_n')'y_m - (ry_m')'y_n = (\lambda_m - \lambda_n)py_n y_m.$$

Integrate both sides of this equation over $[a, b]$ to get

$$\int_a^b [(ry_n')'y_m - (ry_m')'y_n]\,dx = (\lambda_m - \lambda_n)\int_a^b p(x)y_n(x)y_m(x)\,dx.$$

Since $\lambda_m - \lambda_n \neq 0$, we will have shown that the integral on the right side of this equation is zero if we can show that the integral on the left side of the last equation vanishes. Integrate the left side by parts to write the equation as

$$r(b)[y_m(b)y_n'(b) - y_n(b)y_m'(b)] - r(a)[y_m(a)y_n'(a) - y_n(a)y_m'(a)]$$
$$= (\lambda_m - \lambda_n)\int_a^b p(x)y_n(x)y_m(x)\,dx.$$

A systematic use of the boundary values for the problem can now be used to show that

$$r(a)[y_m(a)y_n'(a) - y_n(a)y_m'(a)] = 0$$

and

$$r(b)[y_m(b)y_n'(b) - y_n(b)y_m'(b)] = 0,$$

establishing that $\int_a^b p(x)y_n(x)y_m(x)\,dx = 0$.

By exploiting this weighted orthogonality of distinct eigenfunctions, we can investigate the possibility of expanding a given function $f(x)$ in a series of the form $\sum_{n=1}^{\infty} c_n y_n(x)$. The issue is to determine how to choose the coefficients c_n, and the argument we will pursue follows that used to motivate the choice of

Fourier coefficients for Fourier (trigonometric) series. Suppose for the moment that

$$f(x) = \sum_{m=1}^{\infty} c_m y_m(x)$$

at least on (a, b). Let n be any positive integer and multiply this equation by $p(x)y_n(x)$ to get

$$p(x)f(x)y_n(x) = \sum_{m=1}^{\infty} c_m p(x)y_m(x)y_n(x).$$

Integrate to get

$$\int_a^b p(x)f(x)y_n(x)\,dx = \sum_{m=1}^{\infty} \int_a^b c_m p(x)y_m(x)y_n(x)\,dx.$$

Because of the weighted orthogonality of the eigenfunctions, all the integrals on the right are zero except for $\int_a^b p(x)y_n^2(x)\,dx$, which occurs in the series when $m = n$. Solve the resulting equation for c_n to get

$$c_n = \frac{\int_a^b p(x)f(x)y_n(x)\,dx}{\int_a^b p(x)y_n^2(x)\,dx}. \tag{8.7}$$

With this choice of the coefficients, the series $\sum_{n=1}^{\infty} c_n y_n(x)$ is the *eigenfunction expansion* of $f(x)$ in terms of the eigenfunctions of the given Sturm - Liouville problem on $[a, b]$.

This informal symbol manipulation has led to the intriguing prospect of expanding a function in a series of eigenfunctions. However, we need to understand the relationship between $f(x)$ and this eigenfunction expansion. We will state the result without proof. If f is piecewise smooth on $[a, b]$, then at any x in (a, b), the eigenfunction expansion converges to

$$\frac{1}{2}(f(x+) + f(x-)).$$

The fact that this is the same conclusion that we found for Fourier series should not be surprising, because Fourier series are themselves eigenfunction expansions.

We illustrate these ideas with a specific eigenfunction expansion.

Example 8.3 (*An Eigenfunction Expansion*) *Consider the Sturm - Liouville problem*

$$(xy')' + \frac{\lambda}{x}y = 0; y(1) = y(3) = 0.$$

The differential equation is of Cauchy - Euler type and the general solution is found to be

$$y(x) = A\cos(\sqrt{\lambda}\ln(x)) + B\sin(\sqrt{\lambda}\ln(x)),$$

with A and B arbitrary constants. From the boundary condition at 1 we have $y(1) = A = 0$. Then

$$y(3) = B\sin(\sqrt{\lambda}\ln(3)) = 0.$$

For a nontrivial solution, choose $\sqrt{\lambda}\ln(3)$ to be an integer multiple of π, say $n\pi$ for positive integer n. Denoting λ for each such n by λ_n, the eigenvalues for this problem are

$$\lambda_n = \left(\frac{n\pi}{\ln(3)}\right)^2.$$

Corresponding eigenfunctions are

$$y_n = \sin\left(\frac{n\pi}{\ln(3)}\ln(x)\right).$$

We will write an expansion of $f(x) = 1$ in a series of these eigenfunctions for $1 \le x \le 3$. Notice that in the general form of the Sturm - Liouville differential equation, this problem has weight function $p(x) = 1/x$. The coefficients in the expansion are

$$c_n = \frac{\int_1^3 \frac{1}{x}\sin(n\pi\ln(x)/\ln(3))\,dx}{\int_1^3 \frac{1}{x}\sin^2(n\pi\ln(x)/\ln(3))\,dx}.$$

These integrations are routine and we obtain

$$c_n = \begin{cases} 4/n\pi & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Since the even-indexed coefficients are zero, we may retain only odd-indexed terms. By the convergence theorem, we can write the eigenfunction expansion

$$\sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi}\sin\left(\frac{(2n-1)\pi\ln(x)}{\ln(3)}\right) = 1$$

for $1 < x < 3$. Figure 8.1 shows a graph of the thirtieth partial sum of this series, and Figure 8.2 the one hundredth partial sum. Notice the appearance of the Gibbs phenomenon at the endpoints of the interval. ◇

Solutions Using Legendre Polynomials

In some instances the Sturm - Liouville problem encountered in separating variables in a partial differential equation is easily solved for the eigenvalues and eigenfunctions. Examples 8.1 and 8.2 show that this is not always the case. Such problems may lead us to the use of special functions. These are functions arising in special contexts but which occur sufficiently often that many of their properties have been investigated and documented to facilitate their use. Sines and cosines are perhaps the most familiar special functions, but we also encounter Bessel functions, Hermite, Legendre and Laguerre polynomials, and

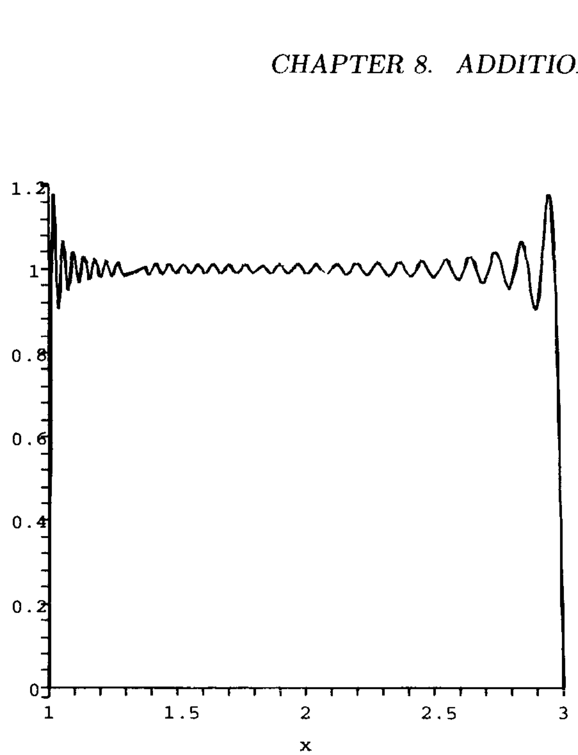

Figure 8.1: Thirtieth partial sum of the expansion in Example 8.3.

many others. In this section we complete the problem begun in Example 8.1 to determine the steady-state temperature distribution in a solid sphere given the temperature at all times on the surface. With the sphere centered at the origin, the steady state-heat equation, in spherical coordinates, is

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial^2 u}{\partial \varphi^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} + \frac{\cot(\varphi)}{\rho^2}\frac{\partial u}{\partial \varphi} = 0.$$

This assumes symmetry of the solution about the $z-$ axis, so the heat equation is independent of θ. Suppose that the sphere has radius κ. The boundary condition is $u(\kappa, \varphi) = f(\varphi)$.

To separate variables in the partial differential equation, substitute $u(\rho, \varphi) = R(\rho)\Phi(\varphi)$ to get

$$\rho^2 R'' + 2\rho R' - \lambda R = 0$$

and

$$\Phi'' + \cot(\varphi)\Phi' + \lambda\Phi = 0,$$

in which λ is the separation constant.

The differential equation for Φ can be written

$$\frac{1}{\sin(\varphi)}[\Phi' \sin(\varphi)]' + \lambda\Phi = 0.$$

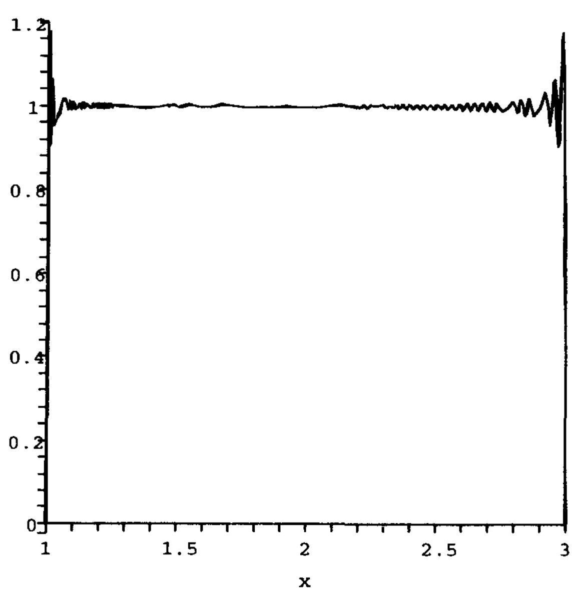

Figure 8.2: One hundredth partial sum.

This looks very close to a Sturm - Liouville differential equation. One more step will give us this type of equation. Change variables by putting $\varphi = \arccos(x)$, so $x = \cos(\varphi)$, and let

$$y(x) = \Phi(\varphi) = \Phi(\arccos(x)).$$

Routine chain rule differentiations transform the differential equation for Φ to

$$[(1 - x^2)y'(x)]' + \lambda y(x) = 0.$$

This is a Sturm - Liouville equation with $r(x) = 1 - x^2$, $p(x) = 1$, and $q(x) = 0$. We also recognize it as Legendre's equation. Since $0 \le \varphi \le \pi$, then $-1 \le x \le 1$, so this Sturm - Liouville problem is defined on $[-1, 1]$. Further, $r(-1) = r(1) = 0$, so there are no boundary conditions on $y(x)$. We seek bounded solutions.

Attempt a power series solution about the origin. Substitute $y(x) = \sum_{k=0}^{\infty} c_k x^k$ into Legendre's equation. After some routine manipulation collecting terms, we obtain a recurrence relation for the coefficients:

$$c_{n+2} = \frac{n(n+1) - \lambda}{(n+1)(n+2)} c_n$$

for $n = 2, 3, \cdots$, while c_0 and c_1 are arbitrary constants. Some of these coefficients are

$$c_2 = -\frac{\lambda}{2} c_0, c_4 = \frac{-\lambda(6 - \lambda)}{4!} c_0, \cdots$$

with each even-indexed c_n a multiple of c_0, and

$$c_3 = \frac{2-\lambda}{3!}c_1, c_5 = \frac{(2-\lambda)(12-\lambda)}{5!}c_1, \cdots$$

with each odd-indexed c_n a multiple of c_1. The first few terms of the series solution thus obtained are

$$y(x) = c_0\left(1 - \frac{\lambda}{2}x^2 - \frac{\lambda(6-\lambda)}{4!}x^4 - \frac{\lambda(6-\lambda)(20-\lambda)}{6!}x^6 + \cdots\right)$$
$$+ c_1\left(x + \frac{2-\lambda}{3!}x^3 + \frac{(2-\lambda)(12-\lambda)}{5!}x^5 + \cdots\right).$$

For each nonnegative integer n, we obtain a polynomial solution of degree n by choosing $\lambda = n(n+1)$ and either c_0 or c_1 equal to zero. The first four polynomial solutions generated in this way are

$y_0(x) = c_0$, with $n = 0$, $\lambda = 0$, and $c_1 = 0$,
$y_1(x) = c_1 x$ with $n = 1$, $\lambda = 2$, and $c_0 = 0$,
$y_2(x) = c_0(1 - 3x^2)$ with $n = 2$, $\lambda = 6$, and $c_1 = 0$,
$y_3(x) = c_1\left(x - \frac{5}{3}x^3\right)$, with $n = 3$, $\lambda = 12$, and $c_0 = 0$.

It is standard to choose c_0 or c_1 in each solution so that the polynomial takes on the value 1 at $x = 1$. In this way we obtain the Legendre polynomials $P_n(x)$, the first six of which are

$$P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1), P_3(x) = \frac{1}{2}(5x^3 - 3x),$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), P_5(x) = \frac{1}{8}(63x^5 - 7 - x^3 + 15x).$$

Graphs of P_1, P_2, and P_3 are shown in Figure 8.3. Notice that for this limited sample, $P_n(x)$ crosses the x - axis exactly n times between -1 and 1. It can be shown that for each positive integer n, P_n has exactly n real zeros, all occurring in $(-1, 1)$.

There are many formulas for $P_n(x)$. One that occurs frequently is

$$P_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k},$$

where $[n/2]$ denotes the largest integer not exceeding $n/2$.

The numbers $\lambda = n(n+1)$ are the eigenvalues of Legendre's equation, and the solutions $y(x) = P_n(x)$ are the corresponding eigenfunctions, for $n = 0, 1, 2, \cdots$.

Armed with these eigenfunctions, return to the equations for $R(\rho)$ and $\Phi(\varphi)$ obtained by separating variables in the spherical coordinate steady-state heat equation. We have, for each nonnegative integer n, a solution

$$\Phi_n(\varphi) = y(\cos(\varphi)) = P_n(\cos(\varphi)).$$

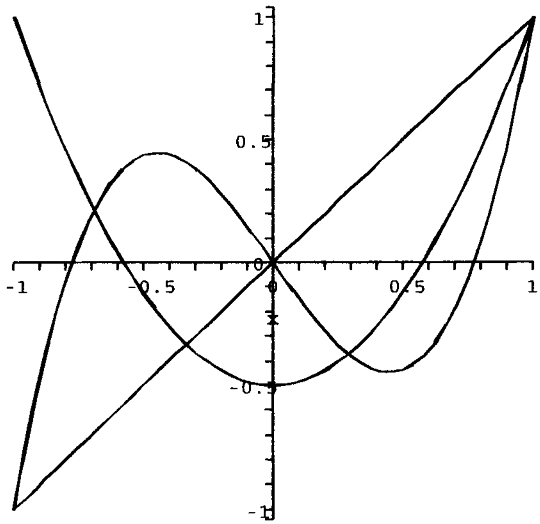

Figure 8.3: Graphs of P_1, P_2, and P_3.

Next put $\lambda = n(n+1)$ into the differential equation for $R(\rho)$ to get

$$\rho^2 R'' + 2\rho R' - n(n+1)R = 0.$$

This Cauchy-Euler type of equation has general solution

$$R(\rho) = a\rho^n + b\rho^{-n-1},$$

in which a and b are arbitrary constants. Since $\rho = 0$ at the center of the sphere, we must choose $b = 0$ to have a bounded solution, making $R(\rho)$ a constant multiple of ρ^n. For each nonnegative integer n, we now have a function

$$u_n(\rho, \varphi) = a_n \rho^n P_n(\cos(\varphi)),$$

with the a_n's to be determined. Each of the u_n's satisfies the steady-state heat equation. To satisfy the boundary condition, use a superposition

$$u(\rho, \varphi) = \sum_{n=0}^{\infty} a_n \rho^n P_n(\cos(\varphi))$$

and require that

$$u(\kappa, \varphi) = \sum_{n=0}^{\infty} a_n \kappa^n P_n(\cos(\varphi)) = f(\varphi).$$

To put this into the context of an eigenfunction expansion, recall that $\varphi = \arccos(x)$ to write

$$\sum_{n=0}^{\infty} a_n \kappa^n P_n(x) = f(\arccos(x)).$$

This is an eigenfunction expansion of $f(\arccos(x))$ in terms of the eigenfunctions of Legendre's equation. The coefficients $a_n \kappa^n$ in this series are given by

$$a_n \kappa^n = \frac{\int_{-1}^{1} f(\arccos(x)) P_n(x)\,dx}{\int_{-1}^{1} P_n^2(x)\,dx}.$$

This determines each a_n. Further, it can be shown that

$$\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}.$$

Therefore,

$$a_n = \frac{2n+1}{2\kappa^n} \int_{-1}^{1} f(\arccos(x)) P_n(x)\,dx.$$

We finally have the solution

$$u(\rho, \varphi) = \sum_{n=0}^{\infty} \frac{2n+1}{2} \left(\int_{-1}^{1} f(\arccos(x))\, P_n(x)\,dx \right) \left(\frac{\rho}{\kappa} \right)^n P_n(\cos(\varphi)).$$

Special functions such as Legendre polynomials are often included in computational software packages, making it possible to compute selected terms of this series solution to approximate the solution.

Example 8.4 *Here is an example in which the solution is a finite sum. Suppose that the sphere has radius $\kappa = 2$ and $f(\varphi) = \sin^2(\varphi)$. Then*

$$\begin{aligned} f(\arccos(\varphi)) &= \sin^2(\arccos(x)) \\ &= 1 - \cos^2(\arccos(x)) = 1 - x^2. \end{aligned}$$

The solution is

$$u(\rho, \varphi) = \sum_{n=0}^{\infty} \frac{2n+1}{2} \left(\int_{-1}^{1} (1 - x^2) P_n(x)\,dx \right) \left(\frac{\rho}{2} \right)^n P_n(\cos(\varphi)).$$

All we need to do is compute the numbers $\int_{-1}^{1} (1 - x^2) P_n(x)\,dx$. First,

$$\int_{-1}^{1} (1 - x^2) P_0(x)\,dx = \int_{-1}^{1} (1 - x^2) dx = \frac{4}{3}$$

and

$$\int_{-1}^{1} (1 - x^2) P_2(x)\,dx = \int_{-1}^{1} (1 - x^2) \frac{1}{2}(3x^2 - 1)\,dx = -\frac{4}{15}.$$

Now evaluate $\int_{-1}^{1}(1-x^2)P_n(x)\,dx$ for some values of n other than 0 and 2. The computational package MAPLE has a Legendre polynomial routine, and it takes only a few seconds to evaluate some of these integrals, obtaining 0 for each. This leads us to conjecture that

$$\int_{-1}^{1}(1-x^2)P_n(x)\,dx = 0$$

if n is a nonnegative integer other than 0 or 2. To verify that this is indeed the case, write

$$x^2 = \frac{2}{3}(3x^2-1)+\frac{1}{3} = \frac{2}{3}P_2(x)+\frac{1}{3}P_0(x).$$

Then

$$1-x^2 = \frac{2}{3}(P_0(x)-P_2(x)).$$

By the orthogonality of the eigenfunctions (Legendre polynomials),

$$\int_{-1}^{1}(1-x^2)P_n(x)\,dx = \frac{2}{3}\int_{-1}^{1}(P_0(x)P_n(x)-P_2(x)P_n(x))\,dx = 0$$

if n is different from 0 and 2. The series solution for $u(\rho,\varphi)$ therefore reduces to two terms:

$$u(\rho,\varphi) = \frac{1}{2}\frac{4}{3}P_0(\cos(\varphi)) - \frac{5}{2}\frac{4}{15}\left(\frac{\rho}{2}\right)^2 P_2(\cos(\varphi)).$$

Then

$$u(\rho,\varphi) = \frac{2}{3}-\frac{\rho^2}{12}(3\cos^2(\varphi)-1).$$

This is the solution for the given f. ◇

Solutions Using Bessel Functions

We will solve the problem of Example 8.2, in which an elastic membrane is stretched over a circular frame, set in motion, and allowed to vibrate. We wrote the wave equation in cylindrical coordinates

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial r^2}+\frac{1}{r}\frac{\partial z}{\partial r}\right),$$

assuming that the motion is symmetric about the z - axis. If the membrane is initially stretched to some position and released from rest, the initial conditions are

$$z(r,0) = f(r) \text{ and } \frac{\partial z}{\partial t}(r,0) = 0.$$

There is also a boundary condition

$$z(\kappa,t) = 0,$$

which states that the boundary of the membrane is fixed on the frame and does not move. Here κ is the radius of the membrane.

Separate variables by putting $z(r,t) = R(r)T(t)$ to obtain

$$R'' + \frac{1}{r}R' + \frac{\lambda}{c^2}R = 0$$

and

$$T'' + \lambda T = 0, \tag{8.8}$$

with λ the separation constant. The equation for R is equation 8.3. We can rewrite it in Sturm-Liouville form as

$$(rR')' + \frac{\lambda}{c^2}rR = 0. \tag{8.9}$$

Now observe that equation 8.9 is Bessel's equation of order ν, equation 8.6, with $\nu = 0$, $R(r)$ in place of $y(x)$, and λ/c^2 in place of λ.

It is known that Bessel's equation 8.6 of order ν has the general solution

$$y(x) = aJ_\nu(\sqrt{\lambda}x) + bY_\nu(\sqrt{\lambda}x),$$

in which a and b are arbitrary constants and J_ν and Y_ν are Bessel functions of the first and second kinds, of order ν, respectively. Since we are interested in the zero-order case, the general solution of 8.9 has the form

$$R(r) = aJ_0\left(\frac{\sqrt{\lambda}}{c}r\right) + bY_0\left(\frac{\sqrt{\lambda}}{c}r\right).$$

Since we are dealing with a specific problem, we need to know J_0 and Y_0 explicitly. It is not difficult to derive (or look up) the expansions

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}n!}x^{2n}$$

and

$$Y_0(x) = J_0(x)\ln(x) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{2n}(n!)^2}\phi(n)x^{2n},$$

where

$$\phi(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}.$$

We now have a general expression for $R(r)$ satisfying equation 8.9. Notice the logarithm term in Y_0. Since the center of the membrane is at $r = 0$, and $Y_0(0)$ is not defined, $Y_0(\sqrt{\lambda}r/c)$ cannot occur in the solution. We must choose $b = 0$. This means that $R(r)$ must have the form

$$R(r) = aJ_0\left(\frac{\sqrt{\lambda}}{c}r\right).$$

Now return to equation 8.8 for T, which has the general solution

$$T(t) = c\cos(\sqrt{\lambda}t) + d\sin(\sqrt{\lambda}t).$$

At this point we have functions

$$z_\lambda(r,t) = a_\lambda J_0\left(\frac{\sqrt{\lambda}}{c}r\right)\cos\left(\sqrt{\lambda}t\right) + b_\lambda J_0\left(\frac{\sqrt{\lambda}}{c}r\right)\sin\left(\sqrt{\lambda}t\right)$$

for each positive number λ and any numbers a_λ and b_λ.

Now use the initial and boundary conditions. First,

$$\frac{\partial z_\lambda}{\partial t}(r,0) = b_\lambda J_0\left(\frac{\sqrt{\lambda}}{c}r\right) = 0$$

for $0 \le r \le \kappa$. Since $J_0(\sqrt{\lambda}r/c)$ is not identically zero, we must choose each $b_\lambda = 0$.

Next,

$$z_\lambda(\kappa,t) = a_\lambda J_0\left(\frac{\sqrt{\lambda}}{c}\kappa\right)\cos(\sqrt{\lambda}t) = 0$$

for all $t \ge 0$. No matter how we choose a positive λ, there will be some time t such that $\cos(\sqrt{\lambda}t) \ne 0$. If we are to have a nontrivial solution, we must allow for at least some a_λ to be nonzero. We therefore seek λ so that

$$J_0\left(\frac{\sqrt{\lambda}}{c}\kappa\right) = 0. \tag{8.10}$$

Values of λ satisfying equation 8.10 are the eigenvalues of this problem. From Sturm-Liouville theory, and also from the extensive literature available on Bessel functions, there is an infinite sequence of positive numbers $j_1, \cdots, j_n, \cdots$ which tend to ∞ as n increases, and satisfying $J_0(j_n) = 0$. These are the positive zeros of J_0. Thus, from equation 8.10, choose the numbers λ to satisfy

$$\frac{\sqrt{\lambda}}{c}\kappa = j_n$$

with n any positive integer. The eigenvalues are the numbers

$$\lambda_n = \left(\frac{j_n c}{\kappa}\right)^2.$$

With these eigenvalues, the corresponding eigenfunctions are $J_0(j_n r/\kappa)$.

Now we have, for each positive integer n, a function

$$z_n(r,t) = a_n J_0\left(\frac{j_n}{\kappa}r\right)\cos\left(\frac{j_n c}{\kappa}t\right)$$

satisfying the partial differential equation, $z(\kappa,t)=0$, and $z_t(r,0)=0$. There remains to choose the coefficients a_n to obtain a solution satisfying $z(r,0)=f(r)$. For this we generally must use a superposition

$$z(r,t)=\sum_{n=1}^{\infty} a_n J_0\left(\frac{j_n}{\kappa}r\right)\cos\left(\frac{j_n c}{\kappa}t\right).$$

As we have done in similar circumstances previously, we will determine the coefficients a_n by using the initial condition to write

$$z(r,0)=\sum_{n=1}^{\infty} a_n J_0\left(\frac{j_n}{\kappa}r\right)=f(r).$$

This is an eigenfunction expansion of the initial position function, in terms of the eigenfunctions of the Sturm-Liouville problem obtained by separation of variables in this initial-boundary value problem. The coefficients are given by equation 8.7, with weight function $p(r)=r$ from equation 8.9:

$$a_n=\frac{\int_0^\kappa rf(r)J_0(j_n r/\kappa)\,dr}{\int_0^\kappa r\,(J_0(j_n r/\kappa))^2\,dr}.$$

These are the Fourier - Bessel coefficients of f. In practice, depending on f, we would need a numerical computation program to approximate the value of the integrals in the numerator and denominator of this expression for a_n, and then approximate the sum of the series by a partial sum. The numbers $j_1, j_2, \cdots$ can be obtained from standard tables of zeros of Bessel functions, or from various software routines.

Example 8.5 *We will illustrate these ideas using MAPLE for the computations. Suppose that $\kappa=c=1$ and $f(r)=\sin(\pi r)$. First generate the coefficients in the series solution. For these, we need the nth positive zero $j(n)$ of J_0. This number is generated by MAPLE for each positive integer n by entering the code*

$$j := n-> evalf(BesselJZeros(0,n));$$

exactly as it is typed here (including the semicolon at the end). The coefficients in the series expansion are given by the quotient

$$a_n=\frac{\int_0^1 r\sin(\pi r)J_0(j(n)r)\,dr}{\int_0^1 r(J_0(j(n)r))^2\,dr}.$$

These can be generated within MAPLE by the code

$$a := n-> int(r*sin(Pi*r)*BesselJ(0,r)(j(n)*r), r=0..1)$$
$$/int(r*((BesselJ(0,r)(j(n)*r))^2), r=0..1);$$

The semicolon at the end of the command is essential. The entire code should be typed continuously, without breaks. In this way we can approximate the coefficients to any reasonable degree of accuracy we want. Note the use of the name

BesselJ(0,r) for J_0, *with r as the designated variable. The infinite series in the solution can be approximated by a partial sum of N terms. If we denote this partial sum as* $Z_N(r,t)$, *then*

$$Z_N(r,t) = \sum_{n=1}^{N} (a_n J_0(j(n)r)\cos(j(n)t)).$$

Keep in mind here that $\kappa = c = 1$ *for this example, so this expression is a little simpler than might occur in general. This partial sum, with the name ZNt, can be entered into MAPLE using the code*

$$ZNt := r- > sum(a(n) * BesselJ(0,n)(j(n) * r) * cos(j(n) * t), n = 1..N);$$

This enables us to plot the approximate solution for different N and t. For example, to plot the solution for $N = 25$ *and* $t = 0.27$, *use*

$$Z25pt27 := r- > sum(a(n) * BesselJ(0,n)(j(n) * r) * cos(j(n) * .27), n = 1..25);$$

followed by

$$plot(Z25pt27(r), r = 0..1);$$

We summed the first 25 terms of the series to approximate the solution for $0 \le r \le 1$ *and times* $t = 0, 0.69, 1.17,$ *and* 1.51. *The use of 25 terms was an estimate based on the fact that, at time zero, this gives an accurate graph of the initial position function. Better accuracy could be achieved with more terms.*

At time $t = 0$ *we get the familiar graph of* $\sin(\pi r)$. *At time* $t = 0.069$ *the center of the membrane* $(r = 0)$ *has moved upward about 0.25 unit. At time* $t = 1.17$ *the center has moved downward to about* -1.19 *units. And at* $t = 1.51$, *the center has moved back upward to be about 0.85 unit below the* x, y *- plane. Figure 8.4 shows the graphs of the approximate solutions at these times.* ◇

Problems for Section 8.1

Some of these problems require the availability of computer software to carry out integrations and summations, and to draw graphs. If MAPLE is used, $P_n(x)$ is identified as $LegendreP(n, x)$.

1. Find the eigenvalues and eigenfunctions for the problem

   $$y'' + \lambda y = 0; y(0) = y'(6) = 0.$$

   Determine the coefficients in the expansion of $f(x) = x$ on $[0, 6]$ in a series of these eigenfunctions. Graph the third and sixth partial sums of this series together with f, for comparison.

2. Find the eigenvalues and eigenfunctions for the problem

   $$y'' + \lambda y = 0; y'(0) = y(3) = 0.$$

   Determine the coefficients in the expansion of $f(x) = x(1 - x)$ on $[0, 3]$ in a series of these eigenfunctions. Graph the third and sixth partial sums of this series, together with f, for comparison.

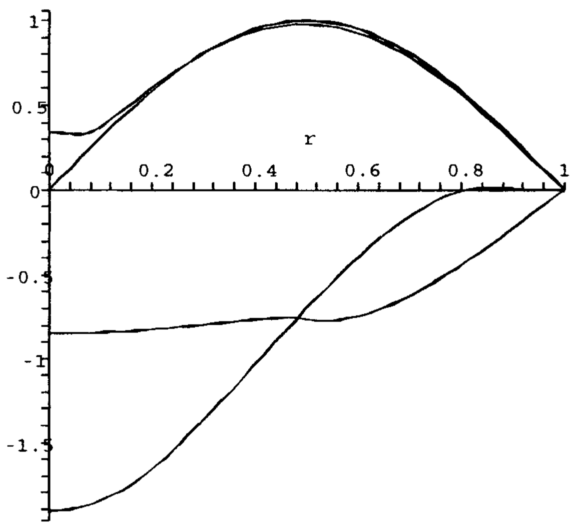

Figure 8.4: Approximate solutions in Example 8.5 at times $0, 0.069, 1.17$, and 1.51.

3. Find the eigenvalues and eigenfunctions for the problem

$$y'' + \lambda y = 0; y'(0) = y'(2) = 0.$$

Determine the coefficients in the expansion of $f(x) = x^2$ on $[0, 2]$ in a series of these eigenfunctions. Graph the third and sixth partial sums of this series, together with f, for comparison.

4. Let

$$L(x,t) = \frac{1}{\sqrt{1 - 2xt + t^2}}.$$

Use the Maclaurin series

$$\frac{1}{\sqrt{1-z}} = 1 + \frac{1}{2}z + \frac{3}{8}z^2 + \frac{15}{48}z^3 + \cdots$$

with $z = 2xt - t^2$ to show that

$$L(x,t) = 1 + \frac{1}{2}(2xt - t^2) + \frac{3}{8}(2xt - t^2)^2 + \frac{15}{48}(2xt - t^2))^3 + \cdots .$$

By expanding these powers of $2xt - t^2$, show that

$$L(x,t) = P_0(x) + P_1(x)t + P_2(x)t^2 + P_3(x)t^3 + \cdots .$$

This suggests that, in general,

$$L(x,t) = \sum_{n=0}^{\infty} P_n(x)t^n.$$

For this reason, $L(x,t)$ is called a generating function for the Legendre polynomials. These polynomials are the coefficients in the expansion of $L(x,t)$ in a Maclaurin series in t.

5. Use the generating function of Problem 4 to derive the recurrence relation

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$$

for $n = 1, 2, \cdots$. Hint: First show that

$$(1 - 2xt + t^2)\frac{\partial L(x,t)}{\partial t} - (x - t)L(x,t) = 0.$$

Substitute $L(x,t) = \sum_{n=0}^{\infty} P_n(x)t^n$ into this equation and determine the coefficient of each power of t.

In each of Problems 6 through 9, graph the given function f and the partial sum of the first six terms of the expansion of this function in a series of Legendre polynomials on $[-1, 1]$.

6. $f(x) = e^x$

7. $f(x) = xe^{-x}$

8. $f(x) = \sin^2(x)$

9. $f(x) = \cos(x)$

   Problems 10, 11, and 12 refer to terms of the solution for the steady-state temperature distribution for a solid sphere (Examples 8.1 and 8.4). Take the radius of the sphere to be $\kappa = 2$.

10. Determine the first five terms (that is, compute the coefficients in the terms of the series) for the case that $f(\varphi) = \varphi$.

11. Determine the first five terms for the case that $f(\varphi) = \cos(\varphi)$.

12. Determine the first five terms for the case that $f(\varphi) = \varphi^2$.

    Problems 13, 14, and 15 refer to the motion of the circular membrane, involving Bessel functions of order 0 (Examples 8.2 and 8.5). Take the radius of the membrane to be $\kappa = 3$, and let $c = 1$.

13. Graph the first five terms of the series solution for the case $f(r) = 1$, for various values of t.

14. Graph the first five terms of the series solution for the case $f(r) = 3 - r$, for various values of t.

15. Graph the first five terms of the series solution for the case $f(r) = r^2$, for various values of t.

8.2 Numerical Approximations of Solutions

Sometimes we need numerical values of solutions of initial-boundary value problems. If these are not readily computable from series or integral solutions, we may want to use a numerical approximation scheme. We will begin with a problem involving the wave equation:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}.$$

The boundary conditions are

$$y(0, t) = y(L, t) = 0$$

and the initial conditions are

$$y(x, 0) = f(x) \text{ and } \frac{\partial y}{\partial t}(x, 0) = g(x).$$

This problem is specified on the x, t - strip S defined by $0 \leq x \leq L$ and $t \geq 0$. The idea underlying an approximation scheme is to first place a grid of points over this strip. We know the values of $y(x,t)$ on two vertical sides of S from the initial data. We also know $y(x,t)$ on the horizontal base interval from the boundary data. Starting at the base of S, $t=0$, move upward to successively higher points of the grid, obtaining approximate values of $y(x,t)$ at each level from values already calculated at lower levels.

To carry out this strategy, we need to be able to approximate the partial derivatives in the wave equation at grid points. To understand how to do this, begin with the derivative of a function $w(x)$ of one variable:

$$w'(x_0) = \lim_{h\to 0} \frac{w(x_0+h)-w(x_0)}{h}.$$

If h is close enough to 0, we can approximate this derivative as nearly as we want by the difference quotient:

$$w'(x_0) \approx \frac{w(x_0+h)-w(x_0)}{h}.$$

In this expression, h can be positive or negative. If we restrict $h>0$, both cases are covered as follows. First,

$$w'(x_0) \approx \frac{w(x_0+h)-w(x_0)}{h}.$$

This is called the *forward difference approximation* (because $h>0$) of $w'(x_0)$. But also

$$w'(x_0) \approx \frac{w(x_0-h)-w(x_0)}{-h},$$

which is the *backward difference approximation* ($-h<0$). The average of the forward and backward difference approximations is the centered difference approximation

$$w'(x_0) \approx \frac{w(x_0+h)-w(x_0-h)}{2h}.$$

If we apply the *centered difference approximation* to the derivative itself, we obtain the centered difference approximation for the second derivative:

$$w''(x_0) \approx \frac{w(x_0+2h)-2w(x_0)+w(x_0-2h)}{4h^2}.$$

Finally, replace $2h$ with h to get

$$w''(x_0) \approx \frac{w(x_0+h)-2w(x_0)+w(x_0-h)}{h^2}.$$

Using these, we can form approximations of the partial derivatives appearing in the wave equation. We have positive increments Δx in x and Δt in t, so

$$\frac{\partial^2 y}{\partial x^2}(x,t) \approx \frac{y(x+\Delta x,t)-2y(x,t)+y(x-\Delta x,t)}{(\Delta x)^2}$$

and

$$\frac{\partial^2 y}{\partial t^2}(x,t) \approx \frac{y(x,t+\Delta t) - 2y(x,t) + y(x,t-\Delta t)}{(\Delta t)^2}.$$

Now construct the grid across the strip S. Choose a positive integer N and define $\Delta x = L/N$. Also choose a positive number Δt. In practice N will be large and Δt small (more about this later). Define the grid points

$$(x_j, t_k) = (j\Delta x, k\Delta t)$$

for $j = 0, 1, 2, \cdots, N$ and $k = 0, 1, 2, \cdots$. Denote

$$y_{j,k} = y(x_j, t_k) = y(j\Delta x, k\,\Delta t).$$

Using this notation, replace the partial derivatives in the wave equation with the difference approximations centered at (x_j, y_k) to get

$$\frac{y_{j,k+1} - 2y_{j,k} + y_{j,k-1}}{(\Delta t)^2} = c^2\frac{y_{j+1,k} - 2y_{j,k} + y_{j-1,k}}{(\Delta x)^2}.$$

Solve this equation for $y_{j,k+1}$ to get

$$y_{j,k+1} = \left(\frac{c\,\Delta t}{\Delta x}\right)^2 (y_{j+1,k} - 2y_{j,k} + y_{j-1,k}) + 2y_{j,k} - y_{j,k-1}.$$

Notice what this equation tells us. It gives the approximation to $y(x,t)$ at the grid point (x_j, t_{k+1}), at horizontal level $k+1$, in terms of certain approximate values at the next two lower layers, the k and $k-1$ levels. This enables us to begin at the lowest level (the lower boundary level $[0, L]$ of the strip), where we know $y(x,t)$, and move upward, deriving approximations at each level from approximations already done in the two levels below.

Specifically, $y(x_j, t_{k+1})$ depends on approximations $y_{j-1,k}$, $y_{j,k}$, $y_{j+1,k}$ at level k, and $y_{j,k-1}$ at level $k-1$. This five-point diamond configuration of dependence is shown in Figure 8.5. Information about the solution at a grid point at level $k+1$ is obtained from previously made approximations at three grid points at level k and one at level $k-1$. This forces us to manufacture an artificial layer below the base $[0, L]$ of the strip (Figure 8.6), in order to have two layers below the $k = 1$ layer from which to make the first approximations. To manufacture this layer, first write

$$\begin{aligned}\frac{\partial y}{\partial t}(x,0) &\approx \frac{y(x_j, -\Delta t) - y(x_j, 0)}{-\Delta t}\\ &= \frac{y_{j,-1} - y_{j,0}}{-\Delta t} = g(x_j) = g(j\,\Delta x)\end{aligned}$$

for $j = 1, 2, \cdots, N-1$. This equation actually has a $y_{j,-1}$ term, leading us to define $y_{j,-1}$ as

$$y_{j,-1} = y_{j,0} - g(j\,\Delta x)\,\Delta t.$$

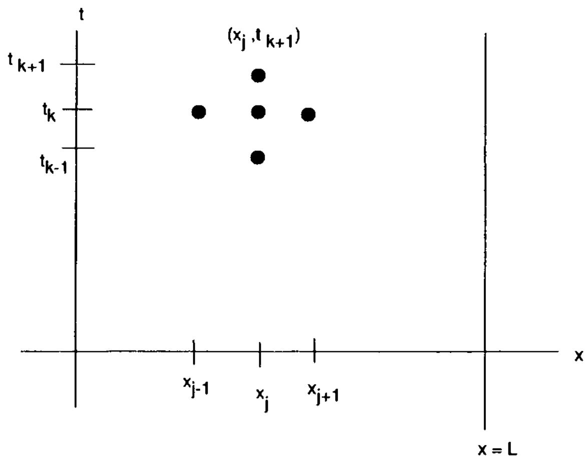

Figure 8.5: Dependence of $y_{j,k+1}$ on approximations at two preceding times.

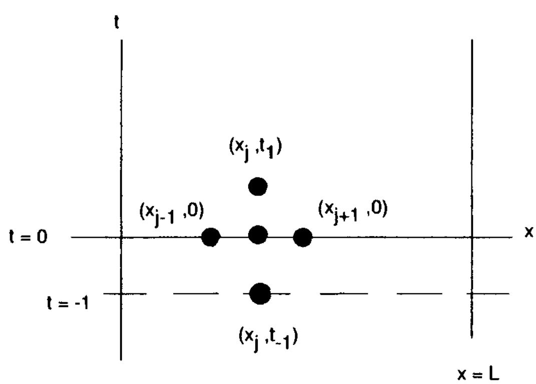

Figure 8.6: Creation of the t_{-1} layer.

We will illustrate some of these calculations for the problem

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2},$$

$$y(0,t) = y(1,t) = 0$$

and

$$y(x,0) = x\sin(\pi x), \frac{\partial y}{\partial t}(x,0) = x(1-x).$$

The interval for this problem is $[0,1]$ and we have taken $c = 1$. Choose $N = 10$, so $\Delta x = 1/10 = 0.1$, and choose $\Delta t = 0.02$. We will approximate values of the solution at layers moving upward from the base of the strip (where the solution is known) in time increments of 0.02 second. Now

$$y_{j,k+1} = 0.04(y_{j+1,k} - 2y_{j,k} + y_{j-1,k}) + 2y_{j,k} - y_{j,k-1}$$

for $j = 1, 2, \cdots, 9$ and $k = 1, 2, 3, \cdots$. Further,

$$y_{j,0} = f\left(\frac{j}{10}\right) = \frac{j}{10}\sin\left(\frac{\pi j}{10}\right)$$

and

$$\begin{aligned} y_{j,-1} &= y_{j,0} - g\left(\frac{j}{10}\right)\Delta t = \frac{j}{10}\sin\left(\frac{\pi j}{10}\right) - \frac{j}{10}\left(1-\frac{j}{10}\right)(0.02) \\ &= \frac{j}{10}\sin\left(\frac{\pi j}{10}\right) - 0.002j\left(1-\frac{j}{10}\right) \end{aligned}$$

for $j = 1, 2, \cdots, 9$.

Compute some approximate values, beginning at the lowest level and working up. First,

$$y_{1,-1} \approx 0.029102, y_{2,-1} \approx 0.11436, y_{3,-1} \approx 0.23851,$$

$$y_{4,-1} \approx 0.37562, y_{5,-1} \approx 0.495, y_{6,-1} \approx 0.56583,$$

$$y_{7,-1} \approx 0.56211, y_{8,-1} \approx 0.46703, y_{9,-1} \approx 0.27632.$$

Moving up to the next level, compute

$$y_{1,0} \approx 0.030902, y_{2,0} \approx 0.11756, y_{3,0} \approx 0.24271,$$

$$y_{4,0} \approx 0.38042, y_{5,0} \approx 0.500000, y_{6,0} \approx 0.57063,$$

$$y_{7,0} \approx 0.56631, y_{8,0} \approx 0.47023, y_{9,0} \approx 0.27812.$$

The next layer gives approximate values of $y(x,t)$ at the time 0.2:

$$y_{1,1} \approx 0.34932, y_{2,1} \approx 0.12230, y_{3,1} \approx 0.24741,$$

$$y_{4,1} \approx 0.38449, y_{5,1} \approx 0.50304, y_{6,1} \approx 0.57243,$$

$$y_{7,1} \approx 0.56684, y_{8,1} \approx 0.46959, y_{9,1} \approx 0.27648.$$

This problem has exact solution

$$y(x,t) = \frac{1}{2}\sin(\pi x)\cos(\pi t) + \sum_{n=2}^{\infty} \frac{-4n(1+(-1)^n)}{\pi^2(n^2-1)^2}\sin(n\pi x)\cos(n\pi t)$$

$$+\sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{n^4\pi^4}\sin(n\pi x)\sin(n\pi t).$$

We can use this expression to compute values of the solution at points of the first ($t = 0.02$) layer of the grid. Of course, since the solution involves infinite series, these values are also approximations. We get (using the equal sign with some informality)

$$y(0.1, 0.2) = 0.03384, y(0.2, 0.2) = 0.12153, y(0.3, 0.2) = 0.24716,$$

$$y(0.4, 0.2) = 0.384856, y(0.5, 0.2) = 0.504011, y(0.6, 0.2) = 0.573918,$$

$$y(0.7, 0.2) = 0.568651, y(0.8, 0.2) = 0.314650, y(0.9, 0.2) = 0.278160.$$

These values of $y(j, 1)$ agree with the approximated values to the third decimal place.

In carrying out numerical approximations (for example, in large projects involving bridge construction or airplane wing design), modern computing power would allow the use of grids with thousands of points, or more, providing high levels of accuracy in the approximations.

Intuitively, we might expect the approximations to improve in accuracy as Δx and Δt are chosen closer to zero. This is true in a sense, but these increments are not entirely independent. One can show that the approximation scheme we have described is stable if $(c\,\Delta t/\Delta x)^2$ remains less than $1/2$. If this condition is violated, the numerical approximations may oscillate and be unreliable.

This method for approximating solutions of a problem involving the wave equation can be adapted to problems involving the heat equation. The significant difference is that in the heat equation the partial derivative with respect to t is a first partial derivative. Consider a typical problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= c\frac{\partial^2 u}{\partial t^2} \\ u(0,t) &= u(L,t) = 0 \\ u(x,0) &= f(x) \end{aligned}$$

for $0 \le x \le L$ and $t \ge 0$.

The centered difference version of the heat equation is

$$\frac{u_{j,k+1} - u_{j,k}}{\Delta t} = c\frac{u_{j+1,k} - 2u_{j,k} + u_{j-1,}}{\Delta x}$$

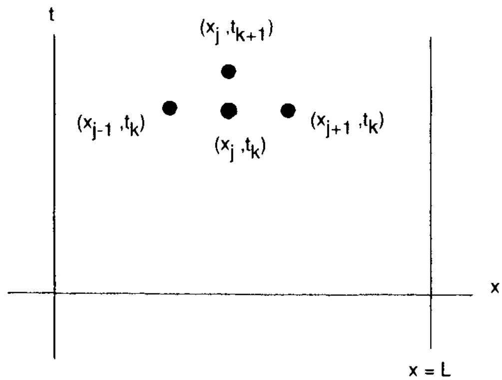

Figure 8.7: Dependence of $u_{j,k+1}$ on approximations at a preceding time.

where $u_{j,k} = u(j\,\Delta x, k\,\Delta t)$. Solve this for $u_{j,k+1}$ to get

$$u_{j,k+1} = \frac{c\,\Delta t}{(\Delta x)^2}(u_{j+1,k} - 2u_{j,k} + u_{j-1,k}) + u_{j,k}.$$

This equation is coupled with the initial condition

$$u_{j,o} = f(j\,\Delta x).$$

The expression for $u_{j,k+1}$ enables us to obtain an approximate value for $u_{j,k+1}$ if we have first approximated $u_{j+1,k}$, $u_{j,k}$, and $u_{j-1,k}$. As shown in Figure 8.7, this gives an approximate value of the solution at a point in the $k+1$ level of the strip S in terms of three values already approximated at the k level. The initial condition provides a starting level (the base of the x, t - strip). While the wave equation requires information on two layers below the one at which the current approximation is being made, for the heat equation we need information only one level below. This means that a fictitious -1 level need not be manufactured for problems involving the heat equation.

Similar to the wave equation, the method does not guarantee that approximate values converge to actual values of the solution as Δx and Δt are chosen closer to 0. However, this will occur if $c\Delta t/(\Delta x)^2 < 1/2$.

Problems for Section 8.2

These problems require the use of software to carry out the computations.

In each of Problems 1, 2, and 3, approximate values $y_{j,1}$ and $y_{j,2}$ of the solution of the wave equation for the given c, L, $y(x, 0)$, $(\partial y/\partial t)(x, 0)$, Δx and Δt. Then, for comparison, write a series solution for $y(x, t)$ and approximate the requested values of $y_{j,k}$ by computing the first 100 terms of the series at $x = x_j, t = t_k$.

1. $c = 1/2$, $L = 1$, $\Delta x = 0.1$, $\Delta t = 0.02$, $y(x, 0) = x(1-x)$, and $(\partial y/\partial t)(x, 0) = 0$.

2. $c = 1/3$, $L = 2$, $\Delta x = 0.1$, $\Delta t = 0.02$, $y(x,0) = 0$, and $(\partial y/\partial t)\partial t(x,0) = \sin(\pi x/2)$.

3. $c = 1, L = 1, \Delta x = 0.1, \Delta t = 0.02, y(x,0) = x(1-x^2)$, and $(\partial y/\partial t)(x,0) = 4$.

   In each of Problems 4, 5, and 6, approximate values $u_{j,1}$ and $u_{j,2}$ of the solution of the heat equation for the given c, L, $u(x,0)$, Δx and Δt. Then write a series solution for $u(x,t)$ and approximate the requested values of $u_{j,k}$ by computing the first 100 terms of the series at $x = x_j, t = t_k$.

4. $c = 1/8$, $L = 1$, $\Delta x = 0.1$, $\Delta t = 0.02$, $u(x,0) = x^2(1-x)$

5. $c = 1/8$, $L = 1$, $\Delta x = 0.1$, $\Delta t = 0.02$, $u(x,0) = \sin(\pi x)$

6. $c = 1/20$, $L = 2$, $\Delta x = 0.1$, $\Delta t = 0.02$, $u(x,0) = x(2-x)^2$

8.3 Burger's Equation

In this section we consider an extended diffusion equation obtained by adding a nonlinear convection term uu_x to the standard heat equation. Burger's equation is

$$u_t + uu_x = ku_{xx},$$

which states that the rate of change of the temperature function with respect to time is a balance of a convection term uu_x and a diffusion term ku_{xx}. This equation has a variety of applications, for example, in mathematical models of fluid flow. We consider Burger's equation over the entire real line, so $-\infty < x < \infty$ and $t \geq 0$. The initial condition is $u(x,0) = f(x)$.

We will show how to write a closed-form solution to this problem as a quotient of integrals. The key is to employ a change of variables called the *Cole-Hopf transformation*. Begin by substituting $u = w_x$ into Burger's equation to get

$$w_{xt} + w_x w_{xx} = kw_{xxx}.$$

We can integrate this equation with respect to x to get

$$w_t + \frac{1}{2}w_x^2 = kw_{xx}.$$

Now define v by

$$v = e^{-w/2k}.$$

Then

$$w = -2k\ln(v)$$

and a routine differentiation yields

$$w_t + \frac{1}{2}w_x^2 - kw_{xx} = 0 = -\frac{2k}{v}(v_t - kv_{xx}).$$

Therefore, v satisfies the standard heat equation $v_t = kv_{xx}$.

The transformation just performed was done in two stages, from u to w, then from w to v. The final transformation may be written

$$u = -\frac{2kv_x}{v}.$$

Using this, transform the initial condition $u(x, 0) = f(x)$ to

$$u(x,0) = -\frac{2kv_x(x,0)}{v(x,0)} = f(x).$$

Integrate this equation to get

$$\int_0^x f(\xi)\, d\xi = -2k \ln(v(x,0)),$$

whereupon

$$v(x,0) = e^{-(1/2k)\int_0^x f(\xi)d\xi} = g(x).$$

We can therefore solve the Burger equation $u_t + uu_x = ku_{xx}$ on the real line, subject to $u(x,0) = f(x)$, by using the Cole-Hopf transformation to arrive at the initial value problem $v_t = kv_{xx}, v(x,0) = g(x)$, which we know how to solve. Write the solution for $v(x,t)$, then transform back to the solution for $u(x,t)$. First,

$$v(x,t) = \frac{1}{2\sqrt{\pi kt}} \int_{-\infty}^{\infty} g(\xi)e^{-(x-\xi)^2/4kt}\, d\xi.$$

Compute

$$v_x(x,t) = -\frac{1}{2\sqrt{\pi kt}} \int_{-\infty}^{\infty} g(\xi)\left(\frac{x-\xi}{2kt}\right) e^{-(x-\xi)^2/4kt}\, d\xi.$$

The analytic solution of the initial value problem involving Burger's equation is therefore

$$\begin{aligned} u(x,t) &= -\frac{2kv_x(x,t)}{v(x,t)} \\ &= \frac{\int_{-\infty}^{\infty} g(\xi)((x-\xi)/t)e^{-(x-\xi)^2/4kt}\, d\xi}{\int_{-\infty}^{\infty} g(\xi)e^{-(x-\xi)^2/4kt}\, d\xi}. \end{aligned}$$

As an example, suppose that $k = 1/2$ and $f(x) = 1$. Then

$$\frac{1}{2k}\int_0^x f(\xi)\, d\xi = x,$$

so

$$g(x) = e^{-x}$$

and the solution is

$$u(x,t) = \frac{\int_{-\infty}^{\infty} e^{-\xi} \left(\frac{x-\xi}{t}\right) e^{-(x-\xi)^2/2t}\, d\xi}{\int_{-\infty}^{\infty} e^{-\xi} e^{-(x-\xi)^2/2t}\, d\xi}.$$

Perhaps surprisingly for an equation associated with diffusion rather than wave motion, Burger's equation may have traveling-wave solutions. By this we mean solutions of the form $u(x,t) = h(x-ct)$, which is called a traveling wave because at time $t > 0$, the graph of $h(x-ct)$ is the graph of $h(x)$ moved ct units to the right.

Under certain conditions, we can determine such a solution explicitly as follows. Look for the function h having two continuous derivatives, and also for a number c that makes $u(x,t) = h(x-ct)$ a solution of $u_t + uu_x = ku_{xx}$. We also require that $h(t)$ have a limit L_+ at ∞, and a limit L_- at $-\infty$, and that $0 < L_+ < L_-$. These limit conditions imply that $h'(x) \to 0$ as x approaches either ∞ or $-\infty$.

To begin, substitute $u(x,t) = h(x-ct)$ into Burger's equation to get

$$-ch' + hh' = kh'',$$

in which h' denotes $h'(\xi)$, with $\xi = x - ct$.

We can integrate this equation directly to obtain

$$-ch + \frac{1}{2}h^2 = kh' + a$$

with a a constant of integration. Solve for h' to get

$$h' = \frac{1}{k}\left(\frac{1}{2}y^2 - cy - a\right). \tag{8.11}$$

From this equation we can determine a and c. Let $x \to \infty$ in equation 8.11 to get

$$\frac{1}{k}\left(\frac{1}{2}L_+^2 - cL_+ - a\right) = 0.$$

Let $x \to -\infty$ in equation 8.11 to get

$$\frac{1}{k}\left(\frac{1}{2}L_-^2 - cL_- - a\right) = 0.$$

Then

$$a = \frac{1}{2}L_+^2 - cL_+ = \frac{1}{2}L_-^2 - cL_-.$$

Then

$$c(L_+ - L_-) = \frac{1}{2}L_+^2 - \frac{1}{2}L_-^2 = \frac{1}{2}(L_+ - L_-)(L_+ + L_-),$$

so

$$c = \frac{1}{2}(L_+ + L_-).$$

This gives us

$$\begin{aligned}a &= \frac{1}{2}L_+^2 - cL_+ = \frac{1}{2}L_+^2 - \frac{1}{2}(L_+ + L_-)L_+ \\ &= \frac{1}{2}L_+^2 - \frac{1}{2}(L_+ + L_-)L_+ = -\frac{1}{2}L_+L_-.\end{aligned}$$

Then

$$\begin{aligned}h' &= \frac{1}{k}\left(\frac{1}{2}h^2 - ch - a\right) \\ &= \frac{1}{k}\left(\frac{1}{2}h^2 - \frac{L_+ + L_-}{2}h + \frac{1}{2}L_+L_-\right) \\ &= \frac{1}{2k}(h - L_+)(h - L_-).\end{aligned}$$

Denoting the independent variable as x, we now have

$$-2k\frac{dh}{dx} = (h - L_+)(L_- - h).$$

This is a separable ordinary differential equation for $h(x)$, which we easily solve. First use a partial fractions decomposition to write

$$\frac{-2}{L_- - L_+}\left(\frac{1}{h - L_+} + \frac{1}{L_- - h}\right) dh = \frac{1}{k}dx$$

and integrate to get

$$\frac{2}{L_- - L_+}\ln\left(\frac{L_- - h}{h - L_+}\right) = \frac{x}{k} + \alpha,$$

in which α is a constant of integration. We can make α vanish (that is, be equal to 0) if we set $h(0) = c$, so

$$\frac{L_- - h(0)}{h(0) - L_+} = 1.$$

Now

$$\ln\left(\frac{L_- - h}{h - L_+}\right) = \frac{(L_- - L_+)x}{2k}.$$

Solve for h to get

$$h(x) = \frac{L_- + L_+e^{(L_- - L_+)x/2k}}{1 + e^{(L_- - L_+)x/2k}}.$$

The traveling-wave solution is therefore

$$\begin{aligned}u(x,t) &= h(x - ct) \\ &= \frac{L_- + L_+e^{(L_- - L_+)(x-ct)/2k}}{1 + e^{(L_- - L_+)(x-ct)/2k}}.\end{aligned}$$

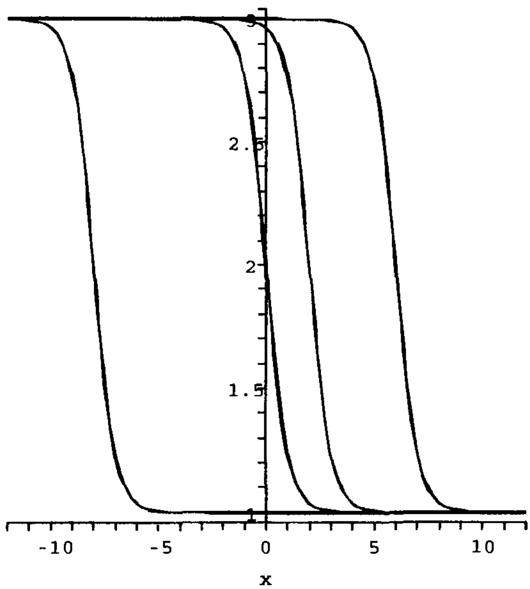

Figure 8.8: Traveling-wave solution of Burger's equation at times $t = -4, 0, 1,$ and 3.

For any time t, the profile of $u(x,t)$ is that of $u(x,0)$ moved ct units to the right. We can interpret c as the wave velocity.

As an example, if we specify that $L_+ = 1$, $L_- = 3$, and $k = 1/2$, then $c = 2$ and the corresponding traveling-wave solution of $u_t + uu_x = \frac{1}{2}u_{xx}$ is

$$u(x,t) = \frac{3 + e^{2(x-2t)}}{1 + e^{2(x-2t)}}.$$

At $t = 0$ this has the profile

$$h(x) = \frac{3 + 2e^{2x}}{1 + e^{2x}},$$

and this graph moves to the right with velocity c as t increases. Figure 8.8 shows a graph of $h(x)$ and some subsequent positions as t increases.

Problems for Section 8.3

In each of Problems 1 through 3, find an expression for the solution of the problem $u_t + uu_x = ku_{xx}$ with $u(x,0) = f(x)$ for all real x, for the given f.

1.

$$f(x) = \begin{cases} 1 - x & \text{if } 0 \le x \le 1 \\ 1 + x & \text{if } -1 \le x \le 0 \\ 0 & \text{if } x < -1 \text{ or if } x > 1 \end{cases}$$

2.
$$f(x) = \begin{cases} 1 & \text{if } -1 \le x \le 1 \\ 0 & \text{if } x < -1 \text{ or if } x > 1 \end{cases}$$

3.
$$f(x) = \begin{cases} -2 & \text{if } -3 \le x \le 0 \\ 5 & \text{if } 0 < x \le 2 \\ 0 & \text{if } x < -3 \text{ or if } x > 2 \end{cases}$$

4. Show that the solution of Burger's equation on the line, subject to $u(x, 0) = f(x)$, can be written as
$$u(x,t) = \frac{\int_{-\infty}^{\infty}((x-\xi)/t)e^{-H(\xi,x,t)/2k}\,d\xi}{\int_{-\infty}^{\infty} e^{-H(\xi,x,t)/2k}\,d\xi},$$
where
$$H(\xi, x, t) = \frac{(x-\xi)^2}{2t} + \int_0^x f(z)\,dz.$$

5. Consider the equation $u_t + uu_x = ku_{xx}$, for $0 < x < 1$, subject to the conditions $u(0, t) = u(1, t) = 0$ and $u(x, 0) = f(x)$. The quantity
$$E(t) = \int_0^1 (u(x,t))^2\,dx$$
is called the *energy* of the system at time t. Show that this energy function is bounded for all $t \ge 0$. Hint: Compute $E'(t)$ as an integral, substitute for u_t in this integral from the differential equation, and integrate by parts, obtaining an ordinary differential equation for $E(t)$.

6. Show that $u(x, t) = c - a\tanh(a(x - ct)/2)$ satisfies the Burger equation $u_t + uu_x = u_{xx}$ on the real line for any positive numbers a and c. Graph this solution with $a = 1$ and $c = 1.5$, for a succession of values of t, say $t = 0, 0.2, 0.3, 0.7, 1, 2, 2.5, 3$ and $t = 5$.

In Problems 7, 8, and 9, determine a traveling-wave solution of Burger's equation on the real line for the given constants. Graph $u(x, 0)$ and $u(x, t)$ for several values of t.

7. $L_+ = 5, L_- = 10, k = 3$

8. $L_+ = 4, L_- = 20, k = 5$

9. $L_+ = 1, L_- = 2, k = 8$

10. Under the conditions assumed in deriving traveling-wave solutions $u(x, t) = h(x - ct)$, attempt to derive solutions $u(x, t) = w(x + ct)$, in which the wave form $u(x, 0)$ moves to the left.

8.4 The Telegraph Equation

Burger's equation is obtained by adding a new term uu_x to the heat equation $u_t = ku_{xx}$. In the same spirit, we obtain the telegraph equation by adding two terms to the wave equation. The form of these new terms is suggested by the analysis of the flow of current through a cable. Let $u(x,t)$ be the current at time t and point x in the cable. It has been found that $u(x,t)$ satisfies the partial differential equation

$$u_{xx} = Cu_{tt} + (RC + SL)u_t + RSu$$

in which C is the capacitance, R the resistance, L the inductance and S the leakage of the cable, per unit length. Because of this background, the partial differential equation

$$u_{tt} + 2bu_t + au = c^2u_{xx} \tag{8.12}$$

is called the *telegraph equation*. Here c is positive and b and a are nonnegative constants. The coefficient of u_t is written as $2b$ for convenience in solving a quadratic equation that will occur shortly.

Our objective is to write a solution of the telegraph equation, subject to the general initial conditions

$$u(x,0) = f(x) \text{ and } u_t(x,0) = g(x).$$

We will assume that f and g have Fourier transforms $F(\omega)$ and $G(\omega)$, respectively. Because x varies over the entire real line, we can attempt a solution by taking the Fourier transform of the telegraph equation, with respect to x. We get

$$U''(\omega,t) + 2bU'(\omega,t) + aU(\omega,t) = -\omega^2c^2U(\omega,t),$$

in which primes denote differentiation with respect to t. On the right we have used the operational formula for the derivative with respect to x of a Fourier transform taken in the x variable. Since x and t are independent, derivatives with respect to t pass through the transform. We now have

$$U''(\omega,t) + 2bU'(\omega,t) + (a + \omega^2c^2)U(\omega,t) = 0.$$

Think of this as an ordinary differential equation in t, with ω carried along as a parameter. This equation has characteristic equation

$$r^2 + 2br + (a + \omega^2c^2) = 0$$

with roots

$$r = -b \pm \sqrt{b^2 - a - \omega^2c^2}.$$

The term under the radical suggests three cases, according to whether $b^2 - a$ is negative, zero, or positive. We will write the solution of the initial value problem in each of these cases.

Case 1: $b^2 - a < 0$.

Now the roots of the characteristic equation are

$$r = -b \pm \sqrt{a - b^2 + (\omega c)^2} i.$$

Then

$$U(\omega, t) =$$
$$e^{-bt}\left[c_1(\omega)\cos\left(\sqrt{a - b^2 + (\omega c)^2} t\right) + c_2(\omega)\sin\left(\sqrt{a - b^2 + (\omega c)^2} t\right)\right].$$

Now

$$c_1(\omega) = U(\omega, 0) = \mathcal{F}(u(x, 0)) = \mathcal{F}(f(x))(\omega) = F(\omega).$$

To obtain $c_2(\omega)$, it is routine to differentiate $U(\omega, t)$ with respect to t and set $t = 0$. This gives us

$$\begin{aligned} U'(\omega, 0) &= \mathcal{F}(u_t(x, 0)) = \mathcal{F}(g(x))(\omega) = G(\omega) \\ &= -bF(\omega) + c_2(\omega)\sqrt{a - b^2 + (\omega c)^2}. \end{aligned}$$

Then

$$c_2(\omega) = \frac{G(\omega) + bF(\omega) \cdot}{\sqrt{a - b^2 + (\omega c)^2}}.$$

This gives us the solution for $U(\omega, t)$:

$$\begin{aligned} U(\omega, t) &= F(\omega)e^{-bt}\cos\left(\sqrt{a - b^2 + (\omega c)^2} t\right) \\ &+ \frac{G(\omega) + bF(\omega)}{\sqrt{a - b^2 + (\omega c)^2}} e^{-bt}\sin\left(\sqrt{a - b^2 + (\omega c)^2} t\right). \end{aligned}$$

Now apply the inverse Fourier transform to obtain the solution of the initial value problem in this case that $b^2 - a < 0$:

$$u(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega, t)e^{i\omega x}\, d\omega.$$

Case 2: $b^2 - a = 0$.

In this case the roots of the characteristic equation are $r = -b \pm \omega c i$, so

$$U(\omega, t) = e^{-bt}\left[c_1(\omega)\cos(\omega c t) + c_2(\omega)\sin(\omega c t)\right].$$

Immediately,

$$U(\omega, 0) = c_1(\omega) = \mathcal{F}(u(x, 0)) = \mathcal{F}(f(x))(\omega) = F(\omega).$$

It is routine to compute $U'(\omega, t)$ and obtain

$$U'(\omega, 0) = -bF(\omega) + c_2(\omega)\omega c = G(\omega).$$

Then

$$c_2(\omega) = \frac{G(\omega) + bF(\omega)}{\omega c}.$$

Now

$$U(\omega, t) = F(\omega)e^{-bt}\cos(\omega ct) + \frac{G(\omega) + bF(\omega)}{\omega c}e^{-bt}\sin(\omega ct),$$

and the solution when $b^2 - a = 0$ is

$$u(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega, t)e^{i\omega x}\, d\omega.$$

Case 3: $b^2 - a > 0$.

This is the most complicated case because now $b^2 - a - (\omega c)^2$ may be positive or less than or equal to zero, depending on the magnitude of ω. We will obtain different expressions for $U(\omega, t)$ in each of two subcases.

Subcase 1: First suppose that $b^2 - a - (\omega c)^2 > 0$. This occurs if

$$|\, \omega \,| < \frac{1}{c}\sqrt{b^2 - a}.$$

Now

$$U_1(\omega, t) = e^{-bt}\left[c_1(\omega)e^{\sqrt{b^2 - a - (\omega c)^2}t} + c_2(\omega)e^{-\sqrt{b^2 - a - (\omega c)^2}t}\right], \tag{8.13}$$

in which the subscript notation U_1 is used to distinguish the solution we are now deriving for $U(\omega, t)$ in subcase 1, from the solution U_2 we will obtain shortly in subcase 2. We immediately have

$$U_1(\omega, 0) = F(\omega) = c_1 + c_2.$$

Further, a routine calculation yields

$$\begin{aligned} U_1'(\omega, 0) &= -b(c_1 + c_2) + \sqrt{b^2 - a - (\omega c)^2}(c_1 - c_2) \\ &= -bF(\omega) + (c_1 - c_2)\sqrt{b^2 - a - (\omega c)^2} = G(\omega). \end{aligned}$$

This gives us two equations for c_1 and c_2, which we solve to get

$$c_1 = \frac{1}{2}F(\omega) + \frac{1}{2}\frac{G(\omega) + bF(\omega)}{\sqrt{b^2 - a - (\omega c)^2}}$$

and

$$c_2 = \frac{1}{2}F(\omega) - \frac{1}{2}\frac{G(\omega) + bF(\omega)}{\sqrt{b^2 - a - (\omega c)^2}}.$$

With these choices of $c_1(\omega)$ and $c_2(\omega)$, $U_1(\omega, t)$ is determined from equation 8.13.

Subcase 2: Now suppose that $b^2 - a - (\omega c)^2 \leq 0$. This occurs if

$$|\omega| \geq \frac{1}{c}\sqrt{b^2 - a}.$$

Now the solution for $U(\omega, t)$ has the form

$$U_2(\omega, t) = e^{-bt}\left[c_3 \cos\left(\sqrt{a - b^2 + (\omega c)^2 t}\right) + c_4 \sin\left(\sqrt{a - b^2 + (\omega c)^2 t}\right)\right]. \tag{8.14}$$

Immediately,

$$U_2(\omega, 0) = c_3 = F(\omega),$$

and we compute that

$$U_2'(\omega, 0) = -bF(\omega) + c_4\sqrt{a - b^2 + (\omega c)^2}.$$

Then

$$c_4 = \frac{G(\omega) + bF(\omega)}{\sqrt{a - b^2 + (\omega c)^2}}.$$

With these choices of $c_3(\omega)$ and $c_4(\omega)$, $U_2(\omega, t)$ is determined from equation 8.14.

Finally, combining subcases 1 and 2, we can write the solution in case 3:

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega, t)e^{i\omega x}\, d\omega \\ &= \frac{1}{2\pi}\left[\int_{|\omega| < \sqrt{b^2 - a}/c} U_1(\omega, t)e^{i\omega x} d\omega + \int_{|\omega| \geq \sqrt{b^2 - a}/c} U_2(\omega, t)e^{i\omega x}\, d\omega\right]. \end{aligned}$$

We now have integral expressions for the solution of the initial value problem involving the telegraph equation over the real line.

An Asymptotic Analysis of the Solution

Sometimes we want to understand the behavior of a function as some parameter or variable becomes arbitrarily large, or arbitrarily small. We illustrate a technique for doing this for the solution of the telegraph equation in Case 3, as the time is allowed to grow without bound. This is called an asymptotic analysis and will yield a qualitative description of how the solution behaves as time extends to infinity.

Because this solution has an integral representation, we begin with a general result about functions defined by integrals. In this context, the O notation ("big O" notation) will be useful. We say that $f(t) = O(g(t))$ as $t \to \infty$ if there are positive numbers C and T such that $|f(t)| \leq Cg(t)$ for $t > T$. For example,

$$\frac{2t^4}{3 + t^6} = O\left(\frac{1}{t^2}\right).$$

This means that, in the sense defined by the notation, $2t^4/(3+t^6)$ behaves like $1/t^2$ for very large t.

Now here is the result we will use. We want to consider functions of the form

$$I(\omega) = \int_a^b Q(\omega)e^{th(\omega)}\,d\omega.$$

In this integral Q and h are smooth functions, and $h'(x) \neq 0$, so h has no maximum on (a, b). The limits of integration may both be real numbers, or one may be infinite (either $a = -\infty$ or $b = \infty$). In these cases we assume that h behaves so that the integral converges. The function Q may be real or complex valued, but h is real valued. We claim that for some number C,

$$I(\omega) = \frac{e^{tM}}{t}\left[C + O\left(\frac{1}{t}\right)\right],$$

in which M is the maximum of $h(x)$ on $[a, b]$, or $[a, \infty)$ or $(-\infty, b]$, whichever applies.

We will sketch a derivation of this result. First, integrate by parts to get

$$\begin{aligned} I(\omega) &= \int_a^b \frac{Q(\omega)}{th'(\omega)}\frac{d}{d\omega}\left(e^{th(\omega)}\right)d\omega \\ &= \left[\frac{Q(\omega)}{th'(\omega)}\right]_a^b - \int_a^b e^{th(\omega)}\frac{1}{t}\frac{d}{d\omega}\left(\frac{Q(\omega)}{h'(\omega)}\right)d\omega. \end{aligned}$$

The last integral on the right can be integrated by parts again:

$$\begin{aligned} &\int_a^b \frac{d}{d\omega}\left(\frac{Q(\omega)}{h'(\omega)}\right)e^{th(\omega)}\,d\omega \\ &= \left[\frac{1}{th'(\omega)}\frac{d}{d\omega}\left(\frac{Q(\omega)}{h'(\omega)}\right)e^{th(\omega)}\right]_a^b \\ &\quad - \int_a^b e^{th(\omega)}\frac{d}{d\omega}\left[\frac{1}{th'(\omega)}\frac{d}{d\omega}\left(\frac{Q(\omega)}{h'(\omega)}\right)\right]d\omega \\ &= \left[\frac{1}{th'(\omega)}\frac{d}{d\omega}\left(\frac{Q(\omega)}{h'(\omega)}\right)e^{th(\omega)}\right]_a^b \\ &\quad - \frac{1}{t}\int_a^b e^{th(\omega)}\frac{d}{d\omega}\left(\frac{1}{h'(\omega)}\frac{d}{d\omega}\left(\frac{Q(\omega)}{h'(\omega)}\right)\right)d\omega. \end{aligned}$$

Now,

$$e^{th(\omega)} \leq e^{tM},$$

so the last part of the preceding equation is $O(e^{tM}/t)$ as $t \to \infty$. Therefore,

$$\begin{aligned} I(\omega) &= \left[\frac{Q(\omega)e^{th(\omega)}}{th'(\omega)}\right]_a^b + \frac{1}{t}O\left(\frac{e^{tM}}{t}\right) \\ &= \frac{e^{tM}}{t}\left[C + O\left(\frac{1}{t}\right)\right] \end{aligned}$$

as $t \to \infty$, for some constant C. This asymptotic estimate of $I(\omega)$ is called *Laplace's method.*

We will apply Laplace's method to the solution of a telegraph equation in case 3. In particular, consider the problem

$$u_{tt} + 2bu_t + au = c^2 u_{xx}; u(x,0) = 0, u_t(x,0) = g(x)$$

with $b^2 - ac > 0$. We know that the solution is

$$u(x,t) = I_1(x,t) + I_2(x,t),$$

where

$$I_1(x,t) = \frac{1}{\sqrt{2\pi}} \int_{|\omega|<\sqrt{b^2-a}/c} U_1(\omega,t)e^{i\omega x}\, d\omega$$

and

$$I_2(x,t) = \frac{1}{\sqrt{2\pi}} \int_{|\omega|\geq\sqrt{b^2-a}/c} U_2(\omega,t)e^{i\omega x}\, d\omega,$$

with

$$U_1(\omega,t) = \frac{G(\omega)}{\sqrt{b^2-a-(\omega c)^2}} e^{-bt} \sinh\left(\sqrt{b^2-a-(\omega c)^2 t}\right) \cdot$$

and

$$U_2(\omega,t) = \frac{G(\omega)}{\sqrt{a-b^2+(\omega c)^2}} e^{-bt} \sin\left(\sqrt{a-b^2+(\omega c)^2 t}\right).$$

Because of the sine term in $U_2(\omega,t)$, we have $I_2(x,t) = O(1)$ as $t \to \infty$. Now look at $I_1(x,t)$, which we will write as

$$\begin{aligned} &I_1(x,t) = \\ &\frac{e^{-bt}}{\sqrt{2\pi}} \int_{|\omega|<\sqrt{b^2-a}/c} \frac{G(\omega)}{2\sqrt{b^2-a-(\omega c)^2}} e^{\sqrt{b^2-a-(\omega c)^2 t}} e^{i\omega x}\, d\omega \\ &- \frac{e^{-bt}}{\sqrt{2\pi}} \int_{|\omega|<\sqrt{b^2-a}/c} \frac{G(\omega)}{2\sqrt{b^2-a-(\omega c)^2}} e^{-\sqrt{b^2-a-(\omega c)^2 t}} e^{i\omega x}\, d\omega. \end{aligned}$$

The second integral on the right is

$$O\left(e^{-\sqrt{b^2-at}}\right)$$

as $t \to \infty$.

For the first integral, apply Laplace's method with

$$Q(\omega) = \frac{G(\omega)}{2\sqrt{b^2-a-(\omega c)^2}} e^{i\omega x}$$

and

$$h(\omega) = \sqrt{b^2-a-(\omega c)^2 t}.$$

It is routine to compute

$$h'(\omega) = \frac{c^2\omega}{\sqrt{b^2 - a - (\omega c)^2}}$$

and

$$h''(\omega) = -\frac{(b^2 - a)c^2}{(b^2 - a - (\omega c)^2)^{3/2}}.$$

Because $h(\omega)$ has its maximum $M = \sqrt{b^2 - a}$ at $\omega = 0$,

$$\int_{|\omega|<\sqrt{b^2-a}/c} e^{th(\omega)}e^{i\omega x}\frac{G(\omega)}{2\sqrt{b^2 - a - (\omega c)^2}}\,d\omega$$
$$= \frac{e^{\sqrt{b^2-a}t}}{\sqrt{-th''(0)}}\left[Q(0) + O\left(\frac{1}{t}\right)\right]$$

as $t \to \infty$. Finally, we conclude that, in this case,

$$u(x,t) = \frac{e^{-(b-\sqrt{b^2-a})}}{\sqrt{t}}\left[C + O\left(\frac{1}{t}\right)\right]$$

as $t \to \infty$.

Problems for Section 8.4

In each of Problems 1, 2, and 3, write the solution of the telegraph equation on the real line for the given constants, with $g(x) = 0$ and $f(x) = 1 - x$ for $0 \le x \le 1$, $f(x) = 1 + x$ for $-1 \le x \le 0$, and $f(x) = 0$ for $x > 1$ and $x < -1$.

1. $b = 2, a = 1, c = 3$

2. $b = 2, a = 6, c = 5$

3. $b = 3, a = 9, c = 4$

   In each of Problems 4, 5, and 6, write the solution of the telegraph equation on the real line for the given constants and for $f(x) = 0$ and $g(x) = 1 - x^2$ for $-1 \le x \le 1$, while $g(x) = 0$ for $x < -1$ and for $x > 1$.

4. $b = 2, a = 4, c = 2$

5. $b = 4, a = 3, c = 1$

6. $b = 2, a = 8, c = 3$

7. Consider the initial value problem for the telegraph equation on the real line, with $f(x) = 0$. Show that the solution in the case $b^2 - a = 0$ can be written
$$u(x,t) = \frac{1}{2c}e^{-bt}\int_{x-ct}^{x+ct} g(\xi)\,d\xi.$$

8. Suppose that $u(x,t)$ is a solution of the telegraph equation. Let $v(x,t) = e^{bt}u(x,t)$. Show that v satisfies

$$v_{tt} + (a - b^2)v = c^2 v_{xx}.$$

9. Solve the problem

$$u_{tt} + 2bu_t + au = c^2 u_{xx}; u(x,0) = f(x), u_t(x,0) = 0$$

for the half-line $x \geq 0$, subject to the boundary condition $u(0,t) = 0$.

8.5 Poisson's Equation

Poisson's equation in two dimensions is

$$u_{xx} + u_{yy} = F(x,y)$$

for (x,y) in some region in the plane. In three dimensions, it is

$$u_{xx} + u_{yy} + u_{zz} = F(x,y,z)$$

for (x,y,z) in some specified region of 3 - space. These reduce to Laplace's equation if F is identically zero.

Poisson's equation is named for Simeon-Denis Poisson (1781-1840), an early pioneer in the development of mathematical physics. The equation occurs in many settings. For example, consider the heat equation in two space dimensions, with an internal energy source term included. This equation is $u_t = k(u_{xx} + u_{yy}) + \sigma(x,y,t)$. The steady-state case occurs in the limit as $t \to \infty$, and this yields Poisson's equation. In a different context, if an electric field is the gradient of a potential function, this potential will satisfy a Poisson equation.

It is obvious that Poisson's equation does not have a unique solution. If u is a solution and h is any harmonic function, $u + h$ is also a solution.

As with Laplace's equation, it is possible to write series and integral solutions of Poisson's equation, in various settings. To illustrate a series solution, we will solve Poisson's equation for a rectangle in the plane, with an assumption on $F(x,y)$. Let the rectangle be given by $0 \leq x \leq a$ and $0 \leq y \leq b$. We will look for a function satisfying Poisson's equation over this rectangle, and vanishing on the horizontal and vertical sides:

$$u(x,0) = u(x,b) = u(0,y) = u(a,y) = 0.$$

The condition we will assume on $F(x,y)$ is that

$$F(x,y) = \sum_{n=1}^{\infty} f_n(y) \sin\left(\frac{n\pi x}{a}\right).$$

This is a Fourier sine expansion in x of $F(x,y)$ on $[0,a]$ for each y in $[0,b]$. The coefficients, which in general will depend on y, are given by

$$f_n(y) = \frac{2}{a}\int_0^a F(\xi, y)\sin\left(\frac{n\pi\xi}{a}\right) d\xi.$$

As a convenience, we will write

$$X_n(x) = \sin\left(\frac{n\pi x}{a}\right).$$

The strategy we will follow is to find functions $u_n(x,y)$ satisfying

$$(u_n)_{xx} + (u_n)_{yy} = f_n(y)X_n(x).$$

If we can do this, then

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y)$$

is a solution of $u_{xx} + u_{yy} = F(x,y)$.

To find $u_n(x,y)$, we will look for $Y_n(y)$ such that

$$u_n(x,y) = Y_n(y)X_n(x).$$

Since $X_n(x)$ satisfies

$$X_n'' + \frac{n^2\pi^2}{a^2}X_n = 0,$$

we have

$$\begin{aligned}(u_n)_{xx} + (u_n)_{yy} &= Y_n''X_n + Y_nX_n'' \\ &= Y_n''X_n - \frac{n^2\pi^2}{a^2}Y_nX_n.\end{aligned}$$

Then

$$\left(Y_n'' - \frac{n^2\pi^2}{a^2}Y_n\right)X_n = f_n(y)X_n(x).$$

Then

$$Y_n''(y) - \frac{n^2\pi^2}{a^2}Y_n(y) = f_n(y). \tag{8.15}$$

At least in theory, $f_n(y)$ is a known function, determined as a Fourier sine coefficient of an expansion of $F(x,y)$ in terms of x on $[0,a]$.

Since $u_n(x,0) = u_n(x,b) = 0$, we must have

$$Y_n(0) = Y_n(b) = 0.$$

This gives us a differential equation 8.15, and two boundary conditions, for $Y_n(y)$. Solve for each $f_n(y)$ and use these functions to solve the boundary value problem for $Y_n(y)$. The solution for u is

$$u(x,y) = \sum_{n=1}^{\infty} Y_n(x)\sin\left(\frac{n\pi x}{a}\right).$$

We illustrate this process for the rectangle $0 \le x \le \pi$, $0 \le y \le 4$, and $F(x, y) = x^2 y$.

First compute

$$\begin{aligned} f_n(y) &= \frac{2}{\pi} \int_0^\pi \xi^2 y \sin(n\xi)\, d\xi \\ &= \frac{2y}{n^3 \pi}(2\cos(n\pi) - 2 - n^2\pi^2 \cos(n\pi)) \\ &= k_n y, \end{aligned}$$

where we have written the coefficient of y as k_n as a convenience. Now the boundary value problem for Y_n is

$$Y_n'' - n^2 Y_n = f_n(y) = k_n y.$$

This problem is easily solved to yield

$$Y_n(y) = \frac{4k_n}{n^2 \sinh(4n)} \sinh(ny) - \frac{k_n y}{n^2}.$$

The solution is

$$u(x, y) = \sum_{n=1}^{\infty} Y_n(y) \sin(nx).$$

This approach to the Poisson problem is reminiscent of the way we solved the heat equation $u_t = ku_{xx} + F(x, t)$ containing a source term. This is no coincidence, since the Laplacian operator $u_{xx} + u_{yy}$ occurring in Poisson's equation is part of the steady-state heat equation in two space dimensions.

It is possible to write a general integral solution for Poisson's equation for a bounded domain (open, connected set) D. In two dimensions this solution is

$$u(x, y) = \iint_D F(\xi, \eta) \ln\left(\frac{1}{\rho((x, y), (\xi, \eta))}\right) d\xi\, d\eta,$$

where (x, y) is an arbitrary point of D and

$$\rho((x, y), (\xi, \eta)) = \sqrt{(x - \xi)^2 + (y - \eta)^2}$$

is the distance between (x, y) and (ξ, η).

In three dimensions, the solution is

$$u(x, y, z) = \iiint_D F(\xi, \eta, \zeta) \frac{1}{\rho((x, y, z), (\xi, \eta, \zeta))} d\rho\, d\eta\, d\zeta.$$

Here

$$\rho((x, y, z), (\xi, \eta, \zeta)) = \sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2}$$

is the distance between (x, y, z) and (ξ, η, ζ) in 3-space.

These integral solutions can be derived by an argument very much like that developed in the proof of the representation theorems used to treat the Dirichlet problem in Chapter 6. As we might expect, these integrals can be evaluated in closed form only for special domains D and functions F.

Problems for Section 8.5

In each of Problems 1 through 6, write a solution of the Poisson problem for the given information. In each problem, $a = \pi$ and b and $F(x, y)$ are given.

1. $b = 3, F(x, y) = \sin(5x)$
2. $b = 4, F(x, y) = \sqrt{2}\sin(12x) - 14\sin(13x)$
3. $b = 1, F(x, y) = x$
4. $b = 2\pi, F(x, y) = \sqrt{3}y$
5. $b = 12, F(x, y) = xy$
6. $b = 7, F(x, y) = x - y$

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by Peter V. O'Neil

Chapter 9

End Materials

9.1 Historical Notes

This section is devoted to some comments on the development of the field of partial differential equations. In this discussion, we may in a broad sense think of the eighteenth century as belonging to the wave equation, and the nineteenth century, to the heat equation.

The first systematic attack on a problem involving a partial differential equation was carried out in a sequence of 1746 papers by Jean Le Rond d'Alembert (1717-1783), who sought the fundamental modes of vibration of a vibrating string. In 1727, John Bernoulli had approximated the vibrating string by imagining a finite number of beads connected by a weightless elastic string and then allowing the number of beads to become infinite. This, however, led to an equation independent of time, and hence not the standard wave equation.

In his papers, d'Alembert derived his formula for the solution in terms of the initial position and velocity functions. Shortly thereafter, the prolific Swiss mathematician Leonhard Euler (1707-1783) extended d'Alembert's results by considering a more general class of functions as possible solutions of the wave equation. It was Euler who, in 1749, obtained special cases of what we would now recognize as Fourier series solutions, resulting in a protracted and sometimes heated debate over the validity of certain kinds of expressions as solutions. The argument engaged some of the leading mathematicians of the day, including Euler, d'Alembert, Joseph-Louis Lagrange (1736-1813), Pierre-Simon de Laplace (1749-1827), and Daniel Bernoulli (1700-1782). Much of the dispute was generated by differing views on the definitions of function and continuity, as well as the absence of the mathematical tools needed to fully understand the convergence of series of functions.

Daniel Bernoulli was perhaps the first mathematical physicist. He was in residence at the St. Petersburg academy established by Catherine the Great of Russia when he did fundamental work on hydrodynamics, elasticity, and the motion of vibrating systems. His 1732 paper gave the higher modes of vibration of a vibrating string.

Some work, particularly by Euler and d'Alembert, was done on a general version of the wave equation. Euler obtained a solution when the mass distribution of the string was allowed to be nonconstant, although he could only treat mass distributions having special forms. D'Alembert also attempted solutions when the thickness of the string varied. Euler considered wave motion in two space dimensions, deriving a series solution for a vibrating membrane. This solution contained functions we would now recognize as Bessel functions.

Another major impetus for work in partial differential equations came from potential theory, motivated by studies of gravitational effects on bodies of different shapes and mass distributions.

First-order partial differential equations began to receive attention about 1739, when Alexis-Claude Clairaut (1713-1765) encountered them in studying the shape of the Earth. In the 1770s Lagrange gave the first systematic treatment of nonlinear first-order equations of the form

$$f(x, y, u, u_x, u_y) = 0.$$

Lagrange had in his possession a form of the method of characteristics, although today Augustin-Louis Cauchy (1789-1857) is credited with this method because he overcame some difficulties that had proved intractable to Lagrange.

Gaspard Monge (1746-1818) was among the first to associate geometry with first-order partial differential equations, developing the concepts of characteristic surfaces and characteristic cones (or Monge cones). Monge also obtained some results on the linear, homogeneous, second-order partial differential equation.

There was also some work on systems of partial differential equations, primarily by Euler and d'Alembert. This was motivated by the fact that Newton's laws of motion are in vector form and lead to systems when written in terms of components.

The major innovation of the nineteenth century was the development of Fourier series and integrals, and the mathematics that arose from studying and understanding these objects. Joseph Fourier (1768-1830) was a fascinating person. In addition to his mathematical contributions, he accompanied Napoleon on his ill-fated Egyptian military campaign, wrote a widely respected history of ancient Egypt, and later, while serving as a government administrator in France, was responsible for a large land reclamation project and the construction of a highway through the Alps. At the beginning of his career, he was condemned to the guillotine by the Robsepierre-led Revolution, but was spared at the last minute and appointed to a faculty position at a newly formed college.

Fourier's main interest was in studying heat conduction, hence the emphasis in the nineteenth century on the heat equation. The paper he submitted to the Academy of Sciences in Paris in 1807 was judged by Lagrange, Laplace, and Legendre and found wanting in rigor. However, these eminent mathematicians were impressed by the originality and importance of Fourier's ideas and established a prize for work on the problem of heat conduction. In 1811, Fourier submitted a revised version of his paper in competition for the prize, which was awarded to him. Nevertheless, the Academy continued to refuse publication of the work. Finally, in 1822 he published his classic *Théorie analytique de la chaleur*, which contained his ideas on separation of variables and the use of Fourier series and integrals. Subsequent work by Riemann, Dirichlet, and others, devoted to proving convergence theorems about Fourier series, opened up whole new areas of mathematics and led to profound developments in analysis and even set theory (Cantor's theory of cardinals).

Potential theory also saw advances in the nineteenth century. Part of the motivation lay in studies of gravitational attraction, but Laplace's equation and Maxwell's field equations gave new initiative to potential theory as well. Major contributors were Simeon-Denis Poisson (1781-1840), Cauchy, Laplace, and Carl Friedrich Gauss (1777-1855), who were all scientists and mathematicians of considerable stature. A surprising figure in potential theory who emerged at this time was the Englishman George Green (1793-1841), who taught himself calculus and in 1828 produced at his own expense a little book entitled *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism.* Because Green was unknown and without university affiliation, the booklet went largely unnoticed until Lord Kelvin happened upon a copy, understood the importance of its contents, and had it published in a journal. Among other results, Green proved several integral relationships which now bear his name (Green's theorem, Green's identities). Green's theorem was proved independently in 1828 by the Ukrainian mathematician Michel Ostrogradsky (1801-1861). Green's functions are also named for George Green.

Much of Green's booklet was devoted to applications of his integral theorems to studies of electricity and magnetism. In one section he anticipated the use of Dirichlet's principle in attempting to prove the existence of a solution of a Dirichlet problem.

As one might expect, early work on partial differential equations focused on attempts to write solutions in terms of series, integrals, or newly developed special functions. Soon it became apparent that subtle existence questions needed to be addressed. Cauchy was among the first to observe that partial differential equations of order two or higher can be exchanged for systems of first-order partial differential equations, and he proved an existence theorem which today bears his name and that of Sophie Kowalevski. Kowalevski (1850-1891) was one of the few women to achieve prominence in mathematics in the nineteenth century. She was a student of the leading German mathematician Karl Weierstrass (1815-1897), and in addition to her results on existence, she did prize-winning work on elasticity and the motion of rotating bodies. In the latter stages of her relatively short life she was a professor of mathematics at the University of Stockholm in Sweden.

The theory of characteristics had been developed by Monge and by André-Marie Ampère (1775-1836) and was later extended by Albert Victor Bäcklund (1845-1922). Jacques Hadamard (1865-1963) adapted the theory to partial differential equations of arbitrary order. Important work was also done on the existence of solutions of Dirichlet problems, particularly by Hermann Amandus Schwarz (1843-1921), who, like Kowalevski, was a student of Weierstrass.

Advances in the understanding of harmonic functions were also made by Henri Poincaré (1854-1912) and David Hilbert (1862-1943), respectively, the leading French and German mathematicians of the last part of the nineteenth century and early part of the twentieth century. It was Hilbert who proved the Dirichlet principle anticipated by Bernhard Riemann and George Green.

Today, partial differential equations encompass a vast body of theory and significant areas of applications. In applied research, such as studies of shock waves and turbulence in fluids, the development of computing power has made it possible to obtain numerical information about solutions of problems that were considered intractable 50 years ago. The need for rigor in establishing results was demonstrated dramatically by the construction of unanticipated examples. In 1957, Hans Lewy produced a linear partial differential equation having no singular points (that is, the coefficient functions are well behaved) and for which no solution exists at any point. In addressing such issues, current work in partial differential equations takes place in a setting of Hilbert and Sobolev spaces, making use of distributions and methods from functional analysis.

9.2 Glossary

$f(x+) = \lim_{h \to 0+} f(x+h)$
$f(x-) = \lim_{h \to 0-} f(x-h)$
$\mathcal{F}$ Fourier transform
$\mathcal{F}^{-1}$ inverse Fourier transform
$\mathcal{F}[f]$ Fourier transform of f
$\hat{f}$ Fourier transform of f
$f * g$ convolution of f with g
$\mathcal{F}_S$ Fourier sine transform
$\hat{f}_S$ Fourier sine transform of f
$\mathcal{F}_C$ Fourier cosine transform
$\hat{f}_C$ Fourier cosine transform of f
∇u gradient of u
$\nabla^2 u$ Laplacian of u
$\mathcal{L}$ Laplace transform
R^n real n - dimensional space
$B(\mathbf{x}_0, r)$ open ball of radius r about $\mathbf{x}_0$
∂A boundary of A
$S(\mathbf{x}_0, r)$ sphere of radius r about $\mathbf{x}_0$ in R^n
$\overline{A}$ closure of A
$\partial u/\partial n$ derivative of u in the direction of a unit vector $\mathbf{n}$
$C^n(A)$ set of functions continuous on A, with continuous partial derivatives on A of orders 1 through n
$f : D \to K$ f is a function mapping elements of D into K
$\mathrm{Re}(z)$ real part of z
$\mathrm{Im}(z)$ imaginary part of z

$f_{K,v}$ a function that is harmonic on a disk K and equal to v on the given domain but outside K
ω_Q a barrier function at a boundary point Q
T_f the distribution associated with a function f

9.3 Answers to Selected Problems

Chapter 1: First-Order Equations

Section 1.1
4. (a) $u(x,t) = \varphi(x + kt)$
(c) Let $p(u) = \cos(u)$, to obtain $u(x,t) = \varphi(x + \cos(u)t)$.
(e) With $p(u) = u\sin(u), u(x,t) = \varphi(x + u\sin(u)t)$.
7. (a) nonlinear, but quasi-linear
(c) nonlinear, but quasi-linear
(e) nonlinear, but quasi-linear
(g) nonlinear and not quasi-linear
(j) nonlinear and not quasi-linear
Section 1.2
1. The characteristics are straight lines $5x - 3y = k$. Let $\xi = x, \eta = 5x - 3y$ to obtain

$$w_\xi + \frac{1}{9}(\xi\eta - 5\xi^2)w = 0.$$

This has the general solution

$$w(\xi,\eta) = g(\eta)e^{(-\eta\xi^2/18)+(5\xi^2/27)}.$$

Then

$$u(x,y) = g(5x - 3y)e^{(yx^2-5x^3/9)/6}.$$

3. The characteristics are lines $y = 4x + k$. Let $\xi = x, \eta = 4x - y$ to obtain

$$w_\xi - \xi w = \xi.$$

This has the general solution

$$w = -1 + e^{\xi^2/2}g(\eta),$$

so

$$u(x,y) = -1 + e^{x^2/2}g(4x - y).$$

5. The characteristics are hyperbolas $xy = k$. Let $\xi = x, \eta = xy$ to get

$$\xi w_\xi + w = \xi,$$

with general solution

$$w = \frac{1}{2}\xi + \frac{1}{\xi}g(\eta).$$

Then
$$u(x,y) = \frac{1}{2}x + \frac{1}{x}g(xy).$$

7. The characteristics are parabolas $y = -(x^2/2)+k$. Let $\xi = x, \eta = y+x^2/2$ to get $w_\xi = 4$, so $w = 4\xi + g(\eta)$ and $u(x,y) = 4x + g(y + x^2/2)$.

9. The characteristics are the lines $x - y = k$. Let $\xi = x, \eta = y - x$ to get
$$w_\xi - w = \xi + \eta$$
with general solution
$$w = -\xi - 1 - \eta + g(\eta)e^\xi.$$
Then
$$u(x,y) = -1 - y + e^x g(y - x).$$

11. The characteristics are defined by $x - \ln(y) = k$, and we get $w_\xi + \xi w = 0$. This has the general solution
$$w = e^{-\xi^2/2}g(\eta),$$
so
$$u(x,y) = e^{-x^2/2}g(x - \ln(y)).$$

13. The characteristics are given by $\alpha x - \ln(y-1) = k$. Let $\xi = x, \eta = \alpha x - \ln(y-1)$. We get
$$w_\xi - \frac{1}{2}\beta f(\xi)e^{\alpha\xi-\eta}w = 0.$$
This yields
$$u(x,y) = g(\alpha x - \ln(y-1))e^{\frac{1}{2}\beta e^{-\eta}\int_0^x f(s)e^{\alpha s}ds}.$$
The solution satisfying $u(0,y) = y^n$ is
$$u(x,y) = \left(1 + e^{-\alpha x+\ln(y-1)}\right)^n e^{\frac{1}{2}\beta e^{-\eta}\int_0^x f(s)e^{\alpha s}ds}.$$

Section 1.3

1. The characteristics are ellipses $x^2 + 3y^2/2 = k$. Let $\xi = x, \eta = x^2 + 3y^2/2$ to get $w_\xi = 0$. Then $w = g(\eta)$ and $u(x,y) = g\left(x^2 + 3y^2/2\right)$.

(a) We need $u(x,x) = x^2 = g(5x^2/2)$, so $g(t) = 2t/5$ and
$$u(x,y) = g\left(x^2 + 3y^2/2\right) = \frac{2}{5}\left(x^2 + \frac{3}{2}y^2\right).$$

(b) Now $u(x,-x) = g(5x^2/2) = 1 - x^2$, so $g(t) = 1 - 2t/5$ and
$$u(x,y) = 1 - \frac{2}{5}\left(x^2 + \frac{3}{2}y^2\right).$$

(c) We want $u(x,y) = 2x$ on $2x^2 + 3y^2 = 4$, which is a characteristic. Now we must choose g so that

$$g\left(x^2 + \frac{3}{2}y^2\right) = g(2) = 2x,$$

which is impossible. There is no solution with data prescribed along this characteristic.

3. The characteristics are lines $y - 2x = k$. We find the general solution $u(x,y) = -1 + g(y-2x)e^{x/4}$.

(a) For a solution satisfying $u(x,3x) = \cos(x)$, we need $u(x,3x) = -1 + g(x)e^{x/4} = \cos(x)$. Choose $g(t) = (1+\cos(t))e^{-t/4}$ to get

$$u(x,y) = -1 + (1+\cos(y-2x))e^{-y/4}e^{3x/4}.$$

(b) There is no solution satisfying the given condition on this characteristic.

(c) We want $u(x,x^2) = 1-x$, so we need $g(x^2-2x) = (2-x)e^{-x/4}$. Let $t = x^2 - 2x$, leading to two possibilities. If $x = 1 + \sqrt{1+t}$, we get

$$g(t) = (1-\sqrt{1+t})e^{-(1+\sqrt{1+t})/4}$$

and

$$u(x,y) = -1 + \left(1 - \sqrt{1+y-2x}\right)e^{x/4}e^{-(1+\sqrt{1+y-2x})/4}.$$

If $x = 1 - \sqrt{1+t}$, we get

$$u(x,y) = -1 + \left(1 + \sqrt{1+y-2x}\right)e^{x/4}e^{-(1-\sqrt{1+y-2x})/4}.$$

5. The characteristics are given by $2x^3 - 3y^2 = k$. Let $\xi = x, \eta = 2x^3 - 3y^2$ to get $w_\xi = \xi$, so $w = \xi^2/2 + g(\eta)$ and

$$u(x,y) = \frac{1}{2}x^2 + g(2x^3 - 3y^2).$$

(a) Choose $g(t) = 4(3t/5)^{1/3} - \frac{1}{2}(3t/5)^{2/3}$ to get

$$u(x,y) = \frac{1}{2}x^2 + 4\left(\frac{3}{5}(2x^3-3y^2)\right)^{1/3} - \frac{1}{2}\left(\frac{3}{5}(2x^3-3y^2)\right)^{2/3}.$$

(b) There is no solution satisfying this condition, since it requires that $u(x,y) = x^3 = \frac{1}{2}x^2 + g(0)$, and this is impossible.

(c) Choose $g(t) = \sin((t/2)^{1/3}) - \frac{1}{2}(t/2)^{2/3}$ to get

$$u(x,y) = \frac{1}{2}x^2 + \sin\left(\left(\frac{2x^3-3y^2}{2}\right)^{1/3}\right) - \frac{1}{2}\left(\frac{2x^3-3y^2}{2}\right)^{2/3}.$$

Section 1.4

1. $u(x,y) = \arcsin\left(\frac{1}{2}\ln(x^3/y)\right)$

3. $u(x,y)$ is implicitly defined by

$$y = \frac{1}{u + 1/((x-u)^2+2)}.$$

5. $u(x,y)$ is implicitly defined by $y = (1 - x + \ln(u))u$.
7.

$$u(x,y) = \frac{3}{y - 2x + \frac{7}{4}}.$$

9. $u(x,y)$ is implicitly defined by

$$\frac{1}{y} = \ln(u/2) + \frac{1}{1-(x-\ln(u/2))^2}.$$

Chapter 2: Linear Second-Order Equations

Section 2.1

3. Here $B^2 - AC = 4x^2 - 7$. The equation is hyperbolic if $4x^2 - 7 > 0$, elliptic if $4x^2 - 7 < 0$, and parabolic if $4x^2 = 7$.

5. $B^2 - AC = -6y$, so the equation is hyperbolic if $y < 0$, elliptic if $y > 0$, and parabolic if $y = 0$.

7. $B^2 - AC = 1 - x^2$, so the equation is hyperbolic if $|x| < 1$ and elliptic if $|x| > 1$.

9. The equation is hyperbolic if $y^2 > 4$, elliptic if $y^2 < 4$, and parabolic if $y^2 = 4$.

Section 2.2

2. (a) The canonical form is $w_{\xi\eta} = 0$, and we get the solution

$$u(x,y) = f\left(y - (2+\sqrt{3})x\right) + g\left(y - (2-\sqrt{3})x\right).$$

(c) The canonical form is $w_{\xi\eta} = 0$, and we get

$$u(x,y) = f\left((1+\sqrt{11})x - 2y\right) + g\left((1-\sqrt{11})x - 2y\right).$$

3. (a)

$$\hat{w}_{\hat{\xi}\hat{\xi}} - \hat{w}_{\hat{\eta}\hat{\eta}} - \frac{2}{9}\hat{w}_{\hat{\xi}} + \frac{1}{6}\hat{w}_{\hat{\eta}} = 0$$

(b)

$$\hat{w}_{\hat{\xi}\hat{\xi}} - \hat{w}_{\hat{\eta}\hat{\eta}} + \frac{1}{216}\hat{\eta}(10\hat{w}_{\hat{\xi}} + 6\hat{w}_{\hat{\eta}}) = 0$$

(c)

$$\hat{w}_{\hat{\xi}\hat{\xi}} - \hat{w}_{\hat{\eta}\hat{\eta}} - \frac{1}{500}(5\hat{\xi} - 3\hat{\eta})\hat{w} = 0$$

(d)

$$\hat{w}_{\hat{\xi}\hat{\xi}} - \hat{w}_{\hat{\eta}\hat{\eta}} - \frac{1}{648}(3\hat{\xi} - 2\hat{\eta})^2\hat{w}_{\hat{\xi}} = 0$$

Section 2.3

2. If we use $\xi = y - \frac{2}{3}x, \eta = x^2$, we get

$$w_{\eta\eta} + \left(\frac{1}{2\eta} + \frac{1}{18\sqrt{\eta}}\right) w_\eta - \frac{1}{54\eta} w_\xi = 0.$$

3. (a) The canonical form is $w_{\eta\eta} = 0$. We get $u(x,y) = xf(4x-y)+g(3x-y)$.

(b) The canonical form is $w_{\eta\eta} = 0$, and the solution is $u(x,y) = xf(2x + y) + g(2x+y)$.

(c) The canonical form is $w_{\eta\eta} = 0$, and the solution is $u(x,y) = xf(2x - 5y) + g(2x-5y)$.

(d) The canonical form is $w_{\eta\eta} = 0$, and the solution is $u(x,y) = xf(2x - 3y) + g(2x-3y)$.

Section 2.4

1. $w_{\xi\xi} + w_{\eta\eta} + \frac{1}{8}(w_\xi + w_\eta) = 0$
3. $w_{\xi\xi} + w_{\eta\eta} - \frac{1}{4}w = 0$
5. $w_{\xi\xi} + w_{\eta\eta} + \frac{1}{5}\cos((2\eta - \xi)/5)w = 0$

Sections 2.1 through 2.4

1. The equation is hyperbolic, with characteristics families of lines

$$y = (-4 + \sqrt{14})x + k, y = (-4 - \sqrt{14})x + c.$$

The canonical form is

$$w_{\xi\eta} = \frac{1}{56\sqrt{14}} \left[\left(4\eta - 4\xi - \sqrt{14}\eta\right) w_\xi + \left(4\eta - 4\xi - \sqrt{14}\xi\right) w_\eta\right] = 0.$$

3. The equation is hyperbolic, with characteristics $y = x + k, y = -x/3 + c$. The canonical form is

$$w_{\xi\eta} + \frac{3}{16}\left(1 + \frac{1}{4}(\xi + 3\eta)\right) w_\xi - \frac{3}{16}\left(\frac{1}{12}(\xi + 3\eta) - 1\right) w_\eta = 0.$$

5. The equation is hyperbolic, with characteristics

$$y = \left(\frac{-4 + \sqrt{10}}{3}\right) x + k, y = \left(\frac{-4 - \sqrt{10}}{3}\right) x + c.$$

The canonical form is

$$w_{\xi\eta} + \frac{3}{16}\frac{1}{2\sqrt{10}}\left[(1 + \sqrt{10})\xi + (-1 + \sqrt{10})\eta\right](w_\xi + w_\eta) = 0.$$

7. The equation is parabolic with characteristics $y = -2x + k$. The canonical form is

$$w_{\eta\eta} + 3w_\xi + w_\eta = 0.$$

9. The equation is hyperbolic, with characteristics $y = -x + k$ and $y = -4x + c$. The canonical form is

$$w_{\xi\xi} - \frac{1}{6}w_\eta = 0.$$

11. Choose $\alpha = a/2$, $\beta = -b/2$, and $h = c + \frac{1}{4}(b^2 - a^2)$.

Section 2.6

1.

$$\varphi(x,y) = y^3 + 4yx + \frac{1}{2}(16 - 4y - 6y^2 - 3y^3)x^2 + \frac{1}{6}(-32 - 48y - 30y^2 + 3y^3)x^3 + \cdots$$

3.

$$\varphi(x,y) = -y^2 + y + \cos(y)x + \frac{1}{2}(-8\cos(y))x^2 + \frac{1}{6}(y\sin(y) + 64\cos(y) - 2)x^3 + \cdots$$

5.

$$\varphi(x,y) = \sin(2y) + 2y^2x + \frac{1}{2}(8y + 6y^2 + 4y^2\sin(2y))x^2 + \frac{1}{6}(16 + 48y + 14y^2 + (12y^2 + 16y - 1)\sin(2y) + 16y^2\cos(2y))x^3 + \cdots$$

7.

$$\varphi(x,y) = x^2 - x + \sin(2x)y + \frac{1}{2}(2x - 2x\cos(2x) - 3)y^2 + \frac{1}{6}(2\cos(2x) - 2x - 4x^2\sin(2x) + 2x\cos(2x) + 4\sin(2x))y^3 + \cdots$$

9.

$$\varphi(x,y) = x\sin(x) + xy + \frac{1}{2}\left(-\frac{2}{x}\cos(x) + \sin(x) - x\right)y^2 + \left(\frac{2}{x}\cos(x) + x\right)y^2 + \cdots$$

Section 2.7

3. Choose $\xi = y - x$ and $\eta = x + y$, to get

$$\varphi_{\xi\xi} + \frac{4}{3}\varphi_{\xi\eta} - \frac{1}{3}\varphi_{\eta\eta} + \frac{1}{6}(\varphi_\xi - \varphi_\eta) = 0.$$

On Γ we have $x(s) = y(s) = s/\sqrt{2}$, so

$$\varphi(x(s), y(s)) = x^2 = s^2/2 = f(s)$$

and

$$\varphi_{\mathbf{n}}(x(s), y(s)) = \frac{1}{\sqrt{2}}\cos(x) = \frac{1}{\sqrt{2}}\cos(s/\sqrt{2}) = g(s).$$

Compute partial derivatives at $(0,0)$, to get

$$\varphi(\xi,\eta) = \frac{1}{2}\xi + \frac{1}{24}\xi^2 + \frac{1}{4}\eta^2 + \cdots.$$

5. Now use $\mathbf{n} = (5\mathbf{i} - \mathbf{j})/\sqrt{26}$, with $\xi = 5x - y$ and $\eta = x + y$, to get

$$\varphi_{\xi\xi} + \frac{22}{19}\varphi_{\xi\eta} - \frac{5}{19}\varphi_{\eta\eta} - \frac{1}{19}(10\varphi_\xi + 2\varphi_\eta) = 0.$$

Now

$$\varphi(x(s), y(s)) = \sin(s/\sqrt{26}) = f(s)$$

and

$$\varphi_{\mathbf{n}}(x(s), y(s)) = \frac{1}{26\sqrt{26}}s^2 = g(s).$$

We get

$$\varphi(\xi, \eta) = -\frac{1}{39}\xi + \frac{1}{6}\eta + \frac{1}{2}\left[\frac{1}{(13)(19)}\xi^2\right] + \cdots.$$

Chapter 3: Elements of Fourier Analysis

Section 3.1

1. $u(x,t) = \sqrt{3}\sin(2x)e^{-4kt}$

3. Look at $\lim_{x\to\pi-}\left(\sum_{n=1}^{N} b_n \sin(nx)\right)$.

Section 3.2

1. The series is

$$\sum_{n=1}^{\infty} \frac{2}{n}\left(1 - (-1)^n\right)\sin(nx).$$

This does not converge to $f(x)$ at $x = 0, \pi, -\pi$.

Section 3.3

3. (a) $\sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^n \sin(n\pi x)$

The series converges to

$$\begin{cases} -x & \text{for } -1 < x < 1 \\ 0 & \text{for } x = 1, -1. \end{cases}$$

(c)

$$\sum_{n=1}^{\infty} \frac{8}{(2n-1)^2\pi^2}\cos\left(\frac{(2n-1)\pi x}{2}\right)$$

The series converges to $1 - \mid x \mid$ for $-2 \le x \le 2$.

(e) This function is its Fourier series on $[-\pi, \pi]$ (but not on other intervals).

(g) The series is

$$\frac{71}{12} + \sum_{n=1}^{\infty}\left(\frac{55\cos(n\pi) - 5}{n^2\pi^2}\right)\cos(n\pi x/5)$$
$$+\left(\frac{50\cos(n\pi) - 50 + (n\pi)^2 - 21(n\pi)^2\cos(n\pi)}{n^3\pi^3}\right)\sin(n\pi x/5).$$

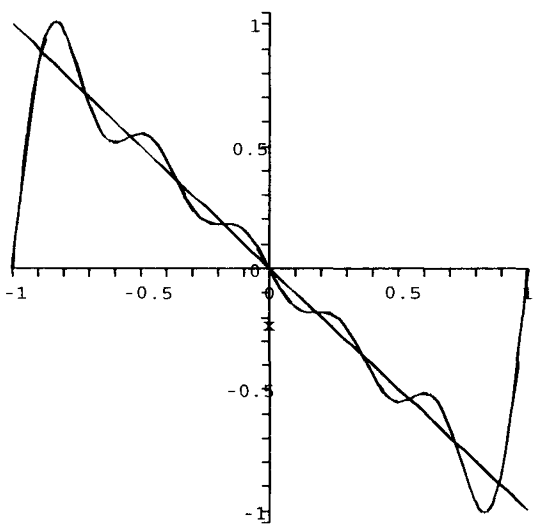

Figure 9.1: Section 3.3, Problem 3(a): fifth partial sum.

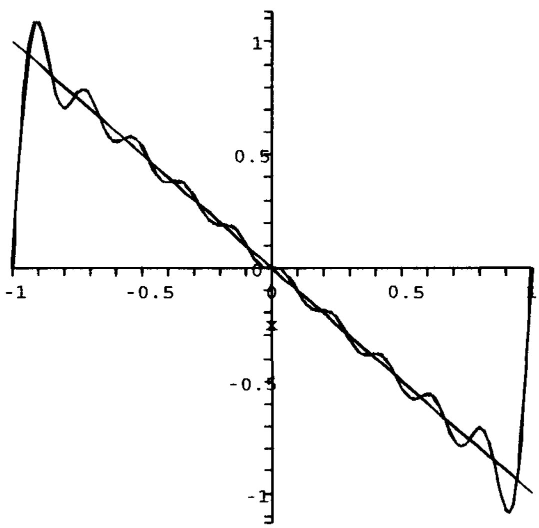

Figure 9.2: Section 3.3, Problem 3(a): tenth partial sum.

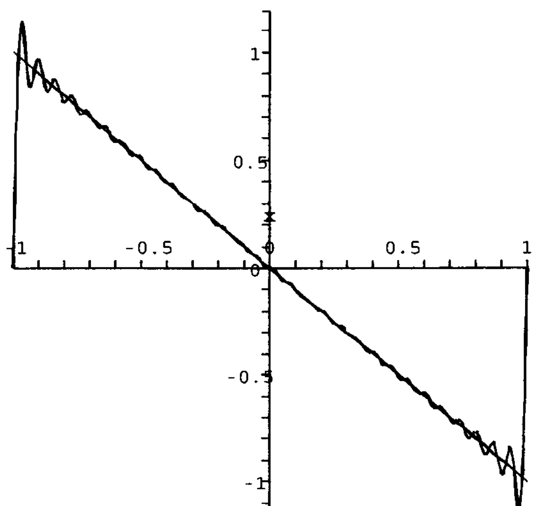

Figure 9.3: Section 3.3, Problem 3(a): thirtieth partial sum.

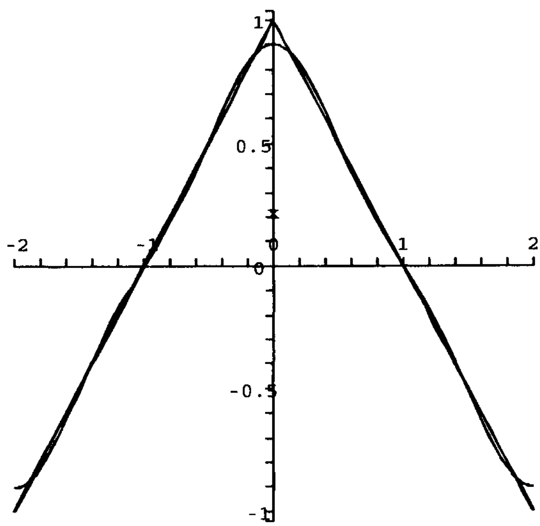

Figure 9.4: Section 3.3, Problem 3(c): second partial sum.

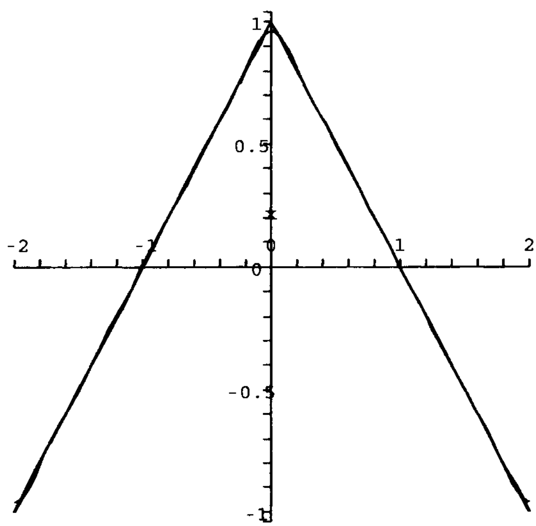

Figure 9.5: Section 3.3, Problem 3(c): fifth partial sum.

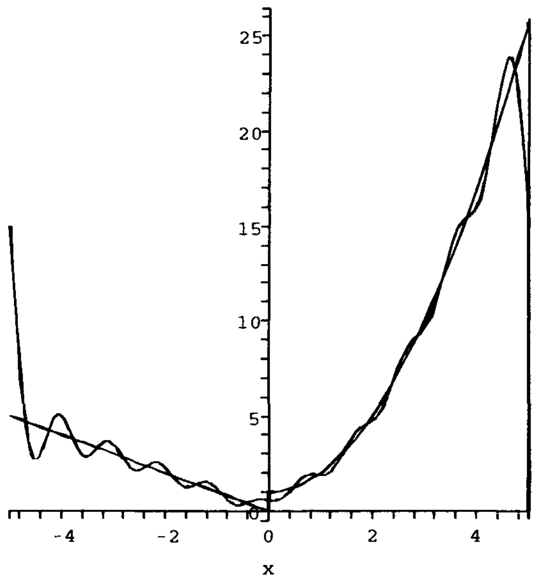

Figure 9.6: Section 3.3, Problem 3(g): tenth partial sum.

Because n is an integer, $\cos(n\pi) = (-1)^n$ in this series. The series converges to

$$\begin{cases} -x & \text{for } -5 < x < 0 \\ 1 + x^2 & \text{for } 0 < x < 5 \\ 1/2 & \text{for } x = 0 \\ 31/2 & \text{for } x = 5 \text{ and } x = -5. \end{cases}$$

(i) The series is

$$\frac{2}{\pi} - \sin(x) + \sum_{n=1}^{\infty} \frac{4}{(4n^2 - 1)\pi}(-1)^{n+1}\cos(nx).$$

This converges to $\cos(x/2) - \sin(x)$ for $-\pi \le x \le \pi$.

4. (a)

$$\begin{cases} 2x & \text{for } -3 < x < -2 \\ 0 & \text{for } -2 < x < 1 \\ x^2 & \text{for } 1 < x < 3 \\ 3/2 & \text{at } x = 3, -3 \\ -2 & \text{at } x = -2 \\ 1/2 & \text{at } x = 1 \end{cases}$$

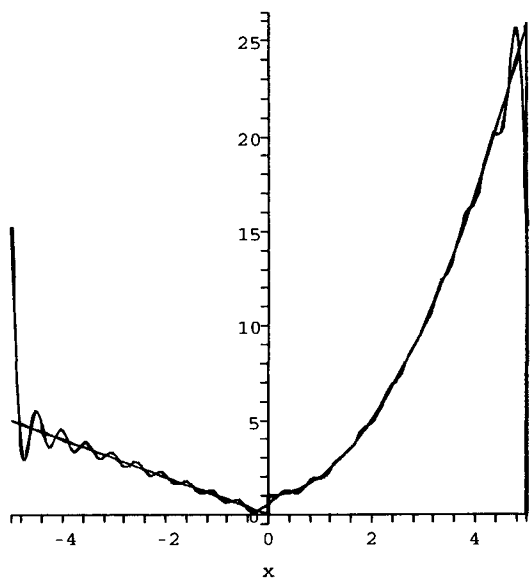

Figure 9.7: Section 3.3, Problem 3(g): twentieth partial sum.

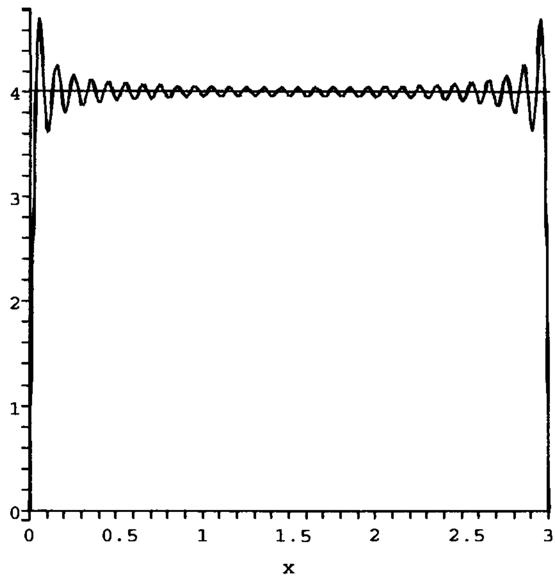

Figure 9.8: Section 3.4, Problem 1: thirtieth partial sum of the sine series.

(c)

$$\begin{cases} -2 & \text{for } -4 < x < 2 \\ 1 + x^2 & \text{for } 2 < x < 3 \\ e^{-x} & \text{for } 3 < x < 4 \\ (-2 + e^{-4})/2 & \text{at } x = 4, -4 \\ 3/2 & \text{at } x = 2 \\ (10 + e^{-3})/2 & \text{at } x = 3 \end{cases}$$

(e)

$$\begin{cases} \cos(\pi x) & \text{for } -2 < x < 0 \\ x & \text{for } 0 < x < 2 \\ 3/2 & \text{at } x = 2, -2 \\ 1/2 & \text{at } x = 0 \end{cases}$$

Section 3.4

1. The sine series is

$$\sum_{n=1}^{\infty} \frac{16}{(2n-1)\pi} \sin((2n-1)\pi x/3).$$

This converges to 4 if $0 < x < 3$ and to 0 for $x = 0, 3$.

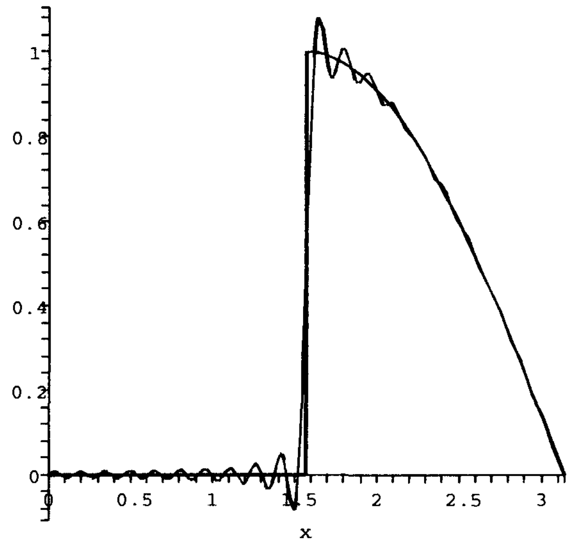

Figure 9.9: Section 3.4, Problem 3: fortieth partial sum of the sine series.

The cosine series is just 4 (the function is its own cosine series on this interval). This converges to 4 for $0 \leq x \leq 3$.

3. The sine series is

$$\frac{1}{2}\sin(x) + \sum_{n=2}^{\infty} \frac{2n\cos(n\pi/2)}{\pi(n^2-1)}\sin(nx),$$

converging to

$$\begin{cases} 0 & \text{for } 0 < x < \pi/2 \\ \sin(x) & \text{for } \pi/2 < x < \pi \\ 1/2 & \text{for } x = \pi/2 \\ 0 & \text{for } x = 0, \pi. \end{cases}$$

The cosine series is

$$\frac{1}{\pi} - \frac{1}{\pi}\cos(x) - \sum_{n=2}^{\infty}\left(\frac{2}{\pi}\frac{\cos(n\pi) + n\sin(n\pi/2)}{n^2-1}\right)\cos(nx).$$

This converges to

$$\begin{cases} 0 & \text{for } 0 \leq x < \pi/2 \\ \sin(x) & \text{for } \pi/2 < x \leq \pi \\ 1/2 & \text{for } x = \pi/2. \end{cases}$$

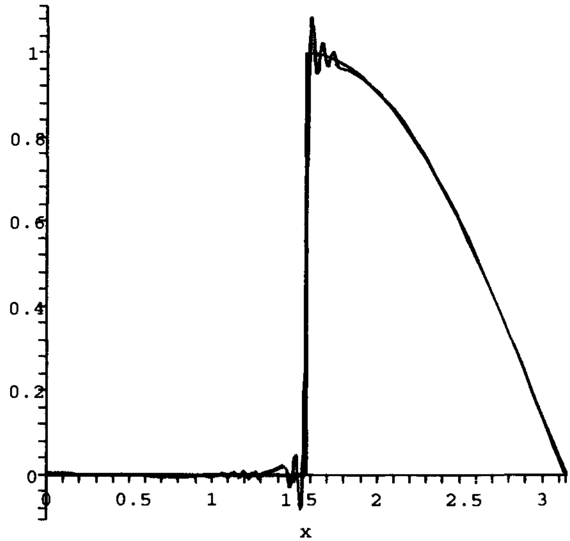

Figure 9.10: Section 3.4, Problem 3: fortieth partial sum of the cosine series.

5. The sine series is

$$\sum_{n=1}^{\infty}\left(-8\frac{n^2\pi^2\cos(n\pi)-2\cos(n\pi)+2}{n^3\pi^3}\right)\sin(n\pi x/2),$$

converging to x^2 for $0 < x < 2$ and to 0 for $x = 0, 2$.

The cosine series is

$$\frac{4}{3}+\sum_{n=1}^{\infty}\frac{16}{n^2\pi^2}(-1)^n\cos(n\pi x/2),$$

converging to x^2 for $0 \le x \le 2$.

7. The sine series is just $\sin(3x)$ for $0 \le x \le \pi$. The cosine series is

$$\frac{2}{3\pi}+\sum_{n=1,n\neq 3}^{\infty}\frac{6}{\pi}\frac{(-1)^n+1}{9-n^2}\cos(nx),$$

converging to $\sin(3x)$ for $0 \le x \le \pi$.

Section 3.5

1. The Fourier integral is

$$\int_0^{\infty}\frac{2}{\pi}\frac{\sin(\pi\omega)-\pi\omega\cos(\pi\omega)}{\omega^2}\sin(\omega x)\,d\omega,$$

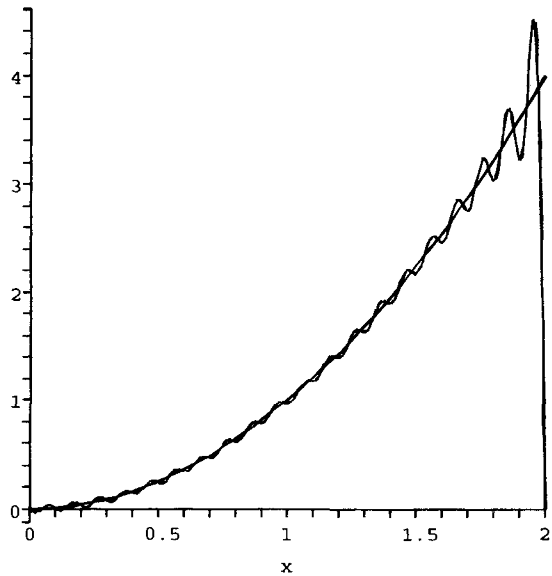

Figure 9.11: Section 3.4, Problem 5: fortieth partial sum of the sine series.

converging to

$$\begin{cases} x & \text{for } \pi < x < \pi \\ -\pi/2 & \text{for } x = -\pi \\ \pi/2 & \text{for } x = \pi \\ 0 & \text{for } \mid x \mid > \pi. \end{cases}$$

3.

$$\int_0^\infty \frac{2}{\pi(1+\omega^2)} \cos(\omega x)\, d\omega,$$

converging to $f(x)$ for all x.

5.

$$\int_0^\infty \frac{2}{\pi\omega^2}(\cos(\alpha\omega) + \alpha\omega \sin(\alpha\omega)) \cos(\omega x)\, d\omega,$$

converging to

$$\begin{cases} \mid x \mid & \text{for } -\alpha < x < \alpha \\ \alpha/2 & \text{for } x = \alpha, -\alpha \\ 0 & \text{for } \mid x \mid > \alpha. \end{cases}$$

9. The sine integral is

$$\frac{e^k}{\pi} \int_0^\pi \frac{1}{1+\omega^2} c_\omega \sin(\omega x)\, d\omega,$$

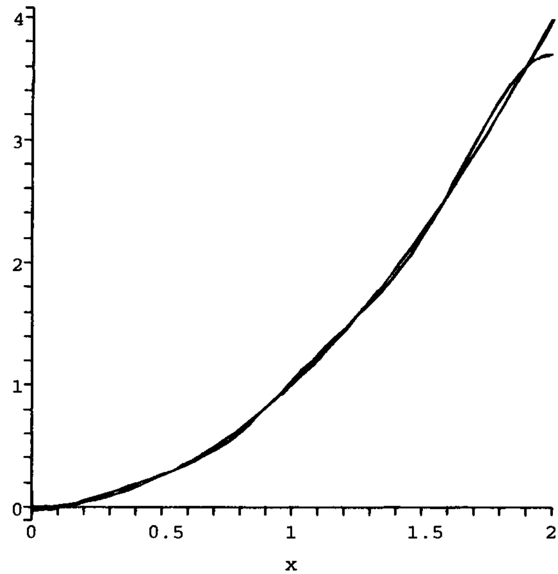

Figure 9.12: Section 3.4, Problem 5: fifth partial sum of the cosine series.

where

$$c_\omega = -\omega\cos(k\omega) + \sin(k\omega) + \omega\cos(k\omega)e^{-2k} + \sin(k\omega)e^{-2k}.$$

This sine integral converges to

$$\begin{cases} \sinh(x) & \text{for } 0 < x < k \\ \sinh(k)/2 & \text{for } x = k \\ 0 & \text{for } x = 0 \text{ and for } x \geq k. \end{cases}$$

The cosine integral is

$$\frac{e^{-k}}{\pi}\int_0^\infty \frac{1}{1+\omega^2} d_\omega \cos(\omega x)\, d\omega,$$

where

$$d_\omega = e^{2k}\cos(k\omega) + \omega e^{2k}\sin(k\omega) + \cos(k\omega) - \omega\sin(k\omega) - 2e^k.$$

This cosine integral converges to

$$\begin{cases} \sinh(x) & \text{for } 0 \leq x < k \\ \sinh(k)/2 & \text{for } x = k \\ 0 & \text{for } x > k. \end{cases}$$

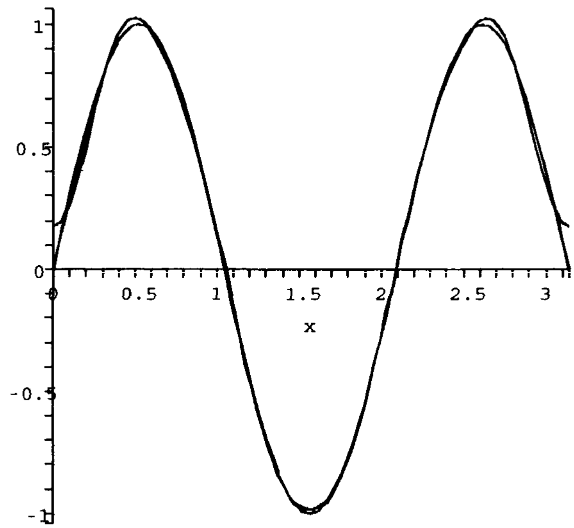

Figure 9.13: Section 3.4, Problem 7: tenth partial sum of the cosine series.

11. The sine integral is

$$\frac{2}{\pi}\int_0^\infty \frac{\omega^3}{(2-2\omega+\omega^2)(2+2\omega+\omega^2)}\sin(\omega x)\,d\omega,$$

converging to $e^{-x}\cos(x)$ for $x > 0$ and to 0 at $x = 0$.

The cosine integral is

$$\frac{2}{\pi}\int_0^\infty \frac{2+\omega^2}{(2-2\omega+\omega^2)(2+2\omega+\omega^2)}\cos(\omega x)\,d\omega,$$

converging to $e^{-x}\cos(x)$ for $x \geq 0$.

13. The sine integral is

$$\frac{2k}{\pi}\int_0^\infty \frac{1-\cos(\alpha\omega)}{\omega}\sin(\omega x)\,d\omega,$$

converging to

$$\begin{cases} k & \text{for } 0 < x < \alpha \\ k/2 & \text{for } x = 0 \\ 0 & \text{for } x = 0 \text{ and } x > \alpha. \end{cases}$$

The cosine integral is

$$\frac{2k}{\pi}\int_0^\infty \frac{\sin(\alpha\omega)}{\omega}\cos(\omega x)\,d\omega,$$

converging to

$$\begin{cases} k & \text{for } 0 \le x < \alpha \\ k/2 & \text{for } x = \alpha \\ 0 & \text{for } x > \alpha. \end{cases}$$

Section 3.6

3. The integral representation is

$$\frac{1}{\pi}\int_{-\infty}^\infty \frac{\omega\cos(\alpha)\sin(\alpha\omega) - \sin(\alpha)\cos(\alpha\omega)}{\omega^2 - 1} e^{i\omega x}\,d\omega.$$

5.

$$\frac{1}{\pi}\int_{-\infty}^\infty -2i\frac{\omega}{(1+\omega^2)^2}e^{i\omega x}\,d\omega$$

7.

$$\frac{ik}{\pi}\int_{-\infty}^\infty \frac{\cos(\alpha\omega)-1}{\omega}e^{i\omega x}\,d\omega$$

9. The Fourier transform of f is

$$\frac{k}{\omega}(\sin(k\omega) - \sin(\alpha\omega)) + i\frac{k}{\omega}(\cos(k\omega) - \cos(\alpha\omega)).$$

11.

$$\frac{2}{1+\omega^2}$$

13.

$$\frac{2 - 2i\omega - \omega^2}{4+\omega^2}$$

Section 3.7

1. $(f * g)(t) = 2e^{-1-t}$

3. $(f * f)(t)$ is given by the following cases:
If $0 \le t < 2k$, then $(f * f)(t) = -\frac{2}{3}k^3 + k^2 t - \frac{1}{6}t^3$.
If $-2k < t < 0$, then $(f * f)(t) = \frac{1}{6}t^3 - k^2 t - \frac{2}{3}k^3$.
If $|\, t \,| \ge 2k$, then $(f * f)(t) = 0$.

Section 3.8

1.

$$\hat{f}_S(\omega) = \frac{\omega}{1+\omega^2}$$
$$\hat{f}_C(\omega) = \frac{1}{1+\omega^2}$$

3.

$$\hat{f}_S(\omega) = \frac{\sin(\alpha\omega)\sin(\alpha) + \omega\cos(\alpha\omega)\cos(\alpha) - \omega}{1 - \omega^2}$$
$$\hat{f}_C(\omega) = \frac{\omega\cos(\alpha)\sin(\alpha\omega) - \sin(\alpha)\cos(\alpha\omega)}{\omega^2 - 1}$$

5.

$$\hat{f}_S(\omega) = \frac{\omega^3}{(2 + 2\omega + \omega^2)(2 - 2\omega + \omega^2)}$$
$$\hat{f}_C(\omega) = \frac{2 + \omega^2}{(2 + 2\omega + \omega^2)(2 - 2\omega + \omega^2)}$$

Chapter 4: The Wave Equation

Section 4.1

1. $u(x,t) = \cos(3x)\cos(21t) + xt$

3.

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(e^{-|x+3t|} + e^{-|x-3t|}\right) \\ &+ \frac{1}{2}t - \frac{1}{24}\sin(2x + 6t) + \frac{1}{24}\sin(2x - 6t) \end{aligned}$$

5.

$$\begin{aligned} u(x,t) &= \frac{1}{2}[\cos(x + 2t) - \sin(x + 2t) + \cos(x - 2t) - \sin(x - 2t)] \\ &+ \frac{1}{2}\sin(x)\sin(2t) \end{aligned}$$

7.

$$\begin{aligned} u(x,t) &= \cos(x)\cos(4t) + \frac{1}{8}x\left(e^{-x+4t} - e^{-x-4t}\right) \\ &- \frac{1}{2}t\left(e^{-x+4t} + e^{-x-4t}\right) + \frac{1}{8}\left(e^{-x+4t} - e^{-x-4t}\right) \end{aligned}$$

9.

$$\begin{aligned} u(x,t) &= x^3 + 27xt^2 + \frac{1}{6}[\cos(x + 3t) - \cos(x - 3t)] \\ &+ \frac{1}{6}(x + 3t)\sin(x + 3t) - \frac{1}{6}(x - 3t)\sin(x - 3t) \end{aligned}$$

Section 4.2

1. The initial position is given by $\varphi(x) = e^{-x^2}$. Of course, e^{-x^2} is positive for all x. However, the graph appears to be zero over most of the axis. This effect occurs because for "most" x, e^{-x^2} is smaller than the scale the graph can display, hence registers as zero in the picture. Figures 9.15 through 9.22 show

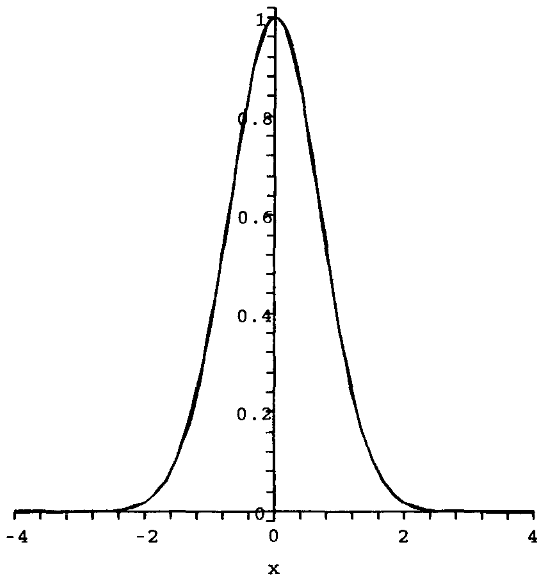

Figure 9.14: Section 4.2, Problem 1: $t = 0$.

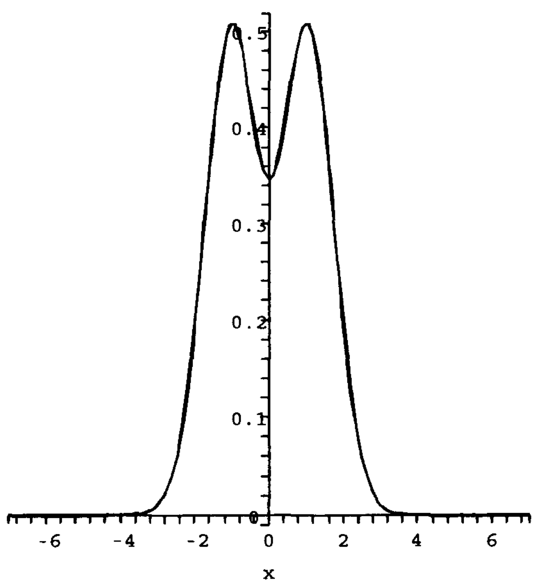

Figure 9.15: Section 4.2, Problem 1: $t = 1.03$.

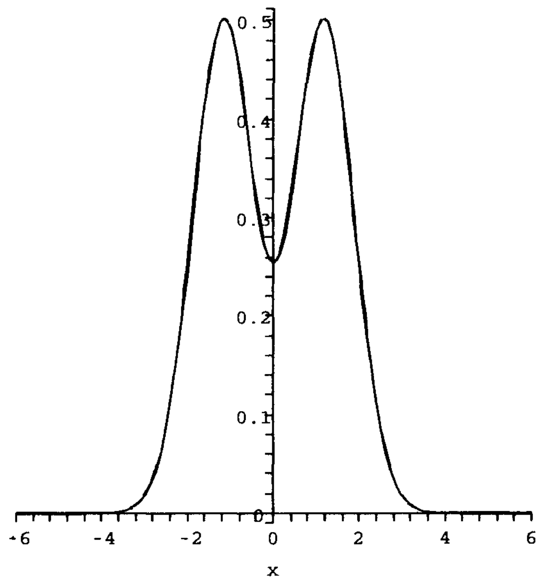

Figure 9.16: Section 4.2, Problem 1: $t = 1.17$.

the progression of the wave over times $t = 1.03, 1.17, 1.36\ 1.97, 2.41, 2.81, 3.11$, and 4.49.

3. The initial position function is

$$\varphi(x) = \begin{cases} x\cos(x) & \text{for } -\pi/2 \le x \le \pi/2 \\ 0 & \text{for } \mid x \mid > \pi/2. \end{cases}$$

Figures 9.23 through 9.29 show positions of the wave at times $t = 0, 0.39, 0.87, 0.97, 1.27, 1.88$, and 3.92, by which time the components moving left and right have separated.

5. The initial position is given by

$$\varphi(x) = \begin{cases} x^3 - x^2 - 2 & \text{for } -1 \le x \le 2 \\ 0 & \text{for } x < -1 \text{ and for } x > 2. \end{cases}$$

In Figures 9.30 through 9.34 the wave is shown at times $t = 0, 0.38, 0.81, 1.43$, and 2.17.

Section 4.3

1. The solution is

$$u(x,t) = \frac{1}{2}[\varphi(x+ct) + \varphi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds.$$

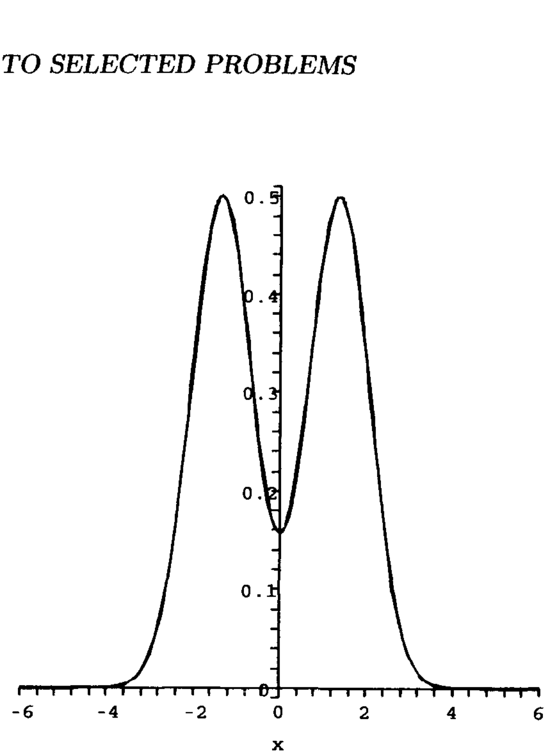

Figure 9.17: Section 4.2, Problem 1: $t = 1.36$.

If $t > 0$ and $x > a + ct$, then $x = a + ct + h$ for some positive h, and then $\varphi(x + ct) = \varphi(x + 2ct + h) = 0$ because $a + 2ct + h > a$. Further, $\varphi(x - ct) = \varphi(a + h) = 0$ because $a + h > a$. Finally,

$$\int_{x-ct}^{x+ct} \psi(s)\, ds = \int_{a+h}^{a+2ct+h} \psi(s)\, ds = 0$$

because $\psi(s) = 0$ if s is in $[a + h, a + 2c + h]$. Then $u(x, t) = 0$ if $x > a + ct$. Similarly, $u(x, t) = 0$ if $x < -a - ct$.

Section 4.4

1.

$$u(x,t) = \begin{cases} x^2 + t^2 + \sin(x)\sin(t) & \text{for } 0 \le t \le x \\ 2xt + \sin(x)\sin(t) & \text{for } 0 < x < t \end{cases}$$

3.

$$u(x,t) = \begin{cases} 1 - e^x \cosh(2t) + x^2 t + 4t^3/3 & \text{if } x - 2t \ge 0 \\ -e^{2t}\sinh(x) + 2xt^2 + x^3/6 & \text{if } x - 2t < 0 \end{cases}$$

5.

$$u(x,t) = \begin{cases} x\sin(x)\cos(2t) + 2t\cos(x)\sin(2t) + x^2 t + 4t^3/3 & \text{for } x - 2t \ge 0 \\ x\cos(x)\sin(2t) + 2t\sin(x)\cos(2t) + x^3/6 + 2xt^2 & \text{for } x - 2t < 0 \end{cases}$$

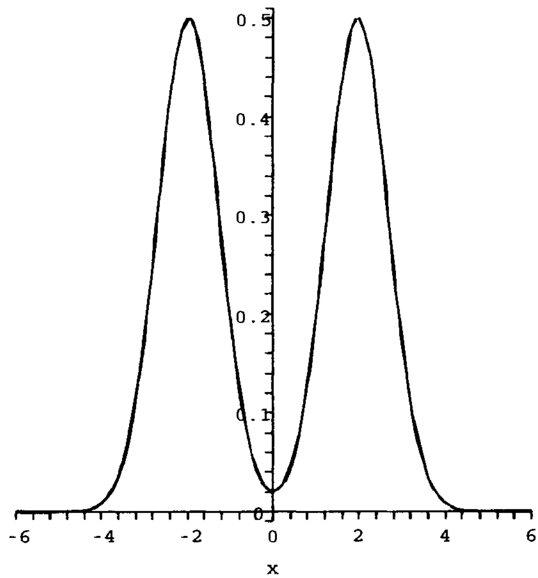

Figure 9.18: Section 4.2, Problem 1: $t = 1.97$.

7.

$$u(x,t) = \begin{cases} x^3 + 27xt^2 + (1/3)e^{-x}\sinh(3t) & \text{for } x - 3t \geq 0 \\ x^3 + 27xt^2 + (1/3)e^{-3t}\sinh(x) & \text{for } x - 3t < 0 \end{cases}$$

9. If $x - 5t \geq 0$,

$$u(x,t) = -1 + \frac{1}{2}[\cosh(x+5t) + \cosh(x-5t)] + \frac{1}{5}\sin(x)\sin(5t).$$

If $x - 5t < 0$,

$$u(x,t) = \frac{1}{2}[\cosh(x+5t) - \cosh(x-5t)] + \frac{1}{5}\sin(x)\sin(5t).$$

Section 4.5

1.

$$u(x,t) = \begin{cases} x + e^{-x}\sinh(t) & \text{for } x - t \geq 0 \\ x^2 + t^2 - 2xt + x + e^{-t}\sinh(x) & \text{for } x - t < 0 \end{cases}$$

3.

$$u(x,t) = \begin{cases} xt + \sin(x)\cos(7t) & \text{for } x - 7t \geq 0 \\ 1 - e^{t-x/7} + \sin(x)\cos(7t) + xt & \text{for } x - 7t < 0 \end{cases}$$

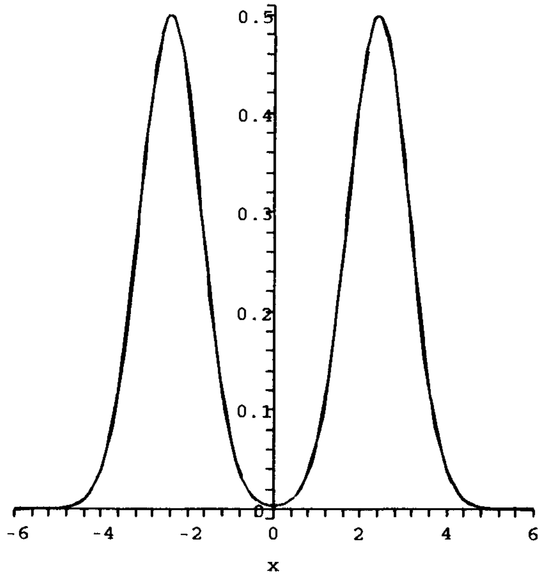

Figure 9.19: Section 4.2, Problem 1: $t = 2.41$.

5.

$$u(x,t) = \begin{cases} x^2 t + 3t^3 + \cos(x)\cos(3t) & \text{for } x - 3t \geq 0 \\ t - x/3 + \cos(t - x/3) - \sin(x)\sin(3t) + \frac{1}{9}x^3 + 3xt^2 & \text{for } x - 3t < 0 \end{cases}$$

7.

$$u(x,t) = \begin{cases} e^{-x}\cosh(3t) + \frac{1}{3}\sin(x)\sin(3t) & \text{for } x - 3t \geq 0 \\ 1 - t + \frac{1}{3}x - e^{-3t}\sinh(x) + \frac{1}{3}\sin(x)\sin(3t) & \text{for } x - 3t < 0 \end{cases}$$

9.

$$u(x,t) = \begin{cases} x + x^2 + 4t^2 + t & \text{for } x - 2t \geq 0 \\ 2t + 4xt + \frac{1}{2}x & \text{for } x - 2t < 0 \end{cases}$$

Section 4.6

1.

$$\begin{aligned} u(x,t) &= \frac{1}{2}[(x+4t) + (x-4t)] + \frac{1}{8}\int_{x-4t}^{x+4t} e^s \, dx \\ &+ \frac{1}{8}\int_0^t \int_{x-4t+4\eta}^{x+4t-4\eta} (\xi + \eta)\, d\xi \, d\eta \\ &= x + \frac{1}{4}e^{-x}\sinh(4t) + \frac{1}{2}xt^2 + \frac{1}{6}t^3 \end{aligned}$$

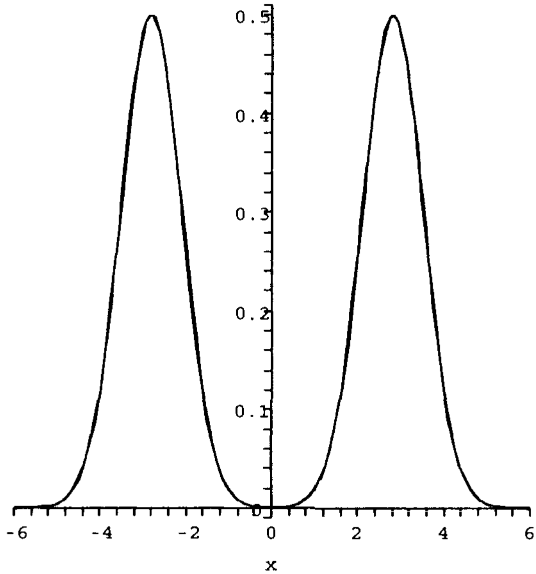

Figure 9.20: Section 4.2, Problem 1: $t = 2.81$.

3.

$$\begin{aligned}
u(x,t) &= \frac{1}{2}((x+8t)^2 - (x+8t) + (x-8t)^2 - (x-8t)) \\
&+ \frac{1}{16}\int_{x-8t}^{x+8t} \cos(2s)\,ds + \frac{1}{16}\int_0^t \int_{x-8t+8\eta}^{x+8t-8\eta} \eta\cos(\xi)\,d\xi\,d\eta \\
&= x^2 + 64t^2 - x + \frac{1}{32}(\sin(2(x+8t)) - \sin(2(x-8t))) \\
&+ \frac{1}{8}\cos(x)\sin(8t)\frac{1}{64}(\cos(8t) + 8t\sin(8t) - 1) \\
&- \frac{1}{8}\cos(x)\cos(8t)\frac{1}{64}(\sin(8t) - 8t\cos(8t))
\end{aligned}$$

5.

$$\begin{aligned}
u(x,t) &= \frac{1}{2}(\cosh(x+3t) + \cosh(x-3t)) \\
&+ \frac{1}{6}\int_{x-3t}^{x+3t} ds + \frac{1}{6}\int_0^t \int_{x-3t+3\eta}^{x+3t-3\eta} 3\xi\eta^2\,d\xi\,d\eta \\
&= \frac{1}{2}(\cosh(x+3t) + \cosh(x-3t)) + t + \frac{1}{4}xt^4
\end{aligned}$$

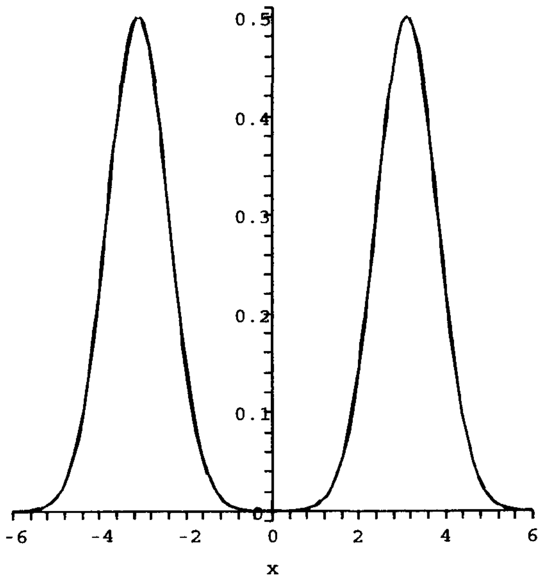

Figure 9.21: Section 4.2, Problem 1: $t = 3.11$.

7.

$$\begin{aligned} u(x,t) &= \frac{1}{2}(\cos(2(x+2t)) + \cos(2(x-2t))) \\ &+ \frac{1}{4}\int_{x-2t}^{x+2t} (1-\cos(s))ds + \frac{1}{4}\int_0^t \int_{x-2t+2\eta}^{x+2t-2\eta} \eta^2 \, d\xi \, d\eta \\ &= \cos(2x)\cos(4t) + t - \cos(x)\sin(t)\cos(t) + \frac{1}{12}t^4 \end{aligned}$$

9.

$$\begin{aligned} u(x,t) &= \frac{1}{2}((x+2t)\sin(x+2t) + (x-2t)\sin(x-2t)) \\ &+ \frac{1}{4}\int_{x-2t}^{x+2t} e^{-s} ds + \frac{1}{4}\int_0^t \int_{x-2t+2\eta}^{x+2t-2\eta} \xi\eta \, d\xi \, d\eta \\ &= \frac{1}{2}((x+2t)\sin(x+2t) + (x-2t)\sin(x-2t)) + \frac{1}{2}e^{-x}\sinh(2t) + \frac{1}{6}xt^3 \end{aligned}$$

Section 4.7

3. In all three regions (by separate calculations), $u(x,t) = 2x - x^2 - t^2$.

5. In all three regions (by separate calculations), $u(x,t) = 2x^2 - x^3 - 27xt^2 + 18t^2$.

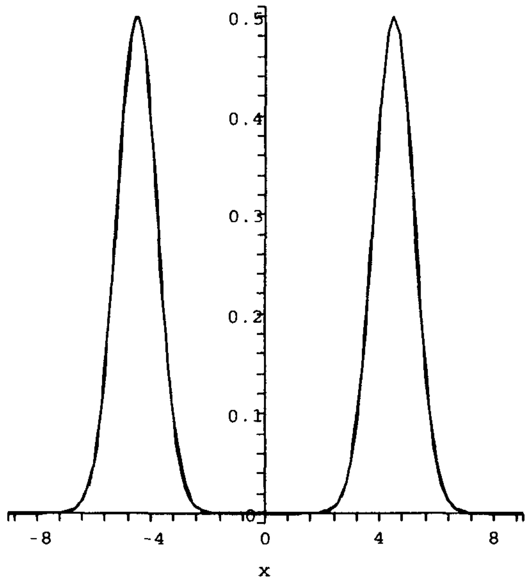

Figure 9.22: Section 4.2, Problem 1: $t = 4.49$.

7. In region I,

$$u(x,t) = 4x - 4x^2 + x^3 + 3xt^2 - 4t^2 + x^2t + \frac{1}{3}t^3.$$

In region II,

$$u(x,t) = -4t^2 - 4x^2 + 4xt^2 + \frac{4}{3}x^3 + 4x.$$

In region III,

$$u(x,t) = -\frac{28}{3} + 10x + 8t - 5x^2 - 5t^2 + \frac{7}{6}x^3 + \frac{7}{2}xt^2 - 2xt.$$

It is routine to check that the solutions in regions I and III agree on their common boundary segments.

9. In region IV,

$$\begin{aligned} u(x,t) = {} & \pi(2x^2 + 18t^2 - 8x - 24t + 16) + \frac{88}{9} \\ & - \frac{1}{2}(x - 3t)\sin(\pi(x - 3t)) - \frac{1}{2}(8 - x - 3t)\sin(\pi(8 - x - 3t)) \\ & + \frac{4}{3}x^2 + 4xt + 12t^2 - x^2t - 3t^3 - \frac{16}{9} - \frac{16}{3}x - 16t. \end{aligned}$$

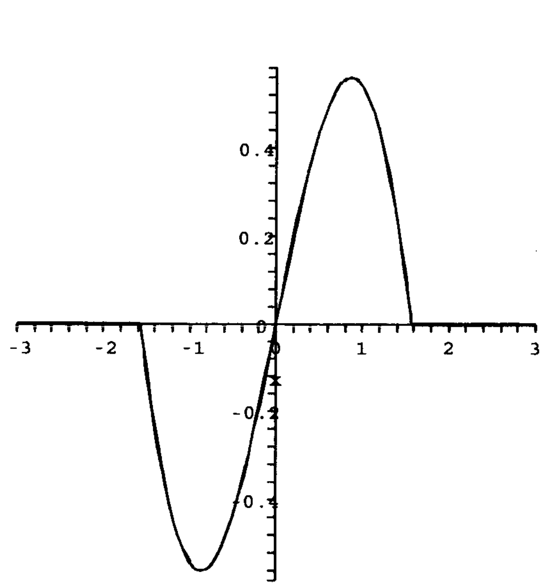

Figure 9.23: Section 4.2, Problem 3: $t = 0$.

Section 4.8

1.

$$u(x,t) = \sum_{n=1,n\neq 2}^{\infty} \frac{8}{\pi}\frac{\cos(n\pi)-1}{n(n^2-4)} \sin(nx/2)\cos(3nt/2)$$

In this solution we can write $\cos(n\pi) = (-1)^n$ and observe that $\cos(n\pi) - 1$ is zero if n is even, hence retain only odd-indexed terms in the summation.

Figure 9.35 shows the wave at times $t = 0, 0.38, 0.61, 0.94$, and 1.27. In this time period, the waves move downward from the initial position at $t = 0$.

3.

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8}{(2n-1)^3\pi^3} \sin((2n-1)\pi x)\cos(6(2n-1)\pi t)$$

Figure 9.36 shows the solution at times $t = 0, 0.047, 0.073, 0.095$, and 0.17. In this time period the waves move downward from the initial parabolic position.

5.

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{3}\frac{1-e^{-2}\cos(n\pi)}{4+n^2\pi^2} \sin(n\pi x/2)\sin(3n\pi t/2)$$

Figure 9.37 shows positions of the wave at times $t = 0.009, 0.035, 0.07, 0.097, 0.38$, and 0.71. Over the first five times, the peaks of the waves move

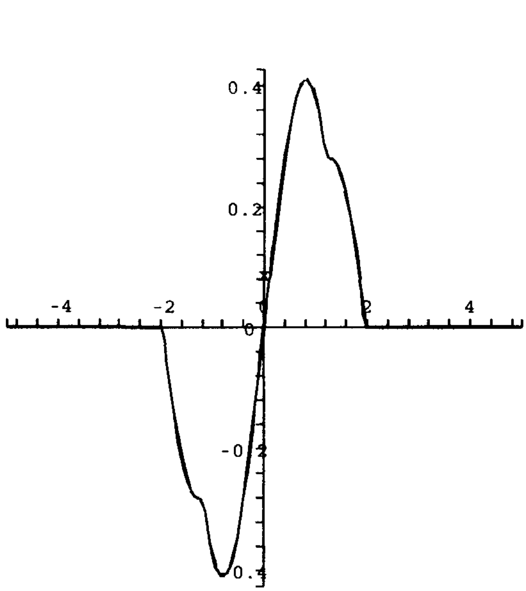

Figure 9.24: Section 4.2, Problem 3: $t = 0.39$.

upward to the right, while at $t = 0.71$ the wave has moved below the horizontal axis.

7.

$$u(x,t) = \sin(x)\cos(3t) + \sum_{n=1}^{\infty} \frac{2}{3n^2}(-1)^{n+1}\sin(nx)\sin(3nt)$$

Figure 9.38 shows wave profiles at times $t = 0, 0.047, 0.087,\ 0.29, 0.41, 0.63,$ and 0.75. Initially, the wave is the sine curve on $[0, \pi]$. For the next three times, the wave moves upward to the right. For the last three times, the wave moves downward, moving partly below the horizontal axis at the last time shown.

9.

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\sin((2n-1)x),$$

where

$$c_n(t) = \frac{-8}{(2n-1)\pi((2n-1)^2-4)}\cos((2n-1)t) + \frac{4}{(2n-1)^2\pi}\sin((2n-1)t).$$

Figure 9.39 shows graphs of this solution at times $t = 0, 0.061, 0.19, 0.41, 0.93,$ and 2.37. At $t = 0$ the initial position is the graph of $\sin^2(x)$ for $0 \le x \le \pi$. For the next three times the wave moves upward. At $t = 0.93$ the "middle part" of

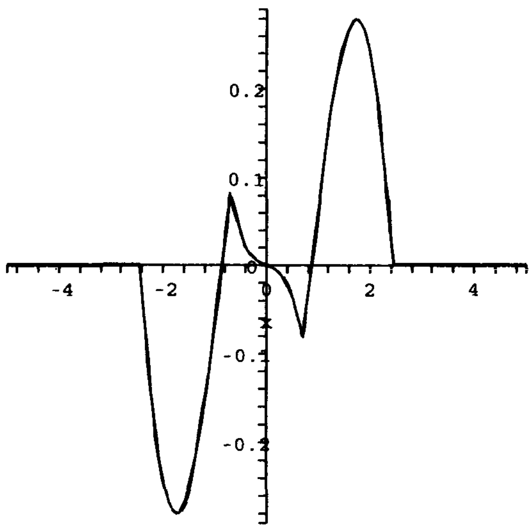

Figure 9.25: Section 4.2, Problem 3: $t = 0.87$.

the wave is starting to move downward, and at $t = 2.37$ part of the wave has moved partially below the horizontal axis.

Section 4.9

1. Figure 9.40 shows graphs of the solution at $t = 0.86$, for the given values of K. The waves increase in magnitude as K is chosen larger. The same effect is seen at time $t = 1.25$ in Figure 9.41.

2.

$$\begin{aligned} u(x,t) = &\sum_{n=1}^{\infty} \frac{A}{9} \frac{2n^2\pi^2(-1)^n - 4(-1)^n + 4}{n^5\pi^5} \cos(3n\pi t)\sin(n\pi x) \\ &+ \frac{A}{108} x(1 - x^2) \end{aligned}$$

Figure 9.42 shows the wave profile at times $t = 0.0035, 0.073, 2.19, 3.47$, and 5.61, with $A = 1.5$. Over these times, the graphs move from the right ($t = 0.0035$) to the left ($t = 5.61$).

3.

$$u(x,t) = U(x,t) + \frac{1}{4}(\cos(x) - 1) + \frac{1}{2\pi} x$$

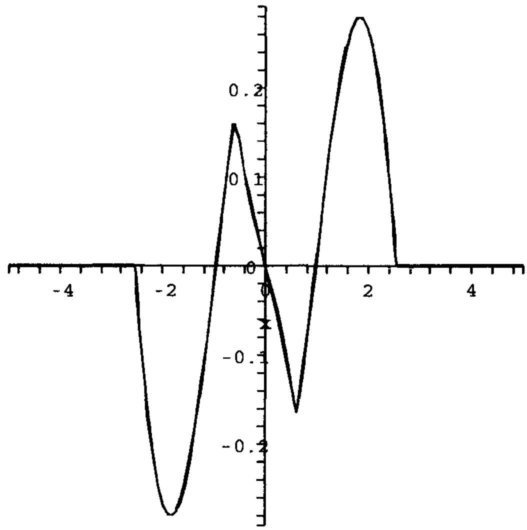

Figure 9.26: Section 4.2, Problem 3: $t = 0.97$.

where

$$U(x,t) =$$
$$\sum_{n=2}^{\infty}\left[-\frac{1}{2\pi}\frac{(-1)^n-1}{n}-\frac{1}{2\pi}\left(\frac{n}{n^2-1}\right)(1+(-1)^n)+\frac{1}{\pi}\frac{(-1)^n}{n}\right]\cos(2nt)\sin(nx).$$

Figure 9.43 shows the string moving upward from its initial position along the horizontal axis, at times very close to zero. To identify the graphs in Figures 9.43 and 9.44, the parts of the graph on the left half of the interval are moving upward as time increases.

5.
$$u(x,t) = U(x,t) + 3e^{-x} + \frac{3}{2}(1-e^{-2})x - 3,$$

where

$$U(x,t) =$$
$$\sum_{n=1}^{\infty}\left[\frac{6}{n\pi}(1-(-1)^n e^{-2})+\frac{6n\pi}{4+n^2\pi^2}((-1)^n e^{-2}-1)\right]\cos(n\pi t/2)\sin(n\pi x/2).$$

Figure 9.45 shows the wave moving downward through times $t = 0.37, 0.71, 0.95,$ and 1.89.

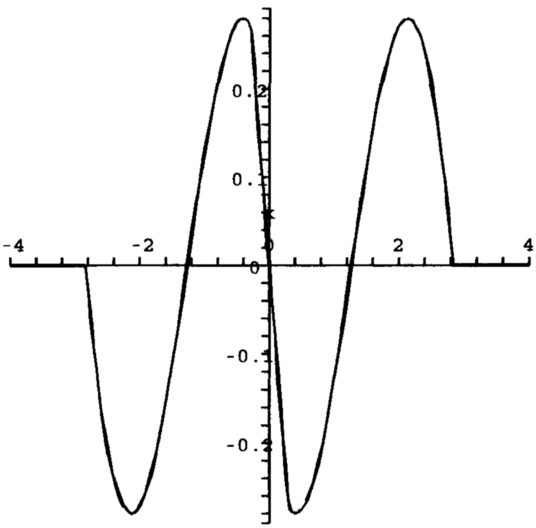

Figure 9.27: Section 4.2, Problem 3: $t = 1.27$.

7.

$$u(x,t) = U(x,t) + \frac{2}{27}x(1-x^2),$$

where

$$U(x,t) = \sum_{n=1}^{\infty}\left[\frac{8}{9}\frac{(-1)^n}{n^3\pi^3}\cos(3n\pi t) + \frac{2}{3n^2\pi^2}(1-(-1)^n)\sin(3n\pi t)\right]\sin(n\pi x).$$

Figure 9.46 shows the solution at five times. At $t = 0.037$ the wave has a nearly trapezoidal shape with height about 0.04; at $t = 081$ the wave has moved up to height about 0.08, retaining a nearly trapezoidal shape. At $t = 0.37$ the wave has moved downward, now lying partly below the horizontal axis. At $t = 0.84$ it has moved back upward to a nearly triangular shape peaking at about 0.2, and then at $t = 0.89$ the wave has moved back downward, now having a height of about 0.15.

9. The method does not directly adapt. If we try $u(x,t) = U(x,t) + g(t)$, we find that $U_t(x,0) = -g'(t)$. But the left side of this equation depends only on x, the right side only on t, and x and t are independent. Some other approach must be devised for this problem.

Section 4.10

1.

$$u(x,t) = \frac{2}{\pi}\int_0^{\infty}\frac{1}{1+\omega^2}\cos(\omega x)\cos(\omega ct)\,d\omega$$

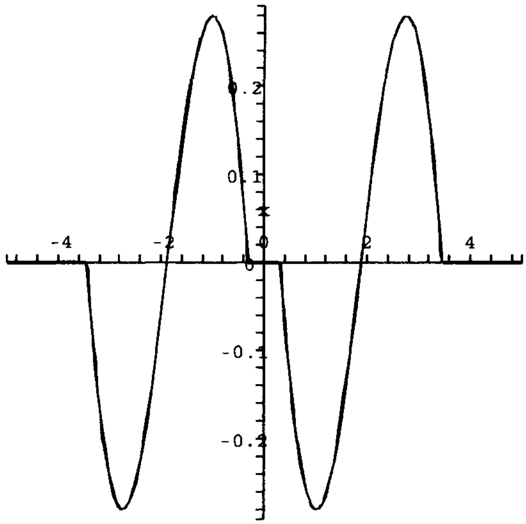

Figure 9.28: Section 4.2, Problem 3: $t = 1.88$.

3.

$$u(x,t) = \frac{2}{\pi} \int_0^\infty \frac{\sin(\pi\omega)}{1-\omega^2} \sin(\omega x) \cos(\omega c t)\, d\omega$$

5.

$$u(x,t) = \frac{1}{\pi} \int_0^\infty \left[\frac{1}{1+\omega^2} \cos(\omega x) + \frac{\omega}{1+\omega^2} \sin(\omega x) \right] \cos(\omega c t)\, d\omega$$

7.

$$u(x,t) = \int_0^\infty [a_\omega \cos(\omega x) + b_\omega \sin(\omega x)] \cos(\omega c t)\, d\omega,$$

where

$$a_\omega = \frac{1}{\pi \omega c} \int_{-\infty}^\infty \psi(s) \cos(\omega s)\, ds$$

and

$$b_\omega = \frac{1}{\pi \omega c} \int_{-\infty}^\infty \psi(s) \sin(\omega s)\, ds.$$

9.

$$u(x,t) = \int_0^\infty \frac{2}{\pi \omega c (1+\omega^2)} \cos(\omega x) \sin(\omega c t)\, d\omega$$

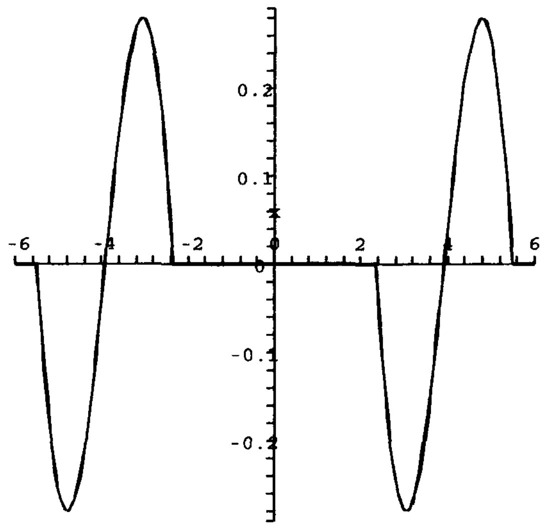

Figure 9.29: Section 4.2, Problem 3: $t = 3.92$.

11.

$$u(x,t) = \int_0^\infty \frac{2}{\pi \omega c} \frac{\cos(\pi\omega/2)}{1-\omega^2} \cos(\omega x)\sin(\omega ct)\, d\omega$$

Section 4.11

1.

$$u(x,y,t) = \sum_{m=1}^{\infty} 72 c_m \sin(\pi x/3)\sin(m\pi y/6)\cos(\alpha_{1m} t),$$

where

$$\alpha_{1m} = \sqrt{\frac{\pi^2}{9} + \frac{m^2\pi^2}{36}}$$

and

$$c_m = \frac{(-1)^{m+1}}{m\pi} + \frac{m^2\pi^2(-1)^m - 2(-1)^m + 2}{m^3\pi^3}.$$

3.

$$u(x,y,t) = \frac{1}{4\sqrt{2}\pi}\sin(\pi x)\sin(\pi y)\sin(4\sqrt{2}\pi t)$$

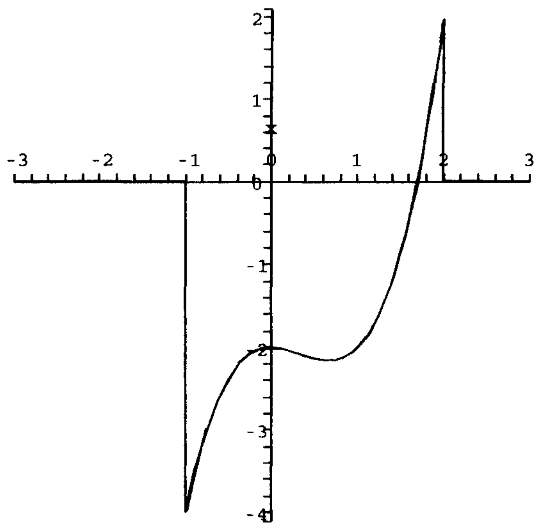

Figure 9.30: Section 4.2, Problem 5: $t = 0$.

5.

$$u(x,y) =$$
$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(a_{nm}\cos(\alpha_{nm}t) + b_{nm}\sin(\alpha_{nm}t))\sin(n\pi x)\sin(m\pi y),$$

where

$$\alpha_{nm} = \pi\sqrt{n^2+m^2},$$
$$a_{nm} = \frac{16}{n^3m^3\pi^6}(2+(-1)^n)((-1)^m - 1)$$

and

$$b_{nm} = \frac{4}{\alpha_{nm}}\frac{(-1)^{n+1}}{n\pi}\left(\frac{m^2\pi^2(-1)^{m+1} + 2(-1)^m - 2}{m^3\pi^3}\right).$$

Section 4.12

1. The solution in this case is

$$u(\mathbf{x},t) = \frac{1}{4\pi t}\int\int_{S_t} k d\sigma_t$$
$$= \frac{k}{4\pi t}A(\sigma_t) = kt,$$

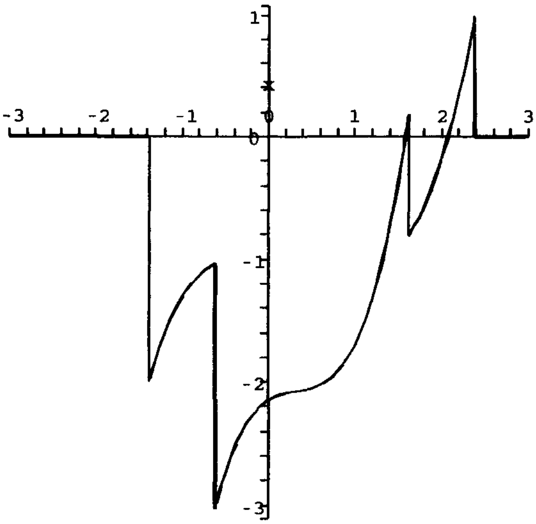

Figure 9.31: Section 4.2, Problem 5: $t = 0.38$.

where $A(\sigma_t)$ is the area of σ_t. Then $u(\mathbf{x}, t)$ is independent of $\mathbf{x}$.

Section 4.13

1. If ψ is a function of x only (one space variable), the solution of Problem VCP becomes

$$u(x,t) = \frac{1}{4\pi t} \iint_{S(x,0,0;t)} \psi(\xi)\, d\sigma_t = \frac{1}{2}\int_{x-t}^{x+t} \psi(\xi)\, d\xi.$$

This gives the solution of Problem CP as

$$\begin{aligned} u(x,t) &= \frac{\partial}{\partial t}\left[\frac{1}{2}\int_{x-t}^{x+t} \varphi(\xi)\, d\xi\right] + \frac{1}{2}\int_{x-t}^{x+t} \psi(\xi)\, d\xi \\ &= \frac{1}{2}(\varphi(x+t) + \varphi(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} \psi(\xi)\, d\xi, \end{aligned}$$

and this is d'Alembert's solution.

Chapter 5: The Heat Equation

Section 5.1

1. Choose $x = n$ and $t = 1/2n^2$ and show that $u_n(x,t)$ can be made as large as we like for certain choices of n.

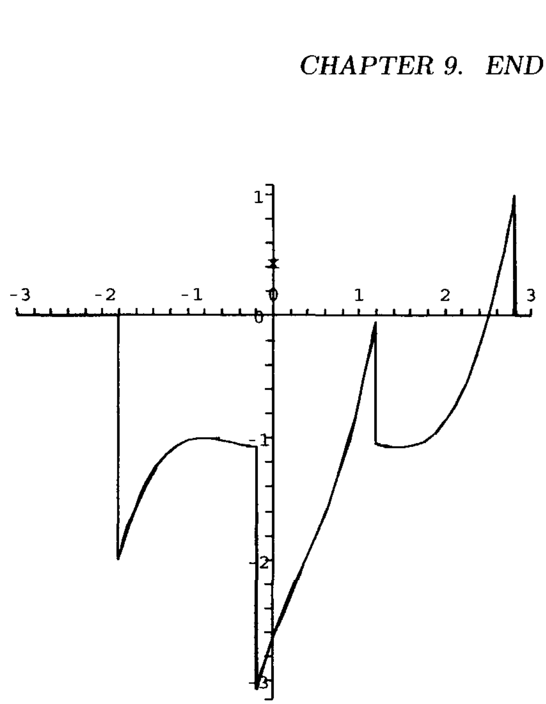

Figure 9.32: Section 4.2, Problem 5: $t = 0.81$.

2. The problem is $u_t = ku_{xx}$ for $0 < x < L, t > 0$, with $u(x,0) = f(x)$ for $0 < x < L$, and $u(0,t) = 0$ and $u_x(L,t) = 0$ for $t > 0$.

Section 5.3

1. The solution reduces to $u(x,t) = \sin(\pi x)e^{-k\pi^2 t}$. Figure 9.47 shows graphs of this function for $t = 0.035, 0.098$, and 0.25. As we might expect in such a simple setting, the temperature function is simply decreasing to zero in a straightforward way.

3.
$$u(x,t) = \frac{16}{3} + \sum_{n=1}^{\infty} 64\frac{(-1)^n}{n^2\pi^2}\cos(n\pi x/4)e^{-n^2\pi^2 kt/16}$$
Figure 9.48 shows graphs of this function at times $t = 0.089, 1.42, 1.97, 2.81, 5.19$, 8, and 16. Because of the initial condition, the temperature at $t = 0.089$ is "close to" the graph of $u = x^2$. The graphs are seen to intersect near $x = 2$. On roughly $[0,2]$, the curves increase corresponding to increasing time. As might be expected, $u(x,t) \to 16/3$, the steady-state value, as $t \to \infty$. At $t = 16$, $u(x,16)$ is close to the graph of this horizontal line.

5.
$$u(x,t) = \frac{1}{6}(1 - e^{-6}) + \sum_{n=1}^{\infty} 12\frac{1 - e^{-6}(-1)^n}{36 + n^2\pi^2}\cos(n\pi x/6)e^{-kn^2\pi^2 t/36}$$

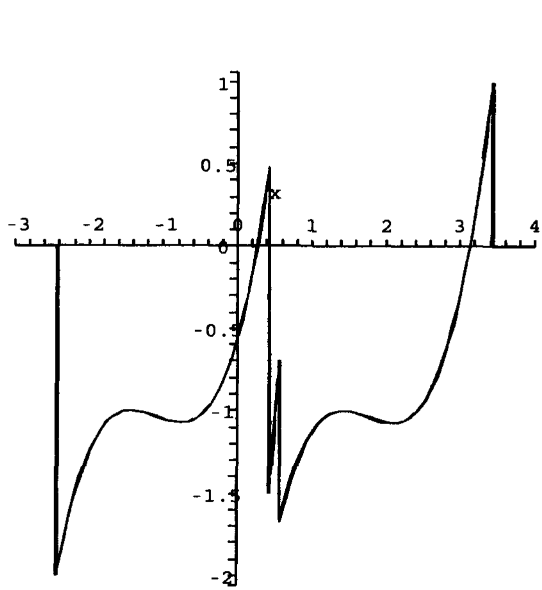

Figure 9.33: Section 4.2, Problem 5: $t = 1.43$.

Figure 9.49 shows the temperature profile at times $t = 0.031, 0.79, 1.27, 2.9, 4.3$, and 8.7. If you look at the interval $[0, 1]$, the graphs move downward as t is chosen larger. Notice that the temperature appears to be leveling off toward its steady-state value of $\frac{1}{6}(1 - e^{-6})$.

7.

$$u(x,t) = L - \frac{1}{3}L^2 + \sum_{n=1}^{\infty} \frac{4L^2(-1)^{n+1}}{n^2\pi^2} \cos(n\pi x/L)e^{-kn^2\pi^2 t/L^2}$$

Figure 9.50 shows graphs of this solution for $L = 1$ and times $t = 0.0041, 0.057, 0.13$, and 0.61. On about $[0, 1/2]$ these graphs are moving downward. By the time $t = 0.61$ the solution is indistinguishable (in the scale of the graph) from its steady-state limit $u = 2/3$.

9. Choose $\alpha = -a/2$ and $\beta = k(b - a^2/4)$.

11.

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin((2n-1)\pi x/2L)e^{-(2n-1)^2\pi^2 kt/4L^2}$$

with

$$b_n = \frac{2}{L}\int_0^L f(x)\sin((2n-1)\pi x/2L)\,dx$$

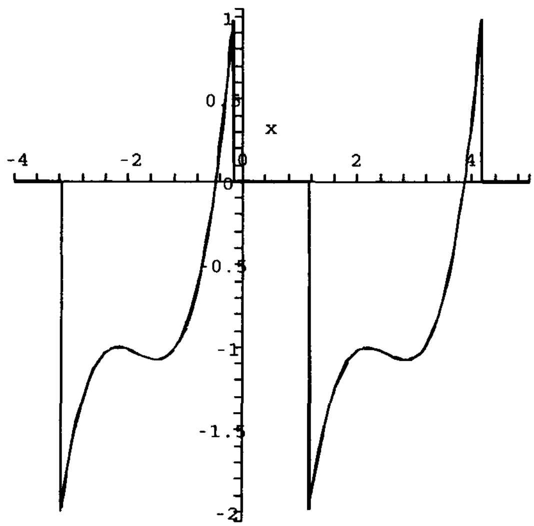

Figure 9.34: Section 4.2, Problem 5: $t = 2.17$.

13.

$$u(x,t) = e^{hx/2k}e^{-h^2t/4k}U(x,t),$$

where

$$U(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/L)e^{-kn^2\pi^2 t/L^2}$$

and

$$b_n = \frac{2}{L}\int_0^L e^{-h\xi/2k}f(\xi)\sin(n\pi\xi/L)\,d\xi.$$

15.

$$u(x,t) = e^{3x-9t}U(x,t)$$

with

$$U(x,t) = \sum_{n=1}^{\infty} \frac{12n}{\pi}\frac{e^{-3\pi}(-1)^n + 1}{100 + 16n^2 + n^4}\sin(nx)e^{-n^2 t}$$

Figure 9.51 shows graphs of the solution at times $t = 0.0029, 0.029, 0.071,$ $0.23, 0.34,$ and 0.46. On the interval $[0,1]$, these graphs are moving downward as time increases. Notice that the solution appears to be flattening out and approaching zero in the limit as t increases.

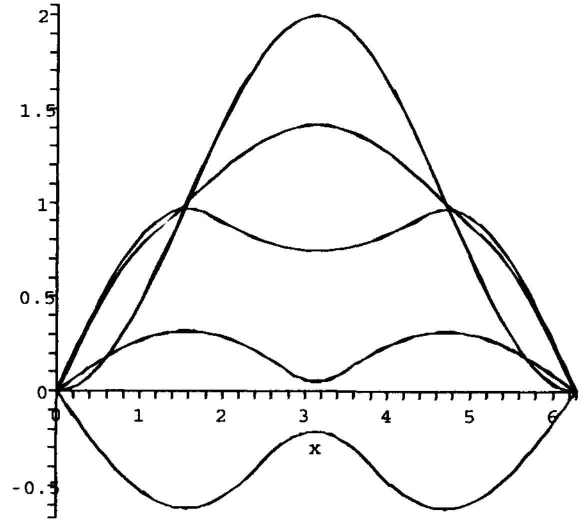

Figure 9.35: Section 4.8, Problem 1: $t = 0, 0.38, 0.61, 0.94$, and 1.27.

17.

$$u(x,t) = \sum_{n=1}^{\infty} \frac{16}{\pi} \frac{1-(-1)^n}{n^3} \sin(nx/2) e^{-8t} e^{-n^2 t}$$

Figure 9.52 shows the temperature profile decreasing toward zero over times $t = 0.0027, 0.031$, and 0.42.

19. $u(x,t) = B$

21. The solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin((2n-1)\pi x/4) e^{-9n^2\pi^2 t/16}$$

with

$$b_n = \frac{-16}{\pi^2} \frac{8n\pi(-1)^n - 4\pi(-1)^n + 8}{8n^3 - 12n^2 + 6n - 1}.$$

Because of the exponential factor, $\lim_{t\to\infty} u(x,t) = 0$.

23.

$$u(x,t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L)) e^{-kn^2\pi^2 t/L^2}$$

with

$$a_n = \frac{1}{L} \int_{-L}^{L} f(\xi) \cos(n\pi\xi/L)\, d\xi$$

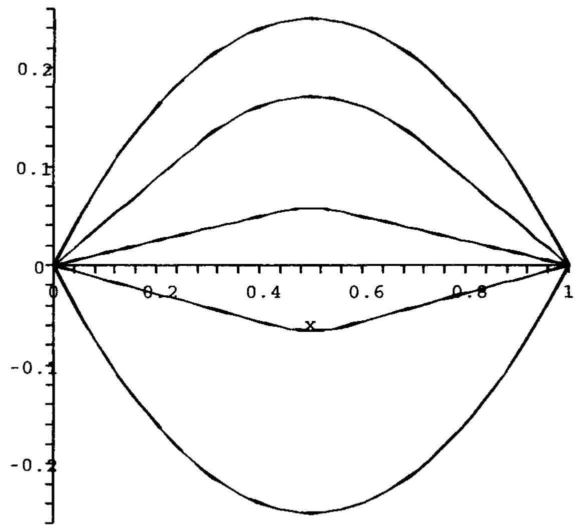

Figure 9.36: Section 4.8, Problem 3: $t = 0, 0.047, 0.073, 0.095$, and 0.17.

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(\xi)\sin(n\pi\xi/L)\,d\xi.$$

25.

$$u(x,t) = A - \frac{A}{\operatorname{erf}(d/2)}\operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)$$

Section 5.4

1. If f is odd, then $f(x)\cos(\omega x)$ is odd, so $a_\omega = 0$ and $u(x,t) = \int_0^\infty b_\omega \sin(\omega x)e^{-\omega^2 kt}$ is odd in the space variable. If f is even, now $b_\omega = 0$ and $u(x,t)$ is even in x.

5. The solution by Fourier integral and transform is

$$u(x,t) = \frac{8}{\pi}\int_0^\infty \frac{1}{16+\omega^2}\cos(\omega x)e^{-\omega^2 kt}\,d\omega.$$

The solution by convolution is

$$u(x,t) = \frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^{\infty} e^{-4|\xi|}e^{-(x-\xi)^2/4kt}d\xi.$$

These are different expressions for the same solution.

7. By Fourier integral we get

$$u(x,t) = \int_0^\infty (a_\omega\cos(\omega x) + b_\omega\sin(\omega x))e^{-\omega^2 kt}\,d\omega,$$

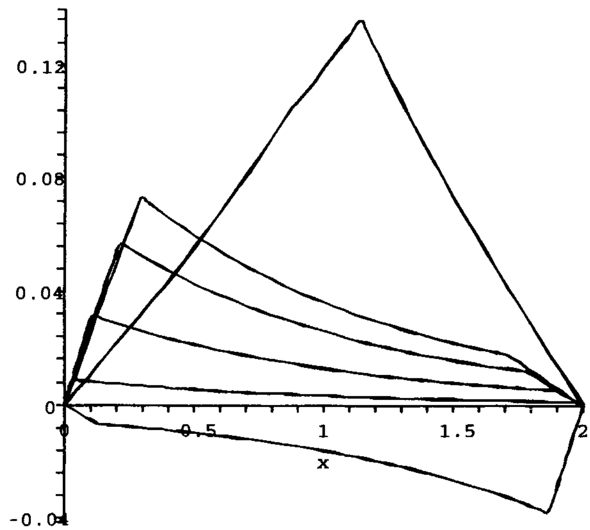

Figure 9.37: Section 4.8, Problem 5: $t = 0.009, 0.035, 0.07, 0.097, 0.38$, and 0.71.

where

$$a_\omega = \frac{8}{\pi}\cos(\omega)\frac{\cos^3(\omega) - \cos(\omega) + 4\omega\sin(\omega)\cos^2(\omega) - 2\omega\sin(\omega)}{\omega^2}$$

and

$$\begin{aligned} b_\omega &= \frac{4}{\pi}\frac{2\sin(\omega)\cos^3(\omega) - \sin(\omega)\cos(\omega)}{\omega^2} \\ &+ \frac{4}{\pi}\frac{-8\omega\cos^4(\omega) + 8\omega\cos^2(\omega) - \omega}{\omega^2} \end{aligned}$$

By Fourier transform,

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^4 \xi\cos(\omega(\xi - x))e^{-\omega^2 kt}\, d\xi\, d\omega$$

and the integral with respect to ξ can be evaluated to give the solution obtained by Fourier integral.

The solution by convolution is

$$u(x,t) = \frac{1}{2\sqrt{\pi kt}}\int_0^4 \xi e^{-(x-\xi)^2/4kt}\, d\xi.$$

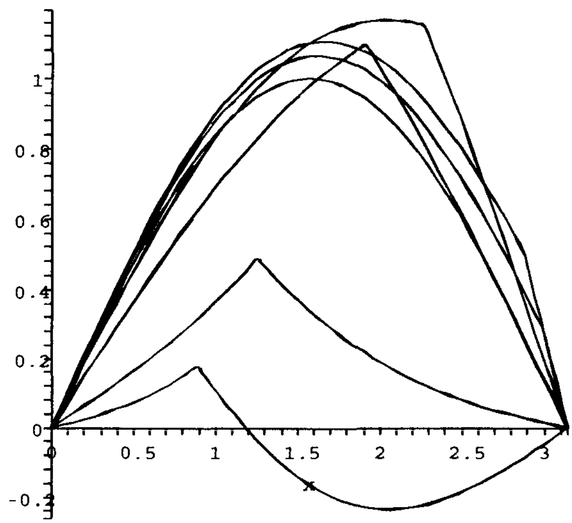

Figure 9.38: Section 4.8, Problem 7: $t = 0, 0.047, 0.087, 0.29, 0.41, 0.63$, and 0.75.

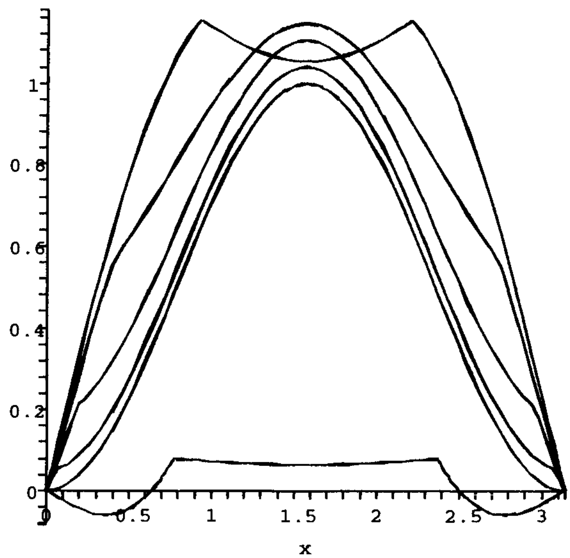

Figure 9.39: Section 4.8, Problem 9: $t = 0, 0.061, 0.19, 0.41, 0.93$, and 2.37.

9. By Fourier integral,

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos(\omega)}{\omega}\sin(\omega x)e^{-\omega^2 kt}\,d\omega.$$

By Fourier transform,

$$u(x,t) = \int_{-\infty}^{\infty}\left[\sin(\omega x)\frac{1-\cos(\omega)}{\pi\omega}\right]e^{\omega^2 kt}\,d\omega.$$

By convolution,

$$u(x,t) = \frac{1}{2\sqrt{\pi kt}}\left[\int_{-1}^{0} -e^{-(x-\xi)^2/4kt}d\xi + \int_0^1 e^{-(x-\xi)^2/4kt}\,d\xi\right].$$

11.

$$u(x,t) =$$

$$\frac{1}{2\sqrt{\pi kt}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(4kt)^n}\sum_{j=0}^{2n}\binom{2n}{j}\frac{(-1)^{2n-j}}{3n-j+1}[2^{3n-j+1}-(-2)^{3n-j+1}]x^j.$$

13.

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\sin(\omega)}{\pi^2-\omega^2}\sin(\omega x)e^{-k\omega^2 t}\,d\omega$$

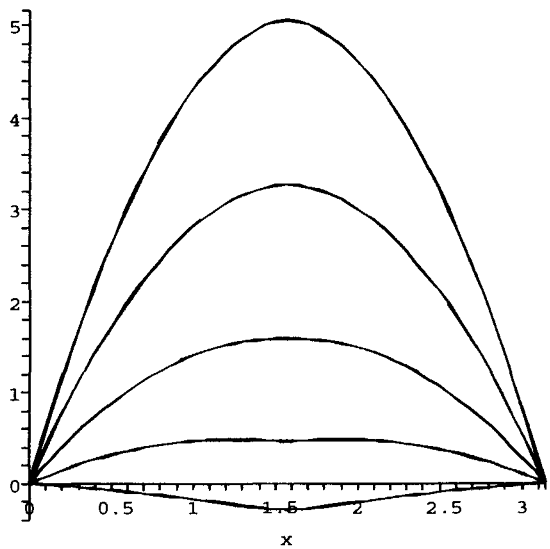

Figure 9.40: Section 4.9, Problem 1: profile at $t = 0.86$ for different values of K.

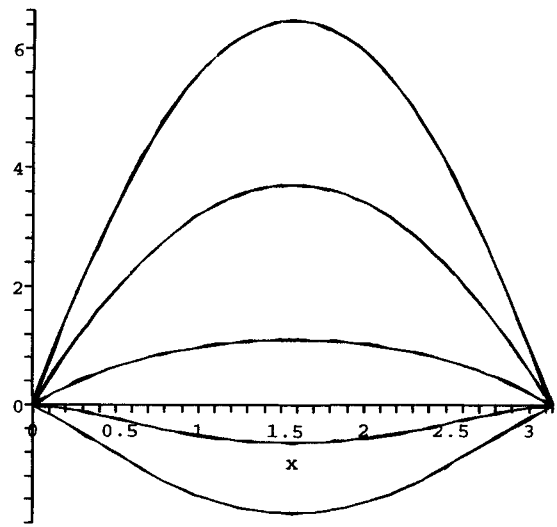

Figure 9.41: Section 4.9, Problem 1: profile at $t = 1.25$ for different values of K.

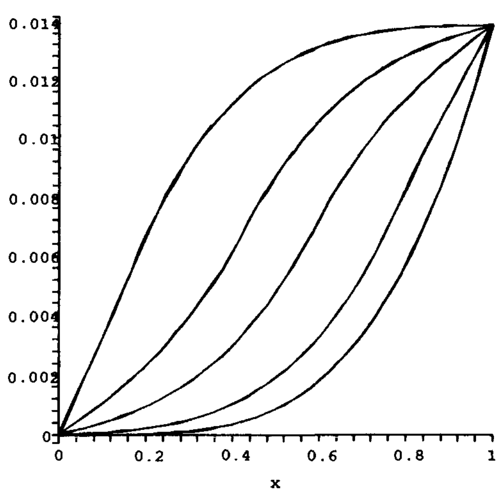

Figure 9.42: Section 4.9, Problem 2: profile for $A = 1.5$ and $t = 0.0035, 0.073, 2.19, 3.47$, and 5.61.

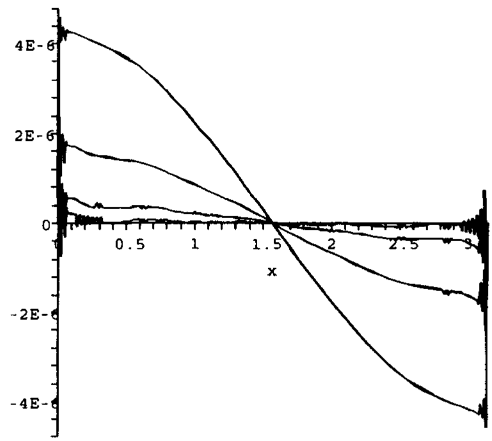

Figure 9.43: Section 4.9, Problem 3: wave moving upward at times $t = 0.00037, 0.00091, 0.0018$, and 0.0029.

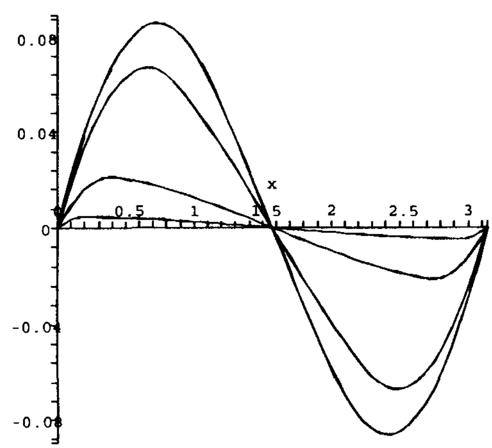

Figure 9.44: Section 4.9, Problem 3: wave moving upward at times $t = 4.81, 4.93, 5.16$, and 5.27.

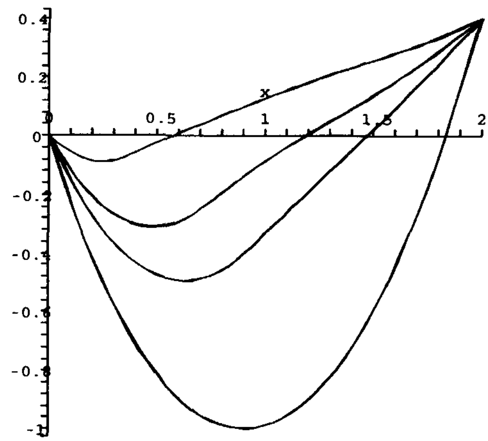

Figure 9.45: Section 4.9, Problem 5: wave moving downward through times $t = 0.37, 0.71, 0.95$, and 1.89.

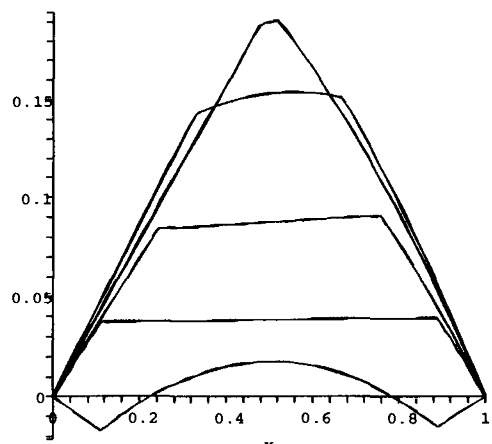

Figure 9.46: Section 4.9, Problem 7: motion of the wave.

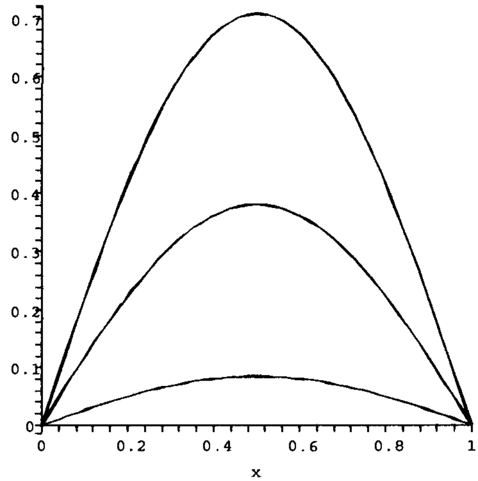

Figure 9.47: Section 5.3, Problem 1: temperature profiles at $t = 0.035, 0.098$, and 0.25.

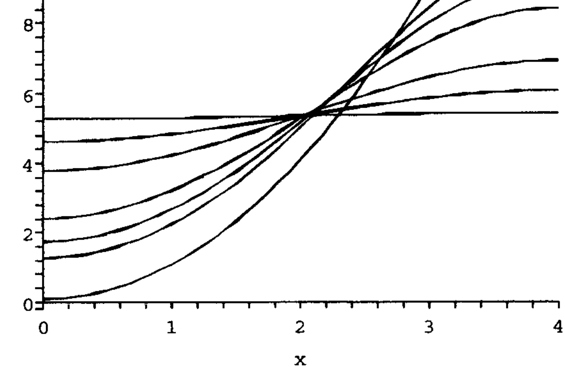

Figure 9.48: Section 5.3, Problem 3: temperature profiles at $t = 0.089, 1.42, 1.97, 2.81, 5.19, 8$, and 16.

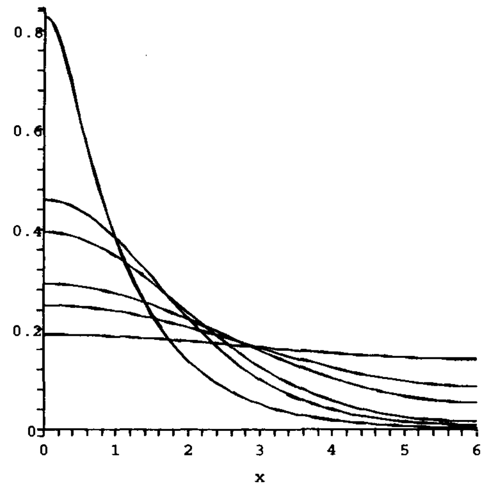

Figure 9.49: Section 5.3, Problem 5: temperature profiles at $t = 0.031, 0.79, 1.27, 2.9, 4.3,$ and 8.7.

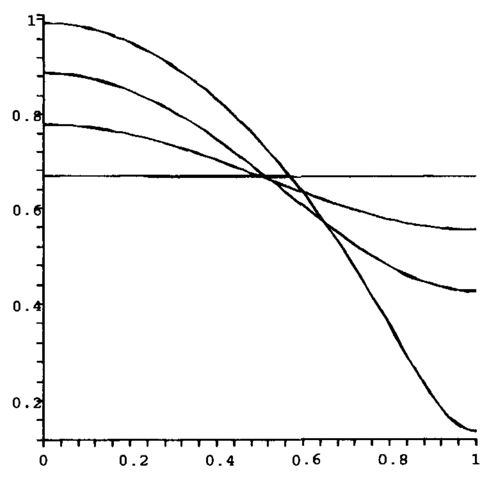

Figure 9.50: Section 5.3, Problem 7: temperature profiles at $t = 0.0041, 0.057, 0.13,$ and 0.61.

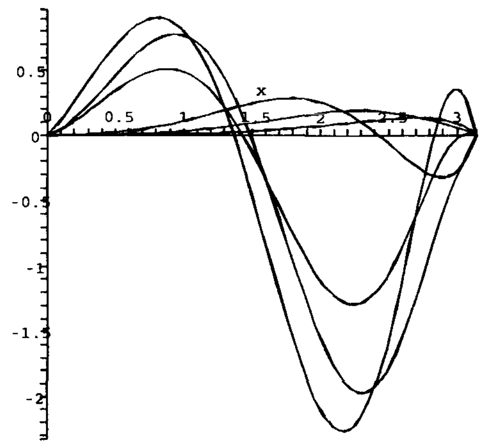

Figure 9.51: Section 5.3, Problem 15: temperature profiles at $t = 0.0029, 0.029, 0.071, 0.23, 0.34$, and 0.46.

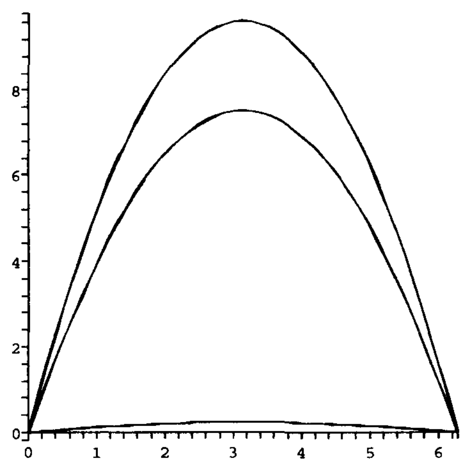

Figure 9.52: Section 5.3, Problem 17: temperature profiles at $t = 0.0027, 0.031$, and 0.42.

15. Write the solution as

$$u(x,t) = \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \cos(\omega(x-\xi)) e^{-k\omega^2 t} \right) f(\xi)\, d\xi.$$

This suggests that we choose

$$G(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos(\omega x) e^{-k\omega^2 t}\, d\omega.$$

To show that $\int_{-\infty}^{\infty} G(x-\xi,t) d\xi = 1$, look at the solution of this problem when $f(\xi) = 1$ for all ξ.

Section 5.5

1. By separation of variables,

$$u(x,t) = \frac{2}{\pi} \int_0^{\infty} \frac{\omega}{\alpha^2 + \omega^2} \sin(\omega x) e^{-k\omega^2 t}\, d\omega.$$

We obtain the same solution by Fourier sine transform. The solution can also be written

$$u(x,t) = \frac{1}{2\sqrt{\pi k t}} \int_0^{\infty} \left(e^{-(x-\xi)^2/4kt} - e^{-(x+\xi)^2/4kt} \right) e^{-\alpha\xi}\, d\xi.$$

3.

$$u(x,t) = \frac{2}{\pi} \int_0^{\infty} \frac{1-\cos(h\omega)}{\omega} \sin(\omega x) e^{-k\omega^2 t}\, d\omega$$

We can also write the solution as

$$u(x,t) = \frac{1}{2\sqrt{\pi k t}} \int_0^{h} \left(e^{-(x-\xi)^2/4kt} - e^{-(x+\xi)^2/4kt} \right) d\xi.$$

5. By separation of variables,

$$u(x,t) = \frac{2}{\pi} \int_0^{\infty} \frac{1}{\omega} (1 - 2\cos(h\omega) + \cos(2h\omega)) e^{-k\omega^2 t}\, d\omega.$$

We can also write

$$\begin{aligned} u(x,t) &= \frac{1}{2\sqrt{\pi k t}} \int_0^{h} \left(e^{-(x-\xi)^2/4kt} - e^{-(x+\xi)^2/4kt} \right) d\xi \\ &\quad - \frac{1}{2\sqrt{\pi k t}} \int_h^{2h} \left(e^{-(x-\xi)^2/4kt} - e^{-(x+\xi)^2/4kt} \right) d\xi. \end{aligned}$$

7.

$$u(x,t) = \frac{2}{\pi} \int_0^{\infty} \frac{-e^{-1}\cos(\omega) + e^{-1}\omega \sin(\omega) + 1}{1+\omega^2} \cos(\omega x) e^{-k\omega^2 t}\, d\omega$$

9.

$$u(x,t) = \int_0^{\infty} (a_\omega \cos(\omega x) + b_\omega \sin(\omega x)) e^{-k\omega^2 t}\, d\omega,$$

with
$$a_\omega = \frac{2}{\pi\omega^3}(-\sin(\omega)\cos(\omega) + 2\omega\cos^2(\omega) - \omega)$$
and
$$b_\omega = \frac{2}{\pi\omega^3}(\cos^2(\omega) - 1 + 2\omega\sin(\omega)\cos(\omega)).$$
This problem can also be solved using the Fourier sine transform.

11. If we try to use the Fourier sine transform, we find that we need $u(0,t)$, about which we have no information. Thus try the cosine transform, obtaining
$$u(x,t) = \frac{2}{\pi}e^{-t}\int_0^\infty \int_0^t f(\xi)e^{(1+\omega^2)\xi}\,d\xi\cos(\omega x)e^{-\omega^2 t}\,d\omega.$$

Section 5.7

1.
$$u(x,t) =$$
$$\frac{2L^2}{k^2\pi^5}\sum_{n=1}^{\infty}\left[\left(\frac{1-(-1)^n}{n^5}\right)(kn^2\pi^2 t + e^{-kn^2\pi^2 t/L^2}L^2 - L^2)\right]\sin(n\pi x/L)$$
$$+\sum_{n=1}^{\infty} 4L^2\frac{1-(-1)^n}{n^3\pi^3}\sin(n\pi x/L)e^{-kn^2\pi^2 t/L^2}.$$

With $L = \pi$ and $k = 1$, the temperature profile is shown in Figure 9.53 for $t = 1.35, 3.35, 5.35$, and 7.35. Because of the source term, the temperature is increasing with time, so the graph with largest maximum is for $t = 7.35$.

3. In the case $L \leq \pi$,
$$u(x,t) = \sum_{n=1}^{\infty} c_n \frac{e^{-kn^2\pi^2 t/L^2} - e^{-kn^2\pi^2(t-L)/L^2}}{kn^2\pi^2}\sin(n\pi x/L)$$
$$+\sum_{n=1}^{\infty}\left(-2L^3\frac{4(-1)^n+2}{n^3\pi^3}\right)\sin(n\pi x\mathrm{L})e^{-kn^2\pi^2 t/L^2},$$
where
$$c_n = 2n\pi L^2\frac{1-(-1)^n\cos(L)}{L^2 - n^2\pi^2}.$$

In the case $L = \pi$,
$$u(x,t) = \sum_{n=2}^{\infty}\left(\frac{2n(1+(-1)^n)}{\pi(n^2-1)}\frac{1-e^{-kn^2 t}}{kn^2}\right)\sin(nx)$$
$$+\sum_{n=1}^{\infty}\left(-4\frac{2(-1)^n+1}{n^3}\right)\sin(nx)e^{-kn^2 t}.$$

Figure 9.54 shows profiles of the solution for $L = \pi$ and times $t = 0.0001, 0.069, 1.42$ and 2.57. Over this time span, the temperature is decreasing, so the graphs are moving downward with time.

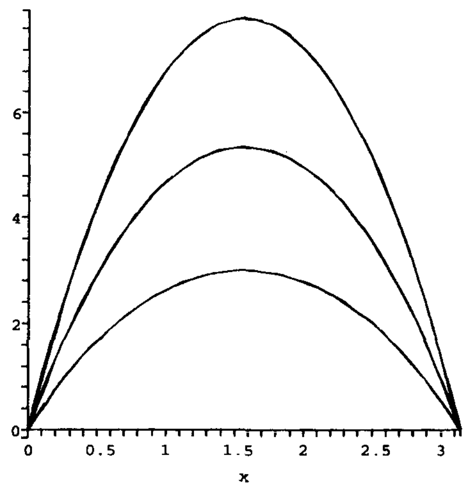

Figure 9.53: Section 5.7, Problem 1: temperature profiles at $t = 1.35, 3.35, 5.35,$ and 7.35.

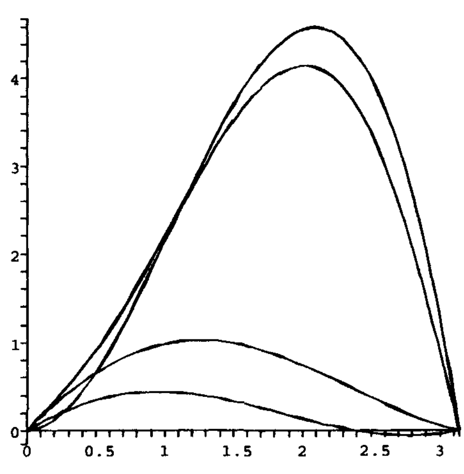

Figure 9.54: Section 5.7, Problem 3: temperature profiles at $t = 0.0001, 0.069, 1.42,$ and 2.57.

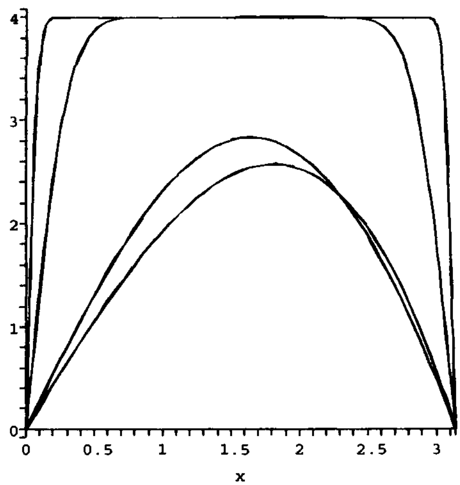

Figure 9.55: Section 5.7, Problem 5: temperature profiles decreasing over $t = 0.0019, 0.027, 0.74$, and 1.53.

5.

$$u(x,t) =$$
$$\sum_{n=1}^{\infty} \frac{2L^3}{k^2 n^5 \pi^5}(-1)^{n+1}\left(kn^2\pi^2 t + L^2 e^{-kn^2\pi^2 t/L^2} - L^2\right)\sin(n\pi x/L)$$
$$+ \frac{2K}{\pi}\sum_{n=1}^{\infty} \frac{1-\cos(n\pi)}{n}\sin(n\pi x/L)e^{-kn^2\pi^2 t/L^2}$$

Graphs of the solution are shown for $K = 4$. Figure 9.55 shows the temperature decreasing for times $t = 0.0019, 0.027, 0.74$, and 1.53, and Figure 9.56 shows the profile increasing for $t = 1.93$ and 2.49.

7.

$$u(x,t) = \frac{1}{4}Lt^2 + 1$$
$$+ \sum_{n=1}^{\infty} \frac{L^2}{k^2 n^4 \pi^4}\left(\frac{2L(\cos(n\pi)-1)}{n\pi}\right)\left(kn^2\pi^2 t + L^2 e^{-kn^2\pi^2 t/L^2} - L^2\right)\cos(n\pi x/L)$$

9.

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin((2n-1)\pi x/2L),$$

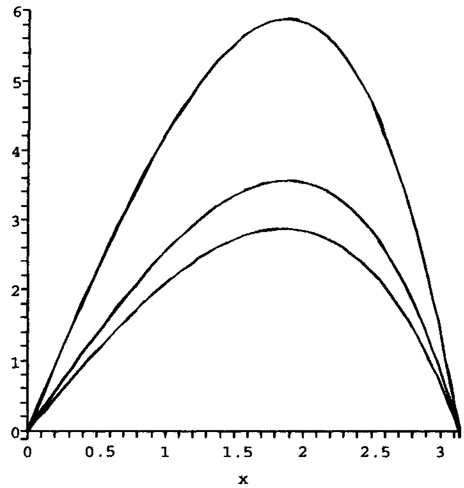

Figure 9.56: Section 5.7, Problem 5: temperature profiles increasing for $t = 1.93, 2.49$, and 3.87.

where

$$T_n(t) = \frac{8L(-1)^{n+1}}{k}\Bigg)\frac{-4L^2 + 4k\pi^2 tn^2 - 4k\pi^2 tn + k\pi^2 t}{(2n-1)^4\pi^4}\Bigg)$$
$$+ \frac{8L(-1)^{n+1}}{k}\left(\frac{4L^2 e^{-k(2n-1)^2\pi^2 t/4L^2}}{(2n-1)^4\pi^4}\right)$$
$$- 8\sqrt{\lambda_n}L\left(\frac{-1 + e^{-k(2n-1)^2\pi^2 t/4L^2}}{(2n-1)^2\pi^2}\right).$$

Section 5.8

1.

$$u(x,y,t) = \sum_{n=1}^{\infty}\frac{4}{\pi n^3}((-1)^n - 1)\sin(x)\sin(ny)e^{-k(1+n^2)t}$$

3.

$$u(x,y,t) = \sum_{m=1}^{\infty} c_{0m}\sin(my)e^{-km^2 t}$$
$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm}\cos(nx)\sin(my)e^{-k(n^2+m^2)},$$

where

$$c_{0m} = \frac{2\pi - 4}{\pi^2}\left(\frac{2 - 2(-1)^m}{m^3}\right)$$

and

$$c_{nm} = \frac{4}{\pi^2}\left(\frac{-\pi(-1)^n + 4\pi(-1)^n n^2 + 2 + 8n^2}{1 - 8n^2 + 16n^4}\right)\left(\frac{4(-1)^m - 4}{m^3}\right).$$

Chapter 6: Dirichlet and Neumann Problems

Section 6.1

1. The interior points are all (x, y) with $0 < x < 1$ and $0 < y < 1$. The boundary points are all $(0, y)$ and $(1, y)$ with $0 \le y \le 1$, and all $(x, 0)$ and $(x, 1)$ with $0 \le x \le 1$. $\overline{S}$ consists of all (x, y) with $0 \le y \le 1$ and $0 \le y \le 1$. S is not open because S contains some boundary points. S is not closed because there are boundary points of S that are not in S (such as $(1, 1)$). S is connected.

3. S is connected but is not a domain because S is not open. For example, the boundary point $(1, 0)$ is in S.

5. S has no interior points, because no disk about a point in the plane can consist only of points with rational coordinates. Every point in the plane is a boundary point of S, because any neighborhood of an arbitrary point will contain points with rational coordinates, and points with at least one irrational coordinate. $\overline{S} = R^2$. S is not open because each point of S is a boundary point, and an open set can contain no boundary point. S is not closed because there are boundary points of S that are not in S. S is not connected. Given two distinct points in the plane, there is no polygonal path connecting them and consisting only of points with rational coordinates.

7. The set of all points on a plane in R^3 is certainly connected, but is not open. A neighborhood (in R^3) of a point in this plane is a sphere that extends outside the plane, hence contains points not in the plane.

Section 6.2

3.

$$\begin{aligned} z\cos(z) &= x\cos(x)\cosh(y) + y\sin(x)\sinh(y) \\ &\quad + i(y\cos(x)\cosh(y) - x\sin(x)\sinh(y)) \end{aligned}$$

Section 6.3

1. Parametrize $\partial\Omega$ by arc length along the curve,with $x = x(s), y = y(s)$. The unit normal to $\partial\Omega$ is

$$\mathbf{n} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}.$$

Show that

$$\begin{aligned} &\int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds \\ &= \int_{\partial\Omega}\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) dx + \left(u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x}\right) dy. \end{aligned}$$

Now apply Green's theorem.

Section 6.6

1.
$$u(x,y) = \frac{1}{1-e^{2\pi^2}} \sin(nx) \left(e^{\pi y} - e^{2\pi^2} e^{-\pi y}\right)$$

3.
$$u(x,y) = \sum_{n=1}^{\infty} \frac{32}{\pi^2 \sinh(4n\pi)} \frac{n(-1)^{n+1}}{(4n^2-1)^2} \sin(n\pi x) \sinh(n\pi y)$$

5. Write the problem as a sum of two problems, one for u, which vanishes on three sides of the square, but $u(x,\pi) = x\sin(\pi x)$, and U vanishes on three sides, but $U(2,y) = \sin(y)$. The solution of the problem is $u + U$, where

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/2) \sinh(n\pi y/2),$$

$$b_n = \begin{cases} \frac{16n}{\pi^2 \sinh(n\pi^2/2)} \frac{(-1)^n - 1}{(n^2-4)^2} & \text{if } n \neq 2 \\ \frac{1}{\sinh(\pi^2)} & \text{if } n = 2 \end{cases}$$

and
$$U(x,y) = \frac{1}{\sinh(2)} \sin(y) \sinh(x).$$

7.
$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{(2n-1)\pi}{2a} x\right) \sinh\left(\frac{(2n-1)\pi}{2a} y\right),$$

where
$$b_n = \frac{2}{a} \frac{1}{\sinh((2n-1)\pi b/2a)} \int_0^a f(\xi) \sin\left(\frac{(2n-1)\pi}{2a}\xi\right) d\xi.$$

Section 6.7

1.
$$u(r,\theta) = \frac{1}{2} + \frac{1}{2}\left(\frac{r}{\rho}\right)^2 \cos(2\theta)$$

3. By inspection,
$$u(r,\theta) = \frac{r}{\rho}\sin(\theta).$$

5.
$$u(x,y) = 32 + x^2y^2 - \frac{1}{8}(x^2+y^2)^2$$

7. Attempt a solution

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta))$$
$$+ \sum_{n=1}^{\infty}(c_n r^{-n}\cos(n\theta) + d_n r^{-n}\sin(n\theta)) + k\ln(r)$$

for $\rho_1 \le r \le \rho_2$ and $-\pi \le \theta \le \pi$. Determine the coefficients:

$$a_0 = \frac{1}{\pi \ln(\rho_2/\rho_1)} \left[\ln(\rho_2) \int_{-\pi}^{\pi} g(\theta)\, d\theta - \ln(\rho_1) \int_{-\pi}^{\pi} f(\theta)\, d\theta\right],$$

$$k = \frac{1}{2\pi \ln(\rho_2/\rho_1)} \int_{-\pi}^{\pi} (f(\theta) - g(\theta))\, d\theta,$$

$$a_n = \frac{1}{\pi} \frac{\rho_1^n \int_{-\pi}^{\pi} g(\theta)\cos(n\theta) d\,\theta - \rho_2^n \int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\, d\theta}{\rho_1^{2n} - \rho_2^{2n}},$$

$$c_n = \frac{1}{\pi} \frac{\rho_2^{-n} \int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\, d\theta - \rho_1^{-n} \int_{-\pi}^{\pi} g(\theta)\cos(n\theta)\, d\theta}{\rho_2^{-2n} - \rho_1^{-2n}},$$

$$b_n = \frac{1}{\pi} \frac{\rho_1^n \int_{-\pi}^{\pi} g(\theta)\sin(n\theta)\, d\theta - \rho_2^n \int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\, d\theta}{\rho_1^{2n} - \rho_2^{2n}},$$

and

$$d_n = \frac{1}{\pi} \frac{\rho_2^{-n} \int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\, d\theta - \rho_1^{-n} \int_{-\pi}^{\pi} g(\theta)\sin(n\theta)\, d\theta}{\rho_2^{-2n} - \rho_1^{-2n}}.$$

9.

$$u(r,\theta) = 1 - \frac{1}{\ln(2)} \ln(r) + \frac{2}{3} r\cos(\theta) - \frac{2}{3} r^{-1}\cos(\theta)$$

11.

$$u(r,\theta) = 1 - \frac{1}{2\ln(2)} \ln(r) + \frac{2}{15} r^2\cos(2\theta) - \frac{2}{15} r^{-2}\cos(2\theta)$$

13.

$$u(r,\theta) = \frac{1}{2} + \frac{1}{24} r^2\cos(2\theta) - \frac{8}{3} r^{-2}\cos(2\theta)$$

Section 6.8

3. By inspection, the problem has solution $u(r,\theta) = 1$. Therefore,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\rho^2 - r^2}{r^2 - 2r\rho\cos(\theta - \xi) + \rho^2} d\xi = 1.$$

5. No, and almost any function harmonic in the plane serves as a counterexample. For example, let $u(x,y) = xy$.

Section 6.9

1.

$$\begin{aligned} u(x,y) &= \frac{y}{\pi} \int_0^1 \frac{\xi}{y^2 + (\xi - x)^2}\, d\xi \\ &= \frac{y}{2\pi} \left[\ln((1-x)^2 + y^2) - \ln(x^2 + y^2)\right] \\ &\quad + \frac{x}{\pi} \left[\arctan\left(\frac{1-x}{y}\right) - \arctan\left(-\frac{x}{y}\right)\right] \end{aligned}$$

3.

$$u(x,y) = \frac{y}{\pi}\int_{-c}^{c} \frac{k}{y^2+(\xi-x)^2}\,d\xi$$
$$= \frac{k}{\pi}\left[\arctan\left(\frac{x+c}{y}\right) - \arctan\left(\frac{x-c}{y}\right)\right]$$

5. $u(x,y) = k$ is a bounded solution. $u(x,y) = k + cy$ is an unbounded solution for any nonzero constant c. If $f(x) = k$, the integral solution derived in the text gives $u(x,y) = k$.

Section 6.10

1.
$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{0}\left[\frac{1}{y^2+(\xi-x)^2} - \frac{1}{y^2+(\xi+x)^2}\right] f(\xi)\,d\xi$$

3.
$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\left[\frac{1}{y^2+(\xi-x)^2} - \frac{1}{y^2+(\xi+x)^2}\right] k\,d\xi$$
$$= -\frac{2k}{\pi}\arctan\left(\frac{x}{y}\right)$$

Section 6.11

1.
$$u(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{4}{\sinh(\beta_{nm})}\frac{\pi-1}{3\pi^2}\sin(n\pi x)\sin(m\pi x)\sinh(\beta_{nm}y),$$

where
$$\beta_{nm} = \pi\sqrt{n^2+m^2}.$$

3. $u = u_1 + u_2$, where u_1 is the solution of the Dirichlet problem specifying $u = 0$ on all sides of the box, except $u(x,y,0) = \sin(3x)\sin(y)$, and u_2 is the solution with $u = 0$ on all sides of the box, except $u(x,y,0) = x(\pi-x)y(\pi-y)$. We find that

$$u_1(x,y,z) = \frac{1}{1-e^{2\sqrt{10}\pi}}\sin(3x)\sin(y)\left[e^{\sqrt{10}z} - e^{2\sqrt{10}\pi}e^{-\sqrt{10}z}\right]$$

and
$$u_2(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_{nm}\sin(nx)\sin(my)\sinh(\beta_{nm}z)$$

with
$$b_{nm} = \frac{16}{\pi^2\sinh(\beta_{nm}\pi)}\frac{(-1)^{n+m}-(-1)^m-(-1)^n+1}{n^3m^3}$$

and
$$\beta_{nm} = \sqrt{n^2+m^2}.$$

Section 6.12

5. Suppose that u and v are solutions, and let $w = u - v$. Show that

$$\frac{\partial w}{\partial n} + hw = 0$$

on $\partial\Omega$. Next show that

$$\int_{\partial\Omega} -hw^2\, ds = \iint_{\Omega} |\nabla w|^2\, dA.$$

Show from this that $|w|$ is constant on Ω, and then that $w = 0$ on Ω.

Section 6.13

1.

$$u(x,y) = \alpha_0 + \frac{4}{\pi(e^{2\pi}-1)}(e^{\pi y} + e^{2\pi}e^{-\pi y})\cos(\pi x)$$

3. Write $u = u_1 + u_2$, where all boundary conditions are zero in Problem 1 except $u_y(x,0) = \cos(3x)$, and all boundary conditions are zero in Problem 2 except for $u_y(x,\pi)6x - 3\pi$. We get

$$u_1(x,y) = \alpha_0 + \frac{1}{3(1-e^6)}(e^{3y} + e^{6\pi}e^{-3y})\cos(3x)$$

and

$$u_2(x,y) = \beta_0 + \sum_{n=1}^{\infty} \frac{12}{\pi n^3 \sinh(n\pi)}((-1)^n - 1)\cosh(ny)\cos(nx).$$

5.

$$u(x,y) = \sum_{n=1}^{\infty} a_n(e^{n\pi x} - e^{2n\pi}e^{-n\pi x})\sin(n\pi y)$$

with

$$a_n = \frac{2}{n\pi(1-e^{2n\pi})}\left(\frac{-n^2\pi^2(-1)^n + 6(-1)^n - 6}{n^3\pi^3}\right)$$

Section 6.14

1.

$$u(t,\theta) = \alpha_0 + \frac{1}{3}\frac{1}{\rho^3}r^3\sin(3\theta)$$

3. Convert to polar coordinates, with $U(r,\theta) = u(x,y)$, to get $U(r,\theta) = 2r^2\sin(2\theta)$. Then $u(x,y) = 4xy$.

5. In polar coordinates, the solution is

$$U(r,\theta) = \frac{1}{4}[r\cos(\theta) - r3\cos^3(\theta) + 3r^3\cos(\theta)\sin^2(\theta)]$$

and this converts in rectangular coordinates to

$$u(x,y) = \frac{1}{4}(x - x^3 + 3xy^2).$$

Section 6.15

1.

$$u(x,y) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \ln(y^2 + (\xi - x)^2) f(\xi)\, d\xi + c$$

3.

$$u(x,y) = \int_0^{\infty} (a_\omega \cos(\omega y) + b_\omega \sin(\omega y)) e^{-\omega x}\, d\omega + c,$$

where

$$a_\omega = -\frac{1}{\pi\omega}\int_{-\infty}^{\infty} g(\xi)\cos(\omega\xi)\, d\xi$$

and

$$b_\omega = -\frac{1}{\pi\omega}\int_{-\infty}^{\infty} g(\xi)\sin(\omega\xi)\, d\xi.$$

Section 6.16

1. For domains in the plane, we get

$$u(\mathbf{x}) = -\int_{\partial\Omega} f(\mathbf{y})\frac{\partial}{\partial n} G(\mathbf{x},\mathbf{y})\, ds_y.$$

13.

$$\frac{1}{4\pi}\frac{1}{|\,\mathbf{y}-\mathbf{x}\,|} - \frac{1}{4\pi}\frac{1}{|\,\frac{a^2}{|y|^2}\mathbf{y}-\mathbf{x}\,|}$$

19.

$$G(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\left[\frac{1}{|\,\mathbf{y}+(0,j)-\mathbf{x}\,|} - \frac{1}{|\,\mathbf{y}-(0,j)-\mathbf{x}\,|}\right]$$

Section 6.17.1

5. $f(z) = \bar{z}$ is not conformal because it reverses orientation. For example, a rotation from 1 to i has positive (counterclockwise) orientation. But a rotation from $f(1) = 1$ to $f(i) = -i$ has negative (clockwise) orientation.

Section 6.17.2

1. Points on the vertical line $z = a \neq 0$ have the form $z = a + iy$ and map to

$$w = T(a+iy) = \frac{a}{a^2+y^2} - i\frac{y}{a^2+y^2} = u + iv.$$

Now check that

$$\left(u - \frac{1}{2a}\right)^2 + v^2 = \frac{1}{4a^2}.$$

3.

$$w = \frac{(1+4i)z + (-3-8i)}{(2+3i)z + (-4-7i)}$$

5.

$$w = 2\left(\frac{(-18+7i)z + 1 - 18i}{(-11+5i)z + 20 - 8i}\right)$$

7.

$$w = -2\left(\frac{(7+4i)z+3-91i}{5z-9+2i}\right)$$

9.

$$T(z) = \frac{iz-4}{z} = -\frac{4}{z} + i$$

and this is a rotation/magnification followed by a translation.

11.

$$T(z) = \frac{-8-i}{2}\frac{1}{2z+i} + \frac{1}{2}$$

13.

$$T(z) = (8-12i)\frac{1}{z+4} - 2 + 3i$$

Section 6.17.5

1. First map the upper half-plane onto the unit disk. There are many mappings that do this. One is

$$w = T(z) = \frac{z-i}{iz-1},$$

obtained as a bilinear transformation mapping $-1 \to 1$, $0 \to i$, and $1 \to -1$. Then

$$u(x,y) = \operatorname{Re}\left[\frac{1}{2\pi i}\int_{\xi=-\infty}^{\infty} g(\xi)\frac{\frac{\xi-i}{i\xi-1}+\frac{z-i}{iz-1}}{\frac{\xi-i}{i\xi-1}-\frac{z-i}{iz-1}}\frac{i\xi-1}{\xi-i}\frac{-2}{(i\xi-1)^2}d\xi\right].$$

After some labor to extract the real part of this expression, we get

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{g(\xi)}{y^2+(\xi-x)^2}\,d\xi$$

consistent with this solution by previous methods.

4. Use the conformal mapping

$$T(z) = -i\left(\frac{e^z-1}{e^z+1}\right)$$

to obtain (after some computation)

$$u(x,y) = \int_{-\infty}^{\infty}\frac{e^{\xi}}{\pi}\frac{e^x\cos(y)}{e^{2x}+e^{2\xi}-2e^{\xi}e^x\sin(y)}g(\xi)\,d\xi.$$

Chapter 8: Additional Topics

Section 8.1

1. Eigenvalues are

$$\lambda_n = \left(\frac{(2n-1)\pi}{12}\right)^2$$

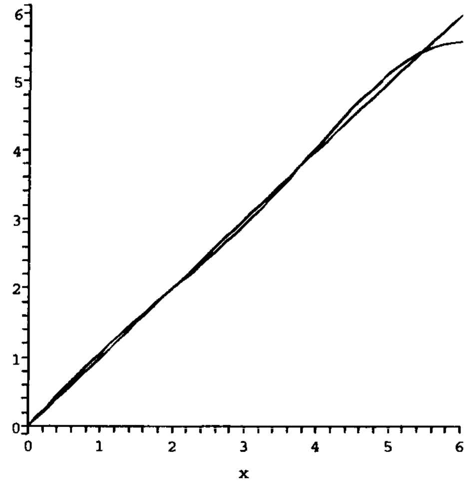

Figure 9.57: Section 8.1, Problem 1: x and the sum of the first three terms.

and eigenfunctions are

$$y_n = \sin\left(\frac{(2n-1)\pi x}{12}\right).$$

The expansion of $f(x) = x$ on $[0, 6]$ is

$$x = \sum_{n=1}^{\infty} \frac{48(-1)^{n+1}}{(2n-1)^2\pi^2} \sin\left(\frac{(2n-1)\pi x}{12}\right).$$

Figure 9.57 compares the function with the sum of the first three terms of this eigenfunction expansion, and Figure 9.58 makes this comparison with the sum of the first six terms.

3. Eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{2}\right)^2$$

and eigenfunctions are

$$y_n = \cos\left(\frac{n\pi x}{2}\right).$$

The eigenfunction expansion of $f(x) = x^2$ on $[0, 2]$ is

$$x^2 = \frac{4}{3} + \sum_{n=1}^{\infty} \frac{16(-1)^n}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right).$$

Figure 9.59 shows graphs of x^2 and the sum of the first three terms of this eigenfunction expansion. Figure 9.60 does this for the first six terms.

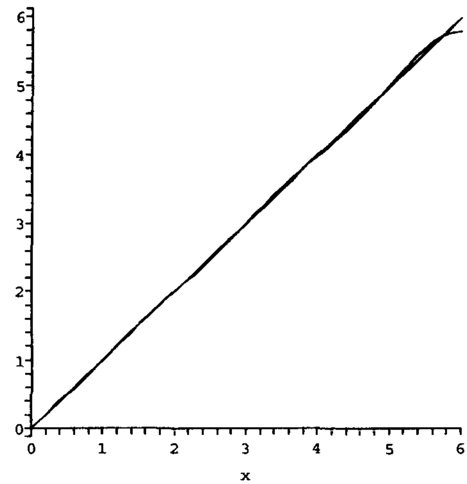

Figure 9.58: Section 8.1, Problem 1: x and the sum of the first six terms.

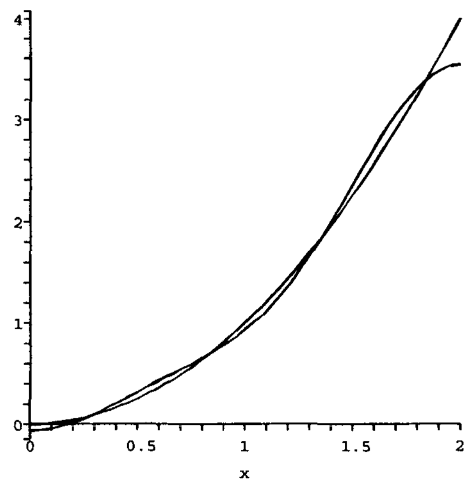

Figure 9.59: Section 8.1, Problem 3: x^2 and the sum of the first three terms.

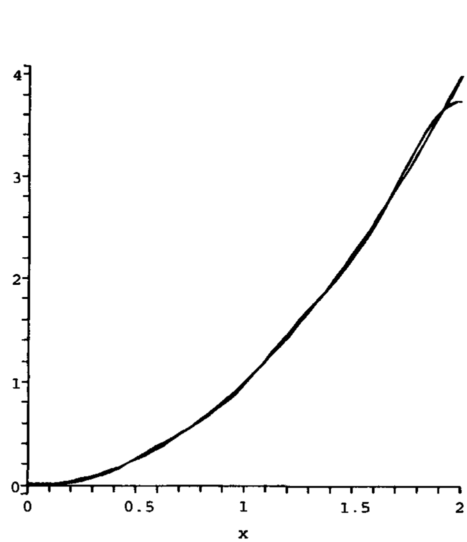

Figure 9.60: Section 8.1, Problem 3: x^2 and the sum of the first six terms.

7. The first six terms of the expansion are

$$\begin{aligned}&-\frac{1}{e}+\left(\frac{3e}{2}-\frac{15}{2e}\right)P_1(x)\\&+\left(\frac{15e}{2}-\frac{115}{2e}\right)P_2(x)+\left(\frac{147e}{2}-\frac{1085}{2e}\right)P_3(x)\\&+\left(\frac{1665e}{2}-\frac{12303}{2e}\right)P_4(x)+\left(11055e-\frac{81686}{e}\right)P_5(x).\end{aligned}$$

For comparison, Figure 9.61 shows a graph of xe^{-x} and the sum of the first six terms of this eigenfunction expansion. Within the scale of the graph, the function and this partial sum are virtually indistinguishable.

9. All the odd-indexed coefficients are zero, because $\cos(x)$ is an even function and P_n is odd if n is odd. The first six terms of the expansion are

$$\begin{aligned}&\sin(1)+(-10\sin(1)+15\cos(1))P_2(x)\\&+(549\sin(1)-855\cos(1))P_4(x)\end{aligned}$$

Figure 9.62 compares a graph of $\cos(x)$ with the first six terms of this eigenfunction expansion. To the scale of the graph, these two curves appear to be the same.

11.

$$u(\rho,\varphi)=\frac{\rho}{2}\cos(\varphi)$$

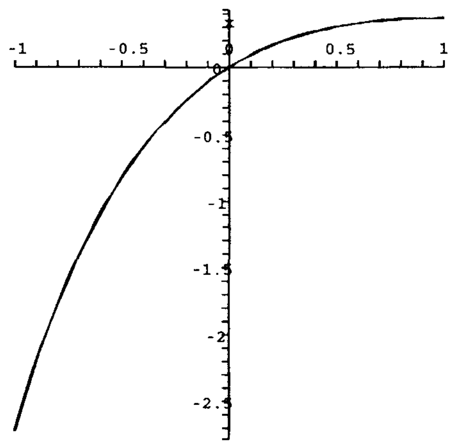

Figure 9.61: Section 8.1, Problem 7: xe^{-x} and the sum of the first six terms.

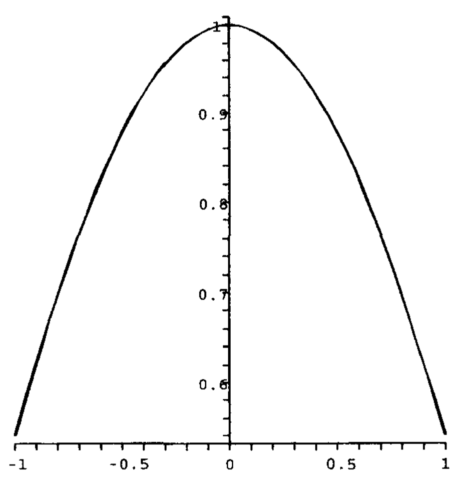

Figure 9.62: Section 8.1, Problem 9: $\cos(x)$ and the sum of the first six terms.

13.

$$\begin{aligned} z(r,t) &= 1.602J_0(1.203r)\cos(1.203ct) - 0.289J_0(2.76r)\cos(2.76ct) \\ &+ 0.851J_0(4.327r)\cos(4.327ct) - 0.272J_0(5.896r)\cos(5.896ct) \\ &+ 0.203J_0(7.466r)\cos(7.466ct) + \cdots \end{aligned}$$

15.

$$\begin{aligned} z(r,t) &= 1.976J_0(1.203r)\cos(1.203ct) - 3.7J_0(2.76r)\cos(2.76ct) \\ &+ 3.223J_0(4.327r)\cos(4.327ct) + 2.834J_0(5.896r)\cos(5.896ct) \\ &+ 2.548J_0(7.466r)\cos(7.466ct) + \cdots \end{aligned}$$

Section 8.2

1. First compute

$$\left(\frac{c\,\Delta t}{\Delta x}\right)^2 = 0.01.$$

Then $y_{1,0} = 0.09$, $y_{2,0} = 0.16$, $y_{3,0} = 0.21$, $y_{4,0} = 0.24$, $y_{5,0} = 0.25$, $y_{6,0} = 0.24$, $y_{7,0} = 0.21$, $y_{8,0} = 0.16$, $y_{9,0} = 0.09$, and $y_{j,-1} = y_{j,0}$.

Next, $y_{1,1} = 0.0898$, $y_{2,1} = 0.1598$, $y_{3,1} = 0.2098$, $y_{4,1} = 0.2398$, $y_{5,1} = 0.2498$, $y_{6,1} = 0.2398$, $y_{7,1} = 0.2098$, $y_{8,1} = 0.1598$, $y_{9,1} = 0.08098$, and $y_{1,2} = 0.0903$, $y_{2,2} = 0.1594$, $y_{3,2} = 0.2196$, $y_{4,2} = 0.2394$, $y_{5,2} = 0.2494$, $y_{6,2} = 0.2394$, $y_{7,2} = 0.2094$, $y_{8,2} = 0.1594$, $y_{9,2} = 0.0894$.

The exact solution is

$$y(x,t) = \sum_{n=1}^{\infty} \frac{8}{(2n-1)^3\pi^3} \sin((2n-1)\pi x)\cos((2n-1)\pi t/2).$$

3. First,

$$\left(\frac{c\,\Delta t}{\Delta x}\right)^2 = 0.04.$$

Then $y_{1,0} = 0.0990$, $y_{2,0} = 0.1920$, $y_{3,0} = 0.2730$, $y_{4,0} = 0.3360$, $y_{5,0} = 0.3750$, $y_{6,0} = 0.3840$, $y_{7,0} = 0.3750$, $y_{8,0} = 0.2880$, $y_{9,0} = 0.1710$, and $y_{1,-1} = 0.0190$, $y_{2,-1} = 0.1120$, $y_{3,-1} = 0.1930$, $y_{4,-1} = 0.2560$, $y_{5,-1} = 0.2950$, $y_{6,-1} \doteq 0.3040$, $y_{7,-1} = 0.2770$, $y_{8,-1} = 0.2080$, $y_{9,-1} = 0.091$.

Next, $y_{1,1} = 0.1788$, $y_{2,1} = 0.2715$, $y_{3,1} = 0.3523$, $y_{4,1} = 0.4504$, $y_{5,1} = 0.4538$, $y_{6,1} = 0.496$, $y_{7,1} = 0.4202$, $y_{8,1} = 0.3661$, $y_{9,1} = 0.2608$.

Finally, $y_{1,2} = 0.2552$, $y_{2,2} = 0.3506$, $y_{3,2} = 0.4323$, $y_{4,2} = 0.5610$, $y_{5,2} = 0.5323$, $y_{6,2} = 0.5142$, $y_{7,2} = 0.4824$, $y_{8,2} = 0.4421$, $y_{9,2} = 0.3387$.

The exact solution is

$$y(x,t) = \sum_{n=1}^{\infty} (a_n \cos(n\pi t) + b_n \sin(n\pi t))\sin(n\pi x),$$

where

$$a_n = \frac{12(-1)^{n+1}}{n^3\pi^3}$$

and

$$b_n = \frac{8(1-(-1)^n)}{n^2\pi^2}.$$

5. First,

$$\frac{c\,\Delta t}{(\Delta x)^2} = 0.25.$$

Begin with $u_{1,0} = 0.3092$, $u_{2,0} = 0.5878$, $u_{3,0} = 0.8090$, $u_{4,0} = 0.9511$, $u_{5,0} = 1$, $u_{6,0} = 0.9511$, $u_{7,0} = 0.8090$, $u_{8,0} = 0.5878$, $u_{9,0} = 0.3090$.

Next, $u_{1,1} = 0.3015$, $u_{2,1} = 0.5734$, $u_{3,1} = 0.7894$, $u_{4,1} = 0.9282$, $u_{5,1} = 0.9758$, $u_{6,1} = 0.9278$, $u_{7,1} = 0.7892$, $u_{8,1} = 0.5734$, $u_{9,1} = 0.3015$.

Finally, $u_{1,2} = 0.2941$, $u_{2,2} = 0.5595$, $u_{3,2} = 0.7701$, $u_{4,2} = 0.9054$, $u_{5,2} = 0.9519$, $u_{6,2} = 0.9051$, $u_{7,2} = 0.7699$, $u_{8,2} = 0.5594$, $u_{9,2} = 0.2941$.

The exact solution is

$$u(x,t) = e^{-\pi^2 t/8}\sin(\pi x).$$

Section 8.3

1. All that is needed is to compute g to insert into the integral formula for $u(x,t)$. We find that

$$g(x) = \begin{cases} e^{-x(1+x/2)/2k} & \text{for } -1 < x < 0 \\ e^{-x(1-x/2)/2k} & \text{for } 0 < x < 1 \\ e^{-1/4k} & \text{for } x < -1 \text{ and for } x > 1. \end{cases}$$

3. Compute

$$g(x) = \begin{cases} e^{-3/k} & \text{for } x < -3 \\ e^{x/k} & \text{for } -3 < x < 0 \\ e^{-5x/2k} & \text{for } 0 < x < 2 \\ e^{-5/k} & \text{for } x > 2. \end{cases}$$

7.

$$u(x,t) = \frac{10 + 5e^{5(x-15t/2)/6}}{1 + e^{5(x-15t/2)/6}}$$

9.

$$u(x,t) = \frac{2 + e^{(x-3t/2)/16}}{1 + e^{(x-3t/2)/16}}$$

Section 8.4

1. Here $b^2 - a = 3 > 0$, so case 3 applies. Here $G(\omega) = 0$ because $g(x) = 0$, and a straightforward calculation yields

$$F(\omega) = \frac{2}{\omega^2}(1 - \cos(\omega)).$$

The solution has the form

$$u(x,t) = \frac{1}{2\pi}\left(\int_{|\omega|<1/\sqrt{3}} U_1(\omega,t)e^{i\omega x}\,d\omega + \int_{|\omega|\geq 1/\sqrt{3}} U_2(\omega,t)e^{i\omega x}\,d\omega\right).$$

Compute

$$U_1(\omega,t) = e^{-2t}\left[c_1 e^{\sqrt{3-(3\omega)^2}t} + c_2 e^{-\sqrt{3-(3\omega)^2}t}\right],$$

where

$$c_1 = \frac{1-\cos(\omega)}{\omega^2}\left(1 + \frac{2}{\sqrt{3-(3\omega)^2}}\right)$$

and

$$c_2 = \frac{1-\cos(\omega)}{\omega^2}\left(1 - \frac{2}{\sqrt{3-(3\omega)^2}}\right).$$

Next,

$$U_2(\omega,t) = e^{-2t}\left[c_3\cos\left(\sqrt{-3+(3\omega)^2}t\right) + c_4\sin\left(\sqrt{-3+(3\omega)^2}t\right)\right],$$

where

$$c_3 = F(\omega) = \frac{2}{\omega^2}(1-\cos(\omega))$$

and

$$\begin{aligned} c_4 &= \frac{2F(\omega)}{\sqrt{-3=(3\omega)^2}} \\ &= \frac{1-\cos(\omega)}{\omega^2}\frac{1}{\sqrt{-3+(3\omega)^2}}. \end{aligned}$$

3. Here $b^2 - a = 0$, so case 2 applies. The solution is

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega,t)e^{i\omega x}\,d\omega,$$

where

$$U_(\omega,t) = e^{-3t}\left[F(\omega)\cos(9\omega t) + \frac{3F(\omega)}{4\omega}\sin(9\omega t)\right].$$

5. Since $b^2 - a = 13 > 0$, look at case 3. Here $F(\omega) = 0$ and we find that

$$G(\omega) = \frac{4}{\omega^2}[\sin(\omega) - \omega\cos(\omega)].$$

Let

$$c_1 = \frac{G(\omega)}{2\sqrt{13-\omega^2}}$$

and $c_2 = -c_1$, and define

$$U_1(\omega,t) = e^{-4t}\left[c_1 e^{\sqrt{13-\omega^2}t} + c_2 e^{-\sqrt{13-\omega^2}t}\right].$$

Let $c_3 = 0$ and

$$c_4 = \frac{G(\omega)}{\sqrt{\omega^2-13}}.$$

Let

$$U_2(\omega,t) = e^{-4t}c_4 \sin\left(\sqrt{\omega^2 - 13}t\right).$$

The solution is

$$u(x,t) = \frac{1}{2\pi}\left[\int_{|\omega|<\sqrt{13}} U_1(\omega,t)e^{i\omega x}\,d\omega + \int_{|\omega|\geq\sqrt{13}} U_2(\omega,t)e^{i\omega x}\,d\omega\right].$$

Section 8.5

1.

$$u(x,y) = Y_5(y)\sin(5x),$$

where

$$Y_5(y) = c_1e^{5y} + c_2e^{-5y} - \frac{1}{25},$$

with

$$c_1 = \frac{\frac{1}{25}(e^{-15} - 1)}{e^{-15} - e^{15}}$$

and

$$c_2 = \frac{\frac{1}{25}(1 - e^{15})}{e^{-15} - e^{15}}.$$

3.

$$u(x,y) = \sum_{n=1}^{\infty} Y_n(y)\sin(nx)$$

with

$$Y_n(y) = \frac{2(-1)^{n+1}}{n^3\sinh(n)}\sinh(ny) + \frac{2(-1)^n y}{n^3}.$$

Index

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