SOLUTIONS

1.1—The flux at the film plane is

$$F_\nu = \int I_\nu \cos\theta \, d\Omega \approx I_\nu \cos\theta \, \Delta\Omega,$$

since $\Delta\Omega \ll 1$. Now we have

$$\Delta\Omega = \frac{\Delta A \cos\theta}{r^2}$$

where $\Delta A = \pi(d/2)^2$ and $r = L/\cos\theta$. Thus

$$F_\nu = \frac{\pi \cos^4\theta}{4f^2} I_\nu,$$

where $f \equiv L/d$.

1.2—We first find the energy absorbed in volume dV and time dt due to radiation in solid angle $d\Omega$ and frequency range $d\nu$. Let dA be the cross-sectional area of the volume normal to the radiation. Then the energy absorbed is $-dI_\nu \, dA \, dt \, d\nu \, d\Omega$, where $dI_\nu = -\alpha_\nu I_\nu \, dl$ by the absorption law (1.20) and where dl is the thickness of the volume along the direction of the radiation. Since $dV = dA \, dl$, the energy absorbed is $\alpha_\nu I_\nu \, dV \, dt \, d\Omega \, d\nu$. Integrating over solid angle gives the energy absorbed per unit volume per unit time per unit frequency range to be $4\pi\alpha_\nu J_\nu$, which is also equal to $c\alpha_\nu u_\nu$ by (1.7). Noting that each photoionization requires an amount $h\nu$ of energy and that $\alpha_\nu = n_a \sigma_\nu$ we obtain the number of photoionizations per unit volume per unit time:

$$4\pi n_a \int_{\nu_0}^{\infty} \frac{\sigma_\nu J_\nu}{h\nu} d\nu = c n_a \int_{\nu_0}^{\infty} \frac{\sigma_\nu u_\nu \, d\nu}{h\nu}.$$

1.3

a. The transfer equation with no absorption is

$$\frac{dI}{ds} = j = \frac{\Gamma}{4\pi}.$$

Here I is defined in terms of photon numbers rather than energy (photons-cm^{-2}-s^{-1}-ster^{-1}). Integrating along a line through the center gives

$$I_0 = j \cdot 2R = \frac{R\Gamma}{2\pi}.$$

b. The observed average intensity $\bar{I}$ is equal to the total flux divided by the solid angle accepted by the detector, $\Delta\Omega_{\text{Det}} = \pi(\Delta\theta_{\text{Det}})^2$, and $\Delta\theta_{\text{Det}}$ is the detector half angle. Since the total luminosity L of the source (photons-s^{-1}) is simply equal to $(4\pi/3)R^3\Gamma$, the flux is

$$F = \frac{L}{4\pi d^2} = \frac{R^3\Gamma}{3d^2},$$

where d is the distance to the source. Thus

$$\bar{I} = \frac{R^3\Gamma}{3\pi d^2(\Delta\theta_{\text{Det}})^2}.$$

Noting that $\Delta\theta_s = R/d$ is the angular size of the source, we can write

$$\frac{\bar{I}}{I_0} = \frac{2}{3}\left(\frac{\Delta\theta_s}{\Delta\theta_{\text{Det}}}\right)^2.$$

For a completely unresolved source $\Delta\theta_s \ll \Delta\theta_{\text{Det}}$, so that $\bar{I} \ll I_0$.

1.4

a. Assume that the luminous object has spherical symmetry, so that the flux F at distance r is just $L/(4\pi r^2)$. From Eq. (1.34) the outward radiation force per unit mass on the cloud is

$$f_{\text{rad}}=\frac{\kappa F}{c}=\frac{\kappa L}{4\pi c r^2}.$$

The inward gravitational force per unit mass due to the object is

$$f_{\text{grav}}=\frac{GM}{r^2}.$$

The condition of ejection is that $f_{\text{grav}}<f_{\text{rad}}$, which can be written

$$\frac{M}{L}<\frac{\kappa}{4\pi Gc}.$$

b. The cloud experiences an inward force per unit mass $G_{\text{eff}}M/r^2$, where the "effective" gravitational constant is given by

$$G_{\text{eff}}=G-\frac{\kappa L}{4\pi Mc}.$$

Thus the effective potential per unit mass is $-G_{\text{eff}}M/r$. Note that G_{eff} is negative under conditions for ejection. Setting up the conservation of energy connecting the state at $r=R$ and $r=\infty$, we obtain

$$-\frac{G_{\text{eff}}M}{R}=\tfrac{1}{2}v^2,$$

which can be written

$$v^2=\frac{2}{R}\left(\frac{\kappa L}{4\pi c}-GM\right).$$

c. The minimum luminosity occurs when the inequality in part (a) becomes an equality. Substitution of $\kappa=\sigma_T/m_H$ then gives the stated result.

1.5

a. The brightness is $I_\nu=F_\nu/\Delta\Omega$, where $\Delta\Omega=\pi(\Delta\theta)^2$. Here $\Delta\theta=\theta/2=2.15$ arc min $=6.25\times10^{-4}$ radian. Thus

$$I_\nu=1.3\times10^{-13}\ \text{erg}\,\text{cm}^{-2}\text{s}^{-1}\text{Hz}^{-1}\text{ster}^{-1}$$

$$T_b=\frac{c^2}{2\nu^2k}I_\nu=4.2\times10^{7}\ K.$$

Since $h\nu \ll kT_b$, the use of the Rayleigh–Jeans approximation is appropriate.

b. $T_b \propto I_\nu \propto (\Delta\theta)^{-2}$. If the true $\Delta\theta$ is smaller, the true T_b will be larger than stated above.

c. From Eq. (1.56b) we find $\nu_{\max} = 2.5 \times 10^{18}$ Hz.

d. The best that can be said is $T \geqslant T_b$. This follows from Eq. (1.62) with $T_b(0) = 0$. In general, the maximum emission of any thermal emitter at given temperature T will occur when the source is optically thick (see Problem 1.8 *d*).

1.6—Since $u(T) = aT^4$, Eqs. (1.40) can be written

$$\left(\frac{\partial S}{\partial T}\right)_V = 4aVT^2, \qquad \left(\frac{\partial S}{\partial V}\right)_T = \frac{4}{3}aT^3.$$

It follows that $S = (4/3)\, aVT^3 + \text{constant}$. The constant must be chosen to be zero, so that $S \to 0$ as $T \to 0$ (third law of thermodynamics).

1.7

a. The equation of statistical equilibrium [Eq. (1.69)] with $B_{21} = 0$ becomes

$$n_1 B_{12} \bar{J} = n_2 A_{21}.$$

With the Boltzmann law (1.70) this implies

$$\bar{J} = \frac{A_{21}}{B_{12}} \frac{g_2}{g_1} e^{-h\nu_0/kT}.$$

This cannot equal the Planck function with A_{21}/B_{12} independent of temperature. However, the choice $(A_{21}/B_{12})(g_2/g_1) = 2h\nu_0^3/c^2$ does yield $\bar{J} = B_{\nu_0}$ in the Wien limit, $h\nu_0 \gg kT$, [cf. Eq. (1.54)].

b. The main difference between the interactions of neutrinos and photons with the atom is that the former particles are fermions, whereas the latter are bosons. Stimulated emission in a fermion field would place two particles in the same state and thus violate the Pauli exclusion principle. This process is replaced by *inhibited emission*, in which an atom in the excited state is prevented from emitting a fermion when one is already present. The analysis using the Einstein coefficient is identical to the photon case, except $B_{21} \to -B_{21}$. One then has, at equilibrium, for atoms of temperature T,

$$n_1 B_{12}\bar{J} = n_2 A_{21} - n_2 B_{21}\bar{J}$$

$$\bar{J} = \frac{A_{21}/B_{21}}{(n_1/n_2)(B_{12}/B_{21})+1}$$

$$\bar{J} = \frac{A_{21}/B_{21}}{(g_1 B_{12}/g_2 B_{21})e^{h\nu/kT}+1} = \frac{2h\nu^3}{c^2(e^{h\nu/kT}+1)},$$

and one obtains the same relationship as before for the Einstein coefficients. These coefficients are properties of the atom alone and clearly must be the same, regardless of the external interactions used to derive them.

1.8

a. Note that $j_\nu = P_\nu/4\pi$ and that, effectively, $\alpha_\nu = 0$, since the cloud is optically thin. Then, using Eq. (1.24),

$$I_\nu(b) = \int j_\nu(z)dz = \frac{P_\nu}{2\pi}\sqrt{R^2-b^2}\ .$$

b. The total power emitted by the cloud is $L = (4/3)\pi R^3 P$, where $P = \int P_\nu \, d\nu$. Then

$$L = 4\pi R^2 \sigma T_{\text{eff}}^4,$$

by definition of T_{eff}, so that

$$T_{\text{eff}} = \left(\frac{PR}{3\sigma}\right)^{1/4}.$$

c. Let d be the distance from the spherical cloud to the earth. Energy conservation gives a relation between F_ν, the flux at the earth, and P_ν:

$$4\pi d^2 F_\nu = \frac{4}{3}\pi R^3 P_\nu,$$

$$F_\nu = P_\nu \frac{R^3}{3d^2}.$$

d. From Eq. (1.30), with $S_\nu = B_\nu(T)$, $I_\nu(0)=0$, $\tau_\nu \ll 1$,

$$I_\nu = B_\nu(T)(1-e^{-\tau_\nu}) \approx \tau_\nu B_\nu(T) \ll B_\nu(T).$$

With the definition of T_b from Eq. (1.59),

$$B_\nu(T_b) \ll B_\nu(T),$$

and the monotoncity of $B_\nu(T)$ with T, we have $T_b \ll T$.

e. For the optically thick case the results are:

a′. From Eq. (1.30) with $\tau_\nu \gg 1$ and with $S_\nu = B_\nu(T)$ we have $I_\nu = B_\nu(T)$ independent of b.

b′. Since $I_\nu = B_\nu$, the flux at the surface is the blackbody flux, so $T_{\text{eff}} = T$.

c′. The monochromatic flux at the surface is $\pi B_\nu(T)$ [cf. Eq. (1.14)], so using the inverse square law gives

$$F_\nu(d) = \pi \left(\frac{R}{d}\right)^2 B_\nu(T).$$

d′. From (a′) and Eq. (1.59) we have $B_\nu(T_b) = B_\nu(T)$, which implies $T_b = T$.

1.9—Ray A starts on the central object with intensity $B_\nu(T_c)$, and this is essentially the observed intensity at $\nu = \nu_1$, where the absorption in the shell is negligible. The observed intensity at $\nu = \nu_0$ depends on whether the source function in the shell, namely, $B_\nu(T_s)$, is greater or smaller than the incident intensity $B_\nu(T_c)$. (See Eq. (1.30) and subsequent discussion.) When $T_s < T_c$ we have $B_\nu(T_s) < B_\nu(T_c)$, and the intensity is reduced by passing through the shell, so that $I^A_{\nu_1}$ is larger than $I^A_{\nu_0}$. When $T_s > T_c$, $I^A_{\nu_0}$ will be larger than $I^A_{\nu_1}$.

Ray B starts with zero intensity, which is the observed intensity at $\nu = \nu_1 : I^B_{\nu_1} = 0$. At $\nu = \nu_0$ the observed intensity will be somewhere between zero and the maximum $B_\nu(T_s)$, depending on the optical depth. In any case, $I^B_{\nu_0} > I^B_{\nu_1}$ always. These cases are illustrated in Fig. S.1.

1.10—The radiative diffusion equation is of the form [cf. Eq. (1.119b)]

$$\frac{1}{3}\frac{\partial^2 J_\nu}{\partial \tau^2} = \epsilon(J_\nu - B_\nu), \tag{1}$$

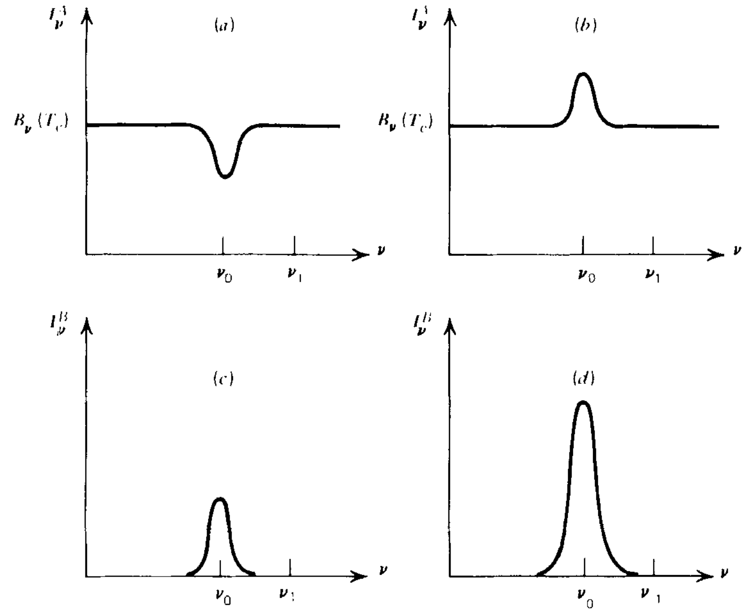

***Figure S.1** Intensity from a blackbody surrounded by a thermal absorbing shell (a) along ray A when $T_s < T_c$, (b) along ray A when $T_s > T_c$, (c) along ray B when $T_s < T_c$, (d) along ray B when $T_s > T_c$.*

where

$$\epsilon \equiv \frac{\alpha_\nu}{\sigma + \alpha_\nu} \tag{2}$$

is the probability per interaction that the photon will be absorbed. The general solution to Eq. (1) is

$$J_\nu - B_\nu = C_1 e^{\tau_*} + C_2 e^{-\tau_*}, \tag{3}$$

where

$$\tau_* \equiv \sqrt{3\epsilon}\ \tau = \sqrt{3\tau_\nu(\tau_\nu + \tau_s)}\ , \tag{4}$$

and C_1 and C_2 are independent of τ and to be determined by boundary conditions. The proper boundary conditions for a semi-infinite half-space are that J_ν remain finite as $\tau \to \infty$ and that there be no incident intensity at $\tau = 0$: $I_\nu(\tau = 0, \mu < 0) = 0$. The first boundary condition requires $C_1 = 0$, that is,

$$J_\nu(\tau) - B_\nu = C_2 e^{-\tau_*}. \tag{5}$$

Note that J_ν approaches B_ν after an effective optical depth τ_* [answer to part (b)].

Now, using the two-stream approximation to the boundary condition at $\tau=0$, Eq. (1.124a)

$$\frac{1}{\sqrt{3}}\frac{\partial J}{\partial \tau} = J \qquad \text{at } \tau = 0,$$

we obtain

$$C_2 = -B_\nu(1+\sqrt{\epsilon}\,)^{-1},$$

$$J_\nu(\tau) = B_\nu\left(1 - \frac{e^{-\tau_*}}{1+\sqrt{\epsilon}}\right). \tag{6}$$

In the Eddington approximation the flux, $F_\nu(\tau)$, satisfies [cf. Eqs. (1.113b) and (1.118)]

$$\begin{aligned} F_\nu(\tau) &= 2\pi\int \mu I_\nu(\mu,\tau)\,d\mu \\ &= 4\pi H = \frac{4\pi}{3}\frac{\partial J}{\partial \tau}. \end{aligned}$$

Thus, we have the result for the emergent flux:

$$F_\nu(0) = \frac{4\pi}{\sqrt{3}}\,\frac{\sqrt{\epsilon}}{1+\sqrt{\epsilon}}\,B_\nu. \tag{7}$$

For small ϵ this differs from Eq. (1.103) only by a factor $1/\sqrt{3}$. (Note $F_\nu(0) = L_\nu/A$). Note, further, that from $J_\nu(\tau)$, one may compute the source function

$$\begin{aligned} S_\nu(\tau) &= (1-\epsilon)J_\nu + \epsilon B_\nu \\ &= B_\nu\left[1-(1-\sqrt{\epsilon}\,)e^{-\tau_*}\right] \end{aligned} \tag{8}$$

and thus the intensity at any optical depth. In the two-stream approximation the intensity in the outward direction ($\mu = 1/\sqrt{3}$) at $\tau=0$ is

$$I_\nu^+(0) = \frac{2B_\nu\sqrt{\epsilon}}{1+\sqrt{\epsilon}} \tag{9}$$

2.1—Writing

$$A(t)=\tfrac{1}{2}(\mathcal{A}e^{-i\omega t}+\mathcal{A}^{*}e^{i\omega t}),$$
$$B(t)=\mathrm{Re}\,\mathcal{B}e^{-i\omega t}=\mathrm{Re}\,\mathcal{B}^{*}e^{i\omega t},$$

and noting the results $\langle 1\rangle=1$ and $\langle e^{\pm 2i\omega t}\rangle=0$, we obtain

$$\langle AB\rangle=\tfrac{1}{2}\mathrm{Re}\mathcal{A}^{*}\mathcal{B}=\tfrac{1}{2}\mathrm{Re}\mathcal{A}\mathcal{B}^{*}.$$

2.2

a. Maxwell's equations (2.6), with $\mathbf{j}=\sigma\mathbf{E}$, are

$$\nabla\cdot\mathbf{E}=4\pi\rho \qquad \nabla\cdot\mathbf{B}=0,$$
$$\nabla\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \qquad \nabla\times\mathbf{B}=\frac{4\pi\sigma\mu}{c}\mathbf{E}+\frac{\epsilon\mu}{c}\frac{\partial\mathbf{E}}{\partial t}.$$

We seek plane-wave solutions, so we assume solutions of the form (2.18a, b) and $\rho=\rho_0\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]$. This gives

$$i\mathbf{k}\cdot\hat{\mathbf{a}}_1E_0=4\pi\rho_0 \qquad i\mathbf{k}\cdot\hat{\mathbf{a}}_2B_0=0$$
$$i\mathbf{k}\times\hat{\mathbf{a}}_1E_0=\frac{i\omega}{c}\hat{\mathbf{a}}_2B_0 \qquad i\mathbf{k}\times\hat{\mathbf{a}}_2=\frac{-i\omega m^2}{c}\hat{\mathbf{a}}_1E_0,$$

where $m^2=\mu\epsilon(1+4\pi\sigma i/\omega\epsilon)$. Dotting the vector $\mathbf{k}$ into the last equation, we find $\mathbf{k}\cdot\hat{\mathbf{a}}_1\equiv0$, which implies $\rho_0=0$ from the first equation. Thus these equations have the same form as Eqs. (2.19), except for the additional m^2 factor. The solution proceeds analogously, leading to the dispersion relation

$$k^2=\frac{\omega^2m^2}{c^2}.$$

b. Take $\mathbf{k}$ along the z-axis, with $\mathbf{E}$ and $\mathbf{B}$ along the x- and y-axes, respectively. Then

$$\mathbf{E}=\hat{\mathbf{x}}E_0e^{-\mathrm{Im}(m)z\omega/c}e^{i[\mathrm{Re}(m)z\omega/c-\omega t]},$$
$$\mathbf{B}=\hat{\mathbf{y}}B_0e^{-\mathrm{Im}(m)z\omega/c}e^{i[\mathrm{Re}(m)z\omega/c-\omega t]},$$

and the Poynting vector is

$$\mathbf{S}=\hat{\mathbf{z}}\frac{c|E_0|^2}{8\pi}e^{-2\mathrm{Im}(m)z\omega/c}=\mathbf{S}(z=0)e^{-\alpha_\nu z},$$

where the absorption coefficient α_ν is given by

$$\alpha_\nu = \frac{2\omega}{c}\,\mathrm{Im}(m).$$

2.3

a. Substituting $\mathbf{F}_{\mathrm{Lorentz}} = Q\,\mathbf{E}$ and $\mathbf{F}_{\mathrm{visc}} = -\beta\mathbf{v}$ into the force equation gives $\mathbf{v} = Q\mathbf{E}/\beta$. The direction of the velocity rotates uniformly in a plane normal to the propagation direction with period $2\pi/\omega$. Thus the radius is found from

$$2\pi r = \oint v\,dt$$

to be $r = QE/\beta\omega$.

b. The power dissipated is $P = -\mathbf{v}\cdot\mathbf{F}_{\mathrm{visc}} = \beta v^2 = Q^2E^2/\beta$. Since the orbit of the charge is constant in time, this is the power transmitted to the fluid.

c. The magnetic force is in the direction of propagation and has magnitude $F_{\mathrm{mag}} = QBv/c = QEv/c = Q^2E^2/\beta c$. Here we have used $|\mathbf{E}| = |\mathbf{B}|$ for a free wave.

d. Using the center of the charge's motion as an origin we find the magnitude of the torque to be $\tau = |\mathbf{F}_{\mathrm{Lorentz}} \times \mathbf{r}| = Q^2E^2/\beta\omega$. For a left-hand circularly polarized wave the E-vector, and thus the charge, rotates counterclockwise as viewed facing the wave. This imparts a torque *along* the direction of propagation. The opposite holds for right-hand polarization. Thus $\tau = \pm Q^2E^2/\beta\omega$.

e. The absorption cross section can be found from $P = \sigma S$ where the Poynting flux is $S = cE^2/4\pi$. Thus $\sigma = 4\pi Q^2/\beta c$.

f. P, F_{mag}, and τ are the rates of energy, momentum, and angular momentum, respectively, given to the fluid. From the results above we have the ratios $F_{\mathrm{mag}}/P = (\text{momentum})/(\text{energy}) = 1/c$ and $\tau/P = (\text{angular momentum})/(\text{energy}) = \pm 1/\omega$. Assuming the quantum relation $E = \hbar\omega$ for a single photon then implies the relations $p = E/c = h/\lambda = \hbar k$ and $J = \pm E/\omega = \pm\hbar$. Since these refer to properties of photons, they are applicable in general, not just to the limited problem considered here.

g. The case of linear polarization leads to the (primed) results:

a′. Again $\mathbf{v}=Q\mathbf{E}/\beta$. Since $\mathbf{E}$ oscillates harmonically along one axis ($E_x=E_0\cos\omega t$), so does the particle. Taking an appropriate origin, we find for the displacement: $x(t)=(QE_0/\omega\beta)\sin\omega t$. The maximum displacement from the origin is $QE_0/\omega\beta$.

b′. The average power dissipated is $\langle P\rangle=-\langle\mathbf{v}\cdot\mathbf{F}_{\mathrm{visc}}\rangle=\beta\langle v^2\rangle=(Q^2E_0^2/\beta)\langle\cos^2\omega t\rangle=Q^2E_0^2/2\beta$.

c′. The average magnetic force is along the direction of propagation and has the magnitude $\langle F_{\mathrm{mag}}\rangle=\langle QvB/c\rangle=(Q/c)\langle\mathbf{v}\cdot\mathbf{E}\rangle=(Q^2E_0^2/\beta c)\langle\cos^2\omega t\rangle=Q^2E_0^2/2\beta c$.

d′. There is no torque on the fluid, since $\mathbf{F}_{\mathrm{visc}}$ always acts along a line through the origin.

e′. The absorption cross section can be found from the relation $\langle P\rangle=\sigma\langle S\rangle$. The average Poynting vector is $cE_0^2/8\pi$, by Eq. (2.24b). Thus $\sigma=4\pi Q^2/\beta c$, as before.

f′. The power and magnetic force are the same as before, with $E^2\to E_0^2/2$. Their ratio is the same, and we conclude that $p=E/c$. The angular momentum is zero, however. Quantum mechanically this comes about because a linearly polarized photon is a superposition of two circularly polarized photons of opposite helicity.

2.4—Suppose that $\nabla\times\mathbf{H}=4\pi c^{-1}\mathbf{j}$. Taking the divergence of both sides yields $\nabla\cdot\mathbf{j}=0$. But the equation of charge conservation is $\nabla\cdot\mathbf{j}=-\partial\rho/\partial t$. Therefore, this form of the field equation applies only to the special case $\dot\rho=0$.

Furthermore, omitting the displacement current form the derivation leading to (2.17) gives:

$$\nabla^2\mathbf{E}=\frac{4\pi\sigma\mu}{c^2}\frac{\partial\mathbf{E}}{\partial t}$$

$$\nabla^2\mathbf{B}=\frac{4\pi\sigma\mu}{c^2}\frac{\partial\mathbf{B}}{\partial t}.$$

That is, the equations for $\mathbf{E}$ and $\mathbf{B}$ become (parabolic) diffusion equations rather than (hyperbolic) wave equations.

3.1

a. By analogy with the Larmor formula for electric dipole radiation, the power radiated by a magnetic dipole is

$$P=\frac{2|\ddot{\mathbf{m}}|^2}{2c^3}. \tag{1}$$

To evaluate this expression, we first note that the component of $\mathbf{m}$ along the rotation axis, $|\mathbf{m}|\cos\alpha$, is a *constant*. Thus

$$|\ddot{\mathbf{m}}| = \omega^2 |\mathbf{m}| \sin\alpha, \tag{2}$$

since $m_x = m \sin\alpha \sin\omega t$ and $m_y = m \sin\alpha \cos\omega t$. Next, we wish to express $|\mathbf{m}|$ in terms of B_0, the magnetic field strength at the star surface. In electrostatics, the electric field due to a dipole of strength $d \equiv el$ is obtained in the following way:

$$\mathbf{E} = \nabla\phi,$$

$$\phi = \frac{e}{|\mathbf{r}|} - \frac{e}{|\mathbf{r}+\mathbf{l}|} \approx \frac{d\cos\theta}{r^2},$$

$$\mathbf{E} = -\frac{2d\cos\theta}{r^3}\hat{\mathbf{r}} - \frac{d\sin\theta}{r^3}\hat{\boldsymbol{\theta}}.$$

At the "pole," $\theta = 0$, the electric field has a magnitude $E = 2d/r^3$. By analogy, the magnetic field B_0 at the magnetic pole has the value

$$B_0 = \frac{2m}{R^3}. \tag{3}$$

Substituting Eqs. (2) and (3) into (1), we obtain

$$P = \frac{\omega^4 R^6 B_0^2 \sin^2\alpha}{6c^3}. \tag{4}$$

b. Assuming the neutron star is a homogeneous body, its rotational energy , E_{rot}, satisfies

$$E_{\mathrm{rot}} = \tfrac{1}{5} M R^2 \omega^2. \tag{5}$$

Now, using $P = -\dot{E}_{\mathrm{rot}} = -2/5 M R^2 \omega\dot{\omega}$ and substituting from Eq. (4), we obtain

$$\tau \equiv \frac{-\omega}{\dot{\omega}} = \frac{12 M c^3}{5 R^4 \omega^2 B_0^2 \sin^2\alpha}. \tag{6}$$

c. To obtain quick quantitative estimates of functions as their arguments assume particular values, astrophysicists frequently write equations with all the quantities normalized to some standard values. Thus, for this problem, we can express Eqs. (4) and (6) in the form

$$P = 3.1\times10^{43}\text{erg s}^{-1}\left(\frac{\omega}{10^4\text{s}^{-1}}\right)^4\left(\frac{R}{10^6\text{ cm}}\right)^6\left(\frac{B_0}{10^{12}\text{ gauss}}\right)^2\sin^2\alpha$$

$$\tau = 42\text{yr}\left(\frac{M}{M_\odot}\right)\left(\frac{R}{10^6\text{ cm}}\right)^{-4}\left(\frac{\omega}{10^4\text{s}^{-1}}\right)^{-2}\left(\frac{B_0}{10^{12}\text{ gauss}}\right)^{-2}\sin^{-2}\alpha.$$

Thus for $\omega = 10^4\text{ s}^{-1}$, 10^3 s^{-1}, 10^2 s^{-1}, $P/3.1\times10^{43}\text{ erg s}^{-1}$ has values 1, 10^{-4}, 10^{-8}, and $\tau/42$ yr has values 1, 10^2, and 10^4.

3.2—Since $v_\perp \ll c$ we can compute the radiation field $\mathbf{E}_{\text{rad}}$ from Eq. (3.15a):

$$\mathbf{E}_{\text{rad}} = \frac{e}{c^2 r}\mathbf{n}\times(\mathbf{n}\times\dot{\mathbf{u}}).$$

Also, since the system is axially symmetric, and we will eventually average the motion over time, no generality is lost by taking $\mathbf{n}$ in the y-z plane (see Fig. S.2). The plane normal to $\mathbf{n}$ contains the unit vectors $\hat{\mathbf{a}}_1 = -\hat{\mathbf{x}}$ and $\hat{\mathbf{a}}_2$,

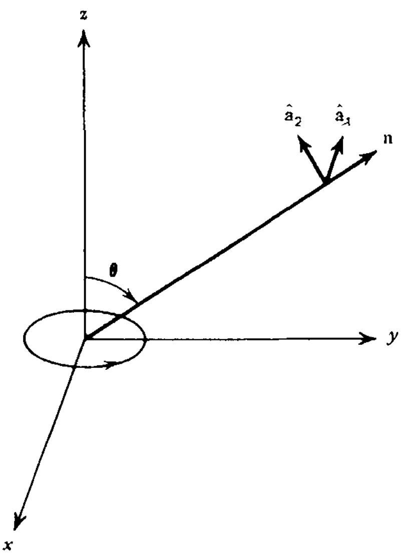

***Figure S.2** Geometry for polarization decomposition of radiation emitted by a circulating charge.*

which lies in the y-z plane. Letting $\omega = v_\perp / a$, we have for the particle

$$\mathbf{r} = a(\hat{\mathbf{x}}\cos\omega t + \hat{\mathbf{y}}\sin\omega t)$$
$$\mathbf{u} = v_\perp(-\hat{\mathbf{x}}\sin\omega t + \hat{\mathbf{y}}\cos\omega t)$$
$$\dot{\mathbf{u}} = -v_\perp\omega(\hat{\mathbf{x}}\cos\omega t + \hat{\mathbf{y}}\sin\omega t).$$

The components of $\mathbf{n}$ and $\hat{\mathbf{a}}_2$ are given by

$$\mathbf{n} = \hat{\mathbf{y}}\sin\theta + \hat{\mathbf{z}}\cos\theta,$$
$$\hat{\mathbf{a}}_2 = -\mathbf{y}\cos\theta + \hat{\mathbf{z}}\sin\theta.$$

We now have

$$\mathbf{n}\times\dot{\mathbf{u}} = -v_\perp\omega(\hat{\mathbf{a}}_1\cos\theta\sin\omega t - \hat{\mathbf{a}}_2\cos\omega t),$$
$$\mathbf{n}\times(\mathbf{n}\times\dot{\mathbf{u}}) = -v_\perp\omega(\hat{\mathbf{a}}_1\cos\omega t + \hat{\mathbf{a}}_2\cos\theta\sin\omega t),$$
$$\mathbf{E}_{\text{rad}} = -\frac{ev_\perp\omega}{rc^2}(\hat{\mathbf{a}}_1\cos\omega t + \hat{\mathbf{a}}_2\cos\theta\sin\omega t).$$

a. The power per solid angle is found from

$$\frac{dP}{d\Omega} = \frac{c}{4\pi}|\mathbf{E}_{\text{rad}}|^2 r^2 = \frac{e^2 v_\perp^2\omega^2}{4\pi c^3}(\cos^2\omega t + \cos^2\theta\sin^2\omega t).$$

Averaging this over time gives

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{e^2 v_\perp^2\omega^2}{8\pi c^3}(1+\cos^2\theta).$$

b. Comparing the formula for $\mathbf{E}_{\text{rad}}$ with Eq. (2.37) (taking the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ directions of that equation to be now $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$, respectively), we find

$$\mathcal{E}_1 = -\frac{ev_\perp\omega}{rc^2}, \qquad \mathcal{E}_2 = -\frac{ev_\perp\omega}{rc^2}\cos\theta$$
$$\phi_1 = 0, \qquad \phi_2 = \pi/2.$$

The Stokes parameters are, therefore, from Eqs. (2.40),

$$I = A(1+\cos^2\theta)$$
$$Q = A(1-\cos^2\theta)$$
$$U = 0$$
$$V = -2A\cos\theta,$$

where $A \equiv (ev_\perp \omega / rc^2)^2$. The radiation is 100% elliptically polarized ($I^2 = U^2 + Q^2 + V^2$), with $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$ being principal axes. Special cases are

$\theta = 0$	:	left-hand circular polarization.
$\theta = \pi/2$	:	linear polarization along $\hat{\mathbf{a}}_1$
$\theta = \pi$	:	right-hand circular polarization

c. Since $\mathbf{E}_{\text{rad}}$ contains only $\sin\omega t$ and $\cos\omega t$ terms, the radiation is monochromatic at frequency ω. (See, however, Problem 3.7 when radiation of higher order than dipole is included.)

d. Setting the magnetic force equal to the centripetal force gives $\omega: m\omega^2 r = e\omega rB/c$,

$$\omega = \frac{eB}{mc}.$$

Using the result of part (a) gives $\langle P \rangle$:

$$\begin{aligned}
\langle P \rangle &= \int \left\langle \frac{dP}{d\Omega} \right\rangle d\Omega \\
&= \frac{e^2 v_\perp^2 \omega^2}{8\pi c^3} \int_0^{2\pi} d\phi \int_0^{\pi} (1 + \cos^2\theta) \sin\theta \, d\theta \\
&= \frac{2}{3} \frac{e^2 v_\perp^2 \omega^2}{c^3}.
\end{aligned}$$

Using $r_0 = e^2/mc^2$ and $\beta_\perp = v_\perp / c$, this becomes

$$\langle P \rangle = \tfrac{2}{3} r_0^2 c \beta_\perp^2 B^2.$$

e. The Lorentz force law for an incident electric field E gives $ma\omega^2 = eE$ or $v_\perp = eE/m\omega$. Thus

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{r_0^2 c E^2}{4\pi} (1 + \cos^2\theta).$$

Now use the results

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \langle S \rangle \frac{d\sigma}{d\Omega}$$

$$\langle S \rangle = \frac{cE^2}{4\pi}$$

to obtain

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{pol}} = \tfrac{1}{2}r_0^2(1+\cos^2\theta).$$

To obtain the cross section for unpolarized radiation we should average this cross section with one for circular polarization of the opposite helicity. But since these cross sections do not depend on helicity the unpolarized results are the same:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} = \tfrac{1}{2}r_0^2(1+\cos^2\theta).$$

This is the same result obtained previously in Eq. (3.40). The total cross section is just equal to the Thomson cross section, independent of the polarization:

$$\sigma = \int \frac{d\sigma}{d\Omega}\,d\Omega = \frac{8\pi r_0^2}{3}.$$

3.3

a. Use Eq. (3.15a) with $q\dot{u} = -\omega^2 d\cos\omega t$ for each dipole, noting that the retarded times for each differ by $\Delta t = (L/c)\sin\theta$ (see Fig. S.3). Then

$$|\mathbf{E}_{\text{rad}}| = -\frac{\omega^2}{rc^2}\left[d_1\cos\omega t + d_2\cos\omega(t-\Delta t)\right]\sin\theta$$

$$= -\frac{\omega^2}{rc^2}\left[(d_1 + d_2\cos\delta)\cos\omega t + d_2\sin\delta\sin\omega t\right]\sin\theta,$$

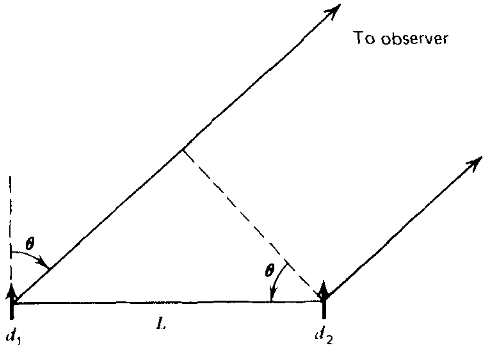

Figure S.3 Geometry for emission from two dipole radiators separated by distance L.

where $\delta = \omega\Delta t = \omega L \sin\theta / c$. Squaring and averaging over time, we find

$$\langle |E_{\rm rad}|^2 \rangle = \frac{\omega^4 \sin^2\theta}{2r^2c^4}\left[(d_1 + d_2\cos\delta)^2 + (d_2 \sin\delta)^2\right]$$

$$= \frac{\omega^4 \sin^2\theta}{2r^2c^4}\left(d_1^2 + 2d_1d_2\cos\delta + d_2^2\right).$$

We have finally,

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{cr^2}{4\pi}\langle |E_{\rm rad}|^2 \rangle$$

$$= \frac{\omega^4 \sin^2\theta}{8\pi c^3}\left(d_1^2 + 2d_1d_2\cos\delta + d_2^2\right).$$

b. When $L \ll \lambda$, we have $\delta \equiv 2\pi L \sin\theta/\lambda \ll 1$, and

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\omega^4 \sin^2\theta}{8\pi c^3}(d_1 + d_2)^2,$$

which is the radiation from an oscillating charge with dipole moment $d_1 + d_2$.

3.4

a. If the cloud is unresolved, then by symmetry there can be no net polarization. Physically, the polarization from different regions of the cloud cancel.

b. Figure S.4*a* shows a typical scattering event in the scattering plane. Radiation from the object can be decomposed into two linearly polarized beams of equal magnitude, one in the plane of scattering and one normal to it. The first produces scattered radiation with polarization direction in the plane of scattering, the second having direction normal to it. These are not of equal magnitude, being in the ratio $\cos^2\theta : 1$ respectively [cf. discussion leading to Eq. (3.41)]. The normal component thus dominates for each value of θ. Integration along the line of sight then gives an observed intensity with dominant component normal to the scattering plane, or, to the observer viewing the plane of the sky, normal to the radial line connecting the object and the point of observation. The plane of the sky with its observed polarization directions is illustrated in Fig. S.4*b*.

c. That the central object can be clearly seen implies that $\tau_e = n_e\sigma_T R \lesssim 1$, and thus $n_e \lesssim (R\sigma_T)^{-1} = 5 \times 10^5\ \mathrm{cm}^{-3}$.

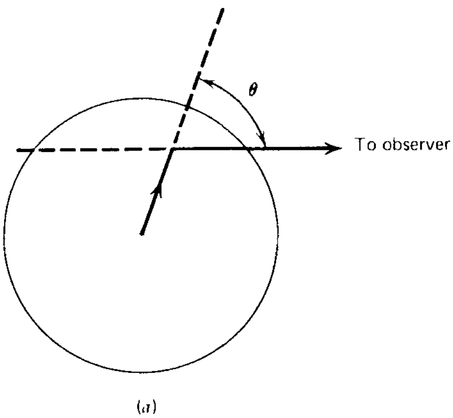

Figure S.4a Scattering event from a spherical cloud.

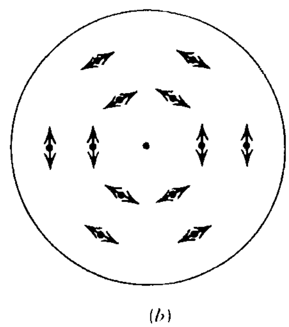

Figure S.4b Observed polarization directions in the plane of the sky.

3.5—Since the field inside the sphere is independent of position, it acts like a single dipole with moment:

$$\mathbf{d}=\frac{4}{3}\pi a^3\alpha E_0\left(1+\frac{4\pi\alpha}{3}\right)^{-1}\cos\omega t\,\hat{\mathbf{x}},$$

where the incident field is $E_0\cos\omega t\,\hat{\mathbf{x}}$. From Eq. (3.23b) with $k=\omega/c$ we obtain the time averaged power:

$$\langle P\rangle=(4\pi\alpha/3)^2(1+4\pi\alpha/3)^{-2}k^4a^6cE_0^2/3.$$

From $\sigma=\langle P\rangle/\langle S\rangle$ with $\langle S\rangle=cE_0^2/8\pi$, we get the required result.

3.6

a. The Fourier transform can be performed explicitly by changing variables in each term of the sum over i:

$$\begin{aligned}\hat{E}(\omega) &= \sum_{i=1}^{N} \frac{1}{2\pi}\int_{-\infty}^{\infty} E_0(u)e^{i\omega u}e^{i\omega t_i}\,du \\ &= \frac{1}{2\pi}\int E_0(u)e^{i\omega u}du \sum e^{i\omega t_i} \\ &= \hat{E}_0(\omega)\sum e^{i\omega t_i}.\end{aligned}$$

b. Explicitly, we have

$$\begin{aligned}\left|\sum_{i=1}^{N} e^{i\omega t_i}\right|^2 &= \sum_{i=1}^{N}\sum_{j=1}^{N} e^{i\omega t_j}e^{-i\omega t_i} \\ &= \sum_{j=1}^{N} e^{i\omega t_j}e^{-i\omega t_j} + \sum_{j\neq i}\sum e^{i\omega(t_j-t_i)} \\ &= N + \sum_{j\neq i}\sum e^{i\omega(t_j-t_i)}.\end{aligned}$$

Now, since t_i and t_j are randomly distributed, the second term averages to zero.

c. Equation (2.33) gives the spectrum:

$$\frac{dW}{dA\,d\omega} = c|\hat{E}(\omega)|^2 = N\left(\frac{dW}{dA\,d\omega}\right)_{\text{single pulse}}.$$

d. In this case we may take each $t_i \approx 0$, because all the pulses have the same arrival time, to order (size of region)/(wavelength). Thus

$$\hat{E}(\omega) \sim N\hat{E}_0(\omega)$$

and

$$\frac{dW}{dA\,d\omega} = N^2\left(\frac{dW}{dA\,d\omega}\right)_{\text{single pulse}}.$$

3.7—Let the charge move in the x-y plane, and its position be denoted by $\mathbf{r}_0(t)$. Then

$$\mathbf{r}_0(t) = r_0(\cos\omega_0 t\,\hat{\mathbf{x}} + \sin\omega_0 t\,\hat{\mathbf{y}})$$

$$\dot{\mathbf{r}}_0(t) = \omega_0 r_0(-\sin\omega_0 t\,\hat{\mathbf{x}} + \cos\omega_0 t\,\hat{\mathbf{y}}).$$

The current is

$$\mathbf{j}(\mathbf{r},t) = e\dot{\mathbf{r}}_0(t)\delta(\mathbf{r}-\mathbf{r}_0(t)). \tag{1}$$

Since $\mathbf{j}(\mathbf{r},t)$ is clearly periodic, we write it as a Fourier series:

$$\mathbf{j}(\mathbf{r},t) = \tfrac{1}{2}\mathbf{j}_0^1(\mathbf{r}) + \sum_{n=1}^{\infty}\left[\mathbf{j}_n^1(\mathbf{r})\cos n\omega_0 t + \mathbf{j}_n^2(\mathbf{r})\sin n\omega_0 t\right] \tag{2}$$

with

$$\mathbf{j}_n^1(\mathbf{r}) = \frac{1}{\pi}\int_0^{2\pi}\mathbf{j}(\mathbf{r},t)\cos n\omega_0 t\, d(\omega_0 t) \tag{3a}$$

$$\mathbf{j}_n^2(\mathbf{r}) = \frac{1}{\pi}\int_0^{2\pi}\mathbf{j}(\mathbf{r},t)\sin n\omega_0 t\, d(\omega_0 t). \tag{3b}$$

From Eq. (3.31), we have the Fourier terms for the l-pole contribution to $\mathbf{A}_n^1, \mathbf{A}_n^2$:

$$\left[\mathbf{A}_n^i(\mathbf{r})\right]_l = C(k,r)\int \mathbf{j}_n^i(\mathbf{r}')(\mathbf{n}\cdot\mathbf{r}')^{(l-1)}\, d^3\mathbf{r}', \tag{4}$$

where $i = 1,2$ for the coefficient of the cosine or sine term, respectively, in the series. Now, substituting Eq. (3) into Eq. (4) and performing the $d^3\mathbf{r}'$ integral first, we have for the dipole contribution ($l = 1$):

$$\left[\mathbf{A}_n^1(\mathbf{r})\right]_1 \propto \int_0^{2\pi}(-\sin\omega_0 t\,\hat{\mathbf{x}} + \cos\omega_0 t\,\hat{\mathbf{y}})\cos n\omega_0 t\, d(\omega_0 t) \propto \delta_{n,1}\hat{\mathbf{y}}, \tag{5a}$$

$$\left[A_n^2(\mathbf{r})\right]_1 \propto \delta_{n,1}\hat{\mathbf{x}}, \tag{5b}$$

where we have used the orthogonality property of sines and cosines and δ is the Kronecker delta. Thus the dipole contribution to the vector potential is nonzero only at $n = 1$ ($\omega = \omega_0$) and the cosine and sine coefficients are vectors along the $\hat{\mathbf{y}}$ and $\hat{\mathbf{x}}$ directions, respectively.

For the quadrupole contribution we obtain, using Eqs. (1), (3), and (4) and performing the $\mathbf{r}'$ integration,

$$[\mathbf{A}_n^1(\mathbf{r})]_2 \propto \int_0^{2\pi} (-\sin\omega_0 t\,\hat{\mathbf{x}} + \cos\omega_0 t\,\hat{\mathbf{y}}) \cdot (n_x \cos\omega_0 t + n_y \sin\omega_0 t)\cos n\omega_0 t\, d(\omega_0 t), \tag{6}$$

where n_x and n_y are the x and y components of the unit vector $\mathbf{n}$. Now, using standard trigonometric identities, we can write this as

$$\begin{aligned}[\mathbf{A}_n^1(\mathbf{r})]_2 &\propto \frac{\hat{\mathbf{x}}}{2}\int_0^{2\pi}[-n_x\sin 2\omega_0 t - n_y(1-\cos 2\omega_0 t)]\cdot\cos n\omega_0 t\, d(\omega_0 t)\\ &\quad + \frac{\hat{\mathbf{y}}}{2}\int_0^{2\pi}[n_x(1+\cos 2\omega_0 t) + n_y \sin 2\omega_0 t]\cdot\cos n\omega_0 t\, d(\omega_0 t)\\ &\propto \frac{\hat{\mathbf{x}}}{2} n_y \delta_{n,2} + \frac{\hat{\mathbf{y}}}{2} n_x \delta_{n,2}.\end{aligned} \tag{7a}$$

Analogously, we have the result,

$$[\mathbf{A}_n^2(\mathbf{r})]_2 \propto \frac{\hat{\mathbf{x}}}{2} n_x \delta_{n,2} + \frac{\hat{\mathbf{y}}}{2} n_y \delta_{n,2}. \tag{7b}$$

Thus the quadrupole contribution is nonzero only at $n=2$ ($\omega = 2\omega_0$). It should be clear that, in the general case, the l-pole contribution is solely at the harmonic $\omega = l\omega_0$.

4.1—The key idea in this problem is that, because of relativistic beaming of radiation, portions of the surface that are at sufficiently large angles from the observer's line of sight never communicate with him. Because the sphere is optically thick, only surface elements can be observed. Referring to Fig. S.5 we see that the cones of emission at any surface point (half-angle of order γ^{-1}) include the observer's direction only for a limited region of the sphere, between points B and B'. Emission from points such as C will not reach the observer.

The observed duration of any pulse has as a lower bound the time delay between the observed radiation from A and B:

$$\Delta t \gtrsim \frac{R}{c}(1-\cos\theta_c) \approx \frac{R\theta_c^2}{2c}.$$

But $\theta_c = \gamma^{-1}$ from the geometry, so $R \lesssim 2\gamma^2 c\,\Delta t$.

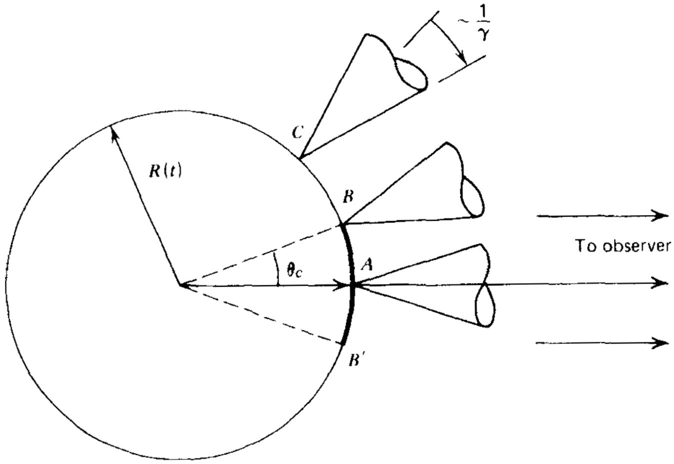

Figure S.5 Geometry of emission cones from points on the surface of a rapidly expanding shell.

4.2—Suppose that K and K' count the same set of M stars. Since M is a scalar, they must agree on the count:

$$P(\theta,\phi)d\Omega = P(\theta',\phi')d\Omega',$$
$$P(\theta',\phi') = P(\theta,\phi)\frac{d\Omega}{d\Omega'}.$$

Now, if we take θ as the angle of the incident light ray with the velocity axis (i.e., $\theta=\pi$ corresponds to the forward direction), we may use Eq. (4.95) for $d\Omega/d\Omega'$ to obtain

$$P(\theta',\phi') = \frac{P(\theta,\phi)(1-\beta^2)}{(1+\beta\cos\theta')^2} = \frac{N}{4\pi}\frac{(1-\beta^2)}{(1+\beta\cos\theta')^2}.$$

Note that $P(\theta',\phi') = P(\theta,\phi)$ if $\beta=0$, and that

$$\int P(\theta',\phi')d\Omega' = \frac{N}{4\pi}\cdot 2\pi\int_0^\pi \frac{(1-\beta^2)}{(1+\beta\cos\theta')^2}\sin\theta' d\theta'$$
$$= N.$$

Finally, since $P(\theta',\phi')$ has a maximum at $\theta'=\pi$, the stars bunch up in the forward direction.

4.3

a. Use Eqs. (4.2) and (4.5) to compute changes in times and velocities measured in different frames:

$$dt=\gamma\left(dt'+\frac{v}{c^2}dx'\right)=\gamma\sigma\,dt',$$

$$du_x=\gamma^{-2}\sigma^{-2}\,du_x',$$

$$du_y=\gamma^{-1}\sigma^{-2}\left(\sigma\,du_y'-\frac{vu_y'}{c^2}\,du_x'\right).$$

Hence

$$a_x=\frac{du_x}{dt}=\gamma^{-3}\sigma^{-3}\frac{du_x'}{dt'}=\gamma^{-3}\sigma^{-3}a_x',$$

$$a_y=\frac{du_y}{dt}=\gamma^{-2}\sigma^{-3}\left(\sigma\frac{du_y'}{dt'}-\frac{vu_y'}{c^2}\frac{du_x'}{dt'}\right),$$

$$=\gamma^{-2}\sigma^{-3}\left(\sigma a_y'-\frac{vu_y'}{c^2}a_x'\right)$$

A similar result holds for a_z.

b. If the particle is at rest instantaneously in K', then $u_x'=u_y'=u_z'=0$. Then $\sigma=1$, and from part (a),

$$a_\parallel'=\gamma^3 a_\parallel,$$

$$a_\perp'=\gamma^2 a_\perp.$$

4.4

a. An inertial frame instantaneously at rest with respect to the rocket measures its acceleration as g. Transforming from this frame (the "primed frame") to the earth frame with problem 4.3 gives

$$a=\frac{d^2x}{dt^2}=\gamma^{-3}a'=\gamma^{-3}g.$$

Note that the choice of which frame is "primed" is not arbitrary, because in only one is the rocket instantaneously at rest.

b. Since $\gamma=(1-\beta^2)^{-1/2}$, the variables are immediately separable:

$$\int \frac{d\beta}{(1-\beta^2)^{3/2}}=\frac{g}{c}\int dt.$$

Let $\beta=\sin u$. Then the integral becomes

$$\int \frac{du}{\cos^2 u}=\tan u=\frac{\beta}{\sqrt{1-\beta^2}},$$

so that

$$\frac{\beta}{\sqrt{1-\beta^2}}=\frac{gt}{c}+\text{constant}.$$

Since $\beta=0$ at $t=0$, the constant vanishes. Inverting the last expression gives

$$\beta(t)=\frac{gt/c}{\sqrt{(gt/c)^2+1}}.$$

c. Set $\beta=c^{-1}dx/dt$ and separate variables again:

$$\int dx=\int \frac{gt\,dt}{\sqrt{(gt/c)^2+1}}.$$

Substituting $u=(gt/c)^2+1$ to transform the integral,

$$x=\frac{c^2}{2g}\int \frac{du}{u^{1/2}},$$

$$x=\frac{c^2}{g}\sqrt{\left(\frac{gt}{c}\right)^2+1}\;+\text{constant}.$$

Since $x=0$ at $t=0$, we obtain

$$x=\frac{c^2}{g}\left[\sqrt{\left(\frac{gt}{c}\right)^2+1}\;-1\right].$$

d. To find the proper time of the rocket we set

$$dt=\gamma(t)d\tau,$$

$$\gamma(t)=(1-\beta^2)^{-1/2}=\sqrt{(gt/c)^2+1}\ ,$$

$$\int\frac{du}{\sqrt{u^2+1}}=\frac{g\tau}{c}+\text{constant},$$

$$\sinh^{-1}\left(\frac{gt}{c}\right)=\frac{g\tau}{c}+\text{constant}.$$

Since $t=0$ when $\tau=0$,

$$\frac{gt}{c}=\sinh\left(\frac{g\tau}{c}\right).$$

e. By symmetry, the journey consists of four segments of equal distance and time. The maximum distance away from earth is twice the result of part (a), which with part (d) can be written

$$d=\frac{2c^2}{g}\left[\cosh\left(\frac{g\tau}{c}\right)-1\right].$$

With $\tau=10$ yr, $g=980$ cm s^{-2}, this yields $d=2.8\times10^{22}$ cm. (One could visit the center of our galaxy).

f. Four times the time given in part (d) is

$$T=\frac{4c}{g}\sinh\left(\frac{g\tau}{c}\right)=5800\text{ yr}$$

Unless their friends have also been exploring, the answer is "no."

g. Changing g, to $2g$ gives $T=8.8\times10^8$ yr and $d=4.2\times10^{26}$ cm. (Were it not for energetic and shielding considerations, round-trip intergalactic travel within one's lifetime would be possible.)

4.5—We demonstrate a simple counterexample: $A^\alpha=B^\alpha=(1,0,0,0)$. Using the boost (4.20) in the transformation law (4.30), we obtain $A'^\alpha=B'^\alpha=\gamma(1,-\beta,0,0)$. Now $A^\alpha B^\alpha=1$, which does not equal $A'^\alpha B'^\alpha=(1+\beta^2)/(1-\beta^2)$, unless $\beta=0$.

4.6—Make an arbitrary boost in a direction that lies in the y-z plane. The photon now may have nonvanishing p^y and p^z as well as p^x. A pure rotation lines up the coordinate frame again so that only p^x is nonvanishing, but p^x does not now have its original magnitude. So, make a final boost along p^x either to redshift it or to blueshift it to the original value. Since $E^2 - p^2 = 0$, E also has its original value. You can easily convince yourself that the product of these transformations is not a pure rotation; there is in general a net boost left over. An example is:

$$\underbrace{\begin{bmatrix} \gamma' & \gamma' v' & 0 & 0 \\ \gamma' v' & \gamma' & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{x\text{-boost}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (1-v^2)^{1/2} & v & 0 \\ 0 & -v & (1-v^2)^{1/2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{x\text{-}y \text{ rotation}}$$

$$\times \underbrace{\begin{bmatrix} \gamma & 0 & \gamma v & 0 \\ 0 & 1 & 0 & 0 \\ \gamma v & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{y\text{-boost}} \begin{bmatrix} E \\ E \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} E \\ E \\ 0 \\ 0 \end{bmatrix}$$

where v' is chosen to satisfy the equality $\gamma\gamma'(1+v') = 1$, which gives $v' = v^2/(v^2 - 2)$.

4.7

a. Suppose that the blob moves from points 1 to 2 in a time Δt (see Fig. S.6). Because 2 is closer to the observer than 1, the apparent time difference between light received by him $(\Delta t)_{\text{app}}$ is

$$(\Delta t)_{\text{app}} = \Delta t\left(1 - \frac{v}{c}\cos\theta\right),$$

(c.f. discussion of Doppler effect, §4.4). The apparent velocity on the sky is

$$v_{\text{app}} = \frac{v\Delta t \sin\theta}{(\Delta t)_{\text{app}}} = \frac{v \sin\theta}{1 - \dfrac{v}{c}\cos\theta}.$$

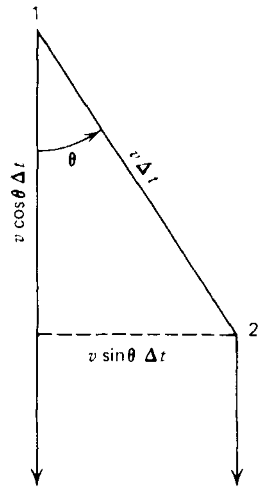

Figure S.6 Geometry of emission from a moving source.

b. Differentiation with respect to θ and setting to zero yields the critical angle θ_c:

$$\cos\theta_c = \frac{v}{c} \equiv \beta; \qquad \sin\theta_c = \sqrt{1-\beta^2} = \gamma^{-1}.$$

The maximum apparent v is thus

$$v_{\max} = \frac{v\sqrt{1-\beta^2}}{1-\beta^2} = \gamma v.$$

This clearly exceeds c when $\gamma \gg 1$.

4.8—Let K be a frame in which the two velocities are $\mathbf{v}_1$ and $\mathbf{v}_2$. The four-velocities are

$$\vec{U}_1 = \gamma(v_1)(1,\mathbf{v}_1), \qquad \vec{U}_2 = \gamma(v_2)(1,\mathbf{v}_2).$$

Let K' be observer 1's rest frame. The four-velocities are:

$$\vec{U}_1' = (1,\mathbf{0}), \qquad \vec{U}_2' = \gamma(v)(1,\mathbf{v}),$$

and $\vec{U}_1'\cdot\vec{U}_2' = \gamma(v)$, so that $v^2 = 1-(\vec{U}_1'\cdot\vec{U}_2')^{-2}$. But $\vec{U}_1\cdot\vec{U}_2 = \vec{U}_1'\cdot\vec{U}_2' =$ scalar, so that $v^2 = 1-(\vec{U}_1\cdot\vec{U}_2)^{-2}$. Using the above expressions for $\vec{U}_1$ and $\vec{U}_2$, we obtain the desired result.

4.9—We must find a tensor expression that reduces to $\mathbf{j}=\sigma\mathbf{E}$ in the fluid rest frame. From Eq. (4.59) for $F_{\mu\nu}$, the "time components" F_{0k}, contain the electric field. We also know that the fluid four-velocity, U^μ, has only time components in the fluid rest frame. Thus we try

$$j^\mu=\sigma F^{\mu\nu}U_\nu. \tag{1}$$

(We use units where $c=1$.) This equation has the right space components in the rest frame

$$j^k=\sigma F^{k\nu}U_\nu=\sigma E^k,$$

but the time component ($\mu=0$) in the rest frame gives $\rho=0$, an unacceptable constraint. Thus we want to subtract out of Eq. (1) its time component in the rest frame, that is, we need to project out only that part which is orthogonal to $\vec{U}$:

$$j^\alpha-j^\beta U_\beta U^\alpha=\sigma F^{\alpha\nu}U_\nu, \tag{2}$$

where we have used $F^{\alpha\nu}U_\alpha U_\nu=0$. Now, Eq. (2) is correct. It is manifestly a tensor equation; its space components give $\mathbf{j}=\sigma\mathbf{E}$ in the rest frame (where $U^k=0$) and its time component in the rest frame gives $0=0$, that is, no constraint on ρ.

4.10

a. If the radiation is isotropic in K', there is no preferred direction in that frame, so by symmetry the particle must remain at rest in K', that is, $a'^\mu=0$. Since $\vec{a}$ is a four-vector that vanishes in K', it also vanishes in K.

b. In K':

$$\vec{P}'_{\text{tot}}=(W',\mathbf{0}),$$

in units where $c=1$. Transforming to K, one obtains

$$\vec{P}_{\text{tot}}=\gamma W'(1,\boldsymbol{\beta}),$$

giving a *spatial* momentum, $\mathbf{P}_{\text{tot}}=\gamma W'\boldsymbol{\beta}$.

c. As shown in (a), the acceleration vanishes in both frames. Momentum is conserved because the particle loses mass, even though its speed in K is unchanged. The mass loss $\Delta m'=W'$ in K' is measured as $\Delta m=\gamma W'$

by an observer in K. Since the particle has speed β, the associated momentum change is $-\beta\Delta m = -\beta\gamma W'$, which just balances the momentum of the radiation.

4.11

a. Let $\gamma' \equiv (1-v'^2)^{-1/2}$ and $\vec{P}$ and $\vec{P}'$ be the total four-momentum vectors before and after absorption. Then ($c=1$),

$$\vec{P} = (m + h\nu, h\nu, 0, 0),$$

$$\vec{P}' = \gamma' m'(1, v', 0, 0).$$

Conservation of energy and momentum gives $\vec{P} = \vec{P}'$, or $m + h\nu = \gamma' m'$ and $h\nu = \gamma' v' m'$. Thus

$$\frac{m'}{m} = \left(1 + \frac{2h\nu}{mc^2}\right)^{1/2}.$$

b. Suppose that in the lab frame K the particle now initially has velocity v. In the frame K' in which the particle is at rest the photon has frequency

$$\nu' = \gamma\nu(1 - \beta\cos\theta).$$

In frame K' we now perform the same computation as in part (a), and we obtain

$$\frac{m}{m'} = \left(1 + \frac{2h\nu'}{mc^2}\right)^{-1/2}$$

Because m/m' is a scalar quantity (the ratio of rest masses), this equation now holds in any frame, including K.

4.12

a. Since the photons carry no angular momentum with respect to the star (unpolarized radiation), none can be given to the particle.

b. By consideration of a Lorentz frame instantaneously at rest with respect to the particle, plus the result of problem 4.10(a), we have immediately that the particle's v and direction cannot change at the instant of emission. Then

$$\frac{l}{l_0}=\frac{(mvr)_{\text{after}}}{(mvr)_{\text{before}}}=\frac{m}{m'}.$$

Thus from Problem 4.11(b) we have

$$\frac{l}{l_0}=\frac{m}{m'}=\left(1+\frac{2\gamma h\nu}{mc^2}\right)^{-1/2},$$

where $\theta=\pi/2$ in the Doppler formula.

c. Expanding to first order in $h\nu/mc^2$,

$$\frac{\Delta l}{l_0}\approx-\frac{h\nu}{mc^2}.$$

d. The net effect of absorbing and reemitting many photons is for the particle to slowly spiral in towards the sun (assuming $\Delta l/l_0 \ll 1$ per orbit), with no change in its mass; the entire effect, in the lab frame, comes from a nonradial redirection of the incident photons. Letting $\mathcal{L}$ be the sun's luminosity and $\dot{N}$ be the rate of photon absorption,

$$\dot{N}=\frac{\mathcal{L}\sigma}{h\nu 4\pi r^2}.$$

Now, using the fact that $l\propto r^{1/2}$ for circular orbits,

$$\frac{1}{l}\frac{dl}{dt}=\frac{1}{2}\frac{1}{r}\frac{dr}{dt}=\left(\frac{\Delta l}{l}\right)_{\text{per photon}}\dot{N}.$$

Combining the above equations with part (c) we have an equation for dr/dt, whose solution is

$$r^2(t)=r_0^2-\left(\frac{\mathcal{L}\sigma}{\pi mc^2}\right)t.$$

Substituting $\mathcal{L}\approx 4\times10^{33}$ erg s^{-1}, $m\approx10^{-11}$ g, $\sigma\sim10^{-8}$ cm^{-2}, $R_\odot\sim7\times10^{10}$ cm, 1 AU$\sim1.5\times10^{13}$ cm, we find that the time for r to decrease from 1 AU to $R_\odot$ is

$$t\sim5\times10^4 \text{ yr}$$

4.13

a. From Eq. (4.110) we have

$$\frac{I_\nu}{\nu^3}=\frac{I'_{\nu'}}{\nu'^3}.$$

Substituting the Planck function $I_\nu = B_\nu(T)$ from Eq. (1.51) we obtain

$$I'_{\nu'}=\frac{2h\nu'^3}{c^2}(e^{h\nu/kT}-1)^{-1},$$

$$=\frac{2h\nu'^3}{c^2}(e^{h\nu'\gamma(1-\beta\cos\theta')/kT}-1)^{-1},$$

using the Doppler formula (4.12b) with $\theta\to\theta+\pi$, since radiation propagates in the direction opposite to the viewing angle. If we define

$$T'\equiv\frac{T}{\gamma(1-\beta\cos\theta')}=T\frac{\sqrt{1-v^2/c^2}}{1-(v/c)\cos\theta'},$$

then $I'_{\nu'}=B_{\nu'}(T')$, and for each direction the observed radiation is blackbody.

b. Expanding for $\beta\ll1$, we obtain $T'\approx T(1+\beta\cos\theta')$, so that $T'_{\max}\approx T(1+\beta)$ and $T'_{\min}\approx T(1-\beta)$. Then

$$I_{\max}\propto(e^{h\nu'/kT'_{\max}}-1)^{-1}\propto 1+\beta,$$

$$I_{\min}\propto(e^{h\nu'/kT'_{\min}}-1)^{-1}\propto 1-\beta,$$

for $h\nu/kT\sim0.18\ll1$. Thus

$$\frac{I_{\max}-I_{\min}}{I_{\max}+I_{\min}}\approx\beta\lesssim10^{-3},$$

so that $v\lesssim300$ km s^{-1}.

4.14—Let the primed frame be the instantaneous rest frame of the particle. Then

$$F_{\parallel}=\frac{dp_{\parallel}}{dt}=\frac{\gamma(dp'_{\parallel}+\beta\,dE')}{\gamma(dt'+\beta/c\,dx')}=\frac{dp'_{\parallel}}{dt'}=F'_{\parallel},$$

because the particle is instantaneously at rest in this frame. Similarly,

$$F_\perp = \frac{dp_\perp}{dt} = \frac{dp'_\perp}{\gamma\, dt'} = \gamma^{-1} F'_\perp .$$

Then, from (4.92),

$$\begin{aligned} P &= \frac{2e^2}{3c^3}\left(a'^2_\parallel + a'^2_\perp\right) \\ &= \frac{2e^2}{3m^2c^3}\left(F^2_\parallel + \gamma^2 F^2_\perp\right). \end{aligned}$$

4.15—The two scalar invariants of the electromagnetic field strengths are $\mathbf{E}\cdot\mathbf{B}$ and $E^2 - B^2$. If we can show that $W^2_{em} - |\mathbf{S}|^2/c^2$ can be written solely in terms of these two invariants, then it must be an invariant. Since $W_{em} = (8\pi)^{-1}(E^2 + B^2)$ and $\mathbf{S} = (c/4\pi)\mathbf{E}\times\mathbf{B}$, we have

$$\begin{aligned} 64\pi^2\left(W^2_{em} - \frac{1}{c^2}|\mathbf{S}|^2\right) &= (E^2 + B^2)^2 - 4|\mathbf{E}\times\mathbf{B}|^2 \\ &= E^4 + 2E^2B^2 + B^4 - 4E^2B^2\sin^2\theta \\ &= E^4 + 2E^2B^2 + B^4 - 4E^2B^2\left(1 - \frac{(\mathbf{E}\cdot\mathbf{B})^2}{E^2B^2}\right) \\ &= (E^2 - B^2)^2 + 4(\mathbf{E}\cdot\mathbf{B})^2. \end{aligned}$$

4.16—The solution to this problem requires some tensor index manipulation plus use of Maxwell's equations in tensor form.

a.

$$\begin{aligned} 4\pi T^{\mu\nu}\eta_{\mu\nu} &= \eta_{\mu\nu} F^{\mu\alpha} F^{\nu}{}_{\alpha} - \tfrac{1}{4}\eta^{\mu\nu}\eta_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \\ &= F^{\mu\alpha} F_{\mu\alpha} - \tfrac{1}{4}\cdot 4 F^{\alpha\beta} F_{\alpha\beta} \\ &= 0. \end{aligned}$$

b.

$$\begin{aligned} 4\pi T^{\mu\nu}{}_{,\nu} &= F^{\mu\alpha}{}_{,\nu} F^{\nu}{}_{\alpha} + F^{\mu\alpha} F^{\nu}{}_{\alpha,\nu} - \tfrac{1}{4}\eta^{\mu\nu}\left(F^{\alpha\beta}{}_{,\nu} F_{\alpha\beta} + F^{\alpha\beta} F_{\alpha\beta,\nu}\right) \\ &= -F^{\mu}{}_{\alpha} F^{\alpha\nu}{}_{,\nu} + F^{\mu\alpha,\nu} F_{\nu\alpha} - \tfrac{1}{2} F^{\alpha\beta,\mu} F_{\alpha\beta} \\ &= -F^{\mu}{}_{\alpha} F^{\alpha\nu}{}_{,\nu} - F_{\alpha\beta}\left(F^{\mu\alpha,\beta} + \tfrac{1}{2} F^{\alpha\beta,\mu}\right). \end{aligned}$$

Now,

$$F_{\alpha\beta}F^{\mu\alpha,\beta}=F_{\beta\alpha}F^{\mu\beta,\alpha}=F_{\alpha\beta}F^{\beta\mu,\alpha},$$

relabeling indices and using antisymmetry of F. Thus

$$4\pi T^{\mu\nu}{}_{,\nu}=-F^{\mu}{}_{\alpha}(F^{\alpha\nu}{}_{,\nu})-\tfrac{1}{2}F_{\alpha\beta}(F^{\mu\alpha,\beta}+F^{\beta\mu,\alpha}+F^{\alpha\beta,\mu})=0,$$

because the two quantities in parentheses vanish according to Maxwell's equations in free space. [See Eqs. (4.60) and (4.61).]

5.1

a. The optically thin luminosity is equal to the volume $V=(4/3)\pi R^3(t)$ times the power radiated per unit volume, Eq. (5.15b):

$$\mathcal{L}_{\text{thin}}=1.7\times10^{-27}n_e n_p T_0^{1/2}V,$$

where we have taken $\bar{g}_B=1.2$. Now, $n_e=n_p=M_0/m_p V$, where $m_p=$ hydrogen mass. Thus

$$\mathcal{L}_{\text{thin}}=1.6\times10^{20}M_0^2T_0^{1/2}R^{-3}(t).$$

b. The optically thick luminosity is equal to the surface area $4\pi R^2(t)$ times the blackbody flux, Eq. (1.43):

$$\mathcal{L}_{\text{thick}}=7.1\times10^{-4}T_0^4R^2(t).$$

c. The transition between thick and thin cases occurs roughly when $\mathcal{L}_{\text{thin}}\approx\mathcal{L}_{\text{thick}}$. Setting the above expressions equal for $t=t_0$ we obtain

$$R(t_0)\approx4.7\times10^4M_0^{2/5}T_0^{-7/10}.$$

[An alternate solution follows by setting $\alpha_R^{ff}R(t_0)\approx1$, using Eq. (5.20). This yields a result of the same form, but with coefficient 2.0×10^4.]

d. See Fig. S.7.

5.2—The knee in the spectrum gives T:

$$T=\frac{E_{\max}}{k}\approx10^9\text{ K}.$$

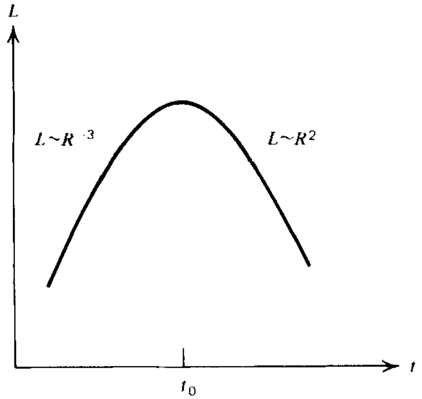

Figure S.7 Luminosity as a function of time, from a collapsing sphere.

From Eq. (5.15b) we obtain

$$F = \frac{1}{4\pi L^2}\frac{4\pi R^3}{3}\left(1.4\times 10^{-27}T^{1/2}n_e n_i Z^2 \bar{g}_B\right).$$

At $T = 10^9$ K the gas is completely ionized. If we can assume it is pure hydrogen, $n_i = n_e$. (Including a typical helium abundance makes only a negligible difference.) Then

$$n_i n_e \approx n_H^2 = \left(\frac{\rho}{m_H}\right)^2 = 3.6\times 10^{47}\rho^2.$$

Taking $Z = 1$ and $\bar{g}_B = 1.2$ gives

$$F = 2.0\times 10^{20}\rho^2 T^{1/2} R^3 L^{-2}. \tag{1}$$

Hydrostatic equilibrium gives another constraint on ρ and R. From the virial theorem we know that $2\times$ (kinetic energy/particle) $=$ $-$(gravitational energy/particle) or

$$3kT \sim \frac{GMm_H}{R}.$$

For $T = 10^9$ K this implies

$$R \approx 5\times 10^8\left(\frac{M}{M_\odot}\right)\ \text{cm}, \tag{2}$$

where $M_\odot$ = mass of sun $\approx 2\times 10^{33}$ g.

Combining Eqs. (1) and (2) gives

$$\rho \approx 4\times 10^{-26} L F^{1/2}\left(\frac{M}{M_\odot}\right)^{-3/2}.$$

Substituting in the measured values of F and L we obtain

$$\rho \approx 1.2\times 10^{-7}\,\mathrm{g\ cm}^{-3}\left(\frac{M}{M_\odot}\right)^{-3/2}. \tag{3}$$

Now, to get an optical depth we must first determine the dominant opacity source, free-free (bremsstrahlung) κ_{ff} or scattering κ_{es}. Using Eq. (5.20) for the Rosseland mean of κ_{ff} we have

$$\frac{\kappa_R^{ff}}{\kappa_{es}} \approx \frac{0.7\times 10^{23}\rho T^{-7/2}}{0.4} \approx 10^{-15}\left(\frac{M}{M_\odot}\right)^{-3/2}.$$

Thus, for $M/M_\odot \gg 10^{-10}$, $\kappa_R^{ff} \ll \kappa_{es}$ and the "effective" opacity coefficient is [cf. Eq. (1.97)],

$$\kappa_* \sim \sqrt{\kappa_R^{ff}\kappa_{es}} \sim 10^{-8}\left(\frac{M}{M_\odot}\right)^{-3/4}\ \mathrm{cm}^2\,\mathrm{g}^{-1}.$$

The effective optical depth τ_* is

$$\tau_* \sim R\rho\kappa_* \sim 6\times 10^{-7}\left(\frac{M}{M_\odot}\right)^{-5/4}.$$

Thus, for $M/M_\odot \gg 10^{-5}$, the source is effectively thin, $\tau_* \ll 1$, and the assumption of bremsstrahlung emission is justified. For complete consistency, however, one must also check to see whether inverse Compton cooling (Chapter 7) is important. [See Problem 7.2.]

6.1—By conservation of energy, $d/dt(\gamma mc^2) = -$power radiated. Therefore, using Eq. (6.5) and $B_\perp = B\sin\alpha$ we have

$$\dot\gamma = \frac{-P}{mc^2} = -A\beta^2\gamma^2 \approx -A\gamma^2,$$

since $\beta \approx 1$. This equation is easily integrated to yield

$$-\gamma^{-1} = -At + \text{constant}.$$

The boundary condition implies constant $= -\gamma_0^{-1}$, so that

$$\gamma = \frac{\gamma_0}{1 + A\gamma_0 t}.$$

At $t = t_{1/2}$ we have $\gamma = \gamma_0/2$. Thus

$$t_{1/2} = (A\gamma_0)^{-1} = \left(\frac{2e^4}{3m^3c^5} \gamma_0 B_\perp^2 \right)^{-1}.$$

To correctly account for the radiation reaction (and decrease of γ) in the particle equation of motion, the electric field of the self-radiation must be added to Eq. (6.1).

6.2

a. We assume that the magnetic field is frozen into the gas. [This is almost always a good approximation for problems on a cosmic scale. See, e.g., Alfven, H., *Cosmical Electrodynamics* (Clarendon, Oxford, 1963)]. The magnetic flux through a loop moving with the gas is then a constant, and since area scales as l^2, the magnetic field is proportional to l^{-2}.

b. The action integral for a particle in a periodic orbit is defined as

$$\mathcal{Q} = \oint \mathbf{p} \cdot d\mathbf{r},$$

where $\mathbf{p}$ = momentum and where the integral is taken over one period. This quantity is an *adiabatic* invariant, that is, it is approximately constant for slow changes in the external parameters, such as the magnetic field [see, e.g., Landau and Lifshitz, *Mechanics* 3rd ed., (Pergamon, New York, 1976)].

The motion of the electron separates into uniform motion along the field and circular motion around the field. We may apply the adiabatic invariant to the circular motion alone. From Eq. (6.3), we find

$$p_\perp c = eBa,$$

where a is the radius of the projected motion on the normal plane. From the adiabatic invariant we have $p_\perp a$ = constant, and from part (a) we have $B \propto l^{-2}$. Thus $p_\perp \propto l^{-1}$ and $a \propto l$. Since the orbit contracts at the same rate as the contraction of the gas, the flux through it remains constant.

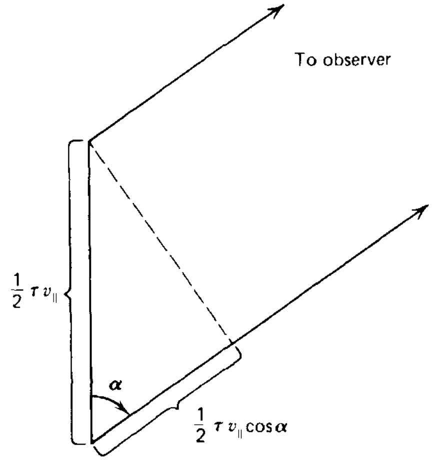

***Figure S.8** Geometry of Doppler shift for a particle spiraling in a magnetic field with pitch angle α.*

c. For relativistic particles $p \propto \gamma$ so that $\gamma \propto l^{-1}$. From Eq. (6.7b) and $B \propto l^{-2}$ we have $P \propto l^{-6}$. From Eq. (6.17) we have $\omega_c \propto l^{-4}$. From Problem 6.1 we have $t_{1/2} \propto l^5$.

6.3—Suppose that the time for the particle to go from 1 to 2 and back from 2 to 1 is τ. For the second half of each cycle, of duration $\tau/2$, *no* radiation from the particle reaches the observer, since the radiation beam, of halfwidth $\gamma^{-1} \ll 1$, is directed at an angle $\pi - 2\alpha$ or greater away from the observer. Now, from simple time delay, the apparent duration of the first half of the cycle is (Fig. S.8)

$$\tau_{\text{app}} = \frac{1}{2}\tau\left(1 - \frac{v_{\parallel}}{c}\cos\theta\right)$$
$$= \frac{1}{2}\tau\left(1 - \frac{v}{c}\cos^2\alpha\right) \approx \frac{1}{2}\tau\sin^2\alpha.$$

Thus the fraction of each cycle in which the particle appears to radiate is

$$\tfrac{1}{2}\sin^2\alpha.$$

6.4

a. When the source is optically thin the observed flux $F_\nu \propto j_\nu = \alpha_\nu S_\nu \propto \nu^{-(p-1)/2}$; when it is optically thick $F_\nu \propto S_\nu \propto \nu^{5/2}$. From the given spectrum, $(p-1)/2 = 1/2$ or $p = 2$. At the critical frequency ν_2, where the synchrotron source becomes optically thick, we have two pieces of information:

$$\int \alpha_{\nu_2}^s \, ds \approx \alpha_{\nu_2}^s R \approx 1,$$

and

$$F_0 = S_{\nu_2}\Omega = \frac{\pi S_{\nu_2} R^2}{d^2}.$$

Therefore,

$$C\left(\frac{B}{B_0}\right)^2\left(\frac{\nu_2}{\nu_0}\right)^{-3} R = 1,$$

$$A\left(\frac{B}{B_0}\right)^{-1/2}(\nu_2/\nu_0)^{5/2}\Omega = F_0,$$

and we have

$$\frac{B}{B_0} = \left[A(\nu_2/\nu_0)^{5/2}\Omega F_0^{-1}\right]^2,$$

$$R = C^{-1}\left(\frac{\nu_2}{\nu_0}\right)^{-7}(A\Omega F_0^{-1})^{-4}.$$

b. The spectrum below ν_1 seems to be dominated by bound-free absorption, exhibiting the clear ν^2 dependence of an optically thick thermal emitter. The frequency ν_1 is thus the frequency at which the hydrogen gas becomes optically thick, so that we have an additional relation for the distance to the source

$$\int \alpha_{\nu_1}^{bf} \, ds \approx \alpha_{\nu_1}^{bf} d \approx 1,$$

or

$$d = D^{-1}\left(\frac{\nu_1}{\nu_0}\right)^3.$$

Using the expressions for R, d, and $\Omega = \pi R^2/d^2$ we have

$$\Omega^9 = \pi A^{-8} C^{-2} D^2 \left(\frac{\nu_1}{\nu_0}\right)^{-6}\left(\frac{\nu_2}{\nu_0}\right)^{-14} F_0^8.$$

6.5

a. Integrating the two intensities of Eqs. (6.32) over the electron distribution, we have for the linear polarization

$$\Pi = \frac{\int G(x)\gamma^{-p}\,d\gamma}{\int F(x)\gamma^{-p}\,d\gamma}.$$

We now write γ in terms of x, $\gamma \propto x^{-1/2}$, obtaining

$$\Pi = \frac{\int G(x)x^{(p-3)/2}\,dx}{\int F(x)x^{(p-3)/2}\,dx}. \tag{1}$$

Now, using Eq. (6.35), and the property of the Γ function, $\Gamma(q+1) = q\Gamma(q)$, we obtain

$$\Pi = \frac{p+1}{p+7/3}. \tag{2}$$

b. The polarization of the frequency-integrated emission is, using Eqs. (6.32),

$$\Pi = \frac{\int G(x)\,dx}{\int F(x)\,dx},$$

where we have used the fact that $\omega \propto x$. Comparing this integral with that of Eq. (1) above, we see that they are equal for $p=3$. Thus substituting this value into Eq. (2) above, we obtain

$$\Pi = \frac{4}{3+\frac{7}{3}} = 75\%.$$

7.1—The energy transfer to a photon of energy ϵ, in a single scattering, has the form [cf. Eq. (7.36)]

$$\Delta\epsilon = \epsilon\left(\frac{4kT}{mc^2} - \frac{\epsilon}{mc^2}\right).$$

Thus, for photons of energy $\epsilon \ll 4kT$, the energy gain per scattering can be put into the approximate differential form

$$\frac{d\epsilon}{dN} \sim \epsilon \frac{4kT}{mc^2},$$

where dN is the differential number of scatterings. After N scatterings, the energy of a photon of initial energy ϵ_i is thus

$$\frac{\epsilon_N}{\epsilon_i} \sim e^{(4kT/mc^2)N} \qquad \text{for } \epsilon_N \ll 4kT.$$

The exponential nature of the energy gain is apparent from the initial equation. In a medium of optical depth $\tau_{es} \gg 1$, the characteristic photon scatters $\sim \tau_{es}^2$ times before escaping, because of the random walk nature of the scattering process. Thus setting $N = \tau_{es}^2$ gives

a.

$$\frac{\epsilon_f}{\epsilon_i} \sim e^y, \qquad y \equiv \frac{4kT}{mc^2}\tau_{es}^2.$$

b. When $\epsilon_f \sim 4kT$, the initial equation shows that photons stop gaining energy from the electrons; the process has saturated. Thus, to obtain τ_{crit},

$$\frac{4kT}{\epsilon_i} \sim e^{y_{\text{crit}}},$$

$$(4kT/mc^2)\tau_{\text{crit}}^2 = \ln\left(\frac{4kT}{\epsilon_i}\right)$$

$$\tau_{\text{crit}} = \left[\frac{mc^2}{4kT}\ln\left(\frac{4kT}{\epsilon_i}\right)\right]^{1/2}.$$

c. From part (a) the parameter is $y \equiv (4kT/mc^2)\tau_{es}^2$.

7.2—From the solution to Problem 5.2, the size R, density ρ, and temperature T, of the emitting region satisfy

$$R = 5 \times 10^8 \text{ cm} \frac{M}{M_\odot}$$

$$\rho = 1.2 \times 10^{-7} \text{ g cm}^{-3}\left(\frac{M}{M_\odot}\right)^{-3/2}$$

$$T = 10^9 \text{ K}.$$

From Problem 7.1, inverse Compton is important if the "Comptonization parameter" $y \equiv (4kT/mc^2)\tau^2$ exceeds unity. Now,

$$\tau_{es} \sim \kappa_{es}\rho R,$$

so that

$$y \sim 400\left(\frac{M}{M_\odot}\right)^{-1}.$$

Thus if $M \gg 400\ M_\odot$, inverse Compton can be ignored, and the determination of T, ρ, and R on the assumption of pure bremsstrahlung cooling is self-consistent. On the other hand, if $M \lesssim 400\ M_\odot$, then the model is self-inconsistent, because inverse Compton cooling was ignored in determining the energy balance.

7.3

a. From Eq. (6.17c) for the characteristic synchrotron frequency, we have, in normalized units, (taking $\sin\alpha = 3^{-1/2}$)

$$h\nu_c \approx 0.10\ \mathrm{eV}\left(\frac{\gamma}{10^4}\right)^2\left(\frac{B}{0.1G}\right).$$

The ratio of the photon's energy to the electron rest mass energy, in the electron rest frame, is then given approximately by

$$\frac{\gamma h\nu_c}{mc^2} \approx 2.0 \times 10^{-3}\left(\frac{\gamma}{10^4}\right)^3\left(\frac{B}{0.1G}\right).$$

b. The energy associated with a temperature of 1 K is $\sim 0.86 \times 10^{-4}$ eV. The blackbody spectrum peaks at $\sim 2.8\ kT$. Thus the characteristic photon in a blackbody spectrum of temperature T has an energy $\sim 2.4 \times 10^{-4} T$ eV. The ratio of a microwave photon energy to electron rest mass in the latter's rest frame is, therefore,

$$\frac{\gamma h\nu}{mc^2} \approx 1.4 \times 10^{-5}\left(\frac{\gamma}{10^4}\right).$$

Note that in both (a) and (b), for the *second scattering*, the relevant ratio is a factor γ^2 higher and no longer less than unity for $\gamma \sim 10^4$!

7.4

a. Computation of Δ, Eq. (7.53):
First we set $c = 1$ in our computations. Let the initial photon four-momentum be $\vec{P}_\gamma = \hbar\omega(1, \mathbf{n})$, final photon four-momentum be $\vec{P}_{\gamma_1} =$

$\hbar\omega_1(1,\mathbf{n}_1)$, initial electron four-momentum be $\vec{P}_e=(E,\mathbf{p})$, and final electron four-momentum be $\vec{P}_{e_1}$. Then, expanding out the expression

$$|\vec{P}_{e_1}|^2=|\vec{P}_e+\vec{P}_\gamma-\vec{P}_{\gamma_1}|^2$$

gives

$$E\hbar\omega-\hbar\omega\mathbf{p}\cdot\mathbf{n}=\hbar^2\omega\omega_1(1-\mathbf{n}\cdot\mathbf{n}_1)+\hbar\omega_1E-\hbar\omega_1\mathbf{p}\cdot\mathbf{n}_1. \tag{1}$$

Here $\mathbf{n}$, $\mathbf{n}_1$, and $\mathbf{p}$ are the initial and final photon directions and the initial electron momentum, respectively. From Eq. (1) we obtain

$$\Delta\equiv\frac{\hbar(\omega_1-\omega)}{kT}=\frac{x\mathbf{p}\cdot(\mathbf{n}_1-\mathbf{n})-x^2kT(1-\mathbf{n}\cdot\mathbf{n}_1)}{E-\mathbf{p}\cdot\mathbf{n}_1+xkT(1-\mathbf{n}\cdot\mathbf{n}_1)} \tag{2}$$

where $x\equiv\hbar\omega/kT$. Now, since $\mathbf{p}/m$ is of order $\alpha\equiv(kT/m)^{1/2}$, to lowest order in α we may replace the denominator of Eq. (2) by $E=m$, where m is the electron mass, and neglect the second term in the numerator, $O(\alpha^2)$, in comparison with the first, thus obtaining (putting back factors of c)

$$\Delta=x\mathbf{p}\cdot\frac{(\mathbf{n}_1-\mathbf{n})}{mc}+O\left(\frac{kT}{mc^2}\right). \tag{3}$$

b. Computation of I_2, Eq. (7.54):
Let χ be the angle between the vector $\mathbf{p}$ and the vector $(\mathbf{n}_1-\mathbf{n})$. Then, using Eq. (7.53) for Δ^2, we obtain

$$\begin{aligned} I_2&\equiv\int\int d^3pf_e\Delta^2\frac{d\sigma}{d\Omega}\,d\Omega\\ &=\left(\frac{x}{mc}\right)^2\int d^3pp^2\cos^2\chi f_e\int|\mathbf{n}_1-\mathbf{n}|^2\frac{d\sigma}{d\Omega}\,d\Omega. \end{aligned} \tag{4}$$

Now, since $d\sigma/d\Omega$ does not depend on $\mathbf{p}$, to lowest order in $v/c\sim\alpha$, the integral over p may be done independently of the integral over photon directions. Next, substitute in Eq. (7.49) for the Maxwellian electron distribution, f_e, and let χ be the polar angle for the d^3p integration, that is, $d^3p=p^2dp\,d\cos\chi\,d\phi$. The integration over d^3p then gives

$$I_2=x^2n_e\frac{kT}{mc^2}\int\frac{d\sigma}{d\Omega}|\mathbf{n}_1-\mathbf{n}|^2d\Omega. \tag{5}$$

Finally, let $\mathbf{n}_1$ lie along the polar axis for the $d\Omega = d\cos\theta\, d\phi$ integration, so that $|\mathbf{n}_1 - \mathbf{n}|^2 = 2(1-\cos\theta)$. Substituting Eq. (7.1b) for $d\sigma/d\Omega$, we obtain the desired result,

$$\begin{aligned} I_2 &= \tfrac{3}{4} x^2 n_e \sigma_T \frac{kT}{mc^2} \int_{-1}^{1} (1 - x + x^2 - x^3)\,dx \\ &= 2x^2 n_e \sigma_T \left(\frac{kT}{mc^2} \right). \end{aligned}$$

c. Computation of $\partial n/\partial t$, Eqs. (7.55):
To conserve the total number of photons, $\partial n/\partial t$ must have the functional form of Eq. (7.55a), since

$$\frac{d}{dt} \int n x^2 \, dx = -\int_0^\infty \frac{\partial}{\partial x} \left[x^2 j(x) \right] dx = -x^2 j(x)\big|_0^\infty, \tag{7}$$

that is, the change in total photon number arises only from a flux through the boundaries of energy space. Next, write Eq. (7.55a) in the form

$$\frac{\partial n}{\partial t} = -\frac{2}{x} j - \frac{\partial j}{\partial x}, \tag{8}$$

and Eq. (7.52) in the form

$$\frac{\partial n}{\partial t} = C_1(x) n'' + C_2(n,x) n' + C_3(n,x). \tag{9}$$

Equations (8) and (9) must be *functionally identical*. Comparing the highest x derivatives in these two equations, we see that j must contain a term linear in n', with coefficient independent of n and no terms in n''. Thus j must be of the functional form

$$j = g(x)\left[n' + h(n,x) \right]. \tag{10}$$

8.1—Consider two media, of refractive indices n_r and n_r'. Let θ and θ' be the angles of incidence and refraction of a beam of radiation incident on an area $d\sigma$ of the surface of separation of the two media. Let I_ν and I_ν' be the intensities of the incident and refracted beam, respectively. Then, assuming that no energy is lost by reflection at the interface, we have

$$I_\nu \cos\theta \, d\sigma \, d\Omega = I_\nu' \cos\theta' \, d\sigma \, d\Omega', \tag{1}$$

where $d\Omega = d\cos\theta\, d\phi$, $d\Omega' = d\cos\theta'\, d\phi'$. Now, $d\phi = d\phi'$, and by Snell's law

$$n_r \sin\theta = n_r' \sin\theta'. \tag{2}$$

Squaring and differentiating Eq. (2) leads to

$$n_r^2 \cos\theta\, d\cos\theta = n_r' \cos\theta'\, d\cos\theta'. \tag{3}$$

Now, combining Eqs. (1) and (3) gives

$$\frac{I_\nu}{n_r^2} = \frac{I_\nu'}{n_r'^2}.$$

In a medium in which the refractive index changes continuously and slowly on the scale of a wavelength, we can imagine that the photon path is made up of a number of short segments in regions of constant refractive index, to which the above result applies. Thus I_ν / n_r^2 is seen to be an invariant over general paths. Note that the assumption of no reflection loss at the interface between media becomes completely valid in the continuous limit.

8.2—The Fourier transform of $\psi(r,t)$ with respect to r is simply $A(k)\exp[-i\omega(k)t]$ from its definition. Therefore, from Parseval's theorem we have

$$\int_{-\infty}^{\infty} |\psi|^2 dr = (2\pi)^{-1} \int_{-\infty}^{\infty} |A\exp(-i\omega t)|^2 dk = (2\pi)^{-1} \int_{-\infty}^{\infty} |A(k)|^2 dk, \tag{1}$$

since ω is real. Thus the normalization of the packet remains constant in time (no absorption). Now consider the result

$$r\psi(r,t) = \int_{-\infty}^{\infty} A(k) e^{-i\omega t} \frac{1}{i} \frac{\partial}{\partial k} e^{ikr}\, dk$$

$$= \int_{-\infty}^{\infty} e^{ikr} i \frac{\partial}{\partial k} \left[A(k) e^{-i\omega t}\right] dk$$

where we have integrated by parts. This shows that the Fourier transform of $r\psi$ is $i\partial/\partial k(Ae^{-i\omega t})$. Using the generalized Parseval's theorem [cf. Eq. (2.31)],

$$\int A^*(r) B(r)\, dr = (2\pi)^{-1} \int \hat{A}^*(k) \hat{B}(k)\, dk,$$

with $A = \psi$ and $B = r\psi$, we obtain

$$\int_{-\infty}^{\infty} r|\psi|^2 dr = \frac{1}{2\pi}\int_{-\infty}^{\infty} A^* e^{i\omega t} i \frac{\partial}{\partial k}(Ae^{-i\omega t})dk,$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(|A|^2 \frac{\partial \omega}{\partial k} t + iA^* \frac{\partial A}{\partial k}\right)dk,$$

which depends on time linearly. Therefore,

$$\frac{d}{dt}\int_{-\infty}^{\infty} r|\psi|^2 dr = \frac{1}{2\pi}\int_{-\infty}^{\infty} |A(k)|^2 \frac{\partial \omega}{\partial k} dk, \tag{2}$$

and dividing by Eq. (1), which is independent of time, yields the desired result:

$$\frac{d}{dt}\langle r(t)\rangle = \left\langle \frac{\partial \omega}{\partial k} \right\rangle. \tag{3}$$

Suppose that the wave packet is localized in both space and wave number, within the restrictions of the uncertainty relation $\Delta k \Delta r \gtrsim 1$, of course. If $\partial\omega/\partial k$ changes slowly over the scale Δk about the central wave number k_0, then the packet will move with the group velocity $(\partial\omega/\partial k)_{k=k_0}$. This is the usual statement of the group velocity property, but Eq. (3) also holds when the packet is spread arbitrarily in space and wave number.

8.3—Taking the derivative of Eq. (8.31) with respect to ω and dividing the resulting equation by Eq. (8.20), we obtain

$$\frac{d\Delta\theta/d\omega}{dt_p/d\omega} = 1.7 \times 10^7 \text{ s}^{-1}\langle B_\parallel \rangle,$$

where $\langle B_\parallel \rangle$ is measured in Gauss. Note the interesting result that the frequency dependence cancels out of the above expression. Substituting $d\Delta\theta/d\omega = 1.9 \times 10^{-4}$ s and $dt_p/d\omega = 1.1 \times 10^{-5}$ s^2, we obtain the result

$$\langle B_\parallel \rangle = 1.0 \times 10^{-6}$$

without needing to know the frequency at which the measurements were made!

9.1

a. The properly normalized *antisymmetric* wave function is:

$$\psi(\mathbf{r}_1,\mathbf{r}_2,s_1,s_2)=\frac{1}{\sqrt{2}}\left[u_a(1)u_b(2)-u_b(1)u_a(2)\right],$$

where 1 and 2 include space and spin coordinates. The operator whose expectation value we want is

$$R^2=(\mathbf{r}_1-\mathbf{r}_2)^2=r_1^2-2\mathbf{r}_1\cdot\mathbf{r}_2+r_2^2.$$

Now, use Dirac notation for integrals,

$$\int u_a^*(\mathbf{r}_1,s_1)\mathbf{r}_1u_b(\mathbf{r}_1,s_1)d^3r_1\equiv\langle a|\mathbf{r}|b\rangle,$$

use the fact that $\mathbf{r}_1$ only operates on functions of $\mathbf{r}_1$,

$$\int u_a^*(\mathbf{r}_1)\mathbf{r}_2u_a(\mathbf{r}_1)d^3r_1=\mathbf{r}_2,$$

and use the orthogonality of orbitals

$$\langle a|b\rangle=0\qquad\text{for } a\neq b,$$

to obtain

$$\begin{aligned}\langle R^2\rangle&=\int\psi^*(\mathbf{r}_1,\mathbf{r}_2)\left[r_1^2-2\mathbf{r}_1\cdot\mathbf{r}_2+r_2^2\right]\psi(\mathbf{r}_1,\mathbf{r}_2)d^3r_1\,d^3r_2\\&=\langle a|\mathbf{r}^2|a\rangle+\langle b|\mathbf{r}^2|b\rangle\\&\quad-2\langle a|\mathbf{r}|a\rangle\cdot\langle b|\mathbf{r}|b\rangle+2|\langle a|\mathbf{r}|b\rangle|^2.\\&=(\mathbf{r}^2)_a+(\mathbf{r}^2)_b-2|\mathbf{r}_a|\,|\mathbf{r}_b|+2|\mathbf{r}_{ab}|^2.\end{aligned}$$

b. Note that the dipole operator vanishes between states of the same parity (see §10.4)

c. Separating space and spin parts,

$$u_a(\mathbf{r}_1,s_1)=u_a(\mathbf{r}_1)|s_1\rangle,$$

we have

$$\langle a|\mathbf{r}|b\rangle=\int u_a^*(r)\mathbf{r}u_b(r)d^3r\langle s_a|s_b\rangle,$$

since **r** does not operate on $|s\rangle$. Since different spin states are orthogonal, for example, $|s_a\rangle = (1,0)$, $|s_b\rangle = (0,1)$, electrons of different spins give $\langle s_a|s_b\rangle = 0 = \langle a|\mathbf{r}|b\rangle$.

d. Since $|\langle a|r|b\rangle|^2$ is a positive definitive quantity, comparison of parts (a) and (c) shows that $\langle R^2\rangle$ is larger for same spin electrons by a term $2|\langle a|r|b\rangle|^2$.

9.2

a. $2s^2$: These are equivalent electrons and must be in opposite spin states, so $S=0$, and only singlets occur. Each orbital has zero orbital angular momentum, so $L=0$, and only S states occur. The parity is even: $(-1)^{0+0}=1$. There is only one possible value for $J=L+S$, that is, zero. The one possible term and level is, therefore, $^1S_0$.

b. $2p3s$: These are nonequivalent electrons, so that all combinations of spins are allowed. Therefore both $S=0$ and $S=1$ are possible, and singlets and triplets occur. The orbital angular momenta of the orbitals are $l=1$ and $l=0$, so that $L=1$ and only P states can occur. The parity is odd: $(-1)^{0+1}=-1$. The angular momentum of the singlet state $(S=0, L=1)$ can be only $J=1$. The triplet state $(S=1)$ can combine with $L=1$ to yield $J=0,1,2$. Therefore, the terms and levels are: $^1P_1^O$ and $^3P_{2,1,0}^O$.

c. $3p4p$: These are nonequivalent electrons, allowing $S=0$ and 1, so that singlets and triplets occur. The values of L, found from combining $l=1$ and $l=1$, are $L=0$, 1 and 2, so that S, P, and D states can occur. The parity is even: $(-1)^{1+1}=1$. There are six ways to pick L and S which lead to the following terms and levels: $^3D_{3,2,1}$, $^3P_{2,1,0}$, $^3S_1$, $^1D_2$, $^1P_1$, $^1S_0$.

d. $2p^43p$: There are four equivalent $2p$ electrons and one $3p$ electron. In cases such as this, find the terms of the equivalent electrons first, then combine each in turn with the remaining nonequivalent electron. The terms of p^4 are the same as for p^2, which are ^1S, ^1D, ^3P. (See §9.4.) The ^1S term plus the $3p$ electron gives rise to $^2P_{3/2,1/2}^O$. The ^1D term plus $3p$ yields $^2F_{7/2,5/2}^O$, $^2D_{5/2,3/2}^O$, and $^2P_{3/2,1/2}^O$. The ^3P term plus $3p$ yields $^4D_{7/2,5/2,3/2,1/2}^O$, $^4P_{5/2,3/2,1/2}^O$, $^4S_{3/2}^O$; and $^2D_{5/2,3/2}^O$, $^2P_{3/2,1/2}^O$ and $^2S_{1/2}^O$.

9.3—Recall the statistical weights: $2(2l+1)$ for each nonequivalent electron in a configuration; $(2L+1)\,(2S+1)$ for a term; and $(2J+1)$ for a level.

a. From $l_1 = l_2 = 0$, we have $2(2l_1+1)\ 2(2l_2+1) = 4$, but two states violate the Pauli principle, and the remaining two are indistinguishable. Thus $N_{\text{conf}} = 1$. From $L=0$ and $S=0$ we obtain $N_{\text{term}} = (2L+1)(2S+1) = 1$. From $J=0$ we obtain $N_{\text{level}} = (2J+1) = 1$.

b. From $l_1 = 0$, $l_2 = 1$ we have $N_{\text{conf}} = 2(2\cdot 0+1)\ 2(2\cdot 1+1) = 12$. The $(L=1, S=0)$ and $(L=1, S=1)$ terms give $N_{\text{term}} = (2\cdot 1+1)(2\cdot 0+1) + (2\cdot 1+1)(2\cdot 1+1) = 3+9 = 12$. The levels have $J = 1,2,1,0$ so that $N_{\text{level}} = 3+5+3+1 = 12$.

c. From $l_1 = 1$, $l_2 = 1$ we have $N_{\text{conf}} = 2(2\cdot 1+1)\ 2(2\cdot 1+1) = 36$. The terms ^3D, ^3P, ^3S, ^1D, ^1P, ^1S yield $N_{\text{term}} = 15+9+3+5+3+1 = 36$. The levels have $J = 3,2,1,2,1,0,1,2,1,0$, so that $N_{\text{level}} = 7+5+3+5+3+1+3+5+3+1 = 36$.

9.4—Using the definitions of λ, ξ, and γ, the Saha equation can be written

$$\frac{\chi_j}{kT} + \ln\left(\frac{N_{j+1}}{N_j}\right) = \ln\left(\frac{2U_{j+1}}{U_j}\right) + \gamma \approx \gamma, \tag{1}$$

since γ is large compared to $\ln(2U_{j+1}/U_j)$.

a. The transition from stage j to $j+1$ is defined by $N_j \approx N_{j+1}$. From Eq. (1) we have $kT \sim \chi_j/\gamma$.

b. From Eq. (1) we have

$$\frac{d\ln(N_{j+1}/N_j)}{d\ln T} = \frac{\chi_j}{kT} + \frac{3}{2},$$

since $\xi \propto T^{-3/2}$. From part (a) we know $\chi/kT \approx \gamma \gg 3/2$, so that

$$\frac{\Delta T}{T} = \left[\frac{d\ln(N_{j+1}/N_j)}{d\ln T}\right]^{-1} \approx \gamma^{-1}.$$

c. The ratio of excited to ground state populations in state j is given by the Boltzmann law:

$$\frac{N_{i,j}}{N_{0,j}} = \frac{g_i}{g_0} e^{-\chi_{i,j}/kT},$$

where g_i and g_0 are statistical weights and $\chi_{i,j}$ is the excitation potential. Using part (a) we have

$$\frac{\chi_{i,j}}{kT} \approx \gamma \frac{\chi_{i,j}}{\chi_j}.$$

Except for very low-lying states, $\chi_{i,j}$ is of order χ_j, so that the exponential term is very small and $N_{0,j} \gg N_{i,j}$.

9.5

a. The Saha equation has the following form:

$$\frac{N_p N_e}{N_H} \equiv \delta N_e = \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \exp\left(-\frac{\chi_H}{kT}\right).$$

The statistical factor $2U_p/U_H$ is unity. Eliminate N_e by writing it in terms of ρ and δ, using $N_e = N_p$ (neglect H^- and H_2):

$$\begin{aligned} \rho &= N_H m_H + N_e(m_e + m_p) \approx (N_H + N_e) \\ &= m_H N_e(1 + \delta^{-1}) \\ N_e &= \frac{\rho\delta}{m_H(\delta+1)}. \end{aligned}$$

Thus

$$\frac{\delta^2}{\delta+1} \equiv \Delta(\rho, T) = \frac{m_H}{\rho} \lambda^{-3} \exp\left(-\frac{\chi_H}{kT}\right),$$

where λ is the thermal de Broglie wavelength of Problem 9.4.

b. Solving the quadratic equation in δ we obtain

$$\delta = \tfrac{1}{2}\left[\Delta + (\Delta^2 + 4\Delta)^{1/2}\right].$$

10.1—The selection rules are:

1. Configuration changes by exactly one orbital, for which $\Delta l = \pm 1$.
2. $\Delta S = 0$.
3. $\Delta L = \pm 1, 0$.
4. $\Delta J = \pm 1, 0$, except $J = 0$ to $J = 0$.

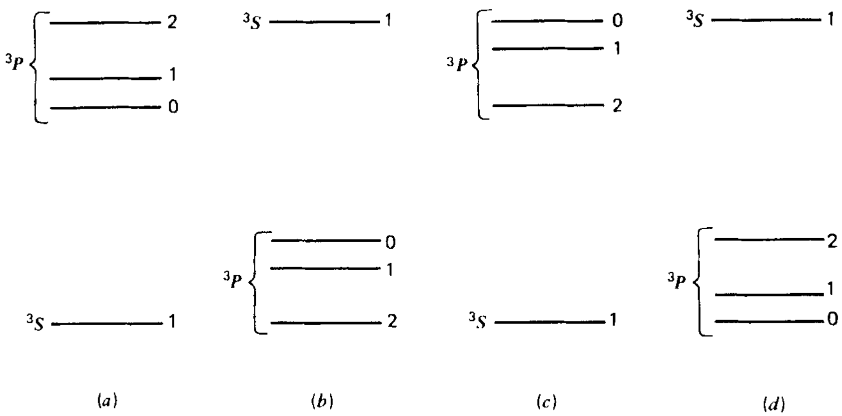

***Figure S.9** Energy spacings for ^3S and ^3P levels when ^3P term is (a) normal, upper, (b) inverted, lower, (c) inverted, upper, (d) normal, lower.*

Without knowing the configurations we can say nothing about (1) because, for example, $1s2s\ ^3S_1 \to 1s2p\ ^3P_{0,1,2}$ is allowed while $1s2s\ ^3S \to 2p3d\ ^3P_{0,1,2}$ is not. Rules (2) and (3) are satisfied, both terms being triplets and $\Delta L = \pm 1$. Finally, all the transitions

$$^3S_1 \to {}^3P_0$$
$$^3S_1 \to {}^3P_1$$
$$^3S_1 \to {}^3P_2$$

satisfy (4) and are therefore allowed.

The four possible arrangements of the two terms with both normal and inverted ^3P term are illustrated in Fig. S.9. Let C be the energy difference between the closest ^3P levels ($J=0$ and $J=1$ levels). Then the energy difference between the other two adjacent levels ($J=1$ and $J=2$) is $2C$, by the Lande interval rule. Also, let ΔE_0 be the energy difference between the $^3S_1$ level and the closest ^3P level, corresponding to the least energetic spectral line.

The energies of the three spectral lines are given in terms of two cases:

1. Normal upper or inverted lower ^3P term (Fig. S.9a and b):

$$^3S_1 \leftrightarrow {}^3P_0, \quad \Delta E = \Delta E_0,$$
$$^3S_1 \leftrightarrow {}^3P_1, \quad \Delta E = \Delta E_0 + C,$$
$$^3S_1 \leftrightarrow {}^3P_2, \quad \Delta E = \Delta E_0 + 3C.$$

2. Normal lower or inverted upper $3P$ term (Fig. S.9c and d):

$$^3S_1 \leftrightarrow {}^3P_2, \quad \Delta E = \Delta E_0,$$

$$^3S_1 \leftrightarrow {}^3P_1, \quad \Delta E = \Delta E_0 + 2C,$$

$$^3S_1 \leftrightarrow {}^3P_0, \quad \Delta E = \Delta E_0 + 3C.$$

10.2

a. $3s\ ^2S_{1/2}$	$\leftrightarrow$	$4s\ ^2S_{1/2}$.	Not allowed. Parity does not change; jumping electron has $\Delta l = 0$.
b. $2p\ ^2P_{1/2}$	$\leftrightarrow$	$3d\ ^2D_{5/2}$.	Not allowed. $\Delta J = 2$.
c. $3s3p\ ^3P_1$	$\leftrightarrow$	$3p^2\ ^1D_2$.	Not allowed. $\Delta S = 1$.
d. $2p3p\ ^3D_1$	$\leftrightarrow$	$3p4d\ ^3F_2$.	Allowed.
e. $2p^2\ ^3P_0$	$\leftrightarrow$	$2p3s\ ^3P_0$.	Not allowed. $J = 0 \rightarrow J = 0$.
f. $3s2p\ ^1P_1$	$\leftrightarrow$	$2p3p\ ^1P_1$.	Allowed.
g. $2s3p\ ^3P_0$	$\leftrightarrow$	$3p4d\ ^3P_1$.	Not allowed. Parity does not change; jumping electron has $\Delta l = 2$.
h. $1s^2\ ^1S_0$	$\leftrightarrow$	$2s2p\ ^1P_1$.	Not allowed. Two electrons jump.
i. $2p3p\ ^3S_1$	$\leftrightarrow$	$2p4d\ ^3D_2$.	Not allowed. $\Delta L = 2$.
j. $2p^3\ ^2D_{3/2}$	$\leftrightarrow$	$2p^3\ ^2D_{1/2}$.	Not allowed. Parity does not change; configuration does not change.

10.3—Comparison of Eqs. (10.10c), (10.23) and (10.29b) shows that the oscillator strength may be written as

$$g_i f_{if} = \sum \frac{2}{3}\frac{m}{\hbar}\omega_{if}|\mathbf{r}_{fi}|^2, \tag{1}$$

where g_i is the statistical weight of the initial state, the sum is over degenerate levels of the initial and final states, ω_{if} is the frequency of the transition, and

$$|\mathbf{r}_{fi}|^2 \equiv \left|\int \psi_f^* \mathbf{r} \psi_i \, d^3r\right|^2. \tag{2}$$

The initial wave function is the $(n,l,m) = (1,0,0)$ ground state of hydrogen, which has the form [cf. Eqs. (9.10) and (9.16)]

$$\psi_{100} = \pi^{-1/2} a_0^{-3/2} e^{-r/a_0}. \tag{3}$$

The final wave function may be any of the three states $(n,l,m)=(2,1,-1)$, $(2,1,0)$, $(2,1,1)$, corresponding to wave functions

$$\begin{aligned}\psi_{21-1}&=r^{-1}R_{21}Y_{1-1},\\ \psi_{210}&=r^{-1}R_{21}Y_{10},\\ \psi_{211}&=r^{-1}R_{21}Y_{11},\end{aligned}\tag{4}$$

where

$$R_{21}\equiv 2^{-3/2}a_0^{-5/2}3^{-1/2}r^2e^{-r/2a_0},\tag{5}$$

and Y_{lm} are the spherical harmonics.

Now, it is easily shown that the operator $|\mathbf{r}_{fi}|^2$ may be written as

$$|\mathbf{r}_{fi}|^2=\tfrac{1}{2}|(x+iy)_{fi}|^2+\tfrac{1}{2}|(x-iy)_{fi}|^2+|z_{fi}|^2,\tag{6}$$

where the latter "spherical operators" are conveniently expressible in terms of spherical harmonics:

$$x\pm iy=r(8\pi/3)^{1/2}Y_{1\pm 1},\tag{7a}$$

$$z=r(4\pi/3)^{1/2}Y_{10}.\tag{7b}$$

Using Eqs. (3) to (7) the position matrix element may then be written as

$$|\mathbf{r}_{fi}|^2=\frac{1}{18}a_0^{-8}\mathcal{R}^2|\mathcal{A}|^2,\tag{8}$$

where

$$\mathcal{R}\equiv\int_0^\infty r^4e^{-3r/2a_0}dr=\left(\tfrac{2}{3}\right)^5 4!\,a_0^5,$$

$$|\mathcal{A}|^2\equiv\left|\int Y^*{}_{1m}Y_{11}d\Omega\right|^2+\left|\int Y^*{}_{1m}Y_{10}d\Omega\right|^2+\left|\int \psi^*{}_{1m}\psi_{1-1}d\Omega\right|^2.$$

Now, by the orthogonality relations of the Y_{lm}, only one term in $|\mathcal{A}|^2$ contributes for each m, and this contribution is unity.

Finally, performing the sum indicated in Eq. (1), that is, summing over $m=0$, ± 1 and multiplying by 2 for the two possible spin states of the initial electron we obtain

$$g_i f_{if} = \frac{2^{17}}{3^{10}} \frac{a_0^2 m\omega_{if}}{\hbar} = \frac{2^{17}}{3^{10}} \frac{\hbar^3 \omega_{if}}{me^4}. \tag{9}$$

Now, ω_{if} is the frequency of transition between the $n=2$ and $n=1$ levels:

$$\omega_{if} = \frac{E_{if}}{\hbar} = \frac{e^2}{2a_0\hbar}\left(1 - \frac{1}{4}\right) = \frac{3}{8}\frac{me^4}{\hbar^3}. \tag{10}$$

Substituting Eq. (10) into (9), we obtain the desired result:

$$g_i f_{if} = \frac{2^{14}}{3^9}.$$

10.4—The initial wave function ψ_i for the $1s$ state of a hydrogen like ion is

$$\psi_i = \left(\frac{Z^3}{\pi a_0^3}\right)^{1/2} e^{-Zr/a_0}.$$

The final wave function ψ_f for the continuum electron in the Born approximation is taken to be the free electron state

$$\psi_f = V^{-1/2} e^{i\mathbf{q}\cdot\mathbf{r}},$$

where $\mathbf{q} = \mathbf{p}/\hbar$ is the wave number of the electron. The normalization $V^{-1/2}$ is consistent with the derivation of Eq. (10.52), the final electron being localized to a volume V.

In the nonrelativistic regime we can neglect the retardation factor exp $(i\mathbf{k}\cdot\mathbf{r})$ in the matrix element (dipole approximation), so we must evaluate $|\langle f|\mathbf{l}\cdot\nabla|i\rangle|^2$. By the Hermitian property this is equivalent to evaluating $|\langle i|\mathbf{l}\cdot\nabla|f\rangle|^2$. Now

$$\langle i|\mathbf{l}\cdot\nabla|f\rangle = \int \psi_i^* \mathbf{l}\cdot\nabla\psi_f \, d^3\mathbf{r}$$

$$= i(\mathbf{l}\cdot\mathbf{q}) \int \psi_i^* \psi_f \, d^3\mathbf{r}.$$

This latter integral can be evaluated by choosing a polar coordinate system with respect to the direction of $\mathbf{q}$. Let θ be the polar angle between $\mathbf{r}$ and $\mathbf{q}$, and let $\mu = \cos\theta$. Then

$$\int \psi_i^* \psi_f d^3x = \left(\frac{Z^3}{\pi V a_0^3}\right)^{1/2} 2\pi \int_0^\infty dr\, r^2 \int_{-1}^{1} d\mu\, e^{-Zr/a_0} e^{iqr\mu},$$

$$= \left(\frac{Z^3}{\pi V a_0^3}\right)^{1/2} \frac{4\pi}{q} \int_0^\infty r\, dr\, e^{-Zr/a_0} \sin qr,$$

$$= 8\pi \left(\frac{Z^5}{\pi V a_0^5}\right)^{1/2} \left[\left(\frac{Z}{a_0}\right)^2 + q^2\right]^{-2},$$

the integrals being elementary. Using these results in the cross-section formula (10.52) we obtain

$$\frac{d\sigma_{bf}}{d\Omega} = \frac{32\alpha\hbar}{m\omega}\left(\frac{Z}{a_0}\right)^5 \frac{q(\mathbf{l}\cdot\mathbf{q})^2}{\left[(Z/a_0)^2 + q^2\right]^4}.$$

When the energy of the electron is large compared to the ionization energy, $\hbar\omega \approx \hbar^2 q^2/2m \gg Z^2 e^2/2a_0$, it follows that $q^2 \gg Z^2/a_0^2$, so that we may write

$$\frac{d\sigma_{bf}}{d\Omega} \to \frac{32\alpha\hbar}{m\omega}\left(\frac{Z}{a_0}\right)^5 \frac{(\mathbf{l}\cdot\mathbf{q})^2}{q^7}.$$

We now integrate this over solid angles to obtain the total cross section. It is convenient to use now polar coordinates with respect to the direction $\mathbf{l}$. Note that

$$\int d\Omega \frac{(\mathbf{l}\cdot\mathbf{q})^2}{q^7} = \frac{2\pi}{q^5}\int_{-1}^{1} d\mu\, \mu^2 = \frac{4\pi}{3q^5}.$$

Therefore, we obtain for the integrated cross section,

$$\sigma_{bf} = \frac{32\alpha\hbar}{m\omega}\left(\frac{Z}{a_0}\right)^5 \frac{4\pi}{3q^5}.$$

Using the relation $\hbar\omega \approx \hbar^2 q^2/2m$ to eliminate q, we obtain finally Eq. (10.53) of the text.

10.5—Since the source is optically thin the spectrum of the emitted radiation is proportional to the emission function (1.73), and thus has the same shape as the profile function $\phi(\nu)$. For this case $\phi(\nu)$ is given by Eq. (10.78), where $\Gamma=\gamma$ is the natural width of Eq. (10.73). In the limiting case of $\Delta\nu_D \ll \gamma/4\pi$ the profile is essentially given by Eq. (10.73), and the observed half-width is independent of temperature. In the limiting case of $\Delta\nu_D \gg \gamma/4\pi$ the profile is essentially given by Eq. (10.68), and the observed half-width grows as the square root of temperature. The critical temperature T_c separating these two cases is found by setting $\Delta\nu_D(T_c)=\gamma/4\pi$. Using Eq. (10.69) we obtain

$$kT_c = \frac{1}{8}(\gamma/\omega_0)^2 m_H c^2, \tag{1}$$

if we assume that hydrogen atoms, $m_a = m_H$, are emitting. For Lyman-α we have $\gamma = A_{21}$ and using Eq. (10.34) we obtain

$$\gamma = \frac{2e^2\omega_0^2}{m_e c^3}\left(\frac{g_1 f_{12}}{g_2}\right)$$

Now $g_1=2$, $g_2=2(2l+1)=6$ for the $2p$ state and $gf=2^{14}/3^9$ from Problem 10.3. The value of ω_0 is found from Eq. (10.42a), with $n=1$ and $n'=2$,

$$\hbar\omega_0 = \frac{3}{4}\frac{e^2}{2a_0} = \frac{3}{8}\frac{m_e e^4}{\hbar^2}.$$

With these values Eq. (1) becomes

$$kT_c = \frac{2^{21}}{3^{18}}\alpha^6 m_H c^2,$$

where $\alpha \equiv e^2/\hbar c$ is the fine structure constant. Numerically,

$$T_c = 8.5\times 10^{-3}\ \mathrm{K}.$$

For most cases of astrophysical interest $T \gg T_c$, and the Doppler broadening dominates, at least near line center. It should be noted, however, that far from line center the Lorentz part of the broadening, which falls off as $(\nu-\nu_0)^{-2}$, will eventually dominate the Doppler part, which falls off extremely rapidly as $\exp[-(\nu-\nu_0)^2/\Delta\nu_D^2]$ [cf. Eqs. (10.77) to (10.79)].

10.6—The dipole matrix elements of $\mathbf{r}\equiv(x,y,z)$ can be conveniently expressed in terms of the matrix elements z_{if} and $(x\pm iy)_{if}$. The matrix element of $z=r\cos\theta$ between states with (l,m) and (l',m') is proportional to

$$\int_{-1}^{+1} P_{l'}^{m'}(\mu)\mu P_l^m(\mu)\,d\mu \int_0^{2\pi} e^{i(m-m')\phi}\,d\phi$$

where $\mu=\cos\theta$ and $P_l^m(\mu)$ is the associated Legendre function. Since the second integral vanishes unless $m'=m$ we need consider only

$$\int_{-1}^{1} P_{l'}^{m}(\mu)\mu P_l^m(\mu)\,d\mu.$$

The recurrence relation

$$(2l+1)\mu P_l^m=(l-m+1)P_{l+1}^m+(l+m)P_{l-1}^m$$

and the orthogonality relations for the P_l^m (see, e.g., Arfken, G. 1970, *Mathematical Methods for Physicists*, Academic, New York) imply that

$$z_{if}=0, \text{ unless } m'=m \text{ and } l'=l+1 \text{ or } l'=l-1.$$

The matrix element of $(x\pm iy)=r\sin\theta\, e^{\pm i\phi}$ is proportional to

$$\int_{-1}^{+1} P_{l'}^{m'}(\mu)\sqrt{1-\mu^2}\; P_l^m\,d\mu \int_0^{2\pi} e^{i(m-m'\pm 1)\phi}\,d\phi,$$

which vanishes unless $m'=m+1$ or $m'=m-1$. If $m'=m+1$ we use the recurrence relation $(2l+1)\sqrt{1-\mu^2}\; P_l^{m-1}=P_{l-1}^m-P_{l+1}^m$ and the orthogonality relations to show that

$$(x+iy)_{if}=0, \text{ unless } m'=m+1 \text{ and } l'=l+1 \text{ or } l'=l-1.$$

Similarly, we show that

$$(x-iy)_{if}=0, \text{ unless } m'=m-1 \text{ and } l'=l+1 \text{ or } l'=l-1.$$

Taken together, these results imply the electric dipole selection rules: $\Delta m=0,\pm 1$ and $\Delta l=\pm 1$.

10.7—Let $\phi_c(t)$ be the collision-induced random phase at time t. Then the electric field will be

$$E(t)=Ae^{i\omega_0 t-\gamma t/2+i\phi_c(t)}, \qquad (1)$$

where A is a constant, ω_0 is the fundamental frequency, and γ is the rate of spontaneous decay. We wish to compute the averaged power spectrum

$$\langle|\hat{E}(\omega)|\rangle^2 = \left\langle \left| \int E(t) e^{i\omega t} dt \right|^2 \right\rangle. \tag{2}$$

From Eq. (1), we have

$$\begin{aligned} |\hat{E}(\omega)|^2 &= |A|^2 \int_0^\infty \int_0^\infty dt_1 \, dt_2 \, e^{i(\omega - \omega_0)(t_1 - t_2) - \frac{\gamma}{2}(t_1 + t_2)} \\ &\quad \cdot e^{i[\phi_c(t_1) - \phi_c(t_2)]} \\ &\equiv |A|^2 \int_0^\infty \int_0^\infty dt_1 \, dt_2 \, G(t_1, t_2) e^{i[\phi_c(t_1) - \phi_c(t_2)]}. \end{aligned} \tag{3}$$

Now, the only random function in $|\hat{E}(\omega)|^2$ is $\phi_c(t)$. Thus we obtain

$$\langle|\hat{E}(\omega)|^2\rangle = |A|^2 \int \int dt_1 \, dt_2 \, G(t_1, t_2) \langle e^{i[\phi_c(t_1) - \phi_c(t_2)]} \rangle. \tag{4}$$

Now, we can write

$$\phi_c(t_1) - \phi_c(t_2) = \Delta\phi_c(t_1 - t_2)$$

where $\Delta\phi_c(t_1 - t_2)$ is the change of phase during the time interval $t_1 - t_2$, and we wish to compute

$$\langle e^{i\Delta\phi_c(t_1 - t_2)} \rangle.$$

Since changes in phase are random, this average vanishes if one or more collisions occurs during $\Delta t \equiv |t_1 - t_2|$ and is unity if no collisions occur during this interval. We are given the mean rate of collisions is ν_{col}, thus implying that the probability for no collisions to occur during Δt is $e^{-|t_1 - t_2|\nu_{\text{col}}}$ (assuming a Poisson distribution for the collisions). Thus

$$\langle e^{i\Delta\phi_c(t_1 - t_2)} \rangle = e^{-|t_1 - t_2|\nu_{\text{col}}}, \tag{5}$$

and Eq. (4) becomes

$$\langle|\hat{E}(\omega)|\rangle^2 = |A|^2 \int_0^\infty dt_1 \int_0^\infty dt_2 \, G(t_1, t_2) e^{-|t_1 - t_2|\nu_{\text{col}}}. \tag{6}$$

Equation (6), using Eq. (3) for $G(t_1 t_2)$, can be integrated in terms of elementary functions and yields Eq. (10.75).

11.1

a. In order of magnitude the equilibrium separation is the Bohr radius of an atom, since the electron binding is what holds the two positive charges together. So

$$\mathbf{r}_0 \sim a_0 \equiv \frac{\hbar^2}{me^2}.$$

b. Since the molecules will be electrically neutral in the temperature range considered, they can be treated approximately as hard spheres of size $\sim r_0 \sim a_0$. The cross section is thus the simple geometrical form for the area

$$\sigma \sim \pi r_0^2 \sim \pi a_0^2.$$

c. For Doppler broadening, the line width, $\Delta\nu_D$, is

$$\Delta\nu_D = \frac{\nu_0}{c}\sqrt{\frac{2kT}{3M_p}} \sim \frac{E_{\text{rot}}}{hc}\sqrt{\frac{2kT}{3M_p}}\ .$$

For collisional broadening, the line width, $\Delta\nu_c$, is the collision frequency ν_{col}. From part (b), we estimate

$$\begin{aligned}\nu_{\text{col}} &= n\sigma_{\text{col}}\langle v\rangle \\ &\sim (\rho/M_p)a_0^2\sqrt{kT/M_p}\ .\end{aligned}$$

For low ρ, the line width is dominated by $\Delta\nu_D$ and is thus independent of ρ. At high ρ, $\nu_{\text{col}} \gg \Delta\nu_D$ and the line width increases in proportion to ρ. The transition occurs at a ρ_0 such that

$$\nu_{\text{col}}(\rho_0) \sim \Delta\nu_D$$

or

$$\frac{\rho_0}{M_p}a_0^2 \sim \frac{E_{\text{rot}}}{hc} \sim \left(\frac{\hbar^2}{M_p a_0^2}\right)\frac{1}{hc}$$

or

$$\rho_0 \sim \frac{\hbar}{ca_0^4} = \alpha\frac{m_e}{a_0^3} \sim 5\times 10^{-5}\ \text{g cm}^{-3}.$$

11.2

$$S(R) \equiv \pi^{-1} \int e^{-|\mathbf{r}-\mathbf{R}_A|} e^{-|\mathbf{r}-\mathbf{R}_B|} r^2 \, dr \, d\cos\theta \, d\phi.$$

Make the changes of variables $\mathbf{R} \equiv \mathbf{R}_A - \mathbf{R}_B$, $y \equiv |\mathbf{r} - \mathbf{R}_A|/R$, $x \equiv \cos\theta$, where θ is the angle between $(\mathbf{r} - \mathbf{R}_A)$ and $\mathbf{R}$. Then, after doing the trivial ϕ integral to obtain a factor 2π, one gets

$$S(R) = 2R^3 \int_{-1}^{1} dx \int_0^{\infty} y^2 \, dy \, e^{-R(y + \sqrt{y^2 + 2yx + 1}\,)}.$$

Now, reverse the order of integration, doing the x integral first, and make the change of variables $a^2 \equiv y^2 + 2yx + 1$, $dx = y^{-1} \, a da$, to obtain

$$S(R) = 2R^3 \int_0^{\infty} e^{-Ry} y \, dy \int_{|y-1|}^{y+1} e^{-Ra} a \, da.$$

This elementary double integral is easily done by (e.g., integration by parts) to yield Eq. (11.12).

11.3—Regard the $H(J)$ term in the expression for $\omega_{n\upsilon J}$ as a function of α_n and $\alpha_{n'}$, written in the form

$$H(J) = j_1(J)(\alpha_n + \alpha_{n'}) + j_2(J)(\alpha_n - \alpha_{n'}),$$

where the "coefficients" $j_1(J)$ and $j_2(J)$ are to be determined by equating the above expression to Eq. (11.42) for $H(J)$. For the P branch, the two resulting equations for $j_1(J)$ and $j_2(J)$ yield

$$j_1(J) = -(J+1) \equiv j$$
$$j_2(J) = j^2.$$

As J ranges over its allowed positive values, j ranges over negative integer values, $j \leqslant -1$. For the R branch the two resulting equations for $j_1(J)$ and $j_2(J)$ yield

$$j_1(J) = J \equiv j,$$
$$j_2(J) = j^2$$

and j ranges over positive integer values $j \geqslant 1$. Combining the two formulae yields

$$H(J) = j(\alpha_n + \alpha_{n'}) + j^2(\alpha_n - \alpha_{n'}),$$

where j ranges over negative and positive integers, excluding zero, the "band origin."

11.4—Regarding j as a continuous variable in the result of Problem 11.3, we find the band head (i.e., the reversal of line spacing with increasing j) by setting the derivative of $H(J)$ equal to 0, $\partial H/\partial j|_{j_{\text{head}}}=0$, giving

$$j_{\text{head}}=\frac{1}{2}\left(\frac{\alpha_n+\alpha_{n'}}{\alpha_{n'}-\alpha_n}\right).$$

If $j_{\text{head}}<0$ (i.e., $\alpha_{n'}<\alpha_n$), the band head clearly falls in the P branch. If $j_{\text{head}}>0$, (i.e., $\alpha_n<\alpha_{n'}$), the band head falls in the R branch. Generally, j_{head}, as deduced above, is not an integer, so that the true value of the band head corresponds to the nearest integer value to j_{head}. From j_{head}, the value of J_{head} may be deduced from the solution to Problem 11.3, that is,

$$\begin{aligned} J_{\text{head}}&\sim j_{\text{head}} && \text{for } j_{\text{head}}>0, \\ &&& (R \text{ branch}) \\ J_{\text{head}}&\sim -(1+j_{\text{head}}) && \text{for } j_{\text{head}}<0 \\ &&& (P \text{ branch}) \end{aligned}$$

The frequency of the band head is found by substituting j_{head} into the expression for $H(j)$:

$$\omega_{nJ}|_{\text{head}}=\omega_0-\frac{1}{4}\frac{(\alpha_n+\alpha_{n'})^2}{(\alpha_n-\alpha_{n'})}.$$

The band head frequency is below or above ω_0, depending on whether $\alpha_n>\alpha_{n'}$ or $\alpha_n<\alpha_{n'}$.

11.5—Using the same arguments as in Problem 11.4, we see that the Q branch would have a band head at the J value satisfying

$$\frac{\partial H(J)}{\partial J}=0,$$

or, using Eq. (11.42b) for the Q branch form of $H(J)$,

$$J_{\text{head}}=-1/2.$$

Since J only has positive integer values, this band head is never actually realized. However, this value is sufficiently near $J=1$ that the Q branch (at low resolution), at $J=1$, resembles a band head.

11.6—Rotational energy levels have energies

$$E_{\text{rot}}=\frac{\hbar^2}{2I}J(J+1),$$

where $I\sim m_p r_0^2 \sim m_p a_0^2$, and vibrational and electronic transitions have energies

$$E_{\text{vib}}\sim\left(\frac{m_p}{m_e}\right)^{\frac{1}{2}}E_{\text{rot}},$$

$$E_{\text{elec}}\sim\left(\frac{m_p}{m_e}\right)E_{\text{rot}}.$$

The probability of a given energy level being occupied is proportional to $\exp(-E/kT)$. Thus if $kT\lesssim E_{\text{rot}}(J=2)$, most molecules will be in the $J=1$ rotational ground states, and few rotational transitions can occur. On the other hand, if $kT\gtrsim E_{\text{vib}}$, vibrational levels will be excited, and rotational-vibrational spectra will be produced. To have pure rotation spectra produced, then, one requires

$$\frac{\hbar^2}{m_p a_0^2}\ll kT\ll\frac{\hbar^2}{m_p a_0^2}\left(\frac{m_p}{m_e}\right)^{\frac{1}{2}}.$$