Differential geometry of surfaces in space

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By definition, free surfaces are not constrained to conform to the level surface of a fixed coordinate system (e.g., planes, cylinders, and spheres). Rather, their position in space is determined such that the governing conservation laws of physics (e.g., that of mass and momentum) are satisfied. It is for this reason that the study of free-surface flow phenomena necessitates a certain level of comfort with the mathematics of differentiable manifolds. Unfortunately, this subject is not commonly taught in modern engineering curricula. Moreover, most of the useful references that exist are written using a notation that, while familiar to mathematical physicists, may appear foreign to engineers and fluid mechanicians. The purpose of this document is to present some useful results from differential geometry using a notation that should be familiar to the reader without having to “water down” the content. For a more complete discussion, the reader is referred to [1, 2].

1 Review of vector and tensor analysis

A working knowledge of the algebra and calculus of three-dimensional vectors and tensors is assumed here. A comprehensive introduction can be found in Chapters 2, 3, and 7 of [3]. In our convention, *scalars* will be denoted by italicized Latin and Greek letters; *vectors* will be denoted by boldfaced, italicized Latin letters; (second-order) *tensors* will be denoted by boldfaced, italicized Greek letters. For example,

$$\begin{aligned} &\psi \quad \text{is a scalar,} \\ &\boldsymbol{v} \quad \text{is a vector (or first-order tensor),} \\ &\boldsymbol{\tau} \quad \text{is a second-order tensor.} \end{aligned}$$

On occasion, we shall also use a boldfaced, calligraphic letter to denote a tensorial quantity, e.g., $\boldsymbol{\mathcal{A}}$ is a second-order tensor.

With any three-dimensional vector space comes a triad of unit basis vectors $\hat{\boldsymbol{e}}_{(1)}, \hat{\boldsymbol{e}}_{(2)}, \hat{\boldsymbol{e}}_{(3)}$ that together define a right-handed, orthonormal coordinate system. Examples of particular relevant to mathematical physics are,

$$\begin{aligned} &\text{the Cartesian basis } (\hat{\boldsymbol{e}}_x, \hat{\boldsymbol{e}}_y, \hat{\boldsymbol{e}}_z), \\ &\text{the cylindrical basis } (\hat{\boldsymbol{e}}_\rho, \hat{\boldsymbol{e}}_\phi, \hat{\boldsymbol{e}}_z), \\ &\text{the spherical basis } (\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\theta, \hat{\boldsymbol{e}}_\phi). \end{aligned}$$

Vectors and tensors may be expressed in any chosen basis and their components converted between bases by the usual conventions of orthogonal transformations [3]. The calculus of vectors and tensors on a differentiable manifold is facilitated by the gradient or "del" operator $\boldsymbol{\nabla}$, which is defined in the usual fashion. A detailed background on basis systems for orthogonal curvilinear coordinates can be found in the appendix of [4, 5].

The most important example of a second-order tensor is the so-called *unit tensor* $\boldsymbol{\delta}$, whose indexed components are defined by the dot product of any two unit basis vectors,

$$\delta_{ij} = \hat{\boldsymbol{e}}_{(i)} \cdot \hat{\boldsymbol{e}}_{(j)} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \tag{1.1}$$

with i, j taking values of 1, 2, or 3. The unit tensor is so named because its dot product with any vector $\boldsymbol{v}$ returns the vector unchanged, e.g., $\boldsymbol{\delta} \cdot \boldsymbol{v} = \boldsymbol{v}$. Furthermore, the inverse of a (nonsingular) second-order tensor $\boldsymbol{\tau}$ is defined such that

$$\boldsymbol{\tau}^{-1} \cdot \boldsymbol{\tau} = \boldsymbol{\tau} \cdot \boldsymbol{\tau}^{-1} = \boldsymbol{\delta}. \tag{1.2}$$

The transpose of a tensor is denoted by $\boldsymbol{\tau}^\dagger$. A second-order tensor is *symmetric* if $\boldsymbol{\tau}^\dagger = \boldsymbol{\tau}$. Any second-order tensor $\boldsymbol{\tau}$ may be characterized by its three principal scalar invariants, namely,

$$\begin{aligned} &\text{the trace } \operatorname{tr}(\boldsymbol{\tau}) = \delta_{ij}\tau_{ji} = \tau_{ii}, \\ &\text{the sum of principal minors } \tfrac{1}{2}\left[\operatorname{tr}^2(\boldsymbol{\tau}) - \operatorname{tr}(\boldsymbol{\tau}^\dagger \cdot \boldsymbol{\tau})\right] = \tfrac{1}{2}(\tau_{ii}\tau_{jj} - \tau_{ij}\tau_{ji}), \\ &\text{the determinant } \det(\boldsymbol{\tau}) = \tfrac{1}{6}\epsilon_{ijk}\epsilon_{lmn}\tau_{il}\tau_{jm}\tau_{kn}, \end{aligned}$$

where summation is implied over repeated indices. Tensors $\boldsymbol{\tau}$ for which $\det(\boldsymbol{\tau}) = 0$ are called *degenerate*, and in such cases the inverse $\boldsymbol{\tau}^{-1}$ appearing in (1.2) does not formally exist.

Tensors of higher order will occasionally appear. Unfortunately, there is no standard convention in Gibbs notation that distinguishes second-order tensors from tensors of higher order. It is usually necessary to switch to Einstein notation, wherein a tensor is represented by its indexed components (making the order of the tensor obvious) and summation is implied over repeated indices. Boldfaced Greek and calligraphic letters are used to represent higher-order tensors in Gibbs notation, and in such cases direct reference is made to the order of the tensor so as to distinguish it from tensors of lower order. The most prevalent example of a third-order tensor is the so-called *alternating tensor* $\boldsymbol{\epsilon}$. The indexed components of $\boldsymbol{\epsilon}$ are defined by the scalar triple product of any three basis vectors,

$$\epsilon_{ijk} = \hat{\boldsymbol{e}}_{(i)} \cdot (\hat{\boldsymbol{e}}_{(j)} \times \hat{\boldsymbol{e}}_{(k)}) = \begin{cases} 1 & \text{if } ijk \text{ is a positive permutation,} \\ -1 & \text{if } ijk \text{ is a negative permutation,} \\ 0 & \text{otherwise,} \end{cases} \tag{1.3}$$

with i, j, k taking values of 1, 2, or 3. Clearly, the alternating tensor $\boldsymbol{\epsilon}$ is a *pseudotensor* – that is, its components change sign under an orientation-reversing coordinate transformation.

We conclude this section with a presentation of key integral theorems from vector and tensor calculus. Consider first a surface $\mathscr{S}$ with bounding contour $\partial\mathscr{S}$. Suppose that a vector field $\boldsymbol{v}$ is everywhere regular in $\mathscr{S}$. *Stokes' curl theorem* states,

$$\int_{\partial\mathscr{S}} \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{s} = \int_{\mathscr{S}} (\boldsymbol{\nabla}\times\boldsymbol{v})\cdot \mathrm{d}\boldsymbol{S}, \tag{1.4}$$

which, in physical terms, relates the flux of material through a contour to the circulation in the surface. Now consider a volume $\mathscr{V}$ with bounding surface $\partial\mathscr{V}$ and let $\boldsymbol{v}$ be regular within the volume. *Gauss' divergence theorem* states,

$$\int_{\partial\mathscr{V}} \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{S} = \int_{\mathscr{V}} (\boldsymbol{\nabla}\cdot\boldsymbol{v})\, \mathrm{d}^3\boldsymbol{x}, \tag{1.5}$$

which, in physical terms, relates the flux of material through a surface to the dilatation in the volume. Although (1.4) and (1.5) are written for a vector $\boldsymbol{v}$, in fact these relations may be trivially generalized to tensors of higher order.

2 Dual bases for a surface

A *surface* is a two-parameter set of contiguous positions in space $\boldsymbol{x}_\mathrm{s}(\xi,\eta)$, where ξ and η are the local surface coordinates. We demarcate the location of the surface in space by the vector equation,

$$\boldsymbol{x} = \boldsymbol{x}_\mathrm{s}(\xi,\eta), \tag{2.1}$$

where $\boldsymbol{x}$ is the position vector. Technically, we may allow the locations of the surface points to also vary parametrically with time t, but for the sake of brevity this dependence is suppressed. As an alternative to (2.1), we may define the level set function $\varphi(\boldsymbol{x})$ such that,*

$$\varphi(\boldsymbol{x}) = 0 \quad \text{at} \quad \boldsymbol{x} = \boldsymbol{x}_\mathrm{s}. \tag{2.2}$$

As an example, the thickness of a falling film adhered to an inclined slab may be easily represented in Cartesian coordinates by the level set function $\varphi(x,y,z) = z - z_\mathrm{s}(x,y)$, with $\xi = x$ and $\eta = y$ (here, z is the coordinate normal to the slab and z_s is the film thickness).

From analyzing how the surface positions $\boldsymbol{x}_\mathrm{s}$ vary with ξ and η, one can derive geometrical properties of the surface. The (generally non-unit) tangential vectors of the surface are obtained by taking partial derivatives of the surface position $\boldsymbol{x}_\mathrm{s}$ with respect to the surface coordinates ξ and η:

$$\boldsymbol{t}_{(1)} = \frac{\partial \boldsymbol{x}_\mathrm{s}}{\partial\xi}, \qquad \boldsymbol{t}_{(2)} = \frac{\partial \boldsymbol{x}_\mathrm{s}}{\partial\eta}. \tag{2.3}$$

Strictly speaking, these are the *covariant* tangential vectors – that is, their magnitude changes in proportion to the change of length scale of the local surface coordinate. As an example, suppose that $\xi = R\theta$ and $\eta = R\phi$ measure the polar and azimuthal arc lengths, respectively, on the surface of a sphere of radius R. The tangential vectors for this surface are $\boldsymbol{t}_{(1)} = R\hat{\boldsymbol{e}}_\theta$ and $\boldsymbol{t}_{(2)} = R\sin\theta\,\hat{\boldsymbol{e}}_\phi$. Now if the radius of the sphere is doubled from R to $R' = 2R$ (that is, the length scale of the surface coordinate has increased by a factor of two), then the tangential vectors $\boldsymbol{t}_{(1)}$ and $\boldsymbol{t}_{(2)}$ will double in magnitude. This response to "stretching" the surface coordinate axes is "co-varying" and hence the term covariant.

Armed solely with the tangential vectors $\boldsymbol{t}_{(1)}$ and $\boldsymbol{t}_{(2)}$, we can already calculate some important geometric properties of the surface. For the purposes of line integration, we define the line elements,

$$\mathrm{d}s_1 = |\boldsymbol{t}_{(1)}|\,\mathrm{d}\xi, \qquad \mathrm{d}s_2 = |\boldsymbol{t}_{(2)}|\,\mathrm{d}\eta. \tag{2.4}$$

* One has to be careful with defining properties of a surface with respect to φ, which is a function of three coordinates! To be clear, the addendum "at $\boldsymbol{x} = \boldsymbol{x}_\mathrm{s}$" is always used to indicate that the property of interest is evaluated on the surface after performing operations in three dimensions (an important example being the gradient $\boldsymbol{\nabla}\varphi$).

For surface integration, we define the surface element,

$$\mathrm{d}S = |\boldsymbol{t}_{(1)} \times \boldsymbol{t}_{(2)}| \,\mathrm{d}\xi \,\mathrm{d}\eta. \tag{2.5}$$

Thus, the covariant tangential vectors provide measures of *distance* on the surface with respect to variations in the surface coordinates.

In order to form a complete basis in three dimensions, one requires a third vector that is orthogonal to both $\boldsymbol{t}_{(1)}$ and $\boldsymbol{t}_{(2)}$. Hence, we define the unit normal vector of the surface as,

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{t}_{(1)} \times \boldsymbol{t}_{(2)}}{|\boldsymbol{t}_{(1)} \times \boldsymbol{t}_{(2)}|}. \tag{2.6}$$

By construction, $|\hat{\boldsymbol{n}}| = 1$. Unlike $\boldsymbol{t}_{(1)}$ and $\boldsymbol{t}_{(2)}$, the unit normal $\hat{\boldsymbol{n}}$ is *invariant* (its magnitude does not change) with respect to a stretching of the surface coordinate axes. In the course of solving problems, it may be desirable to reverse the order of $\boldsymbol{t}_{(1)}$ and $\boldsymbol{t}_{(2)}$ so that $\hat{\boldsymbol{n}}$ points outward from the surface. In some textbooks, it is common to see the unit normal defined with respect to the level set function (2.2),

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{\nabla}\varphi}{|\boldsymbol{\nabla}\varphi|} \quad \text{at} \quad \boldsymbol{x} = \boldsymbol{x}_{\mathrm{s}}, \tag{2.7}$$

which is a perfectly suitable definition.

The covariant basis $(\boldsymbol{t}_{(1)}, \boldsymbol{t}_{(2)}, \hat{\boldsymbol{n}})$ need not be orthonormal (or even orthogonal). Thus, it is necessary to define a reciprocal or *contravariant* basis $(\boldsymbol{t}^{(1)}, \boldsymbol{t}^{(2)}, \hat{\boldsymbol{n}})$. As the term "contravariant" suggests, the magnitude of $\boldsymbol{t}^{(1)}$ and $\boldsymbol{t}^{(2)}$ is reduced by one-half when the scale of the surface coordinate axes is doubled. By construction, the unit normal $\hat{\boldsymbol{n}}$ is orthogonal to $\boldsymbol{t}^{(1)}$ and $\boldsymbol{t}^{(2)}$ and hence is a member of the reciprocal basis. The contravariant tangential vectors are defined by the simultaneous equations,

$$\begin{aligned} \boldsymbol{t}^{(1)} \cdot \boldsymbol{t}_{(1)} &= 1, \qquad & \boldsymbol{t}^{(1)} \cdot \boldsymbol{t}_{(2)} &= 0, \\ \boldsymbol{t}^{(2)} \cdot \boldsymbol{t}_{(1)} &= 0, \qquad & \boldsymbol{t}^{(2)} \cdot \boldsymbol{t}_{(2)} &= 1, \end{aligned} \tag{2.8}$$

whose solution is given by,

$$\boldsymbol{t}^{(1)} = \frac{\boldsymbol{t}_{(2)} \times \hat{\boldsymbol{n}}}{G_{\mathrm{s}}}, \qquad \boldsymbol{t}^{(2)} = \frac{\hat{\boldsymbol{n}} \times \boldsymbol{t}_{(1)}}{G_{\mathrm{s}}}. \tag{2.9}$$

The scalar G_{s} appearing in (2.9) is called the *surface metric* and is defined by the scalar triple product,

$$G_{\mathrm{s}} = (\boldsymbol{t}_{(1)} \times \boldsymbol{t}_{(2)}) \cdot \hat{\boldsymbol{n}} = |\boldsymbol{t}_{(1)} \times \boldsymbol{t}_{(2)}|. \tag{2.10}$$

From (2.5) and (2.10), it is clear that G_{s} provides a measure of the local change in surface area with respect to variation of the surface coordinates. Taking again the example of a sphere of radius R, we find the contravariant tangential vectors to be $\boldsymbol{t}^{(1)} = \hat{\boldsymbol{e}}_\theta / R$ and $\boldsymbol{t}^{(2)} = \hat{\boldsymbol{e}}_\phi/(R\sin\theta)$. The surface metric for the same surface is $G_{\mathrm{s}} = R^2 \sin\theta$.

Taken together, the dual bases $(\boldsymbol{t}_{(1)}, \boldsymbol{t}_{(2)}, \hat{\boldsymbol{n}})$ and $(\boldsymbol{t}^{(1)}, \boldsymbol{t}^{(2)}, \hat{\boldsymbol{n}})$ supply most of the geometrical information about a surface. Namely, one can define the *surface unit tensor* $\boldsymbol{\delta}_{\mathrm{s}}$ by any one of the identities,

$$\boldsymbol{\delta}_{\mathrm{s}} = \boldsymbol{t}^{(1)}\boldsymbol{t}_{(1)} + \boldsymbol{t}^{(2)}\boldsymbol{t}_{(2)} \tag{2.11a}$$

$$= \boldsymbol{t}_{(1)}\boldsymbol{t}^{(1)} + \boldsymbol{t}_{(2)}\boldsymbol{t}^{(2)} \tag{2.11b}$$

$$= \boldsymbol{\delta} - \hat{\boldsymbol{n}}\hat{\boldsymbol{n}} \quad \text{at} \quad \boldsymbol{x} = \boldsymbol{x}_{\mathrm{s}}. \tag{2.11c}$$

Several comments are in order. First, from the last expression (2.11c) it is clear that $\boldsymbol{\delta}_{\mathrm{s}}$ acts as a surface projection operator – that is, the projection $\boldsymbol{\delta}_{\mathrm{s}} \cdot \boldsymbol{v}$ removes the components of $\boldsymbol{v}$ that are normal to the surface. Second, it can be readily verified that $\boldsymbol{\delta}_{\mathrm{s}}$ is symmetric and invariant with respect to stretching of the surface coordinate axes. Third, it is clear that $\boldsymbol{\delta}_{\mathrm{s}}$ degenerates in three dimensions, which precludes the definition of an inverse according to (1.2). We will return to this conundrum in §5 during our discussion of surface curvature.

The surface unit tensor $\boldsymbol{\delta}_{\rm s}$ is the two-dimensional analog of the unit tensor $\boldsymbol{\delta}$, defined by (1.1). There also exists an analog of the alternating tensor $\boldsymbol{\epsilon}$, defined by (1.3). Thus, we define the *surface alternating tensor* $\boldsymbol{\epsilon}_{\rm s}$ as follows:

$$\boldsymbol{\epsilon}_{\rm s} = G_{\rm s}(\boldsymbol{t}^{(1)}\boldsymbol{t}^{(2)} - \boldsymbol{t}^{(2)}\boldsymbol{t}^{(1)}) \tag{2.12a}$$

$$= \frac{1}{G_{\rm s}}(\boldsymbol{t}_{(1)}\boldsymbol{t}_{(2)} - \boldsymbol{t}_{(2)}\boldsymbol{t}_{(1)}) \tag{2.12b}$$

$$= \boldsymbol{\epsilon}\cdot\hat{\boldsymbol{n}} \quad \text{at} \quad \boldsymbol{x} = \boldsymbol{x}_{\rm s}. \tag{2.12c}$$

It should be noticed that $\boldsymbol{\epsilon}_{\rm s}$ is antisymmetric, invariant, and non-invertible according to (1.2).

3 Surface components of vectors and tensors

Having established the dual bases of the surface, we can express vectorial and tensorial quantities in terms of either of these bases. Contraction of a vector $\boldsymbol{v}$ with either $\boldsymbol{t}_{(\alpha)}$ or $\boldsymbol{t}^{(\alpha)}$ ($\alpha = 1$ or 2) returns, respectively, the covariant and contravariant tangential components of that vector. However, we are most interested in the *invariant components* of $\boldsymbol{v}$ (also called the *physical components*) with respect to the local surface basis. Since $\hat{\boldsymbol{n}}$ is an invariant quantity, the invariant component of $\boldsymbol{v}$ in the normal direction is simply given by $\hat{\boldsymbol{n}}\cdot\boldsymbol{v}$. Similar invariant components may be defined in the tangential directions via multiplication by the appropriate Jacobian $\partial s_1/\partial\xi$ or $\partial s_2/\partial\eta$ [see equation (2.4)]. Thus, we define the surface components of $\boldsymbol{v}$ by,

normal: $$v_n = \hat{\boldsymbol{n}}\cdot\boldsymbol{v}, \tag{3.1a}$$

tangent along s_1: $$v_{s_1} = |\boldsymbol{t}_{(1)}|(\boldsymbol{t}^{(1)}\cdot\boldsymbol{v}) = |\boldsymbol{t}_{(1)}|[(\boldsymbol{t}^{(1)}\cdot\boldsymbol{t}^{(1)})(\boldsymbol{t}_{(1)}\cdot\boldsymbol{v}) + (\boldsymbol{t}^{(1)}\cdot\boldsymbol{t}^{(2)})(\boldsymbol{t}_{(2)}\cdot\boldsymbol{v})], \tag{3.1b}$$

tangent along s_2: $$v_{s_2} = |\boldsymbol{t}_{(2)}|(\boldsymbol{t}^{(2)}\cdot\boldsymbol{v}) = |\boldsymbol{t}_{(2)}|[(\boldsymbol{t}^{(2)}\cdot\boldsymbol{t}^{(1)})(\boldsymbol{t}_{(1)}\cdot\boldsymbol{v}) + (\boldsymbol{t}^{(2)}\cdot\boldsymbol{t}^{(2)})(\boldsymbol{t}_{(2)}\cdot\boldsymbol{v})], \tag{3.1c}$$

where v_n, v_{s_1}, and v_{s_2} are all invariant. The vector $\boldsymbol{v}$ is then represented in terms of the surface basis by the vector equation,

$$\boldsymbol{v} = v_n\hat{\boldsymbol{n}} + v_{s_1}\frac{\boldsymbol{t}_{(1)}}{|\boldsymbol{t}_{(1)}|} + v_{s_2}\frac{\boldsymbol{t}_{(2)}}{|\boldsymbol{t}_{(2)}|}. \tag{3.2}$$

Similar arguments lead to the surface components of a second-order tensor $\boldsymbol{\tau}$. We need not list all of them; for our purposes, the surface components of the vector $\boldsymbol{\tau}\cdot\hat{\boldsymbol{n}}$ are of most practical value:

normal: $$\tau_{nn} = \hat{\boldsymbol{n}}\cdot\boldsymbol{\tau}\cdot\hat{\boldsymbol{n}}, \tag{3.3}$$

tangent along s_1: $$\tau_{s_1 n} = |\boldsymbol{t}_{(1)}|(\boldsymbol{t}^{(1)}\cdot\boldsymbol{\tau}\cdot\hat{\boldsymbol{n}}) = |\boldsymbol{t}_{(1)}|[(\boldsymbol{t}^{(1)}\cdot\boldsymbol{t}^{(1)})(\boldsymbol{t}_{(1)}\cdot\boldsymbol{\tau}\cdot\hat{\boldsymbol{n}}) + (\boldsymbol{t}^{(1)}\cdot\boldsymbol{t}^{(2)})(\boldsymbol{t}_{(2)}\cdot\boldsymbol{\tau}\cdot\hat{\boldsymbol{n}})], \tag{3.4}$$

tangent along s_2: $$\tau_{s_2 n} = |\boldsymbol{t}_{(2)}|(\boldsymbol{t}^{(2)}\cdot\boldsymbol{\tau}\cdot\hat{\boldsymbol{n}}) = |\boldsymbol{t}_{(2)}|[(\boldsymbol{t}^{(2)}\cdot\boldsymbol{t}^{(1)})(\boldsymbol{t}_{(1)}\cdot\boldsymbol{\tau}\cdot\hat{\boldsymbol{n}}) + (\boldsymbol{t}^{(2)}\cdot\boldsymbol{t}^{(2)})(\boldsymbol{t}_{(2)}\cdot\boldsymbol{\tau}\cdot\hat{\boldsymbol{n}})]. \tag{3.5}$$

We may represent the vector $\boldsymbol{\tau}\cdot\hat{\boldsymbol{n}}$ as,

$$\boldsymbol{\tau}\cdot\hat{\boldsymbol{n}} = \tau_{nn}\hat{\boldsymbol{n}} + \tau_{s_1 n}\frac{\boldsymbol{t}_{(1)}}{|\boldsymbol{t}_{(1)}|} + \tau_{s_2 n}\frac{\boldsymbol{t}_{(2)}}{|\boldsymbol{t}_{(2)}|}. \tag{3.6}$$

4 Surface gradient

Variations of a field quantity (say ψ, $\boldsymbol{v}$, or $\boldsymbol{\tau}$) with respect to changes in the local surface coordinates ξ and η are measured by the surface gradient operator $\boldsymbol{\nabla}_{\rm s}$:

$$\boldsymbol{\nabla}_{\rm s} = \boldsymbol{t}^{(1)}\frac{\partial}{\partial\xi} + \boldsymbol{t}^{(2)}\frac{\partial}{\partial\eta}. \tag{4.1}$$

This operator is related to the usual gradient operator $\boldsymbol{\nabla}$ by,

$$\boldsymbol{\nabla}_{\rm s} = \boldsymbol{\delta}_{\rm s}\cdot\boldsymbol{\nabla} \quad \text{at} \quad \boldsymbol{x} = \boldsymbol{x}_{\rm s}, \tag{4.2}$$

where the $\boldsymbol{\delta}_\mathrm{s}$-operation has the effect of removing gradients in the direction normal to the surface. An important property of $\boldsymbol{\nabla}_\mathrm{s}$ is that it is invariant with respect to a stretching of the surface coordinate axes. This can be seen from the fact that $\boldsymbol{t}^{(1)}$ and $\boldsymbol{t}^{(2)}$ are both contravariant, whereas the partial derivatives $\partial/\partial\xi$ and $\partial/\partial\eta$ are covariant. The product of a contravariant quantity with a covariant quantity is an invariant quantity; hence, $\boldsymbol{\nabla}_\mathrm{s}$ is invariant.

We now consider variations of fields over the surface. Clearly, the surface gradient of a scalar field ψ is given by,

$$\boldsymbol{\nabla}_\mathrm{s}\psi = \boldsymbol{t}^{(1)}\frac{\partial\psi}{\partial\xi} + \boldsymbol{t}^{(2)}\frac{\partial\psi}{\partial\eta}. \tag{4.3}$$

Now consider the surface divergence of a vector field $\boldsymbol{v} = \boldsymbol{t}_{(1)}(\boldsymbol{t}^{(1)}\cdot\boldsymbol{v}) + \boldsymbol{t}_{(2)}(\boldsymbol{t}^{(2)}\cdot\boldsymbol{v}) + \hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\cdot\boldsymbol{v})$ (p. 187 of [2]):

$$\begin{aligned}\boldsymbol{\nabla}_\mathrm{s}\cdot\boldsymbol{v} &= \boldsymbol{\nabla}_\mathrm{s}\cdot\left[\boldsymbol{t}_{(1)}(\boldsymbol{t}^{(1)}\cdot\boldsymbol{v}) + \boldsymbol{t}_{(2)}(\boldsymbol{t}^{(2)}\cdot\boldsymbol{v}) + \hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\cdot\boldsymbol{v})\right] \\ &= \frac{1}{G_\mathrm{s}}\frac{\partial}{\partial\xi}\left(G_\mathrm{s}\boldsymbol{t}^{(1)}\cdot\boldsymbol{v}\right) + \frac{1}{G_\mathrm{s}}\frac{\partial}{\partial\eta}\left(G_\mathrm{s}\boldsymbol{t}^{(2)}\cdot\boldsymbol{v}\right) + (\boldsymbol{\nabla}_\mathrm{s}\cdot\hat{\boldsymbol{n}})(\hat{\boldsymbol{n}}\cdot\boldsymbol{v}).\end{aligned} \tag{4.4}$$

where the identity $\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}_\mathrm{s} = 0$ has been used. Equation (4.4) expresses the divergence of $\boldsymbol{v}$ in terms of its tangential and normal components and can be trivially extended to tensors of higher order. By combining this expression with (4.3), we can easily derive an expression for the surface Laplacian of a scalar field ψ,

$$\nabla_\mathrm{s}^2\psi = \boldsymbol{\nabla}_\mathrm{s}\cdot\boldsymbol{\nabla}_\mathrm{s}\psi = \frac{1}{G_\mathrm{s}}\frac{\partial}{\partial\xi}\left[G_\mathrm{s}\boldsymbol{t}^{(1)}\cdot\left(\boldsymbol{t}^{(1)}\frac{\partial\psi}{\partial\xi} + \boldsymbol{t}^{(2)}\frac{\partial\psi}{\partial\eta}\right)\right] + \frac{1}{G_\mathrm{s}}\frac{\partial}{\partial\eta}\left[G_\mathrm{s}\boldsymbol{t}^{(2)}\cdot\left(\boldsymbol{t}^{(1)}\frac{\partial\psi}{\partial\xi} + \boldsymbol{t}^{(2)}\frac{\partial\psi}{\partial\eta}\right)\right]. \tag{4.5}$$

We may also consider the surface curl, which has two definitions. First, we may define the surface curl of a scalar field ψ,

$$\hat{\boldsymbol{n}}\times\boldsymbol{\nabla}_\mathrm{s}\psi = -\boldsymbol{\nabla}_\mathrm{s}\times(\psi\hat{\boldsymbol{n}}) = \frac{1}{G_\mathrm{s}}\left(\boldsymbol{t}_{(2)}\frac{\partial\psi}{\partial\xi} - \boldsymbol{t}_{(1)}\frac{\partial\psi}{\partial\eta}\right), \tag{4.6}$$

the result of which is a vector lying in the (ξ,η)-plane. Second, we define the surface curl of a vector field $\boldsymbol{v}$,

$$(\boldsymbol{\nabla}_\mathrm{s}\times\boldsymbol{v})\cdot\hat{\boldsymbol{n}} = \boldsymbol{\nabla}_\mathrm{s}\cdot(\boldsymbol{v}\times\hat{\boldsymbol{n}}) = \frac{1}{G_\mathrm{s}}\frac{\partial}{\partial\xi}\left(\boldsymbol{t}_{(2)}\cdot\boldsymbol{v}\right) - \frac{1}{G_\mathrm{s}}\frac{\partial}{\partial\eta}\left(\boldsymbol{t}_{(1)}\cdot\boldsymbol{v}\right), \tag{4.7}$$

which is a scalar. The latter expression is related to the *surface circulation* of the vector field $\boldsymbol{v}$.

In the last two equations (4.6)-(4.7), we exploited the fact that the unit normal vector $\hat{\boldsymbol{n}}$ is irrotational in the (ξ,η)-plane:

$$\boldsymbol{\nabla}_\mathrm{s}\times\hat{\boldsymbol{n}} = \boldsymbol{0}. \tag{4.8}$$

A corollary of (4.8) is that the surface divergence of the surface alternating tensor $\boldsymbol{\epsilon}_\mathrm{s}$ vanishes identically,

$$\boldsymbol{\nabla}_\mathrm{s}\cdot\boldsymbol{\epsilon}_\mathrm{s} = \boldsymbol{0}. \tag{4.9}$$

By contrast, the surface divergence of $\boldsymbol{\delta}_\mathrm{s}$ does not vanish, viz.,

$$\boldsymbol{\nabla}_\mathrm{s}\cdot\boldsymbol{\delta}_\mathrm{s} = -(\boldsymbol{\nabla}_\mathrm{s}\cdot\hat{\boldsymbol{n}})\hat{\boldsymbol{n}}. \tag{4.10}$$

Physically, this expression gives the magnitude and direction of surface dilatation resulting from variations in the unit normal $\hat{\boldsymbol{n}}$ along the surface. The scalar quantity $-(\boldsymbol{\nabla}_\mathrm{s}\cdot\hat{\boldsymbol{n}})$ is related to the *curvature* of the surface, which is the subject of the next section.

5 Surface curvature

At the end of the last section, we noted that the surface divergence of $\boldsymbol{\delta}_\mathrm{s}$ does not vanish due to the fact that $\hat{\boldsymbol{n}}$ varies with surface position. As a consequence, the quantity $-(\boldsymbol{\nabla}_\mathrm{s}\cdot\hat{\boldsymbol{n}})$ appears on the right-hand side of (4.4) and (4.10). In fact, this quantity is one of the principal invariants of the *surface curvature tensor*,

$$\boldsymbol{\kappa}_\mathrm{s} = -\boldsymbol{\nabla}_\mathrm{s}\hat{\boldsymbol{n}}, \tag{5.1}$$

which, together with the surface unit tensor $\boldsymbol{\delta}_{\mathrm{s}}$, completely species all of the geometric information about the surface.

It is instructive to examine the properties of the surface curvature tensor $\boldsymbol{\kappa}_{\mathrm{s}}$. First, it can be readily verified that $\boldsymbol{\kappa}_{\mathrm{s}}$ is symmetric and invariant with respect to stretching of the surface coordinate axes. Second, by (4.8) it follows that

$$\hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla}_{\mathrm{s}} \hat{\boldsymbol{n}} = \mathbf{0} \quad \text{and} \quad (\boldsymbol{\nabla}_{\mathrm{s}} \hat{\boldsymbol{n}}) \cdot \hat{\boldsymbol{n}} = \mathbf{0}, \tag{5.2}$$

which implies that $\boldsymbol{\kappa}_{\mathrm{s}}$, like $\boldsymbol{\delta}_{\mathrm{s}}$ and $\boldsymbol{\epsilon}_{\mathrm{s}}$, degenerates in three dimensions. Its two non-vanishing eigenvalues κ_1 and κ_2 are called the *principal curvatures* of the surface. They satisfy the characteristic quadratic,

$$\kappa_\alpha^2 - 2H\kappa_\alpha + K = 0 \quad (\alpha = 1 \text{ or } 2), \tag{5.3}$$

where

$$2H = \kappa_1 + \kappa_2 = \operatorname{tr}(\boldsymbol{\kappa}_{\mathrm{s}}) = -\boldsymbol{\nabla}_{\mathrm{s}} \cdot \hat{\boldsymbol{n}}, \tag{5.4a}$$

$$2K = 2\kappa_1\kappa_2 = \operatorname{tr}^2(\boldsymbol{\kappa}_{\mathrm{s}}) - \operatorname{tr}(\boldsymbol{\kappa}_{\mathrm{s}} \cdot \boldsymbol{\kappa}_{\mathrm{s}}) = (\boldsymbol{\nabla}_{\mathrm{s}} \cdot \hat{\boldsymbol{n}})^2 - \boldsymbol{\nabla}_{\mathrm{s}}\hat{\boldsymbol{n}} : \boldsymbol{\nabla}_{\mathrm{s}}\hat{\boldsymbol{n}}. \tag{5.4b}$$

The principal scalar invariants H and K are the *mean and Gaussian curvatures* of the surface, respectively. They may be alternatively expressed in terms of the level set function $\varphi(\boldsymbol{x})$ as follows:

$$2H = -\boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{\nabla}\varphi}{|\boldsymbol{\nabla}\varphi|} \right), \tag{5.5a}$$

$$2K = \left[\boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{\nabla}\varphi}{|\boldsymbol{\nabla}\varphi|} \right) \right]^2 - \boldsymbol{\nabla} \left(\frac{\boldsymbol{\nabla}\varphi}{|\boldsymbol{\nabla}\varphi|} \right) : \boldsymbol{\nabla} \left(\frac{\boldsymbol{\nabla}\varphi}{|\boldsymbol{\nabla}\varphi|} \right). \tag{5.5b}$$

These definitions follow from the fact that $\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}} = 1$ and hence $\boldsymbol{\nabla}(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}}) = 2(\boldsymbol{\nabla}\hat{\boldsymbol{n}}) \cdot \hat{\boldsymbol{n}} = \mathbf{0}$. Making use of (5.2), it follows that

$$\boldsymbol{\nabla}_{\mathrm{s}} \cdot \hat{\boldsymbol{n}} = \boldsymbol{\nabla} \cdot \hat{\boldsymbol{n}}, \quad \boldsymbol{\nabla}_{\mathrm{s}}\hat{\boldsymbol{n}} : \boldsymbol{\nabla}_{\mathrm{s}}\hat{\boldsymbol{n}} = \boldsymbol{\nabla}\hat{\boldsymbol{n}} : \boldsymbol{\nabla}\hat{\boldsymbol{n}} \quad \text{at} \quad \boldsymbol{x} = \boldsymbol{x}_{\mathrm{s}}.$$

Inserting these expressions into (5.4) and making use of (2.7) then leads directly to (5.5).

Since H and K are the only non-vanishing principal scalar invariants of $\boldsymbol{\kappa}_{\mathrm{s}}$, then the Cayleigh-Hamilton theorem states,

$$\boldsymbol{\kappa}_{\mathrm{s}} \cdot \boldsymbol{\kappa}_{\mathrm{s}} - 2H\boldsymbol{\kappa}_{\mathrm{s}} + K\boldsymbol{\delta}_{\mathrm{s}} = \mathbf{0}, \tag{5.6}$$

which is the tensorial analog of (5.3). Taking the trace of (5.6) and rearranging the result leads to the following identity:

$$2D^2 = 2(H^2 - K) = \tfrac{1}{2}(\kappa_1 - \kappa_2)^2 = \boldsymbol{\nabla}_{\mathrm{s}}\hat{\boldsymbol{n}} : \boldsymbol{\nabla}_{\mathrm{s}}\hat{\boldsymbol{n}} - \tfrac{1}{2}(\boldsymbol{\nabla}_{\mathrm{s}} \cdot \hat{\boldsymbol{n}})^2, \tag{5.7}$$

where D is the so-called *deviatoric curvature*. Clearly, D is not independent of H and K and in fact they are related by (5.7). The sign of D may be chosen arbitrarily.

Equation (5.6) implies that a suitable inverse of $\boldsymbol{\kappa}_{\mathrm{s}}$ exists in two dimensions provided that $K \neq 0$, despite the fact that $\boldsymbol{\kappa}_{\mathrm{s}}$ is degenerate in three dimensions. If we restrict our attention to the (ξ, η)-plane, then the inverse tensor $\boldsymbol{\kappa}_{\mathrm{s}}^{-1}$ may be defined to satisfy the relation,

$$\boldsymbol{\kappa}_{\mathrm{s}}^{-1} \cdot \boldsymbol{\kappa}_{\mathrm{s}} = \boldsymbol{\kappa}_{\mathrm{s}} \cdot \boldsymbol{\kappa}_{\mathrm{s}}^{-1} = \boldsymbol{\delta}_{\mathrm{s}}. \tag{5.8}$$

Comparing (5.8) to (1.2) shows this definition to be something of a bastardization of the tensorial inverse. Nevertheless, the quantity $\boldsymbol{\kappa}_{\mathrm{s}}^{-1}$, which had no meaning in three dimensions, is now well defined according to (5.8). Thus, we shall restrict our use of this “liberal” definition of the inverse to our discussion of two-dimensional tensors.

We concluded the previous section with an expression for the surface divergence of $\boldsymbol{\delta}_{\mathrm{s}}$, which may now be rewritten in terms of the mean curvature H:

$$\boldsymbol{\nabla}_{\mathrm{s}} \cdot \boldsymbol{\delta}_{\mathrm{s}} = 2H\hat{\boldsymbol{n}}. \tag{4.10}$$

We conclude this section with an analogous expression for the surface divergence of $\boldsymbol{\kappa}_{\mathrm{s}}$:

$$\boldsymbol{\nabla}_{\mathrm{s}} \cdot \boldsymbol{\kappa}_{\mathrm{s}} = \boldsymbol{\nabla}_{\mathrm{s}}(2H), \tag{5.9}$$

which can be readily verified by exploiting the symmetry of $\boldsymbol{\kappa}_{\mathrm{s}}$.

6 Surface divergence theorem

The last topic in our discussion of surfaces is the analog of the divergence theorem in two dimensions. Consider a vector field $\boldsymbol{v}$ defined (and everywhere regular) on a surface $\mathscr{S}$ with unit normal $\hat{\boldsymbol{n}}$ and bounding contour $\partial\mathscr{S}$. Our starting point is Stokes' curl theorem (1.4) applied to the vector field $\boldsymbol{v}\times\hat{\boldsymbol{n}}$:

$$\int_{\partial\mathscr{S}} (\boldsymbol{v}\times\hat{\boldsymbol{n}})\cdot\left(\boldsymbol{t}_{(1)}\mathrm{d}\xi+\boldsymbol{t}_{(2)}\mathrm{d}\eta\right)=\int_{\mathscr{S}}[\boldsymbol{\nabla}\times(\boldsymbol{v}\times\hat{\boldsymbol{n}})]\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S,$$

where we have substituted $\mathrm{d}\boldsymbol{s}=\boldsymbol{t}_{(1)}\mathrm{d}\xi+\boldsymbol{t}_{(2)}\mathrm{d}\eta$ and $\mathrm{d}\boldsymbol{S}=\hat{\boldsymbol{n}}\,\mathrm{d}S$. Now using elementary vector identities, we may rewrite this expression as,

$$\int_{\partial\mathscr{S}} \boldsymbol{v}\cdot\left[(\hat{\boldsymbol{n}}\times\boldsymbol{t}_{(1)})\mathrm{d}\xi+(\hat{\boldsymbol{n}}\times\boldsymbol{t}_{(2)})\mathrm{d}\eta\right]=\int_{\mathscr{S}}[(\hat{\boldsymbol{n}}\cdot\boldsymbol{v})(\boldsymbol{\nabla}\cdot\hat{\boldsymbol{n}})-\boldsymbol{v}\cdot(\boldsymbol{\nabla}\hat{\boldsymbol{n}})\cdot\hat{\boldsymbol{n}}-(\boldsymbol{\delta}-\hat{\boldsymbol{n}}\hat{\boldsymbol{n}}):\boldsymbol{\nabla}\boldsymbol{v}]\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S.$$

But $\boldsymbol{x}=\boldsymbol{x}_\mathrm{s}$ on $\mathscr{S}$, so we may use (2.9) and (2.11c) as well as the identities $\boldsymbol{\nabla}\cdot\hat{\boldsymbol{n}}=\boldsymbol{\nabla}_\mathrm{s}\cdot\hat{\boldsymbol{n}}$ and $(\boldsymbol{\nabla}\hat{\boldsymbol{n}})\cdot\hat{\boldsymbol{n}}=\frac{1}{2}\boldsymbol{\nabla}(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}})=\boldsymbol{0}$. We obtain,

$$\int_{\partial\mathscr{S}} G_\mathrm{s}\boldsymbol{v}\cdot\left(\boldsymbol{t}^{(1)}\mathrm{d}\eta-\boldsymbol{t}^{(2)}\mathrm{d}\xi\right)=\int_{\mathscr{S}}\left(2H\hat{\boldsymbol{n}}\cdot\boldsymbol{v}+\boldsymbol{\nabla}_\mathrm{s}\cdot\boldsymbol{v}\right)\,\mathrm{d}S. \tag{6.1}$$

The last expression is known as the *surface divergence theorem* [3]. It can be readily generalized to tensors of higher order.

7 Surface geometry in various coordinate systems

7.1 Cartesian coordinates, $z=z_\mathrm{s}(x,y)$

Level set function:

$$\varphi(x,y,z)=z-z_\mathrm{s}(x,y)=0\quad\text{on the surface.} \tag{7.1}$$

Surface position vector:

$$\boldsymbol{x}_\mathrm{s}(x,y)=\hat{\boldsymbol{e}}_x x+\hat{\boldsymbol{e}}_y y+\hat{\boldsymbol{e}}_z z_\mathrm{s}(x,y). \tag{7.2}$$

Surface metric:

$$G_\mathrm{s}=\sqrt{1+\left(\frac{\partial z_\mathrm{s}}{\partial x}\right)^2+\left(\frac{\partial z_\mathrm{s}}{\partial y}\right)^2}. \tag{7.3}$$

Unit normal vector:

$$\hat{\boldsymbol{n}}=\frac{1}{G_\mathrm{s}}\left(\hat{\boldsymbol{e}}_z-\hat{\boldsymbol{e}}_x\frac{\partial z_\mathrm{s}}{\partial x}-\hat{\boldsymbol{e}}_y\frac{\partial z_\mathrm{s}}{\partial y}\right). \tag{7.4}$$

Covariant tangential vectors:

$$\boldsymbol{t}_{(1)}=\hat{\boldsymbol{e}}_x+\hat{\boldsymbol{e}}_z\frac{\partial z_\mathrm{s}}{\partial x}, \tag{7.5a}$$

$$\boldsymbol{t}_{(2)}=\hat{\boldsymbol{e}}_y+\hat{\boldsymbol{e}}_z\frac{\partial z_\mathrm{s}}{\partial y}. \tag{7.5b}$$

Contravariant tangential vectors:

$$\boldsymbol{t}^{(1)}=\frac{1}{G_\mathrm{s}^2}\left\{\hat{\boldsymbol{e}}_x\left[1+\left(\frac{\partial z_\mathrm{s}}{\partial y}\right)^2\right]-\hat{\boldsymbol{e}}_y\frac{\partial z_\mathrm{s}}{\partial x}\frac{\partial z_\mathrm{s}}{\partial y}+\hat{\boldsymbol{e}}_z\frac{\partial z_\mathrm{s}}{\partial x}\right\}, \tag{7.6a}$$

$$\boldsymbol{t}^{(2)}=\frac{1}{G_\mathrm{s}^2}\left\{\hat{\boldsymbol{e}}_y\left[1+\left(\frac{\partial z_\mathrm{s}}{\partial x}\right)^2\right]-\hat{\boldsymbol{e}}_x\frac{\partial z_\mathrm{s}}{\partial x}\frac{\partial z_\mathrm{s}}{\partial y}+\hat{\boldsymbol{e}}_z\frac{\partial z_\mathrm{s}}{\partial y}\right\}. \tag{7.6b}$$

Line elements:

$$\mathrm{d}s_1 = \sqrt{1 + \left(\frac{\partial z_\mathrm{s}}{\partial x}\right)^2}\,\mathrm{d}x, \tag{7.7a}$$

$$\mathrm{d}s_2 = \sqrt{1 + \left(\frac{\partial z_\mathrm{s}}{\partial y}\right)^2}\,\mathrm{d}y. \tag{7.7b}$$

Surface element:

$$\mathrm{d}S = \sqrt{1 + \left(\frac{\partial z_\mathrm{s}}{\partial x}\right)^2 + \left(\frac{\partial z_\mathrm{s}}{\partial y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y. \tag{7.8}$$

Mean curvature:

$$H = \frac{1}{2G_\mathrm{s}^3}\left\{\frac{\partial^2 z_\mathrm{s}}{\partial x^2}\left[1 + \left(\frac{\partial z_\mathrm{s}}{\partial y}\right)^2\right] + \frac{\partial^2 z_\mathrm{s}}{\partial y^2}\left[1 + \left(\frac{\partial z_\mathrm{s}}{\partial x}\right)^2\right] - 2\frac{\partial z_\mathrm{s}}{\partial x}\frac{\partial z_\mathrm{s}}{\partial y}\frac{\partial^2 z_\mathrm{s}}{\partial x \partial y}\right\}. \tag{7.9}$$

Gaussian curvature:

$$K = \frac{1}{G_\mathrm{s}^4}\left[\frac{\partial^2 z_\mathrm{s}}{\partial x^2}\frac{\partial^2 z_\mathrm{s}}{\partial y^2} - \left(\frac{\partial^2 z_\mathrm{s}}{\partial x \partial y}\right)^2\right]. \tag{7.10}$$

Surface components of $\boldsymbol{v}$:

$$v_n = \frac{1}{G_\mathrm{s}}\left(v_z - v_x\frac{\partial z_\mathrm{s}}{\partial x} - v_y\frac{\partial z_\mathrm{s}}{\partial y}\right), \tag{7.11a}$$

$$v_{s_1} = \frac{\sqrt{1 + (\partial z_\mathrm{s}/\partial x)^2}}{G_\mathrm{s}^2}\left\{v_x\left[1 + \left(\frac{\partial z_\mathrm{s}}{\partial y}\right)^2\right] - v_y\frac{\partial z_\mathrm{s}}{\partial x}\frac{\partial z_\mathrm{s}}{\partial y} + v_z\frac{\partial z_\mathrm{s}}{\partial x}\right\}, \tag{7.11b}$$

$$v_{s_2} = \frac{\sqrt{1 + (\partial z_\mathrm{s}/\partial y)^2}}{G_\mathrm{s}^2}\left\{v_y\left[1 + \left(\frac{\partial z_\mathrm{s}}{\partial x}\right)^2\right] - v_x\frac{\partial z_\mathrm{s}}{\partial x}\frac{\partial z_\mathrm{s}}{\partial y} + v_z\frac{\partial z_\mathrm{s}}{\partial y}\right\}. \tag{7.11c}$$

Surface components of $\boldsymbol{\tau}\cdot\hat{\boldsymbol{n}}$:

$$\begin{aligned}\tau_{nn} = \frac{1}{G_\mathrm{s}^2}\Bigg[&\tau_{zz} - \tau_{zx}\frac{\partial z_\mathrm{s}}{\partial x} - \tau_{zy}\frac{\partial z_\mathrm{s}}{\partial y} - \left(\tau_{xz} - \tau_{xx}\frac{\partial z_\mathrm{s}}{\partial x} - \tau_{xy}\frac{\partial z_\mathrm{s}}{\partial y}\right)\frac{\partial z_\mathrm{s}}{\partial x} \\ &- \left(\tau_{yz} - \tau_{yx}\frac{\partial z_\mathrm{s}}{\partial x} - \tau_{yy}\frac{\partial z_\mathrm{s}}{\partial y}\right)\frac{\partial z_\mathrm{s}}{\partial y}\Bigg],\end{aligned} \tag{7.12a}$$

$$\begin{aligned}\tau_{s_1 n} = \frac{\sqrt{1 + (\partial z_\mathrm{s}/\partial x)^2}}{G_\mathrm{s}^3}\Bigg\{&\left(\tau_{xz} - \tau_{xx}\frac{\partial z_\mathrm{s}}{\partial x} - \tau_{xy}\frac{\partial z_\mathrm{s}}{\partial y}\right)\left[1 + \left(\frac{\partial z_\mathrm{s}}{\partial y}\right)^2\right] \\ &- \left(\tau_{yz} - \tau_{yx}\frac{\partial z_\mathrm{s}}{\partial x} - \tau_{yy}\frac{\partial z_\mathrm{s}}{\partial y}\right)\frac{\partial z_\mathrm{s}}{\partial x}\frac{\partial z_\mathrm{s}}{\partial y} + \left(\tau_{zz} - \tau_{zx}\frac{\partial z_\mathrm{s}}{\partial x} - \tau_{zy}\frac{\partial z_\mathrm{s}}{\partial y}\right)\frac{\partial z_\mathrm{s}}{\partial x}\Bigg\},\end{aligned} \tag{7.12b}$$

$$\begin{aligned}\tau_{s_2 n} = \frac{\sqrt{1 + (\partial z_\mathrm{s}/\partial y)^2}}{G_\mathrm{s}^3}\Bigg\{&\left(\tau_{yz} - \tau_{yx}\frac{\partial z_\mathrm{s}}{\partial x} - \tau_{yy}\frac{\partial z_\mathrm{s}}{\partial y}\right)\left[1 + \left(\frac{\partial z_\mathrm{s}}{\partial x}\right)^2\right] \\ &- \left(\tau_{xz} - \tau_{xx}\frac{\partial z_\mathrm{s}}{\partial x} - \tau_{xy}\frac{\partial z_\mathrm{s}}{\partial y}\right)\frac{\partial z_\mathrm{s}}{\partial x}\frac{\partial z_\mathrm{s}}{\partial y} + \left(\tau_{zz} - \tau_{zx}\frac{\partial z_\mathrm{s}}{\partial x} - \tau_{zy}\frac{\partial z_\mathrm{s}}{\partial y}\right)\frac{\partial z_\mathrm{s}}{\partial y}\Bigg\}.\end{aligned} \tag{7.12c}$$

Surface gradient of ψ:

$$\begin{aligned}\boldsymbol{\nabla}_\mathrm{s}\psi = &\frac{1}{G_\mathrm{s}^2}\left\{\hat{\boldsymbol{e}}_x\left[1 + \left(\frac{\partial z_\mathrm{s}}{\partial y}\right)^2\right] - \hat{\boldsymbol{e}}_y\frac{\partial z_\mathrm{s}}{\partial x}\frac{\partial z_\mathrm{s}}{\partial y} + \hat{\boldsymbol{e}}_z\frac{\partial z_\mathrm{s}}{\partial x}\right\}\frac{\partial \psi}{\partial x} \\ &+ \frac{1}{G_\mathrm{s}^2}\left\{\hat{\boldsymbol{e}}_y\left[1 + \left(\frac{\partial z_\mathrm{s}}{\partial x}\right)^2\right] - \hat{\boldsymbol{e}}_x\frac{\partial z_\mathrm{s}}{\partial x}\frac{\partial z_\mathrm{s}}{\partial y} + \hat{\boldsymbol{e}}_z\frac{\partial z_\mathrm{s}}{\partial y}\right\}\frac{\partial \psi}{\partial y}.\end{aligned} \tag{7.13}$$

Surface divergence of $\boldsymbol{v}$:

$$\boldsymbol{\nabla}_{\mathrm{s}} \cdot \boldsymbol{v} = \frac{1}{G_{\mathrm{s}}} \frac{\partial}{\partial x} \left(\frac{1}{G_{\mathrm{s}}} \left\{ v_x \left[1 + \left(\frac{\partial z_{\mathrm{s}}}{\partial y} \right)^2 \right] - v_y \frac{\partial z_{\mathrm{s}}}{\partial x} \frac{\partial z_{\mathrm{s}}}{\partial y} + v_z \frac{\partial z_{\mathrm{s}}}{\partial x} \right\} \right)$$
$$+ \frac{1}{G_{\mathrm{s}}} \frac{\partial}{\partial y} \left(\frac{1}{G_{\mathrm{s}}} \left\{ v_y \left[1 + \left(\frac{\partial z_{\mathrm{s}}}{\partial x} \right)^2 \right] - v_x \frac{\partial z_{\mathrm{s}}}{\partial x} \frac{\partial z_{\mathrm{s}}}{\partial y} + v_z \frac{\partial z_{\mathrm{s}}}{\partial y} \right\} \right)$$
$$- \frac{2H}{G_{\mathrm{s}}} \left(v_z - v_x \frac{\partial z_{\mathrm{s}}}{\partial x} - v_y \frac{\partial z_{\mathrm{s}}}{\partial y} \right), \tag{7.14a}$$
$$= \frac{1}{G_{\mathrm{s}}} \frac{\partial}{\partial x} \left(G_{\mathrm{s}} v_x \right) + \frac{1}{G_{\mathrm{s}}} \frac{\partial}{\partial y} \left(G_{\mathrm{s}} v_y \right) \quad \text{if } v_n = 0. \tag{7.14b}$$

Surface Laplacian of ψ:

$$\nabla_{\mathrm{s}}^2 \psi = \frac{1}{G_{\mathrm{s}}} \frac{\partial}{\partial x} \left(\frac{1}{G_{\mathrm{s}}} \left\{ \left[1 + \left(\frac{\partial z_{\mathrm{s}}}{\partial y} \right)^2 \right] \frac{\partial \psi}{\partial x} - \frac{\partial z_{\mathrm{s}}}{\partial x} \frac{\partial z_{\mathrm{s}}}{\partial y} \frac{\partial \psi}{\partial y} \right\} \right)$$
$$+ \frac{1}{G_{\mathrm{s}}} \frac{\partial}{\partial y} \left(\frac{1}{G_{\mathrm{s}}} \left\{ \left[1 + \left(\frac{\partial z_{\mathrm{s}}}{\partial x} \right)^2 \right] \frac{\partial \psi}{\partial y} - \frac{\partial z_{\mathrm{s}}}{\partial x} \frac{\partial z_{\mathrm{s}}}{\partial y} \frac{\partial \psi}{\partial x} \right\} \right). \tag{7.15}$$

Surface curl of ψ:

$$\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}_{\mathrm{s}} \psi = -\boldsymbol{\nabla}_{\mathrm{s}} \times (\psi \hat{\boldsymbol{n}}) = \frac{1}{G_{\mathrm{s}}} \left[\left(\hat{\boldsymbol{e}}_y + \hat{\boldsymbol{e}}_z \frac{\partial z_{\mathrm{s}}}{\partial y} \right) \frac{\partial \psi}{\partial x} - \left(\hat{\boldsymbol{e}}_x + \hat{\boldsymbol{e}}_z \frac{\partial z_{\mathrm{s}}}{\partial x} \right) \frac{\partial \psi}{\partial y} \right]. \tag{7.16}$$

Surface curl of $\boldsymbol{v}$:

$$(\boldsymbol{\nabla}_{\mathrm{s}} \times \boldsymbol{v}) \cdot \hat{\boldsymbol{n}} = \boldsymbol{\nabla}_{\mathrm{s}} \cdot (\boldsymbol{v} \times \hat{\boldsymbol{n}}) = \frac{1}{G_{\mathrm{s}}} \frac{\partial}{\partial x} \left(v_y + v_z \frac{\partial z_{\mathrm{s}}}{\partial y} \right) - \frac{1}{G_{\mathrm{s}}} \frac{\partial}{\partial y} \left(v_x + v_z \frac{\partial z_{\mathrm{s}}}{\partial x} \right). \tag{7.17}$$

7.2 Cylindrical coordinates, $\rho = \rho_{\mathrm{s}}(\phi, z)$

Level set function:

$$\varphi(\rho, \phi, z) = \rho - \rho_{\mathrm{s}}(\phi, z) = 0 \quad \text{on the surface.} \tag{7.18}$$

Surface position vector:

$$\boldsymbol{x}_{\mathrm{s}}(\phi, z) = \hat{\boldsymbol{e}}_\rho(\phi)\, \rho_{\mathrm{s}}(\phi, z) + \hat{\boldsymbol{e}}_z z. \tag{7.19}$$

Surface metric:

$$G_{\mathrm{s}} = \sqrt{\left(\frac{\partial \rho_{\mathrm{s}}}{\partial \phi} \right)^2 + \rho_{\mathrm{s}}^2 \left[1 + \left(\frac{\partial \rho_{\mathrm{s}}}{\partial z} \right)^2 \right]}. \tag{7.20}$$

Unit normal vector:

$$\hat{\boldsymbol{n}} = \frac{1}{G_{\mathrm{s}}} \left(\hat{\boldsymbol{e}}_\rho \rho_{\mathrm{s}} - \hat{\boldsymbol{e}}_\phi \frac{\partial \rho_{\mathrm{s}}}{\partial \phi} - \hat{\boldsymbol{e}}_z \rho_{\mathrm{s}} \frac{\partial \rho_{\mathrm{s}}}{\partial z} \right). \tag{7.21}$$

Covariant tangential vectors:

$$\boldsymbol{t}_{(1)} = \hat{\boldsymbol{e}}_\phi \rho_{\mathrm{s}} + \hat{\boldsymbol{e}}_\rho \frac{\partial \rho_{\mathrm{s}}}{\partial \phi}, \tag{7.22a}$$
$$\boldsymbol{t}_{(2)} = \hat{\boldsymbol{e}}_z + \hat{\boldsymbol{e}}_\rho \frac{\partial \rho_{\mathrm{s}}}{\partial z}. \tag{7.22b}$$

Contravariant tangential vectors:

$$\boldsymbol{t}^{(1)} = \frac{1}{G_{\mathrm{s}}^2} \left\{ \hat{\boldsymbol{e}}_\phi \rho_{\mathrm{s}} \left[1 + \left(\frac{\partial \rho_{\mathrm{s}}}{\partial z} \right)^2 \right] - \hat{\boldsymbol{e}}_z \frac{\partial \rho_{\mathrm{s}}}{\partial z} \frac{\partial \rho_{\mathrm{s}}}{\partial \phi} + \hat{\boldsymbol{e}}_\rho \frac{\partial \rho_{\mathrm{s}}}{\partial \phi} \right\}, \tag{7.23a}$$
$$\boldsymbol{t}^{(2)} = \frac{1}{G_{\mathrm{s}}^2} \left\{ \hat{\boldsymbol{e}}_z \left[\rho_{\mathrm{s}}^2 + \left(\frac{\partial \rho_{\mathrm{s}}}{\partial \phi} \right)^2 \right] - \hat{\boldsymbol{e}}_\phi \rho_{\mathrm{s}} \frac{\partial \rho_{\mathrm{s}}}{\partial z} \frac{\partial \rho_{\mathrm{s}}}{\partial \phi} + \hat{\boldsymbol{e}}_\rho \rho_{\mathrm{s}}^2 \frac{\partial \rho_{\mathrm{s}}}{\partial z} \right\}. \tag{7.23b}$$

Line elements:

$$\mathrm{d}s_1 = \sqrt{\rho_{\mathrm{s}}^2 + \left(\frac{\partial \rho_{\mathrm{s}}}{\partial \phi}\right)^2}\, \mathrm{d}\phi, \tag{7.24a}$$

$$\mathrm{d}s_2 = \sqrt{1 + \left(\frac{\partial \rho_{\mathrm{s}}}{\partial z}\right)^2}\, \mathrm{d}z. \tag{7.24b}$$

Surface element:

$$\mathrm{d}S = \sqrt{\left(\frac{\partial \rho_{\mathrm{s}}}{\partial \phi}\right)^2 + \rho_{\mathrm{s}}^2 \left[1 + \left(\frac{\partial \rho_{\mathrm{s}}}{\partial z}\right)^2\right]}\, \mathrm{d}\phi\, \mathrm{d}z. \tag{7.25}$$

Mean curvature:

$$H = \frac{1}{2G_{\mathrm{s}}^3} \left\{ \rho_{\mathrm{s}} \frac{\partial^2 \rho_{\mathrm{s}}}{\partial z^2} \left[\rho_{\mathrm{s}}^2 + \left(\frac{\partial \rho_{\mathrm{s}}}{\partial \phi}\right)^2\right] + \rho_{\mathrm{s}} \frac{\partial^2 \rho_{\mathrm{s}}}{\partial \phi^2} \left[1 + \left(\frac{\partial \rho_{\mathrm{s}}}{\partial z}\right)^2\right] - 2\rho_{\mathrm{s}} \frac{\partial \rho_{\mathrm{s}}}{\partial z} \frac{\partial \rho_{\mathrm{s}}}{\partial \phi} \frac{\partial^2 \rho_{\mathrm{s}}}{\partial z \partial \phi} - \left(\frac{\partial \rho_{\mathrm{s}}}{\partial \phi}\right)^2 \right\} - \frac{1}{2G_{\mathrm{s}}}. \tag{7.26}$$

Gaussian curvature:

$$K = \frac{1}{G_{\mathrm{s}}^4} \left\{ \rho_{\mathrm{s}}^2 \left[\frac{\partial^2 \rho_{\mathrm{s}}}{\partial \phi^2} \frac{\partial^2 \rho_{\mathrm{s}}}{\partial z^2} - \left(\frac{\partial^2 \rho_{\mathrm{s}}}{\partial z \partial \phi}\right)^2\right] - \rho_{\mathrm{s}} \frac{\partial^2 \rho_{\mathrm{s}}}{\partial z^2} \left[\rho_{\mathrm{s}}^2 + 2\left(\frac{\partial \rho_{\mathrm{s}}}{\partial \phi}\right)^2\right] + \frac{\partial \rho_{\mathrm{s}}}{\partial z} \frac{\partial \rho_{\mathrm{s}}}{\partial \phi} \left(2\rho_{\mathrm{s}} \frac{\partial^2 \rho_{\mathrm{s}}}{\partial z \partial \phi} - \frac{\partial \rho_{\mathrm{s}}}{\partial z} \frac{\partial \rho_{\mathrm{s}}}{\partial \phi}\right) \right\}. \tag{7.27}$$

Surface components of $\boldsymbol{v}$:

$$v_n = \frac{1}{G_{\mathrm{s}}} \left(v_\rho \rho_{\mathrm{s}} - v_\phi \frac{\partial \rho_{\mathrm{s}}}{\partial \phi} - v_z \rho_{\mathrm{s}} \frac{\partial \rho_{\mathrm{s}}}{\partial z} \right), \tag{7.28a}$$

$$v_{s_1} = \frac{\sqrt{\rho_{\mathrm{s}}^2 + (\partial \rho_{\mathrm{s}}/\partial \phi)^2}}{G_{\mathrm{s}}^2} \left\{ v_\phi \rho_{\mathrm{s}} \left[1 + \left(\frac{\partial \rho_{\mathrm{s}}}{\partial z}\right)^2\right] - v_z \frac{\partial \rho_{\mathrm{s}}}{\partial z} \frac{\partial \rho_{\mathrm{s}}}{\partial \phi} + v_\rho \frac{\partial \rho_{\mathrm{s}}}{\partial \phi} \right\}, \tag{7.28b}$$

$$v_{s_2} = \frac{\sqrt{1 + (\partial \rho_{\mathrm{s}}/\partial z)^2}}{G_{\mathrm{s}}^2} \left\{ v_z \left[\rho_{\mathrm{s}}^2 + \left(\frac{\partial \rho_{\mathrm{s}}}{\partial \phi}\right)^2\right] - v_\phi \rho_{\mathrm{s}} \frac{\partial \rho_{\mathrm{s}}}{\partial z} \frac{\partial \rho_{\mathrm{s}}}{\partial \phi} + v_\rho \rho_{\mathrm{s}}^2 \frac{\partial \rho_{\mathrm{s}}}{\partial z} \right\}. \tag{7.28c}$$

Surface components of $\boldsymbol{\tau} \cdot \hat{\boldsymbol{n}}$:

$$\begin{aligned} \tau_{nn} = \frac{1}{G_{\mathrm{s}}^2} \bigg[& \left(\tau_{\rho\rho} \rho_{\mathrm{s}} - \tau_{\rho\phi} \frac{\partial \rho_{\mathrm{s}}}{\partial \phi} - \tau_{\rho z} \rho_{\mathrm{s}} \frac{\partial \rho_{\mathrm{s}}}{\partial z} \right) \rho_{\mathrm{s}} - \left(\tau_{\phi\rho} \rho_{\mathrm{s}} - \tau_{\phi\phi} \frac{\partial \rho_{\mathrm{s}}}{\partial \phi} - \tau_{\phi z} \rho_{\mathrm{s}} \frac{\partial \rho_{\mathrm{s}}}{\partial z} \right) \frac{\partial \rho_{\mathrm{s}}}{\partial \phi} \\ & - \left(\tau_{z\rho} \rho_{\mathrm{s}} - \tau_{z\phi} \frac{\partial \rho_{\mathrm{s}}}{\partial \phi} - \tau_{zz} \rho_{\mathrm{s}} \frac{\partial \rho_{\mathrm{s}}}{\partial z} \right) \rho_{\mathrm{s}} \frac{\partial \rho_{\mathrm{s}}}{\partial z} \bigg], \end{aligned} \tag{7.29a}$$

$$\begin{aligned} & \tau_{s_1 n} \\ & = \frac{\sqrt{\rho_{\mathrm{s}}^2 + (\partial \rho_{\mathrm{s}}/\partial \phi)^2}}{G_{\mathrm{s}}^3} \left\{ \left(\tau_{\phi\rho} \rho_{\mathrm{s}} - \tau_{\phi\phi} \frac{\partial \rho_{\mathrm{s}}}{\partial \phi} - \tau_{\phi z} \rho_{\mathrm{s}} \frac{\partial \rho_{\mathrm{s}}}{\partial z} \right) \rho_{\mathrm{s}} \left[1 + \left(\frac{\partial \rho_{\mathrm{s}}}{\partial z}\right)^2\right] \right. \\ & \quad \left. - \left(\tau_{z\rho} \rho_{\mathrm{s}} - \tau_{z\phi} \frac{\partial \rho_{\mathrm{s}}}{\partial \phi} - \tau_{zz} \rho_{\mathrm{s}} \frac{\partial \rho_{\mathrm{s}}}{\partial z} \right) \frac{\partial \rho_{\mathrm{s}}}{\partial z} \frac{\partial \rho_{\mathrm{s}}}{\partial \phi} + \left(\tau_{\rho\rho} \rho_{\mathrm{s}} - \tau_{\rho\phi} \frac{\partial \rho_{\mathrm{s}}}{\partial \phi} - \tau_{\rho z} \rho_{\mathrm{s}} \frac{\partial \rho_{\mathrm{s}}}{\partial z} \right) \frac{\partial \rho_{\mathrm{s}}}{\partial \phi} \right\}, \end{aligned} \tag{7.29b}$$

$$\begin{aligned} & \tau_{s_2 n} \\ & = \frac{\sqrt{1 + (\partial \rho_{\mathrm{s}}/\partial z)^2}}{G_{\mathrm{s}}^3} \left\{ \left(\tau_{z\rho} \rho_{\mathrm{s}} - \tau_{z\phi} \frac{\partial \rho_{\mathrm{s}}}{\partial \phi} - \tau_{zz} \rho_{\mathrm{s}} \frac{\partial \rho_{\mathrm{s}}}{\partial z} \right) \left[\rho_{\mathrm{s}}^2 + \left(\frac{\partial \rho_{\mathrm{s}}}{\partial \phi}\right)^2\right] \right. \\ & \quad \left. - \left(\tau_{\phi\rho} \rho_{\mathrm{s}} - \tau_{\phi\phi} \frac{\partial \rho_{\mathrm{s}}}{\partial \phi} - \tau_{\phi z} \rho_{\mathrm{s}} \frac{\partial \rho_{\mathrm{s}}}{\partial z} \right) \rho_{\mathrm{s}} \frac{\partial \rho_{\mathrm{s}}}{\partial z} \frac{\partial \rho_{\mathrm{s}}}{\partial \phi} + \left(\tau_{\rho\rho} \rho_{\mathrm{s}} - \tau_{\rho\phi} \frac{\partial \rho_{\mathrm{s}}}{\partial \phi} - \tau_{\rho z} \rho_{\mathrm{s}} \frac{\partial \rho_{\mathrm{s}}}{\partial z} \right) \rho_{\mathrm{s}}^2 \frac{\partial \rho_{\mathrm{s}}}{\partial z} \right\}. \end{aligned} \tag{7.29c}$$

Surface gradient of ψ:

$$\boldsymbol{\nabla}_{\mathrm{s}}\psi = \frac{1}{G_{\mathrm{s}}^2}\left\{\hat{\boldsymbol{e}}_\phi \rho_{\mathrm{s}}\left[1+\left(\frac{\partial \rho_{\mathrm{s}}}{\partial z}\right)^2\right] - \hat{\boldsymbol{e}}_z \frac{\partial \rho_{\mathrm{s}}}{\partial z}\frac{\partial \rho_{\mathrm{s}}}{\partial \phi} + \hat{\boldsymbol{e}}_\rho \frac{\partial \rho_{\mathrm{s}}}{\partial \phi}\right\}\frac{\partial \psi}{\partial \phi} + \frac{1}{G_{\mathrm{s}}^2}\left\{\hat{\boldsymbol{e}}_z \left[\rho_{\mathrm{s}}^2+\left(\frac{\partial \rho_{\mathrm{s}}}{\partial \phi}\right)^2\right] - \hat{\boldsymbol{e}}_\phi \rho_{\mathrm{s}} \frac{\partial \rho_{\mathrm{s}}}{\partial z}\frac{\partial \rho_{\mathrm{s}}}{\partial \phi} + \hat{\boldsymbol{e}}_\rho \rho_{\mathrm{s}}^2 \frac{\partial \rho_{\mathrm{s}}}{\partial z}\right\}\frac{\partial \psi}{\partial z}. \tag{7.30}$$

Surface divergence of $\boldsymbol{v}$:

$$\boldsymbol{\nabla}_{\mathrm{s}}\cdot\boldsymbol{v} = \frac{1}{G_{\mathrm{s}}}\frac{\partial}{\partial \phi}\left(\frac{1}{G_{\mathrm{s}}}\left\{v_\phi \rho_{\mathrm{s}}\left[1+\left(\frac{\partial \rho_{\mathrm{s}}}{\partial z}\right)^2\right] - v_z \frac{\partial \rho_{\mathrm{s}}}{\partial z}\frac{\partial \rho_{\mathrm{s}}}{\partial \phi} + v_\rho \frac{\partial \rho_{\mathrm{s}}}{\partial \phi}\right\}\right) + \frac{1}{G_{\mathrm{s}}}\frac{\partial}{\partial z}\left(\frac{1}{G_{\mathrm{s}}}\left\{v_z\left[\rho_{\mathrm{s}}^2+\left(\frac{\partial \rho_{\mathrm{s}}}{\partial \phi}\right)^2\right] - v_\phi \rho_{\mathrm{s}}\frac{\partial \rho_{\mathrm{s}}}{\partial z}\frac{\partial \rho_{\mathrm{s}}}{\partial \phi} + v_\rho \rho_{\mathrm{s}}^2 \frac{\partial \rho_{\mathrm{s}}}{\partial z}\right\}\right) - \frac{2H}{G_{\mathrm{s}}}\left(v_\rho \rho_{\mathrm{s}} - v_\phi \frac{\partial \rho_{\mathrm{s}}}{\partial \phi} - v_z \rho_{\mathrm{s}} \frac{\partial \rho_{\mathrm{s}}}{\partial z}\right), \tag{7.31a}$$

$$= \frac{1}{G_{\mathrm{s}}}\frac{\partial}{\partial \phi}\left(\frac{G_{\mathrm{s}} v_\phi}{\rho_{\mathrm{s}}}\right) + \frac{1}{G_{\mathrm{s}}}\frac{\partial}{\partial z}\left(G_{\mathrm{s}} v_z\right) \quad \text{if } v_n = 0. \tag{7.31b}$$

Surface Laplacian of ψ:

$$\nabla_{\mathrm{s}}^2\psi = \frac{1}{G_{\mathrm{s}}}\frac{\partial}{\partial \phi}\left(\frac{1}{G_{\mathrm{s}}}\left\{\left[1+\left(\frac{\partial \rho_{\mathrm{s}}}{\partial z}\right)^2\right]\frac{\partial \psi}{\partial \phi} - \frac{\partial \rho_{\mathrm{s}}}{\partial z}\frac{\partial \rho_{\mathrm{s}}}{\partial \phi}\frac{\partial \psi}{\partial z}\right\}\right) + \frac{1}{G_{\mathrm{s}}}\frac{\partial}{\partial z}\left(\frac{1}{G_{\mathrm{s}}}\left\{\left[\rho_{\mathrm{s}}^2+\left(\frac{\partial \rho_{\mathrm{s}}}{\partial \phi}\right)^2\right]\frac{\partial \psi}{\partial z} - \frac{\partial \rho_{\mathrm{s}}}{\partial z}\frac{\partial \rho_{\mathrm{s}}}{\partial \phi}\frac{\partial \psi}{\partial \phi}\right\}\right). \tag{7.32}$$

Surface curl of ψ:

$$\hat{\boldsymbol{n}}\times\boldsymbol{\nabla}_{\mathrm{s}}\psi = -\boldsymbol{\nabla}_{\mathrm{s}}\times(\psi\hat{\boldsymbol{n}}) = \frac{1}{G_{\mathrm{s}}}\left[\left(\hat{\boldsymbol{e}}_z + \hat{\boldsymbol{e}}_\rho \frac{\partial \rho_{\mathrm{s}}}{\partial z}\right)\frac{\partial \psi}{\partial \phi} - \left(\hat{\boldsymbol{e}}_\phi \rho_{\mathrm{s}} + \hat{\boldsymbol{e}}_\rho \frac{\partial \rho_{\mathrm{s}}}{\partial \phi}\right)\frac{\partial \psi}{\partial z}\right]. \tag{7.33}$$

Surface curl of $\boldsymbol{v}$:

$$(\boldsymbol{\nabla}_{\mathrm{s}}\times\boldsymbol{v})\cdot\hat{\boldsymbol{n}} = \boldsymbol{\nabla}_{\mathrm{s}}\cdot(\boldsymbol{v}\times\hat{\boldsymbol{n}}) = \frac{1}{G_{\mathrm{s}}}\frac{\partial}{\partial \phi}\left(v_z + v_\rho \frac{\partial \rho_{\mathrm{s}}}{\partial z}\right) - \frac{1}{G_{\mathrm{s}}}\frac{\partial}{\partial z}\left(v_\phi \rho_{\mathrm{s}} + v_\rho \frac{\partial \rho_{\mathrm{s}}}{\partial \phi}\right). \tag{7.34}$$

7.3 Cylindrical coordinates, $z = z_{\mathrm{s}}(\rho, \phi)$

Level set function:

$$\varphi(\rho,\phi,z) = z - z_{\mathrm{s}}(\rho,\phi) = 0 \quad \text{on the surface.} \tag{7.35}$$

Surface position vector:

$$\boldsymbol{x}_{\mathrm{s}}(\rho,\phi) = \hat{\boldsymbol{e}}_\rho(\phi)\,\rho + \hat{\boldsymbol{e}}_z z_{\mathrm{s}}(\rho,\phi). \tag{7.36}$$

Surface metric:

$$G_{\mathrm{s}} = \sqrt{\rho^2\left[1+\left(\frac{\partial z_{\mathrm{s}}}{\partial \rho}\right)^2\right] + \left(\frac{\partial z_{\mathrm{s}}}{\partial \phi}\right)^2}. \tag{7.37}$$

Unit normal vector:

$$\hat{\boldsymbol{n}} = \frac{1}{G_{\mathrm{s}}}\left(\hat{\boldsymbol{e}}_z \rho - \hat{\boldsymbol{e}}_\rho \rho \frac{\partial z_{\mathrm{s}}}{\partial \rho} - \hat{\boldsymbol{e}}_\phi \frac{\partial z_{\mathrm{s}}}{\partial \phi}\right). \tag{7.38}$$

Covariant tangential vectors:

$$\boldsymbol{t}_{(1)} = \hat{\boldsymbol{e}}_\rho + \hat{\boldsymbol{e}}_z \frac{\partial z_{\mathrm{s}}}{\partial \rho}, \tag{7.39a}$$

$$\boldsymbol{t}_{(2)} = \hat{\boldsymbol{e}}_\phi \rho + \hat{\boldsymbol{e}}_z \frac{\partial z_{\mathrm{s}}}{\partial \phi}. \tag{7.39b}$$

Contravariant tangential vectors:

$$\boldsymbol{t}^{(1)} = \frac{1}{G_\mathrm{s}^2}\left\{\hat{\boldsymbol{e}}_\rho\left[\rho^2 + \left(\frac{\partial z_\mathrm{s}}{\partial\phi}\right)^2\right] - \hat{\boldsymbol{e}}_\phi\rho\frac{\partial z_\mathrm{s}}{\partial\rho}\frac{\partial z_\mathrm{s}}{\partial\phi} + \hat{\boldsymbol{e}}_z\rho^2\frac{\partial z_\mathrm{s}}{\partial\rho}\right\}, \tag{7.40a}$$

$$\boldsymbol{t}^{(2)} = \frac{1}{G_\mathrm{s}^2}\left\{\hat{\boldsymbol{e}}_\phi\rho\left[1 + \left(\frac{\partial z_\mathrm{s}}{\partial\rho}\right)^2\right] - \hat{\boldsymbol{e}}_\rho\frac{\partial z_\mathrm{s}}{\partial\rho}\frac{\partial z_\mathrm{s}}{\partial\phi} + \hat{\boldsymbol{e}}_z\frac{\partial z_\mathrm{s}}{\partial\phi}\right\}. \tag{7.40b}$$

Line elements:

$$\mathrm{d}s_1 = \sqrt{1 + \left(\frac{\partial z_\mathrm{s}}{\partial\rho}\right)^2}\,\mathrm{d}\rho, \tag{7.41a}$$

$$\mathrm{d}s_2 = \sqrt{\rho^2 + \left(\frac{\partial z_\mathrm{s}}{\partial\phi}\right)^2}\,\mathrm{d}\phi. \tag{7.41b}$$

Surface element:

$$\mathrm{d}S = \sqrt{\rho^2\left[1 + \left(\frac{\partial z_\mathrm{s}}{\partial\rho}\right)^2\right] + \left(\frac{\partial z_\mathrm{s}}{\partial\phi}\right)^2}\,\mathrm{d}\rho\,\mathrm{d}\phi. \tag{7.42}$$

Mean curvature:

$$\begin{aligned} H = {}& \frac{1}{2G_\mathrm{s}^3}\left\{\rho\frac{\partial^2 z_\mathrm{s}}{\partial\rho^2}\left[\rho^2 + \left(\frac{\partial z_\mathrm{s}}{\partial\phi}\right)^2\right] + \rho\frac{\partial^2 z_\mathrm{s}}{\partial\phi^2}\left[1 + \left(\frac{\partial z_\mathrm{s}}{\partial\rho}\right)^2\right] - 2\rho\frac{\partial z_\mathrm{s}}{\partial\rho}\frac{\partial z_\mathrm{s}}{\partial\phi}\frac{\partial^2 z_\mathrm{s}}{\partial\rho\partial\phi} + \frac{\partial z_\mathrm{s}}{\partial\rho}\left(\frac{\partial z_\mathrm{s}}{\partial\phi}\right)^2\right\} \\ & + \frac{1}{2G_\mathrm{s}}\frac{\partial z_\mathrm{s}}{\partial\rho}. \end{aligned} \tag{7.43}$$

Gaussian curvature:

$$K = \frac{1}{G_\mathrm{s}^4}\left\{\rho^2\left[\frac{\partial^2 z_\mathrm{s}}{\partial\phi^2}\frac{\partial^2 z_\mathrm{s}}{\partial\rho^2} - \left(\frac{\partial^2 z_\mathrm{s}}{\partial\rho\partial\phi}\right)^2\right] + \rho^3\frac{\partial z_\mathrm{s}}{\partial\rho}\frac{\partial^2 z_\mathrm{s}}{\partial\rho^2} + \frac{\partial z_\mathrm{s}}{\partial\phi}\left(2\rho\frac{\partial^2 z_\mathrm{s}}{\partial\rho\partial\phi} - \frac{\partial z_\mathrm{s}}{\partial\phi}\right)\right\}. \tag{7.44}$$

Surface components of $\boldsymbol{v}$:

$$v_n = \frac{1}{G_\mathrm{s}}\left(v_z\rho - v_\rho\rho\frac{\partial z_\mathrm{s}}{\partial\rho} - v_\phi\frac{\partial z_\mathrm{s}}{\partial\phi}\right), \tag{7.45a}$$

$$v_{s_1} = \frac{\sqrt{1 + (\partial z_\mathrm{s}/\partial\rho)^2}}{G_\mathrm{s}^2}\left\{v_\rho\left[\rho^2 + \left(\frac{\partial z_\mathrm{s}}{\partial\phi}\right)^2\right] - v_\phi\rho\frac{\partial z_\mathrm{s}}{\partial\rho}\frac{\partial z_\mathrm{s}}{\partial\phi} + v_z\rho^2\frac{\partial z_\mathrm{s}}{\partial\rho}\right\}, \tag{7.45b}$$

$$v_{s_2} = \frac{\sqrt{\rho^2 + (\partial z_\mathrm{s}/\partial\phi)^2}}{G_\mathrm{s}^2}\left\{v_\phi\rho\left[1 + \left(\frac{\partial z_\mathrm{s}}{\partial\rho}\right)^2\right] - v_\rho\frac{\partial z_\mathrm{s}}{\partial\rho}\frac{\partial z_\mathrm{s}}{\partial\phi} + v_z\frac{\partial z_\mathrm{s}}{\partial\phi}\right\}. \tag{7.45c}$$

Surface components of $\boldsymbol{\tau}\cdot\hat{\boldsymbol{n}}$:

$$\begin{aligned} \tau_{nn} = {}& \frac{1}{G_\mathrm{s}^2}\left[\left(\tau_{zz}\rho - \tau_{z\rho}\rho\frac{\partial z_\mathrm{s}}{\partial\rho} - \tau_{z\phi}\frac{\partial z_\mathrm{s}}{\partial\phi}\right)\rho - \left(\tau_{\rho z}\rho - \tau_{\rho\rho}\rho\frac{\partial z_\mathrm{s}}{\partial\rho} - \tau_{\rho\phi}\frac{\partial z_\mathrm{s}}{\partial\phi}\right)\rho\frac{\partial z_\mathrm{s}}{\partial\rho}\right. \\ & \left. - \left(\tau_{\phi z}\rho - \tau_{\phi\rho}\rho\frac{\partial z_\mathrm{s}}{\partial\rho} - \tau_{\phi\phi}\frac{\partial z_\mathrm{s}}{\partial\phi}\right)\frac{\partial z_\mathrm{s}}{\partial\phi}\right], \end{aligned} \tag{7.46a}$$

$$\begin{aligned} & \tau_{s_1 n} \\ & = \frac{\sqrt{1 + (\partial z_\mathrm{s}/\partial\rho)^2}}{G_\mathrm{s}^3}\left\{\left(\tau_{\rho z}\rho - \tau_{\rho\rho}\rho\frac{\partial z_\mathrm{s}}{\partial\rho} - \tau_{\rho\phi}\frac{\partial z_\mathrm{s}}{\partial\phi}\right)\left[\rho^2 + \left(\frac{\partial z_\mathrm{s}}{\partial\phi}\right)^2\right]\right. \\ & \left. - \left(\tau_{\phi z}\rho - \tau_{\phi\rho}\rho\frac{\partial z_\mathrm{s}}{\partial\rho} - \tau_{\phi\phi}\frac{\partial z_\mathrm{s}}{\partial\phi}\right)\rho\frac{\partial z_\mathrm{s}}{\partial\rho}\frac{\partial z_\mathrm{s}}{\partial\phi} + \left(\tau_{zz}\rho - \tau_{z\rho}\rho\frac{\partial z_\mathrm{s}}{\partial\rho} - \tau_{z\phi}\frac{\partial z_\mathrm{s}}{\partial\phi}\right)\rho^2\frac{\partial z_\mathrm{s}}{\partial\rho}\right\}, \end{aligned} \tag{7.46b}$$

$$\tau_{s_2 n} = \frac{\sqrt{\rho^2 + (\partial z_\mathrm{s}/\partial\phi)^2}}{G_\mathrm{s}^3} \left\{ \left(\tau_{\phi z}\rho - \tau_{\phi\rho}\rho\frac{\partial z_\mathrm{s}}{\partial\rho} - \tau_{\phi\phi}\frac{\partial z_\mathrm{s}}{\partial\phi} \right) \rho \left[1 + \left(\frac{\partial z_\mathrm{s}}{\partial\rho} \right)^2 \right] \right.$$

$$\left. - \left(\tau_{\rho z}\rho - \tau_{\rho\rho}\rho\frac{\partial z_\mathrm{s}}{\partial\rho} - \tau_{\rho\phi}\frac{\partial z_\mathrm{s}}{\partial\phi} \right) \frac{\partial z_\mathrm{s}}{\partial\rho}\frac{\partial z_\mathrm{s}}{\partial\phi} + \left(\tau_{zz}\rho - \tau_{z\rho}\rho\frac{\partial z_\mathrm{s}}{\partial\rho} - \tau_{z\phi}\frac{\partial z_\mathrm{s}}{\partial\phi} \right) \frac{\partial z_\mathrm{s}}{\partial\phi} \right\}. \tag{7.46c}$$

Surface gradient of ψ:

$$\boldsymbol{\nabla}_\mathrm{s}\psi = \frac{1}{G_\mathrm{s}^2} \left\{ \hat{\boldsymbol{e}}_\rho \left[\rho^2 + \left(\frac{\partial z_\mathrm{s}}{\partial\phi} \right)^2 \right] - \hat{\boldsymbol{e}}_\phi \rho \frac{\partial z_\mathrm{s}}{\partial z}\frac{\partial z_\mathrm{s}}{\partial\phi} + \hat{\boldsymbol{e}}_z \rho^2 \frac{\partial z_\mathrm{s}}{\partial\rho} \right\} \frac{\partial\psi}{\partial\rho}$$

$$+ \frac{1}{G_\mathrm{s}^2} \left\{ \hat{\boldsymbol{e}}_\phi \rho \left[1 + \left(\frac{\partial z_\mathrm{s}}{\partial\rho} \right)^2 \right] - \hat{\boldsymbol{e}}_\rho \frac{\partial z_\mathrm{s}}{\partial\rho}\frac{\partial z_\mathrm{s}}{\partial\phi} + \hat{\boldsymbol{e}}_z \frac{\partial z_\mathrm{s}}{\partial\phi} \right\} \frac{\partial\psi}{\partial\phi}. \tag{7.47}$$

Surface divergence of $\boldsymbol{v}$:

$$\boldsymbol{\nabla}_\mathrm{s}\cdot\boldsymbol{v} = \frac{1}{G_\mathrm{s}}\frac{\partial}{\partial\rho} \left(\frac{1}{G_\mathrm{s}} \left\{ v_\rho \left[\rho^2 + \left(\frac{\partial z_\mathrm{s}}{\partial\phi} \right)^2 \right] - v_\phi \rho \frac{\partial z_\mathrm{s}}{\partial z}\frac{\partial z_\mathrm{s}}{\partial\phi} + v_z \rho^2 \frac{\partial z_\mathrm{s}}{\partial\rho} \right\} \right)$$

$$+ \frac{1}{G_\mathrm{s}}\frac{\partial}{\partial\phi} \left(\frac{1}{G_\mathrm{s}} \left\{ v_\phi \rho \left[1 + \left(\frac{\partial z_\mathrm{s}}{\partial\rho} \right)^2 \right] - v_\rho \frac{\partial z_\mathrm{s}}{\partial\rho}\frac{\partial z_\mathrm{s}}{\partial\phi} + v_z \frac{\partial z_\mathrm{s}}{\partial\phi} \right\} \right)$$

$$- \frac{2H}{G_\mathrm{s}} \left(v_z\rho - v_\rho\rho\frac{\partial z_\mathrm{s}}{\partial\rho} - v_\phi\frac{\partial z_\mathrm{s}}{\partial\phi} \right), \tag{7.48a}$$

$$= \frac{1}{G_\mathrm{s}}\frac{\partial}{\partial\rho}(G_\mathrm{s}v_\rho) + \frac{1}{\rho G_\mathrm{s}}\frac{\partial}{\partial\phi}(G_\mathrm{s}v_\phi) \quad \text{if } v_n = 0. \tag{7.48b}$$

Surface Laplacian of ψ:

$$\nabla_\mathrm{s}^2\psi = \frac{1}{G_\mathrm{s}}\frac{\partial}{\partial\rho} \left(\frac{1}{G_\mathrm{s}} \left\{ \left[\rho^2 + \left(\frac{\partial z_\mathrm{s}}{\partial\phi} \right)^2 \right] \frac{\partial\psi}{\partial\rho} - \frac{\partial z_\mathrm{s}}{\partial\rho}\frac{\partial z_\mathrm{s}}{\partial\phi}\frac{\partial\psi}{\partial\phi} \right\} \right)$$

$$+ \frac{1}{G_\mathrm{s}}\frac{\partial}{\partial\phi} \left(\frac{1}{G_\mathrm{s}} \left\{ \left[1 + \left(\frac{\partial z_\mathrm{s}}{\partial\rho} \right)^2 \right] \frac{\partial\psi}{\partial\phi} - \frac{\partial z_\mathrm{s}}{\partial\rho}\frac{\partial z_\mathrm{s}}{\partial\phi}\frac{\partial\psi}{\partial\rho} \right\} \right). \tag{7.49}$$

Surface curl of ψ:

$$\hat{\boldsymbol{n}} \times \boldsymbol{\nabla}_\mathrm{s}\psi = -\boldsymbol{\nabla}_\mathrm{s} \times (\psi\hat{\boldsymbol{n}}) = \frac{1}{G_\mathrm{s}} \left[\left(\hat{\boldsymbol{e}}_\phi\rho + \hat{\boldsymbol{e}}_z\frac{\partial z_\mathrm{s}}{\partial\phi} \right) \frac{\partial\psi}{\partial\rho} - \left(\hat{\boldsymbol{e}}_\rho + \hat{\boldsymbol{e}}_z\frac{\partial z_\mathrm{s}}{\partial\rho} \right) \frac{\partial\psi}{\partial\phi} \right]. \tag{7.50}$$

Surface curl of $\boldsymbol{v}$:

$$(\boldsymbol{\nabla}_\mathrm{s} \times \boldsymbol{v}) \cdot \hat{\boldsymbol{n}} = \boldsymbol{\nabla}_\mathrm{s} \cdot (\boldsymbol{v} \times \hat{\boldsymbol{n}}) = \frac{1}{G_\mathrm{s}}\frac{\partial}{\partial\rho} \left(v_\phi\rho + v_z\frac{\partial z_\mathrm{s}}{\partial\phi} \right) - \frac{1}{G_\mathrm{s}}\frac{\partial}{\partial\phi} \left(v_\rho + v_z\frac{\partial z_\mathrm{s}}{\partial\rho} \right). \tag{7.51}$$

7.4 Spherical coordinates, $r = r_\mathrm{s}(\theta,\phi)$

Level set function:[†]

$$\varphi(r,\theta,\phi) = r - r_\mathrm{s}(\theta,\phi) = 0 \quad \text{on the surface.} \tag{7.52}$$

[†]In spherical coordinates, the unit vector in the r direction is simply $\hat{\boldsymbol{e}}_r = \boldsymbol{x}/r$. The usual gradient operator has the convenient decomposition,

$$\boldsymbol{\nabla} = \hat{\boldsymbol{e}}_r\frac{\partial}{\partial r} + \frac{\boldsymbol{\nabla}_\Omega}{r},$$

where Ω is the solid angle and

$$\boldsymbol{\nabla}_\Omega = r(\boldsymbol{\delta} - \hat{\boldsymbol{e}}_r\hat{\boldsymbol{e}}_r) \cdot \boldsymbol{\nabla} = \hat{\boldsymbol{e}}_\theta\frac{\partial}{\partial\theta} + \frac{\hat{\boldsymbol{e}}_\phi}{\sin\theta}\frac{\partial}{\partial\phi}$$

is the surface gradient operator on the unit sphere. The surface element on the unit sphere is $\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$.

Surface position vector:

$$\boldsymbol{x}_{\mathrm{s}}(\theta,\phi) = \hat{\boldsymbol{e}}_r(\theta,\phi)\, r_{\mathrm{s}}(\theta,\phi). \tag{7.53}$$

Surface metric:

$$G_{\mathrm{s}} = r_{\mathrm{s}}\sqrt{\sin^2\theta\left[r_{\mathrm{s}}^2 + \left(\frac{\partial r_{\mathrm{s}}}{\partial\theta}\right)^2\right] + \left(\frac{\partial r_{\mathrm{s}}}{\partial\phi}\right)^2} = r_{\mathrm{s}}\sin\theta\sqrt{r_{\mathrm{s}}^2 + (\boldsymbol{\nabla}_\Omega r_{\mathrm{s}})\cdot(\boldsymbol{\nabla}_\Omega r_{\mathrm{s}})}. \tag{7.54}$$

Unit normal vector:

$$\hat{\boldsymbol{n}} = \frac{r_{\mathrm{s}}}{G_{\mathrm{s}}}\left(\hat{\boldsymbol{e}}_r r_{\mathrm{s}}\sin\theta - \hat{\boldsymbol{e}}_\theta\sin\theta\frac{\partial r_{\mathrm{s}}}{\partial\theta} - \hat{\boldsymbol{e}}_\phi\frac{\partial r_{\mathrm{s}}}{\partial\phi}\right) = \frac{r_{\mathrm{s}}\sin\theta}{G_{\mathrm{s}}}\left(r_{\mathrm{s}}\hat{\boldsymbol{e}}_r - \boldsymbol{\nabla}_\Omega r_{\mathrm{s}}\right). \tag{7.55}$$

Covariant tangential vectors:

$$\boldsymbol{t}_{(1)} = \hat{\boldsymbol{e}}_\theta r_{\mathrm{s}} + \hat{\boldsymbol{e}}_r\frac{\partial r_{\mathrm{s}}}{\partial\theta}, \tag{7.56a}$$

$$\boldsymbol{t}_{(2)} = \hat{\boldsymbol{e}}_\phi r_{\mathrm{s}}\sin\theta + \hat{\boldsymbol{e}}_r\frac{\partial r_{\mathrm{s}}}{\partial\phi}. \tag{7.56b}$$

Contravariant tangential vectors:

$$\boldsymbol{t}^{(1)} = \frac{r_{\mathrm{s}}}{G_{\mathrm{s}}^2}\left\{\hat{\boldsymbol{e}}_\theta\left[r_{\mathrm{s}}^2\sin^2\theta + \left(\frac{\partial r_{\mathrm{s}}}{\partial\phi}\right)^2\right] - \hat{\boldsymbol{e}}_\phi\sin\theta\frac{\partial r_{\mathrm{s}}}{\partial\theta}\frac{\partial r_{\mathrm{s}}}{\partial\phi} + \hat{\boldsymbol{e}}_r r_{\mathrm{s}}\sin^2\theta\frac{\partial r_{\mathrm{s}}}{\partial\theta}\right\}, \tag{7.57a}$$

$$\boldsymbol{t}^{(2)} = \frac{r_{\mathrm{s}}}{G_{\mathrm{s}}^2}\left\{\hat{\boldsymbol{e}}_\phi\sin\theta\left[r_{\mathrm{s}}^2 + \left(\frac{\partial r_{\mathrm{s}}}{\partial\theta}\right)^2\right] - \hat{\boldsymbol{e}}_\theta\frac{\partial r_{\mathrm{s}}}{\partial\theta}\frac{\partial r_{\mathrm{s}}}{\partial\phi} + \hat{\boldsymbol{e}}_r r_{\mathrm{s}}\frac{\partial r_{\mathrm{s}}}{\partial\phi}\right\}. \tag{7.57b}$$

Line elements:

$$\mathrm{d}s_1 = \sqrt{r_{\mathrm{s}}^2 + \left(\frac{\partial r_{\mathrm{s}}}{\partial\theta}\right)^2}\,\mathrm{d}\theta, \tag{7.58a}$$

$$\mathrm{d}s_2 = \sqrt{r_{\mathrm{s}}^2\sin^2\theta + \left(\frac{\partial r_{\mathrm{s}}}{\partial\phi}\right)^2}\,\mathrm{d}\phi. \tag{7.58b}$$

Surface element:

$$\mathrm{d}S = r_{\mathrm{s}}\sqrt{\sin^2\theta\left[r_{\mathrm{s}}^2 + \left(\frac{\partial r_{\mathrm{s}}}{\partial\theta}\right)^2\right] + \left(\frac{\partial r_{\mathrm{s}}}{\partial\phi}\right)^2}\,\mathrm{d}\theta\,\mathrm{d}\phi = r_{\mathrm{s}}\sqrt{r_{\mathrm{s}}^2 + (\boldsymbol{\nabla}_\Omega r_{\mathrm{s}})\cdot(\boldsymbol{\nabla}_\Omega r_{\mathrm{s}})}\,\mathrm{d}\Omega. \tag{7.59}$$

Mean curvature:

$$\begin{aligned} H = \frac{r_{\mathrm{s}}^2\sin\theta}{2G_{\mathrm{s}}^3}&\left\{\frac{\partial^2 r_{\mathrm{s}}}{\partial\theta^2}\left[r_{\mathrm{s}}^2\sin^2\theta + \left(\frac{\partial r_{\mathrm{s}}}{\partial\phi}\right)^2\right] + \frac{\partial^2 r_{\mathrm{s}}}{\partial\phi^2}\left[r_{\mathrm{s}}^2 + \left(\frac{\partial r_{\mathrm{s}}}{\partial\theta}\right)^2\right] - 2\frac{\partial r_{\mathrm{s}}}{\partial\theta}\frac{\partial r_{\mathrm{s}}}{\partial\phi}\frac{\partial^2 r_{\mathrm{s}}}{\partial\theta\partial\phi}\right. \\ &\left. + r_{\mathrm{s}}^3\sin^2\theta + \cot\theta\frac{\partial r_{\mathrm{s}}}{\partial\theta}\left(\frac{\partial r_{\mathrm{s}}}{\partial\phi}\right)^2\right\} - \frac{\sin\theta}{2G_{\mathrm{s}}}\left(3r_{\mathrm{s}} - \cot\theta\frac{\partial r_{\mathrm{s}}}{\partial\theta}\right). \end{aligned} \tag{7.60}$$

Gaussian curvature:

$$\begin{aligned} K = \frac{r_{\mathrm{s}}^3\sin^2\theta}{G_{\mathrm{s}}^4}&\left\{r_{\mathrm{s}}\left[\frac{\partial^2 r_{\mathrm{s}}}{\partial\theta^2}\frac{\partial^2 r_{\mathrm{s}}}{\partial\phi^2} - \left(\frac{\partial^2 r_{\mathrm{s}}}{\partial\theta\partial\phi}\right)^2\right] - \left[\frac{\partial^2 r_{\mathrm{s}}}{\partial\phi^2} - \sin^2\theta\left(r_{\mathrm{s}} - \cot\theta\frac{\partial r_{\mathrm{s}}}{\partial\theta}\right)\right]\left[r_{\mathrm{s}}^2 + 2\left(\frac{\partial r_{\mathrm{s}}}{\partial\theta}\right)^2\right]\right. \\ &- \frac{\partial^2 r_{\mathrm{s}}}{\partial\theta^2}\left[r_{\mathrm{s}}\sin^2\theta\left(r_{\mathrm{s}} - \cot\theta\frac{\partial r_{\mathrm{s}}}{\partial\theta}\right) + 2\left(\frac{\partial r_{\mathrm{s}}}{\partial\phi}\right)^2\right] + 2\frac{\partial r_{\mathrm{s}}}{\partial\phi}\frac{\partial^2 r_{\mathrm{s}}}{\partial\theta\partial\phi}\left(r_{\mathrm{s}}\cot\theta + 2\frac{\partial r_{\mathrm{s}}}{\partial\theta}\right) \\ &\left. - \left(\frac{\partial r_{\mathrm{s}}}{\partial\phi}\right)^2\left(r_{\mathrm{s}}(\cot^2\theta - 2) + 4\cot\theta\frac{\partial r_{\mathrm{s}}}{\partial\theta}\right)\right\}. \end{aligned} \tag{7.61}$$

Surface components of $\boldsymbol{v}$:

$$v_n = \frac{r_\mathrm{s}}{G_\mathrm{s}}\left(v_r r_\mathrm{s}\sin\theta - v_\theta \sin\theta\frac{\partial r_\mathrm{s}}{\partial\theta} - v_\phi\frac{\partial r_\mathrm{s}}{\partial\phi}\right), \tag{7.62a}$$

$$v_{s_1} = \frac{r_\mathrm{s}\sqrt{r_\mathrm{s}^2 + (\partial r_\mathrm{s}/\partial\theta)^2}}{G_\mathrm{s}^2}\left\{v_\theta\left[r_\mathrm{s}^2\sin^2\theta + \left(\frac{\partial r_\mathrm{s}}{\partial\phi}\right)^2\right] - v_\phi\sin\theta\frac{\partial r_\mathrm{s}}{\partial\theta}\frac{\partial r_\mathrm{s}}{\partial\phi} + v_r r_\mathrm{s}\sin^2\theta\frac{\partial r_\mathrm{s}}{\partial\theta}\right\}, \tag{7.62b}$$

$$v_{s_2} = \frac{r_\mathrm{s}\sqrt{r_\mathrm{s}^2\sin^2\theta + (\partial r_\mathrm{s}/\partial\phi)^2}}{G_\mathrm{s}^2}\left\{v_\phi\sin\theta\left[r_\mathrm{s}^2 + \left(\frac{\partial r_\mathrm{s}}{\partial\theta}\right)^2\right] - v_\theta\frac{\partial r_\mathrm{s}}{\partial\theta}\frac{\partial r_\mathrm{s}}{\partial\phi} + v_r r_\mathrm{s}\frac{\partial r_\mathrm{s}}{\partial\phi}\right\}. \tag{7.62c}$$

Surface components of $\boldsymbol{\tau}\cdot\hat{\boldsymbol{n}}$:

$$\begin{aligned}\tau_{nn} = \frac{r_\mathrm{s}^2}{G_\mathrm{s}^2}\Bigg[&\left(\tau_{rr} r_\mathrm{s}\sin\theta - \tau_{r\theta}\sin\theta\frac{\partial r_\mathrm{s}}{\partial\theta} - \tau_{r\phi}\frac{\partial r_\mathrm{s}}{\partial\phi}\right) r_\mathrm{s}\sin\theta \\ &- \left(\tau_{\theta r} r_\mathrm{s}\sin\theta - \tau_{\theta\theta}\sin\theta\frac{\partial r_\mathrm{s}}{\partial\theta} - \tau_{\theta\phi}\frac{\partial r_\mathrm{s}}{\partial\phi}\right)\sin\theta\frac{\partial r_\mathrm{s}}{\partial\theta} \\ &- \left(\tau_{\phi r} r_\mathrm{s}\sin\theta - \tau_{\phi\theta}\sin\theta\frac{\partial r_\mathrm{s}}{\partial\theta} - \tau_{\phi\phi}\frac{\partial r_\mathrm{s}}{\partial\phi}\right)\frac{\partial r_\mathrm{s}}{\partial\phi}\Bigg],\end{aligned} \tag{7.63a}$$

$$\begin{aligned}\tau_{s_1 n} = \frac{r_\mathrm{s}^2\sqrt{r_\mathrm{s}^2 + (\partial r_\mathrm{s}/\partial\theta)^2}}{G_\mathrm{s}^3}\Bigg\{&\left(\tau_{\theta r} r_\mathrm{s}\sin\theta - \tau_{\theta\theta}\sin\theta\frac{\partial r_\mathrm{s}}{\partial\theta} - \tau_{\theta\phi}\frac{\partial r_\mathrm{s}}{\partial\phi}\right)\left[r_\mathrm{s}^2\sin^2\theta + \left(\frac{\partial r_\mathrm{s}}{\partial\phi}\right)^2\right] \\ &- \left(\tau_{\phi r} r_\mathrm{s}\sin\theta - \tau_{\phi\theta}\sin\theta\frac{\partial r_\mathrm{s}}{\partial\theta} - \tau_{\phi\phi}\frac{\partial r_\mathrm{s}}{\partial\phi}\right)\sin\theta\frac{\partial r_\mathrm{s}}{\partial\theta}\frac{\partial r_\mathrm{s}}{\partial\phi} \\ &+ \left(\tau_{rr} r_\mathrm{s}\sin\theta - \tau_{r\theta}\sin\theta\frac{\partial r_\mathrm{s}}{\partial\theta} - \tau_{r\phi}\frac{\partial r_\mathrm{s}}{\partial\phi}\right) r_\mathrm{s}\sin^2\theta\frac{\partial r_\mathrm{s}}{\partial\theta}\Bigg\},\end{aligned} \tag{7.63b}$$

$$\begin{aligned}\tau_{s_2 n} = \frac{r_\mathrm{s}^2\sqrt{r_\mathrm{s}^2\sin^2\theta + (\partial r_\mathrm{s}/\partial\phi)^2}}{G_\mathrm{s}^3}\Bigg\{&\left(\tau_{\phi r} r_\mathrm{s}\sin\theta - \tau_{\phi\theta}\sin\theta\frac{\partial r_\mathrm{s}}{\partial\theta} - \tau_{\phi\phi}\frac{\partial r_\mathrm{s}}{\partial\phi}\right)\sin\theta\left[r_\mathrm{s}^2 + \left(\frac{\partial r_\mathrm{s}}{\partial\theta}\right)^2\right] \\ &- \left(\tau_{\theta r} r_\mathrm{s}\sin\theta - \tau_{\theta\theta}\sin\theta\frac{\partial r_\mathrm{s}}{\partial\theta} - \tau_{\theta\phi}\frac{\partial r_\mathrm{s}}{\partial\phi}\right)\frac{\partial r_\mathrm{s}}{\partial\theta}\frac{\partial r_\mathrm{s}}{\partial\phi} \\ &+ \left(\tau_{rr} r_\mathrm{s}\sin\theta - \tau_{r\theta}\sin\theta\frac{\partial r_\mathrm{s}}{\partial\theta} - \tau_{r\phi}\frac{\partial r_\mathrm{s}}{\partial\phi}\right) r_\mathrm{s}\frac{\partial r_\mathrm{s}}{\partial\phi}\Bigg\}.\end{aligned} \tag{7.63c}$$

Surface gradient of ψ:

$$\begin{aligned}\boldsymbol{\nabla}_\mathrm{s}\psi = &\frac{r_\mathrm{s}}{G_\mathrm{s}^2}\left\{\hat{\boldsymbol{e}}_\theta\left[r_\mathrm{s}^2\sin^2\theta + \left(\frac{\partial r}{\partial\phi}\right)^2\right] - \hat{\boldsymbol{e}}_\phi\sin\theta\frac{\partial r_\mathrm{s}}{\partial\theta}\frac{\partial r_\mathrm{s}}{\partial\phi} + \hat{\boldsymbol{e}}_r r_\mathrm{s}\sin^2\theta\frac{\partial r_\mathrm{s}}{\partial\theta}\right\}\frac{\partial\psi}{\partial\theta} \\ &+ \frac{r_\mathrm{s}}{G_\mathrm{s}^2}\left\{\hat{\boldsymbol{e}}_\phi\sin\theta\left[r_\mathrm{s}^2 + \left(\frac{\partial r_\mathrm{s}}{\partial\theta}\right)^2\right] - \hat{\boldsymbol{e}}_\theta\frac{\partial r_\mathrm{s}}{\partial\theta}\frac{\partial r_\mathrm{s}}{\partial\phi} + \hat{\boldsymbol{e}}_r r_\mathrm{s}\frac{\partial r_\mathrm{s}}{\partial\phi}\right\}\frac{\partial\psi}{\partial\phi}.\end{aligned} \tag{7.64}$$

Surface divergence of $\boldsymbol{v}$:

$$\begin{aligned}\boldsymbol{\nabla}_\mathrm{s}\cdot\boldsymbol{v} &= \frac{1}{G_\mathrm{s}}\frac{\partial}{\partial\theta}\left(\frac{r_\mathrm{s}}{G_\mathrm{s}}\left\{v_\theta\left[r_\mathrm{s}^2\sin^2\theta + \left(\frac{\partial r}{\partial\phi}\right)^2\right] - v_\phi\sin\theta\frac{\partial r_\mathrm{s}}{\partial\theta}\frac{\partial r_\mathrm{s}}{\partial\phi} + v_r r_\mathrm{s}\sin^2\theta\frac{\partial r_\mathrm{s}}{\partial\theta}\right\}\right) \\ &\quad + \frac{1}{G_\mathrm{s}}\frac{\partial}{\partial\phi}\left(\frac{r_\mathrm{s}}{G_\mathrm{s}}\left\{v_\phi\sin\theta\left[r_\mathrm{s}^2 + \left(\frac{\partial r_\mathrm{s}}{\partial\theta}\right)^2\right] - v_\theta\frac{\partial r_\mathrm{s}}{\partial\theta}\frac{\partial r_\mathrm{s}}{\partial\phi} + v_r r_\mathrm{s}\frac{\partial r_\mathrm{s}}{\partial\phi}\right\}\right) \\ &\quad - \frac{2Hr_\mathrm{s}}{G_\mathrm{s}}\left(v_r r_\mathrm{s}\sin\theta - v_\theta\sin\theta\frac{\partial r_\mathrm{s}}{\partial\theta} - v_\phi\frac{\partial r_\mathrm{s}}{\partial\phi}\right),\end{aligned} \tag{7.65a}$$

$$= \frac{1}{G_\mathrm{s}}\frac{\partial}{\partial\theta}\left(\frac{G_\mathrm{s}v_\theta}{r_\mathrm{s}}\right) + \frac{1}{G_\mathrm{s}\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{G_\mathrm{s}v_\phi}{r_\mathrm{s}}\right) \quad \text{if } v_n = 0. \tag{7.65b}$$

Surface Laplacian of ψ:

$$\nabla_{\mathrm{s}}^2\psi = \frac{1}{G_{\mathrm{s}}}\frac{\partial}{\partial\theta}\left(\frac{1}{G_{\mathrm{s}}}\left\{\left[r_{\mathrm{s}}^2\sin^2\theta + \left(\frac{\partial r_{\mathrm{s}}}{\partial\phi}\right)^2\right]\frac{\partial\psi}{\partial\theta} - \frac{\partial r_{\mathrm{s}}}{\partial\theta}\frac{\partial r_{\mathrm{s}}}{\partial\phi}\frac{\partial\psi}{\partial\phi}\right\}\right) + \frac{1}{G_{\mathrm{s}}}\frac{\partial}{\partial\phi}\left(\frac{1}{G_{\mathrm{s}}}\left\{\left[r_{\mathrm{s}}^2 + \left(\frac{\partial r_{\mathrm{s}}}{\partial\theta}\right)^2\right]\frac{\partial\psi}{\partial\phi} - \frac{\partial r_{\mathrm{s}}}{\partial\theta}\frac{\partial r_{\mathrm{s}}}{\partial\phi}\frac{\partial\psi}{\partial\theta}\right\}\right). \quad (7.66)$$

Surface curl of ψ:

$$\hat{\boldsymbol{n}}\times\boldsymbol{\nabla}_{\mathrm{s}}\psi = -\boldsymbol{\nabla}_{\mathrm{s}}\times(\psi\hat{\boldsymbol{n}}) = \frac{1}{G_{\mathrm{s}}}\left[\left(\hat{\boldsymbol{e}}_\phi r_{\mathrm{s}}\sin\theta + \hat{\boldsymbol{e}}_r\frac{\partial r_{\mathrm{s}}}{\partial\phi}\right)\frac{\partial\psi}{\partial\theta} - \left(\hat{\boldsymbol{e}}_\theta r_{\mathrm{s}} + \hat{\boldsymbol{e}}_r\frac{\partial r_{\mathrm{s}}}{\partial\theta}\right)\frac{\partial\psi}{\partial\phi}\right]. \quad (7.67)$$

Surface curl of $\boldsymbol{v}$:

$$(\boldsymbol{\nabla}_{\mathrm{s}}\times\boldsymbol{v})\cdot\hat{\boldsymbol{n}} = \boldsymbol{\nabla}_{\mathrm{s}}\cdot(\boldsymbol{v}\times\hat{\boldsymbol{n}}) = \frac{1}{G_{\mathrm{s}}}\frac{\partial}{\partial\theta}\left(v_\phi r_{\mathrm{s}}\sin\theta + v_r\frac{\partial r_{\mathrm{s}}}{\partial\phi}\right) - \frac{1}{G_{\mathrm{s}}}\frac{\partial}{\partial\phi}\left(v_\theta r_{\mathrm{s}} + v_r\frac{\partial r_{\mathrm{s}}}{\partial\theta}\right). \quad (7.68)$$

7.5 Two-dimensional Cartesian coordinates, $z = z_{\mathrm{s}}(x)$

Level set function:

$$\varphi(x,z) = z - z_{\mathrm{s}}(x) = 0 \quad \text{on the surface.} \quad (7.69)$$

Surface position vector:

$$\boldsymbol{x}_{\mathrm{s}}(x,y) = \hat{\boldsymbol{e}}_x x + \hat{\boldsymbol{e}}_y y + \hat{\boldsymbol{e}}_z z_{\mathrm{s}}(x). \quad (7.70)$$

Surface metric:

$$G_{\mathrm{s}} = \sqrt{1 + \left(\frac{\mathrm{d}z_{\mathrm{s}}}{\mathrm{d}x}\right)^2}. \quad (7.71)$$

Unit normal vector:

$$\hat{\boldsymbol{n}} = \frac{1}{G_{\mathrm{s}}}\left(\hat{\boldsymbol{e}}_z - \hat{\boldsymbol{e}}_x\frac{\mathrm{d}z_{\mathrm{s}}}{\mathrm{d}x}\right). \quad (7.72)$$

Covariant tangential vectors:

$$\boldsymbol{t}_{(1)} = \hat{\boldsymbol{e}}_x + \hat{\boldsymbol{e}}_z\frac{\mathrm{d}z_{\mathrm{s}}}{\mathrm{d}x}, \quad (7.73a)$$

$$\boldsymbol{t}_{(2)} = \hat{\boldsymbol{e}}_y. \quad (7.73b)$$

Contravariant tangential vectors:

$$\boldsymbol{t}^{(1)} = \frac{1}{G_{\mathrm{s}}^2}\left(\hat{\boldsymbol{e}}_x + \hat{\boldsymbol{e}}_z\frac{\mathrm{d}z_{\mathrm{s}}}{\mathrm{d}x}\right), \quad (7.74a)$$

$$\boldsymbol{t}^{(2)} = \hat{\boldsymbol{e}}_y. \quad (7.74b)$$

Line elements:

$$\mathrm{d}s_1 = \sqrt{1 + \left(\frac{\mathrm{d}z_{\mathrm{s}}}{\mathrm{d}x}\right)^2}\,\mathrm{d}x, \quad (7.75a)$$

$$\mathrm{d}s_2 = \mathrm{d}y. \quad (7.75b)$$

Surface element:

$$\mathrm{d}S = \sqrt{1 + \left(\frac{\mathrm{d}z_{\mathrm{s}}}{\mathrm{d}x}\right)^2}\,\mathrm{d}x\,\mathrm{d}y. \quad (7.76)$$

Mean curvature:

$$H = \frac{1}{2G_{\mathrm{s}}^3}\frac{\mathrm{d}^2 z_{\mathrm{s}}}{\mathrm{d}x^2}. \quad (7.77)$$

Gaussian curvature:

$$K = 0. \tag{7.78}$$

Surface components of $\boldsymbol{v}$:

$$v_n = \frac{1}{G_\mathrm{s}}\left(v_z - v_x\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}x}\right), \tag{7.79a}$$

$$v_{s_1} = \frac{1}{G_\mathrm{s}}\left(v_x + v_z\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}x}\right), \tag{7.79b}$$

$$v_{s_2} = v_y. \tag{7.79c}$$

Surface components of $\boldsymbol{\tau}\cdot\hat{\boldsymbol{n}}$:

$$\tau_{nn} = \frac{1}{G_\mathrm{s}^2}\left[\tau_{zz} - (\tau_{zx}+\tau_{xz})\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}x} + \tau_{xx}\left(\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}x}\right)^2\right], \tag{7.80a}$$

$$\tau_{s_1n} = \frac{1}{G_\mathrm{s}^2}\left[\tau_{xz} + (\tau_{zz}-\tau_{xx})\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}x} - \tau_{zx}\left(\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}x}\right)^2\right], \tag{7.80b}$$

$$\tau_{s_2n} = \frac{1}{G_\mathrm{s}}\left(\tau_{yz} - \tau_{yx}\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}x}\right). \tag{7.80c}$$

Surface gradient of ψ:

$$\boldsymbol{\nabla}_\mathrm{s}\psi = \frac{1}{G_\mathrm{s}^2}\left(\hat{\boldsymbol{e}}_x + \hat{\boldsymbol{e}}_z\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}x}\right)\frac{\partial\psi}{\partial x} + \hat{\boldsymbol{e}}_y\frac{\partial\psi}{\partial y}. \tag{7.81}$$

Surface divergence of $\boldsymbol{v}$:

$$\boldsymbol{\nabla}_\mathrm{s}\cdot\boldsymbol{v} = \frac{1}{G_\mathrm{s}}\frac{\partial}{\partial x}\left[\frac{1}{G_\mathrm{s}}\left(v_x + v_z\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}x}\right)\right] + \frac{\partial v_y}{\partial y} - \frac{2H}{G_\mathrm{s}}\left(v_z - v_x\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}x}\right), \tag{7.82a}$$

$$= \frac{1}{G_\mathrm{s}}\frac{\partial}{\partial x}(G_\mathrm{s}v_x) + \frac{\partial v_y}{\partial y} \quad \text{if } v_n = 0. \tag{7.82b}$$

Surface Laplacian of ψ:

$$\nabla_\mathrm{s}^2\psi = \frac{1}{G_\mathrm{s}}\frac{\partial}{\partial x}\left(\frac{1}{G_\mathrm{s}}\frac{\partial\psi}{\partial x}\right) + \frac{\partial^2\psi}{\partial y^2}. \tag{7.83}$$

Surface curl of ψ:

$$\hat{\boldsymbol{n}}\times\boldsymbol{\nabla}_\mathrm{s}\psi = -\boldsymbol{\nabla}_\mathrm{s}\times(\psi\hat{\boldsymbol{n}}) = \frac{1}{G_\mathrm{s}}\left[\hat{\boldsymbol{e}}_y\frac{\partial\psi}{\partial x} - \left(\hat{\boldsymbol{e}}_x + \hat{\boldsymbol{e}}_z\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}x}\right)\frac{\partial\psi}{\partial y}\right]. \tag{7.84}$$

Surface curl of $\boldsymbol{v}$:

$$(\boldsymbol{\nabla}_\mathrm{s}\times\boldsymbol{v})\cdot\hat{\boldsymbol{n}} = \boldsymbol{\nabla}_\mathrm{s}\cdot(\boldsymbol{v}\times\hat{\boldsymbol{n}}) = \frac{1}{G_\mathrm{s}}\frac{\partial v_y}{\partial x} - \frac{1}{G_\mathrm{s}}\frac{\partial}{\partial y}\left(v_x + v_z\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}x}\right). \tag{7.85}$$

7.6 Axisymmetric cylindrical coordinates, $\rho = \rho_\mathrm{s}(z)$

Level set function:

$$\varphi(\rho, z) = \rho - \rho_\mathrm{s}(z) = 0 \quad \text{on the surface.} \tag{7.86}$$

Surface position vector:

$$\boldsymbol{x}_\mathrm{s}(z,\phi) = \hat{\boldsymbol{e}}_\rho(\phi)\,\rho_\mathrm{s}(z) + \hat{\boldsymbol{e}}_z z. \tag{7.87}$$

Surface metric:

$$G_{\mathrm{s}} = \rho_{\mathrm{s}} \sqrt{1 + \left(\frac{\mathrm{d}\rho_{\mathrm{s}}}{\mathrm{d}z}\right)^2}. \tag{7.88}$$

Unit normal vector:

$$\hat{\boldsymbol{n}} = \frac{\rho_{\mathrm{s}}}{G_{\mathrm{s}}} \left(\hat{\boldsymbol{e}}_\rho - \hat{\boldsymbol{e}}_z \frac{\mathrm{d}\rho_{\mathrm{s}}}{\mathrm{d}z}\right). \tag{7.89}$$

Covariant tangential vectors:

$$\boldsymbol{t}_{(1)} = \hat{\boldsymbol{e}}_\phi \rho_{\mathrm{s}}, \tag{7.90a}$$

$$\boldsymbol{t}_{(2)} = \hat{\boldsymbol{e}}_z + \hat{\boldsymbol{e}}_\rho \frac{\mathrm{d}\rho_{\mathrm{s}}}{\mathrm{d}z}. \tag{7.90b}$$

Contravariant tangential vectors:

$$\boldsymbol{t}^{(1)} = \frac{\hat{\boldsymbol{e}}_\phi}{\rho_{\mathrm{s}}}, \tag{7.91a}$$

$$\boldsymbol{t}^{(2)} = \frac{\rho_{\mathrm{s}}^2}{G_{\mathrm{s}}^2} \left(\hat{\boldsymbol{e}}_z + \hat{\boldsymbol{e}}_\rho \frac{\mathrm{d}\rho_{\mathrm{s}}}{\mathrm{d}z}\right). \tag{7.91b}$$

Line elements:

$$\mathrm{d}s_1 = \rho_{\mathrm{s}}\, \mathrm{d}\phi, \tag{7.92a}$$

$$\mathrm{d}s_2 = \sqrt{1 + \left(\frac{\mathrm{d}\rho_{\mathrm{s}}}{\mathrm{d}z}\right)^2}\, \mathrm{d}z. \tag{7.92b}$$

Surface element:

$$\mathrm{d}S = \rho_{\mathrm{s}} \sqrt{1 + \left(\frac{\mathrm{d}\rho_{\mathrm{s}}}{\mathrm{d}z}\right)^2}\, \mathrm{d}\phi\, \mathrm{d}z. \tag{7.93}$$

Mean curvature:

$$H = \frac{\rho_{\mathrm{s}}^3}{2G_{\mathrm{s}}^3} \frac{\mathrm{d}^2\rho_{\mathrm{s}}}{\mathrm{d}z^2} - \frac{1}{2G_{\mathrm{s}}}. \tag{7.94}$$

Gaussian curvature:

$$K = -\frac{\rho_{\mathrm{s}}^3}{G_{\mathrm{s}}^4} \frac{\mathrm{d}^2\rho_{\mathrm{s}}}{\mathrm{d}z^2}. \tag{7.95}$$

Surface components of $\boldsymbol{v}$:

$$v_n = \frac{\rho_{\mathrm{s}}}{G_{\mathrm{s}}} \left(v_\rho - v_z \frac{\mathrm{d}\rho_{\mathrm{s}}}{\mathrm{d}z}\right), \tag{7.96a}$$

$$v_{s_1} = v_\phi, \tag{7.96b}$$

$$v_{s_2} = \frac{\rho_{\mathrm{s}}}{G_{\mathrm{s}}} \left(v_z + v_\rho \frac{\mathrm{d}\rho_{\mathrm{s}}}{\mathrm{d}z}\right). \tag{7.96c}$$

Surface components of $\boldsymbol{\tau} \cdot \hat{\boldsymbol{n}}$:

$$\tau_{nn} = \frac{\rho_{\mathrm{s}}^2}{G_{\mathrm{s}}^2} \left[\tau_{\rho\rho} - (\tau_{z\rho} + \tau_{\rho z}) \frac{\mathrm{d}\rho_{\mathrm{s}}}{\mathrm{d}z} + \tau_{zz} \left(\frac{\mathrm{d}\rho_{\mathrm{s}}}{\mathrm{d}z}\right)^2\right], \tag{7.97a}$$

$$\tau_{s_1 n} = \frac{\rho_{\mathrm{s}}}{G_{\mathrm{s}}} \left(\tau_{\phi\rho} - \tau_{\phi z} \frac{\mathrm{d}\rho_{\mathrm{s}}}{\mathrm{d}z}\right), \tag{7.97b}$$

$$\tau_{s_2 n} = \frac{\rho_{\mathrm{s}}^2}{G_{\mathrm{s}}^2} \left[\tau_{z\rho} + (\tau_{\rho\rho} - \tau_{zz}) \frac{\mathrm{d}\rho_{\mathrm{s}}}{\mathrm{d}z} - \tau_{\rho z} \left(\frac{\mathrm{d}\rho_{\mathrm{s}}}{\mathrm{d}z}\right)^2\right]. \tag{7.97c}$$

Surface gradient of ψ:

$$\boldsymbol{\nabla}_{\mathrm{s}}\psi = \frac{\hat{\boldsymbol{e}}_\phi}{\rho_{\mathrm{s}}}\frac{\partial\psi}{\partial\phi} + \frac{\rho_{\mathrm{s}}^2}{G_{\mathrm{s}}^2}\left(\hat{\boldsymbol{e}}_z + \hat{\boldsymbol{e}}_\rho\frac{\mathrm{d}\rho_{\mathrm{s}}}{\mathrm{d}z}\right)\frac{\partial\psi}{\partial z}. \tag{7.98}$$

Surface divergence of $\boldsymbol{v}$:

$$\boldsymbol{\nabla}_{\mathrm{s}}\cdot\boldsymbol{v} = \frac{1}{\rho_{\mathrm{s}}}\frac{\partial v_\phi}{\partial\phi} + \frac{1}{G_{\mathrm{s}}}\frac{\partial}{\partial z}\left[\frac{\rho_{\mathrm{s}}^2}{G_{\mathrm{s}}}\left(v_z + v_\rho\frac{\mathrm{d}\rho_{\mathrm{s}}}{\mathrm{d}z}\right)\right] - \frac{2H\rho_{\mathrm{s}}}{G_{\mathrm{s}}}\left(v_\rho - v_z\frac{\mathrm{d}\rho_{\mathrm{s}}}{\mathrm{d}z}\right), \tag{7.99a}$$

$$= \frac{1}{\rho_{\mathrm{s}}}\frac{\partial v_\phi}{\partial\phi} + \frac{1}{G_{\mathrm{s}}}\frac{\partial}{\partial z}\left(G_{\mathrm{s}}v_z\right) \quad \text{if } v_n = 0. \tag{7.99b}$$

Surface Laplacian of ψ:

$$\nabla_{\mathrm{s}}^2\psi = \frac{1}{\rho_{\mathrm{s}}^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{1}{G_{\mathrm{s}}}\frac{\partial}{\partial z}\left(\frac{\rho_{\mathrm{s}}^2}{G_{\mathrm{s}}}\frac{\partial\psi}{\partial z}\right). \tag{7.100}$$

Surface curl of ψ:

$$\hat{\boldsymbol{n}}\times\boldsymbol{\nabla}_{\mathrm{s}}\psi = -\boldsymbol{\nabla}_{\mathrm{s}}\times(\psi\hat{\boldsymbol{n}}) = \frac{1}{G_{\mathrm{s}}}\left[\hat{\boldsymbol{e}}_\phi\rho_{\mathrm{s}}\frac{\partial\psi}{\partial\phi} - \left(\hat{\boldsymbol{e}}_z + \hat{\boldsymbol{e}}_\rho\frac{\mathrm{d}\rho_{\mathrm{s}}}{\mathrm{d}z}\right)\frac{\partial\psi}{\partial z}\right]. \tag{7.101}$$

Surface curl of $\boldsymbol{v}$:

$$(\boldsymbol{\nabla}_{\mathrm{s}}\times\boldsymbol{v})\cdot\hat{\boldsymbol{n}} = \boldsymbol{\nabla}_{\mathrm{s}}\cdot(\boldsymbol{v}\times\hat{\boldsymbol{n}}) = \frac{1}{G_{\mathrm{s}}}\frac{\partial}{\partial\phi}\left(v_z + v_\rho\frac{\mathrm{d}\rho_{\mathrm{s}}}{\mathrm{d}z}\right) - \frac{1}{G_{\mathrm{s}}}\frac{\partial}{\partial z}\left(v_\phi\rho_{\mathrm{s}}\right). \tag{7.102}$$

7.7 Axisymmetric cylindrical coordinates, $z = z_{\mathrm{s}}(\rho)$

Level set function:

$$\varphi(\rho, z) = z - z_{\mathrm{s}}(\rho) = 0 \quad \text{on the surface.} \tag{7.103}$$

Surface position vector:

$$\boldsymbol{x}_{\mathrm{s}}(\rho,\phi) = \hat{\boldsymbol{e}}_\rho(\phi)\,\rho + \hat{\boldsymbol{e}}_z z_{\mathrm{s}}(\rho). \tag{7.104}$$

Surface metric:

$$G_{\mathrm{s}} = \rho\sqrt{1 + \left(\frac{\mathrm{d}z_{\mathrm{s}}}{\mathrm{d}\rho}\right)^2}. \tag{7.105}$$

Unit normal vector:

$$\hat{\boldsymbol{n}} = \frac{\rho}{G_{\mathrm{s}}}\left(\hat{\boldsymbol{e}}_z - \hat{\boldsymbol{e}}_\rho\frac{\mathrm{d}z_{\mathrm{s}}}{\mathrm{d}\rho}\right). \tag{7.106}$$

Covariant tangential vectors:

$$\boldsymbol{t}_{(1)} = \hat{\boldsymbol{e}}_\rho + \hat{\boldsymbol{e}}_z\frac{\mathrm{d}z_{\mathrm{s}}}{\mathrm{d}\rho}, \tag{7.107a}$$

$$\boldsymbol{t}_{(2)} = \hat{\boldsymbol{e}}_\phi\rho. \tag{7.107b}$$

Contravariant tangential vectors:

$$\boldsymbol{t}^{(1)} = \frac{\rho^2}{G_{\mathrm{s}}^2}\left(\hat{\boldsymbol{e}}_\rho + \hat{\boldsymbol{e}}_z\frac{\mathrm{d}z_{\mathrm{s}}}{\mathrm{d}\rho}\right), \tag{7.108a}$$

$$\boldsymbol{t}^{(2)} = \frac{\hat{\boldsymbol{e}}_\phi}{\rho}. \tag{7.108b}$$

Line elements:

$$\mathrm{d}s_1 = \sqrt{1 + \left(\frac{\mathrm{d}z_{\mathrm{s}}}{\mathrm{d}\rho}\right)^2}\,\mathrm{d}\rho, \tag{7.109a}$$

$$\mathrm{d}s_2 = \rho\,\mathrm{d}\phi. \tag{7.109b}$$

Surface element:

$$\mathrm{d}S = \rho\sqrt{1+\left(\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}\rho}\right)^2}\,\mathrm{d}\rho\,\mathrm{d}\phi. \tag{7.110}$$

Mean curvature:

$$H = \frac{\rho^3}{2G_\mathrm{s}^3}\frac{\mathrm{d}^2 z_\mathrm{s}}{\mathrm{d}\rho^2} + \frac{1}{2G_\mathrm{s}}\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}\rho} = \frac{1}{2\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{\rho^2}{G_\mathrm{s}}\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}\rho}\right). \tag{7.111}$$

Gaussian curvature:

$$K = \frac{\rho^3}{G_\mathrm{s}^4}\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}\rho}\frac{\mathrm{d}^2 z_\mathrm{s}}{\mathrm{d}\rho^2}. \tag{7.112}$$

Surface components of $\boldsymbol{v}$:

$$v_n = \frac{\rho}{G_\mathrm{s}}\left(v_z - v_\rho\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}\rho}\right), \tag{7.113a}$$

$$v_{s_1} = \frac{\rho}{G_\mathrm{s}}\left(v_\rho + v_z\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}\rho}\right), \tag{7.113b}$$

$$v_{s_2} = v_\phi. \tag{7.113c}$$

Surface components of $\boldsymbol{\tau}\cdot\hat{\boldsymbol{n}}$:

$$\tau_{nn} = \frac{\rho^2}{G_\mathrm{s}^2}\left[\tau_{zz} - (\tau_{z\rho}+\tau_{\rho z})\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}\rho} + \tau_{\rho\rho}\left(\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}\rho}\right)^2\right], \tag{7.114a}$$

$$\tau_{s_1 n} = \frac{\rho^2}{G_\mathrm{s}^2}\left[\tau_{\rho z} + (\tau_{zz}-\tau_{\rho\rho})\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}\rho} - \tau_{z\rho}\left(\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}\rho}\right)^2\right], \tag{7.114b}$$

$$\tau_{s_2 n} = \frac{\rho}{G_\mathrm{s}}\left(\tau_{\phi z} - \tau_{\phi\rho}\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}\rho}\right). \tag{7.114c}$$

Surface gradient of ψ:

$$\boldsymbol{\nabla}_\mathrm{s}\psi = \frac{\rho^2}{G_\mathrm{s}^2}\left(\hat{\boldsymbol{e}}_\rho + \hat{\boldsymbol{e}}_z\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}\rho}\right)\frac{\partial\psi}{\partial\rho} + \frac{\hat{\boldsymbol{e}}_\phi}{\rho}\frac{\partial\psi}{\partial\phi}. \tag{7.115}$$

Surface divergence of $\boldsymbol{v}$:

$$\begin{aligned}\boldsymbol{\nabla}_\mathrm{s}\cdot\boldsymbol{v} &= \frac{1}{G_\mathrm{s}}\frac{\partial}{\partial\rho}\left[\frac{\rho^2}{G_\mathrm{s}}\left(v_\rho + v_z\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}\rho}\right)\right] + \frac{1}{\rho}\frac{\partial v_\phi}{\partial\phi} - \frac{2H\rho}{G_\mathrm{s}}\left(v_z - v_\rho\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}\rho}\right), && (7.116\mathrm{a})\\ &= \frac{1}{G_\mathrm{s}}\frac{\partial}{\partial\rho}\left(G_\mathrm{s}v_\rho\right) + \frac{1}{\rho}\frac{\partial v_\phi}{\partial\phi} \quad \text{if } v_n = 0. && (7.116\mathrm{b})\end{aligned}$$

Surface Laplacian of ψ:

$$\nabla_\mathrm{s}^2\psi = \frac{1}{G_\mathrm{s}}\frac{\partial}{\partial\rho}\left(\frac{\rho^2}{G_\mathrm{s}}\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2}. \tag{7.117}$$

Surface curl of ψ:

$$\hat{\boldsymbol{n}}\times\boldsymbol{\nabla}_\mathrm{s}\psi = -\boldsymbol{\nabla}_\mathrm{s}\times(\psi\hat{\boldsymbol{n}}) = \frac{1}{G_\mathrm{s}}\left[\hat{\boldsymbol{e}}_\phi\rho\frac{\partial\psi}{\partial\rho} - \left(\hat{\boldsymbol{e}}_\rho + \hat{\boldsymbol{e}}_z\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}\rho}\right)\frac{\partial\psi}{\partial\phi}\right]. \tag{7.118}$$

Surface curl of $\boldsymbol{v}$:

$$(\boldsymbol{\nabla}_\mathrm{s}\times\boldsymbol{v})\cdot\hat{\boldsymbol{n}} = \boldsymbol{\nabla}_\mathrm{s}\cdot(\boldsymbol{v}\times\hat{\boldsymbol{n}}) = \frac{1}{G_\mathrm{s}}\frac{\partial}{\partial\rho}(v_\phi\rho) - \frac{1}{G_\mathrm{s}}\frac{\partial}{\partial\phi}\left(v_\rho + v_z\frac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}\rho}\right). \tag{7.119}$$

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