(Local) In[135]:=

```
AppendTo[$Path, NotebookDirectory[]];
<< Matte`
```

```
Matte package loaded. © 2017-2018 Jonas Einarsson (me@jonaseinarsson.se)
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```

Important!!!

Matte unitsphereIntegral only works for 3D! I made a separate one for 2D, make sure you've loaded the right one.

Tensor Spherical Harmonics are made up of *irreducible* rank-n tensors

(Local) In[137]:=

```
On[Assert];
Assert[trace[delta] == 3]
```

These are described nicely in Leal Advanced Transport Phenomena Pg. 527 (within Chapter 8 Creeping Flows)
They can be developed from the solutions to the Laplace equation in spherical coordinates. Where you are taking repeated gradients of (1/r).

I.E. ∇(∇(1/r)) not Laplacian but a gradient of a gradient.

That should be correct, but constants are really important and magnitude of r is equal to 1, and so it is often written in a different method.

$\phi_{-(n+1)} = \frac{(-1)^n}{(2n-1)!!}\ \nabla_1 \ldots \nabla_n\left(\frac{1}{r}\right)$, n = 0, 1, 2,...
That technically gets you the -(n+1)th decaying harmonic, but then the nth order irreducible tensor just has you substitute |r| = 1

First use a recurrence relation to get the repeated gradient of $\frac{1}{r}$ where invr[1] is $\frac{1}{r}$

(Local) In[139]:=

```
RepeatedPartialR[0] := invr[1];
RepeatedPartialR[n_] := partialr[RepeatedPartialR[n - 1], idx[n]] // contract;
```

Now use that to get the decaying harmonics and replace inverse r to any power with 1

(Local) In[141]:=

```
IrreducibleRanknTensor[n_] :=
  ReplaceAll[invr[x_] :→ 1][(-1)^n/(2*n-1)!! RepeatedPartialR[n] // contract];
```

So for *any* vector r, the irreducible rank-n tensor that describes its dyad

(Local) In[142]:=

```
IrreducibleRanknTensor[0] // prettyprint
IrreducibleRanknTensor[1] // prettyprint
IrreducibleRanknTensor[2] // prettyprint
IrreducibleRanknTensor[3] // prettyprint
IrreducibleRanknTensor[4] // prettyprint
```

(Local) Out[142]=

$$1$$

(Local) Out[143]=

$$r_i$$

(Local) Out[144]=

$$r_i\, r_j - \frac{\delta_{i,j}}{3}$$

(Local) Out[145]=

$$r_i\, r_j\, r_k - \frac{1}{5}\, r_k\, \delta_{i,j} - \frac{1}{5}\, r_j\, \delta_{i,k} - \frac{1}{5}\, r_i\, \delta_{j,k}$$

(Local) Out[146]=

$$r_i\, r_j\, r_k\, r_l - \frac{1}{7}\, r_k\, r_l\, \delta_{i,j} - \frac{1}{7}\, r_j\, r_l\, \delta_{i,k} - \frac{1}{7}\, r_j\, r_k\, \delta_{i,l} - \frac{1}{7}\, r_i\, r_l\, \delta_{j,k} + \frac{1}{35}\, \delta_{i,l}\, \delta_{j,k} - \frac{1}{7}\, r_i\, r_k\, \delta_{j,l} + \frac{1}{35}\, \delta_{i,k}\, \delta_{j,l} - \frac{1}{7}\, r_i\, r_j\, \delta_{k,l} + \frac{1}{35}\, \delta_{i,j}\, \delta_{k,l}$$

Here are the repeated gradients of 1/r to show you that they are equivalent, just divided by leading constants.

(Local) In[147]:=

```
x1 = partialr[invr[1], idx[1]] // contract;
x2 = partialr[x1, idx[2]] // contract;
x3 = partialr[x2, idx[3]] // contract;
x1 // prettyprint
x2 // prettyprint
x3 // prettyprint
```

(Local) Out[150]=

$$-\frac{r_i}{r^3}$$

(Local) Out[151]=

$$\frac{3\, r_i\, r_j}{r^5} - \frac{\delta_{i,j}}{r^3}$$

(Local) Out[152]=

$$-\frac{15\, r_i\, r_j\, r_k}{r^7} + \frac{3\, r_k\, \delta_{i,j}}{r^5} + \frac{3\, r_j\, \delta_{i,k}}{r^5} + \frac{3\, r_i\, \delta_{j,k}}{r^5}$$

Part 2: Using as basis functions.

Assert

$$f(x, q, t) = \left[c_0\, n(x, t) * \mathrm{T0} + c_1\, m_\alpha(x, t) * \mathrm{T1}_\alpha + c_2\, Q_{\alpha,\beta}(x, t) * \mathrm{T2}_{\alpha,\beta} + \mathrm{c3}\ C_{\alpha,\beta,\gamma}(x, t)\, \mathrm{T3}_{\alpha,\beta,\gamma} + \ldots\right]$$

Where Tn is the rank n irreducible tensor harmonic.

Note that there are competing definitions for **Q**., one is the traceless version, the other is not. Have to be careful and look at the definition. For this document will be using the traceless version. **Q** = $\int\left(\mathbf{qq} - \frac{\mathbf{I}}{3}\right) f\, \mathrm{dq}$

Also be careful, some people include the prefactor $\frac{1}{4\pi}$ in the integral, others do not. For this document, we will not.

Now you find the constants by recognizing that the tensor harmonics form an orthogonal basis set.

Get c0

$$\int f(x, q, t)\, \mathrm{T0}\, \mathrm{dq} = \int \mathrm{T0} * \left[c_0\, n * \mathrm{T0} + c_1\, m_\alpha * \mathrm{T1}_\alpha + c_2\, Q_{\alpha,\beta} * \mathrm{T2}_{\alpha,\beta} + \mathrm{c3}\ C_{\alpha,\beta,\gamma}\, \mathrm{T3}_{\alpha,\beta,\gamma} + \ldots\right] \mathrm{dq}$$

$\int f(x, q, t)\,\mathrm{T0}\,\mathrm{dq} = c_0\, n * \int(\mathrm{T0} * \mathrm{T0})\,\mathrm{dq}$

On our left hand side is our definition for n

$n = \int(c_0\, n * \mathrm{T0} * \mathrm{T0})\,\mathrm{dq}$

Knowing T0 = 1. We perform the unit sphere integral and see:

(there is a way to do the unit sphere integrals by hand, see attached pdf or this link , but I'll let matte handle it)

(Local) In[153]:=

```
unitsphereIntegral[c0 nfield * IrreducibleRanknTensor[0] * IrreducibleRanknTensor[0]]
```

(Local) Out[153]=

$4\, \mathrm{c0}\, \mathrm{nfield}\, \pi$

$n = \int(c_0\, n * \mathrm{T0} * \mathrm{T0})\,\mathrm{dq} = 4\,\pi\, c_0\, n$

$c_0 = \frac{1}{4\,\pi}$

Get c1

$\int f(x, q, t)\,\mathrm{T1}_n\,\mathrm{dq} = \int \mathrm{T1}_n * \left[c_0\, n * \mathrm{T0} + c_1\, m_\alpha * \mathrm{T1}_\alpha + c_2\, Q_{\alpha,\beta} * \mathrm{T2}_{\alpha,\beta} + \mathrm{c3}\; C_{\alpha,\beta,\gamma}\, \mathrm{T3}_{\alpha,\beta,\gamma} + \ldots\right]\mathrm{dq}$

$\int f(x, q, t)\,\mathrm{T1}_n\,\mathrm{dq} = c_1\, m_\alpha * \int(\mathrm{T1}_\alpha * \mathrm{T1}_n)\,\mathrm{dq}$

On our left hand side is our definition for m

$m_n = c_1\, m_\alpha * \int(\mathrm{T1}_\alpha * \mathrm{T1}_n)\,\mathrm{dq}$

For convenience we are going to rename indices α = i, n = j

$m_j = c_1\, m_i * \int\left(\mathrm{T1}_i * \mathrm{T1}_j\right)\mathrm{dq}$

Knowing T1 = q. We perform the unit sphere integral and see:

(Local) In[154]:=

```
(*The indices need to be different! Set it up that way*)
T1alpha = IrreducibleRanknTensor[1]
T1n = IrreducibleRanknTensor[1] /. idx[1] → idx[2]
unitsphereIntegral[ c1 mfield T1alpha * T1n // Expand // contract] // prettyprint
```

(Local) Out[154]=

r[idx[1]]

(Local) Out[155]=

r[idx[2]]

(Local) Out[156]=

$$\frac{4}{3}\, \mathrm{c1\ mfield}\, \pi\, \delta_{i,j}$$

$m_j = c_1\, m_i * \frac{4}{3}\, \pi\, \delta_{i,j}$

Contract the indices and solve

$m_j = c_1\, m_j * \frac{4}{3}\, \pi$

$c_1 = \frac{3}{4\pi}$

Last example. Find c2. The **trickiest**!

$\int f(x,\, q,\, t)\, \mathrm{T2}_{n,m}\, \mathrm{dq} \;=\; \int \mathrm{T1}_n * \left[c_0\, n * \mathrm{T0} + c_1\, m_\alpha * \mathrm{T1}_\alpha + \; c_2\, Q_{\alpha,\beta} * \mathrm{T2}_{\alpha,\beta} + \mathrm{c3}\; C_{\alpha,\beta,\gamma}\, \mathrm{T3}_{\alpha,\beta,\gamma} + \ldots\right] \mathrm{dq}$

$\int f(x,\, q,\, t)\, \mathrm{T2}_{n,m}\, \mathrm{dq} \;= c_1\, Q_{\alpha,\beta} * \int \left(\mathrm{T2}_{\alpha,\beta} * \mathrm{T1}_{n,m}\right) \mathrm{dq}$

On our left hand side is our definition for m

For convenience we are going to rename indices α = i, β = j , n = k, m = l

$Q_{k,l} = c_2\, Q_{i,j} * \int \left(\mathrm{T2}_{k,l} * \mathrm{T2}_{i,j}\right) \mathrm{dq}$

Knowing T2 = qq-δ/3. We perform the unit sphere integral and see:

(Local) In[157]:=

```
(*The indices need to be different! Set it up that way*)
T2ij = IrreducibleRanknTensor[2]
T2kl = IrreducibleRanknTensor[2] /. idx[1] → idx[3] /. idx[2] → idx[4]
unitsphereIntegral[ c2 Qfield T2ij * T2kl // Expand // contract] // prettyprint
unitsphereIntegral[ c2 Qfield[idx[1], idx[2]] T2ij * T2kl // Expand // contract] // contract //
  prettyprint
```

(Local) Out[157]=

$$-\frac{1}{3}\,\texttt{delta[idx[1], idx[2]]} + \texttt{r[idx[1]]} \times \texttt{r[idx[2]]}$$

(Local) Out[158]=

$$-\frac{1}{3}\,\texttt{delta[idx[3], idx[4]]} + \texttt{r[idx[3]]} \times \texttt{r[idx[4]]}$$

(Local) Out[159]=

$$\frac{4}{15}\,c2\,\pi\,\text{Qfield}\,\delta_{i,l}\,\delta_{j,k} + \frac{4}{15}\,c2\,\pi\,\text{Qfield}\,\delta_{i,k}\,\delta_{j,l} - \frac{8}{45}\,c2\,\pi\,\text{Qfield}\,\delta_{i,j}\,\delta_{k,l}$$

(Local) Out[160]=

$$\frac{4}{15}\,c2\,\pi\,\text{Qfield}_{k,l} + \frac{4}{15}\,c2\,\pi\,\text{Qfield}_{l,k} - \frac{8}{45}\,c2\,\pi\,\delta_{k,l}\,\text{trace[Qfield]}$$

$Q_{k,l} = c_2\, Q_{i,j} * \left[\frac{4}{15}\,\pi\;\delta_{i,l}\,\delta_{j,k} + \frac{4}{15}\,\pi\;\delta_{i,k}\,\delta_{j,l} - \frac{8}{45}\,\pi\;\delta_{i,j}\,\delta_{k,l}\right]$

$Q_{k,l} = c_2\,\frac{4\pi}{15} * \left[Q_{k,l} + Q_{l,k} - \frac{2}{3}\,\delta_{k,l}\,\text{trace}[Q]\right]$

Final step, all these tensors should be symmetric and traceless. trace[Q] = 0, $Q_{k,l} = Q_{l,k}$

$Q_{k,l} = c_2\,\frac{4\pi}{15} * \left[2\,Q_{k,l}\right]$
$c_2 = \frac{15}{8\pi}$

What about rotational gradient?

The rotational gradient is described by
$\mathcal{R} = \nabla_R = q \times \nabla_q$
$\mathcal{R}^2 = \nabla_R{}^2$

And recall the property $\nabla_R{}^2 P_n = -n(n+1)\,P_n$

(Local) In[161]:=

```
RotOpSqrd[ex_] :=
  Module[{j1, j2, j3}, {j1 = getNewIndex[], j2 = getNewIndex[], j3 = getNewIndex[]};
   (delta[j1, j2] - invr[2] * r[j1] * r[j2]) *
     partialr[(delta[j3, j2] - invr[2] * r[j3] * r[j2]) * (partialr[ex, j3]) // Expand //
       contract , j1] // contract] ;
```

(Local) In[162]:=

```
IrreducibleRanknTensor[1] // prettyprint
RotOpSqrd[IrreducibleRanknTensor[1]] // prettyprint

IrreducibleRanknTensor[2] // prettyprint
RotOpSqrd[IrreducibleRanknTensor[2]] // prettyprint;
% /. invr[x_] → 1 // prettyprint
(*Note that r^2 = 1 *)


IrreducibleRanknTensor[3] // prettyprint
RotOpSqrd[IrreducibleRanknTensor[3]] // contract;
% /. invr[x_] → 1 // prettyprint
(*Note that r^2 = 1 *)
```

(Local) Out[162]=

$$r_i$$

(Local) Out[163]=

$$-\frac{2\, r_i}{r^2}$$

(Local) Out[164]=

$$r_i\, r_j - \frac{\delta_{i,j}}{3}$$

(Local) Out[166]=

$$-\frac{6\, r_i\, r_j}{r^2} + 2\,\delta_{i,j}$$

(Local) Out[167]=

$$r_i\, r_j\, r_k - \frac{1}{5}\, r_k\,\delta_{i,j} - \frac{1}{5}\, r_j\,\delta_{i,k} - \frac{1}{5}\, r_i\,\delta_{j,k}$$

(Local) Out[169]=

$$-12\, r_i\, r_j\, r_k + \frac{12}{5}\, r_k\,\delta_{i,j} + \frac{12}{5}\, r_j\,\delta_{i,k} + \frac{12}{5}\, r_i\,\delta_{j,k}$$