Continuum Mechanics

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2.2 Index Notation for Vector and Tensor Operations

Operations on Cartesian components of vectors and tensors may be expressed very efficiently and clearly using *index notation***.**

2.1. Vector and tensor components.

Let **x** be a (three dimensional) vector and let **S** be a second order tensor. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a Cartesian basis. Denote the components of **x** in this basis by (x_1, x_2, x_3), and denote the components of **S** by

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

Using index notation, we would express **x** and **S** as

$$\mathbf{x} \equiv x_i \qquad \mathbf{S} \equiv S_{ij}$$

2.2. Conventions and special symbols for index notation

- **Range Convention:** Lower case Latin subscripts *(i, j, k...)* have the range $(1,2,3)$. The symbol x_i denotes three components of a vector x_1, x_2 and x_3. The symbol S_{ij} denotes nine components of a second order tensor, S_{11}, S_{12}, S_{13}, S_{21} $\ldots$ S_{33}

- **Summation convention (Einstein convention)**: If an index is repeated in a product of vectors or tensors, summation is implied over the repeated index. Thus

$$\lambda = a_i b_i \quad \equiv \quad \lambda = \sum_{i=1}^{3} a_i b_i \quad \equiv \quad \lambda = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$c_i = S_{ik} x_k \quad \equiv \quad c_i = \sum_{k=1}^{3} S_{ik} x_k \equiv \quad \begin{cases} c_1 = S_{11} x_1 + S_{12} x_2 + S_{13} x_3 \\ c_2 = S_{21} x_1 + S_{22} x_2 + S_{23} x_3 \\ c_3 = S_{31} x_1 + S_{32} x_2 + S_{33} x_3 \end{cases}$$

$$\lambda = S_{ij} S_{ij} \quad \equiv \quad \lambda = \sum_{i=1}^{3} \sum_{j=1}^{3} S_{ij} S_{ij} \quad \equiv \quad \lambda = S_{11} S_{11} + S_{12} S_{12} + \ldots + S_{31} S_{31} + S_{32} S_{32} + S_{33} S_{3:}$$

$$C_{ij} = A_{ik} B_{kj} \quad \equiv \quad C_{ij} = \sum_{k=1}^{3} A_{ik} B_{kj} \quad \equiv \quad [C] = [A]\,[B]$$

$$C_{ij} = A_{ki} B_{kj} \quad \equiv \quad C_{ij} = \sum_{k=1}^{3} A_{ki} B_{kj} \quad \equiv \quad [C] = [A]^T\,[B]$$

In the last two equations, $[A]$, $[B]$ and $[C]$ denote the (3×3) component matrices of **A, B** and **C**.

- **The Kronecker Delta**: The symbol δ_{ij} is known as the Kronecker delta, and has the properties

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

thus

$$\delta_{11} = \delta_{22} = \delta_{33} = 1 \qquad \delta_{12} = \delta_{21} = \delta_{13} = \delta_{31} = \delta_{32} = \delta_{32} = 0$$

You can also think of δ_{ij} as the components of the identity tensor, or a (3×3) identity matrix. Observe the following useful results

$$\begin{aligned} &\delta_{ij} = \delta_{ji} \\ &\delta_{kk} = 3 \\ &a_i = \delta_{ik} a_k \\ &A_{ij} = \delta_{ik} A_{kj} \end{aligned}$$

- **The Permutation Symbol:** The symbol $\in_{ijk}$ has properties

$$\in_{ijk} = \begin{cases} 1 & i, j, k = 1, 2, 3; \quad 2, 3, 1 \quad \text{or} \quad 3, 1, 2 \\ -1 & i, j, k = 3, 2, 1; \quad 2, 1, 3 \quad \text{or} \quad 1,3,2 \\ 0 & \text{otherwise} \end{cases}$$

thus

$$\begin{aligned} &\in_{123} = \in_{231} = \in_{312} = 1 \\ &\in_{321} = \in_{213} = \in_{132} = -1 \\ &\in_{111} = \in_{112} = \in_{113} = \in_{121} = \in_{122} = \in_{131} = \in_{133} = 0 \\ &\in_{211} = \in_{212} = \in_{221} = \in_{222} = \in_{223} = \in_{232} = \in_{233} = 0 \\ &\in_{311} = \in_{313} = \in_{322} = \in_{323} = \in_{321} = \in_{332} = \in_{333} = 0 \end{aligned}$$

Note that

$$\begin{aligned} &\in_{ijk} = \in_{kij} = \in_{jki} = -\in_{jik} = -\in_{kji} = -\in_{kji} \\ &\in_{kki} = 0 \\ &\in_{ijk} \in_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{mk} \\ &\in_{ijk} \in_{lmn} = \delta_{il} \left(\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km} \right) - \delta_{im} \left(\delta_{jl} \delta_{kn} - \delta_{jn} \delta_{kl} \right) + \delta_{in} \left(\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl} \right) \end{aligned}$$

2.3. Rules of index notation

1. The same index (subscript) may not appear more than twice in a product of two (or more) vectors or tensors. Thus

$$A_{ik} x_k, \quad A_{ik} B_{kj}, \quad A_{ij} B_{ik} C_{nk}$$

are valid, but

$$A_{kk} x_k, \quad A_{ik} B_{kk}, \quad A_{ij} B_{ik} C_{ik}$$

are meaningless

2. Free indices on each term of an equation must agree. Thus

$$\begin{aligned} &x_i = u_i + c_i \qquad \equiv \qquad \mathbf{x} = \mathbf{u} + \mathbf{c} \\ &a_i = A_{ki} B_{kj} x_j + C_{ik} u_k \qquad \equiv \qquad \mathbf{a} = \mathbf{A}^T \mathbf{B}\, \mathbf{x} + \mathbf{C} \mathbf{u} \end{aligned}$$

are valid, but

$$\begin{aligned} &x_i = A_{ij} \\ &x_j = A_{ik} u_k \\ &x_i = A_{ik} u_k + c_j \end{aligned}$$

are meaningless.

3. Free and dummy indices may be changed without altering the meaning of an expression, provided that rules 1 and 2 are not violated. Thus

$$x_i = A_{ik} x_k \Leftrightarrow x_j = A_{jk} x_k \Leftrightarrow x_j = A_{ji} x_i$$

2.4. Vector operations expressed using index notation

- **Addition.** $\mathbf{c} = \mathbf{a} + \mathbf{b} \quad \equiv \quad c_i = a_i + b_i$
- **Dot Product** $\lambda = \mathbf{a} \cdot \mathbf{b} \quad \equiv \quad \lambda = a_i b_i$
- **Vector Product** $\mathbf{c} = \mathbf{a} \times \mathbf{b} \quad \equiv \quad c_i = \in_{ijk} a_j b_k$
- **Dyadic Product** $\mathbf{S} = \mathbf{a} \otimes \mathbf{b} \quad \equiv \quad S_{ij} = a_i b_j$
- **Change of Basis**. Let **a** be a vector. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a Cartesian basis, and denote the components of **a** in this basis by a_i. Let $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ be a second basis, and denote the components of **a** in this basis by α_i. Then, define

$$Q_{ij} = \mathbf{m}_i \cdot \mathbf{e}_j = \cos\theta(\mathbf{m}_i, \mathbf{e}_j)$$

where $\theta(\mathbf{m}_i, \mathbf{e}_j)$ denotes the angle between the unit vectors $\mathbf{m}_i$ and $\mathbf{e}_j$. Then

$$\alpha_i = Q_{ij} a_j$$

2.5. Tensor operations expressed using index notation.

- **Addition.** $\mathbf{C} = \mathbf{A} + \mathbf{B} \quad \equiv \quad C_{ij} = A_{ij} + B_{ij}$
- **Transpose** $\mathbf{A} = \mathbf{B}^T \quad \equiv \quad A_{ij} = B_{ji}$
- **Scalar Products** $\begin{array}{l} \lambda = \mathbf{A} : \mathbf{B} \quad \equiv \quad \lambda = A_{ij} B_{ij} \\ \lambda = \mathbf{A} \cdot\cdot \mathbf{B} \quad \equiv \lambda = A_{ji} B_{ij} \end{array}$
- **Product of a tensor and a vector** $\begin{array}{l} \mathbf{c} = \mathbf{A}\mathbf{b} \quad \equiv \quad c_i = A_{ij} b_j \\ \mathbf{c} = \mathbf{A}^T \mathbf{b} \quad \equiv \quad c_i = A_{ji} b_j \end{array}$
- **Product of two tensors** $\begin{array}{l} \mathbf{C} = \mathbf{A}\mathbf{B} \quad \equiv \quad C_{ij} = A_{ik} B_{kj} \\ \mathbf{C} = \mathbf{A}^T \mathbf{B} \quad \equiv \quad C_{ij} = A_{ki} B_{kj} \end{array}$
- **Determinant** $\begin{array}{l} \lambda = \det \mathbf{A} \quad \equiv \quad \lambda = \frac{1}{6} \in_{ijk} \in_{lmn} A_{li} A_{mj} A_{nk} = \in_{ijk} A_{i1} A_{j2} A_{k3} \\ \Leftrightarrow \ \in_{lmn} \lambda = \ \in_{ijk} A_{li} A_{mj} A_{nk} = \in_{ijk} A_{il} A_{jm} A_{kn} \end{array}$
- **Inverse** $S^{-1}_{ji} = \frac{1}{2\det(\mathbf{S})} \in_{ipq} \in_{jkl} S_{pk} S_{ql}$
- **Change of Basis**. Let **A** be a second order tensor. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a Cartesian basis, and denote the components of **A** in this basis by A_{ij}. Let $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ be a second basis, and denote the components of **A** in this basis by Λ_{ij}. Then, define

$$Q_{ij} = \mathbf{m}_i \cdot \mathbf{e}_j = \cos\theta(\mathbf{m}_i, \mathbf{e}_j)$$

where $\theta(\mathbf{m}_i, \mathbf{e}_j)$ denotes the angle between the unit vectors $\mathbf{m}_i$ and $\mathbf{e}_j$. Then

$$\Lambda_{ij} = Q_{ik} A_{km} Q_{jm}$$

2.6. Calculus using index notation

The derivative $\partial x_i / \partial x_j$ can be deduced by noting that $\partial x_i / \partial x_j = 1 \ \ i = j$ and $\partial x_i / \partial x_j = 0 \ \ i \neq j$. Therefore

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

The same argument can be used for higher order tensors

$$\frac{\partial A_{ij}}{\partial A_{kl}} = \delta_{ik}\delta_{jl}$$

2.7. Examples of algebraic manipulations using index notation

1. Let **a, b, c, d** be vectors. Prove that

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d})$$

Express the left hand side of the equation using index notation (check the rules for cross products and dot products of vectors to see how this is done)

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) \quad\equiv\quad \in_{ijk}\, a_j b_k \in_{imn}\, c_m d_n$$

Recall the identity

$$\in_{ijk}\in_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{mk}$$

so

$$\in_{ijk}\, a_j b_k \in_{imn}\, c_m d_n = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{mk})\, a_j b_k c_m d_n$$

Multiply out, and note that

$$\delta_{jm} a_j = a_m \qquad \delta_{kn} b_k = b_n$$

(multiplying by a Kronecker delta has the effect of switching indices…) so

$$(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{mk})\, a_j b_k c_m d_n = a_m b_n c_m d_n - a_n b_m c_m d_n$$

Finally, note that

$$a_m c_m \quad \equiv \mathbf{a}\cdot\mathbf{c}$$

and similarly for other products with the same index, so that

$$a_m b_n c_m d_n - a_n b_m c_m d_n = a_m c_m b_n d_n - b_m c_m a_n d_n \equiv (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d})$$

2. The stress—strain relation for linear elasticity may be expressed as

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right)$$

where σ_{ij} and ε_{ij} are the components of the stress and strain tensor, and E and ν denote Young's modulus and Poisson's ratio. Find an expression for strain in terms of stress.

Set *i=j* to see that

$$\sigma_{ii} = \frac{E}{1+\nu}\left(\varepsilon_{ii} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ii}\right)$$

Recall that $\delta_{ii} = 3$, and notice that we can replace the remaining *ii* by *kk*

$$\sigma_{kk} = \frac{E}{1+\nu}\left(\varepsilon_{kk} + \frac{\nu}{1-2\nu}3\varepsilon_{kk}\right) = \frac{E}{1-2\nu}\varepsilon_{kk}$$

$$\Leftrightarrow \quad \varepsilon_{kk} = \frac{1-2\nu}{E}\sigma_{kk}$$

Now, substitute for ε_{kk} in the given stress—strain relation

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{E}\sigma_{kk}\delta_{ij}\right)$$

$$\Leftrightarrow \varepsilon_{ij} = \frac{1+\nu}{E}\left(\sigma_{ij} - \frac{\nu}{1+\nu}\sigma_{kk}\delta_{ij}\right)$$

3. Solve the equation

$$\mu\left\{\delta_{kj} a_i a_i + \frac{1}{1-2\nu} a_k a_j\right\} U_k = P_j$$

for U_k in terms of P_i and a_i

Multiply both sides by a_j to see that

$$\mu\left\{a_j\delta_{kj}a_ia_i + \frac{1}{1-2\nu}a_ka_ja_j\right\}U_k = P_ja_j$$
$$\Leftrightarrow\ \mu\left\{a_ka_ia_i + \frac{1}{1-2\nu}a_ka_ja_j\right\}U_k = P_ja_j$$
$$\Leftrightarrow \mu U_ka_k\frac{2(1-\nu)}{1-2\nu}a_ia_i = P_ja_j \quad \Leftrightarrow U_ka_k = \frac{(1-2\nu)P_ja_j}{2\mu(1-\nu)a_ia_i}$$

Substitute back into the equation given for U_ka_k to see that

$$\mu U_ja_ia_i + \frac{P_ka_k}{2(1-\nu)a_ia_i}a_j = P_j \quad \Rightarrow U_j = \frac{1}{\mu a_ia_i}\left(P_j - \frac{P_ka_k}{2(1-\nu)a_na_n}a_j\right)$$

4. Let $r = \sqrt{x_kx_k}$. Calculate $\frac{\partial r}{\partial x_i}$

We can just apply the usual chain and product rules of differentiation

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}\frac{1}{\sqrt{x_kx_k}}\left(x_k\frac{\partial x_k}{\partial x_i} + \frac{\partial x_k}{\partial x_i}x_k\right) = \frac{1}{\sqrt{x_kx_k}}x_k\delta_{ik} = \frac{x_i}{\sqrt{x_kx_k}} = \frac{x_i}{r}$$

5. Let $\lambda = A_{ij}A_{ij}$. Calculate $\partial\lambda/\partial A_{kl}$

Using the product rule

$$\frac{\partial\lambda}{\partial A_{kl}} = A_{ij}\delta_{ik}\delta_{jl} + \delta_{ik}\delta_{jl}A_{ij} = 2A_{kl}$$