A Table of Spherical Harmonic Lore

	Property	Formula	Comments
1.	Laplacian in polar coordinates	$\nabla^2 = \frac{1}{r^2}\nabla_1^2 + \frac{1}{r}\frac{\partial^2 r}{\partial r^2}$	∇_1^2 is angular part of familiar Laplacian
2.	Eigenvalue	$\nabla_1^2 Y_l^m = -l(l+1)Y_l^m$, $l = 0, 1, 2, \cdots$	There are $2l+1$ linearly independent eigenfunctions per l
3.	Orthogonality	$\int d^2\hat{\mathbf{s}}\; Y_l^m(\hat{\mathbf{s}})\, Y_n^k(\hat{\mathbf{s}})^* = 0$, unless $l = n$ and $m = k$	True for every normalization
4.	Theoretician's normalization	$\int d^2\hat{\mathbf{s}}\; \lvert Y_l^m(\hat{\mathbf{s}})\rvert^2 = 1$	Other choices: 4π or $4\pi/(2l+1)$
5.	Completeness	$f(\hat{\mathbf{s}}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_{lm} Y_l^m(\hat{\mathbf{s}})$	Works for any reasonable function f on $S(1)$
6.	Expansion coefficients	$c_{lm} = \int d^2\hat{\mathbf{s}}\; f(\hat{\mathbf{s}})\, Y_l^m(\hat{\mathbf{s}})^*$	Requires property 4
7.	Addition Theorem	$\frac{2l+1}{4\pi} P_l(\hat{\mathbf{s}}\cdot\hat{\mathbf{r}}) = \sum_{m=-l}^{l} Y_l^m(\hat{\mathbf{s}})\, Y_l^m(\hat{\mathbf{r}})^*$	Requires property 4
8.	Wavelength of Y_l^m	$\frac{2\pi}{l+\frac{1}{2}}$	Depends only on degree l, not on order m or $\hat{\mathbf{s}}$
9.	Appearance	Re Y_l^m vanishes on $2m$ meridians and $l-m$ parallels	Im Y_l^m the same but rotated about $\hat{\mathbf{z}}$
10.	Parseval's Theorem	$\int d^2\hat{\mathbf{s}}\; \lvert f(\hat{\mathbf{s}})\rvert^2 = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \lvert c_{lm}\rvert^2$	Requires property 4. Get RMS value of f by dividing by 4π and taking square root
11.	Generating function	$\frac{1}{(1-2\mu r+r^2)^{\frac{1}{2}}} = \sum_{l=0}^{\infty} r^l P_l(\mu)$	Often used in conjunction with property 7
12.	Another orthogonality	$\int d^2\hat{\mathbf{s}}\; \nabla_1 Y_l^m(\hat{\mathbf{s}})\cdot\nabla_1 Y_n^k(\hat{\mathbf{s}})^* = l(l+1)\,\delta_{ln}\,\delta_{mk}$	Requires property 4.