(Local) In[3]:=

```
AppendTo[$Path, NotebookDirectory[]];
<< Matte`
```

```
Matte package loaded. © 2017-2018 Jonas Einarsson (me@jonaseinarsson.se)
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```

Tensor Spherical Harmonics are made up of *irreducible* rank-n tensors

These are described nicely in Leal Advanced Transport Phenomena Pg. 527 (within Chapter 8 Creeping Flows)
They can be developed from the solutions to the Laplace equation in spherical coordinates. Where you are taking repeated gradients of (1/r).

I.E. ∇(∇(1/r)) not Laplacian but a gradient of a gradient.

That should be correct, but constants are really important and magnitude of r is equal to 1, and so it is often written in a different method.

$\phi_{-(n+1)} = \frac{(-1)^n}{(2n-1)!!} \nabla_1 \ldots \nabla_n \left(\frac{1}{r}\right)$, n = 0, 1, 2,...

That technically gets you the -(n+1) th decaying harmonic, but then the nth order irreducible tensor just has you substitute $|r| = 1$

First use a recurrence relation to get the repeated gradient of $\frac{1}{r}$ where invr[1] is $\frac{1}{r}$

(Local) In[5]:=

```
RepeatedPartialR[0] := invr[1];
RepeatedPartialR[n_] := partialr[RepeatedPartialR[n - 1], idx[n]] // contract;
```

Now use that to get the decaying harmonics and replace inverse r to any power with 1

```
IrreducibleRanknTensor[n_] :=
  ReplaceAll[invr[x_] :> 1][(-1)^n/(2*n - 1)!! RepeatedPartialR[n] // contract];
```

So for *any* vector r, the irreducible rank-n tensor that describes it

```
IrreducibleRanknTensor [0] // prettyprint
IrreducibleRanknTensor [1] // prettyprint
IrreducibleRanknTensor [2] // prettyprint
IrreducibleRanknTensor [3] // prettyprint
IrreducibleRanknTensor [4] // prettyprint
IrreducibleRanknTensor [5] // prettyprint
```

(Local) Out[21]=

$$1$$

(Local) Out[22]=

$$r_i$$

(Local) Out[23]=

$$r_i\, r_j - \frac{\delta_{i,j}}{3}$$

(Local) Out[24]=

$$r_i\, r_j\, r_k - \frac{1}{5}\, r_k\, \delta_{i,j} - \frac{1}{5}\, r_j\, \delta_{i,k} - \frac{1}{5}\, r_i\, \delta_{j,k}$$

(Local) Out[25]=

$$r_i\, r_j\, r_k\, r_l - \frac{1}{7}\, r_k\, r_l\, \delta_{i,j} - \frac{1}{7}\, r_j\, r_l\, \delta_{i,k} - \frac{1}{7}\, r_j\, r_k\, \delta_{i,l} - \frac{1}{7}\, r_i\, r_l\, \delta_{j,k} + \frac{1}{35}\, \delta_{i,l}\, \delta_{j,k} - \frac{1}{7}\, r_i\, r_k\, \delta_{j,l} + \frac{1}{35}\, \delta_{i,k}\, \delta_{j,l} - \frac{1}{7}\, r_i\, r_j\, \delta_{k,l} + \frac{1}{35}\, \delta_{i,j}\, \delta_{k,l}$$

(Local) Out[26]=

$$\begin{aligned}
&r_i\, r_j\, r_k\, r_l\, r_p - \frac{1}{9}\, r_k\, r_l\, r_p\, \delta_{i,j} - \frac{1}{9}\, r_j\, r_l\, r_p\, \delta_{i,k} - \frac{1}{9}\, r_j\, r_k\, r_p\, \delta_{i,l} - \frac{1}{9}\, r_j\, r_k\, r_l\, \delta_{i,p} - \frac{1}{9}\, r_i\, r_l\, r_p\, \delta_{j,k} + \\
&\frac{1}{63}\, r_p\, \delta_{i,l}\, \delta_{j,k} + \frac{1}{63}\, r_l\, \delta_{i,p}\, \delta_{j,k} - \frac{1}{9}\, r_i\, r_k\, r_p\, \delta_{j,l} + \frac{1}{63}\, r_p\, \delta_{i,k}\, \delta_{j,l} + \frac{1}{63}\, r_k\, \delta_{i,p}\, \delta_{j,l} - \\
&\frac{1}{9}\, r_i\, r_k\, r_l\, \delta_{j,p} + \frac{1}{63}\, r_l\, \delta_{i,k}\, \delta_{j,p} + \frac{1}{63}\, r_k\, \delta_{i,l}\, \delta_{j,p} - \frac{1}{9}\, r_i\, r_j\, r_p\, \delta_{k,l} + \frac{1}{63}\, r_p\, \delta_{i,j}\, \delta_{k,l} + \\
&\frac{1}{63}\, r_j\, \delta_{i,p}\, \delta_{k,l} + \frac{1}{63}\, r_i\, \delta_{j,p}\, \delta_{k,l} - \frac{1}{9}\, r_i\, r_j\, r_l\, \delta_{k,p} + \frac{1}{63}\, r_l\, \delta_{i,j}\, \delta_{k,p} + \frac{1}{63}\, r_j\, \delta_{i,l}\, \delta_{k,p} + \\
&\frac{1}{63}\, r_i\, \delta_{j,l}\, \delta_{k,p} - \frac{1}{9}\, r_i\, r_j\, r_k\, \delta_{l,p} + \frac{1}{63}\, r_k\, \delta_{i,j}\, \delta_{l,p} + \frac{1}{63}\, r_j\, \delta_{i,k}\, \delta_{l,p} + \frac{1}{63}\, r_i\, \delta_{j,k}\, \delta_{l,p}
\end{aligned}$$

Here are the repeated gradients of 1/r to show you that they are equivalent, just divided by leading constants.

(Local) In[14]:=

```
x1 = partialr[invr[1], idx[1]] // contract;
x2 = partialr[x1, idx[2]] // contract;
x3 = partialr[x2, idx[3]] // contract;
x1 // prettyprint
x2 // prettyprint
x3 // prettyprint
```

(Local) Out[17]=

$$-\frac{r_i}{r^3}$$

(Local) Out[18]=

$$\frac{3\, r_i\, r_j}{r^5} - \frac{\delta_{i,j}}{r^3}$$

(Local) Out[19]=

$$-\frac{15\, r_i\, r_j\, r_k}{r^7} + \frac{3\, r_k\, \delta_{i,j}}{r^5} + \frac{3\, r_j\, \delta_{i,k}}{r^5} + \frac{3\, r_i\, \delta_{j,k}}{r^5}$$