Optimization of a Complex Shape Spar under Uncertain Loading

UNIQUE ID: 1075

Executive Summary

The objective of this project was to optimize a quasi-three dimensional spar so as to minimize its weight in a similar fashion as was done in *Project 2: Optimization of a Complex Shape Spar*. Much of the project's assumptions, limitations, and methods were kept constant, and in effect, this project is just an extension of *Project 2*. The key differentiating factor is the fact that the loading of the spar is taken to be uncertain, and as such, necessitates that the spar geometry will need to accommodate for this uncertainty. The methods under which this is done are elaborated later in this paper.

The goal of this project was to achieve a spar weight 70% lower than the nominal weight of 287.63 N, or 29.32 kg (Corresponding to 86.29 N or 8.80 Kg). Though the results and methods through which an optimal geometry was achieved are detailed in this report, a greater emphasis was placed on the analysis methods through which the uncertain loads could be accommodated into the stress constraints in the spar. In addition to this, the effects of different number of Gauss-Hermite Quadrature Points were also analyzed with respect to computation time, and percent improvement over nominal mass.

Some of the key findings are as follows:

1. With greater number of elements, the probability of solutions that violate Euler-Bernoulli Beam Theory assumptions greatly increases (In particular the geometry becomes non-smooth)
2. A number of 3 Gauss-Hermite Quadrature Points was sufficient for the accurate optimization of the complex spar, while maintaining a reasonable computation time
3. A number of 40 elements was determined to be ideal for reducing error in stress calculations
4. The minimum weight achieved by this optimization was 84.28 N (or 8.5909 Kg), using 40 spar elements and 3 Gauss-Hermite Quadrature Points

Analysis Method

As was done in *Project 2: Optimization of a Complex Shape Spar*, a quasi-three-dimensional spar was optimized in order to minimize the weight given geometric and stress constraints. The spar was modelled as a circular annulus, with inner and outer radii which would change along the length of the spar. These radii acted as design variables in the optimization.

The inner and outer radii of the spar would be bounded by a minimum and maximum radii ($\boldsymbol{r_{min}} = \mathbf{0.01}\boldsymbol{m}$, $\boldsymbol{r_{max}} = \mathbf{0.05}\boldsymbol{m}$), along with a minimum thickness ($\boldsymbol{t_{min}} = \mathbf{0.0025}\boldsymbol{m}$). In addition, the maximum stress at each point in the spar is bounded by an uncertain constraint, where the mean plus 6 standard deviations of the stress must be lower than the ultimate strength of the spar material.

$$\mathbf{E[s(x,\xi)] + 6\ p\ Var[s(x,\xi)] \leq syield}$$

In this equation, $\mathrm{E[s(x,\xi)]}$ is the expected value of the stress at a position, x, along the spar, and with an uncertain parameter ξ. The uncertain parameter comes about due to an uncertain loading on the spar that is characterized by:

$$\mathbf{f(x,\xi)} = \boldsymbol{f_{nom}}\mathbf{(x)} + \boldsymbol{\delta_f}\mathbf{(x,\xi)}$$

$$\boldsymbol{f_{nom}(x)} = \frac{\mathbf{2.5}\boldsymbol{W}}{\boldsymbol{L}}\left(\mathbf{1} - \frac{\boldsymbol{x}}{\boldsymbol{L}}\right)$$

$$\boldsymbol{\delta_f(x,\xi)} = \sum_{n=1}^{4} \boldsymbol{\xi_n} \cos\left(\frac{(\mathbf{2}\boldsymbol{n} - \mathbf{1})\boldsymbol{\pi x}}{\mathbf{2}\boldsymbol{L}}\right)$$

- $\boldsymbol{\xi_n} \sim \mathbf{N}\left(\mathbf{0}, \frac{\mathrm{f}_{nom}(\mathbf{0})}{\mathbf{10}\boldsymbol{n}}\right)$: Normally distributed random number with $\mu = 0$ and $\sigma = \frac{\mathrm{f}_{nom}(0)}{10n}$
- $\boldsymbol{f_{nom}(x)}$: The loading on the spar under with no uncertainty
- $\boldsymbol{\delta_f}\mathbf{(x,\xi)}$: The probabilistic perturbation of loading on the spar
- $\mathbf{f}_{\boldsymbol{nom}}\mathbf{(0)}$: The loading at the root of the spar
- L: The total length of the spar

As was done in *Project 2*, the calculation of stress for use in the constraints is done using Euler-Bernoulli Beam Theory. Some inherent assumptions are that there is planar symmetry along the spar, the cross-section of the spar varies smoothly with x, and that the internal strain within the spar accounts only for bending moment deformations. It is also assumed that the spar material is elastic and isotropic.

Though it is assumed that the cross section varies smoothly along the spar length, it was shown in experiments that solutions that violate this assumption can be obtained for certain numbers of spar elements.

Geometry

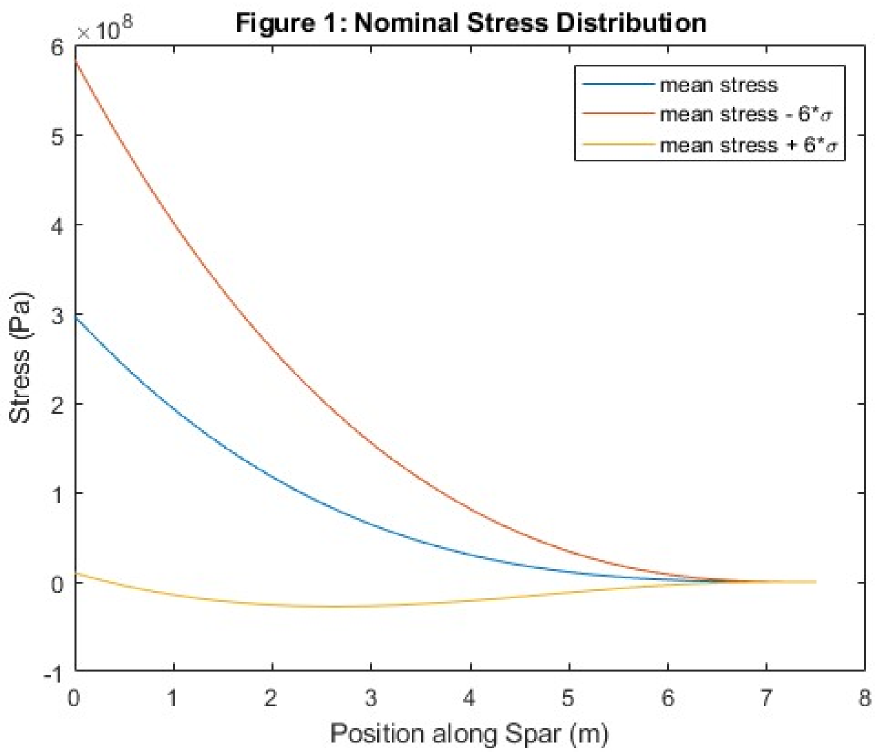
The spar geometry was defined by a set of inner and outer radii, located at nodes along the spar. The nodes discretize the spar into elements which are then used in calculation of the beam displacements and corresponding stress. There are *Nelem+1* evenly spaced nodes along the spar, where '*Nelem'* refers to the number of elements in the spar.

Gauss-Hermite Quadrature

In order to optimize with these uncertain parameters, ξ_n, stochastic collocation is used (as compared to Monte Carlo Simulation) due to the fact that the number of uncertain parameters is relatively small. In particular, Gauss-Hermite Quadrature is used, with determination of the ideal number of quadrature points being an emphasis of one of the studies conducted in this project. An inherent assumption in the usage of this method is that the objective function (the weight function) is sufficiently smooth.

Confirmation of Analysis Method

In order to confirm the analysis methods and the methods used to implement the uncertainty of the spar loading were accurate, the stress distribution on the spar was plotted for the nominal design and compared to the predicted values.



As can be seen in **Figure 1: Nominal Stress Distribution**, the stress distribution along the spar given this analysis model accurately represents the nominal design for $r_{in} = 0.415m$ and $r_{out} = 0.05m$. The spar stress is at a maximum at the root of the spar, while the stress is approximately zero at the tip if the spar, which corresponds to the assumptions made with Euler-Bernoulli Beam Theory when considering a cantilevered beam (spar).

Optimization Methods

The optimization of the spar was done using Matlab's built-in function *fmincon*. Using the nonlinear stress constraint, linear thickness constraint, and the radii bound constraints in the optimization, a solution could be achieved that satisfied all system constraints. The *'SQP'* algorithm was used as this proved to be a significant improvement over the default algorithm, and is the same algorithm that was used in *Project 2*.

An initial guess for the optimizer was kept near the upper and lower bound constraints of the geometry, as these initial values satisfied all the constraints. As was determined in *Project 2*, randomized initial guesses tended to result in assumption-violating optimized geometry at a much higher rate than with constant inner and outer radii, so these were not implemented.

As would be seen in initial testing, as the number of elements increases, the probability that optimized solution contained non-smooth geometry also increases. As such, for both decreased computation time, and a decrease in this probability, the number of elements used in the initial optimization procedure was set at 15 elements.

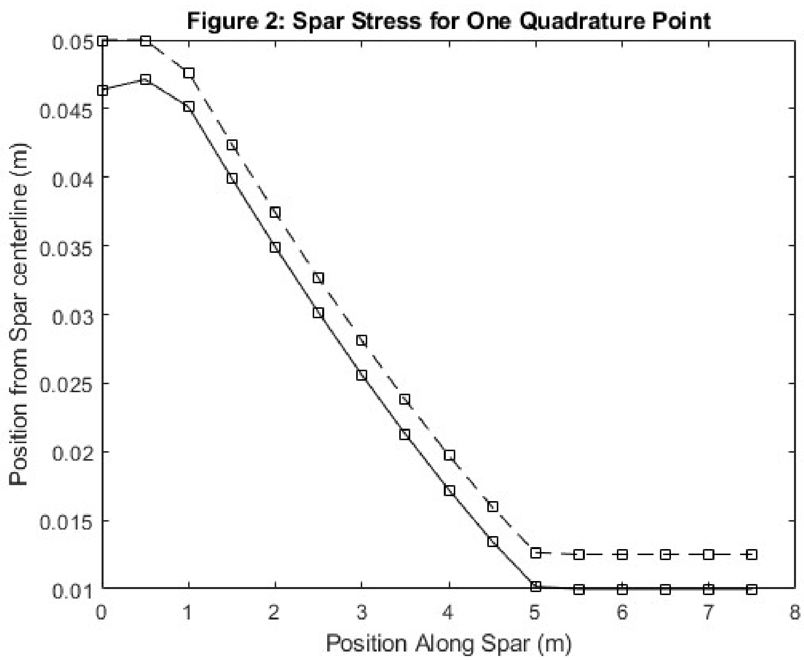
Results

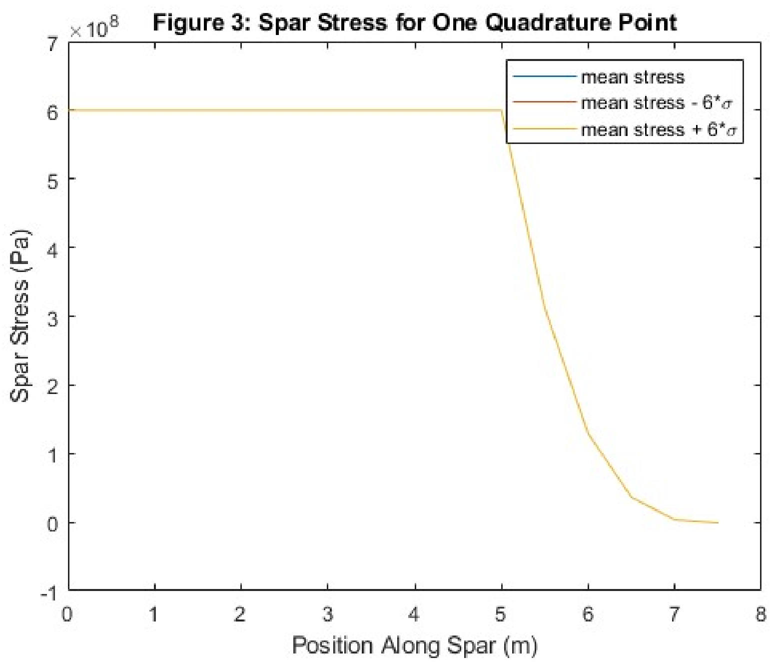
Initial experimentation showed that with higher numbers of elements, the higher the probability that the optimizer would converge to a solution that had non-smooth geometry. In order to combat this, a lower number of elements was used to optimize spar geometry to minimize weight. Studies were conducted in order to determine the optimal number of Gauss-Hermite quadrature points for optimization, and the effect of the number of elements on tip stress convergence.

Effect of Number of Quadrature Points on Optimization

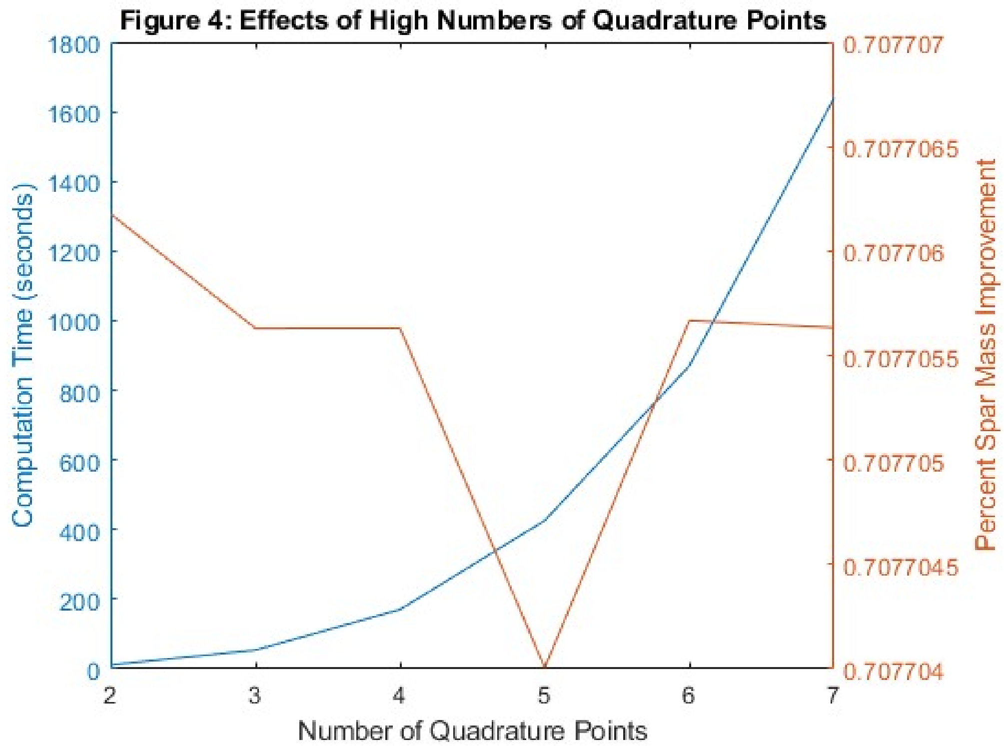
In initial testing, it was concluded the number of Gauss-Hermite quadrature points needed to optimize the spar would need to be determined keeping in mind computation time, accuracy, and optimization convergence.

With only one quadrature point, the geometry obtained was the optimal solution from *Project 2*. With this solution, the standard deviation of the mean stress was zero. Though this is a valid solution in terms of constraints being fulfilled, by only using one quadrature point, the solution entirely neglects the uncertainty in the loading. In order to get a valid solution in terms of the uncertainty, more than one quadrature point will need to be used. The plots of the geometry and stress for one quadrature point is seen in **Figure 2: Spar Geometry for One Quadrature Point** and **Figure 3: Spar Stress for One Quadrature Point**.





With more than one quadrature point, the optimized geometry satisfies all constraints, however, a study was done in order compare higher numbers of quadrature points in terms of optimized solution convergence and computation time. As shown below in **Figure 4: Effects of High Numbers of Quadrature Points**, the percent spar mass improvement remains relatively constant over the number of quadrature points tested when greater than 2. An exception is when using 5 quadrature points, however, further optimization was not done using this due to the significantly higher computation time as compared to lower numbers of quadrature points.

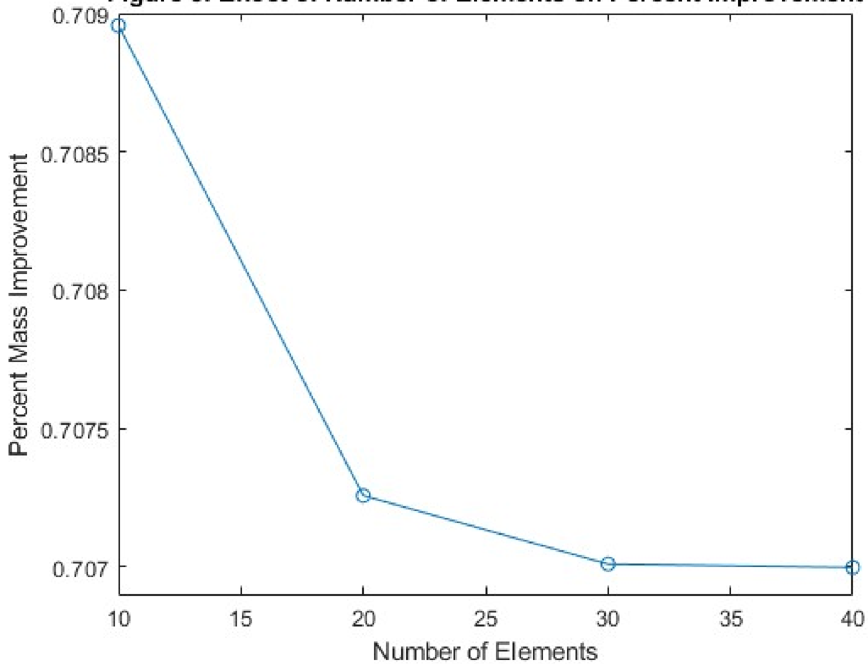


As can be seen in the figure, using more quadrature points exponentially increases the computation time, therefore it is reasonable to choose a lower number of quadrature points, especially when the study shows minimal gains in accuracy with higher number of points. For all future experiments, 3 quadrature points are used.

Effect of Number of Elements on Mass and Root-Normalized Tip Stress

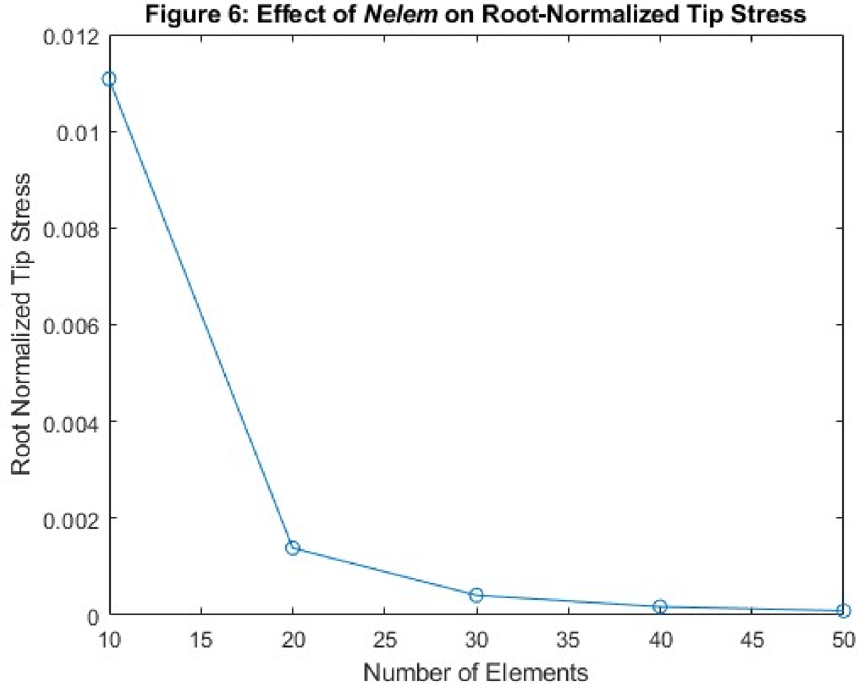
This study was done in order to determine the effect of increasing the number of spar elements on the root-normalized tip stress and the percent mass improvement over the nominal design. As can be seen in **Figure 5: Effect of Number of Elements on Percent Improvement**, as the number of elements increases, the percent mass improvement converges to approximately 70.7%, with the solution being sufficiently converged at 30 elements. Notably, at only 15 elements, the solution has a mass that is less than 2% difference in value to this converged value.

Figure 5: Effect of Number of Elements on Percent Improvement



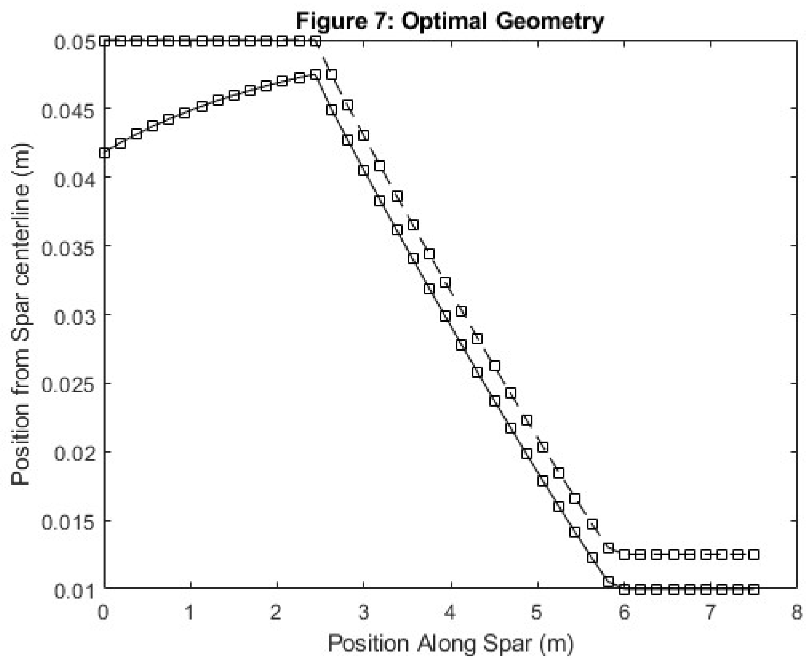
The study on root-normalized tip convergence was conducted in order to determine the accuracy of the optimization as the tip stress should be zero. As the number of elements increases, this values approaches zero. It can be seen from **Figure 6: Effect of Nelem on Root-Normalized Tip Stress** that at 30 elements, the tip stress has converged to less than 0.2% difference in the expected stress and the experimental stress.

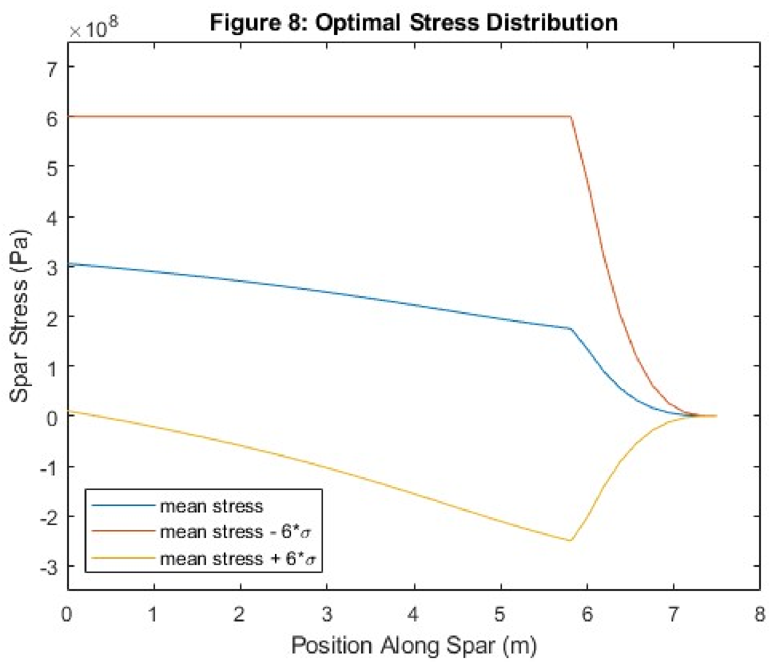
Figure 6: Effect of *Nelem* on Root-Normalized Tip Stress



Spar weight optimization

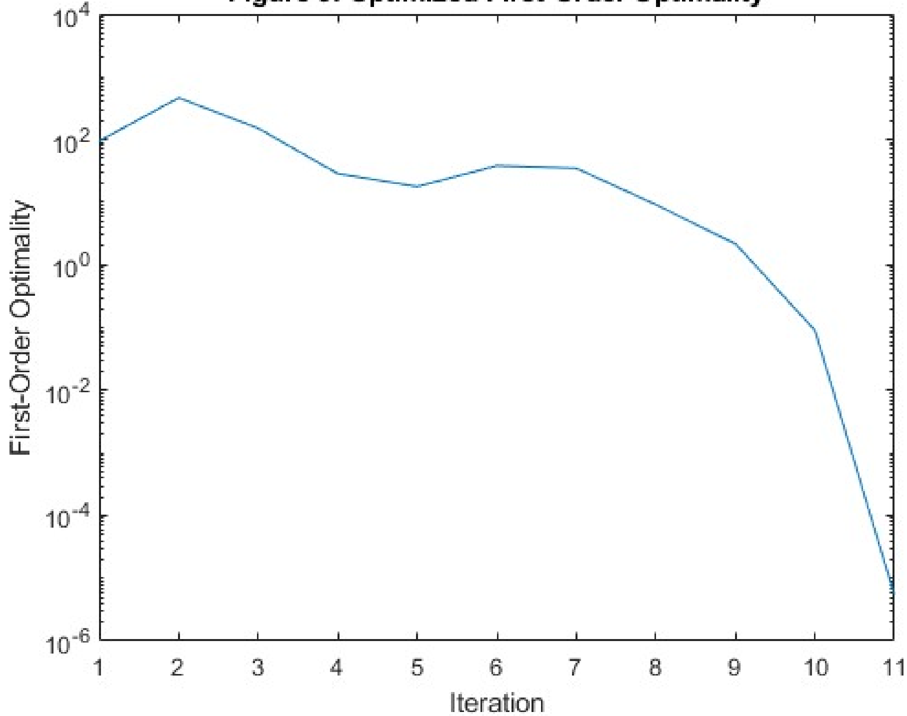
Using the solutions achieved through the study on element convergence, an optimal solution was achieved. Using 40 spar elements and 3 Gauss-Hermite Quadrature Points, an optimal mass of **8.5909 Kg** (or 84.28 N) was achieved. As shown below in **Figure 7: Optimal Geometry**, the geometry converges to solution that is similar to what was obtained in Project 2, except for a larger section at the root of the spar. Also shown below in **Figure 8: Optimized Stress Distribution** is the mean stress distribution along the spar, along with bands corresponding to the mean plus or minus six times the standard deviation of the stress along the spar.





It can be shown that this solution is the optimal solution through the convergence history of the first-order optimality. As shown in **Figure 9: Optimized First-Order Optimality**, the optimality reduces by more than seven orders of magnitude. From this significant reduction, it can be inferred that this solution is a local minimum.

Figure 9: Optimized First-Order Optimality



Conclusions

As concluded from the element convergence study, the ideal number of spar elements for optimization that sufficiently reduces tip error within the spar while maintaining a low computation cost is 40 elements. Though increasing this value further may further reduce error, it will also exponentially increase computation cost and increase the probability that a non-smooth solution will result. The final optimized geometry resulted in a weight reduction from nominal of 70.7 % - to an optimal mass of 8.5909 Kg (or 84.28 N).

Code Appendix

Quadrature Point Study

```
% minimize wing spar weight subject to stress constraints at 2.5g maneuver. This code contains
% a study to determine the ideal number of Gauss-Hermite Quadrature Points
clear all;
close all;
clc

global std_stress
global mean_stress

% carbon fiber values from http://www.performance-
composites.com/carbonfibre/mechanicalproperties_2.asp
Nelem = 15;
L = 7.5; % semi-span in meters
rho = 1600; % density of standard carbon fiber, kg/m^3
yield = 600e6; % tensile strength of standard carbon fiber, Pa
E = 70e9; % Young's modulus, Pa
W = 0.5*500*9.8; % half of the operational weight, N
f_nom = (2*(2.5*W)/(L^2))*[L:-L/Nelem:0].'; % loading at manueuver
nominal_mass = 29.32;
% define function and constraints
fun = @(r) SparWeight(r, L, rho, Nelem);
lb = 0.01*ones(2*(Nelem+1),1);
up = 0.05*ones(2*(Nelem+1),1);
A = zeros(Nelem+1,2*(Nelem+1));
b = -0.0025*ones(Nelem+1,1);
for k = 1:(Nelem+1)
    A(k,k) = 1.0;
    A(k,Nelem+1+k) = -1.0;
end

% define initial guess (the nominal spar)
r0 = zeros(2*(Nelem+1),1);
r0(1:Nelem+1) = 0.0415*ones(Nelem+1,1);
r0(Nelem+2:2*(Nelem+1)) = 0.05*ones(Nelem+1,1);

options = optimset('GradObj','on','GradConstr','on', 'TolCon', 1e-4, ...
    'TolX', 1e-8, 'Display','iter','Algorithm','sqp'); %, 'DerivativeCheck','on');

data_fval = [];
data_opt_dvs = [];
mean_stresses = [];
stdevs_stresses = [];
t_end = []; % comp times for each opt (with differing quad points)

for i = 1:7 % up to 7 quadrature points
t_start = tic; % start timer for calculating computation time
nQuadPts = i; % Number of Hermite Quadrature Points
nonlcon = @(r) WingConstraints(r, L, E, f_nom, yield, Nelem,nQuadPts);
[ropt,fval,exitflag,output] = fmincon(fun, r0, A, b, [], [], lb, up, ...
    nonlcon, options);

% plot optimal radii
r_in = ropt(1:Nelem+1);
r_out = ropt(Nelem+2:2*(Nelem+1));
x = [0:L/Nelem:L].';
figure(2*i - 1)
plot(x, r_in, '-ks');
hold on;
plot(x, r_out, '--ks');
title_str = 'Spar Geometry for Uncertain Loading with %d Quadrature Points';
title(sprintf(title_str,i))
xlabel('Position Along Spar (m)')
ylabel('Position from Spar centerline (m)')
dim = [.6 .5 .3 .3];
```

```
str = sprintf('mass = %d',fval);
annotation('textbox',dim,'String',str,'FitBoxToText','on');


% display weight and stress constraints
[f,~] = fun(ropt);
[c,~,~,~] = nonlcon(ropt);

% Plot optimal stress
figure(2*i)
plot(x,real(mean_stress))
hold on
plot(x,(real(mean_stress) + 6*real(std_stress)))
hold on
plot(x,(real(mean_stress) - 6*real(std_stress)))
title_str = 'Mean Spar Stress with Uncertainty for %d Quadrature Points';
title(sprintf(title_str,i))
xlabel('Position Along Spar (m)')
ylabel('Spar Stress (Pa)')
legend('mean stress','mean stress - 6*\sigma','mean stress + 6*\sigma')

data_fval(i) = f;
data_opt_dvs(:,i) = ropt;
mean_stresses(:,i) = real(mean_stress);
stdevs_stresses(:,i) = real(std_stress);

t_end(i) = toc(t_start);

end

% Plot effect of quad points on computation time and percent improvement
n_quadPts = 1:1:length(t_end);
figure(25)
yyaxis left
plot(n_quadPts(2:end),t_end(2:end))
title('Effect of Number of Quadrature Points on Computation Time')
xlabel('Number of Quadrature Points')
ylabel('Computation Time (seconds)')
yyaxis right
plot(n_quadPts(2:end),abs(data_fval(2:end)-nominal_mass)/nominal_mass)
ylabel('Percent Spar Mass Improvement')
```

Wing Constraints

```
function [c, ceq, dcdx, dceqdx] = WingConstraints(x, L, E, fnom, yield, Nelem,nQuadPts)
% Computes the nonlinear inequality constraints for the wing-spar problem
% Inputs:
%   x - the DVs; x(1:Nelem+1) inner and x(Nelem+2:2*(Nelem+1) outer radius
%   L - length of the beam
%   E - longitudinal elastic modulus
%   force - force per unit length along the beam axis x
%   yield - the yield stress for the material
%   Nelem - number of finite elements to use
% Outputs:
%   c, ceq - inequality (stress) and equality (empty) constraints
%   dcdx, dceqdx - Jacobians of c and ceq
%--------------------------------------------------------------------------

global std_stress
global mean_stress

assert( size(fnom,1) == (Nelem+1) );
assert( size(x,1) == (2*(Nelem+1)) );

c = CalcInequality(x);
ceq = [];
dcdx = zeros(2*(Nelem+1),Nelem+1);
dceqdx = [];
for k = 1:2*(Nelem+1)
    xc = x;
    xc(k) = xc(k) + complex(0.0, 1e-30);
    dcdx(k,:) = imag(CalcInequality(xc))/1e-30;
end

    function cineq = CalcInequality(dvar)
        % compute the displacements and the stresses
        r_in = dvar(1:Nelem+1);
        r_out = dvar(Nelem+2:2*(Nelem+1));
        Iyy = CalcSecondMomentAnnulus(r_in, r_out);

        node_locs = linspace(0,L,Nelem+1);
        mu = 0;
        sigman = @(n) fnom(1,1)/(10*n);

        if nQuadPts == 1
            % n=1 point
            xi = [0];
            wts = [sqrt(pi)]./sqrt(pi);
        elseif nQuadPts == 2
            % n=2 points
            xi = [0.707107 -0.707107];
            wts = [0.886227 0.886227]./sqrt(pi);
        elseif nQuadPts == 3
            % n=3 points
            xi = [-1.22474487 0 1.22474487];
            wts = [0.295408975 1.1816359 0.295408975]./sqrt(pi);
        elseif nQuadPts == 4
            % n=4 points
            xi = [-1.6506801 -0.52464762 0.52464762 1.6506801];
            wts = [0.081312835 0.80491409 0.80491409 0.081312835]./sqrt(pi);
        elseif nQuadPts == 5
            % n=5 points
            xi = [0 -0.958572 0.958572 -2.02018 2.02018];
            wts = [0.945309 0.393619 0.393619 0.0199532 0.0199532]./sqrt(pi);
       elseif nQuadPts == 6
            % n=6 points
            xi = [-2.3506050 -1.3358491 -0.43607741 0.43607741 1.3358491 2.3506050];
            wts = [0.0045300099 0.15706732 0.72462960 0.72462960 0.15706732
0.0045300099]./sqrt(pi);
        elseif nQuadPts == 7
            % n=7 points
```

```
            xi = [-2.651961357, -1.673551629, -0.8162878829, 0, 0.816287883, 1.673551629,
2.651961357];
            wts = [9.71781245E-4, 0.05451558282, 0.4256072526, 0.8102646176, 0.4256072526,
0.0545155828, 9.71781245E-4]./sqrt(pi);
        else
            % Error
        end

        f = @(x,xi1,xi2,xi3,xi4) fnom.' + xi1*cos((2*1-1)*pi.*x./(2*L)) + xi2*cos((2*2-
1)*pi.*x./(2*L)) + xi3*cos((2*3-1)*pi.*x./(2*L)) + xi4*cos((2*4-1)*pi.*x./(2*L));
        mean_stress = 0;
        mean_squared_stress = 0;
        for i1 = 1:size(xi,2)
            pt1 = sqrt(2)*sigman(1)*xi(i1) + mu;
            for i2 = 1:size(xi,2)
                pt2 = sqrt(2)*sigman(2)*xi(i2) + mu;
                for i3 = 1:size(xi,2)
                    pt3 = sqrt(2)*sigman(3)*xi(i3) + mu;
                    for i4 = 1:size(xi,2)
                        pt4 = sqrt(2)*sigman(4)*xi(i4) + mu;
                        mean_f = f(node_locs,pt1,pt2,pt3,pt4);
                        u = CalcBeamDisplacement(L, E, Iyy, mean_f, Nelem);
                        stress = CalcBeamStress(L, E, r_out, u, Nelem);
                        mean_stress = mean_stress + wts(i1)*wts(i2)*wts(i3)*wts(i4).*stress;
                        mean_squared_stress = mean_squared_stress +
wts(i1)*wts(i2)*wts(i3)*wts(i4).*(stress.^2);
                    end
                end
            end
        end

        std_stress = sqrt(abs(mean_squared_stress - (mean_stress.^2)));
        cineq = (mean_stress+(6*std_stress))./yield - ones(Nelem+1,1);

    end
end
```

Nelem Convergence Study

```
% This code is used to conduct a study on the number of elements in the spar and the effect of
% the number of elements on the convergence of percent improvement and root-normalized tip stress
clear all;
close all;
clc

global std_stress
global mean_stress

% carbon fiber values from http://www.performance-
composites.com/carbonfibre/mechanicalproperties_2.asp
L = 7.5; % semi-span in meters
rho = 1600; % density of standard carbon fiber, kg/m^3
yield = 600e6; % tensile strength of standard carbon fiber, Pa
E = 70e9; % Young's modulus, Pa
W = 0.5*500*9.8; % half of the operational weight, N
nominal_mass = 29.32;

Nelem = 10;

data_fval = [];
data_opt_dvs = [];
mean_stresses = [];
stdevs_stresses = [];
data_tip_stress = [];
t_end = []; % comp times for each opt (with differing quad points)


for i = Nelem:10:60

Nelem = i;

f_nom = (2*(2.5*W)/(L^2))*[L:-L/Nelem:0].'; % loading at manueuver

% define function and constraints
fun = @(r) SparWeight(r, L, rho, Nelem);
lb = 0.01*ones(2*(Nelem+1),1);
up = 0.05*ones(2*(Nelem+1),1);
A = zeros(Nelem+1,2*(Nelem+1));
b = -0.0025*ones(Nelem+1,1);
for k = 1:(Nelem+1)
    A(k,k) = 1.0;
    A(k,Nelem+1+k) = -1.0;
end

% define initial guess (the nominal spar)
r0 = zeros(2*(Nelem+1),1);
r0(1:Nelem+1) = 0.0415*ones(Nelem+1,1);
r0(Nelem+2:2*(Nelem+1)) = 0.05*ones(Nelem+1,1);

options = optimset('GradObj','on','GradConstr','on', 'TolCon', 1e-4, ...
    'TolX', 1e-8, 'Display','iter','Algorithm','sqp'); %, 'DerivativeCheck','on');


% t_start = tic; % start timer for calculating computation time
nQuadPts = 3; % Number of Hermite Quadrature Points
nonlcon = @(r) WingConstraints(r, L, E, f_nom, yield, Nelem,nQuadPts);
[ropt,fval,exitflag,output] = fmincon(fun, r0, A, b, [], [], lb, up, ...
    nonlcon, options);

% plot optimal radii
r_in = ropt(1:Nelem+1);
r_out = ropt(Nelem+2:2*(Nelem+1));
x = [0:L/Nelem:L].';
figure(2*i - 1)
plot(x, r_in, '-ks');
hold on;
plot(x, r_out, '--ks');
```

```
title_str = 'Spar Geometry for Uncertain Loading with 3 Quadrature Points';
title(sprintf(title_str,i))
xlabel('Position Along Spar (m)')
ylabel('Position from Spar centerline (m)')

% display weight and stress constraints
[f,~] = fun(ropt);
[c,~,~,~] = nonlcon(ropt);

data_fval(i/10) = fval;
data_tip_stress(i/10) = real(mean_stress(end));
end

n_Elems = 10:10:60;
figure(10)
plot(n_Elems,abs(data_fval-nominal_mass)/nominal_mass,'-o')
ylabel('Percent Spar Mass Improvement')
xlabel('Number of Spar Elements')
title('Effect of Number of Elements on Spar Mass Optimization')
ylim([0.7069,0.709])

figure(11)
plot(x,real(mean_stress))
hold on
plot(x,(real(mean_stress) + 6*real(std_stress)))
hold on
plot(x,(real(mean_stress) - 6*real(std_stress)))
title_str = 'Mean Spar Stress with Uncertainty for 3 Quadrature Points and 40 Elements';
title(sprintf(title_str,i))
xlabel('Position Along Spar (m)')
ylabel('Spar Stress (Pa)')
legend('mean stress','mean stress - 6*\sigma','mean stress + 6*\sigma','location','southwest')
ylim([-3.5e8, 7.5e8])

figure(12)
plot(x, ropt(1:Nelem+1), '-ks');
hold on;
plot(x, ropt(Nelem+2:end), '--ks');
title_str = 'Spar Geometry for Uncertain Loading with 3 Quadrature Points and 40 Elements';
title(sprintf(title_str,3))
xlabel('Position Along Spar (m)')
ylabel('Position from Spar centerline (m)')
```

Nominal Stress Plotting

```
% This code is used to plot the nominal stress and corresponding standard deviation bands
% given the nominal design, using any number of quadrature points (though this has little effect)
clear all;
close all;
clc

global std_stress
global mean_stress
% carbon fiber values from http://www.performance-
composites.com/carbonfibre/mechanicalproperties_2.asp
Nelem = 50;
L = 7.5; % semi-span in meters
rho = 1600; % density of standard carbon fiber, kg/m^3
yield = 600e6; % tensile strength of standard carbon fiber, Pa
E = 70e9; % Young's modulus, Pa
W = 0.5*500*9.8; % half of the operational weight, N
fnom = (2*(2.5*W)/(L^2))*[L:-L/Nelem:0].'; % loading at manueuver

nQuadPts = 3; % Number of Hermite Quadrature Points

% compute the displacements and the stresses
r_in = 0.0415*ones(1,Nelem+1).';
r_out = 0.05*ones(1,Nelem+1).';
Iyy = CalcSecondMomentAnnulus(r_in, r_out);

x = linspace(0,L,Nelem+1);
mu = 0;
sigman = @(n) fnom(1)/(10*n);

if nQuadPts == 1
    % n=1 point
    xi = [0];
    wts = [sqrt(pi)]./sqrt(pi);
elseif nQuadPts == 2
    % n=2 points
    xi = [0.707107 -0.707107];
    wts = [0.886227 0.886227]./sqrt(pi);
elseif nQuadPts == 3
    % n=3 points
    xi = [-1.22474487 0 1.22474487];
    wts = [0.295408975 1.1816359 0.295408975]./sqrt(pi);
elseif nQuadPts == 4
    % n=4 points
    xi = [-1.6506801 -0.52464762 0.52464762 1.6506801];
    wts = [0.081312835 0.80491409 0.80491409 0.081312835]./sqrt(pi);
elseif nQuadPts == 5
    % n=5 points
    xi = [0 -0.958572 0.958572 -2.02018 2.02018];
    wts = [0.945309 0.393619 0.393619 0.0199532 0.0199532]./sqrt(pi);
elseif nQuadPts == 6
    % n=6 points
    xi = [-2.3506050 -1.3358491 -0.43607741 0.43607741 1.3358491 2.3506050];
    wts = [0.0045300099 0.15706732 0.72462960 0.72462960 0.15706732 0.0045300099]./sqrt(pi);
elseif nQuadPts == 7
    % n=7 points
    xi = [-2.651961357, -1.673551629, -0.8162878829, 0, 0.816287883, 1.673551629, 2.651961357];
    wts = [9.71781245E-4, 0.05451558282, 0.4256072526, 0.8102646176, 0.4256072526, 0.0545155828,
9.71781245E-4]./sqrt(pi);
else
    % Error
end

mean_stress = 0;
mean_squared_stress = 0;
f = @(x,xi1,xi2,xi3,xi4) fnom.' + xi1*cos((2*1-1)*pi.*x./(2*L)) + xi2*cos((2*2-1)*pi.*x./(2*L)) +
xi3*cos((2*3-1)*pi.*x./(2*L)) + xi4*cos((2*4-1)*pi.*x./(2*L));
```

```
for i1 = 1:size(xi,2)
    pt1 = sqrt(2)*sigman(1)*xi(i1) + mu;
    for i2 = 1:size(xi,2)
        pt2 = sqrt(2)*sigman(2)*xi(i2) + mu;
        for i3 = 1:size(xi,2)
            pt3 = sqrt(2)*sigman(3)*xi(i3) + mu;
            for i4 = 1:size(xi,2)
                pt4 = sqrt(2)*sigman(4)*xi(i4) + mu;
                mean_f = f(x,pt1,pt2,pt3,pt4);
                u = CalcBeamDisplacement(L, E, Iyy, mean_f, Nelem);
                stress = CalcBeamStress(L, E, r_out, u, Nelem);
                mean_stress = mean_stress + wts(i1)*wts(i2)*wts(i3)*wts(i4).*stress;
                mean_squared_stress = mean_squared_stress +
wts(i1)*wts(i2)*wts(i3)*wts(i4).*(stress.^2);
            end
        end
    end
end

std_stress = sqrt(mean_squared_stress - (mean_stress.^2));

figure(1)
plot(x,mean_stress)
hold on
plot(x,mean_stress + 6*std_stress)
hold on
plot(x,mean_stress - 6*std_stress)

xlabel('Position along Spar (m)')
ylabel('Stress (Pa)')
legend('mean stress','mean stress - 6*\sigma','mean stress + 6*\sigma')
title('Nominal Stress Distribution Along Spar')
```