DẠNG : BẤT ĐẲNG THỨC AM-GM (hay gọi là BĐT Cauchy)

1) Cho n số thực không âm : $a_1, a_2, .., a_n$

* Dạng 1: $\dfrac{a_1 + a_2 + ... + a_n}{n} \ge \sqrt[n]{a_1 a_2 ... a_n}$

* Dạng 2: $a_1 + a_2 + ... + a_n \ge n.\sqrt[n]{a_1 a_2 ... a_n}$

* Dạng 3: $\left(\dfrac{a_1 + a_2 + ... + a_n}{n}\right)^n \ge a_1 a_2 ... a_n$

Đẳng thức xảy ra khi và chỉ khi $a_1 = a_2 = ... = a_n$

2) Cho n số thực dương : $a_1, a_2, .., a_n$

* Dạng 4: $\dfrac{1}{a_1} + \dfrac{1}{a_2} + ... + \dfrac{1}{a_n} \ge \dfrac{n^2}{a_1 + a_2 + ... + a_n}$

* Dạng 5: $(a_1 + a_2 + ... + a_n)\left(\dfrac{1}{a_1} + \dfrac{1}{a_2} + ... + \dfrac{1}{a_n}\right) \ge n^2$

Đẳng thức xảy ra khi và chỉ khi $a_1 = a_2 = ... = a_n$

3) Cho hai số a, b không âm : $\dfrac{a+b}{2} \ge \sqrt{ab} \Leftrightarrow a + b \ge 2\sqrt{ab} \Leftrightarrow ab \le \left(\dfrac{a+b}{2}\right)^2$

Đẳng thức xảy ra khi và chỉ khi $a = b$

4) Cho ba số a,b,c không âm : $\dfrac{a+b+c}{3} \ge \sqrt[3]{abc} \Leftrightarrow a + b + c \ge 3\sqrt[3]{abc} \Leftrightarrow abc \le \left(\dfrac{a+b+c}{3}\right)^3$

Đẳng thức xảy ra khi và chỉ khi $a = b = c$

1) Cho a, b, c > 0 . Chứng minh :

a) $\dfrac{1}{a+b} \le \dfrac{1}{4}\left(\dfrac{1}{a} + \dfrac{1}{b}\right)$

b) $\dfrac{bc}{a} + \dfrac{ca}{b} + \dfrac{ab}{c} \ge a + b + c$

c) $\dfrac{a}{bc} + \dfrac{b}{ca} + \dfrac{c}{ab} \ge \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$

d) $\sqrt{2(a+b)} \ge \sqrt{a} + \sqrt{b}$

e) $\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a} \ge \sqrt{2}\left(\sqrt{a} + \sqrt{b} + \sqrt{c}\right)$

f) $a^2 + b^2 + \dfrac{1}{a} + \dfrac{1}{b} \ge 2\left(\sqrt{a} + \sqrt{b}\right)$

g) $\dfrac{1}{a^3} + \dfrac{a^3}{b^3} + b^3 \ge \dfrac{1}{a} + \dfrac{a}{b} + b$

h) $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} \ge \dfrac{2}{a+b} + \dfrac{2}{b+c} + \dfrac{2}{c+a}$

i) $\dfrac{b+c}{a} + \dfrac{c+a}{b} + \dfrac{a+b}{c} \ge 4\left(\dfrac{a}{b+c} + \dfrac{b}{c+a} + \dfrac{c}{a+b}\right)$

j) $\dfrac{1}{a^2+bc} + \dfrac{1}{b^2+ca} + \dfrac{1}{c^2+ab} \le \dfrac{a+b+c}{2abc}$

k) $\dfrac{2\sqrt{a}}{a^3+b^2} + \dfrac{2\sqrt{b}}{b^3+c^2} + \dfrac{2\sqrt{c}}{c^3+a^2} \le \dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2}$

l) $\dfrac{a^3}{b} + \dfrac{b^3}{c} + \dfrac{c^3}{a} \ge a^2 + b^2 + c^2$

m) $(abc+1)\left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right) + \dfrac{a}{c} + \dfrac{c}{b} + \dfrac{b}{a} \ge a + b + c + 6$

2) Cho $a, b \ge 0 : a + b = 5$. Chứng minh : a) $ab \le \dfrac{25}{4}$ b) $a^2 b \le \dfrac{500}{27}$ c) $a^2 b^3 \le 108$

3) Cho $a,b,c\ge 0: a+b^2+c^3=11$. Chứng minh : a) $ab^2c^3\le \frac{1331}{27}$ b) $abc\le 6\sqrt[6]{108}$

4) Chứng minh rằng :

a) nếu $a,b\in[0;1]$ thì $(1-a)(1-b)(a+b)\le \frac{8}{27}$

b) nếu $a\in[-2;2], b\in\left[\frac{1}{3};3\right], c\in[0;4]$ thì $(2-a)(3-b)(4-c)(2a+3b+4c+3)\le \frac{512}{3}$

5) Chứng minh rằng :

a) với $a>1$ thì : $a+\frac{27}{2(a-1)(a+1)^3}\ge \frac{5}{2}$

b) với $a>b\ge 0$ thì : $a+\frac{4}{(a-b)(b+1)^2}\ge 3$

c) với $a,b,c>0$ 0 và $a>b, a>c$ thì : $2a+\frac{1}{(a-b)(a-c)(b+c)}\ge 4$

6) Cho $a,b,c\ge 0$. Chứng minh :

a) $a^2b^2c^2(a+b)(b+c)(c+a)\le 8\left(\frac{a+b+c}{3}\right)^9$

b) $abc(a+b)^2(b+c)^2(c+a)^2\le 64\left(\frac{a+b+c}{3}\right)^9$

7) Cho $a,b>0$ .Chứng minh : a) $\left(\sqrt{a}+\sqrt{b}\right)^8\ge 64ab(a+b)^2$ b) $\left(\sqrt[3]{a}+\sqrt[3]{b}\right)^9\ge 2^8ab(a+b)$

8) Cho $a,b\ge 0: a+b=1$. Chứng minh : a) $\frac{1}{ab}+\frac{1}{a^2+b^2}\ge 6$ b) $\frac{2}{ab}+\frac{3}{a^2+b^2}\ge 14$

9) Cho a,b,c > 0 và a + b + c = 1 . Chứng minh :

a) $b+c\ge 16abc$ b) $\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)\ge 64$ c) $0<ab+bc+ca-2abc\le \frac{7}{27}$

d) $a^2+b^2+c^2+2\sqrt{3abc}\le 1$ e) $\frac{a+bc}{b+c}+\frac{b+ca}{c+a}+\frac{c+ab}{a+b}\ge 2$

10) Cho a, b, c > 0 . Chứng minh:

a) $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge \frac{3}{2}$ (bđt **Nesbit**)

b) $\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\ge \frac{a+b+c}{2}$

c) $\frac{a^2}{xb+yc}+\frac{b^2}{xc+ya}+\frac{c^2}{xa+yb}\ge \frac{a+b+c}{x+y}$ với x,y > 0

d) $\frac{1}{a+2b+c}+\frac{1}{b+2c+a}+\frac{1}{c+2a+b}\le \frac{1}{4}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

e) $\frac{a}{2a+b+c}+\frac{b}{2b+c+a}+\frac{c}{2c+a+b}\le \frac{3}{4}$

f) $\frac{a}{a+\sqrt{(a+b)(a+c)}}+\frac{b}{b+\sqrt{(b+c)(b+a)}}+\frac{c}{c+\sqrt{(c+a)(c+b)}}\le 1$

g) $\frac{1}{a+3b}+\frac{1}{b+3c}+\frac{1}{c+3a}\ge \frac{1}{a+2b+c}+\frac{1}{b+2c+a}+\frac{1}{c+2a+b}$

h) $\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}\le \frac{a^3+b^3+c^3}{2abc}$

i) $\dfrac{bc}{2a+b+c}+\dfrac{ca}{2b+c+a}+\dfrac{ab}{2c+a+b}\le\dfrac{a+b+c}{4}$

j) $\dfrac{1}{a+3b}+\dfrac{1}{b+3c}+\dfrac{1}{c+3a}\le\dfrac{1}{4}\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)$

k) $\dfrac{a^2}{a+b}+\dfrac{b^2}{b+c}+\dfrac{c^2}{c+a}\ge\dfrac{\sqrt{2}}{4}\left(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\right)$

l) $\dfrac{1}{a(b+1)}+\dfrac{1}{b(c+1)}+\dfrac{1}{c(a+1)}\ge\dfrac{3}{abc+1}$

m) $\dfrac{ab}{a+3b+2c}+\dfrac{bc}{b+3c+2a}+\dfrac{ca}{c+3a+2b}\le\dfrac{a+b+c}{6}$

11) Cho a, b, c > 0 : abc = 1 . Chứng minh :

a) $\dfrac{1}{a^2+2b^2+3}+\dfrac{1}{b^2+2c^2+3}+\dfrac{1}{c^2+2a^2+3}\le\dfrac{1}{2}$

b) $\dfrac{1}{a^3(b+c)}+\dfrac{1}{a^3(c+a)}+\dfrac{1}{c^3(a+b)}\ge\dfrac{3}{2}$ (HD : đặt x = 1/a, y = 1/b, x = 1/c)

c) $\dfrac{1}{a^4(a+b)}+\dfrac{1}{b^4(b+c)}+\dfrac{1}{c^4(c+a)}\ge\dfrac{3}{2}$

d) $\dfrac{1}{\sqrt{a^3+2b^3+6}}+\dfrac{1}{\sqrt{b^3+2c^3+6}}+\dfrac{1}{\sqrt{c^3+2a^3+6}}\le 1$

e) $\dfrac{1}{\sqrt{a^5-a^2+3ab+6}}+\dfrac{1}{\sqrt{b^5-b^2+3bc+6}}+\dfrac{1}{\sqrt{c^5-c^2+3ca+6}}\le 1$

12) Cho a, b, c ,d > 0 .Chứng minh rằng :

a) Nếu $\dfrac{1}{a+1}+\dfrac{1}{b+1}+\dfrac{1}{c+1}+\dfrac{1}{d+1}\ge 3$ thì $abcd\le\dfrac{1}{81}$

b) Nếu $\dfrac{1}{a+6}+\dfrac{1}{b+6}+\dfrac{1}{c+6}+\dfrac{1}{d+6}\ge\dfrac{1}{2}$ thì $abcd\le 16$

13) a) Chứng minh rằng: với $\alpha,x,y,z>0$ thì $(\alpha+x)(\alpha+y)(\alpha+z)\ge\left(\alpha+\sqrt[3]{xyz}\right)^3$

b) Cho a,b,c > 0 : $a+b+c=\sqrt{3}$. Chứng minh : $\left(\sqrt{3}+\dfrac{1}{a}\right)\left(\sqrt{3}+\dfrac{1}{b}\right)\left(\sqrt{3}+\dfrac{1}{c}\right)\ge 24\sqrt{3}$

c) Cho a,b,c > 0 :a+ b + c = 6 . Chứng minh : $\left(1+\dfrac{1}{a^3}\right)\left(1+\dfrac{1}{b^3}\right)\left(1+\dfrac{1}{c^3}\right)\ge\dfrac{729}{512}$

14) Cho a,b,c > 0 .Chứng minh :

a) $\dfrac{a}{\sqrt{(a+b)(a+c)}}+\dfrac{b}{\sqrt{(b+c)(b+a)}}+\dfrac{c}{\sqrt{(c+a)(c+b)}}\le\dfrac{3}{2}$

b) $\dfrac{\sqrt{bc}}{\sqrt{(a+b)(a+c)}}+\dfrac{\sqrt{ca}}{\sqrt{(b+c)(b+a)}}+\dfrac{\sqrt{ab}}{\sqrt{(c+a)(c+b)}}\le\dfrac{3}{2}$

c) $\dfrac{1}{\sqrt{(a+b)(a+c)}}+\dfrac{1}{\sqrt{(b+c)(b+a)}}+\dfrac{1}{\sqrt{(c+a)(c+b)}}\le\dfrac{1}{2}\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)$

15) Cho a, b, c > 0 : a + b + c = 6. Chứng minh : $\dfrac{a}{\sqrt{b^3+1}}+\dfrac{b}{\sqrt{c^3+1}}+\dfrac{c}{\sqrt{a^3+1}}\ge 2$

16) Cho a, b, c > 0 : a + b + c = 1 .Chứng minh : $\left(a+\dfrac{1}{b}\right)\left(b+\dfrac{1}{c}\right)\left(c+\dfrac{1}{a}\right)\ge\left(\dfrac{10}{3}\right)^3$

17) Cho a, b, c > 0 : a + b + c = 3 . Chứng minh :

$$\frac{1}{4a^2+b^2+c^2}+\frac{1}{4b^2+c^2+a^2}+\frac{1}{4c^2+a^2+b^2}\le\frac{1}{2}$$

18) Chứng minh rằng với $a,b\in[0;1]$ thì $\frac{a}{1+b}+\frac{b}{1+a}+(1-a)(1-b)\le 1$

Đẳng thức xảy ra khi nào ? HD: giả sử $a\ge b\,;a=0,b\in[0;1]\vee a=b=1$

19) Chứng minh rằng với $a,b\in[0;2]$ thì $\frac{a}{2+b+c}+\frac{b}{2+c+a}+\frac{c}{2+a+b}+\frac{(2-a)(2-b)(2-c)}{8}\le 1$

Đẳng thức xảy ra khi nào ? HD: giả sử $a=\max(a;b;c)\,;a=0,b,c\in[0;2]\vee a=b=c=2$

20) Chứng minh rằng với $a,b\in[0;2]$ thì $\frac{8}{a+b}\ge 2+(2-a)(2-b)$