Sheafification in HoTT

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***Abstract*—Abstract**

I. Introduction

Plan of the paper :

- In section II, we define the foundations on which our worked is based.
- In section III, we explain the construction of the sheafification functor.
- Finally, we discuss related works and present future works.

II. Homotopy Type Theory

A. Homotopy Type Theory

We refer to [Uni13]. Truncation levels ?

If x is an object of a product or a sigma type, we denote the i-th projection of x by either $\pi_i x$ or x_i.

- We use the subobjects classification seen in [RS13] : for all type B, Type_n classifies subobjects of B with n-truncated fibers ; we have an equivalence

$$\sum_{A:\mathrm{Type}} \sum_{f:A\to B} \prod_{b\in B} \mathrm{IsTrunc}\ n\ \mathrm{fib}_f(b) \simeq (B \to \mathrm{Type}_n)$$

and a pullback (with $\mathrm{Type}_n^\bullet$ the universe of pointed n-truncated types) :

$$\begin{tikzcd} A \arrow[r, "t_f"] \arrow[d, "f"'] & \mathrm{Type}_n^\bullet \arrow[d, "\mathrm{pr}_1"] \\ B \arrow[r, "\chi_f"'] & \mathrm{Type}_n \end{tikzcd}$$

forall f with n-truncated fibers, with

$$t_f = \lambda a,\ (\mathrm{fib}_f(f(a)), (a, \mathrm{idpath})).$$

B. Left exact Modalities

We use the following definition of truncated left exact modalities :

Definition 1. *Let p be a truncation index. Then a left exact modality is the data of*

(i) A predicate $P : \mathrm{Type}_p \to \mathrm{HProp}$
(ii) For every p-truncated type A, a p-truncated type $\bigcirc A$ such that $P(\bigcirc A)$
(iii) For every p-truncated type A, a map $\eta_A : A \to \bigcirc A$
such that
(iv) For every p-truncated types A and B, if $P(B)$ then

$$\begin{cases} (\bigcirc A \to B) & \to & (A \to B) \\ f & \mapsto & f \circ \eta_A \end{cases}$$

is an equivalence.
(v) For all $A : \mathrm{Type}_p$ and $B : A \to \mathrm{Type}_p$ such that $P(A)$ and $\Pi_{x:A} P(Bx)$, then $P\left(\sum_{x:A} B(x)\right)$
(vi) For all $A : \mathrm{Type}_p$ and $x, y : A$, if $\bigcirc A$ is contractible, then $\bigcirc(x = y)$ is contractible.

Conditions (i) to (iv) define a reflective subuniverse*, (i) to (v) a* modality.

Since basic operations (dependent products, products, sigma types) are stable for truncation levels, all theorem in [Uni13], chapter 7.7 remains true.

Moreover, left exactness implies in particular fibers preservations :

Proposition 2. *For any n-truncated types X and Y, and any map $f : X \to Y$, the modalisation of fiber of f above any element $y : Y$ is the fiber of $\bigcirc f$ above $\eta_Y y$:*

$$\bigcirc\left(\sum_{x:X} (fx = y)\right) = \sum_{x:\bigcirc X} (\bigcirc fx = \eta_Y y).$$

Moreover the following diagram commute

$$\begin{tikzcd} \sum_{x:X}(fx = y) \arrow[r, "\eta"] \arrow[d, "\gamma"'] & \bigcirc\left(\sum_{x:X}(fx = y)\right) \arrow[dl, equal] \\ \sum_{x:\bigcirc X}(\bigcirc fx = \eta_Y y) & \end{tikzcd}$$

where $\pi_1 \circ \gamma = \eta_X$, and $\pi_2 \circ \gamma$ is the usual modalisation of paths.

C. Generalized pullbacks

Definition 3. *Let $f : X \to Y$ be a map, and $p : \mathrm{nat}$. The p-pullback of f, noted $X \times_Y \cdots \times_Y X$ is the limit of the diagram*

$$\begin{tikzcd} X \arrow[dr, "f"'] & \cdots & X \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$

with p copies of X. The 0-pullback of f is Unit.

In homotopy type theory, we have

$$X \times_Y \cdots \times_Y X = \sum_{x:X^n} (fx_1 = fx_2) \wedge \cdots \wedge (fx_{n-1} = fx_n).$$

Definition 4. *If $f : X \to Y$ is a map, then the* Čech nerve *of f is the diagram*

$$C(f) := \cdots X \times_Y X \times_Y X \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} X \times_Y X \rightrightarrows X$$

with canonical projections.

The Giraud axiom asserts that, if $f : X \to Y$ is a surjection, then the colimit of $C(f)$ is Y. See [HTT].

III. Sheaves

We define sheaves by induction on the homotopical truncation level of types : we define sheaves in HProp as the classical types (*ie* implied by their double negation), and we define sheaves on Type_{n+1} *via* usual definition of sheaves, as seen in [MM92] and sheaves on Type_n. Of course, the definition must be cumulative.

We will call $\mathcal{S}_i$ the universe of sheaves on Type_i.

A. Definitions and first properties

- Density
- Separated Type / Sheaf / closure : a subobject is always dense into its closure.
- $\Omega_n^{\bigcirc}$ is a sheaf.
- Sheaves are stable by dependent products.

Definition 5. *Let E be a type. The* closure *of a subobject of E classified by χ is the subobject of E classified by $\bigcirc \circ \chi$.*

The subobject of E classified by χ is said closed in *E if its closure is itself : $\chi = \bigcirc \circ \chi$.*

Definition 6. *Let E be a type, and $\chi : E \to \mathrm{Type}_n$. The subobject A of E classified by χ is* dense *in E if its $\bigcirc$-closure is E seen as a subobject of E,* ie

$$\forall e : E,\ \left(\sum_{e' : E} e = e'\right) \simeq (\bigcirc(\chi\ e)).$$

Moreover, for all $x : A$ we need the following coherence diagram to commute

$$\begin{array}{ccc} \sum_{e':A} x = e' & = & (\chi\ x) \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \eta_{(\chi\ x)}} \\ \sum_{e':E} x = e' & = & \bigcirc(\chi\ x) \end{array}$$

where $\iota : x \mapsto (x_{11}; x_2)$.

It follows from fibers preservation that any n-subobject of a type seen as a n-subobject of its closure is closed.

B. Construction of sheafification

We mimic the construction in [MM92] Two steps : separation, sheafification.

1) For h-propositions:

- Any left exact modality on HProp.
- We use $\neg\neg$

2) From Type to Separated Type:

- Definition of the separation reflective subuniverse $(\square,\ \mu)$. $\square$ + $\square T$ is separated for all $(n+1)$-truncated type T

  Theorem 7. *A $(n+1)$-truncated type T with an embedding $f : T \to U$ into a separated $(n+1)$-truncated type U is itself separated.*
- If $a, b : T : \mathrm{Type}_{n+1}$ such that $\mu_T a = \mu_T b$, then $\bigcirc(a = b)$.
- Cech nerve of μ is the diagram of closure of diagonals, hence their colimits are equal. Since colimits are universal, so is $\square$.
- $(\square,\ \mu)$ is a modality.

3) From Separated Type to Sheaf:

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  Theorem 8. *Any type $A : \mathrm{Type}_{n+1}$ closed in a sheaf is itself a sheaf.*
- Definition of sheafification reflexive subuniverse $(\star,\ \nu)$
- left exactness
- cumulativity

IV. Future Works

Prove that the Giraud axiom holds in homotopy type theory.

V. Conclusion

The conclusion goes here.

Acknowledgment

The authors would like to thank...

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