Sheafification in HoTT

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***Abstract*—Abstract**

I. Introduction

II. Homotopy Type Theory

A. Homotopy Type Theory

We refer to [Uni13].

- We use the subobjects classification seen in [RS13] : for all type B, Type_n classifies subobjects of B with n-truncated fibers ; we have an equivalence

$$\sum_{A:\mathrm{Type}} \sum_{f:A\to B} \prod_{b\in B} \mathrm{IsTrunc}\ n\ \mathrm{fib}_f(b) \simeq (B \to \mathrm{Type}_n)$$

and a pullback (with $\mathrm{Type}_n^\bullet$ the universe of pointed n-truncated types) :

$$\begin{tikzcd} A \arrow[r, "t_f"] \arrow[d, "f"'] & \mathrm{Type}_n^\bullet \arrow[d, "\mathrm{pr}_1"] \\ B \arrow[r, "\chi_f"'] & \mathrm{Type}_n \end{tikzcd}$$

forall f with n-truncated fibers, with

$$t_f = \lambda a,\ (\mathrm{fib}_f(f(a)), (a, \mathrm{idpath})).$$

B. Left exact Modalities

We use the following definition of truncated left exact modalities :

Definition 1. *Let p be a truncation index. Then a left exact modality is the data of*

- *A predicate $P\ :\ \mathrm{Type}_p \to \mathrm{HProp}$*
- *For every p-truncated type A, a p-truncated type $\bigcirc A$ such that $P(\bigcirc A)$*
- *For every p-truncated type A, a map $\eta_A\ :\ A \to \bigcirc A$*

such that

- *For every p-truncated types A and B, if $P(B)$ then*

$$\begin{cases} (\bigcirc A \to B) & \to & (A \to B) \\ f & \mapsto & f \circ \eta_A \end{cases}$$

is an equivalence.

- *For all $A\ :\ \mathrm{Type}_p$ and $B\ :\ A \to \mathrm{Type}_p$ such that $P(A)$ and $\Pi_{x:A}P(Bx)$, then $P\left(\Sigma_{x:A}B(x)\right)$*
- *For all $A\ :\ \mathrm{Type}_p$ and $x, y\ :\ A$, if $\bigcirc A$ is contractible, then $\bigcirc(x = y)$ is contractible.*

All theorem in [Uni13], chapter 7.7 remains true + fibers preservation :

for all n-truncated types, and all map $f\ :\ X \to Y$, the modalisation of fiber of f above any element $y : Y$ is the fiber of $\bigcirc f$ above ηy :

$$\bigcirc\{x : X\ \&\ f\ x = y\} = \{x : \bigcirc X\ \&\ \bigcirc f\ x = \eta_Y y\}.$$

We want moreover that the following diagram commute

$$\begin{tikzcd} \{x : X\ \&\ f\ x = y\} \arrow[r, "\eta"] \arrow[d, "\gamma"'] & \bigcirc\{x : X\ \&\ f\ x = y\} \arrow[dl, equal] \\ \{x : \bigcirc X\ \&\ \bigcirc f\ x = \eta_Y y\} & \end{tikzcd}$$

where $\Pi_1 \circ \gamma = \eta_X$, and $\Pi_2 \circ \gamma$ is the usual modalisation of paths.

C. Generalized pullbacks

- definition
- giraud axiom

III. Sheaves

We try to define sheaves inductively : we define sheaves in HProp as the classical types (*ie* implied by their double negation), and we define sheaves on Type_{n+1} *via* usual definition of sheaves, as seen in maclanemoerdijk and sheaves on Type_n. Of course, the definition must be cumulative.

We will call $\mathcal{S}_i$ the universe of sheaves on Type_i.

A. Definitions and forst properties

- Density
- Separated Type / Sheaf / closure : a subobject is always dense into its closure.
- $\Omega_n^\bigcirc$ is a sheaf.
- Sheaves are stable by dependent products.

B. Construction of sheafification

We mimic the construction in [MM92] Two steps : separation, sheafification.

1) For h-propositions:

- Any left exact modality on HProp.
- We use $\neg\neg$

2) From Type to Separated Type:

- Definition of the separation reflective subuniverse $(\Box,\ \mu)$.
  $\Box$ + $\Box T$ is separated for all $(n+1)$-truncated type T

  Theorem 2. *A $(n+1)$-truncated type T with an embedding $f : T \to U$ into a separated $(n+1)$-truncated type U is itself separated.*
- If $a, b\ :\ T\ : \mathrm{Type}_{n+1}$ such that $\mu_T a = \mu_T b$, then $\bigcirc(a = b)$.
- Cech nerve of μ is the diagram of closure of diagonals, hence their colimits are equal. Since colimits are universal, so is $\Box$.
- $(\Box,\ \mu)$ is a modality.

3) From Separated Type to Sheaf:

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  Theorem 3. *Any type $A\ :\ \mathrm{Type}_{n+1}$ closed in a sheaf is itself a sheaf.*
- Definition of sheafification reflective subuniverse $(\star,\ \nu)$

IV. Future Works

V. Conclusion

The conclusion goes here.

Acknowledgment

The authors would like to thank...

References

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