# Sheafification in HoTT

Kevin Quirin and Nicolas Tabareau
INRIA
École des Mines de Nantes
Nantes, France

***Abstract*—Abstract**

## I. Introduction

Plan of the paper :

- In section II, we define the foundations on which our worked is based.
- In section III, we explain the construction of the sheafification functor.
- Finally, we discuss related works and present future works.

## II. Homotopy Type Theory

### A. Homotopy Type Theory

We refer to [Uni13]. Truncation levels + Im ?

If $x$ is an object of a product or a sigma type, we denote the $i$-th projection of $x$ by either $\pi_i x$ or $x_i$.

- We use the subobjects classification seen in [RS13] : for all type $B$, $\mathrm{Type}_n$ classifies subobjects of $B$ with $n$-truncated fibers ; we have an equivalence

$$\sum_{A:\mathrm{Type}} \sum_{f:A\to B} \prod_{b\in B} \mathrm{IsTrunc}\ n\ \mathrm{fib}_f(b) \simeq (B \to \mathrm{Type}_n)$$

and a pullback (with $\mathrm{Type}_n^\bullet$ the universe of pointed $n$-truncated types) :

$$\begin{tikzcd} A \arrow[r, "t_f"] \arrow[d, "f"'] & \mathrm{Type}_n^\bullet \arrow[d, "\mathrm{pr}_1"] \\ B \arrow[r, "\chi_f"'] & \mathrm{Type}_n \end{tikzcd}$$

forall $f$ with $n$-truncated fibers, with

$$t_f = \lambda a,\ (\mathrm{fib}_f(f(a)), (a, \mathrm{idpath})).$$

### B. Left exact Modalities

We use the following definition of truncated left exact modalities :

**Definition 1.** *Let $p$ be a truncation index. Then a left exact modality is the data of*

*(i) A predicate $P : \mathrm{Type}_p \to \mathrm{HProp}$*
*(ii) For every $p$-truncated type $A$, a $p$-truncated type $\bigcirc A$ such that $P(\bigcirc A)$*
*(iii) For every $p$-truncated type $A$, a map $\eta_A : A \to \bigcirc A$ such that*
*(iv) For every $p$-truncated types $A$ and $B$, if $P(B)$ then*

$$\begin{cases} (\bigcirc A \to B) & \to & (A \to B) \\ f & \mapsto & f \circ \eta_A \end{cases}$$

*is an equivalence.*
*(v) For all $A : \mathrm{Type}_p$ and $B : A \to \mathrm{Type}_p$ such that $P(A)$ and $\prod_{x:A} P(Bx)$, then $P\left(\sum_{x:A} B(x)\right)$*
*(vi) For all $A : \mathrm{Type}_p$ and $x, y : A$, if $\bigcirc A$ is contractible, then $\bigcirc(x = y)$ is contractible.*

*Conditions (i) to (iv) define a* reflective subuniverse*, (i) to (v) a* modality.

The type of all $p$-types such that $P$ will be noted $\mathrm{Type}_p^\bigcirc$

Since basic operations (dependent products, products, sigma types) are stable for truncation levels, all theorem in [Uni13], chapter 7.7 remains true.

Moreover, left exactness implies in particular fibers preservations :

**Proposition 2.** *For any $n$-truncated types $X$ and $Y$, and any map $f : X \to Y$, the modalisation of fiber of $f$ above any element $y : Y$ is the fiber of $\bigcirc f$ above $\eta_Y y$ :*

$$\bigcirc \left( \sum_{x:X} (fx = y) \right) = \sum_{x:\bigcirc X} (\bigcirc f x = \eta_Y y).$$

*Moreover the following diagram commute*

$$\begin{tikzcd} \sum_{x:X}(fx = y) \arrow[r, "\eta"] \arrow[d, "\gamma"'] & \bigcirc\left(\sum_{x:X}(fx = y)\right) \arrow[dl, equal] \\ \sum_{x:\bigcirc X}(\bigcirc f x = \eta_Y y) & \end{tikzcd}$$

*where $\pi_1 \circ \gamma = \eta_X$, and $\pi_2 \circ \gamma$ is the usual modalisation of paths.*

### C. Generalized pullbacks

**Definition 3.** *Let $f : X \to Y$ be a map, and $p : \mathrm{nat}$. The $p$-pullback of $f$, noted $X \times_Y \cdots \times_Y X$ is the limit of the diagram*

$$\begin{tikzcd} X \arrow[dr, "f"'] & \cdots & X \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$

*with $p$ copies of $X$. The $0$-pullback of $f$ is* Unit.

In homotopy type theory, we have

$$X \times_Y \cdots \times_Y X = \sum_{x:X^n} (fx_1 = fx_2) \wedge \cdots \wedge (fx_{n-1} = fx_n).$$

**Definition 4.** *If $f : X \to Y$ is a map, then the* Čech nerve *of $f$ is the diagram*

$$C(f) := \cdots \; X \times_Y X \times_Y X \rightrightarrows X \times_Y X \rightrightarrows X$$

*with canonical projections.*

The Giraud axiom asserts that, if $f : X \to Y$ is a surjection, then the colimit of $C(f)$ is $Y$. See [HTT].

## III. Sheaves

We define sheaves by induction on the homotopical truncation level of types : we define sheaves in HProp as the classical types (*ie* implied by their double negation), and we define sheaves on $\mathrm{Type}_{n+1}$ *via* usual definition of sheaves, as seen in [MM92] and sheaves on $\mathrm{Type}_n$. Of course, the definition must be cumulative.

We will call $\mathcal{S}_i$ the universe of sheaves on $\mathrm{Type}_i$.

In this section, we suppose given a truncation index $n \geqslant -1$, and a left exact modality $\bigcirc$ on $\mathrm{Type}_n$.

**Definition 5.** *Let $E$ be a type. The* closure *of a subobject of $E$ classified by $\chi$ is the subobject of $E$ classified by $\bigcirc \circ \chi$.*

*The subobject of $E$ classified by $\chi$ is said* closed in *$E$ if its closure is itself : $\chi = \bigcirc \circ \chi$.*

**Definition 6.** *Let $E$ be a type, and $\chi : E \to \mathrm{Type}_n$. The subobject $A$ of $E$ classified by $\chi$ is* dense *in $E$ if its $\bigcirc$-closure is $E$ seen as a subobject of $E$,* ie

$$\forall e : E, \;\; \left( \sum_{e':E} e = e' \right) \simeq (\bigcirc(\chi\ e)).$$

*Moreover, for all $x : A$ we need the following coherence diagram to commute*

$$\begin{tikzcd}
\sum_{e':A} x = e' \arrow[r, equal] \arrow[d, "\iota"'] & (\chi\ x) \arrow[d, "\eta_{(\chi\ x)}"] \\
\sum_{e':E} x = e' \arrow[r, equal] & \bigcirc(\chi\ x)
\end{tikzcd}$$

*where $\iota : x \mapsto (x_{11}; x_2)$.*

It follows from fibers preservation that any $n$-subobject of a type seen as a $n$-subobject of its closure is closed.

**Definition 7** (Restriction)**.** *For any $A, E$ : Type and $F$ : $\mathrm{Type}_{n+1}$, we define :*

- *if $A \hookrightarrow E$, $M_E^\chi$ is the map sending an arrow $f : E \to F$ to $\upharpoonright fA$ the restriction of $f$ to $A$.*
- *if $A \to E$ with $n$-truncated fibers, $\Phi_E^\chi$ is the map sending $f : E \to f$ to $\upharpoonright fA$.*

**Definition 8** (Separated Type)**.** *A type $F$ in $\mathrm{Type}_{n+1}$ is* separated *if for all type $E$, and all dense subobject of $E$ classified by $\chi : E \to \mathrm{Type}_n$, $\Phi_E^\chi$ is a monomorphism. In other words, the dotted arrow, if exists, is unique.*

$$\begin{tikzcd}
\sum_{e:E}(\chi\ e) \arrow[r, "f"] \arrow[d, "\pi_1"'] & F \\
E \arrow[ur, dashed] &
\end{tikzcd}$$

**Definition 9** (Sheaves)**.** *A type $F$ of $\mathrm{Type}_{n+1}$ is a $(n+1)$-*sheaf *if it is separated, and for all type $E$ and all dense subobject of $E$ classified by $\chi : E \to \mathrm{Type}_{-1}$, $M_E^\chi$ is an equivalence. In other words, the dotted arrow exists and is unique.*

$$\begin{tikzcd}
\sum_{e:E}(\chi\ e) \arrow[r, "f"] \arrow[d, "\pi_1"'] & F \\
E \arrow[ur, dashed] &
\end{tikzcd}$$

From these definitions, one can show that

**Proposition 10.**
- *$\mathrm{Type}_n^\bigcirc$ is a sheaf.*
- *If $A : \mathrm{Type}_{n+1}$ and $B : A \to \mathrm{Type}_{n+1}$ such that for all $a : A$, $(B\ a)$ is a sheaf, then $\prod_{a:A} B\ a$ is a sheaf.*

## IV. Construction of sheafification

We mimic the construction in [MM92]

### A. For h-propositions

For the case $n = -1$, one can take any left exact modality on HProp. Here we took the $\neg\neg$ modality :

$$\forall P : \mathrm{HProp}, \;\; \bigcirc_j P = \neg\neg P.$$

One can easily show that it defines a left exact modality.

### B. From Type to Separated Type

In this section, we suppose given a truncation index $n \geqslant -1$, and a left exact modality $\bigcirc$ on $\mathrm{Type}_n$, compatible with the modality on HProp.

Let $T$ be in $\mathrm{Type}_{n+1}$. We define $\square T$ as the image of $\bigcirc^T \circ \{\cdot\}_T$, as in

$$\begin{tikzcd}
T \arrow[r, "\{\cdot\}_T"] \arrow[d, "\mu_T"'] & \mathrm{Type}_n^T \arrow[d, "\bigcirc^T"] \\
\square T \arrow[r] & \left(\mathrm{Type}_n^\bigcirc\right)^T
\end{tikzcd},$$

where $\{\cdot\}_T$ is the singleton map $\lambda(t : T),\ \lambda(t' : T),\ t = t'$. In type theory words,

$$\begin{aligned}
\square T &= \mathrm{Im}(\lambda\ t : T,\ \lambda\ t',\ \bigcirc (t = t')) \\
&= \sum_{u:T \to \mathrm{Type}_n^\bigcirc} \left\| \sum_{a:X} (\lambda t,\ \bigcirc (a = t)) = u \right\|
\end{aligned}$$

**Proposition 11.** *For all $T : \mathrm{Type}_{n+1}$, $\square T$ is separated.*

*Proof.* We will use the following lemma :

**Lemma 12.** *A $(n+1)$-truncated type $T$ with an embedding $f : T \to U$ into a separated $(n+1)$-truncated type $U$ is itself separated.*

As $\square T$ embeds in $\left(\mathrm{Type}_n^{\bigcirc}\right)^T$, we only have to show that the latter is separated. But it is the case with both parts proposition 10. $\square$

We now need to show that $\square$ is universal. The following lemma is central in the proof :

**Lemma 13.** *Let $T : \mathrm{Type}_{n+1}$. Then $\square T$ is the colimit of the closed diagonal diagram*

$$\cdots \overline{\Delta_3} \rightrightarrows \overline{\Delta_2} \rightrightarrows \overline{\Delta_1}$$

*where $\Delta_k$ is the $k$-pullback of* $\mathrm{id} : T \to T$.

*Proof.* As $\mu_T$ is an embedding, we know by Giraud axiom that $\square T$ is the colimit of $C(\mu_T)$. If we can show that $C(\mu_T) = C(\mathrm{id})$, the the result will follow.

Let $k : \mathrm{nat}$, let's show that

$$\sum_{t:T^k}(\mu_T t_1 = \mu_T t_2 \wedge \cdots \wedge \mu_T t_{k-1}$$
$$= \mu_T t_k) = \sum_{t:T^k} \bigcirc(t_1 = t_2 \wedge \cdots \wedge t_{k-1} = t_k)$$

By induction on $k$, and the preservation of products by $\bigcirc$, it suffices to show that for all $a, b : T$, $\bigcirc(a = b) = (\mu_T a = \mu_T b)$, which is true by univalence. $\square$

Now that $\square T$ is a colimit, we can easily define an inverse to $\lambda\,(f : \square T \to U),\ x \circ \mu_T$ for any separated type $U$, using the universal property of the colimit.

We now have to prove point *(v)* in the definition of modality. Let $A : \mathrm{Type}_{n+1}$ be a sheaf and $B : A \to \mathrm{Type}_{n+1}$ be a sheaf family. We want to show that $\sum_{x:A}(Bx)$ is separated. Let $E$ be a type, and $\sum_{e:E}(\chi\, e)$ a dense subobject of E.

Let $f, g$ be two maps from $\sum_{e:E}(\chi\, e)$ to $\sum_{x:A}(Bx)$, equal when precomposed with $\pi_1$.

As $A$ is separated, for any $x : E$, the first components of $fx$ and $gx$ are equal. For the second components, notice that for all $x : E$, $\sum_{y:E} x = y$ has a dense subobject, $\sum_{y:\sum_{e:E}(\chi\, e)} x = y_1$. Using the separation property with these type, one can show the second components of $fx$ and $gx$ are equals too, with respect to the first path.

### *C. From Separated Type to Sheaf*

If $T$ is already a separated type, the following lemma alllows us to build a sheaf :

**Lemma 14.** *Let $T : \mathrm{Type}_{n+1}$ be separated, and $U$ be a sheaf. If $T$ embeds in $U$, and is closed in $U$, then $T$ is a sheaf.*

As any separated type $T$ embeds in $\left(\mathrm{Type}_n^{\bigcirc}\right)^T$, it suffices to take the closure of $T$ to get a sheaf.

### *D. Summary*

If $T : \mathrm{Type}_{n+1}$, the sheafification of $T$ is

$$\star T = \sum_{u:T\to\mathrm{Type}_n^{\bigcirc}} \neg\neg \left\| \sum_{a:X} (\lambda t,\ \bigcirc(a = t)) = u \right\|$$

We already know that $\star$ defines a modality. It remains to show that this modality is left exact.

Let $T : \mathrm{Type}_{n+1}$ such that $\star X$ is contractible. Let $x, y : T$. We want to show that $\star(x = y)$ is contractible.

As $\star T$ is contractible, we have a center $u_0 : T \to \mathrm{Type}_n^{\bigcirc}$ with an hypothesis $H_0 : \neg\neg \|\sum_{a:X}(\lambda t,\ \bigcirc(a = t)) = u_0\|$, and a contractibility condition $c : \prod_{t:\star T} x = (u_0, H_0)$. As $\mathrm{Contr}(\star(x = y))$ is a HProp-sheaf, we can change $H_0$ into a $a : T$ and a proof $u_0 = (\lambda t,\ \bigcirc(a = t))$. Finally, $c$ type becomes $\prod_{v:\star T} v_1 = (\lambda t,\ \bigcirc(a = t))$.

The last thing to prove is the cumulativity of our sheafifications :

**Proposition 15.** *If $T : \mathrm{Type}_n$, then $\bigcirc T = \star\widehat{T}$ (where $\widehat{T}$ is $T$ seen as a $\mathrm{Type}_{n+1}$.*

## V. Future Works

Prove that the Giraud axiom holds in homotopy type theory.

Extend the sheafification to all types, even if non truncated.

## VI. Conclusion

The conclusion goes here.


## Acknowledgment

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## References


[RS13] Egbert Rijke and Bas Spitters. *Sets in homotopy type theory*. 2013.

[Uni13] Univalent Foundations Project. *Homotopy Type Theory: Univalent Foundations for Mathematics*. 2013.

[MM92] Saunders MacLane and Ieke Moerdijk. *Sheaves in Geometry and Logic*. Springer-Verlag, 1992.

[HTT] Jacob Lurie *Higher Topos Theory*

[lumsdaine] Peter LeFanu Lumsdaine, Jeremy Avigad and Chris Kapulkin *Homotopy limits in type theory*