Sheafification in HoTT

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***Abstract*—Abstract**

I. Introduction

Plan of the paper :

- In section II, we define the foundations on which our worked is based.
- In section III, we explain the construction of the sheafification functor.
- Finally, we discuss related works and present future works.

II. Homotopy Type Theory

A. Homotopy Type Theory

We refer to [Uni13]. Truncation levels + Im ?

If x is an object of a product or a sigma type, we denote the i-th projection of x by either $\pi_i x$ or x_i.

- We use the subobjects classification seen in [RS13] : for all type B, Type_n classifies subobjects of B with n-truncated fibers ; we have an equivalence

  $$\sum_{A:\mathrm{Type}} \sum_{f:A\to B} \prod_{b\in B} \mathrm{IsTrunc}\ n\ \mathrm{fib}_f(b) \simeq (B \to \mathrm{Type}_n)$$

  and a pullback (with $\mathrm{Type}_n^\bullet$ the universe of pointed n-truncated types) :

  $$\begin{tikzcd} A \arrow[r, "t_f"] \arrow[d, "f"'] & \mathrm{Type}_n^\bullet \arrow[d, "\mathrm{pr}_1"] \\ B \arrow[r, "\chi_f"'] & \mathrm{Type}_n \end{tikzcd}$$

  forall f with n-truncated fibers, with

  $$t_f = \lambda a,\ (\mathrm{fib}_f(f(a)), (a, \mathrm{idpath})).$$

B. Left exact Modalities

We use the following definition of truncated left exact modalities :

Definition 1. *Let p be a truncation index. Then a left exact modality is the data of*

(i) A predicate $P : \mathrm{Type}_p \to \mathrm{HProp}$
(ii) For every p-truncated type A, a p-truncated type $\bigcirc A$ such that $P(\bigcirc A)$
(iii) For every p-truncated type A, a map $\eta_A : A \to \bigcirc A$
such that
(iv) For every p-truncated types A and B, if $P(B)$ then

$$\begin{cases} (\bigcirc A \to B) & \to \quad (A \to B) \\ \qquad\quad f & \mapsto \quad f \circ \eta_A \end{cases}$$

is an equivalence.
(v) For all $A : \mathrm{Type}_p$ and $B : A \to \mathrm{Type}_p$ such that $P(A)$ and $\prod_{x:A} P(Bx)$, then $P\left(\sum_{x:A} B(x)\right)$
(vi) For all $A : \mathrm{Type}_p$ and $x, y : A$, if $\bigcirc A$ is contractible, then $\bigcirc(x = y)$ is contractible.

Conditions (i) to (iv) define a reflective subuniverse*, (i) to (v) a* modality*.*

The type of all p-types such that P will be noted $\mathrm{Type}_p^\bigcirc$

Since basic operations (dependent products, products, sigma types) are stable for truncation levels, all theorem in [Uni13], chapter 7.7 remains true.

Moreover, left exactness implies in particular fibers preservations :

Proposition 2. *For any n-truncated types X and Y, and any map $f : X \to Y$, the modalisation of fiber of f above any element $y : Y$ is the fiber of $\bigcirc f$ above $\eta_Y y$:*

$$\bigcirc\left(\sum_{x:X} (fx = y)\right) = \sum_{x:\bigcirc X} (\bigcirc f x = \eta_Y y).$$

Moreover the following diagram commute

$$\begin{tikzcd} \sum_{x:X}(fx = y) \arrow[r, "\eta"] \arrow[d, "\gamma"'] & \bigcirc\left(\sum_{x:X}(fx = y)\right) \arrow[dl, equal] \\ \sum_{x:\bigcirc X}(\bigcirc f x = \eta_Y y) & \end{tikzcd}$$

where $\pi_1 \circ \gamma = \eta_X$, and $\pi_2 \circ \gamma$ is the usual modalisation of paths.

Another useful properties of modalities is the following :

Proposition 3. *If $P : \mathrm{HProp}$, then $\bigcirc \widehat{P} : \mathrm{HProp}$, where $\widehat{P}$ is P seen as a p-type.*

A left exact modality leads to a new theory with new principles. A basic example is the *open modality of P* for a $P : \mathrm{HProp}$, $\bigcirc_P T = P \to T$. When one works in the subuniverse defined by this modality, defining an inhabitant of a type T is actually an inhabitant of $P \to T$. Thus, being in this subuniverse is just adding the axiom P to the theory. Sadly, adding an axiom using the modality does not add any computational content to P, thus it is not completely satisfactory.

Another example is the $\neg\neg$-modality, $\bigcirc_{\neg\neg} T = (T \to \bot) \to \bot$. Although we might think it enables us to use classical reasonings in the corresponding reflective subuniverse, the main problem is that every type in the new universe is an HProp, which is very restrictive.

C. Generalized pullbacks

Definition 4. *Let $f : X \to Y$ be a map, and $p : \mathrm{nat}$. The p-pullback of f, noted $X \times_Y \cdots \times_Y X$ is the limit of the diagram*

$$\begin{tikzcd} X \arrow[dr, "f"] & \cdots & X \arrow[dl, "f"'] \\ & Y & \end{tikzcd}$$

with p copies of X. The 0-pullback of f is Unit.

In homotopy type theory, we have

$$X \times_Y \cdots \times_Y X = \sum_{x:X^n} (fx_1 = fx_2) \wedge \cdots \wedge (fx_{n-1} = fx_n).$$

Definition 5. *If $f : X \to Y$ is a map, then the* Čech nerve *of f is the diagram*

$$C(f) := \cdots \; X \times_Y X \times_Y X \rightrightarrows X \times_Y X \rightrightarrows X$$

with canonical projections.

The Giraud axiom asserts that, if $f : X \to Y$ is a surjection, then the colimit of $C(f)$ is Y. See [HTT].

III. Sheaves

We will define sheaves by induction on the homotopical truncation level of types : the base case will be any left exact modality on HProp, and we give in this section some definitions useful for th inductive case.

We will call $\mathcal{S}_i$ the universe of sheaves on Type_i.

In this section, we suppose given a truncation index $n \geqslant -1$, and a left exact modality $\bigcirc$ on Type_n.

Definition 6. *Let E be a type. The* closure *of a subobject of E classified by χ is the subobject of E classified by $\bigcirc \circ \chi$.*

The subobject of E classified by χ is said closed *in E if its closure is itself : $\chi = \bigcirc \circ \chi$.*

Definition 7. *Let E be a type, and $\chi : E \to \mathrm{Type}_n$. The subobject A of E classified by χ is* dense *in E if its $\bigcirc$-closure is E seen as a subobject of E,* ie

$$\forall e : E, \; \left(\sum_{e':E} e = e' \right) \simeq (\bigcirc(\chi \; e)).$$

Moreover, for all $x : A$ we need the following coherence diagram to commute

$$\begin{tikzcd} \sum_{e':A} x = e' \arrow[r, equal] \arrow[d, "\iota"'] & (\chi \; x) \arrow[d, "\eta_{(\chi \; x)}"] \\ \sum_{e':E} x = e' \arrow[r, equal] & \bigcirc(\chi \; x) \end{tikzcd}$$

where $\iota : x \mapsto (x_{11}; x_2)$.

It follows from fibers preservation that any n-subobject of a type seen as a n-subobject of its closure is closed.

Definition 8 (Restriction)**.** *For any $A, E : \mathrm{Type}$ and $F : \mathrm{Type}_{n+1}$, we define :*

- *if $A \hookrightarrow E$, M_E^χ is the map sending an arrow $f : E \to F$ to $\restriction fA$ the restriction of f to A.*
- *if $A \to E$ with n-truncated fibers, Φ_E^χ is the map sending $f : E \to f$ to $\restriction fA$.*

Definition 9 (Separated Type)**.** *A type F in Type_{n+1} is* separated *if for all type E, and all dense subobject of E classified by $\chi : E \to \mathrm{Type}_n$, Φ_E^χ is a monomorphism. In other words, the dotted arrow, if exists, is unique.*

$$\begin{tikzcd} \sum_{e:E}(\chi \; e) \arrow[r, "f"] \arrow[d, "\pi_1"'] & F \\ E \arrow[ur, dashed] & \end{tikzcd}$$

Definition 10 (Sheaves)**.** *A type F of Type_{n+1} is a* $(n+1)$-sheaf *if it is separated, and for all type E and all dense subobject of E classified by $\chi : E \to \mathrm{Type}_{-1}$, M_E^χ is an equivalence. In other words, the dotted arrow exists and is unique.*

$$\begin{tikzcd} \sum_{e:E}(\chi \; e) \arrow[r, "f"] \arrow[d, "\pi_1"'] & F \\ E \arrow[ur, dashed] & \end{tikzcd}$$

Note that these definitions are almost the same as the ones in [MM92]. The main difference is that *separated* is defined for n-subobjects, while *sheaf* only for -1-subobject.

From these definitions, one can show that

Proposition 11.

- *$\mathrm{Type}_n^\bigcirc$ is a sheaf.*
- *If $A : \mathrm{Type}_{n+1}$ and $B : A \to \mathrm{Type}_{n+1}$ such that for all $a : A$, $(B \; a)$ is a sheaf, then $\prod_{a:A} B \; a$ is a sheaf.*

IV. Construction of sheafification

We mimic the construction in [MM92]

A. For h-propositions

For the case $n = -1$, one can take any left exact modality on HProp. Here we took the $\neg\neg$ modality :

$$\forall P : \mathrm{HProp}, \;\; \bigcirc_j P = \neg\neg P.$$

One can easily show that it defines a left exact modality.

B. From Type to Separated Type

In this section, we suppose given a truncation index $n \geqslant -1$, and a left exact modality $\bigcirc$ on Type_n, compatible with the modality on HProp.

Let T be in Type_{n+1}. We define $\Box T$ as the image of $\bigcirc^T \circ \{\cdot\}_T$, as in

$$\begin{tikzcd} T \arrow[r, "\{\cdot\}_T"] \arrow[d, "\mu_T"'] & \mathrm{Type}_n^T \arrow[d, "\bigcirc^T"] \\ \Box T \arrow[r] & \left(\mathrm{Type}_n^{\bigcirc}\right)^T \end{tikzcd} ,$$

where $\{\cdot\}_T$ is the singleton map $\lambda(t : T),\ \lambda(t' : T),\ t = t'$. In type theory words,

$$\begin{aligned} \Box T &= \mathrm{Im}(\lambda\ t : T,\ \lambda\ t',\ \bigcirc(t = t')) \\ &= \sum_{u:T\to\mathrm{Type}_n^{\bigcirc}} \left\| \sum_{a:X} (\lambda t,\ \bigcirc(a = t)) = u \right\| \end{aligned}$$

There again, the separation step has the same definition as in [MM92], using $\mathrm{Type}_n^{\bigcirc}$ instead of the j-subobject classifier.

Proposition 12. *For all $T : \mathrm{Type}_{n+1}$, $\Box T$ is separated.*

Proof. We will use the following lemma :

Lemma 13. *A $(n+1)$-truncated type T with an embedding $f : T \to U$ into a separated $(n+1)$-truncated type U is itself separated.*

As $\Box T$ embeds in $\left(\mathrm{Type}_n^{\bigcirc}\right)^T$, we only have to show that the latter is separated. But it is the case with both parts proposition 11. $\Box$

We now need to show that $\Box$ is universal. The following lemma is central in the proof :

Lemma 14. *Let $T : \mathrm{Type}_{n+1}$. Then $\Box T$ is the colimit of the closed diagonal diagram*

$$\cdots \overline{\Delta_3} \Rrightarrow \overline{\Delta_2} \rightrightarrows \overline{\Delta_1}$$

where Δ_k is the k-pullback of $\mathrm{id} : T \to T$.

This lemma is an adaptation of the sheafification process in [MM92], where they consider only the kernel pair of μ_T instead of the Čech nerve.

Proof. As μ_T is an embedding, we know by Giraud axiom that $\Box T$ is the colimit of $C(\mu_T)$. If we can show that $C(\mu_T) = C(\mathrm{id})$, the the result will follow.

Let $k : \mathrm{nat}$, let's show that

$$\begin{aligned} &\sum_{t:T^k} (\mu_T t_1 = \mu_T t_2 \wedge \cdots \wedge \mu_T t_{k-1} \\ =&\mu_T t_k) = \sum_{t:T^k} \bigcirc(t_1 = t_2 \wedge \cdots \wedge t_{k-1} = t_k) \end{aligned}$$

By induction on k, and the preservation of products by $\bigcirc$, it suffices to show that for all $a, b : T$, $\bigcirc(a = b) = (\mu_T a = \mu_T b)$, which is true by univalence. $\Box$

Now that $\Box T$ is a colimit, we can easily define an inverse to $\lambda\,(f : \Box T \to U),\ x \circ \mu_T$ for any separated type U, using the universal property of the colimit.

We now have to prove point *(v)* in the definition of modality. Let $A : \mathrm{Type}_{n+1}$ be a sheaf and $B : A \to \mathrm{Type}_{n+1}$ be a sheaf family. We want to show that $\sum_{x:A}(Bx)$ is separated. Let E be a type, and $\sum_{e:E}(\chi\, e)$ a dense subobject of E.

Let f, g be two maps from $\sum_{e:E}(\chi\, e)$ to $\sum_{x:A}(Bx)$, equal when precomposed with π_1.

As A is separated, for any $x : E$, the first components of fx and gx are equal. For the second components, notice that for all $x : E$, $\sum_{y:E} x = y$ has a dense subobject, $\sum_{y:\sum_{e:E}(\chi\, e)} x = y_1$. Using the separation property with these type, one can show the second components of fx and gx are equals too, with respect to the first path.

C. From Separated Type to Sheaf

If T is already a separated type, the following lemma alllows us to build a sheaf :

Lemma 15. *Let $T : \mathrm{Type}_{n+1}$ be separated, and U be a sheaf. If T embeds in U, and is closed in U, then T is a sheaf.*

As any separated type T embeds in $\left(\mathrm{Type}_n^{\bigcirc}\right)^T$, it suffices to take the closure of T to get a sheaf.

D. Summary

If $T : \mathrm{Type}_{n+1}$, the sheafification of T is

$$\star T = \sum_{u:T\to\mathrm{Type}_n^{\bigcirc}} \neg\neg \left\| \sum_{a:X} (\lambda t,\ \bigcirc(a = t)) = u \right\|$$

We already know that $\star$ defines a modality. It remains to show that this modality is left exact.

Let $T : \mathrm{Type}_{n+1}$ such that $\star X$ is contractible. Let $x, y : T$. We want to show that $\star(x = y)$ is contractible.

As $\star T$ is contractible, we have a center $u_0 : T \to \mathrm{Type}_n^{\bigcirc}$ with an hypothesis $H_0 : \neg\neg \left\|\sum_{a:X}(\lambda t,\ \bigcirc(a = t)) = u_0\right\|$, and a contractibility condition $c : \prod_{t:\star T} x = (u_0, H_0)$. As $\mathrm{Contr}(\star(x = y))$ is a HProp-sheaf, we can change H_0 into a $a : T$ and a proof $u_0 = (\lambda t,\ \bigcirc(a = t))$. Finally, c type becomes $\prod_{v:\star T} v_1 = (\lambda t,\ \bigcirc(a = t))$.

The last thing to prove is the cumulativity of our sheafifications :

Proposition 16. *If $T : \mathrm{Type}_n$, then $\bigcirc T = \star\widehat{T}$ (where $\widehat{T}$ is T seen as a Type_{n+1}.*

V. Future Works

Prove that the Giraud axiom holds in homotopy type theory.
Extend the sheafification to all types, even if non truncated.

VI. Conclusion

The conclusion goes here.

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