

# Sheafification in HoTT

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## Abstract—Abstract

### I. INTRODUCTION

Plan of the paper :

- In section II, we define the foundations on which our worked is based.
- In section III, we explain the construction of the sheafification functor.
- Finally, we discuss related works and present future works.

### II. HOMOTOPY TYPE THEORY

#### A. Homotopy Type Theory

We refer to [Uni13]. Truncation levels + Im ?

If  $x$  is an object of a product or a sigma type, we denote the  $i$ -th projection of  $x$  by either  $\pi_i x$  or  $x_i$ .

- We use the subobjects classification seen in [RS13] : for all type  $B$ ,  $\text{Type}_n$  classifies subobjects of  $B$  with  $n$ -truncated fibers ; we have an equivalence

$$\sum_{A:\text{Type}} \sum_{f:A \rightarrow B} \prod_{b \in B} \text{IsTrunc } n \text{ fib}_f(b) \simeq (B \rightarrow \text{Type}_n)$$

and a pullback (with  $\text{Type}_n^\bullet$  the universe of pointed  $n$ -truncated types) :

$$\begin{array}{ccc} A & \xrightarrow{t_f} & \text{Type}_n^\bullet \\ f \downarrow & & \downarrow \text{pr}_1 \\ B & \xrightarrow{\chi_f} & \text{Type}_n \end{array}$$

for all  $f$  with  $n$ -truncated fibers, with

$$t_f = \lambda a, (\text{fib}_f(f(a)), (a, \text{idpath})).$$

#### B. Left exact Modalities

We use the following definition of truncated left exact modalities :

**Definition 1.** Let  $p$  be a truncation index. Then a left exact modality is the data of

- (i) A predicate  $P : \text{Type}_p \rightarrow \text{HProp}$
- (ii) For every  $p$ -truncated type  $A$ , a  $p$ -truncated type  $\circ A$  such that  $P(\circ A)$
- (iii) For every  $p$ -truncated type  $A$ , a map  $\eta_A : A \rightarrow \circ A$  such that

(iv) For every  $p$ -truncated types  $A$  and  $B$ , if  $P(B)$  then

$$\left\{ \begin{array}{ll} (\circ A \rightarrow B) & \rightarrow (A \rightarrow B) \\ f & \mapsto f \circ \eta_A \end{array} \right.$$

is an equivalence.

- (v) For all  $A : \text{Type}_p$  and  $B : A \rightarrow \text{Type}_p$  such that  $P(A)$  and  $\prod_{x:A} P(Bx)$ , then  $P(\sum_{x:A} B(x))$
- (vi) For all  $A : \text{Type}_p$  and  $x, y : A$ , if  $\circ A$  is contractible, then  $\circ(x = y)$  is contractible.

Conditions (i) to (iv) define a reflective subuniverse, (i) to (v) a modality.

The type of all  $p$ -types such that  $P$  will be noted  $\text{Type}_p^\circ$

Since basic operations (dependent products, products, sigma types) are stable for truncation levels, all theorem in [Uni13], chapter 7.7 remains true.

Moreover, left exactness implies in particular fibers preservations :

**Proposition 2.** For any  $n$ -truncated types  $X$  and  $Y$ , and any map  $f : X \rightarrow Y$ , the modalisation of fiber of  $f$  above any element  $y : Y$  is the fiber of  $\circ f$  above  $\eta_Y y$  :

$$\circ \left( \sum_{x:X} (fx = y) \right) = \sum_{x:\circ X} (\circ fx = \eta_Y y).$$

Moreover the following diagram commute

$$\begin{array}{ccc} \sum_{x:X} (fx = y) & \xrightarrow{\eta} & \circ \left( \sum_{x:X} (fx = y) \right) \\ \gamma \downarrow & & \swarrow \\ \sum_{x:\circ X} (\circ fx = \eta_Y y) & & \end{array}$$

where  $\pi_1 \circ \gamma = \eta_X$ , and  $\pi_2 \circ \gamma$  is the usual modalisation of paths.

#### C. Generalized pullbacks

**Definition 3.** Let  $f : X \rightarrow Y$  be a map, and  $p : \text{nat}$ . The  $p$ -pullback of  $f$ , noted  $X \times_Y \cdots \times_Y X$  is the limit of the diagram

$$\begin{array}{ccccc} X & & \cdots & & X \\ & \searrow f & & \swarrow f & \\ & & Y & & \end{array}$$

with  $p$  copies of  $X$ . The 0-pullback of  $f$  is  $\text{Unit}$ .

In homotopy type theory, we have

$$X \times_Y \cdots \times_Y X = \sum_{x: X^n} (fx_1 = fx_2) \wedge \cdots \wedge (fx_{n-1} = fx_n).$$

**Definition 4.** If  $f : X \rightarrow Y$  is a map, then the Čech nerve of  $f$  is the diagram

$$C(f) := \cdots X \times_Y X \times_Y X \rightrightarrows X \times_Y X \rightrightarrows X$$

with canonical projections.

The Giraud axiom asserts that, if  $f : X \rightarrow Y$  is a surjection, then the colimit of  $C(f)$  is  $Y$ . See [HTT].

### III. SHEAVES

We define sheaves by induction on the homotopical truncation level of types : we define sheaves in  $\mathbf{HProp}$  as the classical types (ie implied by their double negation), and we define sheaves on  $\mathbf{Type}_{n+1}$  via usual definition of sheaves, as seen in [MM92] and sheaves on  $\mathbf{Type}_n$ . Of course, the definition must be cumulative.

We will call  $\mathcal{S}_i$  the universe of sheaves on  $\mathbf{Type}_i$ .

In this section, we suppose given a truncation index  $n \geq -1$ , and a left exact modality  $\circ$  on  $\mathbf{Type}_n$ .

**Definition 5.** Let  $E$  be a type. The closure of a subobject of  $E$  classified by  $\chi$  is the subobject of  $E$  classified by  $\circ \circ \chi$ .

The subobject of  $E$  classified by  $\chi$  is said closed in  $E$  if its closure is itself :  $\chi = \circ \circ \chi$ .

**Definition 6.** Let  $E$  be a type, and  $\chi : E \rightarrow \mathbf{Type}_n$ . The subobject  $A$  of  $E$  classified by  $\chi$  is dense in  $E$  if its  $\circ$ -closure is  $E$  seen as a subobject of  $E$ , ie

$$\forall e : E, \left( \sum_{e' : E} e = e' \right) \simeq (\circ(\chi e)).$$

Moreover, for all  $x : A$  we need the following coherence diagram to commute

$$\begin{array}{ccc} \sum_{e' : A} x = e' & \equiv & (\chi x) \\ \downarrow \iota & & \downarrow \eta_{(\chi x)} \\ \sum_{e' : E} x = e' & \equiv & \circ(\chi x) \end{array}$$

where  $\iota : x \mapsto (x_{11}; x_2)$ .

It follows from fibers preservation that any  $n$ -subobject of a type seen as a  $n$ -subobject of its closure is closed.

**Definition 7 (Restriction).** For any  $A, E : \mathbf{Type}$  and  $F : \mathbf{Type}_{n+1}$ , we define :

- if  $A \hookrightarrow E$ ,  $M_E^\chi$  is the map sending an arrow  $f : E \rightarrow F$  to  $\upharpoonright f A$  the restriction of  $f$  to  $A$ .
- if  $A \rightarrow E$  with  $n$ -truncated fibers,  $\Phi_E^\chi$  is the map sending  $f : E \rightarrow F$  to  $\upharpoonright f A$ .

**Definition 8 (Separated Type).** A type  $F$  in  $\mathbf{Type}_{n+1}$  is separated if for all type  $E$ , and all dense subobject of  $E$  classified by  $\chi : E \rightarrow \mathbf{Type}_n$ ,  $\Phi_E^\chi$  is a monomorphism. In other words, the dotted arrow, if exists, is unique.

$$\begin{array}{ccc} \sum_{e:E} (\chi e) & \xrightarrow{f} & F \\ \pi_1 \downarrow & \nearrow & \\ E & & \end{array}$$

**Definition 9 (Sheaves).** A type  $F$  of  $\mathbf{Type}_{n+1}$  is a  $(n+1)$ -sheaf if it is separated, and for all type  $E$  and all dense subobject of  $E$  classified by  $\chi : E \rightarrow \mathbf{Type}_{-1}$ ,  $M_E^\chi$  is an equivalence. In other words, the dotted arrow exists and is unique.

$$\begin{array}{ccc} \sum_{e:E} (\chi e) & \xrightarrow{f} & F \\ \pi_1 \downarrow & \nearrow & \\ E & & \end{array}$$

From these definitions, one can show that

**Proposition 10.** •  $\mathbf{Type}_n^\circ$  is a sheaf.

- If  $A : \mathbf{Type}_{n+1}$  and  $B : A \rightarrow \mathbf{Type}_{n+1}$  such that for all  $a : A$ ,  $(B a)$  is a sheaf, then  $\prod_{a:A} B a$  is a sheaf.

### IV. CONSTRUCTION OF SHEAFIFICATION

We mimic the construction in [MM92]

A. For  $h$ -propositions

For the case  $n = -1$ , one can take any left exact modality on  $\mathbf{HProp}$ . Here we took the  $\neg\neg$  modality :

$$\forall P : \mathbf{HProp}, \circ_j P = \neg\neg P.$$

One can easily show that it defines a left exact modality.

B. From Type to Separated Type

In this section, we suppose given a truncation index  $n \geq -1$ , and a left exact modality  $\circ$  on  $\mathbf{Type}_n$ , compatible with the modality on  $\mathbf{HProp}$ .

Let  $T$  be in  $\mathbf{Type}_{n+1}$ . We define  $\square T$  as the image of  $\circ^T \circ \{\cdot\}_T$ , as in

$$\begin{array}{ccc} T & \xrightarrow{\{\cdot\}_T} & \mathbf{Type}_n^T \\ \mu_T \downarrow & & \downarrow \circ^T \\ \square T & \longrightarrow & (\mathbf{Type}_n^\circ)^T \end{array}$$

where  $\{\cdot\}_T$  is the singleton map  $\lambda(t : T), \lambda(t' : T), t = t'$ . In type theory words,

$$\begin{aligned} \square T &= \text{Im}(\lambda t : T, \lambda t', \circ(t = t')) \\ &= \sum_{u : T \rightarrow \mathbf{Type}_n^\circ} \left\| \sum_{a : X} (\lambda t, \circ(a = t)) = u \right\| \end{aligned}$$

**Proposition 11.** For all  $T : \mathbf{Type}_{n+1}$ ,  $\square T$  is separated.

*Proof.* We will use the following lemma :

**Lemma 12.** A  $(n+1)$ -truncated type  $T$  with an embedding  $f : T \rightarrow U$  into a separated  $(n+1)$ -truncated type  $U$  is itself separated.

As  $\Box T$  embeds in  $(\text{Type}_n^\circ)^T$ , we only have to show that the latter is separated. But it is the case with both parts proposition 10.  $\square$

We now need to show that  $\Box$  is universal. The following lemma is central in the proof :

**Lemma 13.** *Let  $T : \text{Type}_{n+1}$ . Then  $\Box T$  is the colimit of the closed diagonal diagram*

$$\cdots \overline{\Delta_3} \rightrightarrows \overline{\Delta_2} \rightrightarrows \overline{\Delta_1},$$

where  $\Delta_k$  is the  $k$ -pullback of  $\text{id} : T \rightarrow T$ .

*Proof.* As  $\mu_T$  is an embedding, we know by Giraud axiom that  $\Box T$  is the colimit of  $C(\mu_T)$ . If we can show that  $C(\mu_T) = C(\text{id})$ , the result will follow.

Let  $k : \text{nat}$ , let's show that

$$\begin{aligned} & \sum_{t:T^k} (\mu_T t_1 = \mu_T t_2 \wedge \cdots \wedge \mu_T t_{k-1} \\ & = \mu_T t_k) = \sum_{t:T^k} \circ(t_1 = t_2 \wedge \cdots \wedge t_{k-1} = t_k) \end{aligned}$$

By induction on  $k$ , and the preservation of products by  $\circ$ , it suffices to show that for all  $a, b : T$ ,  $\circ(a = b) = (\mu_T a = \mu_T b)$ , which is true by univalence.  $\square$

Now that  $\Box T$  is a colimit, we can easily define an inverse to  $\lambda(f : \Box T \rightarrow U)$ ,  $x \circ \mu_T$  for any separated type  $U$ , using the universal property of the colimit.

We now have to prove point (v) in the definition of modality. Let  $A : \text{Type}_{n+1}$  be a sheaf and  $B : A \rightarrow \text{Type}_{n+1}$  be a sheaf family. We want to show that  $\sum_{x:A} (Bx)$  is separated. Let  $E$  be a type, and  $\sum_{e:E} (\chi e)$  a dense subobject of  $E$ .

Let  $f, g$  be two maps from  $\sum_{e:E} (\chi e)$  to  $\sum_{x:A} (Bx)$ , equal when precomposed with  $\pi_1$ .

As  $A$  is separated, for any  $x : E$ , the first components of  $fx$  and  $gx$  are equal. For the second components, notice that for all  $x : E$ ,  $\sum_{y:E} x = y$  has a dense subobject,  $\sum_{y:\sum_{e:E} (\chi e)} x = y_1$ . Using the separation property with these type, one can show the second components of  $fx$  and  $gx$  are equals too, with respect to the first path.

### C. From Separated Type to Sheaf

If  $T$  is already a separated type, the following lemma alllows us to build a sheaf :

**Lemma 14.** *Let  $T : \text{Type}_{n+1}$  be separated, and  $U$  be a sheaf. If  $T$  embeds in  $U$ , and is closed in  $U$ , then  $T$  is a sheaf.*

As any separated type  $T$  embeds in  $(\text{Type}_n^\circ)^T$ , it suffices to take the closure of  $T$  to get a sheaf.

### D. Summary

If  $T : \text{Type}_{n+1}$ , the sheafification of  $T$  is

$$\star T = \sum_{u:T \rightarrow \text{Type}_n^\circ} \neg\neg \left\| \sum_{a:X} (\lambda t, \circ(a = t)) = u \right\|$$

We already know that  $\star$  defines a modality. It remains to show that this modality is left exact.

The last thing to prove is the cumulativity of our sheafifications :

**Proposition 15.** *If  $T : \text{Type}_n$ , then  $\circ T = \star \hat{T}$  (where  $\hat{T}$  is  $T$  seen as a  $\text{Type}_{n+1}$ ).*

## V. FUTURE WORKS

Prove that the Giraud axiom holds in homotopy type theory. Extend the sheafification to all types, even if non truncated.

## VI. CONCLUSION

The conclusion goes here.

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