Sheafification in HoTT

Kevin Quirin and Nicolas Tabareau
INRIA
École des Mines de Nantes
Nantes, France

***Abstract*—Abstract**

I. Introduction

Plan of the paper :

- In section II, we define the foundations on which our worked is based.
- In section III, we explain the construction of the sheafification functor.
- Finally, we discuss related works and present future works.

II. Homotopy Type Theory

A. Homotopy Type Theory

We refer to [Uni13]. Truncation levels + Im ?

If x is an object of a product or a sigma type, we denote the i-th projection of x by either $\pi_i x$ or x_i.

- We use the subobjects classification seen in [RS13] : for all type B, Type_n classifies subobjects of B with n-truncated fibers ; we have an equivalence

$$\sum_{A:\mathrm{Type}} \sum_{f:A\to B} \prod_{b\in B} \mathrm{IsTrunc}\ n\ \mathrm{fib}_f(b) \simeq (B \to \mathrm{Type}_n)$$

and a pullback (with $\mathrm{Type}_n^\bullet$ the universe of pointed n-truncated types) :

$$\begin{tikzcd} A \arrow[r, "t_f"] \arrow[d, "f"'] & \mathrm{Type}_n^\bullet \arrow[d, "\mathrm{pr}_1"] \\ B \arrow[r, "\chi_f"'] & \mathrm{Type}_n \end{tikzcd}$$

forall f with n-truncated fibers, with

$$t_f = \lambda a,\ (\mathrm{fib}_f(f(a)), (a, \mathrm{idpath})).$$

B. Left exact Modalities

We use the following definition of truncated left exact modalities :

Definition 1. *Let p be a truncation index. Then a left exact modality is the data of*

(i) A predicate $P : \mathrm{Type}_p \to \mathrm{HProp}$
(ii) For every p-truncated type A, a p-truncated type $\bigcirc A$ such that $P(\bigcirc A)$
(iii) For every p-truncated type A, a map $\eta_A : A \to \bigcirc A$ such that
(iv) For every p-truncated types A and B, if $P(B)$ then

$$\begin{cases} (\bigcirc A \to B) & \to & (A \to B) \\ f & \mapsto & f \circ \eta_A \end{cases}$$

is an equivalence.
(v) For all $A : \mathrm{Type}_p$ and $B : A \to \mathrm{Type}_p$ such that $P(A)$ and $\prod_{x:A} P(Bx)$, then $P\left(\sum_{x:A} B(x)\right)$
(vi) For all $A : \mathrm{Type}_p$ and $x, y : A$, if $\bigcirc A$ is contractible, then $\bigcirc(x = y)$ is contractible.

Conditions (i) to (iv) define a reflective subuniverse*, (i) to (v) a* modality.

The type of all p-types such that P will be noted $\mathrm{Type}_p^\bigcirc$

Since basic operations (dependent products, products, sigma types) are stable for truncation levels, all theorem in [Uni13], chapter 7.7 remains true.

Moreover, left exactness implies in particular fibers preservations :

Proposition 2. *For any n-truncated types X and Y, and any map $f : X \to Y$, the modalisation of fiber of f above any element $y : Y$ is the fiber of $\bigcirc f$ above $\eta_Y y$:*

$$\bigcirc\left(\sum_{x:X}(fx = y)\right) = \sum_{x:\bigcirc X}(\bigcirc f x = \eta_Y y).$$

Moreover the following diagram commute

$$\begin{tikzcd} \sum_{x:X}(fx = y) \arrow[r, "\eta"] \arrow[d, "\gamma"'] & \bigcirc\left(\sum_{x:X}(fx = y)\right) \arrow[dl, equal] \\ \sum_{x:\bigcirc X}(\bigcirc f x = \eta_Y y) & \end{tikzcd}$$

where $\pi_1 \circ \gamma = \eta_X$, and $\pi_2 \circ \gamma$ is the usual modalisation of paths.

C. Generalized pullbacks

Definition 3. *Let $f : X \to Y$ be a map, and $p : \mathrm{nat}$. The p-pullback of f, noted $X \times_Y \cdots \times_Y X$ is the limit of the diagram*

$$\begin{tikzcd} X \arrow[dr, "f"] & \cdots & X \arrow[dl, "f"'] \\ & Y & \end{tikzcd}$$

with p copies of X. The 0-pullback of f is Unit.

In homotopy type theory, we have

$$X \times_Y \cdots \times_Y X = \sum_{x:X^n} (fx_1 = fx_2) \wedge \cdots \wedge (fx_{n-1} = fx_n).$$

Definition 4. *If $f : X \to Y$ is a map, then the* Čech nerve *of f is the diagram*

$$C(f) := \cdots X \times_Y X \times_Y X \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; X \times_Y X \rightrightarrows X$$

with canonical projections.

The Giraud axiom asserts that, if $f : X \to Y$ is a surjection, then the colimit of $C(f)$ is Y. See [HTT].

III. Sheaves

We define sheaves by induction on the homotopical truncation level of types : we define sheaves in HProp as the classical types (*ie* implied by their double negation), and we define sheaves on Type_{n+1} *via* usual definition of sheaves, as seen in [MM92] and sheaves on Type_n. Of course, the definition must be cumulative.

We will call $\mathcal{S}_i$ the universe of sheaves on Type_i.

A. Definitions and first properties

In this section, we suppose given a truncation index $n \geqslant -1$, and a left exact modality $\bigcirc$ on Type_n.

Definition 5. *Let E be a type. The* closure *of a subobject of E classified by χ is the subobject of E classified by $\bigcirc \circ \chi$.*

The subobject of E classified by χ is said closed in E *if its closure is itself : $\chi = \bigcirc \circ \chi$.*

Definition 6. *Let E be a type, and $\chi : E \to \mathrm{Type}_n$. The subobject A of E classified by χ is* dense *in E if its $\bigcirc$-closure is E seen as a subobject of E,* ie

$$\forall e : E, \ \left(\sum_{e':E} e = e' \right) \simeq (\bigcirc(\chi\ e)).$$

Moreover, for all $x : A$ we need the following coherence diagram to commute

$$\begin{tikzcd}
\sum_{e':A} x = e' \arrow[r, equal] \arrow[d, "\iota"'] & (\chi\ x) \arrow[d, "\eta_{(\chi\ x)}"] \\
\sum_{e':E} x = e' \arrow[r, equal] & \bigcirc(\chi\ x)
\end{tikzcd}$$

where $\iota : x \mapsto (x_{11}; x_2)$.

It follows from fibers preservation that any n-subobject of a type seen as a n-subobject of its closure is closed.

Definition 7 (Restriction). *For any A, E : Type and F : Type_{n+1}, we define :*

- *if $A \hookrightarrow E$, M_E^χ is the map sending an arrow $f : E \to F$ to $\restriction fA$ the restriction of f to A.*
- *if $A \to E$ with n-truncated fibers, Φ_E^χ is the map sending $f : E \to f$ to $\restriction fA$.*

Definition 8 (Separated Type). *A type F in Type_{n+1} is* separated *if for all type E, and all dense subobject of E classified by $\chi : E \to \mathrm{Type}_n$, Φ_E^χ is a monomorphism. In other words, the dotted arrow, if exists, is unique.*

$$\begin{tikzcd}
\sum_{e:E}(\chi\ e) \arrow[r, "f"] \arrow[d, "\pi_1"'] & F \\
E \arrow[ur, dashed]
\end{tikzcd}$$

Definition 9 (Sheaves). *A type F of Type_{n+1} is a $(n+1)$-*sheaf *if it is separated, and for all type E and all dense subobject of E classified by $\chi : E \to \mathrm{Type}_{-1}$, M_E^χ is an equivalence. In other words, the dotted arrow exists and is unique.*

$$\begin{tikzcd}
\sum_{e:E}(\chi\ e) \arrow[r, "f"] \arrow[d, "\pi_1"'] & F \\
E \arrow[ur, dashed]
\end{tikzcd}$$

From these definitions, one can show that

Proposition 10.

- *$\mathrm{Type}_n^\bigcirc$ is a sheaf.*
- *If $A : \mathrm{Type}_{n+1}$ and $B : A \to \mathrm{Type}_{n+1}$ such that for all $a : A$, $(B\ a)$ is a sheaf, then $\prod_{a:A} B\ a$ is a sheaf.*

B. Construction of sheafification

We mimic the construction in [MM92]

1) For h-propositions: For the case $n = -1$, one can take any left exact modality on HProp. Here we took the $\neg\neg$ modality :

$$\forall P : \mathrm{HProp}, \ \ \bigcirc P = \neg\neg P.$$

One can easily show that it defines a left exact modality.

2) From Type to Separated Type: In this section, we suppose given a truncation index $n \geqslant -1$, and a left exact modality $\bigcirc$ on Type_n, compatible with the modality on HProp.

Let T be in Type_{n+1}. We define $\Box T$ as the image of $\bigcirc^T \circ \{\cdot\}_T$, as in

$$\begin{tikzcd}
T \arrow[r, "\{\cdot\}_T"] \arrow[d, "\mu_T"'] & \mathrm{Type}_n^T \arrow[d, "\bigcirc^T"] \\
\Box T \arrow[r] & \left(\mathrm{Type}_n^\bigcirc\right)^T
\end{tikzcd} ,$$

where $\{\cdot\}_T$ is the singleton map $\lambda(t : T),\ \lambda(t' : T),\ t = t'$. In type theory words,

$$\Box T = \mathrm{Im}(\lambda\ t : T,\ \lambda\ t',\ \bigcirc(t = t')).$$

Proposition 11. *For all $T : \mathrm{Type}_{n+1}$, $\Box T$ is separated.*

Proof. We will use the following lemma :

Lemma 12. *A $(n+1)$-truncated type T with an embedding $f : T \to U$ into a separated $(n+1)$-truncated type U is itself separated.*

As $\Box T$ embeds in $\left(\mathrm{Type}_n^{\bigcirc}\right)^T$, we only have to show that the latter is separated. But it is the case with both parts proposition 10. $\Box$

We now need to show that $\Box$ is universal. The following lemma is central in the proof :

Lemma 13. *Let* $T : \mathrm{Type}_{n+1}$. *Then* $\Box T$ *is the colimit of the closed diagonal diagram*

$$\cdots \overline{\Delta_3} \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \overline{\Delta_2} \substack{\longrightarrow \\ \longrightarrow} \overline{\Delta_1},$$

where Δ_k *is the* k*-pullback of* $\mathrm{id} : T \to T$.

Proof. As μ_T is an embedding, we know by Giraud axiom that $\Box T$ is the colimit of $C(\mu_T)$. If we can show that $C(\mu_T) = C(\mathrm{id})$, the the result will follow.

Let $k : \mathrm{nat}$, let's show that

$$\begin{aligned} &\sum_{t:T^k} (\mu_T t_1 = \mu_T t_2 \wedge \cdots \wedge \mu_T t_{k-1} \\ =&\mu_T t_k) = \sum_{t:T^k} \bigcirc(t_1 = t_2 \wedge \cdots \wedge t_{k-1} = t_k) \end{aligned}$$

By induction on k, and the preservation of products by $\bigcirc$, it suffices to show that for all $a, b : T$, $\bigcirc(a = b) = (\mu_T a = \mu_T b)$, which is true by univalence. If U is a separated type, we can now easily define an inverse to $\lambda\,(f : \Box T \to f),\ x \circ \mu_T$. $\Box$

- $(\Box,\ \mu)$ is a modality.

3) From Separated Type to Sheaf:

- **Theorem 14.** *Any type* $A : \mathrm{Type}_{n+1}$ *closed in a sheaf is itself a sheaf.*
- Definition of sheafification reflextive subuniverse $(\star,\ \nu)$
- left exactness
- cumulativity

IV. Future Works

Prove that the Giraud axiom holds in homotopy type theory.

V. Conclusion

The conclusion goes here.

Acknowledgment

The authors would like to thank...

References

[RS13] Egbert Rijke and Bas Spitters. Sets in homotopy type theory. 2013.
[Uni13] Univalent Foundations Project. *Homotopy Type Theory: Univalent Foundations for Mathematics*. 2013.
[MM92] Saunders MacLane and Ieke Moerdijk. *Sheaves in Geometry and Logic*. Springer-Verlag, 1992.
[HTT] Jacob Lurie *Higher Topos Theory*
[lumsdaine] Lumsdaine Diagrams and limits