Lawvere-Tierney Sheafification in Homotopy Type Theory

Kevin Quirin
Mines de Nantes
Nantes, France

29 June 2015

Lawvere-Tierney Sheafification in Homotopy Type Theory
Kevin Quirin
Mines de Nantes
Nantes, France
29 June 2015

Thank you all for coming, especially the members of the committee. It is a great honour for me to have such a great comitee for my defence.

I will present you my work on sheafification in homotopy type theory

Introduction to type theory

- Formalizing proofs
- The Curry-Howard isomorphism

Forcing and Sheafification

Lawvere-Tierney Sheafification in Type Theory

Kevin Quirin
Mines de Nantes
Nantes, France

Introduction to type theory
Forcing and Sheafification
Lawvere-Tierney Sheafification in Type Theory

Foo

Bar

Kevin Quirin
Mines de Nantes
Nantes, France

Foo
Bar

Errors in mathematics

One issue with mathematics: it is hard to check proofs.

..

Errors in mathematics

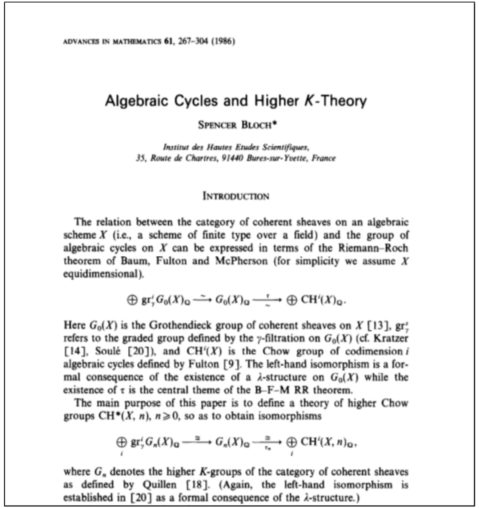
One issue with mathematics: it is hard to check proofs.

..

1. In a paper by Spencer Bloch, Suslin found an error in lemma 1.1. Almost all the paper ws relying on this lemma

Errors in mathematics

One issue with mathematics: it is hard to check proofs.



ADVANCES IN MATHEMATICS 61, 267–304 (1986)

Algebraic Cycles and Higher K-Theory

SPENCER BLOCH*

Institut des Hautes Etudes Scientifiques, 35, Route de Chartres, 91440 Bures-sur-Yvette, France

INTRODUCTION

The relation between the category of coherent sheaves on an algebraic scheme X (i.e., a scheme of finite type over a field) and the group of algebraic cycles on X can be expressed in terms of the Riemann–Roch theorem of Baum, Fulton and McPherson (for simplicity we assume X equidimensional).

$$\bigoplus \mathrm{gr}^i_\gamma G_0(X)_{\mathbb{Q}} \xrightarrow{\sim} G_0(X)_{\mathbb{Q}} \xrightarrow[\sim]{\tau} \bigoplus \mathrm{CH}^i(X)_{\mathbb{Q}}.$$

Here $G_0(X)$ is the Grothendieck group of coherent sheaves on X [13], gr^i_γ refers to the graded group defined by the γ-filtration on $G_0(X)$ (cf. Kratzer [14], Soulé [20]), and $\mathrm{CH}^i(X)$ is the Chow group of codimension i algebraic cycles defined by Fulton [9]. The left-hand isomorphism is a formal consequence of the existence of a λ-structure on $G_0(X)$ while the existence of τ is the central theme of the B–F–M RR theorem.

The main purpose of this paper is to define a theory of higher Chow groups $\mathrm{CH}^*(X, n)$, $n \geq 0$, so as to obtain isomorphisms

$$\bigoplus_i \mathrm{gr}^i_\gamma G_n(X)_{\mathbb{Q}} \xrightarrow{\cong} G_n(X)_{\mathbb{Q}} \xrightarrow[\tau_n]{\cong} \bigoplus_i \mathrm{CH}^i(X, n)_{\mathbb{Q}},$$

where G_n denotes the higher K-groups of the category of coherent sheaves as defined by Quillen [18]. (Again, the left-hand isomorphism is established in [20] as a formal consequence of the λ-structure.)

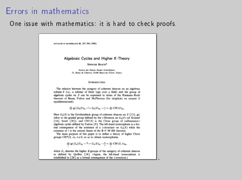
.

Lawvere-Tierney Sheafification in Homotopy Type Theory

Kevin Quirin
Mines de Nantes
Nantes, France

Type Theory
Proof
Curry-Howard
Forcing
Sheafification

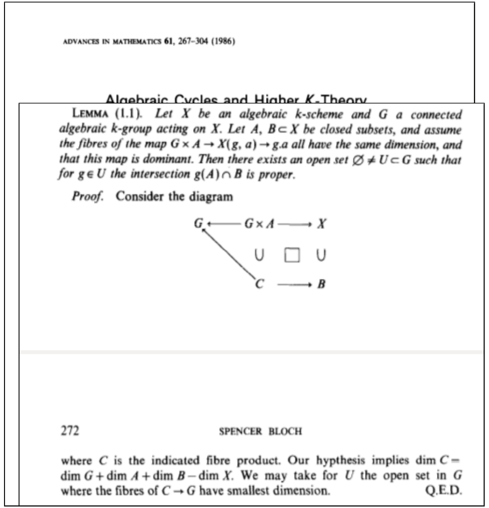
2016-11-26



1. In a paper by Spencer Bloch, Suslin found an error in lemma 1.1. Almost all the paper ws relying on this lemma
2. While the original - false - proof was only a few lines long, the new proof is about thirty pages long, and contains complex arguments.

Errors in mathematics

One issue with mathematics: it is hard to check proofs.


ADVANCES IN MATHEMATICS 61, 267–304 (1986)

Algebraic Cycles and Higher K-Theory

LEMMA (1.1). *Let X be an algebraic k-scheme and G a connected algebraic k-group acting on X. Let $A, B \subset X$ be closed subsets, and assume the fibres of the map $G \times A \to X(g, a) \to g.a$ all have the same dimension, and that this map is dominant. Then there exists an open set $\varnothing \neq U \subset G$ such that for $g \in U$ the intersection $g(A) \cap B$ is proper.*

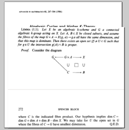
Proof. Consider the diagram

$$\begin{array}{ccccc} G & \longleftarrow & G \times A & \longrightarrow & X \\ & \nwarrow & \cup & \square & \cup \\ & & C & \longrightarrow & B \end{array}$$

272 SPENCER BLOCH

where C is the indicated fibre product. Our hypthesis implies $\dim C = \dim G + \dim A + \dim B - \dim X$. We may take for U the open set in G where the fibres of $C \to G$ have smallest dimension. Q.E.D.

Errors in mathematics
One issue with mathematics: it is hard to check proofs.



1. In a paper by Spencer Bloch, Suslin found an error in lemma 1.1. Almost all the paper ws relying on this lemma
2. While the original - false - proof was only a few lines long, the new proof is about thirty pages long, and contains complex arguments.

Errors in mathematics

One issue with mathematics: it is hard to check proofs.

..

Proofs are more and more complicated.

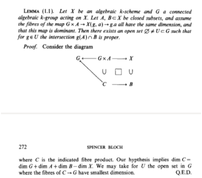
Errors in mathematics

One issue with mathematics: it is hard to check proofs.

..

Proofs are more and more complicated.

1. In a paper by Spencer Bloch, Suslin found an error in lemma 1.1. Almost all the paper ws relying on this lemma
2. While the original - false - proof was only a few lines long, the new proof is about thirty pages long, and contains complex arguments.



Lemma (1.1). *Let X be an algebraic k-scheme and G a connected algebraic k-group acting on X. Let $A, B \subset X$ be closed subsets, and assume the fibres of the map $G \times A \to X (g, a) \to g.a$ all have the same dimension, and that this map is dominant. Then there exists an open set $\varnothing \neq U \subset G$ such that for $g \in U$ the intersection $g(A) \cap B$ is proper.*

Proof. Consider the diagram

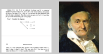
$$\begin{tikzcd} G & G \times A \arrow[l] \arrow[r] & X \\ & \cup \arrow[r, phantom, "\square"] & \cup \\ & C \arrow[uul] \arrow[r] & B \end{tikzcd}$$

272 SPENCER BLOCH

where C is the indicated fibre product. Our hypthesis implies $\dim C = \dim G + \dim A + \dim B - \dim X$. We may take for U the open set in G where the fibres of $C \to G$ have smallest dimension. Q.E.D.

Lawvere-Tierney Sheafification in Homotopy Type Theory

Kevin Quirin
Mines de Nantes
Nantes, France



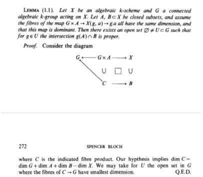
1. If you give a proof to the best mathematician

Ich weiß nicht

Kevin Quirin
Mines de Nantes
Nantes, France

Ich weiß nicht

1. If you give a proof to the best mathematician
2. He will probably don't know if it is true or not



LEMMA (1.1). *Let X be an algebraic k-scheme and G a connected algebraic k-group acting on X. Let $A, B \subset X$ be closed subsets, and assume the fibres of the map $G \times A \to X (g, a) \to g.a$ all have the same dimension, and that this map is dominant. Then there exists an open set $\varnothing \neq U \subset G$ such that for $g \in U$ the intersection $g(A) \cap B$ is proper.*

Proof. Consider the diagram

$$\begin{tikzcd} G & G \times A \arrow[l] \arrow[r] & X \\ & \cup \arrow[r, phantom, "\square"] & \cup \\ & C \arrow[uul] \arrow[r] & B \end{tikzcd}$$

272 SPENCER BLOCH

where C is the indicated fibre product. Our hypthesis implies $\dim C = \dim G + \dim A + \dim B - \dim X$. We may take for U the open set in G where the fibres of $C \to G$ have smallest dimension. Q.E.D.

1. If you give a proof to the best mathematician
2. He will probably don't know if it is true or not
3. Our hope is to give to rather to a computer

False

Error on line 4

Kevin Quirin
Mines de Nantes
Nantes, France

1. If you give a proof to the best mathematician
2. He will probably don't know if it is true or not
3. Our hope is to give to rather to a computer
4. Who can decide if it is right or wrong, and in the latter case, where is the error

Curry-Howard

As the previous section suggests, it is a good idea to know what is a correct proof.

This part of mathematics is called *proof theory*.

It describes how to be sure that a proof is correct.

Curry-Howard
As the previous section suggests, it is a good idea to know what is a correct proof.
This part of mathematics is called *proof theory*.
It describes how to be sure that a proof is correct.

$$\frac{}{\Gamma \vdash \top} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

Lawvere-Tierney Sheafification in Homotopy Type Theory

Kevin Quirin
Mines de Nantes
Nantes, France

$$\frac{}{\Gamma \vdash \top} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$

These rules really look like the ones of lambda-calculus, the most simple programming language :

$$\frac{}{\Gamma \vdash \quad \top} \qquad \frac{\Gamma, \quad A \vdash \quad B}{\Gamma \vdash A \Rightarrow B}$$

$$\frac{\Gamma \vdash \quad A \qquad \Gamma \vdash \quad B}{\Gamma \vdash \quad A \wedge B}$$

$$\frac{\Gamma \vdash \quad A}{\Gamma \vdash \quad A \vee B} \qquad \frac{\Gamma \vdash \quad B}{\Gamma \vdash \quad A \vee B}$$

Kevin Quirin
Mines de Nantes
Nantes, France

These rules really look like the ones of lambda-calculus, the most simple programming language :

These rules really look like the ones of lambda-calculus, the most simple programming language :

$$\frac{}{\Gamma \vdash \star : \top} \qquad \frac{\Gamma, a : A \vdash b : B}{\Gamma \vdash A \Rightarrow B}$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash (a, b) : A \wedge B}$$

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash \mathrm{inl}\, a : A \vee B} \qquad \frac{\Gamma \vdash b : B}{\Gamma \vdash \mathrm{inr}\, b : A \vee B}$$

These rules really look like the ones of lambda-calculus, the most simple programming language :

$$\frac{}{\Gamma \vdash \star : \top} \qquad \frac{\Gamma, a : A \vdash b : B}{\Gamma \vdash A \Rightarrow B}$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash (a, b) : A \wedge B}$$

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash \mathrm{inl}\, a : A \vee B} \qquad \frac{\Gamma \vdash b : B}{\Gamma \vdash \mathrm{inr}\, b : A \vee B}$$

The Curry-Howard isomorphism states that if a term of a type can be seen as a program, it can also be seen as a proof of a formula.

The Curry-Howard isomorphism states that if a term of a type can be seen as a program, it can also be seen as a proof of a formula.

Introduction to type theory

Formalizing proofs

The Curry-Howard isomorphism

Forcing and Sheafification

Lawvere-Tierney Sheafification in Type Theory

Introduction to type theory

Formalizing proofs

The Curry-Howard isomorphism

Forcing and Sheafification

Lawvere-Tierney Sheafification in Type Theory