- inner product: $\langle x,y\rangle = x_1y_1 + \cdots + x_ny_n = \sum x_iy_i$
  - → in matrix: $\langle x,y\rangle = x^Ty = [x_1 \cdots x_n]\begin{bmatrix} y_1 \\ y_n \end{bmatrix}$
  - $\langle x,y\rangle = \langle y,x\rangle$ $\langle x, cy+dz\rangle = c\langle x,y\rangle + d\langle y,z\rangle$
  - $\langle x, Ay\rangle = \langle A^Tx, y\rangle$ $\langle x,x\rangle = \|x\|_2^2$
  - $|\langle x,y\rangle| \leq \|x\|_2 \cdot \|y\|_2$ $\langle x,y\rangle = \|x\|\|y\|\cos\theta$
- orthog vectors: $v_1 \perp v_2$ if $\langle v_1, v_2\rangle = 0$
- orthog sets: $\{x_1, x_2 \dots x_n\}$ is orthog if $\langle x_i, x_j\rangle = 0$ → $i \neq j$
- orthonormal sets: set is orthog + unit vectors (normalize!)
  - $\langle x_i, x_j\rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$
- orthogonal subspace: 2 subspace S_1 & S_2 is orthog if $\forall u \in S_1$ and $\forall v \in S_2$ → $\langle u,v\rangle = 0$ → $S_1 \perp S_2$
  - ⇒ check if their basis sets are orthog
  - → $B^TC = 0$ where B & C are basis matrix
- ortho-complement: if U is the subspace of W, $U^\perp$ is the set of all vec that's orthog to U
  - → $U^\perp$ is largest subspace that's ortho to U
  - → $\dim(U^\perp) + \dim(U) = \dim(W)$ → $(U^\perp)^\perp = U$
  - → $\text{basis}(U^\perp) \cup \text{basis}(U) = \text{basis}(W)$
  - ⇒ $N(A) = R(A^T)^\perp$ where A is $m\times n$ matrix
  - $N(A^T) = R(A)^\perp$
- ortho-proj onto vectors: projecting x onto u (vectors)
  - $\text{proj}_U(x) = \frac{\langle x,u\rangle}{\langle u,u\rangle}u = \left(\frac{uu^T}{\|x\|^2}\right)x = (P_u)x$
  - → NB: you can't proj smt out of u once it's in it alr
- orthonormal basis (ONB): a set $\{w_1 \dots w_m\}$ is ONB if it's a basis + it is orthonormal (normalize!!)
  - → if $\{w_i\}$ is ONB for U and $x \in U$, there's c_i s.t
  - 1) $x = \sum c_jw_j$ 2) $c_j = \langle w_j, x\rangle$ 3) $\|x\| = \sum \|c_j\|^2$
- Gram Schmidt: given basis $\{v_1, v_2 \dots v_n\}$ of (U), find ONB
  - $u_1 = v_1$ $u_2 = v_2 - \frac{\langle v_2\, u_1\rangle}{\langle u_1, u_1\rangle}u_1$ ⇒ $e_i = \frac{u_i}{\|u_i\|}$
  - $u_3 = v_3 - \frac{\langle v_3\, u_1\rangle}{\langle u_1, u_1\rangle}u_1 - \frac{\langle v_3\, u_2\rangle}{\langle u_2\, u_2\rangle}u_2$ → normalize!!
  - → $\{u_1 \dots u_n\}$ is orthogonal basis for U
  - → $\{e_1 \dots e_n\}$ is orthonormal basis for U
- proj onto subspace: given ONB of U $\{w_1 \dots w_n\}$
  - $\text{proj}_U(x) = \text{proj}_{w_1}(x) + \dots + \text{proj}_{w_n}(x)$
  - $= (w_1w_1^T + \dots w_nw_n^T)x$
  - $= P_Ux$ → P_U is ortho-projector onto U
  - → let $B = [w_1 \cdots w_n]$ (ONB as col)
  - $P_U = BB^T$ (only works for ONB)

- ortho-projector matrix: matrix P is an ortho-proj matrix iff $P^2 = P$ and $P^T = P$ (also $P^k = P$)
  - ↳ let P_U be ortho-proj onto U, $P_{U^\perp} = I - P_U$
  - ↳ $x - P_U(x) \in P_{U^\perp}$ → $x = P_U(x) + P_{U^\perp}(x)$
  - ↳ $\|x - P_U(x)\|^2 = \|P_{U^\perp}(x)\|^2$ = min dist from x to subspace U
- orthogonal matrix: mat A is orthog if $A^TA = AA^T = I$
  - → A is square + inv $(A^{-1} = A^T)$ → $\|Ax\| = \|x\|$ (norm preserving)
  - → rows of A are orthonormal → col of A also orthonormal
- reflection matrix: reflection of vec x across subspace U
  - $\text{ref}_U(x) = (I - 2P_{U^\perp})x$ → $I - 2P_{U^\perp}$ is ref matrix
  - → for any ortho-proj P, $I - 2P$ is orthogonal
- QR decomp: given $m\times n$ mat A, write it as $A = QR$ where Q is $m\times n$ orthog mat & R is ~~$m\times n$~~ $n\times n$ upper triangular
  - 1) write A in columns: $A = [a_1 \cdots a_n]$ NORMALIZE!
  - 2) apply Gram Schmidt to $\{a_1 \dots a_n\}$ to get $\{w_1, \dots w_n\}$
  - 3) get thin QR decomp: $A = Q_1R_1$
  - $Q_1 = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}$ $(m\times n)$ $R_1 = \begin{bmatrix} \langle w_1\, a_1\rangle & \langle w_1\, a_2\rangle & \cdots & \langle w_1\, a_n\rangle \\ & \langle w_2\, a_2\rangle & \cdots & \langle w_2\, a_n\rangle \\ & & \ddots & \\ & & & \langle w_m\, a_n\rangle \end{bmatrix}$ $(n\times n)$
  - 4) get full QR decomp $A = QR$
  - $Q = [Q_1 \;\; Q_2]$ $R = \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$ → Q_1 is ONB for $R(A)$ → Q_2 is ONB for $R(A)^\perp$
  - → $Q_2 = N(A^T) = N(Q_1^T)$ → $\text{proj}_{R(A)}x = (Q_1Q_1^T)x$ → $\text{proj}_{R(A)^\perp}x = (Q_2Q_2^T)x$
- least squares: A is $m\times n$ w/ $m > n$ & $\text{rank}(A) \geq n$ - least squares sol to problem $Ax \approx b$ ($\text{rank}(A) = n$ give unique sol)
  - $A^TAx^* = A^Tb$ → $x^* = (A^TA)^{-1}A^Tb$
  - $R_1x^* = Q^Ty$ → $x^* = R^{-1}Q_1^Ty$ if given thin QR decomp of A
  - residual $= \|Ax - b\| = \|Q_2^Ty\|$
- fitting models: apply $f(t)$ to each t_i and construct A
- eigval/eigvec: a pair if $Av = \lambda v$, v is non-zero vec, λ is scalar
  - → characteristic poly: $c_A(\lambda) = \det(A - I\lambda)$ $E_{\lambda_j} = N(A - \lambda_jI)$ = eigenspace of λ_j
  - → roots of $c_A(\lambda)$ is eigvals
  - → solve $N(A - \lambda_jI)$ to find corresp eigvector(s)
  - → will have n eigval (may be 0, ⊖ or repeated)
- fact: there's an eigbasis (eigvec span $\mathbb{R}^n$) if $d_j = m_j$ for all λ_j (m_j = # times eigval repeat: $d_j = \dim(E_{\lambda_j})$)
- diagonalizability: matrix is diagonalizable if there exist inv matrix P and diagonal mat D s.t $A = PDP^{-1}$ → A is $n\times n$
  - → A diagonalizable iff it has n distinct eigvec (eigbasis)
  - set $P = [v_1 \dots v_n]$ and $D = [\lambda_1 \dots \lambda_n]$ → λ_j, v_j on eigvec/val cols
  - → application: $A^k = PD^kP^{-1}$
- symmetric matrix: matrix is symmetric if $A^T = A$
  - → all eigval of real symm matrix are real
  - → if λ_1 & λ_2 are distinct eigval of symm matrix, $v_1 \perp v_2$
- if A is real symmetric; it is orthogonally diagonalizable
  - $A = PDP^{-1}$ ⇒ PDP^T b/c P is an orthogonal matrix (need to normalize)

- let A be any real $m\times n$ matrix
  - → A^TA and AA^T are real symmetric ∴ they are orthogonal diagonalizable
  - → if λ is non-zero eigval in AA^T, it'll be eigval in A^TA
  - → if λ " " of AA^T, λ has same lvl of repetition in A^TA
  - ⇒ all eigvals of AA^T and A^TA are non-negative
- SVD: let A be a real $m\times n$ matrix, then we can write $A = P\Sigma Q^T$ P, Q are orthog matrix, Σ is "diagonal" matrix
  - 1) Construct Σ $(m\times n)$ important
    - → find eigval of A^TA or AA^T, order them
    - → set $\sigma_k = \sqrt{\lambda_k}$ (singular values) → put on diag
  - 2) Construct Q $(n\times n)$
    - → find corresponding eigvec for each eigval
    - → if missing eigvec, assume additional eigval = 0
    - → set normalized corresp eigvec as col of Q
    - $Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}$ give first r col of P
  - 3) Construct P $(m\times m)$
    - → let p_k be col of P, then $p_k = \frac{1}{\sigma_k}Aq_k$
    - → for remaining $m-r$ col, complete it to get ONB (recall thin → full QR, solve $N(Q_1^T)$)
- → $\|A\|_{op} = \sigma_1$ (largest singular val) $\|A^{-1}\| = 1/\sigma_r$
  - $\|A\|_F = (\sigma_1^2 + \dots + \sigma_r^2)^{1/2}$ $\text{rank}(A) = r$
  - $\text{cond}(A) = \sigma_1/\sigma_r$
- → if A is $n\times n$ and inv, $A^{-1} = Q\Sigma^{-1}P^T$
  - → reorder columns to get SVD of A^{-1}
- → let $P = [p_1 | \dots | p_m]$ and $Q = [q_1 | \dots | q_n]$
  - $\{p_1 \dots p_r\}$ is ONB of $R(A)$
  - $\{p_{r+1} \dots p_m\}$ is ONB of $N(A^T)$
  - $\{q_1 \dots q_r\}$ is ONB of $R(A^T)$
  - $\{q_{r+1} \dots q_n\}$ is ONB of $N(A)$