MATH 307 Midterm Exam 1

June 2, 2022

- No calculators, cellphones, laptops or notes
- Time allowed: 50 minutes
- 45 total marks
- Write your name and student number in the space below
- Notation: $N(A)$ is the nullspace of A and $R(A)$ is the range of A

Name: SOLUTIONS

Student Number:

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1. Determine if the statement is **True** or **False**. No justification required.

   (a) (2 marks) If $A^3 = 0$ then $R(A) \subseteq N(A)$.

   $\boxed{\text{False}}$ $A^3 = 0 \Rightarrow R(A^2) \subseteq N(A)$

   (b) (2 marks) Let A and B be $m \times n$ matrices. Then the set

   $$U = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = B\boldsymbol{x}\}$$

   is a subspace of $\mathbb{R}^n$.

   $\boxed{\text{True}}$ $U = N(A - B)$

   (c) (2 marks) If $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is a basis of a subspace $U \subseteq \mathbb{R}^n$ then

   $$\{\ \boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3\ ,\ \boldsymbol{u}_1 + \boldsymbol{u}_3\ ,\ \boldsymbol{u}_1 - \boldsymbol{u}_2 + \boldsymbol{u}_3\ \}$$

   is also a basis of U.

   $\boxed{\text{False}}$ $(\underline{u}_1 + \underline{u}_2 + \underline{u}_3) + (\underline{u}_1 - \underline{u}_2 + \underline{u}_3) = 2(\underline{u}_1 + \underline{u}_3)$

   (d) (2 marks) Suppose A_1 and A_2 are $m \times n$ matrices such that $A_1 = LU_1$ and $A_2 = LU_2$. In other words, the unit lower triangular matrix L is the same in both LU decompositions (however the upper triangular matrices U_1 and U_2 are different). If $\text{rank}(A_1) \leq \text{rank}(A_2)$ then $R(A_1) \subseteq R(A_2)$.

   $\boxed{\text{True}}$

   $R(A_1) = \text{span}\{\underline{l}_1, \ldots, \underline{l}_{r_1}\}$ $r_1 = \text{rank}(A_1)$

   $R(A_2) = \text{span}\{\underline{l}_1, \ldots, \underline{l}_{r_2}\}$ $r_2 = \text{rank}(A_2)$

   $L = \begin{bmatrix} \underline{l}_1 & \cdots & \underline{l}_m \end{bmatrix}$ $r_1 \leq r_2$

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2. Short answer questions. Give a brief justification. Parts (a) and (b) are independent.

(a) (3 marks) Let A be a $n \times n$ matrix, and let $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$ such that $\|A\boldsymbol{x}_1\| = 5$, $\|A\boldsymbol{x}_2\| = 15$, $\|\boldsymbol{x}_1\| = 2$ and $\|\boldsymbol{x}_2\| = 1$. Find a value $C > 1$ such that $C \leq \mathrm{cond}(A)$.

$$\frac{\|A\underline{x}_1\|}{\|\underline{x}_1\|} = \frac{5}{2} \Rightarrow \|A\| \geq 5/2 \text{ and } \|A^{-1}\| \geq \frac{1}{5/2}$$

$$\frac{\|A\underline{x}_2\|}{\|\underline{x}_2\|} = \frac{15}{1} \Rightarrow \|A\| \geq 15 \text{ and } \|A^{-1}\| \geq \frac{1}{15}.$$

$$\Rightarrow \mathrm{cond}(A) = \|A\|\|A^{-1}\| \geq 15 \cdot \frac{1}{5/2} = \boxed{6}$$

(b) (3 marks) Consider $N+1$ points $(t_0, y_0), \ldots, (t_N, y_N)$. Suppose we want to construct an interpolating function $p(t)$ defined piecewise by N functions $p_1(t), \ldots, p_N(t)$ such that each $p_k(t)$ is defined on the interval $[t_{k-1}, t_k]$. Determine the number of equations the functions $p_1(t), \ldots, p_N(t)$ must satisfy to guarantee that $p(t)$ interpolates the data and $p'(t)$, $p''(t)$ and $p'''(t)$ are continuous.

Interpolation: $p_k(t_{k-1}) = y_{k-1}$ and $p_k(t_k) = y_k$ $k = 1, \ldots, N$

$\Rightarrow 2N$ equations

Continuity $p'(t)$: $p_k'(t_k) = p_{k+1}'(t_k)$ $k = 1, \ldots, N-1$

$\Rightarrow N-1$ equations

Continuity $p''(t)$: $N-1$ equations

Continuity $p'''(t)$: $N-1$ equations

$\Rightarrow \boxed{5N-3 \text{ equations}}$

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3. (5 marks) Find the unique function of the form

$$p(t) = a + bt^2 + ct^3$$

such that $p(-1) = 1$, $p'(1) = 0$ and $p''(2) = -1$.

$p(-1) = 1 \Rightarrow a + b - c = 1$

$p'(1) = 0 \Rightarrow 2b + 3c = 0$

$p''(2) = -1 \Rightarrow 2b + 6c(2) = -1$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 2 & 3 & 0 \\ 0 & 2 & 12 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 9 & -1 \end{array}\right]$$

$\Rightarrow c = -1/9 \quad b = -3\left(\frac{-1}{9}\right)\frac{1}{2} = \frac{1}{6}$

$a = 1 + \frac{-1}{9} - \frac{1}{6} = \frac{13}{18}$

$$\Rightarrow \boxed{p(t) = \frac{13}{18} + \frac{1}{6}t^2 - \frac{1}{9}t^3}$$

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4. Suppose $A = LU$ where L is a unit lower triangular matrix and

$$U = \begin{bmatrix} 2 & 1 & 0 & -3 & -2 & 3 \\ 0 & 0 & 1 & -1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Let $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5, \boldsymbol{a}_6 \in \mathbb{R}^4$ be the columns of A

$$A = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 & \boldsymbol{a}_6 \end{bmatrix}$$

Determine if the statement is **True** or **False**. No justification required.

(a) (2 marks) The dimension of span$\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\}$ is 2.

$\boxed{\text{True}}$

(b) (2 marks) The vectors $\boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5$ are linearly independent.

$\boxed{\text{True}}$

Columns 3,4,5 $\Rightarrow \begin{bmatrix} 0 & -3 & -2 \\ 1 & -1 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 4 \\ 0 & -3 & -2 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$

(c) (2 marks) $\dim(N(A)) = 3$.

$\boxed{\text{False}}$ $\dim(N(A)) = 2$

(d) (2 marks) The set $\{\boldsymbol{a}_2, \boldsymbol{a}_4, \boldsymbol{a}_5, \boldsymbol{a}_6\}$ is a basis of $R(A)$.

rank$(A) = 4$ Columns 2,4,5,6 $\begin{bmatrix} 1 & -3 & -2 & 3 \\ 0 & -1 & 4 & 1 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

$\boxed{\text{True}}$

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5. (5 marks) Suppose $p(t)$ is a natural cubic spline with coefficient matrix

$$C = \begin{bmatrix} 2 & -3 & -2 & 2 & 0 & 1 \\ 0 & 6 & -3 & -9 & -3 & -3 \\ -1 & 5 & c_3 & c_4 & -16 & -22 \\ 1 & 2 & d_3 & d_4 & 2 & -17 \end{bmatrix}$$

such that $p(t)$ interpolates $(t_0, y_0), \dots, (t_6, y_6)$ where

$$t_0 = 0\ ,\ t_1 = 1\ ,\ t_2 = 2\ ,\ t_3 = 3\ ,\ t_4 = 4\ ,\ t_5 = 5\ ,\ t_6 = 6$$

and

$$p_k(t) = a_k(t - t_{k-1})^3 + b_k(t - t_{k-1})^2 + c_k(t - t_{k-1}) + d_k$$

Determine the coefficients c_3, d_3, c_4, d_4.

$p_k(t_k) = y_k \Rightarrow a_k + b_k + c_k + d_k = d_{k+1}$

$p_k'(t_k) = p_{k+1}'(t_k) \Rightarrow 3a_k + 2b_k + c_k = c_{k+1}$

$p_k''(t_k) = p_{k+1}''(t_k) \Rightarrow 6a_k + 2b_k = 2b_{k+1}$

① $a_2 + b_2 + c_2 + d_2 = d_3 \Rightarrow -3 + 6 + 5 + 2 = d_3 \Rightarrow \boxed{d_3 = 10}$

② $3a_2 + 2b_2 + c_2 = c_3 \Rightarrow 3(-3) + 2(6) + 5 = c_3 \Rightarrow \boxed{c_3 = 8}$

③ $3a_3 + 2b_3 + c_3 = c_4 \Rightarrow 3(-2) + 2(-3) + 8 = c_4 \Rightarrow \boxed{c_4 = -4}$

④ $a_4 + b_4 + c_4 + d_4 = d_5 \Rightarrow 2 - 9 + (-4) + d_4 = 2 \Rightarrow \boxed{d_4 = 13}$

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6. Consider the matrix

$$A = \begin{bmatrix} -1 & 1 & 0 & -6 & 2 \\ -1 & -1 & 3 & -8 & 3 \\ 3 & 3 & -9 & 29 & -6 \\ 0 & 12 & -18 & 7 & -9 \end{bmatrix}$$

(a) (5 marks) Find the LU decomposition of A.

(b) (3 marks) Find a basis for $R(A)$.

(a)

$$\begin{bmatrix} -1 & 1 & 0 & -6 & 2 \\ -1 & -1 & 3 & -8 & 3 \\ 3 & 3 & -9 & 29 & -6 \\ 0 & 12 & -18 & 7 & -9 \end{bmatrix} \xrightarrow[3R_1+R_3]{-R_1+R_2} \begin{bmatrix} -1 & 1 & 0 & -6 & 2 \\ 0 & -2 & 3 & -2 & 1 \\ 0 & 6 & -9 & 11 & 0 \\ 0 & 12 & -18 & 7 & -9 \end{bmatrix}$$

$$\xrightarrow[6R_2+R_4]{3R_2+R_3} \begin{bmatrix} -1 & 1 & 0 & -6 & 2 \\ 0 & -2 & 3 & -2 & 1 \\ 0 & 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & -5 & -3 \end{bmatrix} \xrightarrow{R_3+R_4} \begin{bmatrix} -1 & 1 & 0 & -6 & 2 \\ 0 & -2 & 3 & -2 & 1 \\ 0 & 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -3 & -3 & 1 & 0 \\ 0 & -6 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & -6 & 2 \\ 0 & -2 & 3 & -2 & 1 \\ 0 & 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Columns 1,2,4 of A :

$$\left\{ \begin{bmatrix} -1 \\ -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \\ 12 \end{bmatrix}, \begin{bmatrix} -6 \\ -8 \\ 29 \\ 7 \end{bmatrix} \right\}.$$

or columns 1,3,4
1,3,5
1,2,5

Columns 1,2,3 of L :

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

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7. (5 marks) Let $a, b \in \mathbb{R}$ such that $0 < a < b$. Find $\mathbf{c}$ such that $A\mathbf{c} = \boldsymbol{y}$ where

$$A = \begin{bmatrix} 1 & -b & b^2 & -b^3 & b^4 \\ 1 & -a & a^2 & -a^3 & a^4 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & a & a^2 & a^3 & a^4 \\ 1 & b & b^2 & b^3 & b^4 \end{bmatrix} \qquad \boldsymbol{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

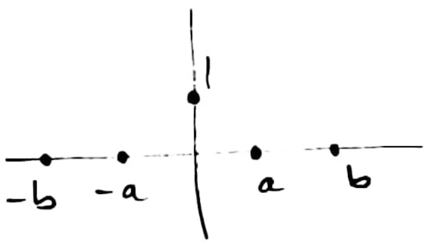
Hints: (1) A is a Vandermonde matrix; (2) it is possible to determine $\mathbf{c}$ without using Gaussian elimination to solve the system.

Polynomial interpolation for $(-b,0), (-a,0), (0,1), (a,0), (b,0)$.

$\Rightarrow$ $p(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^3$ where

$p(-b) = p(-a) = p(a) = p(b) = 0$ and $p(0) = 1$

$\Rightarrow$



$\Rightarrow$ $p(t) = k(t+b)(t+a)(t-a)(t-b)$ for some k.

Plug in $t=0$ to find $1 = ka^2b^2$

$\Rightarrow k = \frac{1}{a^2b^2}$

$$\Rightarrow p(t) = \frac{1}{a^2b^2}(t^2-b^2)(t^2-a^2)$$

$$= \frac{1}{a^2b^2}\left(t^4 - (a^2+b^2)t^2 + a^2b^2\right)$$

$$= 1 - \frac{(a^2+b^2)}{a^2b^2}t^2 + \frac{1}{a^2b^2}t^4$$

$$\Rightarrow \boxed{\underline{c} = \begin{bmatrix} 1 \\ 0 \\ \frac{-(a^2+b^2)}{a^2b^2} \\ 0 \\ \frac{1}{a^2b^2} \end{bmatrix}}$$

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