MATH 307 Practice Midterm Exam 1

October 2021

- No calculators, cellphones, laptops or notes
- Time allowed: 45 minutes
- 45 total marks
- Write your name and student number in space below

Name:

Student Number:

1. Short answer questions. Each part is independent of the others.

   (a) (3 marks) **True** or **False**: If A is an invertible matrix, then $\|A\| \geq \|A^{-1}\|$. Justify your answer.

   (b) (3 marks) Suppose the finite difference method yields a solution

$$\boldsymbol{y} = \begin{bmatrix} 1.0 \\ 1.5 \\ 1.8 \\ 1.1 \end{bmatrix}$$

   for 6 equally spaced points from $t_0 = 0$ to $t_5 = 1$. Find an approximation of $y''(0.6)$.

(c) (3 marks) Find the LU decomposition of

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

(d) (3 marks) Let $\boldsymbol{b} \in \mathbb{R}^n$ such that $\boldsymbol{b} \neq \mathbf{0}$. Determine whether the set

$$U = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \boldsymbol{b}\}$$

is a subspace of $\mathbb{R}^n$. Justify your answer.

(e) (3 marks) Find α and β such that the linear system

$$\begin{array}{rcrcrcl} 2x & + & \alpha y & - & z & = & \beta \\ x & - & y & & & = & 2 \\ 2x & - & 3y & + & z & = & 4 \end{array}$$

has infinitely many solutions.

(f) (3 marks) Sketch the natural cubic spline represented by the coefficient matrix

$$C = \begin{bmatrix} -1 & 2 & 1 & -3 & 1 \\ 0 & -3 & 3 & 6 & -3 \\ -2 & -5 & -5 & 4 & 7 \\ 1 & -2 & -8 & -9 & -2 \end{bmatrix}$$

where $t_0 = 0, t_1 = 1, t_2 = 2, t_3 = 3, t_4 = 4, t_5 = 5$.

2. Let $N \geq 4$ and consider the $N \times N$ matrix

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$$

Let $A = LU$ be the LU decomposition of A.

(a) (3 marks) Find the first 3 rows of L and the first 3 rows of U.

(b) (3 marks) Find a recursive formula for the diagonal entries $u_{n,n}$ of U.

(c) (3 marks) Find $\det(A)$ for $N = 5$.

3. (6 marks) Suppose we have 4 points $(0, y_0), (1, y_1), (2, y_2), (3, y_3)$ and we want to interpolate the data using a spline $p(t)$ constructed from polynomials p_1, p_2, p_3 where

$$p_k(t) = a_k(t - t_{k-1})^3 + b_k(t - t_{k-1}) + c_k \quad , \quad t \in [t_{k-1}, t_k]$$

We require that $p(t)$ and $p'(t)$ are continuous and $p''(t_3) = 0$. Setup (but do **not** solve) a linear system $A\boldsymbol{x} = \boldsymbol{b}$ where the solution is

$$\boldsymbol{x} = \begin{bmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \\ a_3 \\ b_3 \end{bmatrix}$$

The system depends on y_0, y_1, y_2, y_3.

4. Let $U = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\} \subseteq \mathbb{R}^4$ where

$$\boldsymbol{u}_1 = \begin{bmatrix} 2 \\ 4 \\ 4 \\ 2 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} 3 \\ 5 \\ 3 \\ 1 \end{bmatrix} \qquad \boldsymbol{u}_3 = \begin{bmatrix} 3 \\ 3 \\ -1 \\ -11 \end{bmatrix} \qquad \boldsymbol{u}_4 = \begin{bmatrix} 0 \\ 3 \\ 11 \\ -2 \end{bmatrix}$$

(a) (4 marks) Find a basis and the dimension of U.

(b) (2 marks) Is $\{\boldsymbol{u}_1, \boldsymbol{u}_3, \boldsymbol{u}_4\}$ a basis of U? Explain.

5. Consider the natural cubic spline $p(t)$ represented by the coefficient matrix

$$C = \begin{bmatrix} 3 & a_2 & 2 & -4 & -2 & -2 & 8 \\ 0 & 9 & b_3 & b_4 & b_5 & -18 & -24 \\ 0 & 9 & 12 & 6 & c_5 & -36 & -78 \\ 0 & 3 & 16 & 24 & d_5 & 6 & -50 \end{bmatrix}$$

where $t_0 = 0, t_1 = 1, t_2 = 2, t_3 = 3, t_4 = 4, t_5 = 5, t_6 = 6, t_7 = 7$.

(a) (4 marks) Find the value b_4.

(b) (2 marks) Find $p''(5.5)$

Extra workspace. Do not write in the table below.

Q1	/18
Q2	/9
Q3	/6
Q4	/6
Q5	/6
Total	/45