MATH 307 Final Exam

December 14, 2021

- No calculators, cellphones, laptops or notes
- Time allowed: 2 hours 30 minutes
- 75 total marks
- Write your name and student number in the space below
- Notation:
  - A^T is the *transpose* of the matrix A
  - $R(A)$ is the *range* of the matrix A (also called the *column space*)
  - $N(A)$ is the *nullspace* of the matrix A
  - $U^\perp$ is the *orthogonal complement* of the subspace $U \subseteq \mathbb{R}^n$
  - $\omega_N = e^{2\pi i/N}$

Name: Solutions

Student Number:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Total
10	12	10	7	8	10	5	6	7	75

1. Determine whether the statement is **True** or **False**. No justification required.

(a) (2 marks) The matrix

$$\begin{bmatrix} 5 & -1 & -1 & 2 & 1 \\ 1 & 3 & 3 & -1 & 2 \\ -4 & 5 & 4 & 0 & 0 \\ 3 & 2 & 6 & 8 & -7 \end{bmatrix}$$

is the coefficient matrix of a natural cubic spline for 6 data points $(t_0, y_0), \dots, (t_5, y_5)$ such that $t_k - t_{k-1} = 1$ for each $k = 1, \dots, 5$.

$\boxed{\text{False}}$ Coefficients do not satisfy spline equations such as $a_1 + b_1 + c_1 + d_1 = d_2$

$\Rightarrow 5 + 1 - 4 + 3 \neq 2$

(b) (2 marks) If A is a complex hermitian matrix then the diagonal entries of A are real numbers.

$\boxed{\text{True}}$ $A = \bar{A}^T \Rightarrow a_{i,i} = \overline{a_{i,i}}$ for each i

(c) (2 marks) Let X be a (normalized) data matrix, let $\boldsymbol{x}$ be a row of X, let $\boldsymbol{w}_1$ be the first weight vector of X and let $\boldsymbol{w}_2$ be the second weight vector of X. If $\langle \boldsymbol{x}, \boldsymbol{w}_1 \rangle \neq 0$ then $|\langle \boldsymbol{x}, \boldsymbol{w}_2 \rangle| < \|\boldsymbol{x}\|$.

X is $n \times p$

$\boxed{\text{True}}$ $\underline{x} = \langle \underline{x}, \underline{w}_1 \rangle \underline{w}_1 + \dots + \langle \underline{x}, \underline{w}_p \rangle \underline{w}_p$

$\Rightarrow |\langle \underline{x}, \underline{w}_2 \rangle|^2 \leq \|\underline{x}\|^2 - |\langle \underline{x}, \underline{w}_1 \rangle|^2 < \|\underline{x}\|^2$

(d) (2 marks) If $\mathrm{DFT}(\boldsymbol{x}) = \mathrm{DFT}(\boldsymbol{y})$ then $\boldsymbol{x} = \boldsymbol{y}$.

$\boxed{\text{True}}$ DFT is <u>invertible</u>.

(e) (2 marks) Let U_1 and U_2 be subspaces of $\mathbb{R}^n$. Let P_1 be the projection onto U_1 and let P_2 be the projection onto U_2. If $P_1 P_2 = 0$ then U_1 and U_2 are orthogonal subspaces.

$\boxed{\text{True}}$ $P_1 P_2 \Rightarrow R(P_2) \subset N(P_1)$

$\Rightarrow U_2 \subset U_1^\perp$

$\Rightarrow U_1 \perp U_2$.

2. Short answer questions. Each part is independent of the others. **Justify your answers.**

(a) (3 marks) Determine all values c such that the matrix

$$A = \begin{bmatrix} 3 & c \\ -1 & 5 \end{bmatrix}$$

is orthogonally diagonalizable. In other words, find all possible values c such that there exists a diagonal matrix D and orthogonal matrix P such that $A = PDP^T$.

A is orthogonally diagonalizable if and only if A is symmetric therefore $\boxed{c = -1}$

(b) (3 marks) Let A be a 3×3 matrix (*not* a diagonal matrix) such that the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 1/2$ and $\lambda_3 = -6$. Let $\boldsymbol{x}_0 \in \mathbb{R}^3$ be a random nonzero vector and let $\boldsymbol{x}_k = A^{-k}\boldsymbol{x}_0$. Determine the (most likely) value c such that

$$\frac{\langle \boldsymbol{x}_k, \boldsymbol{x}_{k+1} \rangle}{\langle \boldsymbol{x}_k, \boldsymbol{x}_k \rangle} \to c \text{ as } k \to \infty$$

This is the power method applied to A^{-1}.

The dominant eigenvalue of A^{-1} is 2 since eigenvalues of A^{-1} are $1, 2, -1/6$.

$\Rightarrow \boxed{c = 2}$

(c) (3 marks) Determine all values k such that $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & k \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & k & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & k-1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & k-7 & 0 \end{array}\right]$$

$\Rightarrow$ Unique solution for $\boxed{k \neq 7}$

(d) Compute DFT($\boldsymbol{x}$) where $\boldsymbol{x} = 5\cos(4\pi \boldsymbol{t} + \pi/2) \in \mathbb{C}^8$. Recall, the vector $\boldsymbol{t}$ is

$$\boldsymbol{t} = \begin{bmatrix} 0 & 1/8 & 1/4 & 3/8 & 1/2 & 5/8 & 3/4 & 7/8 \end{bmatrix}^T$$

$\underline{x} = 5\cos(4\pi \underline{t} + \pi/2) \Rightarrow A = 5 \quad k = 2 \quad \phi = \pi/2$

$$\mathrm{DFT}(\underline{x}) = \frac{AN}{2} e^{i\phi} \underline{e}_k + \frac{AN}{2} e^{-i\phi} \underline{e}_{N-k}$$

$$= 20 e^{i\pi/2} \underline{e}_2 + 20 e^{-i\pi/2} \underline{e}_6$$

$$= 20 i \, \underline{e}_2 - 20 i \, \underline{e}_6$$

$$= \begin{bmatrix} 0 \\ 0 \\ 20i \\ 0 \\ 0 \\ 0 \\ -20i \\ 0 \end{bmatrix}$$

3. Consider the matrix

$$A = \begin{bmatrix} -1 & -2 & -2 & -1 \\ 4 & 10 & 11 & 2 \\ -3 & -2 & 0 & -7 \end{bmatrix}$$

(a) (5 marks) Compute the LU decomposition of A.

(b) (2 marks) Determine the dimension of $N(A^T)$.

(c) (3 marks) Find a basis of $N(A)$.

(a)
$$\begin{bmatrix} -1 & -2 & -2 & -1 \\ 4 & 10 & 11 & 2 \\ -3 & -2 & 0 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -2 & -2 & -1 \\ 0 & 2 & 3 & -2 \\ 0 & 4 & 6 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -2 & -2 & -1 \\ 0 & 2 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} -1 & -2 & -2 & -1 \\ 0 & 2 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{U}$$

(b) $N(A^T) = R(A)^\perp$ Since $\operatorname{rank}(A) = 2$ we have

$$\dim(R(A)^\perp) = 3 - 2 = 1$$

$$\Rightarrow \boxed{\dim(N(A^T)) = 1}$$

(c) $x_4 = t$

$x_3 = s$

$x_2 = (2t - 3s)/2 = t - \frac{3}{2}s$

$x_1 = -(t + 2s + 2(t - \frac{3}{2}s))$

$= -t - 2s - 2t + 3s$

$= -3t + s$

$$\Rightarrow \underline{x} = \begin{bmatrix} -3t + s \\ t - \frac{3}{2}s \\ s \\ t \end{bmatrix}$$

$$\Rightarrow \boxed{N(A) = \operatorname{Span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3/2 \\ 1 \\ 0 \end{bmatrix} \right\}}$$

4. (7 marks) Find the shortest distance from $\boldsymbol{x}$ to $U = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\} \subseteq \mathbb{R}^4$ where

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

The shortest distance is $\| \underline{x} - \text{proj}_U(\underline{x}) \|$.

Note $\dim(U) = \dim(U^\perp) = 2$. Find an orthonormal basis of U by Gram-Schmidt:

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \underline{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} - \frac{\left\langle \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} - \frac{4}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

Compute the projection:

$$\text{proj}_U(\underline{x}) = \frac{\left\langle \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{\left\langle \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\rangle} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

$$= \frac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{-1}{2} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \| \underline{x} - \text{proj}_U(\underline{x}) \| = \left\| \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \\ 0 \end{bmatrix} \right\|$$

$$= \frac{1}{2}\sqrt{2} = \boxed{\frac{1}{\sqrt{2}}}$$

5. Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

(a) (5 marks) Compute the *thin* QR decomposition $A = Q_1 R_1$.

(b) (3 marks) Use the thin QR decomposition to find the least squares approximation $A\boldsymbol{x} \cong \boldsymbol{b}$ for the vector

$$\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

(a) Find an orthogonal basis of $R(A)$. Note that columns 1 and 2 are orthogonal, and columns 2 and 3 also.

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad \underline{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \quad \underline{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow Q_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \qquad R_1 = \begin{bmatrix} \sqrt{3} & 0 & \sqrt{3} \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$$

(b) $A\underline{x} \approx \underline{b} \Rightarrow R_1 \underline{x} = Q_1^T \underline{b} \Rightarrow Q_1^T \underline{b} = \frac{1}{\sqrt{3}}\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

$$\Rightarrow \left[\begin{array}{ccc|c} \sqrt{3} & 0 & \sqrt{3} & 2/\sqrt{3} \\ 0 & \sqrt{3} & 0 & 1/\sqrt{3} \\ 0 & 0 & \sqrt{3} & 3/\sqrt{3} \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 3 & 0 & 3 & 2 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

$$\Rightarrow \boxed{\underline{x} = \begin{bmatrix} -1/3 \\ 1/3 \\ 1 \end{bmatrix}}$$

6. Consider the matrix

$$A = \frac{1}{2}\begin{bmatrix} 1 & 2 & 3 & 1 \\ -1 & 2 & -1 & -3 \\ -1 & 2 & 1 & 3 \\ 1 & 2 & -3 & -1 \end{bmatrix}$$

(a) (5 marks) Compute the condition number of A. (Hint: consider A^TA not AA^T.)

(b) (5 marks) Find a *unit* vector $\boldsymbol{x}$ such that $\|A\boldsymbol{x}\| = \|A\|$.

(a) $A^TA = \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 \\ 0 & 0 & 20 & 12 \\ 0 & 0 & 12 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 3 & 5 \end{bmatrix}$

$\Rightarrow$ Characteristic polynomial $c_A(x) = (x-1)(x-4)\underbrace{(x^2 - \cancel{10}x + 16)}_{(x-8)(x-2)}$

$\Rightarrow$ Eigenvalues of A^TA are $1, 4, 2, 8$

$\Rightarrow$ Singular values of A are $1, \sqrt{2}, 2, 2\sqrt{2}$

$\Rightarrow$ $\|A\| = 2\sqrt{2}$ and $\|A^{-1}\| = 1/1 = 1$.

$\Rightarrow$ $\boxed{\mathrm{cond}(A) = 2\sqrt{2}}$

(b) Find singular vector $\underline{q}_1$. Eigenvector of A^TA for $\lambda = 8$ is q_1 therefore solve $(A^TA - 8I)\underline{v} = \underline{0}$ $\left[\begin{array}{cccc|c} -7 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & -3 & 3 & 0 \\ 0 & 0 & 3 & -3 & 0 \end{array}\right]$

$\Rightarrow$ $\boxed{q_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}}$ $\cancel{\Rightarrow q_1 = \frac{1}{\sigma_1} A^T p_1}$

$\cancel{q_1 = \frac{1}{8}\cdot\frac{1}{4}\begin{bmatrix} 1 & -1 & -1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & -1 & 1 & -3 \\ 1 & -3 & 3 & -1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{32\sqrt{2}}\begin{bmatrix} 0 \\ 4 \\ -2 \\ 2 \end{bmatrix}}$ $\cancel{q_1 = \frac{1}{16\sqrt{2}}\begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}}$

7. (5 marks) Use at least 3 iterations of the power method to approximate the dominant eigenvalue and corresponding eigenvector of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Let $x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$x_1 = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$ $x_2 = \begin{bmatrix} 5 \\ 7 \\ 5 \end{bmatrix}$ $\underline{x}_3 = \begin{bmatrix} 12 \\ 17 \\ 12 \end{bmatrix}$

$$\Rightarrow \lambda \cong \frac{\left\langle \begin{bmatrix} 5 \\ 7 \\ 5 \end{bmatrix}, \begin{bmatrix} 12 \\ 17 \\ 12 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 5 \\ 7 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 5 \end{bmatrix} \right\rangle} = \frac{2(5)(12) + 7(17)}{2(5)^2 + 7^2}$$

$$= \frac{120 + 119}{99} = \frac{239}{99} \approx 2.4$$

$$\Rightarrow \lambda \approx 2.4 \quad \underline{v} = \begin{bmatrix} 12/17 \\ 1 \\ 12/17 \end{bmatrix}.$$

8. (6 marks) Let $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbb{R}^3$ such that $\|\boldsymbol{u}_1\| = a > 0$, $\|\boldsymbol{u}_2\| = b > 0$, and $\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle = 0$. Determine the eigenvalues and eigenvectors of the matrix

$$A = \boldsymbol{u}_1\boldsymbol{u}_1^T + \boldsymbol{u}_2\boldsymbol{u}_2^T$$

Let $\underline{u}_3$ be any vector that is orthogonal to $\underline{u}_1$ and $\underline{u}_2$.

Then the eigenvalues of A are $a^2, b^2, 0$ with corresponding eigenvectors $\underline{u}_1, \underline{u}_2, \underline{u}_3$. Since

$$A\underline{u}_1 = (\underline{u}_1\underline{u}_1^T + \underline{u}_2\underline{u}_2^T)\underline{u}_1 = \underline{u}_1\underline{u}_1^T\underline{u}_1 + \underline{u}_2\underline{u}_2^T\underline{u}_1$$

$$= \|\underline{u}_1\|^2\underline{u}_1 + \cancel{\langle \underline{u}_2, \underline{u}_1\rangle}^{0}\underline{u}_2 = a^2\underline{u}_1$$

$$A\underline{u}_2 = b^2\underline{u}_2$$

$$A\underline{u}_3 = \underline{u}_1\underbrace{\underline{u}_1^T\underline{u}_3}_{=0} + \underline{u}_2\underbrace{\underline{u}_2^T\underline{u}_3}_{=0} = 0.$$

9. (7 marks) Let $\boldsymbol{y} \in \mathbb{C}^8$ such that

$$\boldsymbol{y} = \begin{bmatrix} 1 \\ 0 \\ 2-2i \\ 1+i\sqrt{3} \\ 0 \\ 1-i\sqrt{3} \\ 2+2i \\ 0 \end{bmatrix}$$

Find values A_0, A_1, A_2, k_1, k_2, ϕ_1, ϕ_2 such that $\boldsymbol{y} = \mathrm{DFT}(\boldsymbol{x})$ where $\boldsymbol{x}$ is of the form

$$\boldsymbol{x} = A_0 + A_1 \cos(2\pi k_1 \boldsymbol{t} + \phi_1) + A_2 \cos(2\pi k_2 \boldsymbol{t} + \phi_2) \quad , \quad k_1 < k_2$$

Recall $\boldsymbol{t} \in \mathbb{C}^N$ is the vector

$$\boldsymbol{t} = \begin{bmatrix} 0 \\ 1/N \\ 2/N \\ \vdots \\ (N-1)/N \end{bmatrix}$$

If $\underline{x} = A\cos(2\pi k \underline{t} + \phi)$ then $\mathrm{DFT}(\underline{x}) = \frac{AN}{2} e^{i\phi} e_k + \frac{AN}{2} e^{-i\phi} \underline{e}_{N-k}$

$\Rightarrow k_1 = 2 \qquad \frac{A_1(8)}{2} e^{i\phi_1} = 2-2i = 2\sqrt{2}\, e^{-i\pi/4}$

$k_2 = 3 \qquad \frac{A_2(8)}{2} e^{i\phi_2} = 1+i\sqrt{3} = 2e^{i\pi/3}$

$\Rightarrow \boxed{\begin{matrix} k_1 = 2 & A_1 = \frac{1}{\sqrt{2}} & \phi_1 = \frac{-\pi}{4} \\ k_2 = 3 & A_2 = \frac{1}{2} & \phi_2 = \frac{\pi}{3} \end{matrix}}$

$A_0 N = 1$

$\boxed{A_0 = \frac{1}{8}}$