# MATH 307 Midterm Exam 2

*June 16, 2022*

- No calculators, cellphones, laptops or notes
- Time allowed: 50 minutes
- 45 total marks
- Write your name and student number in the space below
- Notation:
  - $N(A)$ is the nullspace of $A$ and $R(A)$ is the range of $A$
  - $U^{\perp}$ is the orthogonal complement of a subspace $U$
  - $I$ is the identity matrix

**Name:**

**Student Number:**

1. Determine if the statement is **True** or **False**. No justification required.

   (a) (2 marks) If $A$ is a symmetric matrix then $N(A)^{\perp} = R(A)$.

   (b) (2 marks) Let $U \subset \mathbb{R}^5$ be a subspace such that $\dim(U) = 2$. There exists a subspace $V \subset \mathbb{R}^5$ such that $\dim(V) = 4$ and $V$ is orthogonal to $U$.

   (c) (2 marks) Let $A$ be an $m \times n$ matrix with $\operatorname{rank}(A) = n$. Let $A = QR$ be the QR decomposition with

$$Q = [\ Q_1\ Q_2\ ] \qquad\qquad R = \begin{bmatrix} R_1 \\ \mathbf{0} \end{bmatrix}$$

   where $A = Q_1 R_1$ is the thin QR decomposition. The projection of $\boldsymbol{x} \in \mathbb{R}^m$ onto $R(A)^{\perp}$ is equal to $Q_2 Q_2^T \boldsymbol{x}$.

   (d) (2 marks) Let $P$ be the projection matrix onto a subspace $U \subset \mathbb{R}^6$ with $\dim(U) = 4$. Then the rank of the matrix $I - P$ is 4.

2. Short answer questions. Give a brief justification. Parts (a) and (b) are independent.

   (a) (3 marks) If $P$ is a $5 \times 5$ projection matrix such that $\text{rank}(P) = 2$ then determine the dimension of the nullspace $N(P)$.

   (b) (4 marks) Suppose $A$ is a $m \times n$ matrix with $m > n$ such that $\det(A^T A) \neq 0$. Determine the algebraic multiplicity of the eigenvalue $\lambda = 0$ for $AA^T$.

3. (5 marks) Find the projection matrix $P$ which projects onto $U = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ where

$$\boldsymbol{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} \qquad \boldsymbol{u}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

4. (5 marks) Find the thin QR decomposition of

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

5. (5 marks) Determine the dimension of $R(A)^{\perp}$ where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \end{bmatrix}$$

6. (5 marks) Find the singular values of the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \\ -2 & -1 & 0 \end{bmatrix}$$

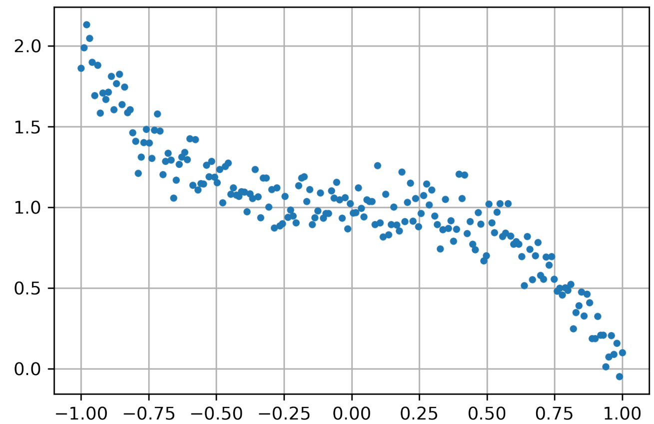
7. (5 marks) Suppose $A = QR$ is the QR decomposition of $A$ where

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 2\sqrt{2} & -\sqrt{2} \\ 0 & -2\sqrt{2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Determine the shortest distance from $\boldsymbol{x}$ to $R(A)$ where

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \end{bmatrix}$$

8. (5 marks) The figure below shows 200 data points $(t_1, y_1), \ldots, (t_{200}, y_{200})$




Determine (approximately) the least squares approximation $A\boldsymbol{c} \cong \boldsymbol{y}$ where

$$A = \begin{bmatrix} 1 & t_1 & t_1^2 & t_1^3 \\ 1 & t_2 & t_2^2 & t_2^3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & t_{199} & t_{199}^2 & t_{199}^3 \\ 1 & t_{200} & t_{200}^2 & t_{200}^3 \end{bmatrix} \qquad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{199} \\ y_{200} \end{bmatrix}$$

*Extra workspace.*

*Extra workspace. Do not write in the table below.*

| | |
|---|---|
| Q1 | /8 |
| Q2 | /7 |
| Q3 | /5 |
| Q4 | /5 |
| Q5 | /5 |
| Q6 | /5 |
| Q7 | /5 |
| Q8 | /5 |
| Total | /45 |