MATH 307 Midterm Exam 2

November 18, 2021

- No calculators, cellphones, laptops or notes
- Time allowed: 45 minutes
- 35 total marks
- Write your name and student number in the space below

Name:

Student Number:

1. Short answer questions. Each part is independent of the others.

   (a) (3 marks) Let A be a 3×3 matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and corresponding eigenvectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

   True or **False**: A is a symmetric matrix. Justify your answer.

   (b) (3 marks) Consider a matrix A with LU decomposition $A = LU$ where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   Determine the dimension of $N(A)^{\perp}$.

(c) (3 marks) Determine the values of a, b and c such that

$$Q = \begin{bmatrix} 1/\sqrt{18} & a & 2/3 \\ 1/\sqrt{18} & 1/\sqrt{2} & b \\ -4/\sqrt{18} & 0 & c \end{bmatrix}$$

is an orthogonal matrix.

(d) (3 marks) **True** or **False**: Suppose $A = P\Sigma Q^T$ is the singular value decomposition of A such that Q is a permutation matrix. Then the columns of A are orthogonal. (Recall that a permutation matrix is any matrix obtained from the identity matrix by permuting the rows.) Justify your answer.

2. (6 marks) Consider the matrix $A = Q_1 R_1$ where

$$Q_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \qquad R_1 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Compute the orthogonal projection of $\boldsymbol{v}$ onto $N(A^T)$ where

$$\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

3. (5 marks) Suppose $A = Q_1 R_1$ where

$$Q_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \qquad R_1 = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Find the least squares approximation $A\boldsymbol{x} \approx \boldsymbol{b}$ for

$$\boldsymbol{b} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

4. (6 marks) Find the shortest distance from $\boldsymbol{x} = \begin{bmatrix} 5 \\ 3 \\ 7 \end{bmatrix}$ to the plane in $\mathbb{R}^3$ given by

$$U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : 2x - y + 3z = 0 \right\}.$$

5. (6 marks) Consider the matrix

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}.$$

(a) (4 marks) Find matrices P and D such that $A = PDP^{-1}$.

(b) (2 marks) Compute the limit

$$\lim_{k \to \infty} \lambda_1^{-k} A^k$$

where λ_1 is the largest eigenvalue (in absolute value). In other words, if λ_1 and λ_2 are the eigenvalues of A then $|\lambda_1| > |\lambda_2|$.

Extra workspace

Extra workspace. Do not write in the table below.

Q1	/12
Q2	/6
Q3	/5
Q4	/6
Q5	/6
Total	/35