Instructions: No notes, books, or calculators are allowed. A list of MATLAB/Octave formula commands is provided.

1. Let $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \sqrt{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

   (a) (3 points) For what real values of a (if any) is $||A|| = 4$?

   Solution: The norm of the diagonal matrix A is the largest entry (in absolute value) of its diagonal elements. This means that $||A|| = 4$ if and only if $|a| \leq 4$.

   (b) (3 points) For what real values of a (if any) is $\text{cond}(A) = 4$?

   Solution: The condition number of a matrix A is defined by $\text{cond}(A) = ||A||||A^{-1}||$. In this case, $A^{-1} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/a & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$. Each of A, A^{-1} is diagonal so each of their norms is the largest diagonal entry in absolute value. We consider three possible cases for the value of a:

   - If $|a| > 4$, $\text{cond}(A) = \frac{|a|}{2}$, so $\text{cond}(A) = 4$ if and only if $|a| = 8$.
   - If $2 \leq |a| \leq 4$, then $\text{cond}(A) = \frac{4}{2} = 2$, so $\text{cond}(A) = 4$ is impossible in this case.
   - If $|a| < 2$, then $\text{cond}(A) = \frac{4}{|a|}$, so $\text{cond}(A) = 4$ if and only if $|a| = 1$.

   So $\text{cond}(A) = 4$ if and only if $|a| = 8$ or $|a| = 1$.

   (c) (3 points) Compute the stretching ratio $\dfrac{||B\mathbf{x}||}{||\mathbf{x}||}$ where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

   Solution: Since $B\mathbf{x} = \sqrt{2}\begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$, then

   $$||B\mathbf{x}|| = \sqrt{2}\sqrt{(x_1 - x_2)^2 + (x_1 + x_2)^2} = \sqrt{2}\sqrt{2x_1^2 + 2x_2^2} = 2\sqrt{x_1^2 + x_2^2} = 2||\mathbf{x}||.$$

   Therefore, the stretching ratio $\frac{||B\mathbf{x}||}{||\mathbf{x}||}$ is 2.

   (d) (3 points) Use the calculation in the previous part to determine $||B||$ and $\text{cond}(B)$.

   Solution: The matrix norm is defined by $||B|| = \max_{\mathbf{x}:||\mathbf{x}||\neq 0} \dfrac{||B\mathbf{x}||}{||\mathbf{x}||}$, or in other words, this is the maximum possible stretching ratio for the matrix B. We have computed in part (c) that the streching ratio for B is constant over all non-zero vectors, so $||B|| = 2$.
   To compute the condition number, we also need to compute $||B^{-1}||$. The matrix for B^{-1} is $B^{-1} = \dfrac{1}{2\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Hence, $B^{-1}\mathbf{x} = \dfrac{1}{2\sqrt{2}}\begin{bmatrix} x_1 + x_2 \\ -x_1 + x_2 \end{bmatrix}$. Therefore,

   $$||B^{-1}\mathbf{x}|| = \frac{1}{2\sqrt{2}}\sqrt{(x_1 + x_2)^2 + (-x_1 + x_2)^2} = \frac{1}{2\sqrt{2}}\sqrt{(x_1 + x_2)^2 + (x_1 - x_2)^2} = \frac{\sqrt{2}||\mathbf{x}||}{2\sqrt{2}} = \frac{||\mathbf{x}||}{2}.$$

   Therefore, $||B^{-1}|| = \frac{1}{2}$ and hence, $\text{cond}(B) = 2 * \frac{1}{2} = 1$.

(e) (3 points) Suppose C is a 3×3 matrix with $\text{cond}(C) = 10$. If $C \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $C \begin{bmatrix} 1 \\ 1+a \\ 1 \end{bmatrix} = \begin{bmatrix} 1.1 \\ 0 \\ 0 \end{bmatrix}$, what are the possible values of a?

Solution: If $C\mathbf{x} = \mathbf{b}$, then if $\mathbf{x} + \Delta\mathbf{x}$ satisfies $C(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{b} + \Delta\mathbf{b}$ then

$$\frac{||\Delta\mathbf{x}||}{||\mathbf{x}||} \leq \text{cond}(C)\frac{||\Delta\mathbf{b}||}{||\mathbf{b}||}$$

must be satisfied. In this case, $\mathbf{x} = \begin{bmatrix}1, 1, 1\end{bmatrix}^T, \mathbf{b} = \begin{bmatrix}1, 0, 0\end{bmatrix}^T, \Delta\mathbf{x} = \begin{bmatrix}0, a, 0\end{bmatrix}^T$, and $\Delta\mathbf{b} = \begin{bmatrix}0.1, 0, 0\end{bmatrix}^T$. Hence, $||\mathbf{x}|| = \sqrt{3}, ||\mathbf{b}|| = 1, ||\boldsymbol{\Delta}\mathbf{x}|| = |a|$, and $||\boldsymbol{\Delta}\mathbf{b}|| = 0.1$. Plugging these norms into the above inequality imply that $\frac{|a|}{\sqrt{3}} \leq 10\frac{0.1}{1}$ must hold. Hence, all a satisfying $|a| \leq \sqrt{3}$ are possible values of a.

2. Let (x_i, y_i) be four points in the plane with $x_1 < x_2 < x_3 < x_4$.

(a) (3 points) If the polynomial $p(x) = a_1x^3 + a_2x^2 + a_3x + a_4$ interpolates the four points, then the coefficient vector $\mathbf{a} = \begin{bmatrix}a_1, a_2, a_3, a_4\end{bmatrix}^T$ satisfies an equation of the form $A\mathbf{a} = \mathbf{d}$. Write down A and $\mathbf{d}$.

Solution: The polynomial interpolates the points so $a_1x_i^3 + a_2x_i^2 + a_3x_i + a_4 = y_i$ must be satisfied for $i = 1, 2, 3, 4$. Therefore, $\mathbf{a}$ satisfies

$$\begin{bmatrix} x_1^3 & x_1^2 & x_1 & 1 \\ x_2^3 & x_2^2 & x_2 & 2 \\ x_3^3 & x_3^2 & x_3 & 1 \\ x_4^3 & x_4^2 & x_4 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}.$$

(b) (3 points) If the polynomial $p(x) = b_1x^4 + b_2x^3 + b_3x^2 + b_4x + b_5$ interpolates the four points and also satisfies $p'(x_4) = 0$, then the coefficient vector $\mathbf{b} = \begin{bmatrix}b_1, b_2, b_3, b_4, b_5\end{bmatrix}^T$ satisfies an equation of the form $B\mathbf{b} = \mathbf{d}$. Write down B and $\mathbf{d}$.

Solution: The polynomial interpolates the points so $b_1x_i^4 + b_2x_i^3 + b_3x_i^2 + b_4x_i + b_5 = y_i$ must be satisfied for $i = 1, 2, 3, 4$. Furthermore, the condition $p'(x_4) = 0$ implies that $4b_1x_4^3 + 3b_2x_4^2 + 2b_3x_4 + b_4 = 0$ should be satisfied. Therefore, $\mathbf{b}$ satisfies

$$\begin{bmatrix} x_1^4 & x_1^3 & x_1^2 & x_1 & 1 \\ x_2^4 & x_2^3 & x_2^2 & x_2 & 1 \\ x_3^4 & x_3^3 & x_3^2 & x_3 & 1 \\ x_4^4 & x_4^3 & x_4^2 & x_4 & 1 \\ 4x_4^3 & 3x_4^2 & 2x_4 & 1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ 0 \end{bmatrix}.$$

(c) (4 points) If the polynomial $p(x) = c_1x^2 + c_2x + c_3$ interpolates the four points, then the coefficient vector $\mathbf{c} = \begin{bmatrix}c_1, c_2, c_3\end{bmatrix}^T$ satisfies an equation of the form $C\mathbf{c} = \mathbf{e}$. Write down C and $\mathbf{e}$.

Solution: The polynomial interpolates the points so $c_1x_i^2 + c_2x_i + c_3 = y_i$ must be satisfied for $i = 1, 2, 3, 4$. Therefore, $\mathbf{c}$ satisfies

$$\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \\ x_4^2 & x_4 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}.$$

(d) (4 points) For each of the equations in parts (a), (b), and (c), say whether you expect there to be a solution or not.

For the cases where you do not expect a solution, write down the least squares equation. Do these have a solution? Give a reason. What quantity is minimized when the least square equation is satisfied?

Solution: We expect the equations (a) and (b) to have a solution since each x_i is distinct, so the matrices A and B are expected to be non-singular.

We do not expect equation (c) to have a solution since this system of equations is overdetermined. The least squares equation for the coefficient vector $\mathbf{c}$ is $C^TC\mathbf{c} = C^T\mathbf{e}$, which has a solution since C^TC is expected to be invertible (again because each x_i is distinct). The vector $\mathbf{c}$ obtained through the solution of the least square equation minimizes the distance $||C\mathbf{c} - \mathbf{e}||^2$.

3. Consider the chemical system consisting of the species $H_2SO_4, HSO_4^-, SO_4^{--}$ and H^+. In addition to the species H, S and O, we also regard the charge as a species q. Thus the formula matrix of this system is

$$A = \begin{array}{c} \\ H \\ S \\ 0 \\ q \end{array} \begin{array}{c} \begin{array}{cccc} H_2SO_4 & HSO_4^- & SO_4^{--} & H^+ \end{array} \\ \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 4 & 4 & 4 & 0 \\ 0 & -1 & -2 & 1 \end{pmatrix} \end{array}$$

After defining A in MATLAB/Octave, the following output is obtained.

$$\texttt{rref(A)} = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \texttt{rref(A')} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) (4 points) Write down a basis for $N(A)$ and $N(A^T)$.

Solution: From examining the row-reduced form of A, variables x_3 and x_4 can be any real number, and for $A\mathbf{x} = 0$ to be satisfied, $x_1 = x_3 - x_4$ and $x_2 = -2x_3 + x_4$. Hence, solutions of $A\mathbf{x} = 0$ satisfy

$$\mathbf{x} = \begin{bmatrix} x_3 - x_4 \\ -2x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

where x_3, x_4 is any real number. Therefore, a basis for $N(A)$ is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Similarly, from examining the row-reduced form of A^T, variables x_3 and x_4 can be any real number, and for $A^T\mathbf{x} = 0$ to be satisfied, $x_1 = -x_4$ and $x_2 = -4x_3 + 2x_4$. Hence solutions of $A^T\mathbf{x} = 0$ satisfy

$$\mathbf{x} = \begin{bmatrix} -x_4 \\ -4x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

for any real x_3, x_4, meaning that a basis for $N(A^T)$ is $\left\{ \begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(b) (4 points) Write down the possible reactions for this system.

Solution: The possible reactions for this system are determined by basis vectors in $N(A)$. The basis vector $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ corresponds to the reaction $H_2SO_4 + SO_4^{--} = 2HSO_4^-$ and the basis vector $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ corresponds to the reaction $HSO_4^- + H^+ = H_2SO_4$.

(c) (4 points) If a sample contains 350 atoms of H, 200 atoms of S, and 800 atoms of O, what is the total charge q? (Hint: what subspace contains $\left[350, 200, 800, q\right]^T$?)

Solution: The vector $\mathbf{v} = \begin{bmatrix} 350 \\ 200 \\ 800 \\ q \end{bmatrix}$ belongs to the range space $R(A)$.

Method 1: The range space $R(A)$ is orthogonal to the $N(A^T)$. In particular, it must be orthogonal to each of the basis vectors of $N(A^T)$. The given vector is already orthogonal to $\begin{bmatrix} 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}$. If the vector is orthogonal to $\begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$, then $-350 + 400 + q = 0$. This means that the charge is $q = -50$.

Method 2: A basis for the range space is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \\ -1 \end{bmatrix} \right\}$ and this is obtained from the pivot columns of the row-reduced form of A. Hence, to express $\mathbf{v}$ in terms of these basis vectors, we must solve the system of equations $2a + b = 350$, $a + b = 200$, which yield $a = 150, b = 50$. This implies that the total charge is $q = 150(0) + 50(-1) = -50$.

4. (a) (4 points) Write down two conditions that must be satisfied in order for the functions $\phi_n(t)$ for $t \in [0, 1]$ and $n \in \mathbb{Z}$ to form an orthonormal set. Do the functions $e^{2\pi i n t}$ satisfy these conditions?

Solution: The functions $\phi_n(t)$ must satisfy $\int_0^1 |\phi_n(t)|^2\,dt = 1$ and $\int_0^1 \overline{\phi_m(t)}\phi_n(t)\,dt = 0$ for all $m \neq n$. The functions $e^{2\pi int}$ satisfy these conditions since

$$\int_0^1 |\phi_n(t)|^2\,dt = \int_0^1 1\,dt = 1$$

and if $m \neq n$

$$\int_0^1 e^{2\pi i-mt}e^{2\pi int}\,dt = \int_0^1 e^{2\pi i(n-m)t}\,dt = \frac{e^{2\pi i(n-m)t}}{n-m}\Big|_0^1 = 0$$

since $e^{2\pi ik} = 1$ for every integer k.

(b) (5 points) Do the functions $\phi_n(t) = \frac{1}{\sqrt{2}}e^{2\pi int}$ on the larger interval $t \in [0,2]$ form an orthonormal set? Can we expand any (sufficiently nice) function defined for $t \in [0,2]$ into a series $\sum_{n=-\infty}^{\infty} c_n\phi_n(t)$?

Solution: Yes, the functions are an orthonormal set since

$$\int_0^2 |\phi_n(t)|^2\,dt = \frac{1}{2}\int_0^2 1\,dt = 1$$

and if $n \neq m$,

$$\int_0^2 \frac{1}{2}e^{2\pi i(n-m)t} = \frac{1}{2}\frac{e^{2\pi i(n-m)t}}{n-m}\Big|_0^2 = 0.$$

If a function can be expanded into a series of this form, the Fourier coefficient c_n is given by $c_n = \int_0^2 \frac{1}{\sqrt{2}}f(x)e^{-2\pi int}$. It may **not** be possible to expand any function with $\int_0^2 |f(t)|^2\,dt < \infty$. into a series of this form. For instance, if $f(t) = e^{\pi it}$, then

$$c_n = \frac{1}{\sqrt{2}}\int_0^2 e^{\pi i(1-2n)t}\,dt = \frac{1}{\sqrt{2}}\frac{e^{\pi i(1-2n)t}}{1-2n}\Big|_0^2 = 0$$

Since $e^0 = 1$ and $e^{2i\pi(1-2n)} = 1$. So, $f(t) = e^{i\pi t}$ defined on the interval $[0,2]$ is not in the space spanned by these functions, although $\int_0^2 |e^{i\pi t}|^2\,dt = 2 < \infty$.

(c) (4 points) Find the coefficients c_n in the Fourier series $e^{i\pi t} = \sum_{n=-\infty}^{\infty} c_n e^{2\pi int}$ where $t \in [0,1]$.

Solution: The Fourier coefficients c_n are given by the integral $c_n = \int_0^1 e^{i\pi t}e^{-2\pi int}\,dt$. Therefore,

$$c_n = \int_0^1 e^{i\pi(1-2n)t}\,dt = \frac{e^{i\pi(1-2n)t}}{i\pi(1-2n)}\Big|_0^1 = \frac{e^{i\pi(1-2n)}-1}{i\pi(1-2n)} = \frac{-2}{i\pi(1-2n)} = \frac{2}{i\pi(2n-1)}.$$

(d) (4 points) What does Parseval's formula say for the series in part (c)?

Solution: The Parseval's formula states that if $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi int}$, then $\sum_{n=-\infty}^{\infty} |c_n|^2 = \int_0^1 |f(t)|^2\,dt$. Since $\int_0^1 |e^{i\pi t}|^2\,dt = 1$, then applying the Parseval's formula to the series

$$e^{i\pi t} = \sum_{n=-\infty}^{\infty} \frac{2}{i\pi(2n-1)}e^{2\pi int}$$

implies

$$\sum_{n=-\infty}^{\infty} \frac{4}{\pi^2(2n-1)^2} = 1$$

or equivalently

$$\sum_{n=-\infty}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{4}.$$

5. Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$.

(a) (4 points) Is A diagonalizable? Give a reason.

Solution: To determine if A is diagonalizable or not, we must compute its eigenvalues and the dimension of the corresponding eigenspaces. The characteristic equation of A is $(2-\lambda)(-\lambda)+1 = (\lambda-1)^2 = 0$, meaning that $\lambda = 1$ is its only eigenvalue with algebraic multiplicity 2. Next, since $A - I = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, then $\dim(N(A-I)) = 1$. This means that the dimension of the eigenspace corresponding to $\lambda = 1$ is 1, which is not its algebraic multiplicity 2. This means that A is **not** diagonalizable.

(b) (4 points) Schur's Lemma states that there is a unitary matrix U and an upper triangular matrix T such that $A = UTU^*$ Find U and T.

Solution: To construct U, we need to find an orthogonal basis of $\mathbb{R}^2$ starting from an eigenvector of A. We can see from the matrix $A - I$ in the previous part that $\mathbf{v_1} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ is a normalized eigenvector of A of eigenvalue 1. Next, $\mathbf{v_2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ is normalized, orthogonal to $\mathbf{v_1}$, and together $\{\mathbf{v_1}, \mathbf{v_2}\}$ span $\mathbb{R}^2$. Hence we let

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

and furthermore

$$T = U^*AU = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

is upper triangular.

(c) (4 points) What are T^2 and T^3? Guess a formula for T^n for any positive integer and use it to compute A^n.

Solution: The matrices for T^2 and T^3 are $T^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$ and $T^3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$. In general, $T^n = \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix}$.

$$A^n = UT^nU^* = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{2n+1}{\sqrt{2}} & \frac{1-2n}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$
$$= \begin{bmatrix} n+1 & -n \\ n & 1-n \end{bmatrix}$$

(d) (4 points) Write down a formula for the n^{th} term, x_n in the sequence defined by the recurrence relation $x_0 = a, x_1 = b$ and $x_{n+1} = 2x_n - x_{n-1}$ for $n \geq 1$.

Solution: The recurrence equation in matrix form is

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix},$$

and in general

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} b \\ a \end{bmatrix}.$$

Using the result from the previous part,

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = \begin{bmatrix} n+1 & -n \\ n & 1-n \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix}$$

so $x_n = nb + (1-n)a$.

6. Suppose A is an 5×5 real symmetric matrix defined in MATLAB/Octave. The commands

```
x = rand(5,1)
for k=1:50 y = (A - 3 * eye(5)) \ x; x = y / norm(y) end
```

yield output ending in

```
x = [0.12692; -0.32566; 0.80572; -0.11046; -0.46524]
x = [-0.12692; 0.32566; -0.80572; 0.11046; 0.46524]
x = [0.12692; -0.32566; 0.80572; -0.11046; -0.46524]
```

If the eigenvalues of A are 0, 0.5, 1.5, 2.5, and 4, what output would you get for (a) `dot(x,A*x)` (4 points), (b) `dot(x,x)` (3 points) and (c) `dot(y,x)` (3 points)?

Solution: The code performs inverse power iteration on A in order to find an eigenvector of A of eigenvalue closest to 3. Since the method has converged, $\mathbf{x}$ is an normalized eigenvector of A (up to numeric error) of eigenvalue 2.5. This immediately tells us that $\mathbf{x} \cdot A\mathbf{x} = 2.5\mathbf{x} \cdot \mathbf{x} = 2.5$, since $\mathbf{x} \cdot \mathbf{x} = 1$.

Next,

$$\mathbf{y} \cdot \mathbf{x} = \mathbf{y} \cdot \frac{\mathbf{y}}{||\mathbf{y}||} = ||\mathbf{y}||.$$

The vector $\mathbf{y}$ also satisfies $\mathbf{y} = (A - 3I)^{-1}\mathbf{x}_{prev}$, where $\mathbf{x}_{prev}$ is the vector $\mathbf{x}$ used in the second last iteration of the main loop. Since $\mathbf{x}_{prev}$ is also an eigenvector of A of eigenvalue 2.5 from the given output, then $\mathbf{y} = \dfrac{\mathbf{x}_{prev}}{2.5 - 3} = -2\mathbf{x}_{prev}$. Hence

$$\mathbf{y} \cdot \mathbf{x} = ||\mathbf{y}|| = \sqrt{4\mathbf{x}_{prev} \cdot \mathbf{x}_{prev}} = 2$$

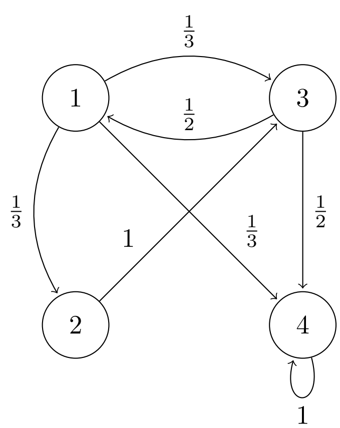
since $\mathbf{x}_{prev}$ is normalized.

7. (a) (4 points) Let $P = [p_{i,j}]$ be an $n \times n$ matrix. What does it mean to say that P is a stochastic matrix? If P is stochastic, what can you say about the eigenvalues and eigenvectors? What additional information do you have about the eigenvalues and eigenvectors if P^k has all positive entries for some k?

Solution: If a matrix P is stochastic, its columns sum to 1 and each entry is non-negative. A stochastic matrix P has an eigenvector of eigenvalues 1 with non-negative entries, and all other eigenvalues λ_k satisfy $|\lambda_k| < 1$. Furthermore, the entries of P^k are all positive for some k, then the eigenvector corresponding to the eigenvalue $\lambda = 1$ has all positive entries.

(b) (4 points) Draw the 4-site internet represented by the stochastic matrix $P = \begin{bmatrix} 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1 & 0 & 0 \\ 1/3 & 0 & 1/2 & 1 \end{bmatrix}$.

Solution:



(c) (3 points) By examining this internet or otherwise, find an eigenvector of P with eigenvalue 1.

Solution: Method 1: Recall that if $\mathbf{v}$ is an eigenvector of P of eigenvalue 1. We see an eigenvector whose entries sum to one, since in this case, each entry v_i corresponds to the long-term probability a random walk will be in state i. The state space can be divided into the sets $A = \{1, 2, 3\}$ and $B = \{4\}$ and the probability of going from a state in B to any state in A is 0. This implies that the eigenvector $\mathbf{v}$ will have 0 everywhere which corresponds to a state in site A. Therefore, $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ must be an eigenvector of P of eigenvalue 1.

Method 2: We can also directly solve the system of equations $P\mathbf{v} = \mathbf{v}$ subject to the constraint

$v_1 + v_2 + v_3 + v_4 = 1$, which are

$$\frac{1}{2}v_3 = v_1 \tag{1}$$
$$\frac{1}{3}v_1 = v_2 \tag{2}$$
$$\frac{1}{3}v_1 + v_2 = v_3 \tag{3}$$
$$\frac{1}{3}v_1 + \frac{1}{2}v_3 + v_4 = v_4 \tag{4}$$

From equation (4), $\frac{1}{3}v_1 + \frac{1}{2}v_3 = 0$, so $v_1 = -\frac{3}{2}v_3$. Hence plugging this into equation (1) yields $\frac{1}{2}v_3 = -\frac{3}{2}v_3$. This shows that $v_3 = 0$. Therefore using equation (1) and (2), $v_1 = 0$ and $v_2 = 0$. The constraint $v_1 + v_2 + v_3 + v_4 = 1$ now implies $v_4 = 1$ must hold. So again we have found that $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ must be an eigenvector of P of eigenvalue 1.

(d) (3 points) Given that the eigenvalues of P satisfy $\lambda_1 = 1, |\lambda_2| = 0.65034, |\lambda_3| = |\lambda_4| = 0.50624$, what is $\lim_{k\to\infty} P^k \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$?

Solution: The limit $\lim_{k\to\infty} P^k\mathbf{v}$ where $\mathbf{v}$ is any probability distribution over the possible states converges to $\mathbf{v_1}$, the eigenvector of P of eigenvalue $\lambda = 1$. This limit represents the long-run proportion of times a random walk on the Internet spends in each state, given that the initial state is drawn from the distribution $\mathbf{v}$. This implies

$$\lim_{k\to\infty} P^k\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

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(e) (4 points) If we add damping with a damping factor $\alpha = \frac{1}{2}$, what is the new stochastic matrix S? What can you say about $\lim_{k\to\infty} S^k \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$?

Solution: The new stochastic matrix S is $S = \frac{1}{2}J + \frac{1}{2}P$ where J is the 4×4 matrix where each entry is $\frac{1}{4}$. The entries of S are therefore

$$S = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & \frac{3}{8} & \frac{1}{8} \\ \frac{7}{24} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{7}{24} & \frac{5}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{7}{24} & \frac{1}{8} & \frac{3}{8} & \frac{5}{8} \end{bmatrix}.$$

Since each entry of S is positive, the limit $\lim_{k\to\infty} S^k \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ is an eigenvector of S of eigenvalue 1 (normalized to be a probability distribution) and has all positive entries since each entry of S is positive. This can be confirmed via a MATLAB computation, which shows that

$$\lim_{k\to\infty} S^k \begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0.1833\\0.1556\\0.2333\\0.4278\end{bmatrix}.$$