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Math 307: 202 — Midterm 2 — 50 minutes

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- The test consists of 10 pages and 4 questions worth a total of 50 marks.
- You are allowed 1 page of notes (single-sided, in your handwriting).
- Aside from that is a closed-book examination. **None of the following are allowed:** documents, or electronic devices of any kind (including calculators, cell phones, etc.)
- No work on this page will be marked.

Please do not write on this page — it will not be marked.

Additional instructions

- Please use the spaces indicated.
- **Unless it is specified not to do so, justify your answers.**
- If you require extra paper then put up your hand and ask your instructor.
  - You must put your name and student number on any extra pages.
  - You must indicate the test-number and question-number.
  - Please do this **on both sides** of any extra pages.
- Please do not dismember your test. You must submit all pages.

1	4	10
2	5+9	21
3	7	10
4	9	9
Total	34	50

1. $\boxed{\text{10 marks}}$ Let $A = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}$.

2

(a) Compute the Singular value decomposition of A. (Hint: *you are allowed* to compute the SVD in any way you want, but one of the two ways is a lot easier than the other).

$$\cancel{A^TA = \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -3 & 5 \end{bmatrix}}$$

$$\cancel{\det(A^TA - I\lambda) = (6-\lambda)(5-\lambda) - 6}$$
$$= \lambda^2 - 11\lambda + 24$$
$$= (\lambda - 3)(\lambda - 8) = 0$$
$$\lambda_1 = 8,\ \lambda_2 = 3$$

$$\cancel{\sigma_1 = \sqrt{8} \quad \sigma_2 = \sqrt{2}}$$

ignore

$$AA^T = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}$$

λ of AA^T is $8, 2 \Rightarrow \lambda_1 = 8,\ \lambda_2 = 2$

$\therefore\ \sigma_1 = \sqrt{8} \quad \sigma_2 = \sqrt{2}$

2

(b) Find $||A||$.

$$||A|| = \sigma_1 = \sqrt{8}$$

2. $\boxed{\text{21 marks}}$ Short answer questions, each question 3 marks. **For True or False questions**, if true, provide a short justification. If false, show a counter-example that contradicts the statement. For other questions, justify your answer by showing your work.

2 (a) Find an orthogonal basis for the subspace $U = \text{span}\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\0\\-1\end{bmatrix}\right\}$.

GS:

$u_1 = \begin{bmatrix}1\\2\\0\end{bmatrix}$

$u_2 = \begin{bmatrix}2\\0\\-1\end{bmatrix} - \frac{2}{5}\begin{bmatrix}1\\2\\0\end{bmatrix}$

$= \begin{bmatrix}5/5\\-4/5\\-7/5\end{bmatrix}$

x

u_1 and u_2 are orthogonal basis

3 (b) **True or False:** If A is a real matrix with orthogonal columns, then A^TA is a diagonal matrix.

$A = \begin{bmatrix} | & | & | \\ c_1 & c_2 & c_3 \\ | & | & | \end{bmatrix}$

This only works for 3x3!

$A^TA = \begin{bmatrix} -c_1^T- \\ -c_2^T- \\ -c_3^T- \end{bmatrix}\begin{bmatrix} | & | & | \\ c_1 & c_2 & c_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} c_1^Tc_1 & c_1^Tc_2 & \cdots \\ \vdots & \ddots & \\ & & \end{bmatrix}$

$= \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & 0 & c \end{bmatrix}$ where a b c d are non zero

entries of

→ True. A^TA is the dot product of each column and $\langle c_i, c_j \rangle = 0$ when $i \neq j$ ✓ ← This a great proof!

$S^{-1} = \begin{pmatrix} 1 & ? \\ 1 & 1 \end{pmatrix}$

3 (c) Let $A = SDS^{-1}$ where $S = \begin{pmatrix} 1 & 4 \\ 2 & 4 \end{pmatrix}$ and $D = \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix}$, find the eigenvalues of A^2 and a basis for $N(A^2 - 4I)$.

$A^2 = SD^2S^{-1}$ hence the eigen values of A^2 is the eigenvalue of A (diagonal of D) squared

$\rightarrow \lambda = 4, 16$ ✓ $\Rightarrow A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 4 \end{bmatrix}^{-1}$ ✓

$N(A^2 - 4I)$ = eigvec corresponding to $\lambda_1 = 4$, which is the first column is S

$\therefore N(A^2 - 4I) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ✓

0 (d) **True or False:** Let $A = \begin{pmatrix} 2 & 0 & 2 & 4 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$, then A is diagonalizable.

A is full rank, thus its eigenvectors span $\mathbb{R}^3$ and form eigen basis, so it is diagonalizable

True ✗

3 (e) **True or False:** Let A be an $n \times n$ matrix. If λ is an eigenvalue of A, then $\lambda + 1$ is an eigenvalue of $A + I$.

$Av = \lambda v$ for some $v \neq 0$

$$(A+I)v = Av + Iv$$
$$= Av + v$$
$$= \lambda v + v$$
$$= \underbrace{v}_{\text{eigvec}}\underbrace{(\lambda+1)}_{\text{eigval}} \Rightarrow \text{True}$$

✓

3 (f) **True or False:** Let A be a **non-zero** $n \times n$ **orthogonal projection matrix**. If y is a non-zero vector in $R(A)$, then y is **not** in $N(A)$.

$A^2 = A$ and $A^T = A$

$$N(A) = R(A^T)^\perp$$
$$= R(A)^\perp \quad \text{since } A \text{ is ortho-projector}$$

which by definition a non-zero vector in $R(A)$ will not be in $R(A)^\perp$ $\Rightarrow$ True

✓

0

(g) **True or False:** Let A be an $n \times n$ matrix. If all eigenvalues of A are positive real numbers, then $||A||$ equals its largest eigenvalue.

Given A as above, this means that A is real symmetric and thus orthogonally diagonalizable so

$$A = P\bar{D}P^{-1} = PDP^{T}$$

From class, we know that P is orthog matrix and thus norm-preserving, so is P^T, thus

$||A|| = ||D||$ which will be the largest value on the diag of D which is the largest eigenvalue

$\Rightarrow$ True

x

3. $\boxed{10 \text{ marks}}$ Let $U = \text{span}\{\mathbf{x}\} \subseteq \mathbb{R}^3$ where $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

2 (a) Construct an orthogonal projection matrix which projects onto U.

$$\text{ONB}(U) = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \vec{u}_1$$

$$P_u = \vec{u}\,\vec{u}^T = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \checkmark$$

1 (b) Find an orthogonal projection matrix P such that $N(P) = U$.
[**Hint:** Such a matrix P must satisfy $R(P^T) = N(P)^\perp = U^\perp$.]

Let $A = P^T$, $\quad R(A) = U^\perp = N(\vec{u}_1^T)$

$$= \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \end{array}\right] \rightarrow \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\} \leftarrow \text{basis of } U^\perp$$

$$\therefore A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = P^T$$

$$N(P): = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \checkmark$$

$$P = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

↑

P is not a projection!

4 (c) Find a basis for $U^\perp$.

as above, basis of $U^\perp$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ ✓

9 4. $\boxed{\text{9 marks}}$ Let $A = Q_1 R_1$ where

$$Q_1 = \begin{bmatrix} 1/3 & -2/3 \\ 2/3 & -1/3 \\ 2/3 & 2/3 \end{bmatrix}, \text{ and } R_1 = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}.$$

Find the least squares approximation $A\mathbf{x} \approx \mathbf{b}$ where

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}.$$

$R_1 c^* = Q_1^T b$

$1+2+6$
$-2-1+6$

$Q_1 = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}$

$Q_1^T b = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$R_1 c^* = Q_1^T b$

$\begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 3 & 4 & 3 \\ 0 & 2 & 1 \end{array}\right] -$

$3c_1 = 3 - 4c_2 = 3 - 2 = 1$

$c_2 = 1/2 \qquad c_1 = 1/3$

$\therefore c^* = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/2 \end{bmatrix}$ ✓