MATH 307 Practice Final Exam

8 December 2021

- No calculators, cellphones, laptops or notes
- Time allowed: 2 hours 30 minutes
- 60 total marks
- Write your name and student number in the space below

Name:

Student Number:

1. True or false questions. Each part is independent of the others.

   (a) (3 marks) **True** or **False**: If A is an $n \times n$ singular matrix (i.e., not invertible), then 0 is an eigenvalue of A. Justify your answer.

   (b) (3 marks) **True** or **False**: If A is an $n \times n$ matrix and all n eigenvalues of A are equal, then A is a diagonal matrix. Justify your answer.

(c) (3 marks) **True** or **False**: If U and V are the two subspaces of $\mathbb{R}^4$ given by

$$U = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \\ -1 \end{bmatrix} \right\} \quad \text{and} \quad V = \text{span}\left\{ \begin{bmatrix} 0 \\ -1 \\ 4 \\ 2 \end{bmatrix} \right\},$$

then $V = U^{\perp}$. Justify your answer.

(d) (3 marks) **True** or **False**: The matrix A is Hermitian when

(i) $A = \begin{bmatrix} 1 & 3-i \\ 3+i & i \end{bmatrix}$.

(ii) $A = \begin{bmatrix} 3 & 2-i & -3i \\ 2+i & 0 & 1-i \\ 3i & 1+i & 1 \end{bmatrix}$.

Justify your answers.

2. Short answer questions. Each part is independent of the others.

   (a) (3 marks) The matrix A has the LU decomposition $A = LU$ where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & -1 & -2 & 4 \\ 0 & 1 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

   Determine $\dim(R(A))$ and $\dim(R(A^T))$.

   (b) (3 marks) Consider 10 data points $(t_0, y_0), \ldots, (t_9, y_9)$ such that $t_k - t_{k-1} = 1$ for each $k = 1, \ldots, 9$. Suppose the coefficient matrix of the corresponding natural cubic spline is given by

$$\begin{bmatrix} 2 & -1 & 1 & a & -1 & 0 & 3 & 1 & -1 \\ 0 & 6 & 3 & b & -6 & -9 & -9 & 0 & 3 \\ -5 & 1 & 10 & c & 19 & 4 & -14 & -23 & -20 \\ 3 & 0 & 6 & 20 & 41 & 53 & 48 & 28 & 6 \end{bmatrix}$$

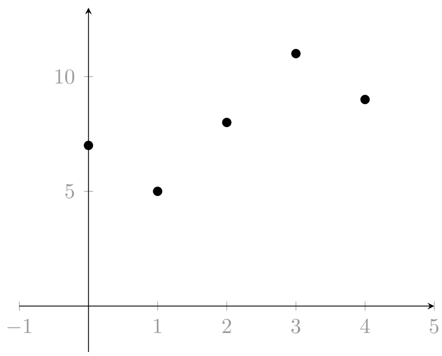
   Determine the missing values a, b and c.

(c) (3 marks) Determine whether $\operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\} = \operatorname{span}\{\boldsymbol{u}_3, \boldsymbol{u}_4\}$ for

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ -5 \\ 4 \\ 5 \end{bmatrix}, \quad \boldsymbol{u}_3 = \begin{bmatrix} 2 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_4 = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 5 \end{bmatrix}.$$

(d) (3 marks) Suppose A is a 3×3 matrix which depends on a parameter $c > 0$ such that the singular values of A are given by 2, 5 and c. Determine the minimum possible value for $\operatorname{cond}(A)$ for all values of c.

3. (6 marks) Use least squares linear regression to find the linear function $f(t) = c_0 + c_1 t$ that best fits the data points $(0, 7), (1, 5), (2, 8), (3, 11)$ and $(4, 9)$.



4. (6 marks) Let $A = \begin{bmatrix} -2 & 1 \\ 1 & -3 \\ 0 & 1 \end{bmatrix}$.

   (a) (3 marks) Compute the thin QR decomposition of A.

   (b) (3 marks) Compute the projection of $\boldsymbol{b}$ onto $R(A)$ for $\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

5. (6 marks) Consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$$

(a) (4 marks) Find the orthogonal projection matrix P which projects onto $R(A)$.

(b) (2 marks) Find the shortest distance from $\boldsymbol{x}$ to $R(A)$ where

$$\boldsymbol{x} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}.$$

6. (6 marks) Use the power method (at least 3 iterations) to approximate the dominant eigenvalue and a corresponding eigenvector of the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$$

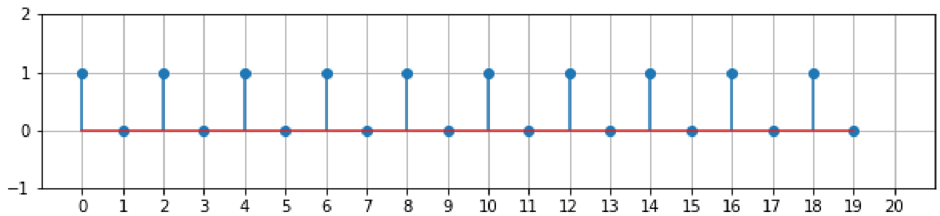
7. (6 marks) Let U be the subspace of $\mathbb{R}^4$ spanned by $\boldsymbol{u_1} = (1,1,1,1)^T$ and $\boldsymbol{u_2} = (1,1,-1,1)^T$.

   (a) (4 marks) Find the pseudo-inverse A^+ of

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

   (b) (2 marks) Find the linear combination $c_1\boldsymbol{u_1}+c_2\boldsymbol{u_2}$ which is nearest to $\boldsymbol{x} = (2,1,4,1)^T$.

8. (6 marks) The stemplot of $\boldsymbol{x} \in \mathbb{C}^{20}$ is shown below:



(a) (2 marks) Find A_1, A_2, k_1 and k_2 such that

$$\boldsymbol{x} = A_1 \cos(2\pi k_1 \boldsymbol{t}) + A_2 \cos(2\pi k_2 \boldsymbol{t}).$$

(b) (4 marks) Compute $\mathrm{DFT}(\boldsymbol{x})$.

Extra workspace.

Extra workspace. Do not write in the table below.

Q1	/12
Q2	/12
Q3	/6
Q4	/6
Q5	/6
Q6	/6
Q7	/6
Q8	/6
Total	/60