1. Short answer questions. Each part below is independent of the others.

(a) (3 marks) **True** or **False**: If $A$ is an invertible matrix, then $\|A^{-1}\| = \|A\|^{-1}$. Justify your answer.

$\boxed{\text{False}}$

$\|A\| = \max_{\|\vec{x}\|=1} \|A\vec{x}\| \quad \Rightarrow \quad \|A\|^{-1} = \dfrac{1}{\max_{\|\vec{x}\|=1} \|A\vec{x}\|}$

$\|A^{-1}\| = \dfrac{1}{\min_{\|\vec{x}\|=1} \|A\vec{x}\|} \neq \dfrac{1}{\max_{\|\vec{x}\|=1} \|A\vec{x}\|} = \|A\|^{-1}$ in general

( We have equality only if all unit vectors are stretched by the same amount by $A$ )

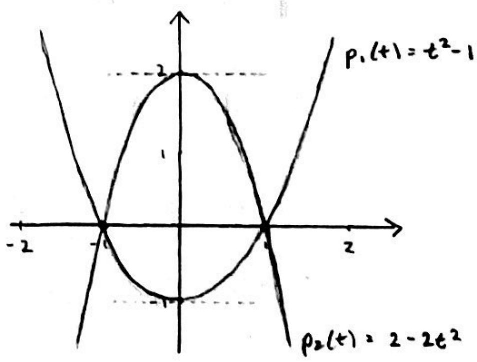
(b) (3 marks) **True** or **False**: There exists a unique polynomial $p(t)$ of degree 2 (or less) such that

$$p(-1) = p'(0) = p(1) = 0$$

Justify your answer.

$\boxed{\text{False}}$

There are infinitely many polynomials of degree $\leq 2$ satisfying the given conditions.




Let $p(t) = at^2 + bt + c$.

The conditions give the system:

$$\begin{cases} a - b + c = 0 \\ b = 0 \\ a + b + c = 0 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ infinitely many solutions! ( Any $p(t) = a(t^2 - 1)$, $a \in \mathbb{R}$ )

(c) (3 marks) Determine (approximately) the condition number of the matrix

$$A = \begin{bmatrix} c & 1 & & & \\ 1 & c & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & c & 1 \\ & & & 1 & c \end{bmatrix}$$

where $c$ is very large positive number. Justify your answer.

When $c$ is very large, $A\vec{x} = \begin{bmatrix} c & 1 & & \\ 1 & c & 1 & \\ & \ddots & \ddots & \\ & & 1 & c \end{bmatrix} \vec{x} \approx \begin{bmatrix} c & & & \\ & c & & \\ & & \ddots & \\ & & & c \end{bmatrix} \vec{x} = c\vec{x}$

for a unit vector $\vec{x}$ (since adding $x_{k-1} + x_{k+1}$ has minor impact compared to $c x_k$).

Therefore, $\|A\| = \max_{\|\vec{x}\|=1} \|A\vec{x}\| \approx \max_{\|\vec{x}\|=1} \|c\vec{x}\| = c$

and $\|A^{-1}\| \approx \frac{1}{c}$ $\Rightarrow$ $\underline{\text{cond}(A) = \|A\| \|A^{-1}\| \approx 1}$

(d) (3 marks) Consider 11 data points $(t_0, y_0), \ldots, (t_{10}, y_{10})$ such that $t_k - t_{k-1} = 1$ for each $k = 1, \ldots, 10$. Suppose the coefficient matrix of the corresponding natural cubic spline is given by

$$\begin{bmatrix} -1 & 2 & 6 & -9 & -1 & 3 & 2 & 8 & 3 & -13 \\ 0 & -3 & 3 & 21 & -6 & -9 & 0 & 6 & 30 & 39 \\ 2 & -1 & -1 & 23 & 38 & 23 & \square & 20 & 56 & 125 \\ -7 & -6 & -8 & 0 & 35 & 66 & 83 & 99 & 133 & 222 \end{bmatrix}$$

Determine the missing value $\square$.

$p_k(t) = a_k (t-t_{k-1})^3 + b_k (t-t_{k-1})^2 + c_k (t-t_{k-1}) + d_k$

$\Rightarrow p_k'(t) = 3a_k (t-t_{k-1})^2 + 2b_k (t-t_{k-1}) + c_k$ , $t \in [t_{k-1}, t_k]$

Since $p'(t)$ should be continuous, $p_k'(t_k) = p_{k+1}'(t_k)$ , $k=1, \ldots, 9$

$\Rightarrow 3a_k \underbrace{(t_k - t_{k-1})^2}_{=1} + 2b_k \underbrace{(t_k - t_{k-1})}_{=1} + c_k = c_{k+1}$

$k=7$: $3a_7 + 2b_7 + c_7 = c_8 \Leftrightarrow 3 \cdot 2 + 2 \cdot 0 + \square = 20 \Leftrightarrow \square = 14$

or from $p_k(t_k) = p_{k+1}(t_k)$ with $t_k - t_{k-1} = 1$, we get: $a_k + b_k + c_k + d_k = d_{k+1}$

$\Rightarrow 2 + 0 + \square + 83 = 99 \Leftrightarrow \square = 14$

2. (5 marks) Determine all values $c$ such that the vectors

$$u_1 = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 4 \\ 5 \\ c-1 \end{bmatrix}$$

are linearly independent.

The vectors $\vec{u}_1$, $\vec{u}_2$ and $\vec{u}_3$ are linearly independent if

$a_1\vec{u}_1 + a_2\vec{u}_2 + a_3\vec{u}_3 = \vec{0}$ if and only if $a_1 = a_2 = a_3 = 0$

We want to find $c$ such that the system

$$\begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ has a unique solution $(a_1 = a_2 = a_3 = 0)$.

Gaussian elimination:

$$\left[\begin{array}{ccc|c} -1 & 2 & 4 & 0 \\ -2 & 3 & 5 & 0 \\ 5 & -6 & c-1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} -1 & 2 & 4 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & 4 & c+19 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} -1 & 2 & 4 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & 0 & c+7 & 0 \end{array}\right]$$

We see that the system has a unique solution if $c \neq -7$ and infinitely many solutions if $c = -7$

$\therefore$ $\vec{u}_1$, $\vec{u}_2$ and $\vec{u}_3$ are linearly independent if $\boxed{c \neq -7}$

3. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 8 & 3 & 8 & 2 \\ -4 & -3 & 5 & -1 \\ 2 & -2 & 7 & 11 \end{bmatrix}$$

(a) (4 marks) Find the $LU$ decomposition of $A$.

(b) (2 mark) Compute $\det(A)$.

(a)

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 8 & 3 & 8 & 2 \\ -4 & -3 & 5 & -1 \\ 2 & -2 & 7 & 11 \end{bmatrix} \xrightarrow[R_4 - R_1]{R_2 - 4R_1,\ R_3 + 2R_1} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -1 & 4 & 2 \\ 0 & -1 & 7 & -1 \\ 0 & -3 & 6 & 11 \end{bmatrix} \xrightarrow[R_4 - 3R_2]{R_3 - R_2} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -1 & 4 & 2 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & -6 & 5 \end{bmatrix}$$

$$\xrightarrow{R_4 + 2R_3} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -1 & 4 & 2 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$\Rightarrow A = LU$ where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & 3 & -2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & -1 & 4 & 2 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

(b) $\det(A) = \det(U) =$ product of elements on diagonal

$$= 2 \cdot (-1) \cdot 3 \cdot (-1) = 6$$

4. (6 marks) Setup (but do **not** solve) a linear system $A\boldsymbol{x} = \boldsymbol{b}$ such that the solution

$$\boldsymbol{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

determines the unique function of the form

$$f(t) = a\sin(\pi t) + b\cos(\pi t) + c\sin(2\pi t) + d\cos(2\pi t)$$

which interpolates the data $(0, y_0)$, $(1/4, y_1)$, $(1/2, y_2)$, $(3/4, y_3)$. The system depends on $y_0, y_1, y_2, y_3$.

We should have:

$f(0) = y_0 \Rightarrow a\sin(0) + b\cos(0) + c\sin(0) + d\cos(0) = y_0$

$\Leftrightarrow b + d = y_0$

$f(\frac{1}{4}) = y_1 \Rightarrow a\sin(\frac{\pi}{4}) + b\cos(\frac{\pi}{4}) + c\sin(\frac{\pi}{2}) + d\cos(\frac{\pi}{2}) = y_1$

$\Leftrightarrow \frac{a}{\sqrt{2}} + \frac{b}{\sqrt{2}} + c = y_1$

$f(\frac{1}{2}) = y_2 \Rightarrow a\sin(\frac{\pi}{2}) + b\cos(\frac{\pi}{2}) + c\sin(\pi) + d\cos(\pi) = y_2$

$\Leftrightarrow a - d = y_2$

$f(\frac{3}{4}) = y_3 \Rightarrow a\sin(\frac{3\pi}{4}) + b\cos(\frac{3\pi}{4}) + c\sin(\frac{3\pi}{2}) + d\cos(\frac{3\pi}{2}) = y_3$

$\Leftrightarrow \frac{a}{\sqrt{2}} - \frac{b}{\sqrt{2}} - c = y_3$

$$\Rightarrow \underbrace{\begin{bmatrix} 0 & 1 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & 0 \\ 1 & 0 & 0 & -1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\vec{b}}$$

5. (6 marks) Find all polynomials $p(t)$ of degree 3 (or less) such that

$$p(1) = p(-1) \qquad p(-2) = -7p(0) \qquad p'(1) = 3p'(-1) \qquad 5p''(1) = -7p''(-1)$$

$p(t) = at^3 + bt^2 + ct + d$

$p'(t) = 3at^2 + 2bt + c$

$p''(t) = 6at + 2b$

The conditions give:

- $p(1) = p(-1) \iff a+b+c+d = -a+b-c+d \iff a+c=0$
- $p(-2) = -7p(0) \iff -8a+4b-2c+d = -7d \iff -4a+2b-c+4d=0$
- $p'(1) = 3p'(-1) \iff 3a+2b+c = 3(3a-2b+c) \iff 3a-4b+c=0$
- $5p''(1) = -7p''(-1) \iff 5(6a+2b) = -7(-6a+2b) \iff a-2b=0$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ -4 & 2 & -1 & 4 & 0 \\ 3 & -4 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\substack{R_3-3R_1 \\ R_4-R_1}]{R_2+4R_1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 3 & 4 & 0 \\ 0 & -4 & -2 & 0 & 0 \\ 0 & -2 & -1 & 0 & 0 \end{array}\right] \xrightarrow[R_4+R_2]{R_3+2R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 3 & 4 & 0 \\ 0 & 0 & 4 & 8 & 0 \\ 0 & 0 & 2 & 4 & 0 \end{array}\right]$$

$$\xrightarrow[\frac{1}{4}R_3]{R_4-\frac{1}{2}R_3} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 3 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let $d = r \in \mathbb{R}$.

Then, $c + 2d = 0 \Rightarrow c = -2r$

$2b + 3c + 4d = 0 \Rightarrow b = r$

$a + c = 0 \Rightarrow a = 2r$

$$\Rightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = r \begin{bmatrix} 2 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\therefore \boxed{p(t) = r\left(2t^3 + t^2 - 2t + 1\right), \ r \in \mathbb{R}}$$