MATH 307 Midterm Exam 2

November 17, 2021

- No calculators, cellphones, laptops or notes
- Time allowed: 45 minutes
- 35 total marks
- Write your name and student number in the space below

Name:

Student Number:

1. Short answer questions. Each part is independent of the others.

   (a) (3 marks) **True** or **False**: If A is an $n \times n$ symmetric matrix such that $A^2 = I$, then A is orthogonal. Justify your answer.

   (b) (3 marks) Let $a, b \in \mathbb{R}$ such that $a \neq b$ and consider the matrix

$$A = \begin{bmatrix} a & b & a & b & a & b \\ b & a & b & a & b & a \\ a & b & a & b & a & b \\ b & a & b & a & b & a \\ a & b & a & b & a & b \end{bmatrix}$$

   Determine the dimension of $N(A)^{\perp}$. Justify your answer.

(c) (3 marks) Suppose A is a 4×4 matrix with characteristic polynomial

$$c_A(x) = (x^2 + x - 2)(x^2 - x - 2)$$

Is A diagonalizable? Justify your answer.

(d) (3 marks) **True** or **False**: Suppose $A = P\Sigma Q^T$ is the singular value decomposition of A such that Q is a permutation matrix. Then the columns of A are orthogonal. (Recall that a permutation matrix is any matrix obtained from the identity matrix by permuting the rows.) Justify your answer.

2. (6 marks) Find the orthogonal projection matrix P which projects onto $U = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ where

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

3. (6 marks) Consider the matrix

$$A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$$

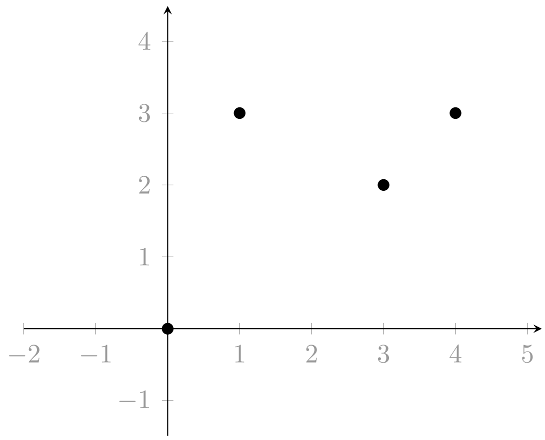
   (a) (4 marks) Find matrices P and D such that $A = PDP^{-1}$.

   (b) (2 marks) Compute the limit

$$\lim_{k \to \infty} \lambda_1^{-k} A^k$$

   where λ_1 is the largest eigenvalue (in absolute value). In other words, if λ_1 and λ_2 are the eigenvalues of A then $|\lambda_1| > |\lambda_2|$.

4. (5 marks) Use least squares linear regression to find the linear function $f(t) = c_0 + c_1 t$ that best fits the data points $(0, 0), (1, 3), (3, 2)$ and $(4, 3)$.



5. (6 marks) Suppose $A = Q_1 R_1$ where

$$Q_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \qquad R_1 = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Compute the projection of $\boldsymbol{v}$ onto $R(A)^\perp$ for

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$$

Extra workspace. Do not write in the table below.

Extra workspace. Do not write in the table below.

Q1	/12
Q2	/6
Q3	/6
Q4	/5
Q5	/6
Total	/35