MATH 307 Final Exam

June 29, 2022

- No calculators, cellphones, laptops or notes
- Time allowed: 150 minutes
- 70 total marks
- Write your name and student number in the space below
- Notation:
  - A^T is the transpose of A
  - $N(A)$ is the nullspace of A and $R(A)$ is the range of A
  - $U^\perp$ is the orthogonal complement of a subspace U
  - I is the identity matrix
  - $\text{proj}_U(\boldsymbol{x})$ is the projection of $\boldsymbol{x}$ onto U

Name: SOLUTION

Student Number:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Total
12	8	8	6	6	6	5	6	6	7	70

1. Determine if the statement is **True** or **False**. No justification required.

(a) (2 marks) If U_1 and U_2 are orthogonal subspaces of $\mathbb{R}^n$, then the orthogonal complements $U_1^\perp$ and $U_2^\perp$ are orthogonal. In other words, if $U_1 \perp U_2$ then $U_1^\perp \perp U_2^\perp$.

$\boxed{\text{False}}$ Example: $U_1, U_2 \subset \mathbb{R}^3$ $\dim(U_1) = \dim(U_2) = 1$

$\Rightarrow \dim(U_1^\perp) = \dim(U_2^\perp) = 2$

(b) (2 marks) If A is any 4×4 matrix with characteristic polynomial

$$c_A(x) = (x^2 + 2x - 2)(x^2 - 2x - 2)$$

then A is diagonalizable.

$$x^2 + 2x - 2 \Rightarrow \frac{-2 \pm \sqrt{4 - 4(-2)}}{2} = -1 \pm \sqrt{3}$$

$$x^2 - 2x - 2 \Rightarrow \frac{2 \pm \sqrt{4 - 4(-2)}}{2} = 1 \pm \sqrt{3}$$

$\boxed{\text{True}}$ $\Rightarrow$ 4 distinct eigenvalues.

(c) (2 marks) Let $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1 \neq \lambda_2$, and let $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^2$ be nonzero vectors such that $\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = 0$. There is a unique 2×2 symmetric matrix A such that $\cancel{A\boldsymbol{v}_1 = \lambda_1}$ and $\cancel{A\boldsymbol{v}_2 = \lambda_2}$.

$A\underline{v}_1 = \lambda_1 \underline{v}_1$ $A\underline{v}_2 = \lambda_2 \underline{v}_2$

$\boxed{\text{True}}$ $A = \begin{bmatrix} \frac{1}{\|\underline{v}_1\|}\underline{v}_1 & \frac{1}{\|\underline{v}_2\|}\underline{v}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\|\underline{v}_1\|}\underline{v}_1 & \frac{1}{\|\underline{v}_2\|}\underline{v}_2 \end{bmatrix}^T$.

(d) (2 marks) The matrix

$$P = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

is the projection matrix onto a subspace U such that $\dim(U) = 2$.

$\boxed{\text{False}}$ $R(P) = U$ and $\text{rank}(P) = 1$

$\Rightarrow \dim(U) = 1$.

(e) (2 marks) Let A be a $m \times n$ matrix such that $\text{rank}(A) = n$ and let $A = Q_1 R_1$ be the thin QR decomposition. Then $Q_1^T Q_1 = I$.

$\boxed{\text{True}}$ Columns of Q_1 are orthonormal.

(f) (2 marks) Let $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^N$ and let $\boldsymbol{y}_1 = \text{DFT}(\boldsymbol{x}_1)$ and $\boldsymbol{y}_2 = \text{DFT}(\boldsymbol{x}_2)$. If $\langle \boldsymbol{x}_1, \boldsymbol{x}_2 \rangle = 0$ then $\langle \boldsymbol{y}_1, \boldsymbol{y}_2 \rangle = 0$.

$\boxed{\text{True}}$ $\langle \underline{y}_1, \underline{y}_2 \rangle = \langle F_N \underline{x}_1, F_N \underline{x}_2 \rangle$

$= \langle \underline{x}_1, \overline{F}^T F_N \underline{x}_2 \rangle$

$= N \langle \underline{x}_1, \underline{x}_2 \rangle = 0$.

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 4 & -1 & 0 \\ -1 & -5 & 1 & 3 \\ 2 & 8 & 0 & -2 \\ 1 & 3 & -2 & 4 \end{bmatrix}$$

(a) (4 marks) Find the LU decomposition of A.

(b) (4 marks) Find a basis of $R(A)^{\perp}$.

(a) $$A \xrightarrow[\substack{-2R_1+R_3 \\ -R_1+R_4}]{R_1+R_2} \begin{bmatrix} 1 & 4 & -1 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 2 & -2 \\ 0 & -1 & -1 & 4 \end{bmatrix} \xrightarrow{-R_2+R_4} \begin{bmatrix} 1 & 4 & -1 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_3+R_4} \begin{bmatrix} 1 & +4 & -1 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & -1/2 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 4 & -1 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) $$R(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1/2 \end{bmatrix} \right\}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/2 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_4 = t \quad x_3 = \frac{1}{2}t \quad x_2 = -t \\ x_1 = -t - 2\frac{1}{2}t + (-t) = -3t \end{array}$$

$$\Rightarrow R(A)^{\perp} = \text{span}\left\{ \begin{bmatrix} -3 \\ -1 \\ 1/2 \\ 1 \end{bmatrix} \right\}$$

3. Let $A = Q_1R_1$ be the thin QR decomposition of A, and let $\boldsymbol{b} \in \mathbb{R}^4$ where

$$Q_1 = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad R_1 = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

(a) (4 marks) Find the projection of $\boldsymbol{b}$ onto $R(A)^\perp$.

(b) (4 marks) Find the least squares approximation $A\boldsymbol{x} \cong \boldsymbol{b}$.

(a) $$\mathrm{proj}_{R(A)^\perp}(\underline{b}) = \underline{b} - Q_1Q_1^T\underline{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix} - Q_1\begin{bmatrix} 5/2 \\ -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 6 \\ 4 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \\ -1 \\ 1/2 \end{bmatrix}$$

(b) $$A\underline{x} \cong \underline{b} \Rightarrow R_1\underline{x} = Q_1^T\underline{b} = \frac{1}{2}\begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c} 2 & 1 & 5/2 \\ 0 & 5 & -1/2 \end{array}\right]$$

$$\Rightarrow \underline{x} = \begin{bmatrix} (5/2 + 1/10)/2 \\ -1/10 \end{bmatrix}$$

$$\underline{x} = \begin{bmatrix} 13/10 \\ -1/10 \end{bmatrix}$$

4. (6 marks) Find the orthogonal diagonalization $A = PDP^T$ of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The characteristic polynomial of A is $c_A(x) = x^3 - 5x^2 + 4x$.

$$x^3 - 5x^2 + 4x = x(x-4)(x-1) \Rightarrow \lambda_1 = 4 \quad \lambda_2 = 1 \quad \lambda_3 = 0$$

$$\lambda_1 = 4 \quad \left[\begin{array}{ccc|c} -3 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & -3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 2 & -4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \underline{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 1 \quad \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \underline{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\lambda_3 = 0 \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \underline{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\Rightarrow A = PDP^T \qquad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix}$$

5. Let a and b be nonzero numbers and consider the matrix

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

(a) (4 marks) Compute $\|A\|$.

(b) (2 marks) Compute $\mathrm{cond}(A)$.

(a) $A^T A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix}$

$\Rightarrow \sigma_1 = \sigma_2 = \sqrt{a^2+b^2}$.

$\Rightarrow \boxed{\|A\| = \sqrt{a^2+b^2}}$

(b) $\boxed{\mathrm{cond}(A) = \frac{\sigma_1}{\sigma_2} = 1}$

6. (6 marks) Find the shortest distance from $\boldsymbol{x}$ to $U = \operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ where

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \qquad \boldsymbol{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Find a basis of $U^\perp$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow \quad U^\perp = \operatorname{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\frac{\langle \underline{x}, \underline{v} \rangle}{\langle \underline{v}, \underline{v} \rangle} \underline{v} = \quad + \frac{1}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ +1/2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \text{ Shortest distance is } \left\| \operatorname{proj}_{U^\perp}(\underline{x}) \right\| = \sqrt{\left(\tfrac{-1}{2}\right)^2 + 0^2 + \left(\tfrac{1}{2}\right)^2 + 0^2}$$

$$= \frac{1}{\sqrt{2}}.$$

7. (5 marks) Let A be a 4×4 matrix with singular value decomposition $A = P\Sigma Q^T$ where

$$P = \begin{bmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3 & \boldsymbol{p}_4 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 & \boldsymbol{q}_3 & \boldsymbol{q}_4 \end{bmatrix}$$

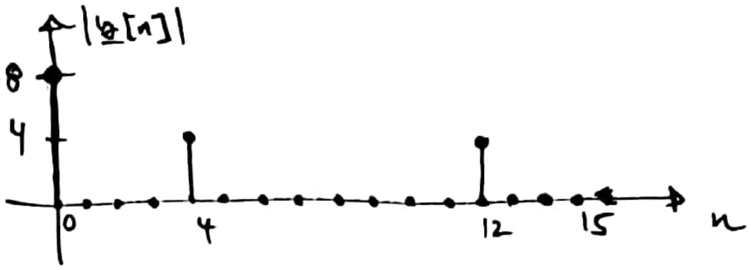
Let $\boldsymbol{x} = \boldsymbol{q}_1 + \boldsymbol{q}_2 + \boldsymbol{q}_3 + \boldsymbol{q}_4$. Compute $\|A\boldsymbol{x}\|$.

$$A\underline{x} = P\Sigma Q^T \underline{x} = P\Sigma \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = P \begin{bmatrix} 5 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$= 5\underline{P}_1 + 2\underline{P}_2 + \underline{P}_3$$

$$\Rightarrow \boxed{\|A\underline{x}\| = \sqrt{30}}$$

8. (6 marks) Let $\boldsymbol{x} \in \mathbb{R}^{16}$ such that $\boldsymbol{y} = \mathrm{DFT}(\boldsymbol{x})$ where $\boldsymbol{y}[0] = 8$, $\boldsymbol{y}[4] = \boldsymbol{y}[12] = 4$, and all other entries of $\boldsymbol{y}$ are zero. Sketch the stemplot of $\boldsymbol{x}$.

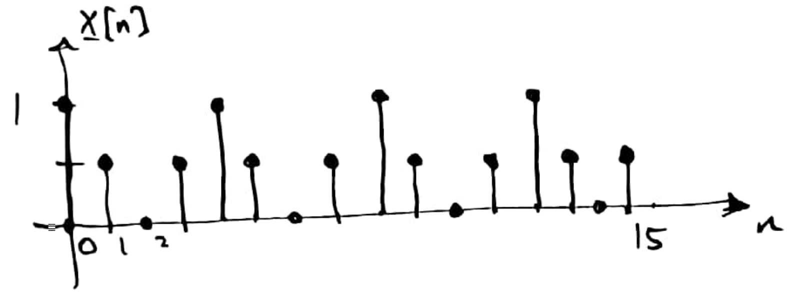


$$\Rightarrow \quad \underline{x} = A_0 \underline{f}_0 + A_1 \cos(2\pi k_1 \underline{t} + \phi_1)$$

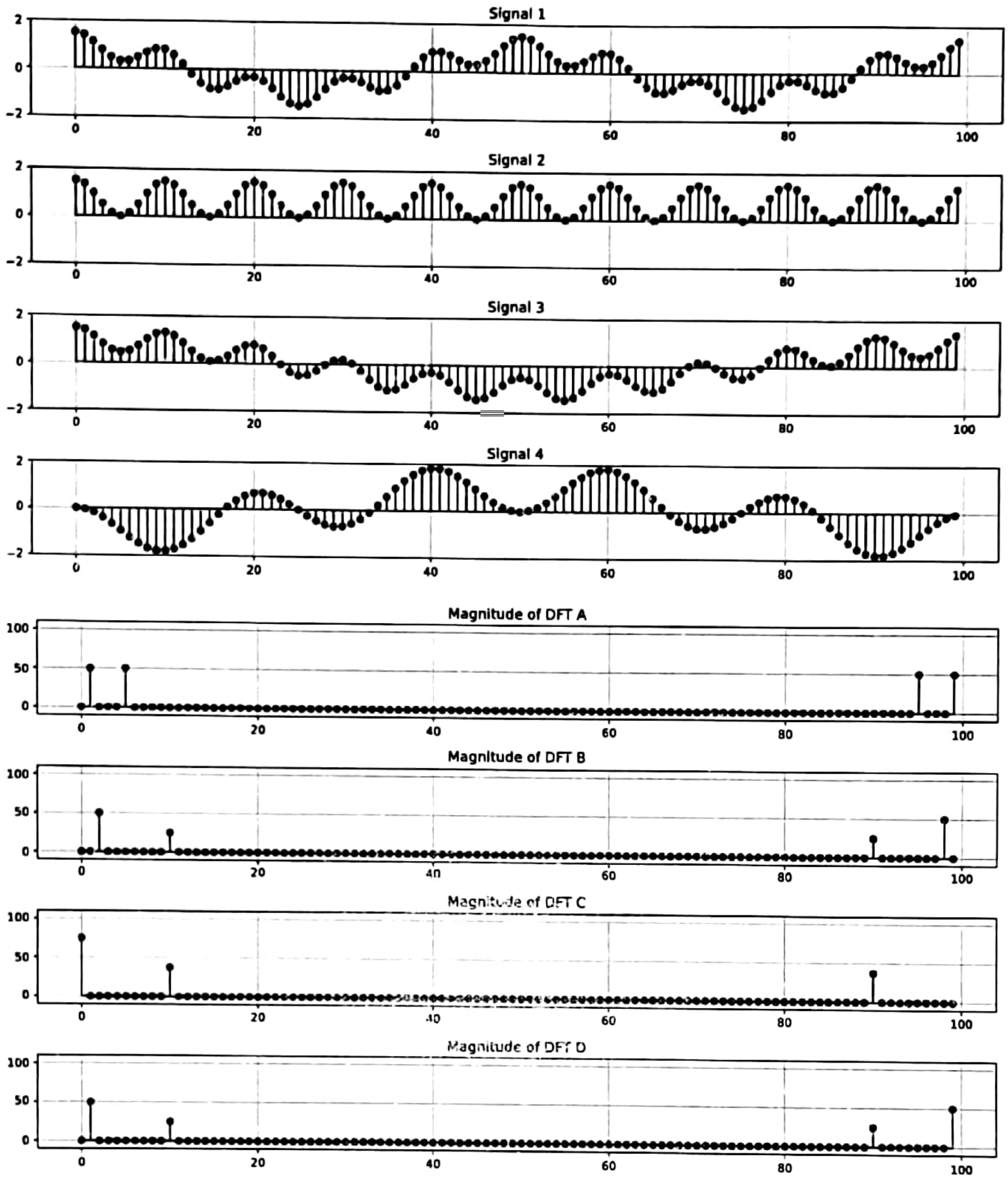
$$\Rightarrow \quad A_0 N = 8 \qquad \frac{A_1 N}{2} = 4 \qquad e^{i\phi_1} = 1$$

$$A_0 = \frac{1}{2} \qquad A_1 = \frac{1}{2} \qquad \phi_1 = 0 \qquad k_1 = 4$$

$$\Rightarrow \quad \boxed{\underline{x} = \frac{1}{2}\underline{f}_0 + \frac{1}{2}\cos(2\pi(4)\underline{t})}$$



9. (6 marks) Match the signal with its discrete Fourier transform.



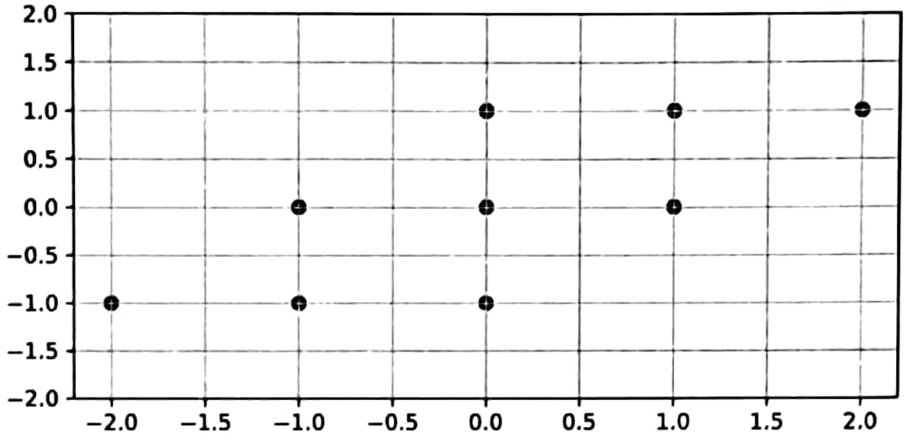
Signal 1 = DFT B

Signal 2 = DFT C

Signal 3 = DFT D

Signal 4 = DFT A

10. (7 marks) Find the weight vectors $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ for the dataset displayed below



$$X = \begin{bmatrix} -2 & -1 \\ -1 & -1 \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad X^T X = \begin{bmatrix} 12 & 6 \\ 6 & 6 \end{bmatrix}$$

$$\lambda^2 - 18\lambda + 36 = \cancel{}$$

$$\Rightarrow \lambda = \frac{18 \pm \sqrt{18^2 - 4(36)}}{2}$$

$$= 9 \pm \sqrt{9^2 - 36} = 9 \pm 3\sqrt{5}$$

$$\lambda_1 = 9 + 3\sqrt{5} \qquad \left[\begin{array}{cc|c} 3-3\sqrt{5} & 6 & 0 \\ 6 & -3-3\sqrt{5} & 0 \end{array}\right] \Rightarrow \underline{w}_1 = \begin{bmatrix} 2 \\ -1+\sqrt{5} \end{bmatrix}$$

$$\Rightarrow \underline{v}_2 = \begin{bmatrix} -1+\sqrt{5} \\ -2 \end{bmatrix}$$

$$\Rightarrow \boxed{\underline{w}_1 = \frac{1}{\sqrt{4 + (-1+\sqrt{5})^2}} \begin{bmatrix} 2 \\ -1+\sqrt{5} \end{bmatrix} \qquad \underline{w}_2 = \frac{1}{\sqrt{10 - 2\sqrt{5}}} \begin{bmatrix} -1+\sqrt{5} \\ -2 \end{bmatrix}}$$