Math 307: 202 — Midterm 1 — 50 minutes

Last Name ______________________ **First** ______________________

Student Number ______________________ **Signature** ______________________

- The test consists of 12 pages and 4 questions worth a total of 50 marks.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- No work on this page will be marked.

Please do not write on this page — it will not be marked.

Additional instructions

- Please use the spaces indicated.
- **Unless it is specified not to do so, justify your answers.**
- If you require extra paper then put up your hand and ask your instructor.
  - You must put your name and student number on any extra pages.
  - You must indicate the test-number and question-number.
  - Please do this **on both sides** of any extra pages.
- Please do not dismember your test. You must submit all pages.

1		8
2		24
3		10
4		8
Total		50

1. $\boxed{\text{8 marks}}$ Let $A = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 1 \\ 2 & 6 & 6 & 1 \\ 0 & -3 & 6 & 4 \end{pmatrix}$.

(a) Compute the LU decomposition of A.

(b) Compute $\det(A)$.

Solution: (a) $A \xrightarrow{R_3 = R_3 - 2R_1} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 1 \\ 0 & 2 & 0 & -3 \\ 0 & -3 & 6 & 4 \end{pmatrix} \xrightarrow[R_4 = R_4 - 3R_2]{R_3 = R_3 + 2R_2} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

So $L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(b) $\det(A) = \det(L)\det(U) = \det(U) = 1 \cdot (-1) \cdot 4 \cdot 1 = -4$.

2. $\boxed{\text{24 marks}}$ Short answer questions. **For True or False questions**, if true, provide a short justification. If false, show a counter-example that contradicts the statement. **For other questions**, justify your answer by showing your work.

(a) Let $A = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Calculate (no part marks):

(i) Cond(A), i.e., the condition number of A

(ii) $\|Ax\|_2$

(iii) $\|Ax\|_1$

(b) **True or False:** Let A be an $n \times n$ matrix with LU decomposition $A = LU$. Then $\text{rank}(A) = \text{rank}(L)$.

(c) **Find** a 2×2 **diagonal** matrix A with $\mathrm{cond}(A) = \|A^2\| = 2$; or **explain why such a matrix cannot exist**.

(d) **True or False:** Let U and V be two subspaces of $\mathbb{R}^n$. Their union (that is, the set of vectors belonging to at least one of U and V) is also a subspace of $\mathbb{R}^n$.

(e) **True or False:** Let A and B be $n \times n$ matrices. Then $\mathcal{R}(BA) = \mathcal{R}(A)$.

(f) **True or False:** Let $A \in \mathbb{R}^{n\times n}$ be invertible and let $\mathbf{x} \in \mathbb{R}^n$. Then $||A\mathbf{x}|| \geq ||A^{-1}||^{-1} \cdot ||\mathbf{x}||$.

(g) **True or False:**. Let $A, B \in \mathbb{R}^{n\times n}$ both be invertible. Then $||AB|| \leq ||A|| \cdot ||B||$.

(h) **True or False:** Let $A \in \mathbb{R}^{n\times n}$ be invertible. If $||A|| = ||A^{-1}|| = cond(A) = 1$, then $A = I$.

Solution:

(a) (i) For any $y = (y_1, y_2, y_3)^t$, since $||Ay|| = ||(y_2, -2y_1, -5y_3)|| = ||(-2y_1, y_2, -5y_3)^t|| = ||diag(-2, 1, -5)y||$, so $||A|| = ||diag(-2, 1, -5)|| = 5$. Similarly, $||A^{-1}|| = 1$. So $cond(A) = 5$.
(ii) $||Ax||_2 = ||(1, -2, -5)^t||_2 = \sqrt{1^2 + (-2)^2 + (-5)^2} = \sqrt{30}$.
(iii) $||Ax||_1 = ||(1, -2, -5)^t||_1 = 1 + |-2| + |-5| = 8$.

(b) False. When $A = 0$, we have $L = I$, $U = 0$. So $rank(A) = 0 \neq n = rank(L)$.

(c) Let $A = diag(\sqrt{2}, \sqrt{1/2})$, then $cond(A) = \frac{\sqrt{2}}{\sqrt{1/2}} = 2$ and $||A^2|| = ||diag(2, 1/2)|| = 2$.

(d) False. Let $n = 2$. Consider

$$U = \text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\} \text{ and } V = \text{span}\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}.$$

Then,

$$\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \in U \cup V, \text{ but } \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \notin U \cup V.$$

This violates the "closed under addition" condition of a subspace. So $U \cup V$ is not a subspace.

(e) False. Let $n = 2$. Consider

$$A = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \text{ (the identity), and } B = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} \text{ (the zero matrix).}$$

Then, $\mathcal{R}(A) = \mathbb{R}^2$ and $\mathcal{R}(BA) = \{\mathbf{0}\}$.

(f) True. We know that $||x|| = ||A^{-1}Ax|| \leq ||A^{-1}|| \cdot ||Ax||$. So $||Ax|| \geq ||A^{-1}||^{-1} \cdot ||x||$.

(g) True. For any $\mathbf{x} \in \mathbb{R}^n$, we have $||AB\mathbf{x}|| = ||A(B\mathbf{x})|| \leq ||A|| \cdot ||B\mathbf{x}|| \leq ||A|| \cdot ||B|| \cdot ||\mathbf{x}||$, so $||AB|| \leq ||A|| \cdot ||B||$.

(h) False. When A is a non-identity rotation matrix, for example $\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$, then we have $||A|| = ||A^{-1}|| = 1$ so $cond(A) = 1$. But $A \neq I$.

3. $\boxed{\text{10 marks}}$ Let

$$A = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 2 & 5 & 3 \\ 1 & 0 & -1 & 2 \\ 1 & 4 & 9 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find the dimensions of $N(A)$ and $R(A)$.

Solution: Let

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & 2 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since L is non-singular, $\dim N(A) = \dim N(U) = 2$ and $\dim R(A) = \dim R(U) = 2$.

(b) Find a basis for $R(A)$.

Solution: The first $\operatorname{rank} A$ columns of L form a basis for $R(A)$. Since $\operatorname{rank} A = \dim R(A) = 2$,

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

is a basis for $R(A)$.

Alternative solution: Since the pivots of A are in the first 2 columns, the first two columns of A also form a basis for $R(A)$.

(c) Find a basis for $N(A)$.

Solution: By theorem, $N(A) = N(U)$, so we want to compute $N(U)$. By definition,

$$N(U) = \left\{ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : \begin{matrix} x_1 + 2x_2 + 4x_3 + 5x_4 = 0, \text{ and} \\ 2x_2 + 5x_3 + 3x_4 = 0 \end{matrix} \right\}.$$

We know $\dim N(U) = 2$, so we want to write $N(U)$ using two parameters $t, s \in \mathbb{R}$. Set $x_3 = t$ and $x_4 = s$. Solving for x_1 and x_2 gives

$$x_1 = t - 2s \text{ and } x_2 = -\frac{5}{2}t - \frac{3}{2}s.$$

So

$$\mathbf{x} = \begin{bmatrix} t - 2s \\ -\frac{5}{2}t - \frac{3}{2}s \\ t \\ s \end{bmatrix} = t \begin{bmatrix} 1 \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}$$

where the two vectors on the right hand side forms a basis for $N(U)$.

4. $\boxed{\text{8 marks}}$ Parts (a) and (b) are independent.

(a) Find a polynomial $p(t)$ of degree at most 3 such that

$$p(0) = p(1) = 0, p'(0) = -2, p'(1) = 3.$$

Solution: Let $p(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$. The conditions given in the question can be rewritten in the following way:

$$\begin{aligned} p(0) = 0 &\iff c_0 = 0 \\ p(1) = 0 &\iff c_0 + c_1 + c_2 + c_3 = 0 \\ p'(0) = -2 &\iff c_1 = -2 \\ p'(1) = 3 &\iff c_1 + 2c_2 + 3c_3 = 3 \end{aligned}$$

Solving this system for c_0, c_1, c_2, c_3 gives

$$c_0 = 0, c_1 = -2, c_2 = 1, c_3 = 1.$$

So $p(t) = -2t + t^2 + t^3$.

(b) Given $d+1$ points $(t_0, y_0), \ldots, (t_d, y_d)$ such that $t_i \neq t_j$ for $i \neq j$. Does there exist a polynomial of degree at most $d+2$ which interpolates the data? If such a polynomial exists, is it unique? Justify both of your answers.

Solution: Yes, there exists a polynomial, but not a unique one. To see this, let

$$p(t) = c_0 + c_1 t + \cdots + c_{d+2} t^{d+2}$$

be the polynomial which interpolates the data. Finding $c_0, \ldots, c_{d+2}$ amounts to solving the linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & t_0 & \cdots & t_0^{d-1} & t_0^d & t_0^{d+1} & t_0^{d+2} \\ 1 & t_1 & \cdots & t_1^{d-1} & t_1^d & t_1^{d+1} & t_1^{d+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_d & \cdots & t_d^{d-1} & t_d^d & t_d^{d+1} & t_d^{d+2} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} c_0 \\ \vdots \\ c_{d+2} \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} y_0 \\ \vdots \\ y_d \end{bmatrix}$$

Notice A is a $(d+1) \times (d+3)$ matrix which contains (as a submatrix) the $(d+1) \times (d+1)$ Vandermonde matrix

$$\begin{bmatrix} 1 & t_0 & \cdots & t_0^{d-1} & t_0^d \\ 1 & t_1 & \cdots & t_1^{d-1} & t_1^d \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & t_d & \cdots & t_d^{d-1} & t_d^d \end{bmatrix}.$$

Since Vandermonde matrices are non-singular (when $t_i \neq t_j$ for $i \neq j$), the first $d+1$ columns of A are linearly independent. So A has rank $d+1$. Hence, the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

Common (Subtle) Mistake: If you said that the above system has $d+1$ equations and $d+3$ variables (or equivalently A is a $(d+1) \times (d+3)$ matrix), this is not enough to justify why there **exists** a polynomial. A system with $d+1$ equations and $d+3$ can have no solution or infinitely many. You would need some way of justifying why a solution exists in the first place. i.e., because A has rank $d+1$.

Alternative Solution: (Inspired by a student solution.) Consider adding two extra points (t_{d+1}, y_{d+1}), (t_{d+2}, y_{d+2}) to our data such that $t_{d+1} \neq t_i \neq t_{d+2}$ for $i = 0, \ldots, d$. Then, there exists a unique polynomial $p(t)$ of degree at most $d+2$ which interpolates the $d+3$ points $(t_0, y_0), \ldots, (t_{d+2}, y_{d+2})$. In particular, $p(t)$ also interpolates our original $d+1$ points $(t_0, y_0), \ldots, (t_d, y_d)$. However, there are infinitely many choices of y_{d+1} and y_{d+2}, and each choice results in a different polynomial, this process can generate infinitely many different polynomials $p(t)$. So there are infinitely many polynomials of degree at most $d+2$ which interpolates the data.