MATH 307 Final Exam

June 29, 2022

- No calculators, cellphones, laptops or notes
- Time allowed: 150 minutes
- 70 total marks
- Write your name and student number in the space below
- Notation:
  - A^T is the transpose of A
  - $N(A)$ is the nullspace of A and $R(A)$ is the range of A
  - $U^\perp$ is the orthogonal complement of a subspace U
  - I is the identity matrix
  - $\text{proj}_U(\boldsymbol{x})$ is the projection of $\boldsymbol{x}$ onto U

Name:

Student Number:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Total
12	8	8	6	6	6	5	6	6	7	70

1. Determine if the statement is **True** or **False**. No justification required.

   (a) (2 marks) If U_1 and U_2 are orthogonal subspaces of $\mathbb{R}^n$, then the orthogonal complements $U_1^\perp$ and $U_2^\perp$ are orthogonal. In other words, if $U_1 \perp U_2$ then $U_1^\perp \perp U_2^\perp$.

   (b) (2 marks) If A is <u>any</u> 4×4 matrix with characteristic polynomial

   $$c_A(x) = (x^2 + 2x - 2)(x^2 - 2x - 2)$$

   then A is diagonalizable.

   (c) (2 marks) Let $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1 \neq \lambda_2$, and let $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^2$ be nonzero vectors such that $\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = 0$. There is a <u>unique</u> 2×2 symmetric matrix A such that $A\boldsymbol{v}_1 = \lambda_1$ and $A\boldsymbol{v}_2 = \lambda_2$.

(d) (2 marks) The matrix

$$P = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

is the projection matrix onto a subspace U such that $\dim(U) = 2$.

(e) (2 marks) Let A be a $m \times n$ matrix such that $\mathrm{rank}(A) = n$ and let $A = Q_1 R_1$ be the thin QR decomposition. Then $Q_1^T Q_1 = I$.

(f) (2 marks) Let $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^N$ and let $\boldsymbol{y}_1 = \mathrm{DFT}(\boldsymbol{x}_1)$ and $\boldsymbol{y}_2 = \mathrm{DFT}(\boldsymbol{x}_2)$. If $\langle \boldsymbol{x}_1, \boldsymbol{x}_2 \rangle = 0$ then $\langle \boldsymbol{y}_1, \boldsymbol{y}_2 \rangle = 0$.

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 4 & -1 & 0 \\ -1 & -5 & 1 & 3 \\ 2 & 8 & 0 & -2 \\ 1 & 3 & -2 & 4 \end{bmatrix}$$

(a) (4 marks) Find the LU decomposition of A.

(b) (4 marks) Find a basis of $R(A)^{\perp}$.

3. Let $A = Q_1 R_1$ be the thin QR decomposition of A, and let $\boldsymbol{b} \in \mathbb{R}^4$ where

$$Q_1 = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad R_1 = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

(a) (4 marks) Find the projection of $\boldsymbol{b}$ onto $R(A)^{\perp}$.

(b) (4 marks) Find the least squares approximation $A\boldsymbol{x} \cong \boldsymbol{b}$.

4. (6 marks) Find the orthogonal diagonalization $A = PDP^T$ of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The characteristic polynomial of A is $c_A(x) = x^3 - 5x^2 + 4x$.

5. Let a and b be nonzero numbers and consider the matrix

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

(a) (4 marks) Compute $\|A\|$.

(b) (2 marks) Compute $\operatorname{cond}(A)$.

6. (6 marks) Find the shortest distance from $\boldsymbol{x}$ to $U = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ where

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \qquad \boldsymbol{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

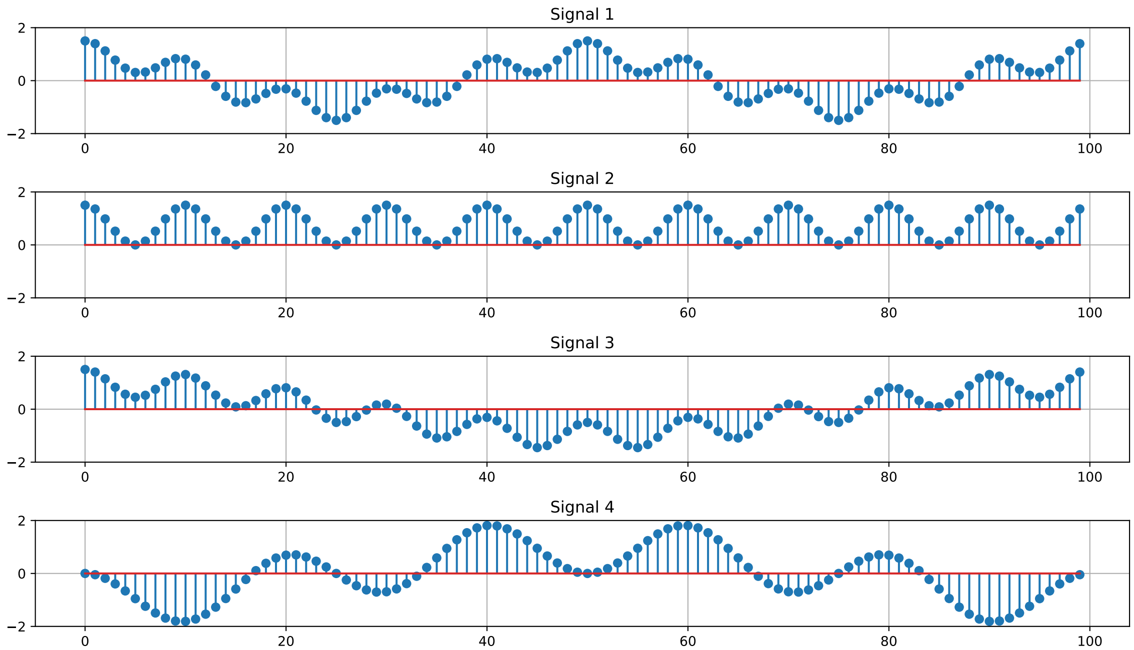
7. (5 marks) Let A be a 4×4 matrix with singular value decomposition $A = P\Sigma Q^T$ where

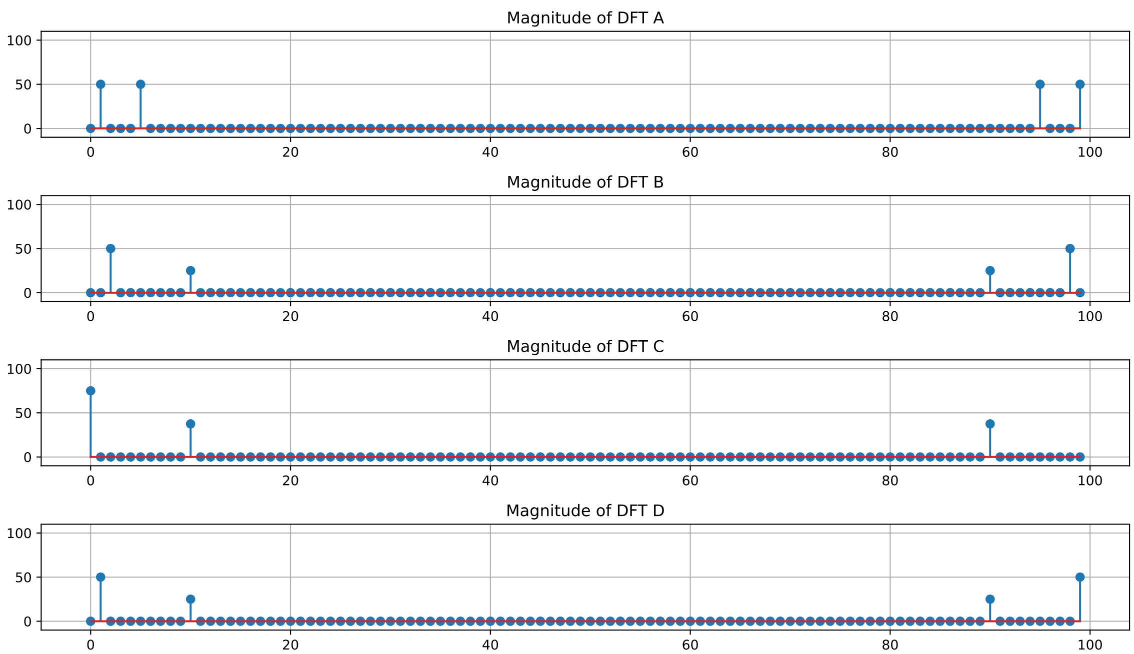
$$P = \begin{bmatrix} \boldsymbol{p}_1 & \boldsymbol{p}_2 & \boldsymbol{p}_3 & \boldsymbol{p}_4 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 & \boldsymbol{q}_3 & \boldsymbol{q}_4 \end{bmatrix}$$

Let $\boldsymbol{x} = \boldsymbol{q}_1 + \boldsymbol{q}_2 + \boldsymbol{q}_3 + \boldsymbol{q}_4$. Compute $\|A\boldsymbol{x}\|$.

8. (6 marks) Let $\boldsymbol{x} \in \mathbb{R}^{16}$ such that $\boldsymbol{y} = \mathrm{DFT}(\boldsymbol{x})$ where $\boldsymbol{y}[0] = 8$, $\boldsymbol{y}[4] = \boldsymbol{y}[12] = 4$, and all other entries of $\boldsymbol{y}$ are zero. Sketch the stemplot of $\boldsymbol{x}$.

9. (6 marks) Match the signal with its discrete Fourier transform.

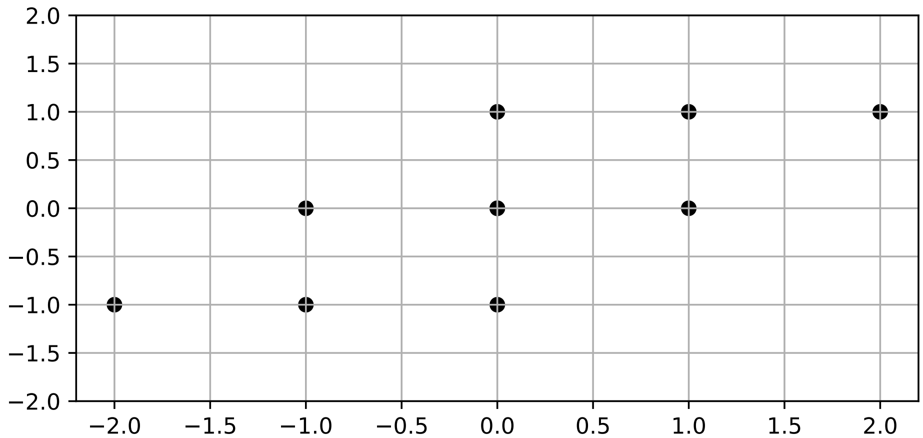




Signal 1 = DFT ________ Signal 2 = DFT ________

Signal 3 = DFT ________ Signal 4 = DFT ________

10. (7 marks) Find the weight vectors $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ for the dataset displayed below



Extra workspace

Extra workspace

Extra workspace