MATH 307 Practice Final Exam

6 December 2021

- No calculators, cellphones, laptops or notes
- Time allowed: 2 hours 30 minutes
- 60 total marks
- Write your name and student number in the space below

Name:

Student Number:

1. True or false questions. Each part is independent of the others.

(a) (3 marks) **True** or **False**: If A is an $n \times n$ singular matrix (i.e., not invertible), then 0 is an eigenvalue of A. Justify your answer.

since A is singular, $\det(A) = 0$

$\Rightarrow \det(A - 0 \cdot I) = 0$

Therefore 0 is an eigenvalue of A.

$\boxed{\text{True}}$

(b) (3 marks) **True** or **False**: If A is an $n \times n$ matrix and all n eigenvalues of A are equal, then A is a diagonal matrix. Justify your answer.

$\boxed{\text{False}}$ For example, let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

then eigenvalues are $\lambda_1 = \lambda_2 = 1$, however

A is not diagonal.

(c) (3 marks) **True** or **False**: If U and V are the two subspaces of $\mathbb{R}^4$ given by

$$U = \text{span}\left\{\begin{bmatrix}1\\0\\-1\\2\end{bmatrix}, \begin{bmatrix}3\\2\\1\\-1\end{bmatrix}\right\} \quad \text{and} \quad V = \text{span}\left\{\begin{bmatrix}0\\-1\\4\\2\end{bmatrix}\right\},$$

then $V = U^{\perp}$. Justify your answer.

$\boxed{\text{False}}$

Since $\dim\{U\} + \dim\{U^{\perp}\} = 4$, and $\dim\{U\} = 2$. However $\dim\{V\} = 1$, so V is not orthogonal complement of U.

(d) (3 marks) **True** or **False**: The matrix A is Hermitian when

(i) $A = \begin{bmatrix}1 & 3-i\\3+i & i\end{bmatrix}$.

(ii) $A = \begin{bmatrix}3 & 2-i & -3i\\2+i & 0 & 1-i\\3i & 1+i & 1\end{bmatrix}$.

Justify your answers.

We check if $\bar{A}^T = A$:

(i) $\bar{A}^T = \begin{pmatrix}1 & 3-i\\3+i & -i\end{pmatrix} \neq A \Rightarrow \boxed{\text{False}}$

(ii) $\bar{A}^T = \begin{pmatrix}3 & 2-i & -3i\\2+i & 0 & 1-i\\3i & 1+i & 1\end{pmatrix} = A \Rightarrow \boxed{\text{True}}$

2. Short answer questions. Each part is independent of the others.

(a) (3 marks) Let

$$A = \begin{bmatrix} 1 & 2 & -1 & -2 & 4 \\ 2 & -3 & 3 & 4 & -11 \\ 3 & 5 & -4 & -6 & 16 \end{bmatrix}.$$

(i) Compute the LU decomposition of A.

(ii) Determine $\dim(R(A))$ and $\dim(R(A^T))$.

(ii) $\dim(R(A)) = \text{Rank}(A)$
$= \text{Rank}(U) = 3$
$\dim(R(A^T)) = \dim(R(A))$
$= 3$

(i) $$\begin{pmatrix} 1 & 2 & -1 & -2 & 4 \\ 2 & -3 & 3 & 4 & -11 \\ 3 & 5 & -4 & -6 & 16 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 3R_1 \searrow E_2]{\nearrow E \; R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 2 & -1 & -2 & 4 \\ 0 & -7 & 5 & 8 & -19 \\ 0 & -1 & -1 & 0 & 4 \end{pmatrix} \xrightarrow{\nearrow E_3 \; R_3 \to R_3 - \frac{1}{7}R_2} \underbrace{\begin{pmatrix} 1 & 2 & -1 & -2 & 4 \\ 0 & -7 & 5 & 8 & -19 \\ 0 & 0 & -\frac{12}{7} & -\frac{8}{7} & \frac{47}{7} \end{pmatrix}}_{U}$$

$\Rightarrow E_3 E_2 E_1 A = U.$

where $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$, $E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{7} & 1 \end{pmatrix}$

$$\Rightarrow A = (E_1^{-1} E_2^{-1} E_3^{-1}) U = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{7} & 1 \end{pmatrix} \cdot U = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1/7 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 & -2 & 4 \\ 0 & -7 & 5 & 8 & -19 \\ 0 & 0 & -\frac{12}{7} & -\frac{8}{7} & \frac{47}{7} \end{pmatrix}$$

(b) (3 marks) Consider 10 data points $(t_0, y_0), \ldots, (t_9, y_9)$ such that $t_k - t_{k-1} = 1$ for each $k = 1, \ldots, 9$. Suppose the coefficient matrix of the corresponding natural cubic spline is given by

$$\begin{bmatrix} 2 & -1 & 1 & a & -1 & 0 & 3 & 1 & -1 \\ 0 & 6 & 3 & b & -6 & -9 & -9 & 0 & 3 \\ -5 & 1 & 10 & c & 19 & 4 & -14 & -23 & -20 \\ 3 & 0 & 6 & 20 & 41 & 53 & 48 & 28 & 6 \end{bmatrix}$$

Determine the missing values a, b and c.

$P_3''(t_3) = P_4''(t_3)$ yields : $6a_3 + 2b_3 = 2b_4 \Rightarrow b = b_4 = 3 \cdot 1 + 3 = 6$

$P_3'(t_3) = P_4'(t_3)$ yields : $3a_3 + 2b_3 + c_3 = c_4 \Rightarrow c = c_4 = 3 \cdot 1 + 2 \cdot 3 + 10 = 19$

$P_4'(t_4) = P_5'(t_4)$ yields: $3a_4 + 2b_4 + c_4 = c_5 \Rightarrow a = a_4 = \frac{19 - 19 - 2 \cdot 6}{3}$

$= -4$

(c) (3 marks) Determine whether $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\} = \text{span}\{\boldsymbol{u}_3, \boldsymbol{u}_4\}$ for

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ -5 \\ 4 \\ 5 \end{bmatrix}, \quad \boldsymbol{u}_3 = \begin{bmatrix} 2 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_4 = \begin{bmatrix} 4 \\ 0 \\ 1 \\ 5 \end{bmatrix}.$$

First see that u_1, u_2 are linearly independent (since $\underline{u}_1$ cannot be multiple of $\underline{u}_2$) so $\dim(\text{span}\{\underline{u}_1, \underline{u}_2\}) = 2$, also u_3, u_4 are linearly independent, so $\dim(\text{span}\{u_3, u_4\}) = 2$. Then we check $\dim(\text{span}\{u_1, u_2, u_3, u_4\})$ by reducing the following matrix:

$$\begin{pmatrix} 1 & 1 & 2 & 4 \\ -1 & -5 & 2 & 0 \\ 1 & 4 & -1 & 1 \\ 2 & 5 & 1 & 5 \end{pmatrix} \xrightarrow[\substack{R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 2R_1}]{R_2 \to R_2 + R_1} \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & -4 & 4 & 4 \\ 0 & 3 & -3 & -3 \\ 0 & 3 & -3 & -3 \end{pmatrix} \xrightarrow{R_2 \to \frac{R_2}{4}} \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & -1 & 1 & 1 \\ 0 & 3 & -3 & -3 \\ 0 & 3 & -3 & -3 \end{pmatrix} \xrightarrow[R_4 \to R_4 + 3R_2]{R_3 \to R_3 + 3R_2} \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \dim(\text{span}\{u_1, u_2, u_3, u_4\}) = 2 = \dim(\text{span}\{u_1, u_2\}) = \dim(\text{span}\{u_3, u_4\})$

Therefore $\text{span}\{u_1, u_2\} = \text{span}\{u_3, u_4\}$.

(d) (3 marks) Suppose A is a 3×3 matrix which depends on a parameter $c > 0$ such that the singular values of A are given by 2, 5 and c. Determine the minimum possible value for $\text{cond}(A)$ for all values of c.

Case 1: $0 < c \leq 2$

$\sigma_1 = 5, \quad \sigma_3 = c$

then $\text{cond}(A) = \frac{5}{c}$

Case 2: $2 < c \leq 5$

then $\sigma_1 = 5, \quad \sigma_3 = 2$, so $\text{cond}(A) = \frac{5}{2}$

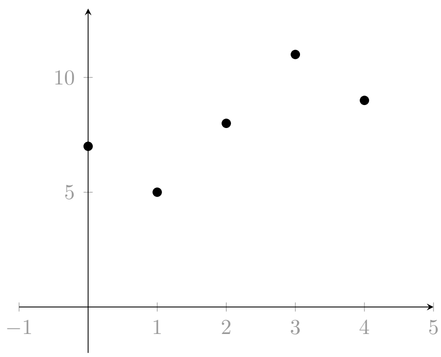
Case 3: $c > 5$

then $\sigma_1 = c, \quad \sigma_3 = 2 \Rightarrow \text{cond}(A) = \frac{c}{2}$

since $\frac{5}{2} < \frac{5}{c}$ when $c < 2$ and $\frac{5}{2} < \frac{c}{2}$ when $c > 5$

the minimum possible value for $\text{cond}(A)$ is $\frac{5}{2}$.

3. (6 marks) Use least squares linear regression to find the linear function $f(t) = c_0 + c_1 t$ that best fits the data points $(0, 7), (1, 5), (2, 8), (3, 11)$ and $(4, 9)$.



plug data points into $f(t) = c_0 + c_1 t$ yields:

$A\underline{x} = \underline{b}$, where $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}$, $\underline{x} = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$

$\underline{b} = \begin{pmatrix} 7 \\ 5 \\ 8 \\ 11 \\ 9 \end{pmatrix}$

we solve $A^T A \underline{x} = A^T \underline{b}$.

$$A^T A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 10 & 30 \end{pmatrix}$$

$$A^T \underline{b} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 7 \\ 5 \\ 8 \\ 11 \\ 9 \end{pmatrix} = \begin{pmatrix} 40 \\ 90 \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{cc|c} 5 & 10 & 40 \\ 10 & 30 & 90 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 5 & 10 & 40 \\ 0 & 10 & 10 \end{array}\right)$$

$\Rightarrow \quad c_1 = 1 \quad c_0 = 6$

Therefore $f(t) = 6 + t$

4. (6 marks) Let $A = \begin{bmatrix} -2 & 1 \\ 1 & -3 \\ 0 & 1 \end{bmatrix}$.

(a) Compute the thin QR decomposition of A.

(b) Compute the projection of $\boldsymbol{b}$ onto $R(A)$ for $\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

(a) Apply Gram - schmidt algorithm

let $u_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$,

then $u_2 = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} - \dfrac{\langle \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \rangle}{\langle \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \rangle} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$,

normalize u_1 u_2 to get Q_1:

$$Q_1 = \begin{pmatrix} -2/\sqrt{5} & -1/\sqrt{6} \\ 1/\sqrt{5} & -2/\sqrt{6} \\ 0 & 1/\sqrt{6} \end{pmatrix}$$

$$\langle q_1, a_1 \rangle = \langle \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \rangle = \sqrt{5}$$

$$R_1 = \begin{pmatrix} \sqrt{5} & -\sqrt{5} \\ 0 & \sqrt{6} \end{pmatrix}$$

$$\langle q_1, a_2 \rangle = \langle \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \rangle = -\sqrt{5}$$

$$\langle q_2, a_2 \rangle = \langle \begin{pmatrix} -1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \rangle = \sqrt{6}$$

$$\Rightarrow A = Q_1 R_1 = \begin{pmatrix} -2/\sqrt{5} & -1/\sqrt{6} \\ 1/\sqrt{5} & -2/\sqrt{6} \\ 0 & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} \sqrt{5} & -\sqrt{5} \\ 0 & \sqrt{6} \end{pmatrix}$$

(b) $$\operatorname{proj}_{R(A)}(\underline{b}) = Q_1 Q_1^T \underline{b} = \begin{pmatrix} -2/\sqrt{5} & -1/\sqrt{6} \\ 1/\sqrt{5} & -2/\sqrt{6} \\ 0 & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -2/\sqrt{5} & -1/\sqrt{6} \\ 1/\sqrt{5} & -2/\sqrt{6} \\ 0 & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 0 \\ -3/\sqrt{6} \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \\ -1/2 \end{pmatrix}$$

5. (6 marks) Consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$$

(a) Find the orthogonal projection matrix P which projects onto $R(A)$.

(b) Find the shortest distance from $\boldsymbol{x}$ to $R(A)$ where

$$\boldsymbol{x} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Apply Gram-Schmidt algorithm:

let $u_1 = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} - \dfrac{\left\langle \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix} \right\rangle} \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1/2 \\ 3/2 \end{pmatrix}$

$$P = \frac{1}{\|u_1\|^2} u_1 u_1^T + \frac{1}{\|u_2\|^2} u_2 u_2^T$$

$$= \frac{1}{6} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \end{pmatrix} + \frac{2}{9} \begin{pmatrix} 1 & -1 & 1/2 & 3/2 \\ -1 & 1 & -1/2 & -3/2 \\ 1/2 & -1/2 & 1/4 & 3/4 \\ 3/2 & -3/2 & 3/4 & 9/4 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 2 & -2 & 1 & 3 \\ -2 & 8 & 2 & 0 \\ 1 & 2 & 2 & 3 \\ 3 & 0 & 3 & 6 \end{pmatrix}$$

(b) Since $\begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix} \in R(A)$. so the shortest distance from $\underline{x}$ to $R(A)$ is 0.

6. (6 marks) Use the power method to approximate the dominant eigenvalue and a corresponding eigenvector of the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$$

choose $\underline{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\quad \underline{x}_1 = \dfrac{A\underline{x}_0}{\|A\underline{x}_0\|_\infty} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$

$A\begin{pmatrix} 1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7/2 \\ 3 \end{pmatrix}$, $\quad \underline{x}_2 = \dfrac{A\underline{x}_1}{\|A\underline{x}_1\|_\infty} = \begin{pmatrix} 1 \\ 6/7 \end{pmatrix}$

$A\begin{pmatrix} 1 \\ 6/7 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 6/7 \end{pmatrix} = \begin{pmatrix} 25/7 \\ 26/7 \end{pmatrix}$, $\quad \underline{x}_3 = \dfrac{A\underline{x}_2}{\|A\underline{x}_2\|_\infty} = \begin{pmatrix} 25/26 \\ 1 \end{pmatrix}$

$A\begin{pmatrix} 25/26 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 25/26 \\ 1 \end{pmatrix} = \begin{pmatrix} 103/26 \\ 102/26 \end{pmatrix}$ $\quad \underline{x}_4 = \dfrac{A\underline{x}_3}{\|A\underline{x}_3\|_\infty} = \begin{pmatrix} 1 \\ 102/103 \end{pmatrix}$

Therefore we get approximations:

dominant eigenvalue $\lambda_1 \approx 103/26 \approx 4$,

eigenvector $\underline{v} \approx \begin{pmatrix} 1 \\ 102/103 \end{pmatrix} \approx \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

7. (6 marks) Let U be the subspace of $\mathbb{R}^4$ spanned by $\boldsymbol{u_1} = (1,1,1,1)^T$ and $\boldsymbol{u_2} = (1,1,-1,1)^T$.

(a) Find the pseudo-inverse A^+ of

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

(b) Find the linear combination $c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2}$ which is nearest to $\boldsymbol{x} = (2,1,4,1)^T$.

(a) First find SVD of A : $A = P\Sigma Q$

$$A^TA = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

compute eigenvalues of A^TA :

$$\det\begin{pmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix} = (4-\lambda)^2 - 4 = 0 \Rightarrow \begin{matrix} \lambda_1 = 6 \\ \lambda_2 = 2 \end{matrix}$$

compute eigenvectors:

$$\lambda_1 = 6 : \quad (A - \lambda_1 I)\underline{v_1} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\underline{v_1} = \underline{0} \Rightarrow \underline{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 2 : \quad (A - \lambda_2 I)\underline{v_2} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\underline{v_2} = \underline{0} \Rightarrow v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Therefore $\Sigma = \begin{pmatrix} \sqrt{6} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

Now find column vectors $\underline{p_i}$ in P :

$$\underline{p_1} = \frac{1}{\sigma_1}A\underline{q_1} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \end{pmatrix}$$

$$\underline{p_2} = \frac{1}{\sigma_2}A\underline{q_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$A^+ = Q\Sigma^+P^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1/\sqrt{6} & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 0 \end{pmatrix}\begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 0 & 1/\sqrt{3} \\ 0 & 0 & 1 & 0 \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix}$$

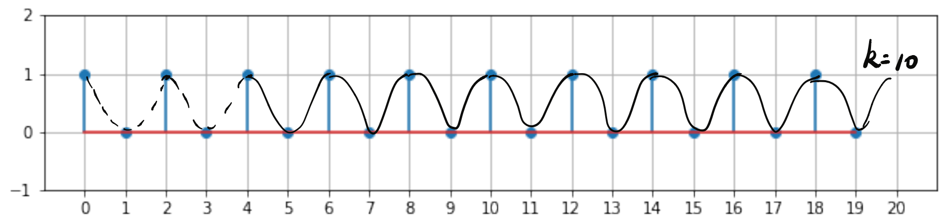
$$= \frac{1}{6}\begin{pmatrix} 1 & 1 & 3 & 1 \\ 1 & 1 & -3 & 1 \end{pmatrix}$$

(b) This can be considered a least square problem

$$A\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \cong \begin{pmatrix} 2 \\ 1 \\ 4 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = A^+\begin{pmatrix} 2 \\ 1 \\ 4 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 8 \\ -4 \end{pmatrix}$$

8. (6 marks) The stemplot of $\boldsymbol{x} \in \mathbb{C}^{20}$ is shown below:



(a) Find A_1, A_2, k_1 and k_2 such that

$$\boldsymbol{x} = A_1 \cos(2\pi k_1 \boldsymbol{t}) + A_2 \cos(2\pi k_2 \boldsymbol{t}).$$

(b) Compute DFT($\boldsymbol{x}$).

(a) From the picture, we can see $\underline{x}$ is a sinusoid with shift up, and frequency is 10.

so $k_1 = 0$, $k_2 = 10$. then

$$\underline{x} = A_1 + A_2 \cos(20\pi \underline{t})$$

The amplitude of the sinusoid is $A_2 = \frac{1-0}{2} = \frac{1}{2}$.

and $A_1 = \frac{1}{2}$.

(b) $$\text{DFT}(\underline{x}) = \text{DFT}(A_1) + \text{DFT}(A_2 \cos(20\pi \underline{t}))$$

$$= (A_1 N \underline{e}_0) + \left(\frac{A_2 N}{2} \underline{e}_{10} + \frac{A_2 N}{2} \underline{e}_{20-10}\right)$$

$$= 10\, \underline{e}_0 + 10\, \underline{e}_{10}$$

Extra workspace.

Extra workspace. Do not write in the table below.

Q1	/12
Q2	/12
Q3	/6
Q4	/6
Q5	/6
Q6	/6
Q7	/6
Q8	/6
Total	/60