MATH 307 Midterm Exam 1

June 2, 2022

- No calculators, cellphones, laptops or notes
- Time allowed: 50 minutes
- 45 total marks
- Write your name and student number in the space below
- Notation: $N(A)$ is the nullspace of A and $R(A)$ is the range of A

Name:

Student Number:

1. Determine if the statement is **True** or **False**. No justification required.

   (a) (2 marks) If $A^3 = 0$ then $R(A) \subseteq N(A)$.

   (b) (2 marks) Let A and B be $m \times n$ matrices. Then the set

   $$U = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = B\boldsymbol{x}\}$$

   is a subspace of $\mathbb{R}^n$.

   (c) (2 marks) If $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is a basis of a subspace $U \subseteq \mathbb{R}^n$ then

   $$\{\ \boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3\ ,\ \boldsymbol{u}_1 + \boldsymbol{u}_3\ ,\ \boldsymbol{u}_1 - \boldsymbol{u}_2 + \boldsymbol{u}_3\ \}$$

   is also a basis of U.

   (d) (2 marks) Suppose A_1 and A_2 are $m \times n$ matrices such that $A_1 = LU_1$ and $A_2 = LU_2$. In other words, the unit lower triangular matrix L is the same in both LU decompositions (however the upper triangular matrices U_1 and U_2 are different). If $\text{rank}(A_1) \leq \text{rank}(A_2)$ then $R(A_1) \subseteq R(A_2)$.

2. Short answer questions. Give a brief justification. Parts (a) and (b) are independent.

   (a) (3 marks) Let A be a $n \times n$ matrix, and let $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$ such that $\|A\boldsymbol{x}_1\| = 5$, $\|A\boldsymbol{x}_2\| = 15$, $\|\boldsymbol{x}_1\| = 2$ and $\|\boldsymbol{x}_2\| = 1$. Find a value $C > 1$ such that $C \leq \mathrm{cond}(A)$.

   (b) (3 marks) Consider $N+1$ points $(t_0, y_0), \ldots, (t_N, y_N)$. Suppose we want to construct an interpolating function $p(t)$ defined piecewise by N functions $p_1(t), \ldots, p_N(t)$ such that each $p_k(t)$ is defined on the interval $[t_{k-1}, t_k]$. Determine the number of equations the functions $p_1(t), \ldots, p_N(t)$ must satisfy to guarantee that $p(t)$ interpolates the data and $p'(t)$, $p''(t)$ and $p'''(t)$ are continuous.

3. (5 marks) Find the unique function of the form

$$p(t) = a + bt^2 + ct^3$$

such that $p(-1) = 1$, $p'(1) = 0$ and $p''(2) = -1$.

4. Suppose $A = LU$ where L is a unit lower triangular matrix and

$$U = \begin{bmatrix} 2 & 1 & 0 & -3 & -2 & 3 \\ 0 & 0 & 1 & -1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Let $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5, \boldsymbol{a}_6 \in \mathbb{R}^4$ be the columns of A

$$A = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 & \boldsymbol{a}_4 & \boldsymbol{a}_5 & \boldsymbol{a}_6 \end{bmatrix}$$

Determine if the statement is **True** or **False**. No justification required.

(a) (2 marks) The dimension of $\text{span}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\}$ is 2.

(b) (2 marks) The vectors $\boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5$ are linearly independent.

(c) (2 marks) $\dim(N(A)) = 3$.

(d) (2 marks) The set $\{\boldsymbol{a}_2, \boldsymbol{a}_4, \boldsymbol{a}_5, \boldsymbol{a}_6\}$ is a basis of $R(A)$.

5. (5 marks) Suppose $p(t)$ is a natural cubic spline with coefficient matrix

$$C = \begin{bmatrix} 2 & -3 & -2 & 2 & 0 & 1 \\ 0 & 6 & -3 & -9 & -3 & -3 \\ -1 & 5 & c_3 & c_4 & -16 & -22 \\ 1 & 2 & d_3 & d_4 & 2 & -17 \end{bmatrix}$$

such that $p(t)$ interpolates $(t_0, y_0), \ldots, (t_6, y_6)$ where

$$t_0 = 0 \ , \ t_1 = 1 \ , \ t_2 = 2 \ , \ t_3 = 3 \ , \ t_4 = 4 \ , \ t_5 = 5 \ , \ t_6 = 6$$

and

$$p_k(t) = a_k(t - t_{k-1})^3 + b_k(t - t_{k-1})^2 + c_k(t - t_{k-1}) + d_k$$

Determine the coefficients c_3, d_3, c_4, d_4.

6. Consider the matrix

$$A = \begin{bmatrix} -1 & 1 & 0 & -6 & 2 \\ -1 & -1 & 3 & -8 & 3 \\ 3 & 3 & -9 & 29 & -6 \\ 0 & 12 & -18 & 7 & -9 \end{bmatrix}$$

(a) (5 marks) Find the LU decomposition of A.

(b) (3 marks) Find a basis for $R(A)$.

7. (5 marks) Let $a, b \in \mathbb{R}$ such that $0 < a < b$. Find $\boldsymbol{c}$ such that $A\boldsymbol{c} = \boldsymbol{y}$ where

$$A = \begin{bmatrix} 1 & -b & b^2 & -b^3 & b^4 \\ 1 & -a & a^2 & -a^3 & a^4 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & a & a^2 & a^3 & a^4 \\ 1 & b & b^2 & b^3 & b^4 \end{bmatrix} \qquad \boldsymbol{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Hints: (1) A is a Vandermonde matrix; (2) it is possible to determine $\boldsymbol{c}$ without using Gaussian elimination to solve the system.

Extra workspace. Do not write in the table below.

Q1	/8
Q2	/6
Q3	/5
Q4	/8
Q5	/5
Q6	/8
Q7	/5
Total	/45