Math 307: 202 — Midterm 1 — 50 minutes

Last Name Nguyen First Minh Khoi (Kevin)

Student Number 97703045 Signature Kl...

- The test consists of 8 pages and 4 questions worth a total of 50 marks.
- This is a closed-book examination. **None of the following are allowed:** documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- No work on this page will be marked.

Please do not write on this page — it will not be marked.

Additional instructions

- Please use the spaces indicated.
- **Unless it is specified not to do so, justify your answers.**
- If you require extra paper then put up your hand and ask your instructor.
  - You must put your name and student number on any extra pages.
  - You must indicate the test-number and question-number.
  - Please do this **on both sides** of any extra pages.
- Please do not dismember your test. You must submit all pages.

1	8	8
2	13	24
3	10	10
4	4	8
Total	35	50

1. $\boxed{\text{8 marks}}$ Let $A = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 1 \\ 2 & 6 & 6 & 1 \\ 0 & -3 & 6 & 4 \end{pmatrix}$.

(a) Compute the LU decomposition of A.

$$\xrightarrow{R_3 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 1 \\ 0 & 2 & 0 & -3 \\ 0 & -3 & 6 & 4 \end{bmatrix} \xrightarrow[R_4 - 3R_2]{R_3 + 2R_2} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Compute $\det(A)$.

$$\det(A) = \det(U)$$
$$= (1)(-1)(4)(1)$$
$$= -4$$

2. $\boxed{\text{24 marks}}$ Short answer questions. **For True or False questions**, if true, provide a short justification. If false, show a counter-example that contradicts the statement. **For other questions**, justify your answer by showing your work.

3 (a) Let $A = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Calculate (no part marks):

(i) Cond(A), i.e., the condition number of A

condition # is not affected by row swap (cond(PA) = cond (A))

$PA = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix} \rightarrow$ $\|A\| = 5$ (max stretch

$\|A^{-1}\| = 1/1 = 1$

$\therefore$ cond $(A) = \|A\| \, \|A^{-1}\| = 5$

(ii) $\|Ax\|_2$

$Ax = \begin{bmatrix} 0+1+0 \\ -2+0+0 \\ 0+0-5 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} \Rightarrow \|Ax\|_2 = \sqrt{1+4+25}$

$= \sqrt{30}$

(iii) $\|Ax\|_1$

$\|Ax\|_1 = 1+2+5 = 8$

0 (b) **True or False:** Let A be an $n \times n$ matrix with LU decomposition $A = LU$. Then rank(A) = rank(L).

True

1 (c) **Find** a 2×2 **diagonal** matrix A with $\text{cond}(A) = \|A^2\| = 2$; or **explain why such a matrix cannot exist.**

$A = \begin{bmatrix} 1 & \\ & 1/2 \end{bmatrix}$

$\begin{bmatrix} a & \\ & b \end{bmatrix}$ $A^2 = \begin{bmatrix} a^2 & \\ & b^2 \end{bmatrix}$

$\therefore \|A^2\| = a^2$

$\text{cond}(A) \cancel{=} a \times b$

$a \times b = a^2 = 2$

$\therefore b = a = \sqrt{2}$

$$\boxed{\begin{array}{l} A = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \\ \|A\| = \sqrt{2} \quad \|A^{-1}\| = \sqrt{2} \\ \therefore \text{cond}(A) = 2 \\ A^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \therefore \|A^2\| = 2 \end{array}}$$

0 (d) **True or False:** Let U and V be two subspaces of $\mathbb{R}^n$. Their union (that is, the set of vectors belonging to at least one of U and V) is also a subspace of $\mathbb{R}^n$.

✗ True, we can simply use all of the vectors that were originally in U, that itself will make a subspace. Any additional vectors are irrelevant

0 (e) **True or False:** Let A and B be $n \times n$ matrices. Then $\mathcal{R}(BA) = \mathcal{R}(A)$.

X True. If we treat BA as linear combinations, for example we have

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} [a_1 | a_2 | a_3] = [a_1 \cdot r_1 | a_2 \cdot r_2 | a_3 \cdot r_3]$$

→ All of the resulting elements are linear combinations of the columns of A

3 (f) **True or False:** Let $A \in \mathbb{R}^{n \times n}$ be invertible and let $\mathbf{x} \in \mathbb{R}^n$. Then $||A\mathbf{x}|| \geq ||A^{-1}||^{-1} \cdot ||\mathbf{x}||$.

$\|Ax\|$ = magnitude of x after A is applied (after it's been stretched by A)

$\|A^{-1}\|^{-1}$ = min stretch of A

$\therefore \|A^{-1}\|^{-1} \cdot \|x\|$ = length of x after being stretch by the minimum stretch of A

by definition, of minimum, we cannot stretch a vector by a factor of less than its minimum stretch factor, it'll be a least equal

→ so it's True

$\|Ax\| \geq$ min stretch $(A) \times \|x\|$

3 (g) **True or False:.** Let $A, B \in \mathbb{R}^{n\times n}$ both be invertible. Then $||AB|| \leq ||A|| \cdot ||B||$.

True. This is by property of the operator norm

3 (h) **True or False:** Let $A \in \mathbb{R}^{n\times n}$ be invertible. If $||A|| = ||A^{-1}|| = cond(A) = 1$, then $A = I$.

$cond(A) = ||A|| \, ||A^{-1}||$

$= ||A||^2 = 1 \quad$ sin $||A|| = ||A^{-1}||$

$||A|| = \pm 1$

$\rightarrow$ false. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is a counter example

CamScanner

3. $\boxed{\text{10 marks}}$ Let

$$A = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 2 & 5 & 3 \\ 1 & 0 & -1 & 2 \\ 1 & 4 & 9 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

2 (a) Find the dimensions of $N(A)$ and $R(A)$.

$\dim(R(A)) = \text{rank}(A) = 2$ ✓

$\dim(N(A)) = n - \text{rank}(A) = 2$ ✓

4 (b) Find a basis for $R(A)$.

→ first 2 col of L

✓

$R(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

4 (c) Find a basis for $N(A)$.

→ note: $N(A) = N(U)$ → let $x = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ and set $Ux = 0$

$c = t \quad d = s$

$2b + 5c + 3d = 0$

$2b = -5c - 3d$

$b = -\frac{5}{2}t - \frac{3}{2}s$

$\therefore x = \begin{bmatrix} 1 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} -2 \\ -3/2 \\ 0 \\ 1 \end{bmatrix} s$

$a + 2b + 4c + 5d = 0$

$a = -2\left(-\frac{5}{2}t - \frac{3}{2}s\right) - 4(t) - 5s$

$= 5t + 3s - 4t - 5s$

$= t - 2s$

$\therefore$ basis of $N(A)$ is $\left\{ \begin{bmatrix} 1 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3/2 \\ 0 \\ 1 \end{bmatrix} \right\}$ ✓

4. $\boxed{\text{8 marks}}$ **Parts (a) and (b) are independent.**

4 (a) Find a polynomial $p(t)$ of degree at most 3 such that

$$p(0) = p(1) = 0, p'(0) = -2, p'(1) = 3.$$

$p(t) = at^3 + bt^2 + ct + d$

$p(0) = p(1) \Rightarrow d = a + b + c + d \Rightarrow d = 0$

$p'(t) = 3at^2 + 2bt + c$

$p'(0) = c = -2$

$p'(1) = 3a + 2b + c = 3$

$\therefore a + b = -c = 2 \Rightarrow b = 1$

$3a + 2b = 5 \qquad a = 1$

$\therefore p(t) = t^3 + t^2 - 2t$ ✓

0 (b) Given $d + 1$ points $(t_0, y_0), \ldots, (t_d, y_d)$ such that $t_i \neq t_j$ for $i \neq j$. Does there exist a polynomial of degree at most $d + 2$ which interpolates the data? If such a polynomial exists, is it unique? Justify both of your answers.

No, we have too many equations $(d+2)$ compared to variables $(d+1)$

→ whether there is a unique solution or no solution (inconsistent) depends on the y_i's ×