Math 307-201
Practice Final Exam Problems

These practice final exam questions should be done without your notes if possible. We will meet at 1pm, Thursday April 15 at the classroom to go over the solutions.

Problem 1 Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & -3 \end{pmatrix}.$$

(i) Put A into the form $LPDU$, where P is a permutation matrix, D is diagonal, L is lower triangular unipotent, and U is upper triangular unipotent.

(ii) Write A as a product of elementary matrices.

(iii) Find the determinant of A using (i) or (ii).

Problem 2 Find an orthonormal basis of $\mathbb{R}^3$ that consists of eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

and use it to express A in the form QDQ^T.

Problem 3 Let Q be an orthogonal $n \times n$ real matrix.

(i) Verify that $||Qx||^2 = ||x||^2$.

(ii) Use (i) to show that complex eigenvalues λ of Q satisfy $|\lambda| = 1$.

(iii) Suppose all complex eigen-values of Q are 1. True or false: $Q = I_n$.

Problem 4 Let A and M be $n \times n$ real matrices such that M is invertible.

(i) Prove that A and MAM^{-1} have the same eigen-values.

(ii) True or false: AM and MA have the same eigenvalues. Explain.

(iii) True or false: If A is diagonalizable, then MAM^T is also diagonalizable. Explain.

(iv) True or false: If the determinant of A is nonzero, then A^TA is positive definite.

Problem 5 Let A be a real symmetric matrix such that $(1, 1, 1)^T$ is an eigenvector for eigenvalue 1, $(1, -2, 1)^T$ is an eigenvector for eigenvalue -1, and $\det(A) = 2$. Find A.

Problem 6. Let V be a vector space over a field $\mathbb{F}$, and suppose W is a subspace of V. Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be a basis of W. Suppose $\mathbf{v}$ is a vector in V that isn't in W. Show that $\mathbf{v}_1, \ldots, \mathbf{v}_k$ and $\mathbf{v}$ are linearly independent.

Problem 7 Let $\mathbb{F} = \mathbb{F}_2$ be the field with 2 elements. Let W be the row space of

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

(i) Compute the dimension of W.

(ii) How many vectors does W contain?

(iii) Find a basis of W.

Problem 8 Consider the matrix

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

(i) Find the inverse of M as a matrix over the reals.

(ii) Find the inverse of M as a matrix over $\mathbb{F}_2$.

Problem 9 Let W be the subspace of $\mathbb{R}^4$ spanned by $\mathbf{a}_1 = (1, 1, 1, 1)^T$ and $\mathbf{a}_2 = (1, -1, 1, 1)^T$.

(i) Find the matrix of the projection of $\mathbb{R}^4$ onto W.

(ii) Find the pseudo-inverse A^+ of

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

(iii) Find the linear combination $r_1\mathbf{a}_1 + r_2\mathbf{a}_2$ which is nearest to $(2, 1, 3, 1)^T$.

Problem 10 Which of the following statements are always true, which are sometimes true and which are always false? Discuss your reasoning.

(i) For any square matrices A and B over $\mathbb{R}$, $e^{A+B} = e^A e^B$.

(ii) Every unitary matrix is diagonalizable.

(iii) The eigenvalues of a unitary matrix are real.

(iv) If A is skew symmetric, e^A is orthogonal.

(iv) If A is skew Hermitian, e^A is unitary.

Problem 11 Consider the symmetric matrix A of Problem 1. Sylvester's Law of Inertia says that if A is a real symmetric matrix and M is an invertible real matrix, and both are $n \times n$, then the eigenvalues of A and MAM^T have the same signs. Use this fact to discover the number of positive eigenvalues of A. (Suggestion: begin by replacing A by PAP^T for a well chosen permutation matrix P.)