MATH 307 Final Exam

December 14, 2021

- No calculators, cellphones, laptops or notes
- Time allowed: 2 hours 30 minutes
- 75 total marks
- Write your name and student number in the space below
- Notation:
  - A^T is the *transpose* of the matrix A
  - $R(A)$ is the *range* of the matrix A (also called the *column space*)
  - $N(A)$ is the *nullspace* of the matrix A
  - $U^\perp$ is the *orthogonal complement* of the subspace $U \subseteq \mathbb{R}^n$
  - $\omega_N = e^{2\pi i/N}$

Name:

Student Number:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Total
10	12	10	7	8	10	5	6	7	75

1. Determine whether the statement is **True** or **False**. No justification required.

(a) (2 marks) The matrix

$$\begin{bmatrix} 5 & -1 & -1 & 2 & 1 \\ 1 & 3 & 3 & -1 & 2 \\ -4 & 5 & 4 & 0 & 0 \\ 3 & 2 & 6 & 8 & -7 \end{bmatrix}$$

is the coefficient matrix of a natural cubic spline for 6 data points $(t_0, y_0), \ldots, (t_5, y_5)$ such that $t_k - t_{k-1} = 1$ for each $k = 1, \ldots, 5$.

(b) (2 marks) If A is a complex hermitian matrix then the diagonal entries of A are real numbers.

(c) (2 marks) Let X be a (normalized) data matrix, let $\boldsymbol{x}$ be a row of X, let $\boldsymbol{w}_1$ be the first weight vector of X and let $\boldsymbol{w}_2$ be the second weight vector of X. If $\langle \boldsymbol{x}, \boldsymbol{w}_1 \rangle \neq 0$ then $|\langle \boldsymbol{x}, \boldsymbol{w}_2 \rangle| < \|\boldsymbol{x}\|$.

(d) (2 marks) If $\mathrm{DFT}(\boldsymbol{x}) = \mathrm{DFT}(\boldsymbol{y})$ then $\boldsymbol{x} = \boldsymbol{y}$.

(e) (2 marks) Let U_1 and U_2 be subspaces of $\mathbb{R}^n$. Let P_1 be the projection onto U_1 and let P_2 be the projection onto U_2. If $P_1 P_2 = 0$ then U_1 and U_2 are orthogonal subspaces.

2. Short answer questions. Each part is independent of the others. **Justify your answers.**

   (a) (3 marks) Determine all values c such that the matrix

$$A = \begin{bmatrix} 3 & c \\ -1 & 5 \end{bmatrix}$$

   is orthogonally diagonalizable. In other words, find all possible values c such that there exists a diagonal matrix D and orthogonal matrix P such that $A = PDP^T$.

   (b) (3 marks) Let A be a 3×3 matrix (*not* a diagonal matrix) such that the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 1/2$ and $\lambda_3 = -6$. Let $\boldsymbol{x}_0 \in \mathbb{R}^3$ be a random nonzero vector and let $\boldsymbol{x}_k = A^{-k}\boldsymbol{x}_0$. Determine the (most likely) value c such that

$$\frac{\langle \boldsymbol{x}_k, \boldsymbol{x}_{k+1} \rangle}{\langle \boldsymbol{x}_k, \boldsymbol{x}_k \rangle} \to c \quad \text{as} \quad k \to \infty$$

(c) (3 marks) Determine all values k such that $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & k \end{bmatrix} \qquad \boldsymbol{b} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

(d) Compute DFT($\boldsymbol{x}$) where $\boldsymbol{x} = 5\cos(4\pi\boldsymbol{t} + \pi/2) \in \mathbb{C}^8$. Recall, the vector $\boldsymbol{t}$ is

$$\boldsymbol{t} = \begin{bmatrix} 0 & 1/8 & 1/4 & 3/8 & 1/2 & 5/8 & 3/4 & 7/8 \end{bmatrix}^T$$

3. Consider the matrix

$$A = \begin{bmatrix} -1 & -2 & -2 & -1 \\ 4 & 10 & 11 & 2 \\ -3 & -2 & 0 & -7 \end{bmatrix}$$

(a) (5 marks) Compute the LU decomposition of A.

(b) (2 marks) Determine the dimension of $N(A^T)$.

(c) (3 marks) Find a basis of $N(A)$.

4. (7 marks) Find the shortest distance from $\boldsymbol{x}$ to $U = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\} \subseteq \mathbb{R}^4$ where

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

5. Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

(a) (5 marks) Compute the *thin* QR decomposition $A = Q_1 R_1$.

(b) (3 marks) Use the thin QR decomposition to find the least squares approximation $A\boldsymbol{x} \cong \boldsymbol{b}$ for the vector

$$\boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

6. Consider the matrix

$$A = \frac{1}{2}\begin{bmatrix} 1 & 2 & 3 & 1 \\ -1 & 2 & -1 & -3 \\ -1 & 2 & 1 & 3 \\ 1 & 2 & -3 & -1 \end{bmatrix}$$

(a) (5 marks) Compute the condition number of A. (Hint: consider $A^T A$ not AA^T.)

(b) (5 marks) Find a *unit* vector $\boldsymbol{x}$ such that $\|A\boldsymbol{x}\| = \|A\|$.

7. (5 marks) Use at least 3 iterations of the power method to approximate the dominant eigenvalue and corresponding eigenvector of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

8. (6 marks) Let $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbb{R}^3$ such that $\|\boldsymbol{u}_1\| = a > 0$, $\|\boldsymbol{u}_2\| = b > 0$, and $\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle = 0$. Let $\boldsymbol{u}_3 \in \mathbb{R}^3$ such that $\|\boldsymbol{u}_3\| = 1$ and $\langle \boldsymbol{u}_1, \boldsymbol{u}_3 \rangle = \langle \boldsymbol{u}_2, \boldsymbol{u}_3 \rangle = 0$. Determine the eigenvalues and eigenvectors of the matrix

$$A = \boldsymbol{u}_1 \boldsymbol{u}_1^T + \boldsymbol{u}_2 \boldsymbol{u}_2^T$$

9. (7 marks) Let $\boldsymbol{y} \in \mathbb{C}^8$ such that

$$\boldsymbol{y} = \begin{bmatrix} 1 \\ 0 \\ 2-2i \\ 1+i\sqrt{3} \\ 0 \\ 1-i\sqrt{3} \\ 2+2i \\ 0 \end{bmatrix}$$

Find values A_0, A_1, A_2, k_1, k_2, ϕ_1, ϕ_2 such that $\boldsymbol{y} = \mathrm{DFT}(\boldsymbol{x})$ where $\boldsymbol{x}$ is of the form

$$\boldsymbol{x} = A_0 + A_1 \cos(2\pi k_1 \boldsymbol{t} + \phi_1) + A_2 \cos(2\pi k_2 \boldsymbol{t} + \phi_2) \quad , \quad k_1 < k_2$$

Recall $\boldsymbol{t} \in \mathbb{C}^N$ is the vector

$$\boldsymbol{t} = \begin{bmatrix} 0 \\ 1/N \\ 2/N \\ \vdots \\ (N-1)/N \end{bmatrix}$$