MATH 307 Midterm Exam 2

November 17, 2021

- No calculators, cellphones, laptops or notes
- Time allowed: 45 minutes
- 35 total marks
- Write your name and student number in the space below

Name: Solutions.

Student Number:

1. Short answer questions. Each part is independent of the others.

(a) (3 marks) **True** or **False**: If A is an $n \times n$ symmetric matrix such that $A^2 = I$, then A is orthogonal. Justify your answer.

$$\left.\begin{array}{l} A \text{ is symmetric} \Rightarrow A = A^T \\ A^2 = I \Rightarrow A = A^{-1} \end{array}\right\} \Rightarrow A^{-1} = A^T$$

$\Rightarrow A A^T = A^T A = I$

$\Rightarrow$ A is orthogonal $\boxed{\text{True}}$

(b) (3 marks) Let $a, b \in \mathbb{R}$ such that $a \neq b$ and consider the matrix

$$A = \begin{bmatrix} a & b & a & b & a & b \\ b & a & b & a & b & a \\ a & b & a & b & a & b \\ b & a & b & a & b & a \\ a & b & a & b & a & b \end{bmatrix}.$$

Determine the dimension of $N(A)^\perp$. Justify your answer.

rank(A) = 2 since there are only 2 linearly independent rows. ~~$\Rightarrow R(A) \neq R(A^T)$~~.

$\Rightarrow \dim(R(A)) = \dim(R(A^T)) = 2.$

$\Rightarrow \dim(N(A)^\perp) = \dim(R(A^T)) = 2.$

or $\dim(N(A)) = 6 - 2 = 4$

and $\dim(N(A)^\perp) = 6 - 4 = 2$

(c) (3 marks) Suppose A is a 4×4 matrix with characteristic polynomial

$$c_A(x) = (x^2 + x - 2)(x^2 - x - 2)$$

Is A diagonalizable? Justify your answer.

$x^2 + x - 2 = (x+2)(x-1) \qquad x^2 - x - 2 = (x-2)(x+1)$

$\Rightarrow$ A has 4 distinct eigenvalues

$\Rightarrow$ A has 4 linearly independent eigenvectors

$\Rightarrow$ A is diagonalizable

(d) (3 marks) **True** or **False**: Suppose $A = P\Sigma Q^T$ is the singular value decomposition of A such that Q is a permutation matrix. Then the columns of A are orthogonal. (Recall that a permutation matrix is any matrix obtained from the identity matrix by permuting the rows.) Justify your answer.

Q is a permutation matrix $\Rightarrow$ eigenvectors of A^TA are standard basis: $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$.

$\Rightarrow$ A^TA is a diagonal matrix

$\Rightarrow$ Columns of A^TA are orthogonal $\boxed{\text{True}}$

2. (6 marks) Find the orthogonal projection matrix P which projects onto $U = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ where

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Find $\underline{u}_4 \in U^\perp \Rightarrow \underline{u}_4 \in N(A^T)$ for $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 0 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \underline{u}_4 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$P_\perp = \frac{1}{\langle \underline{u}_4, \underline{u}_4 \rangle} \underline{u}_4 \underline{u}_4^T = \frac{1}{2} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow P = I - P_\perp = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{P = \begin{bmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix}}$$

3. (6 marks) Consider the matrix

$$A = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$$

(a) (4 marks) Find matrices P and D such that $A = PDP^{-1}$.

(b) (2 marks) Compute the limit

$$\lim_{k\to\infty} \lambda_1^{-k} A^k$$

where λ_1 is the largest eigenvalue (in absolute value). In other words, if λ_1 and λ_2 are the eigenvalues of A then $|\lambda_1| > |\lambda_2|$.

(a) $c_A(x) = x^2 - 2x \cancel{+} - 24 = (x-6)(x+4)$ $\boxed{\lambda_1 = 6 \quad \lambda_2 = -4}$

$\lambda_1 = 6 \quad (A - 6I)\underline{v}_1 = \underline{0} \Rightarrow \left[\begin{array}{cc|c} -5 & 5 & 0 \\ 5 & -5 & 0 \end{array}\right] \Rightarrow \underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\langle \underline{v}_2, \underline{v}_1 \rangle = 0 \Rightarrow \underline{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ since A is symmetric.

$$\Rightarrow A = \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix}}_{D} \underbrace{\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right)^T}_{P^T}.$$

(b) $$\lambda_1^{-1} A = \frac{1}{6}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 6 & \\ & -4 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

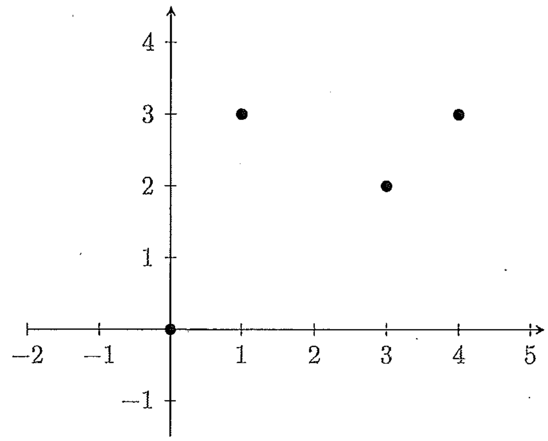
$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & \\ & (-2/3) \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\Rightarrow \lambda_1^{-k} A^k = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & \\ & (-2/3)^k \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\to \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{as } k \to \infty$$

$$\Rightarrow \lambda_1^{-k} A^k \to \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{as } k \to \infty.$$

4. (5 marks) Use least squares linear regression to find the linear function $f(t) = c_0 + c_1 t$ that best fits the data points $(0,0), (1,3), (3,2)$ and $(4,3)$.



$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \qquad \underline{y} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}$$

$$A^T \underline{y} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 21 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c} 4 & 8 & 8 \\ 8 & 26 & 21 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 10 & 5 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 1 & 1/2 \end{array}\right]$$

$$\Rightarrow c_1 = 1/2 \qquad c_0 = 2 - 2(1/2) = 1.$$

$$\boxed{\Rightarrow f(t) = 1 + \tfrac{1}{2} t}$$

5. (6 marks) Suppose $A = Q_1R_1$ where

$$Q_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \qquad R_1 = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Compute the projection of $\boldsymbol{v}$ onto $R(A)^\perp$ for

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$$

$$R(A) = R(Q_1) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}.$$

$$R(A)^\perp = N(A^T) = N(Q_1^T)$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow \underline{u} \in N(Q_1^T) \quad \underline{u} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow R(A)^\perp = \text{span}\{\underline{u}\}.$$

$$\Rightarrow \text{proj}_{\underline{u}}(\underline{v}) = \frac{\langle \underline{v}, \underline{u} \rangle}{\langle \underline{u}, \underline{u} \rangle}\underline{u} = \frac{-1+2+1+3}{4}\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\boxed{\text{proj}_{\underline{u}}(\underline{v}) = \frac{5}{4}\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}}$$

Extra workspace. Do not write in the table below.

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Q1	/12
Q2	/6
Q3	/6
Q4	/5
Q5	/6
Total	/35