MATH 307 Midterm Exam 1

October 13, 2021

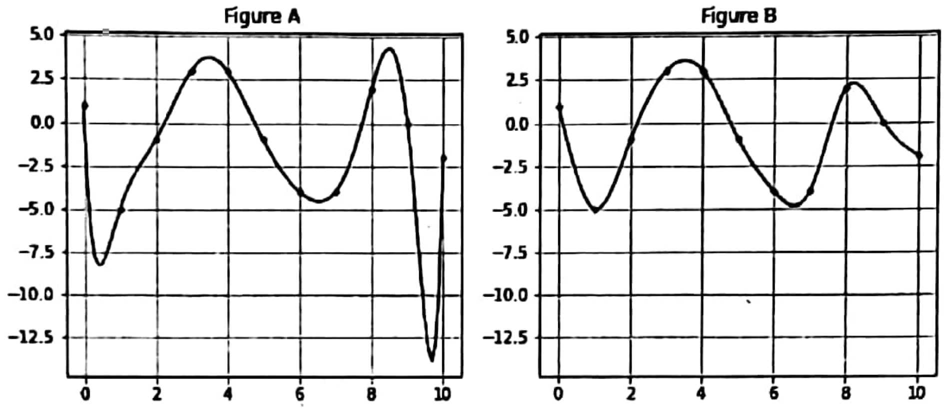
- No calculators, cellphones, laptops or notes
- Time allowed: 45 minutes
- 35 total marks
- Write your name and student number in the space below

Name: SOLUTIONS

Student Number:

1. Short answer questions. Each part is independent of the others.

   (a) (3 marks) The figures show different interpolating functions for the same dataset:



Determine which figure corresponds to polynomial interpolation and which corresponds to cubic spline interpolation. Justify your answer.

Figure A $\Rightarrow$ Polynomial interpolation
Figure B $\Rightarrow$ Cubic Spline interpolation

because we expect cubic to be smoother and better fit the data.

   (b) (3 marks) **True or False:** If A is an invertible $n \times n$ matrix such that $\|A\boldsymbol{x}\| \leq \|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \mathbb{R}^n$, then $\|A^{-1}\| \geq 1$. Justify your answer.

$\boxed{\text{True}}$ $\quad \|A^{-1}\| = \dfrac{1}{\min\limits_{\|\underline{x}\|=1} \|A\underline{x}\|}$ and we know

$\|A\underline{x}\| \leq 1$ for all $\|\underline{x}\| = 1$ therefore

$\min\limits_{\|\underline{x}\|=1} \|A\underline{x}\| \leq 1$ and so $\|A^{-1}\| \geq 1$.

Scanned with CamScanner

(c) (3 marks) Determine whether or not the set

$$U = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 : abc = 0 \right\}$$

is a subspace of $\mathbb{R}^3$. Justify your answer.

Not a subspace since $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \in U$

however $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \notin U$.

(d) (3 marks) Consider the boundary value problem

$$y'' = \cos(\pi t^2) \ , \ y(0) = 0 \ , \ y(2) = 0$$

If we know that $y(0.9) \approx 0.018$ and $y(1.1) \approx 0.023$, find an approximation of $y(1)$.

$$y''(t) \approx \frac{y(t+h) - 2y(t) + y(t-h)}{h^2} \qquad t = 1 \quad h = 0.1$$

$$\Rightarrow y''(1) \approx \frac{y(1.1) - 2y(1) + y(0.9)}{0.1^2} \quad \text{and} \quad y''(1) = \cos(\pi) = -1$$

$$\Rightarrow y(1) \approx \frac{0.023 + 0.018 + 0.01}{2}$$

$$\boxed{y(1) \approx 0.0255}$$

2. Let $p(t)$ be the natural cubic spline which interpolates the data

$$(0,1)\ ,\ (1,3)\ ,\ (2,8)\ ,\ (3,10)\ ,\ (4,9)\ ,\ (5,-1)\ ,\ (6,-17)$$

Suppose the coefficient matrix of $p(t)$ is

$$\begin{bmatrix} 1 & -2 & 1 & a_4 & 1 & 1 \\ 0 & 3 & -3 & b_4 & -6 & -3 \\ 1 & 4 & 4 & c_4 & -5 & -14 \\ 1 & 3 & 8 & 10 & 9 & -1 \end{bmatrix}$$

(a) (4 marks) Determine the coefficients a_4, b_4, c_4.

(b) (2 mark) Determine the value $p''(2.5)$.

(a) $p_3''(t_3) = p_4''(t_3) \Rightarrow 3a_3 + b_3 = b_4$

$3(1) + (-3) = b_4$

$\Rightarrow \boxed{b_4 = 0}$

$p_3'(t_3) = p_4'(t_3) \Rightarrow 3a_3 + 2b_3 + c_3 = c_4$

$3(1) + 2(-3) + 4 = c_4$

$\Rightarrow \boxed{c_4 = 1}$

$p_4(t_4) = p_5(t_4) \Rightarrow a_4 + b_4 + c_4 + d_4 = d_5$

$a_4 + 0 + 1 + 10 = 9$

$\Rightarrow \boxed{a_4 = -2}$

(b) $p''(2.5) = p_3''(2.5) = 6a_3(2.5-2) + 2b_3$

$= 6(1)(\frac{1}{2}) + 2(-3)$

$\boxed{p''(2.5) = -3}$

3. Consider the matrix

$$A = \begin{bmatrix} -3 & 1 & 2 & 0 \\ 3 & 1 & -2 & 1 \\ -6 & 2 & 5 & 1 \\ -9 & 3 & 4 & 2 \end{bmatrix}$$

(a) (4 marks) Find the LU decomposition of A.

(b) (2 mark) Compute $\det(A)$.

(a) $A = \begin{bmatrix} -3 & 1 & 2 & 0 \\ 3 & 1 & -2 & 1 \\ -6 & 2 & 5 & 1 \\ -9 & 3 & 4 & 2 \end{bmatrix} \xrightarrow[\substack{R_3-2R_1 \\ R_4-3R_1}]{R_2+R_1} \begin{bmatrix} -3 & 1 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 2 \end{bmatrix}$

$\xrightarrow[R_4+2R_3]{} \begin{bmatrix} -3 & 1 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

$\Rightarrow A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$

(b) $\det(A) = \det(U) = (-3)(2)(1)(4)$

$\Rightarrow \boxed{\det(A) = -24}$

4. (6 marks) Determine whether span$\{u_1, u_2\}$ = span$\{u_3, u_4\}$ where

$$u_1 = \begin{bmatrix} 2 \\ -3 \\ 1 \\ -1 \end{bmatrix} \quad u_2 = \begin{bmatrix} -5 \\ 1 \\ 2 \\ -2 \end{bmatrix} \quad u_3 = \begin{bmatrix} -1 \\ -5 \\ 4 \\ -4 \end{bmatrix} \quad u_4 = \begin{bmatrix} 3 \\ -11 \\ 6 \\ -10 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -5 & -1 & 3 \\ -3 & 1 & -5 & -11 \\ 1 & 2 & 4 & 6 \\ -1 & -2 & -4 & -10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 4 & 6 \\ 0 & -9 & -9 & -9 \\ 0 & 7 & 7 & 7 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 4 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \{\underline{u}_1, \underline{u}_2, \underline{u}_4\}$$

are linearly independent

therefore span$\{\underline{u}_1, \underline{u}_2\} \neq$ span$\{\underline{u}_3, \underline{u}_4\}$.

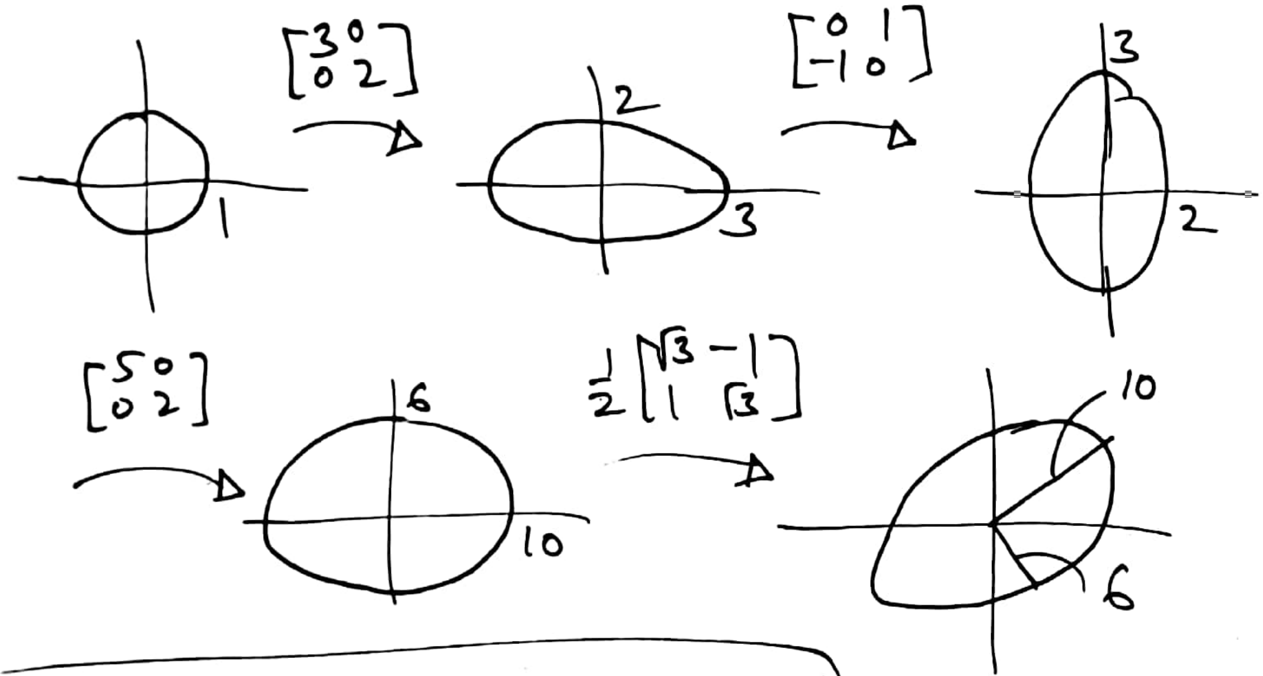
5. (5 marks) Determine $\|A\|$ for the matrix

$$A = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

Hint: counterclockwise rotation in $\mathbb{R}^2$ by angle θ corresponds to matrix multiplication by

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Sketch the image of the unit circle under the linear transformation A:



$$\boxed{\Rightarrow \quad \|A\| = \max_{\|x\|=1} \|Ax\| = 10}$$