Math 307: 201 — Midterm 1 — 50 minutes

Last Name ______________________ **First** ______________________

Student Number ______________________ **Signature** ______________________

- The test consists of 12 pages and 4 questions worth a total of 50 marks.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- No work on this page will be marked.

Please do not write on this page — it will not be marked.

Additional instructions

- Please use the spaces indicated.
- **Unless it is specified not to do so, justify your answers.**
- If you require extra paper then put up your hand and ask your instructor.
  - You must put your name and student number on any extra pages.
  - You must indicate the test-number and question-number.
  - Please do this **on both sides** of any extra pages.
- Please do not dismember your test. You must submit all pages.

1		8
2		24
3		10
4		8
Total		50

1. $\boxed{\text{8 marks}}$ Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 3 & 3 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$.

(a) Find a vector $\mathbf{x} \in \mathbb{R}^4$ such that $A\mathbf{x} = \mathbf{b}$.

Solution: Denote the two matrices by L and U respectively.
First we find $\mathbf{y} \in \mathbb{R}^4$ s.t. $L\mathbf{y} = \mathbf{b}$. Let $\mathbf{y} = (y_1, y_2, y_3, y_4)^t$.
Then $y_1 = 1$,
and $y_1 + y_2 = 2$, which gives $y_2 = 1$,
and $2y_1 + y_3 = 3$, which gives $y_3 = 1$,
and $2y_1 + 2y_2 - y_3 + y_4 = 4$, which gives $y_4 = 1$.
So $\mathbf{y} = (1, 1, 1, 1)^t$.
Now we find $\mathbf{x} \in \mathbb{R}^4$ s.t. $U\mathbf{x} = \mathbf{y}$. Let $\mathbf{x} = (x_1, x_2, x_3, x_4)^t$.
Then $x_4 = 1$,
and $-2x_3 + x_4 = 1$, which gives $x_3 = 0$,
and $-x_2 + 3x_3 + 2x_4 = 1$, which gives $x_2 = 1$,
and $2x_1 + 2x_2 + 3x_3 + 3x_4 = 1$, which gives $x_1 = -2$.
So $\mathbf{x} = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$.

(b) Find $\det(A)$.

Solution: $\det(A) = \det(L)\det(U) = \det(U) = 2 \cdot (-1) \cdot (-2) \cdot 1 = 4.$

2. $\boxed{\text{24 marks}}$ Short answer questions, each question 3 marks. **For True or False questions**, if true, provide a short justification. If false, show a counter-example that contradicts the statement. For other questions, justify your answer by showing your work.

(a) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 3 & 0 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Calculate (no part marks):

(i) $\|A\|$, i.e. the operator norm of A

(ii) $\|A\|_{\mathrm{FR}}$, i.e. the Frobenius norm of A

(iii) $\|Ax\|_2$

(b) **True or False:** Let A be an $n \times n$ matrix with LU decomposition $A = LU$. Then $U\boldsymbol{x} = \boldsymbol{y}$ has a unique solution for every $\boldsymbol{y} \in \mathbb{R}^n$.

(c) **Find** a 2×2 **diagonal** matrix A with $\|A\| = 2$ and $\|A\|_F = 1$, where $\| \cdot \|_F$ is the Frobenius matrix norm; or **explain why such a matrix cannot exist**.

(d) **True or False:** Let U and V be two subspaces of $\mathbb{R}^n$. Their intersection (that is, the set of vectors belonging to both U and V) is also a subspace of $\mathbb{R}^n$.

(e) **True or False:** Let A and B be $n \times n$ matrices. Then $\mathcal{N}(A) = \mathcal{N}(BA)$.

(f) **True or False:** Let $A \in \mathbb{R}^{n \times n}$ be invertible. Then $||A^2|| = ||A||^2$.

(g) **True or False:** Let $A \in \mathbb{R}^{n\times n}$ be invertible. Then $||A|| \cdot ||A^{-1}|| \geq 1$.

(h) **True or False:** Let $A, B \in \mathbb{R}^{n\times n}$ be invertible. Then $cond(A+B) \leq cond(A) + cond(B)$.

Solution:

(a) (i) For any $y = (y_1, y_2, y_3)^t$, since $||Ay|| = ||(y_1, 2y_3, 3y_2)|| = ||(y_1, 3y_2, 2y_3)^t|| = ||diag(1,3,2)y||$, so $||A|| = ||diag(1,3,2)|| = 3$.
(ii) $||A||_{FR} = \sqrt{1^2+2^2+3^2} = \sqrt{14}$.
(iii) $||A\mathbf{x}||_2 = ||(1,2,3)^t||_2 = \sqrt{1^2+2^2+3^2} = \sqrt{14}$.

(b) False. When $A = 0$, $\mathbf{y} = (1, \ldots, 1)^t$, we have $L = I$ and $U = 0$, so there's no solution.

(c) It doesn't exist. Let $A = diag(a, b)$. Then $||A|| = \max(|a|, |b|)$ and $||A||_{FR} = \sqrt{a^2+b^2} \geq \max(|a|,|b|) = ||A||$. So it's not possible that $||A||_{FR} = 1$ but $||A|| = 2$.

(d) True. All conditions of a subspace remain true in the intersection:

1. Since U and V are subspaces, $\mathbf{0} \in U$ and $\mathbf{0} \in V$. So $\mathbf{0} \in U \cap V$.
2. Let $\mathbf{x}_1, \mathbf{x}_2 \in U \cap V$. Then, $\mathbf{x}_1, \mathbf{x}_2 \in U$ and $\mathbf{x}_1, \mathbf{x}_2 \in V$. Since U and V are subspaces, $\mathbf{x}_1 + \mathbf{x}_2 \in U$ and $\mathbf{x}_1 + \mathbf{x}_2 \in V$. So $\mathbf{x}_1 + \mathbf{x}_2 \in U \cap V$.

3. Let $\mathbf{x} \in U \cap V$ and $c \in \mathbb{R}$. Then, $\mathbf{x} \in U$ and $\mathbf{x} \in V$. Since U and V are subspaces, $c\mathbf{x} \in U$ and $c\mathbf{x} \in V$. So $c\mathbf{x} \in U \cap V$.

(e) False. Let $n = 2$. Consider

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ (the identity), and } B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ (the zero matrix).}$$

Then, $\mathcal{N}(A) = \{\mathbf{0}\}$ and $\mathcal{N}(BA) = \mathbb{R}^2$.

(f) False. Let $A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, then $||A|| = 2$, $A^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $||A^2|| = 2 \neq ||A||^2$.

(g) True, because

$$||A|| \cdot ||A^{-1}|| = \frac{\max_{||x||=1} ||Ax||}{\min_{||x||=1} ||Ax||} \geq 1.$$

(h) False. Let $A = \begin{pmatrix} -0.9 & 0 \\ 0 & -0.99 \end{pmatrix}$, $B = I$, then $A + B = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.01 \end{pmatrix}$, so $cond(A + B) = \frac{0.1}{0.01} = 10$. But $cond(A) + cond(B) = \frac{0.99}{0.9} + 1 < 10$.

3. $\boxed{\text{10 marks}}$ Let

$$A = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 2 & 5 & 3 \\ 1 & 0 & -1 & 2 \\ 1 & 4 & 9 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find the dimensions of $N(A)$ and $R(A)$.

Solution: Let

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & 2 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since L is non-singular, $\dim N(A) = \dim N(U) = 2$ and $\dim R(A) = \dim R(U) = 2$.

(b) Find a basis for $R(A)$.

Solution: The first $\operatorname{rank} A$ columns of L form a basis for $R(A)$. Since $\operatorname{rank} A = \dim R(A) = 2$,

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

is a basis for $R(A)$.

Alternative solution: Since the pivots of A are in the first 2 columns, the first two columns of A also form a basis for $R(A)$.

(c) Find a basis for $N(A)$.

Solution: By theorem, $N(A) = N(U)$, so we want to compute $N(U)$. By definition,

$$N(U) = \left\{ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : \begin{matrix} x_1 + 2x_2 + 4x_3 + 5x_4 = 0, \text{ and} \\ 2x_2 + 5x_3 + 3x_4 = 0 \end{matrix} \right\}.$$

We know $\dim N(U) = 2$, so we want to write $N(U)$ using two parameters $t, s \in \mathbb{R}$. Set $x_3 = t$ and $x_4 = s$. Solving for x_1 and x_2 gives

$$x_1 = t - 2s \text{ and } x_2 = -\frac{5}{2}t - \frac{3}{2}s.$$

So

$$\mathbf{x} = \begin{bmatrix} t - 2s \\ -\frac{5}{2}t - \frac{3}{2}s \\ t \\ s \end{bmatrix} = t \begin{bmatrix} 1 \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}$$

where the two vectors on the right hand side forms a basis for $N(U)$.

4. $\boxed{\text{8 marks}}$ Suppose we have 4 points $(0,2),(1,3),(2,2),(3,5)$ and we want to interpolate the data using a polynomial

$$p(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4$$

of degree at most 4 such that $p^{(4)}(0) = 0$.
(Note: $p^{(4)}(0)$ is the fourth derivative evaluated at 0) .

(a) Setup (but do **not** solve) a linear system $A\mathbf{x} = \mathbf{b}$ where the solution is

$$\mathbf{x} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}.$$

Solution: The condition of $p(t)$ interpolating the 4 points is equivalent to the following 4 equations:

$$\begin{aligned} p(0) = 2 &\iff c_0 = 2 \\ p(1) = 3 &\iff c_0 + c_1 + c_2 + c_3 + c_4 = 3 \\ p(2) = 2 &\iff c_0 + 2c_1 + 2^2 c_2 + 2^3 c_3 + 2^4 c_4 = 2 \\ p(3) = 5 &\iff c_0 + 3c_1 + 3^2 c_2 + 3^3 c_3 + 3^4 c_4 = 5. \end{aligned}$$

The final condition $p^{(4)}(0) = 0$ is equivalent the following equation:

$$p^{(4)}(0) = 0 \iff 24c_4 = 0.$$

Writing in matrix form, the linear system is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 & 2^4 \\ 1 & 3 & 3^2 & 3^3 & 3^4 \\ 0 & 0 & 0 & 0 & 24 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 5 \\ 0 \end{bmatrix}.$$

(b) Does there exist a polynomial of degree at most 4 satisfying the above conditions? If it exists, is it unique? Justify both of your answers.

Solution: Yes, and it is unique. Notice the condition $24c_4 = 0$ forces $c_4 = 0$. So the $c_0, \ldots, c_3$ are given by the following simplified linear system:

$$\begin{aligned} c_0 &= 2 \\ c_0 + c_1 + c_2 + c_3 &= 3 \\ c_0 + 2c_1 + 2^2c_2 + 2^3c_3 &= 2 \\ c_0 + 3c_1 + 3^2c_2 + 3^3c_3 &= 5 \end{aligned}$$

Writing in matrix form, the simplified linear system is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 5 \end{bmatrix}.$$

The matrix in this system is a Vandermonde matrix, so non-singular. Hence, there exists a unique solution for $c_0, \ldots, c_4$. i.e., $c_4 = 0$ and $c_0, \ldots, c_3$ are the unique solution to the above linear system.

Alternative Solution 1: You can deduce the matrix from part (a) is non-singular using any method. i.e., the determinant is non-zero, row reduce and show it is has rank 5, notice the row/column vectors are linearly independent, etc. In any case, this implies there is a unique solution to the linear system in part (a).

Alternative Solution 2: (Inspired by a student solution.) The condition $24c_4 = 0$ forces $c_4 = 0$. So $p(t)$ is a polynomial of degree at most 3. By theorem, there exists a unique polynomial of degree at most 3 which interpolates 4 points. This gives a unique polynomial of degree at most 4 interpolating the 4 points with $p^{(4)}(0) = 0$.