MATH 307 Midterm Exam 2

November 18, 2021

- No calculators, cellphones, laptops or notes
- Time allowed: 45 minutes
- 35 total marks
- Write your name and student number in the space below

Name: Solutions.

Student Number:

1. Short answer questions. Each part is independent of the others.

 (a) (3 marks) Let A be a 3×3 matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and corresponding eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

 True or **False**: A is a symmetric matrix. Justify your answer.

$\underline{v}_1, \underline{v}_2, \underline{v}_3$ are orthogonal: $\langle \underline{v}_1, \underline{v}_2 \rangle = \langle \underline{v}_1, \underline{v}_3 \rangle = \langle \underline{v}_2, \underline{v}_3 \rangle = 0$

$\Rightarrow A = PDP^{-1} = [\underline{v}_1\ \underline{v}_2\ \underline{v}_3] \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} [\underline{v}_1\ \underline{v}_2\ \underline{v}_3]^{-1}$

Normalize the vectors to create $P = \left[\frac{\underline{v}_1}{\|\underline{v}_1\|}\ \frac{\underline{v}_2}{\|\underline{v}_2\|}\ \frac{\underline{v}_3}{\|\underline{v}_3\|} \right]$

then $P^{-1} = P^T$ and $A = PDP^T \Rightarrow A = A^T$. $\boxed{\text{True}}$

 (b) (3 marks) Consider a matrix A with LU decomposition $A = LU$ where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

 Determine the dimension of $N(A)^\perp$.

There are 3 nonzero rows in $U \Rightarrow \text{rank}(A) = 3$

$\Rightarrow \dim(N(A)) = 5 - 3 = 2$

$\Rightarrow \dim(N(A)^\perp) = 5 - 2 = \underline{\underline{3}}$

(c) (3 marks) Determine the values of a, b and c such that

$$Q = \begin{bmatrix} 1/\sqrt{18} & a & 2/3 \\ 1/\sqrt{18} & 1/\sqrt{2} & b \\ -4/\sqrt{18} & 0 & c \end{bmatrix}$$

is an orthogonal matrix.

Columns are <u>orthonormal</u>

$$\Rightarrow \left\langle \begin{bmatrix} 1/\sqrt{18} \\ 1/\sqrt{18} \\ -4\sqrt{18} \end{bmatrix}, \begin{bmatrix} a \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \right\rangle = 0 \Rightarrow \boxed{a = -1/\sqrt{2}}$$

$$\left\langle \frac{1}{\sqrt{18}}\begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 2/3 \\ b \\ c \end{bmatrix} \right\rangle = 0 \Rightarrow \frac{2}{3} + b - 4c = 0$$

$$\left\langle \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 2/3 \\ b \\ c \end{bmatrix} \right\rangle = 0 \Rightarrow -\frac{2}{3} + b = 0$$

$$\Rightarrow \boxed{\begin{aligned} b &= 2/3 \\ c &= 1/3 \end{aligned}}$$

(d) (3 marks) **True** or **False**: Suppose $A = P\Sigma Q^T$ is the singular value decomposition of A such that Q is a permutation matrix. Then the columns of A are orthogonal. (Recall that a permutation matrix is any matrix obtained from the identity matrix by permuting the rows.) Justify your answer.

Q is a permutation matrix $\Rightarrow$ eigenvectors of A^TA are standard basis: $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$.

$\Rightarrow$ A^TA is a diagonal matrix

$\Rightarrow$ Columns of A^TA are orthogonal $\boxed{\text{True}}$

2. (6 marks) Consider the matrix $A = Q_1 R_1$ where

$$Q_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \qquad R_1 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Compute the orthogonal projection of $\boldsymbol{v}$ onto $N(A^T)$ where

$$\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

$$N(A^T) = R(A)^\perp = R(Q_1)^\perp = N(Q_1^T)$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\underline{u} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow N(A^T) = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right\}$$

$$\text{proj}_{\underline{u}}(\underline{v}) = \frac{\left\langle \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\rangle} \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \boxed{\frac{3}{4}\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}}$$

3. (5 marks) Suppose $A = Q_1 R_1$ where

$$Q_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \qquad R_1 = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Find the least squares approximation $A\boldsymbol{x} \approx \boldsymbol{b}$ for

$$\boldsymbol{b} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$A\underline{x} \approx \underline{b} \Rightarrow R\underline{x} = \underline{c}_1$ where $Q^T \underline{b} = \begin{bmatrix} \underline{c}_1 \\ \underline{c}_2 \end{bmatrix}$

$\Rightarrow \underline{c}_1 = Q_1^T \underline{b} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$

$R\underline{x} = \underline{c}_1 \Rightarrow \left[\begin{array}{ccc|c} -1 & 1 & 2 & 3/2 \\ 0 & -2 & -1 & 3/2 \\ 0 & 0 & 1 & 3/2 \end{array}\right]$

$x_3 = 3/2$

$x_2 = (3/2 + 3/2)/(-2)$

$= -3/2$

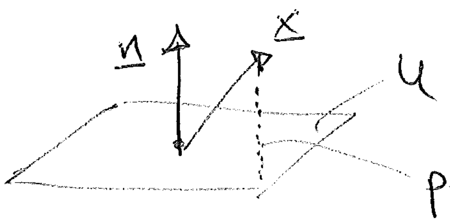
$x_1 = (3/2 - 2(3/2) - (-3/2))/(-1)$

$= 0$

$\Rightarrow \boxed{\underline{x} = \begin{bmatrix} 0 \\ -3/2 \\ 3/2 \end{bmatrix}}$

4. (6 marks) Find the shortest distance from $\boldsymbol{x} = \begin{bmatrix} 5 \\ 3 \\ 7 \end{bmatrix}$ to the plane in $\mathbb{R}^3$ given by

$$U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : 2x - y + 3z = 0 \right\}.$$



$$\underline{n} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$\|\text{proj}_{\underline{n}}(\underline{x})\| = \frac{|\langle \underline{n}, \underline{x} \rangle|}{|\langle \underline{n}, \underline{n} \rangle|} \|\underline{n}\| = \frac{|\langle \underline{n}, \underline{x} \rangle|}{\|\underline{n}\|}.$$

$$= \frac{5(2) + 3(-1) + 7(3)}{\sqrt{2^2 + (-1)^2 + 3^2}}$$

$$\boxed{= \frac{28}{\sqrt{14}} = 2\sqrt{14}}$$

5. (6 marks) Consider the matrix

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}.$$

(a) (4 marks) Find matrices P and D such that $A = PDP^{-1}$.

(b) (2 marks) Compute the limit

$$\lim_{k\to\infty} \lambda_1^{-k} A^k$$

where λ_1 is the largest eigenvalue (in absolute value). In other words, if λ_1 and λ_2 are the eigenvalues of A then $|\lambda_1| > |\lambda_2|$.

(a) $C_A(x) = x^2 - 7x + 6 = (x-1)(x-6)$ $\boxed{\lambda_1 = 6 \quad \lambda_2 = 1}$

$\lambda_1 = 6$ $(A - \lambda_1 I)\underline{v}_1 = \underline{0}$ $\left[\begin{array}{cc|c} -4 & -2 & 0 \\ -2 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow$ $\underline{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\lambda_2 = 1$ $(A - \lambda_2 I)\underline{v}_2 = \underline{0}$ $\left[\begin{array}{cc|c} 1 & -2 & 0 \\ -2 & 4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow$ $\underline{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\langle \underline{v}_1, \underline{v}_2 \rangle = 0$ ✓

$\Rightarrow$ $A = \underbrace{\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}^T}_{P^T}$

(b) $\lambda_1^{-k} A^k = \lambda_1^{-k}(PDP^T)^k = \lambda_1^{-k} P D^k P^T = P \begin{bmatrix} 1 & \\ & (1/6)^k \end{bmatrix} P^T$

$\Rightarrow$ $\lim\limits_{k\to\infty} \lambda_1^{-k} A^k = P\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^T = \underline{\underline{\frac{1}{5}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}}}$

Extra workspace

Extra workspace. Do not write in the table below.

Q1	/12
Q2	/6
Q3	/5
Q4	/6
Q5	/6
Total	/35