MATH 307 Midterm Exam 2

June 16, 2022

- No calculators, cellphones, laptops or notes
- Time allowed: 50 minutes
- 45 total marks
- Write your name and student number in the space below
- Notation:
  - $N(A)$ is the nullspace of A and $R(A)$ is the range of A
  - $U^{\perp}$ is the orthogonal complement of a subspace U

Name: SOLUTIONS

Student Number:

1. Determine if the statement is **True** or **False**. No justification required.

   (a) (2 marks) If A is a symmetric matrix then $N(A)^\perp = R(A)$.

   $\boxed{\text{TRUE}}$ $N(A)^\perp = R(A^T) = R(A)$ since $A = A^T$.

   (b) (2 marks) Let $U \subset \mathbb{R}^5$ be a subspace such that $\dim(U) = 2$. There exists a subspace $V \subset \mathbb{R}^5$ such that $\dim(V) = 4$ and V is orthogonal to U.

   $\boxed{\text{FALSE}}$ $\dim(U) + \dim(U^\perp) = 5$

   $\Rightarrow$ The largest subspace orthogonal to U is $U^\perp$ with $\dim(U^\perp) = 3$.

   (c) (2 marks) Let A be an $m \times n$ matrix. Let $A = QR$ be the QR decomposition

   $$Q = [\, Q_1 \; Q_2 \,] \qquad R = \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$$

   where $A = Q_1 R_1$ is the thin QR decomposition. The projection of $x \in \mathbb{R}^m$ onto $R(A)^\perp$ is equal to $Q_2 Q_2^T x$. (Assume $\mathrm{rank}(A) = n$.)

   $\boxed{\text{TRUE}}$ Columns of Q are orthonormal basis of $\mathbb{R}^m$.

   Columns of Q_1 are orthonormal basis of $R(A)$.

   Columns of Q_1 are orthogonal to columns of Q_2.

   Columns of Q_2 are orthonormal basis of $R(A)^\perp$.

   $\Rightarrow \mathrm{proj}_{R(A)^\perp}(x) = Q_2 Q_2^T x$.

   (d) (2 marks) Let P be the projection matrix onto a subspace $U \subset \mathbb{R}^6$ with $\dim(U) = 4$. Then the rank of the matrix $I - P$ is 4.

   $\boxed{\text{FALSE}}$ $\mathrm{rank}(P) = 4$

   $P_\perp = I - P$ is projection onto $U^\perp$

   $\Rightarrow \mathrm{rank}(P_\perp) = \dim(U^\perp) = 6 - 4 = 2$

2. Short answer questions. Give a brief justification. Parts (a) and (b) are independent.

(a) (3 marks) If P is a 5×5 projection matrix such that $\text{rank}(P) = 2$ then determine the dimension of the nullspace $N(P)$.

Let $U = R(P)$. Then $N(P) = U^\perp$ and so

$$\dim(N(P)) = \dim(U^\perp) = 5 - \dim(U) = 5 - 2 = \boxed{3}$$

(b) (4 marks) Suppose A is a $m \times n$ matrix with $m > n$ such that $\det(A^T A) \neq 0$. Determine the algebraic multiplicity of the eigenvalue $\lambda = 0$ for AA^T.

If $\det(A^TA) \neq 0$ then ~~[illegible]~~ all n eigenvalues of A^TA are nonzero. A^TA is $n \times n$. AA^T is $m \times m$. A^TA and AA^T have same set of nonzero eigenvalues. Therefore AA^T has eigenvalue 0 with multiplicity $\boxed{m-n}$.

3. (5 marks) Find the projection matrix P which projects onto $U = \operatorname{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ where

$$\boldsymbol{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} \qquad \boldsymbol{u}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$\dim U = 3 \quad \Rightarrow \quad \dim U^{\perp} = 4 - 3 = 1.$

Find $P_{\perp}$ then compute $P = I - P_{\perp}$.

$U^{\perp} = N(A)$ where $A = \begin{bmatrix} -1 & 1 & 1 & 2 \\ 1 & 0 & 2 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

$$A \rightarrow \begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$x_4 = 0 \qquad x_2 = -0 - 3t = -3t$

$x_3 = t \qquad x_1 = -(-2(0) - t + 3t)$

$= -2t$

$\Rightarrow N(A) = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix} \right\}.$

$$P_{\perp} = \frac{1}{14} \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -2 & -3 & 1 & 0 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 4 & 6 & -2 & 0 \\ 6 & 9 & -3 & 0 \\ -2 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{P = I - P_{\perp} = \frac{1}{14} \begin{bmatrix} 10 & -6 & 2 & 0 \\ -6 & 5 & 3 & 0 \\ 2 & 3 & 13 & 0 \\ 0 & 0 & 0 & 14 \end{bmatrix}}$$

4. (5 marks) Find the <u>thin</u> QR decomposition of

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$\underline{v}_1 = \underline{a}_1 \qquad \underline{v}_2 = \underline{a}_2 - \cancel{\frac{\langle \underline{a}_2, \underline{v}_1 \rangle}{\langle \underline{v}_1, \underline{v}_1 \rangle}}^{0} \underline{v}_1 = \underline{a}_2$

$\underline{v}_3 = \underline{a}_3 - \frac{\langle \underline{a}_3, \underline{v}_1 \rangle}{\langle \underline{v}_1, \underline{v}_1 \rangle} \underline{v}_1 - \cancel{\frac{\langle \underline{a}_3, \underline{v}_2 \rangle}{\langle \underline{v}_2, \underline{v}_2 \rangle}}^{0} \underline{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \frac{-1}{4} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \underline{0}$

$= \frac{1}{4} \begin{bmatrix} 3 \\ 1 \\ -3 \\ 5 \end{bmatrix}$

$\Rightarrow Q_1 = \begin{bmatrix} -1/2 & 1/\sqrt{6} & 3/\sqrt{44} \\ 1/2 & 2/\sqrt{6} & 1/\sqrt{44} \\ 1/2 & 0 & -3/\sqrt{44} \\ 1/2 & -1/\sqrt{6} & 5/\sqrt{44} \end{bmatrix}$

$R_1 = \begin{bmatrix} \langle \underline{w}_1, \underline{a}_1 \rangle & \langle \underline{w}_1, \underline{a}_2 \rangle & \langle \underline{w}_1, \underline{a}_3 \rangle \\ 0 & \langle \underline{w}_2, \underline{a}_2 \rangle & \langle \underline{w}_2, \underline{a}_3 \rangle \\ 0 & 0 & \langle \underline{w}_3, \underline{a}_3 \rangle \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1/2 \\ 0 & \sqrt{6} & 0 \\ 0 & 0 & 11/\sqrt{44} \end{bmatrix}$

$\Rightarrow \underline{\underline{A = Q_1 R_1}}$

$11/\sqrt{44} = \frac{\sqrt{11}}{2}.$

5. (5 marks) Determine the dimension of $R(A)^{\perp}$ where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \end{bmatrix}$$

$$A \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & -2 & -4 & -6 & -8 \\ 0 & -3 & -6 & -9 & -12 \\ 0 & -4 & -8 & -12 & -16 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \operatorname{rank}(A) = 2 \quad \Rightarrow \dim\left(R(A)^{\perp}\right) = 5 - 2 = \boxed{3}$$

6. (5 marks) Find the singular values of the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \\ -2 & -1 & 0 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 2 & -1 & 1 & -2 \\ 0 & 1 & -1 & -1 \\ 1 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \\ -2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 3 \end{bmatrix}$$

$$C_A(x) = (x-10)\left((x-3)^2-4\right) = (x-10)(x^2-6x+5)$$
$$= (x-10)(x-5)(x-1)$$

$$\Rightarrow \boxed{\sigma_1 = \sqrt{10} \quad \sigma_2 = \sqrt{5} \quad \sigma_3 = 1}$$

7. (5 marks) Suppose $A = QR$ is the QR decomposition of A where

$$Q = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 2\sqrt{2} & -\sqrt{2} \\ 0 & -2\sqrt{2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Determine the shortest distance from $\boldsymbol{x}$ to $R(A)$ where

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \end{bmatrix}$$

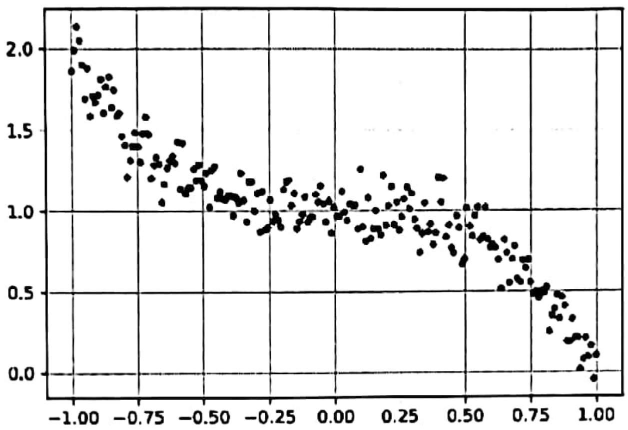
$$\|\text{proj}_{R(A)^\perp}(\underline{x})\| = \|\langle \underline{x}, \underline{q}_3\rangle \underline{q}_3 + \langle \underline{x}, \underline{q}_4\rangle \underline{q}_4\|$$

$$= \sqrt{|\langle \underline{x}, \underline{q}_3\rangle|^2 + |\langle \underline{x}, \underline{q}_4\rangle|^2}$$

$$= \sqrt{\left(\frac{1}{\sqrt{2}}(1-2)\right)^2 + \left(\frac{1}{\sqrt{2}}(2+3)\right)^2}$$

$$= \frac{1}{\sqrt{2}}\sqrt{1+5^2} = \boxed{\sqrt{13}}$$

8. (5 marks) The figure below shows 200 data points $(t_1, y_1), \ldots, (t_{200}, y_{200})$



Determine (approximately) the least squares approximation $A\mathbf{c} \cong \mathbf{y}$ where

$$A = \begin{bmatrix} 1 & t_1 & t_1^2 & t_1^3 \\ 1 & t_2 & t_2^2 & t_2^3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & t_{199} & t_{199}^2 & t_{199}^3 \\ 1 & t_{200} & t_{200}^2 & t_{200}^3 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{199} \\ y_{200} \end{bmatrix}$$

The ~~line of~~ cubic which best fits looks like $y = 1 - t^3$.

$$\Rightarrow \quad \underline{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$