Math 307: 202 — Midterm 2 — 50 minutes

Last Name ______________________ **First** ______________________

Student Number ______________________ **Signature** ______________________

- The test consists of 13 pages and 4 questions worth a total of 50 marks.
- You are allowed 1 page of notes (single-sided, in your handwriting).
- Aside from that is a closed-book examination. **None of the following are allowed**: documents, or electronic devices of any kind (including calculators, cell phones, etc.)
- No work on this page will be marked.

Please do not write on this page — it will not be marked.

Additional instructions

- Please use the spaces indicated.
- **Unless it is specified not to do so, justify your answers.**
- If you require extra paper then put up your hand and ask your instructor.
  - You must put your name and student number on any extra pages.
  - You must indicate the test-number and question-number.
  - Please do this **on both sides** of any extra pages.
- Please do not dismember your test. You must submit all pages.

1		10
2		21
3		10
4		9
Total		50

1. $\boxed{\text{10 marks}}$ Let $A = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}$.

(a) Compute the Singular value decomposition of A. (Hint: you are allowed to compute the SVD in any way you want, but one of the two ways is a lot easier than the other).

Solution: $AA^t = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}$, so the eigenvalues of AA^t are $8, 2$. So $\sigma_1 = \sqrt{8}$, $\sigma_2 = \sqrt{2}$.

Computing the nullspaces of $AA^t - 8I$ and $AA^t - 2I$, we get $p_1 = (1, 0)^t$ and $p_2 = (0, 1)^t$.

So $q_1 = \frac{1}{\sigma_1} A^t p_1 = \frac{1}{\sqrt{8}}(2, -2)^t = \frac{1}{\sqrt{2}}(1, -1)^t$,
and $q_2 = \frac{1}{\sigma_2} A^t p_2 = \frac{1}{\sqrt{2}}(1, 1)^t$.

So $A = P\Sigma Q^T$ where $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\Sigma = \begin{pmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$, and $Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

(b) Find $||A||$.

Solution: We know that $||A||$ is the largest singular value, so $||A|| = \sqrt{8}$.

2. $\boxed{\text{21 marks}}$ Short answer questions, each question 3 marks. **For True or False questions**, if true, provide a short justification. If false, show a counter-example that contradicts the statement. For other questions, justify your answer by showing your work.

(a) Find an orthogonal basis for the subspace $U = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \right\}$.

> **Solution:** Denote the two vectors by u, v. Let $w = v - \frac{\langle u,v \rangle}{||u||^2} u$, then $w = \frac{1}{5}(8, -4, -5)^t$.
> So $\{(1, 2, 0)^t, \frac{1}{5}(8, -4, -5)^t\}$ is a basis for U.

(b) **True or False:** If A is a real matrix with orthogonal columns, then $A^T A$ is a diagonal matrix.

> **Solution:** True. The (i, j)-th entry of $A^T A$ is the inner product of the i-th column and the j-th column of A, so it is zero when $i \neq j$.

(c) Let $A = SDS^{-1}$ where $S = \begin{pmatrix} 1 & 4 \\ 2 & 4 \end{pmatrix}$ and $D = \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix}$, find the eigenvalues of A^2 and a basis for $N(A^2 - 4I)$.

Solution: $A^2 = SD^2S^{-1} = S\begin{pmatrix} 4 & 0 \\ 0 & 16 \end{pmatrix}S^{-1}$, so the eigenvalues of A^2 are 4 and 16. Since the eigenvalue 4 of A^2 has eigenvector $(1,2)^t$, so $N(A^2 - 4I) = span\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$.

(d) **True or False:** Let $A = \begin{pmatrix} 2 & 0 & 2 & 4 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$, then A is diagonalizable.

Solution: False. The only eigenvalue of A is 2, but $A - 2I$ has rank 2, so its geometric multiplicity is $4 - 2 = 2$.

(e) **True or False:** Let A be an $n \times n$ matrix. If λ is an eigenvalue of A, then $\lambda + 1$ is an eigenvalue of $A + I$.

> **Solution:** True. Suppose that $Av = \lambda v$ for $v \neq 0$, then $(A + I)v = Av + v = \lambda v + v = (\lambda + 1)v$. So v is an eigenvector for $A + I$ with eigenvalue $\lambda + 1$.

(f) **True or False:** Let A be a **non-zero** $n \times n$ **orthogonal projection matrix**. If y is a non-zero vector in $R(A)$, then y is **not** in $N(A)$.

Solution: True. Let $y \in R(A)$ and $y \neq 0$, then $y = Au$ for some $u \in \mathbb{R}^n$. Since A is an orthogonal projector, we have $A^2 = A$, so $Ay = A^2u = Au = y \neq 0$. So $y \notin N(A)$.

Updated Mar 23 after grading: Alternative solution from students: We know that $R(A) = N(A^T)^{\perp}$. Since A is orthogonal projection, so $A = A^T$. So $R(A) \perp N(A)$. Therefore $R(A) \cap N(A) = \{0\}$.

(g) **True or False:** Let A be an $n \times n$ matrix. If all eigenvalues of A are positive real numbers, then $||A||$ equals its largest eigenvalue.

Solution: False. Let $A = \begin{pmatrix} 1 & 2024 \\ 0 & 2 \end{pmatrix}$, then $det(A - \lambda I) = (\lambda - 1)(\lambda - 2)$, A has two positive real eigenvalues. However, $||A|| \geq ||A(0,1)^t|| = ||(2024, 2)^t|| > 2024 > 2$, so $||A||$ is greater than its largest eigenvalue.

3. $\boxed{\text{10 marks}}$ Let $U = \operatorname{span}\{\mathbf{x}\} \subseteq \mathbb{R}^3$ where $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

(a) Construct an orthogonal projection matrix which projects onto U.

(b) Find an orthogonal projection matrix P such that $N(P) = U$.
[**Hint:** Such a matrix P must satisfy $R(P^T) = N(P)^\perp = U^\perp$.]

(c) Find a basis for $U^\perp$.

Solution:

(a) The orthogonal projection matrix onto U is

$$Q = \frac{1}{\langle \mathbf{x}, \mathbf{x} \rangle} \mathbf{x}\mathbf{x}^T = \frac{1}{3} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}.$$

(b) We try to figure out what matrix we need for P. Notice that

$$N(P) = U \iff R(P^T) = U^\perp.$$

But since P is an orthogonal projection matrix, $R(P^T) = R(P) = U^\perp$. So P should be the orthogonal projection matrix onto $U^\perp$. Since Q is the orthogonal projection matrix onto U, we have

$$P = I - Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

(c) By part (b), we have $R(P) = U^\perp$. So we want to find a basis for $R(P)$. To do this, row-reduce P:

$$\frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \to \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & \frac{3}{2} & \frac{3}{2} \end{bmatrix} \to \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Since the pivots are in the first 2 columns, the first 2 columns form a basis for $R(P)$. i.e., a basis for $U^\perp$ is

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

4. $\boxed{\text{9 marks}}$ Let $A = Q_1 R_1$ where

$$Q_1 = \begin{bmatrix} 1/3 & -2/3 \\ 2/3 & -1/3 \\ 2/3 & 2/3 \end{bmatrix}, \text{ and } R_1 = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}.$$

Find the least squares approximation $A\mathbf{x} \approx \mathbf{b}$ where

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}.$$

Solution: The least squares approximation of the system $A\mathbf{x} \approx \mathbf{b}$ is given by $R_1\mathbf{x} = Q_1^T\mathbf{b}$. Notice that

$$Q_1^T\mathbf{b} = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

So we want to solve the linear system

$$R_1\mathbf{x} = Q_1^T\mathbf{b} \iff \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Solving this linear system by hand, we have

$$\begin{aligned} 2x_2 = 1 &\implies x_2 = \frac{1}{2} \\ 3x_1 + 4x_2 = 3 &\implies x_1 = \frac{1}{3} \end{aligned}$$

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