Math 307 Practice Midterm Exam 2

1.(a) True/false: If A is diagonalizable, then A^3 is diagonalizable.

- A diagonalizable $\Rightarrow$ $A = PDP^{-1}$ where D is diagonal
- Then $A^3 = P\underbrace{DP^{-1}P}_{I}\underbrace{DP^{-1}P}_{I}DP^{-1} = PD^3P^{-1}$

and D^3 is a diagonal matrix $\left(D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \Rightarrow D^3 = \begin{bmatrix} \lambda_1^3 & & \\ & \ddots & \\ & & \lambda_n^3 \end{bmatrix} \right)$

$\therefore$ A^3 is diagonalizable

$\Rightarrow$ $\boxed{\text{True}}$

(b) Let $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 \neq \lambda_2$ and let $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ be linearly independent. How many 2×2 matrices are there with eigenvalues λ_1 and λ_2 and corresponding eigenvectors $\vec{v}_1$ and $\vec{v}_2$?

- Any 2×2 matrix with 2 linearly independent eigenvectors is diagonalizable.
- If A is diagonalizable, $A = PDP^{-1}$, then the entries in D are the eigenvalues of A, and the columns of P are corresponding eigenvectors.
- So $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}^{-1}$ is the only possible matrix

$\Rightarrow$ Exactly $\boxed{\text{one}}$

(Note that P and D are not unique; we can switch column order in D and P or scale the vectors $\vec{v}_1$ and $\vec{v}_2$ in P, but that will still give the same matrix A.)

(c) Find the QR decomposition of A given the thin QR decomposition

$$A = Q_1 R_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{1}{3} \\ \frac{1}{\sqrt{2}} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & -\sqrt{2} \\ 0 & 3 \end{bmatrix}$$

- The columns of Q_1 form an orthonormal basis of $R(A) \subset \mathbb{R}^3$. In order to get the orthogonal matrix Q we need to extend it to an orthonormal basis of the whole $\mathbb{R}^3$.

- Find $\text{span}\{\vec{q}_1, \vec{q}_2\}^\perp = R(Q_1)^\perp = N(Q_1^T)$

$\Rightarrow$ solve $\left[\begin{array}{ccc|c} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right]$

$x_3 = t \Rightarrow x_2 = -4t, \; x_1 = -t$

$\Rightarrow$ any vector $\vec{x} = t\begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}$ is orthogonal to both $\vec{q}_1$ and $\vec{q}_2$.

- Normalize to get $\vec{q}_3 = \frac{1}{3\sqrt{2}}\begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}$ $\left(\text{or } \vec{q}_3 = \frac{1}{3\sqrt{2}}\begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} \right)$

$$\Rightarrow A = QR = [\vec{q}_1 \; \vec{q}_2 \; \vec{q}_3] \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{2}{3} & -\frac{1}{3\sqrt{2}} \\ 0 & \frac{1}{3} & -\frac{4}{3\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{2}{3} & \frac{1}{3\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & -\sqrt{2} \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$$

(d) A is <u>symmetric</u> 3×3 with eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$.

Let $\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ be eigenvectors corresponding to λ_1.

Find an eigenvector $\vec{v_3}$ for λ_2.

- For any symmetric matrix, eigenvectors corresponding to distinct eigenvalues are orthogonal.
- Thus, any eigenvector $\vec{v_3}$ corresponding to $\lambda_2 = -1$ must be orthogonal to both $\vec{v_1}$ and $\vec{v_2}$ (corresponding to $\lambda_1 = 3$).

$$\Rightarrow \begin{cases} \langle \vec{v_1}, \vec{v_3} \rangle = \vec{v_1}^T \vec{v_3} = 0 \\ \langle \vec{v_2}, \vec{v_3} \rangle = \vec{v_2}^T \vec{v_3} = 0 \end{cases} \iff \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \end{bmatrix} \vec{v_3} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Solve $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 3 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & -3 & 0 \end{array}\right]$ $\quad x_3 = t \Rightarrow x_2 = -3t$, $x_1 = 5t$

$$\therefore \vec{v_3} = t \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} \text{ for any } t \in \mathbb{R}.$$

2. Find the shortest distance from $(2,-6,-4)$ to the line $(x,y,z) = (-4t, 5t, -2t)$.

- The line is a subspace $U = \text{span}\left\{\begin{bmatrix}-4\\5\\-2\end{bmatrix}\right\}$ of $\mathbb{R}^3$, and we want to find the shortest distance between $\vec{x} = \begin{bmatrix}2\\-6\\-4\end{bmatrix}$ and U.

- The projection theorem states that this shortest distance is

$$\|\vec{x} - \text{proj}_U(\vec{x})\| = \left\| \vec{x} - \frac{\langle \vec{x}, \vec{u}\rangle}{\|\vec{u}\|} \vec{u} \right\|$$

$$= \left\| \begin{bmatrix}2\\-6\\-4\end{bmatrix} - \frac{\left\langle \begin{bmatrix}2\\-6\\-4\end{bmatrix}, \begin{bmatrix}-4\\5\\-2\end{bmatrix}\right\rangle}{(-4)^2 + 5^2 + (-2)^2} \begin{bmatrix}-4\\5\\-2\end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix}2\\-6\\-4\end{bmatrix} - \frac{(-30)}{45} \begin{bmatrix}-4\\5\\-2\end{bmatrix} \right\| = \left\| -\frac{1}{3} \begin{bmatrix}2\\8\\16\end{bmatrix} \right\|$$

$$= \frac{1}{3}\sqrt{2^2 + 8^2 + 16^2} = \frac{1}{3}\sqrt{324} = \frac{18}{3} = 6$$

$\therefore$ The shortest distance is $\boxed{6}$

3. Find the orthogonal projection matrix P that projects a vector in $\mathbb{R}^3$ onto the plane spanned by

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}.$$

Alternative 1

- Find an orthogonal basis $\{\vec{v}_1, \vec{v}_2\}$ of the plane.

Use Gram-Schmidt:

$$\vec{v}_1 = \vec{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} - \frac{10}{5} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$$

- Now compute P as the sum of the projection matrices for each orthogonal basis vector:

$$P = \frac{1}{\|\vec{v}_1\|^2} \vec{v}_1 \vec{v}_1^T + \frac{1}{\|\vec{v}_2\|^2} \vec{v}_2 \vec{v}_2^T = \frac{1}{5} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{30} \begin{bmatrix} 4 & -2 & 10 \\ -2 & 1 & -5 \\ 10 & -5 & 25 \end{bmatrix}$$

$$= \frac{1}{30} \begin{bmatrix} 10 & 10 & 10 \\ 10 & 25 & -5 \\ 10 & -5 & 25 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & -1 \\ 2 & -1 & 5 \end{bmatrix}$$

Alternative 2

- Find the projection matrix $P_\perp$ for the orthogonal complement of the plane, and then compute $P = I - P_\perp$

- $$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 4 & 3 & 5 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -5 & 5 & 0 \end{array}\right] \Rightarrow \vec{x} = t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$ is orthogonal to the plane

$$\Rightarrow P_\perp = \frac{1}{6} \begin{bmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{bmatrix} \Rightarrow P = I - P_\perp = \frac{1}{6} \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & -1 \\ 2 & -1 & 5 \end{bmatrix}.$$

4.

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 2 & 2 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$

a) Find the singular values of A.

- The singular values are the square roots of the eigenvalues of AA^T (or A^TA) in decreasing order.
- In our case, we use A^TA since it is smaller (3×3).
- $$A^TA = \begin{bmatrix} 0 & -1 & -1 & -1 \\ 1 & 2 & -1 & -1 \\ -1 & 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ -1 & 2 & 2 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 5 \\ 0 & 5 & 7 \end{bmatrix}$$
- $$\det(A^TA - \lambda I) = 0 \iff \det\left(\begin{bmatrix} 3-\lambda & 0 & 0 \\ 0 & 7-\lambda & 5 \\ 0 & 5 & 7-\lambda \end{bmatrix}\right) = 0$$

$$\iff (3-\lambda)\left((7-\lambda)^2 - 25\right) = 0 \iff (3-\lambda)(\lambda^2 - 14\lambda + 24) = 0$$

$$\iff (3-\lambda)(\lambda-12)(\lambda-2) = 0$$

$\Rightarrow \lambda_1 = 12,\ \lambda_2 = 3$ and $\lambda_3 = 2$ are the eigenvalues of A^TA

$\therefore$ The singular values of A are:

$$\sigma_1 = \sqrt{12} = 2\sqrt{3}, \quad \sigma_2 = \sqrt{3}, \quad \sigma_3 = \sqrt{2}$$

b) The norm $\|A\|$ of the matrix A is given by the largest singular value of A.

$$\therefore \quad \|A\| = \sigma_1 = 2\sqrt{3}$$

5. Find the line $f(t) = c_0 + c_1 t$ that best fits the data points $(-6, 23)$, $(0, -1)$ and $(6, -37)$ using least squares.

- We want to find the least squares solution to the system

$$\begin{bmatrix} 1 & t_0 \\ 1 & t_1 \\ 1 & t_2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix},$$

i.e. $A\vec{c} = \vec{y}$ where $A = \begin{bmatrix} 1 & -6 \\ 1 & 0 \\ 1 & 6 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 23 \\ -1 \\ -37 \end{bmatrix}$

Alternative 1 : Normal equation

- Solve $A^T A \vec{c} = A^T \vec{y}$ $\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ -6 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & -6 \\ 1 & 0 \\ 1 & 6 \end{bmatrix} \vec{c} = \begin{bmatrix} 1 & 1 & 1 \\ -6 & 0 & 6 \end{bmatrix} \begin{bmatrix} 23 \\ -1 \\ -37 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 3 & 0 \\ 0 & 72 \end{bmatrix} \vec{c} = \begin{bmatrix} -15 \\ -360 \end{bmatrix} \Rightarrow \begin{aligned} c_0 &= -\frac{15}{3} = -5 \\ c_1 &= -\frac{360}{72} = -5 \end{aligned}$$

$\therefore$ $\boxed{f(t) = -5 - 5t}$ is the least squares solution

Alternative 2 : QR decomposition

- In this case, the columns of A are already orthogonal, so it is easy to find the thin QR decomposition (just normalizing the columns) :

$$Q_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad R_1 = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 6\sqrt{2} \end{bmatrix}$$

- We find the least squares solution by solving $R_1 \vec{c} = Q_1^T \vec{y}$

$$\Rightarrow \left[\begin{array}{cc|c} \sqrt{3} & 0 & -\frac{15}{\sqrt{3}} \\ 0 & 6\sqrt{2} & -\frac{60}{\sqrt{2}} \end{array}\right] \Rightarrow c_0 = c_1 = -5$$

$$\therefore \boxed{f(t) = -5 - 5t}$$