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# MATH 307 Midterm Exam 1

*October 13, 2021*

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- No calculators, cellphones, laptops or notes
- Time allowed: 45 minutes
- 35 total marks
- Write your name and student number in the space below

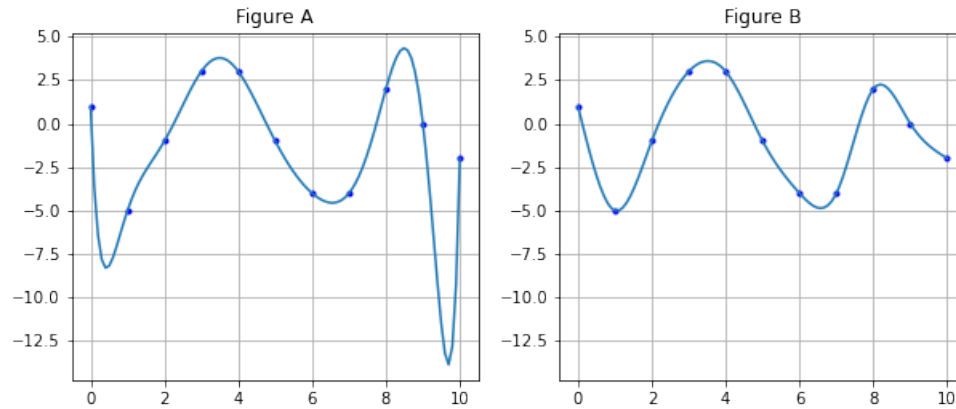
**Name:**

**Student Number:**



1. Short answer questions. Each part is independent of the others.

(a) (3 marks) The figures show different interpolating functions for the same dataset:



Determine which figure corresponds to polynomial interpolation and which corresponds to cubic spline interpolation. Justify your answer.

(b) (3 marks) **True or False:** If  $A$  is an invertible  $n \times n$  matrix such that  $\|A\mathbf{x}\| \leq \|\mathbf{x}\|$  for all  $\mathbf{x} \in \mathbb{R}^n$ , then  $\|A^{-1}\| \geq 1$ . Justify your answer.

- (c) (3 marks) Determine whether or not the set

$$U = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 : abc = 0 \right\}$$

is a subspace of  $\mathbb{R}^3$ . Justify your answer.

- (d) (3 marks) Consider the boundary value problem

$$y'' = \cos(\pi t^2) , \quad y(0) = 0 , \quad y(2) = 0$$

If we know that  $y(0.9) \approx 0.018$  and  $y(1.1) \approx 0.023$ , find an approximation of  $y(1)$ .

2. Let  $p(t)$  be the natural cubic spline which interpolates the data

$$(0, 1), (1, 3), (2, 8), (3, 10), (4, 9), (5, -1), (6, -17)$$

Suppose the coefficient matrix of  $p(t)$  is

$$\begin{bmatrix} 1 & -2 & 1 & a_4 & 1 & 1 \\ 0 & 3 & -3 & b_4 & -6 & -3 \\ 1 & 4 & 4 & c_4 & -5 & -14 \\ 1 & 3 & 8 & 10 & 9 & -1 \end{bmatrix}$$

- (a) (4 marks) Determine the coefficients  $a_4, b_4, c_4$ .
- (b) (2 mark) Determine the value  $p''(2.5)$ .

3. Consider the matrix

$$A = \begin{bmatrix} -3 & 1 & 2 & 0 \\ 3 & 1 & -2 & 1 \\ -6 & 2 & 5 & 1 \\ -9 & 3 & 4 & 2 \end{bmatrix}$$

- (a) (4 marks) Find the  $LU$  decomposition of  $A$ .
- (b) (2 mark) Compute  $\det(A)$ .

4. (6 marks) Determine whether  $\text{span}\{\mathbf{u}_1, \mathbf{u}_2\} = \text{span}\{\mathbf{u}_3, \mathbf{u}_4\}$  where

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ -3 \\ 1 \\ -1 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} -5 \\ 1 \\ 2 \\ -2 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} -1 \\ -5 \\ 4 \\ -4 \end{bmatrix} \quad \mathbf{u}_4 = \begin{bmatrix} 3 \\ -11 \\ 6 \\ -10 \end{bmatrix}$$

5. (5 marks) Determine  $\|A\|$  for the matrix

$$A = \frac{1}{2} \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

Hint: counterclockwise rotation in  $\mathbb{R}^2$  by angle  $\theta$  corresponds to matrix multiplication by

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$



*Extra workspace. Do not write in the table below.*

Q1	/12
Q2	/6
Q3	/6
Q4	/6
Q5	/5
Total	/35