MATH 307 Midterm Exam 1

October 14, 2021

- No calculators, cellphones, laptops or notes
- Time allowed: 45 minutes
- 35 total marks
- Write your name and student number in the space below

Name:

Student Number:

1. Short answer questions. Each part below is independent of the others.

   (a) (3 marks) **True** or **False**: If A is an invertible matrix, then $\|A^{-1}\| = \|A\|^{-1}$. Justify your answer.

   (b) (3 marks) **True** or **False**: There exists a unique polynomial $p(t)$ of degree 2 (or less) such that

   $$p(-1) = p'(0) = p(1) = 0$$

   Justify your answer.

(c) (3 marks) Determine (approximately) the condition number of the matrix

$$A = \begin{bmatrix} c & 1 & & & \\ 1 & c & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & c & 1 \\ & & & 1 & c \end{bmatrix}$$

where c is very large positive number. Justify your answer.

(d) (3 marks) Consider 11 data points $(t_0, y_0), \ldots, (t_{10}, y_{10})$ such that $t_k - t_{k-1} = 1$ for each $k = 1, \ldots, 10$. Suppose the coefficient matrix of the corresponding natural cubic spline is given by

$$\begin{bmatrix} -1 & 2 & 6 & -9 & -1 & 3 & 2 & 8 & 3 & -13 \\ 0 & -3 & 3 & 21 & -6 & -9 & 0 & 6 & 30 & 39 \\ 2 & -1 & -1 & 23 & 38 & 23 & \square & 20 & 56 & 125 \\ -7 & -6 & -8 & 0 & 35 & 66 & 83 & 99 & 133 & 222 \end{bmatrix}$$

Determine the missing value $\square$.

2. (5 marks) Determine all values c such that the vectors

$$\boldsymbol{u}_1 = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix} \qquad \boldsymbol{u}_3 = \begin{bmatrix} 4 \\ 5 \\ c-1 \end{bmatrix}$$

are linearly independent.

3. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 8 & 3 & 8 & 2 \\ -4 & -3 & 5 & -1 \\ 2 & -2 & 7 & 11 \end{bmatrix}$$

(a) (4 marks) Find the LU decomposition of A.

(b) (2 mark) Compute $\det(A)$.

4. (6 marks) Setup (but do **not** solve) a linear system $A\boldsymbol{x} = \boldsymbol{b}$ such that the solution

$$\boldsymbol{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

determines the unique function of the form

$$f(t) = a\sin(\pi t) + b\cos(\pi t) + c\sin(2\pi t) + d\cos(2\pi t)$$

which interpolates the data $(0, y_0)$, $(1/4, y_1)$, $(1/2, y_2)$, $(3/4, y_3)$. The system depends on y_0, y_1, y_2, y_3.

5. (6 marks) Find all polynomials $p(t)$ of degree 3 (or less) such that

$$p(1) = p(-1) \qquad p(-2) = -7p(0) \qquad p'(1) = 3p'(-1) \qquad 5p''(1) = -7p''(-1)$$

Extra workspace. Do not write in the table below.

Q1	/12
Q2	/5
Q3	/6
Q4	/6
Q5	/6
Total	/35