MATH 307 Practice Midterm Exam 1 (October 2021)

SOLUTIONS

1. (a) $\boxed{\text{False}}$. Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$ for $0 < c < 1$.

Then $\|A\| = 1$ and $\|A^{-1}\| = \frac{1}{c} > 1$.

(b) $y''(t) \approx \dfrac{y(t+h) - 2y(t) + y(t-h)}{h^2}$ $\quad h = 0.2$

$$\Rightarrow y''(0.6) \approx \frac{y(0.8) - 2y(0.6) + y(0.4)}{0.2^2} = \frac{1.1 - 2(1.8) + 1.5}{0.04}$$

$$\Rightarrow \boxed{y''(0.6) \approx -25}$$

(c) $$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \boxed{A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}$$

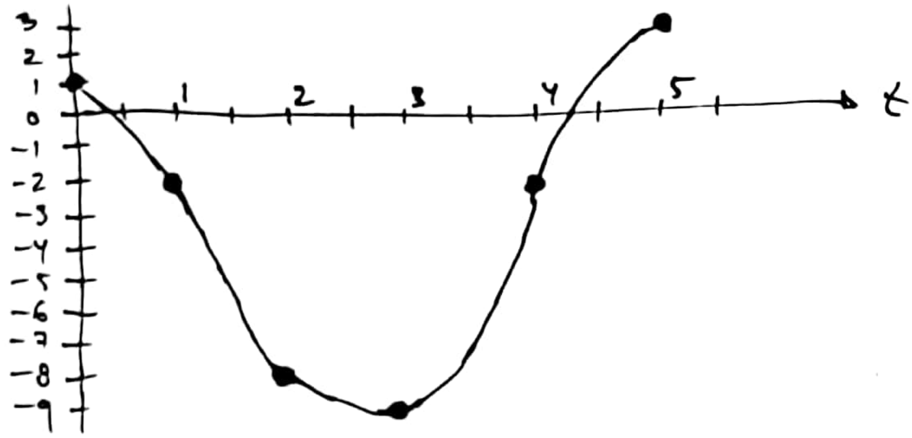
(d) $U = \{ \underline{x} \in \mathbb{R}^n : A\underline{x} = \underline{b} \}$ is <u>not</u> a subspace of $\mathbb{R}^n$ since $\underline{0} \notin U$ because $A\underline{0} = \underline{0} \neq \underline{b}$. Also $\underline{y}_1 + \underline{y}_2 \notin U$ for $\underline{y}_1, \underline{y}_2 \in U$ since $A(\underline{y}_1 + \underline{y}_2) = A\underline{y}_1 + A\underline{y}_2 = \underline{b} + \underline{b} = 2\underline{b}$

(e) $\left[\begin{array}{ccc|c} 2 & \alpha & -1 & \beta \\ 1 & -1 & 0 & 2 \\ 2 & -3 & 1 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & \alpha+2 & -1 & \beta-4 \\ 0 & -1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & \alpha+1 & \beta-4 \end{array}\right]$

Infinitely many solutions if $\boxed{\alpha = -1 \text{ and } \beta = 4}$.

(f) From coefficient matrix, we see the y values

$y_0 = 1 \quad y_1 = -2 \quad y_2 = -8 \quad y_3 = -9 \quad y_4 = -2 \quad y_5 = 3$



2.(a) $\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \ddots & \end{bmatrix} \xrightarrow{R_2 + \frac{1}{2}R_1} \begin{bmatrix} 2 & -1 & 0 & & \\ 0 & 3/2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \ddots & \end{bmatrix}$

$\xrightarrow{R_3 + \frac{2}{3}R_2} \begin{bmatrix} 2 & -1 & 0 & 0 & \\ 0 & 3/2 & -1 & 0 & \\ 0 & 0 & 4/3 & -1 & \\ & & & \ddots & \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -\frac{1}{2} & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\frac{2}{3} & 1 & 0 & \cdots & 0 \\ 0 & 0 & * & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & * & 1 \end{bmatrix}$

$$
U = \begin{bmatrix}
2 & -1 & 0 & \cdots & \cdots & 0 \\
0 & 3/2 & -1 & 0 & & \\
0 & 0 & 4/3 & -1 & \ddots & \\
\vdots & & \ddots & \ddots & \ddots & 0 \\
\vdots & & & \ddots & \ddots & -1 \\
0 & \cdots & \cdots & \cdots & 0 & *
\end{bmatrix}
$$

(b) $u_{1,1} = 2$

$u_{2,2} = 2 - \frac{1}{u_{1,1}} = 2 - \frac{1}{2} = \frac{3}{2}$ $\Rightarrow \boxed{u_{n+1,n+1} = 2 - \frac{1}{u_{n,n}}}$

$u_{3,3} = 2 - \frac{1}{u_{2,2}} = 2 - \frac{2}{3} = \frac{4}{3}$

(c) $u_{4,4} = 2 - \frac{3}{4} = \frac{5}{4}$ For $N=5$

$u_{5,5} = 2 - \frac{4}{5} = \frac{6}{5}$

$\det(A) = \det(U)$

$= (2)(\frac{3}{2})(\frac{4}{3})(\frac{5}{4})(\frac{6}{5})$

$\Rightarrow \boxed{\det(A) = 6}$

In fact, $\det(A) = N+1$ for any N.

3. $P_k(t) = a_k(t-t_{k-1})^3 + b_k(t-t_{k-1}) + c_k$, $t \in [t_{k-1}, t_k]$.

$P_k(t_{k-1}) = y_{k-1} \Rightarrow \boxed{c_k = y_{k-1}}$ $k = 1, 2, 3$

$P_k(t_k) = y_k \Rightarrow \boxed{a_k + b_k + c_k = y_k}$ $k = 1, 2, 3$

$$P_k'(t_k) = P_{k+1}'(t_k) \Rightarrow \boxed{3a_k + b_k = b_{k+1}} \quad k = 1, 2$$

$$P_3''(t_3) = 0 \Rightarrow \boxed{6a_3 = 0}$$

$$\Rightarrow \begin{array}{ll} a_1 + b_1 = y_1 - y_0 & 3a_1 + b_1 - b_2 = 0 \\ a_2 + b_2 = y_2 - y_1 & 3a_2 + b_2 - b_3 = 0 \\ a_2 + b_3 = y_3 - y_2 & a_3 = 0 \end{array}$$

$$\Rightarrow \left[\begin{array}{cccccc|c} 1 & 1 & 0 & 0 & 0 & 0 & y_1 - y_0 \\ 0 & 0 & 1 & 1 & 0 & 0 & y_2 - y_1 \\ 0 & 0 & 0 & 0 & 1 & 1 & y_3 - y_2 \\ 3 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

4. (a)

$$A = \begin{bmatrix} 2 & 3 & 3 & 0 \\ 4 & 5 & 3 & 3 \\ 4 & 3 & -1 & 11 \\ 2 & 1 & -11 & -2 \end{bmatrix} \to \begin{bmatrix} 2 & 3 & 3 & 0 \\ 0 & -1 & -3 & 3 \\ 0 & -3 & -7 & 11 \\ 0 & -2 & -14 & -2 \end{bmatrix}$$

$$\to \begin{bmatrix} 2 & 3 & 3 & 0 \\ 0 & -1 & -3 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -8 & -8 \end{bmatrix} \to \begin{bmatrix} 2 & 3 & 3 & 0 \\ 0 & -1 & -3 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \text{rank}(A) = 3 \Rightarrow \boxed{\dim(U) = 3}$ and a basis of U is $\{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$

(b) Part (a) shows that $[\underline{u}_1\ \underline{u}_3\ \underline{u}_4] \rightarrow \begin{bmatrix} 2 & 3 & 0 \\ 0 & -3 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore $\{\underline{u}_1, \underline{u}_3, \underline{u}_4\}$ are linearly independent and form a basis of U.

5. (a) $p_4''(t_4) = p_5''(t_4) \Rightarrow 3a_4 + b_4 = b_5$

$$-12 + b_4 = b_5$$

$p_5''(t_5) = p_6''(t_5) \Rightarrow 3a_5 + b_5 = b_6$

$$-6 + b_5 = -18$$

$\Rightarrow b_5 = -12 \Rightarrow \boxed{b_4 = 0}$

(b) $p''(5.5) = p_6''(5.5) = 6a_6(5.5-5) + 2b_6$

$$= 6(-2)(\tfrac{1}{2}) + 2(-18)$$

$$= -6 - 36 = -42$$

$\Rightarrow \boxed{p''(5.5) = -42}$