Math 307: 201 — Midterm 2 — 50 minutes

Last Name ______________________ **First** ______________________

Student Number ______________________ **Signature** ______________________

- The test consists of 13 pages and 4 questions worth a total of 50 marks.
- You are allowed 1 page of notes (single-sided, in your handwriting).
- Aside from that is a closed-book examination. **None of the following are allowed**: documents, or electronic devices of any kind (including calculators, cell phones, etc.)
- No work on this page will be marked.

Please do not write on this page — it will not be marked.

Additional instructions

- Please use the spaces indicated.
- **Unless it is specified not to do so, justify your answers.**
- There is a blank page at the end of the exam that you can use as scratch paper.
- Please do not dismember your test. You must submit all pages.

1		10
2		21
3		10
4		9
Total		50

1. $\boxed{\text{10 marks}}$ Let $A = \begin{pmatrix} 3 & -1 \\ 3 & 1 \end{pmatrix}$.

(a) Compute the Singular value decomposition of A. (Hint: you are allowed to compute the SVD in any way you want, but one of the two ways is a lot easier than the other).

Solution: $A^tA = \begin{pmatrix} 18 & 0 \\ 0 & 2 \end{pmatrix}$, so the eigenvalues of A^tA are $18, 2$. So $\sigma_1 = \sqrt{18}$, $\sigma_2 = \sqrt{2}$.
Computing the nullspaces of $A^tA - 18I$ and $A^tA - 2I$, we get $q_1 = (1,0)^t$ and $q_2 = (0,1)^t$.
So $p_1 = \frac{1}{\sigma_1}Aq_1 = \frac{1}{\sqrt{18}}(3,3)^t = \frac{1}{\sqrt{2}}(1,1)^t$,
and $p_2 = \frac{1}{\sigma_2}Aq_2 = \frac{1}{\sqrt{2}}(-1,1)^t$.

So $A = P\Sigma Q^T$ where $P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $\Sigma = \begin{pmatrix} \sqrt{18} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$, and $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(b) Find $||A||$.

Solution: We know that $||A||$ is the largest singular value, so $||A|| = \sqrt{18}$.

2. $\boxed{\text{21 marks}}$ Short answer questions, each question 3 marks. **For True or False questions**, if true, provide a short justification. If false, show a counter-example that contradicts the statement. For other questions, justify your answer by showing your work.

(a) Let A have the singular value decomposition $A = P\Sigma Q^T$ where

$$P = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Calculate (no part marks):

(i) $||P||$, i.e., the operator norm of P. (**Not** $||A||$.)

(ii) The dimension of the null space of A.

(iii) $||A^TAA^TA||$.

Solution: (i) Since P is orthogonal, $||P|| = 1$. (ii) r=rank(A) = number of singular values = 3. $\dim(\mathrm{N}(A)) = n - r = 4 - 3 = 1$. (iii) The eigenvalues of the symmetric matrix A^TA are the squares of the singular values. The eigenvalues of $(A^TA)^2$ are the squares of the eigenvalues of A^TA. The operator norm of $(A^TA)^2$ is the magnitude of the largest eigenvalue, that is $(3^2)^2 = 3^4 = 81$.

(b) Let

$$A = \begin{pmatrix} 1 & 2 \\ 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Calculate the least squares solution to $Ax \approx b$.

Solution: Since b is orthogonal to the columns of A, it is orthogonal to the range of A. Thus the projection of b onto the range of A returns 0, so the solution is $x = (0, 0)^T$. Alternatively, use the formula $x = (A^T A)^{-1} A^T b$.

(c) Let A be a 2×2 matrix. Suppose that $N(A - I) = span \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$ and $N(A + 2I) = span \left\{ \begin{pmatrix} 3 \\ 3 \end{pmatrix} \right\}$, find an invertible 2×2 matrix S and a diagonal matrix D such that $A = SDS^{-1}$.

Solution: Since A has eigenvalue 1 and -2 with eigenvectors $(1, 2)^t$ and $(3, 3)^t$, so $A = SDS^{-1}$ where $S = \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$.

(d) **True or False:** Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 2024 \end{pmatrix}$, then A is diagonalizable.

> **Solution:** True. The eigenvalues of A are $1, 5, 8, 2024$, which are all distinct. So A is diagonalizable.

(e) **True or False:** Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & -2024 & -2024 & 3 \\ 3 & -2024 & -2024 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}$. Then all eigenvalues of A are real, and A is diagonalizable.

> **Solution:** True. Since A is symmetric, and symmetric matrices have real eigenvalues and are diagonalizable.

(f) **True or False:** Let A, B be $n \times n$ matrices. If v is an eigenvector for both A, B, then v is an eigenvector for AB.

Solution: True. Let $Av = \lambda v$ and $Bv = \mu v$, then $ABv = A(\mu v) = \mu Av = \mu\lambda v$.

(g) **True or False:** Let A, B be $n \times n$ matrices that are orthogonal projections. If $R(A) \perp R(B)$, then $A + B$ is an orthogonal projection.

Solution: True. We have $AB = BA = 0$ because $R(A) \perp R(B)$. So we have $(A+B)^2 = A^2 + AB + BA + B^2 = A + 0 + 0 + B = A + B$. We also have $(A+B)^t = A^t + B^t = A + B$.

Updated Mar 27: Alternative solution from students: Let $\{u_i\}, \{v_i\}$ be orthonormal bases for U and V. Then $A = \sum u_i u_i^T$ and $B = \sum v_i v_i^T$. Since $R(A) \perp R(B)$, the union of these two bases is an orthonormal basis. And since $A + B = \sum u_i u_i^T + \sum v_i v_i^T$, so $A + B$ is orthogonal projection.

3. $\boxed{\text{10 marks}}$ Let $\{\mathbf{x}_1, \mathbf{x}_2\}$ be a basis of the subspace $U \subseteq \mathbb{R}^3$ where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

(a) Find a 3×2 matrix A such that $U = N(A^T)^\perp$.

(b) Find a basis for $U^\perp$.

(c) Construct an orthogonal projection matrix which projects onto U.

Solution:

(a) Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

chosen so that $R(A) = U$. Then, $U = R(A) = N(A^T)^\perp$.

(b) We want to find a basis for $U^\perp = N(A^T)$, so we need to solve the linear system $A^T\mathbf{x} = 0$. First, row reduce A^T:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

This tells us the solution to the linear system $A^T\mathbf{x} = 0$ has a free variable in the 3th coordinate of $\mathbf{x}$. In particular,

$$\mathbf{x} = s \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

for some $s \in \mathbb{R}$. So a basis for $U^\perp$ is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(c) From part (b), a basis for U is $\{\mathbf{x}\}$ where

$$\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

The orthogonal projection matrix onto $U^\perp$ is

$$P = \frac{1}{\langle \mathbf{x}, \mathbf{x} \rangle} \mathbf{x}\mathbf{x}^T = \frac{1}{3} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

Now, the orthogonal projection matrix onto U is

$$I - P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

4. $\boxed{\text{9 marks}}$ Find a <u>thin</u> QR decomposition of

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 2 & 2 \end{bmatrix}.$$

Solution: Let $\mathbf{x}_1, \mathbf{x}_2$ be the column vectors of A. We use Gram-Schmidt to find an orthogonal basis $\{\mathbf{y}_1, \mathbf{y}_2\}$ for $R(A)$:

$$\mathbf{y}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\mathbf{y}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{y}_1, \mathbf{x}_2 \rangle}{||\mathbf{y}_1||^2}\mathbf{y}_1 = \mathbf{x}_2 - \frac{3}{9}\mathbf{y}_1 = \frac{1}{3}\begin{bmatrix} -4 \\ -2 \\ 4 \end{bmatrix}$$

Normalizing, an orthonormal basis for $R(A)$ is

$$\{\mathbf{u}_1, \mathbf{u}_2\} = \left\{ \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \frac{1}{3}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \right\}.$$

Then, $A = Q_1 R_1$ is a thin QR decomposition where

$$Q_1 = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} 1/3 & -2/3 \\ 2/3 & -1/3 \\ 2/3 & 2/3 \end{bmatrix}$$

and

$$\begin{aligned} R_1 &= \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{x}_1 \rangle & \langle \mathbf{u}_1, \mathbf{x}_2 \rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{x}_2 \rangle \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}. \end{aligned}$$

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