STAT305: Introduction to Inference **Fall 2021-22**

Assignment 4

Due date: see Piazza post.

Instructions

Academic integrity policy: We encourage you to discuss verbally with other students about the assignment. However, you should write your answers by yourself (in particular, without ever putting your eyes on someone else's answer or draft answer). If you use online resources (including asking questions on math exchange or similar), you *have* to cite them. If you have not done se already, read: `http://learningcommons.ubc.ca/guide-to-academic-integrity/` Please also make sure to read the audit process described during lecture 1.

Instructions: read this right away!

1. Due date: refer to the Piazza post announcing the assignment.
2. Justify all your answers. Use precise mathematical notation. Define all symbols you introduce. Marks will be deducted for imprecise, incorrect or insufficient justification.
3. Write your answers in order, and clearly label the question you are answering, for example 1.1 for part 1 of question 1.
4. We only accept submissions via Gradescope. Please make sure at least 24 hours in advance that you are confortable with the submission process, to give us enough time to fix any problems with the submission process.
5. See slide in lecture 1 for late assignment policy.

1 Likelihood ratio test

Let $Y_1, ..., Y_n$ be i.i.d random variables following $N(\mu, \sigma^2)$, with known μ. We want to test:

$$H_0 : \sigma^2 = \sigma_0^2 \text{ versus } H_a : \sigma^2 = \sigma_a^2, \text{ with } \sigma_a^2 < \sigma_0^2$$

1. Write down and simplify the likelihood ratio.

   [2 mark(s)]

2. Is the likelihood ratio an increasing or decreasing function of the test statistic $\sum_{i=1}^{n}(y_i - \mu)^2$?

   [2 mark(s)]

3. What is the distribution of $\frac{\sum_{i=1}^{n}(Y_i - \mu)^2}{\sigma_0^2}$ under H_0? Do not forget to state its parameters. Is this an exact or an asymptotic result?

   [2 mark(s)]

4. Provide the rejection region for testing H_0 versus H_a with significance level α.

[2 mark(s)]

5. Assume $n = 50, \sum_{i=1}^{n} y_i = 60, \sum_{i=1}^{n} y_i^2 = 200, \mu = 1, \sigma_0 = 5$. Do you reject H_0 at a significance level $\alpha = 0.05$? What if $\alpha = 0.01$?

[1 mark(s)]

2 Likelihood ratio test

Suppose $y_1, y_2, \cdots, y_n$ are realizations of independent random variables $Y_1, Y_2, \cdots, Y_n$ each following the Normal distribution, $Y_i \sim N(\mu, \sigma^2)$ with PDF

$$f_{Y_i}(y_i|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y_i-\mu)^2} (-\infty < y_i < \infty; -\infty < \mu < \infty; \sigma^2 > 0) \tag{1}$$

we want to test

$$H_0 : \mu = 0 \quad \text{versus} \quad H_a : \mu > 0 \tag{2}$$

1. Suppose σ^2 is know, show that the likelihood ratio test for the simple hypothesis $H_0 : \mu = 0$ versus $H_a : \mu = \mu_a$(where $\mu_a > 0$) leads to a decision rule which is equivalent to $\bar{y} > c$ for some constant c and the observed sample mean $\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$.
[2 mark(s)]

2. Briefly explain why the decision rule developed in last question leads to the most powerful test (given a fixed significance level) for the 1-sided alternative hypothesis $H_a : \mu > 0$.

[2 mark(s)]

3. Now suppose σ^2 is unknown, what is the decision rule of the generalized likelihood ratio test for the hypotheses?
where $s^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} (y_i - \bar{y})^2$ for some constant d, and $\bar{y}$ and s^2 being the observed sample mean and observed sample variance, respectively. **[2 mark(s)]**