

## STAT 305 Assignment 2

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### Question 1

- From Example 4.3 we have

$$\begin{aligned}\hat{\lambda} &= \frac{1}{\bar{y}} \\ \therefore \hat{\theta} &= \frac{1}{\hat{\lambda}} = \bar{y} \\ &= 168.3\end{aligned}$$

$$\begin{aligned}Var(\theta) &= Var(\bar{y}) \\ &= Var\left(\frac{\sum_{i=1}^n y}{n}\right) \\ &= \frac{1}{n^2} Var\left(\sum_{i=1}^n y\right) && \text{via linearity of the Variance} \\ &= \frac{1}{n^2} \sum_{i=1}^n Var(y) && \text{via sum of variance of independent random variables} \\ &= \frac{1}{n^2} \sum_{i=1}^n \frac{1}{\lambda^2} && \text{via Variance of the Exponential} \\ &= \frac{1}{n^2} \cdot \frac{n}{\lambda^2} \\ &= \frac{1}{n\lambda} \\ \therefore se(\hat{\theta}) &= \sqrt{\frac{1}{n\hat{\lambda}^2}} = \sqrt{\frac{1}{n}(\hat{\theta})^2} && \text{since } \theta = \frac{1}{\lambda} \\ &= \sqrt{\frac{(168.3)^2}{10}} = 53.22\end{aligned}$$

- Likewise, we can compute the standard deviation necessary for the 80% confidence interval

$$\begin{aligned}CI &= \hat{\theta} \pm z_{90} \cdot se \\ &= \hat{\theta} \pm qnorm(0.90) \cdot 53.22 \\ &= 168.3 \pm 1.281552 \cdot 53.22 \\ &= [100.0958, 236.5402]\end{aligned}$$

3. We can once again apply the MLE method by using the joint distribution:

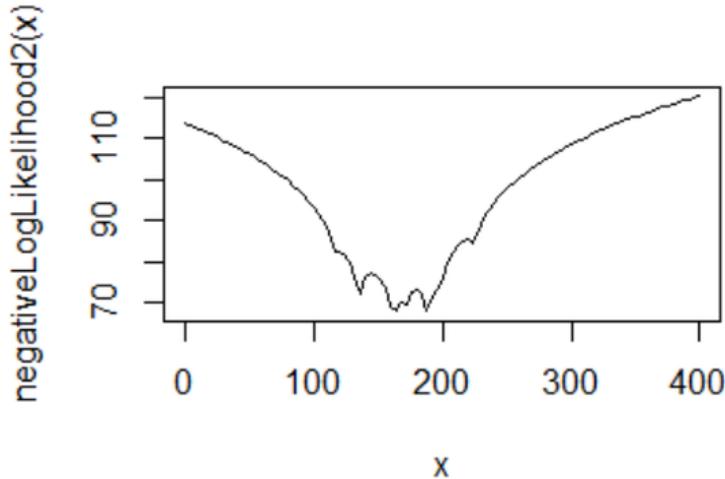
$$\begin{aligned}
 f_{X_1, \dots, X_n}(x_1, \dots, x_n) &= \prod_{i=1}^n [\pi(1 + (x - \theta)^2)]^{-1} \\
 \ln f_{X_1, \dots, X_n}(x_1, \dots, x_n) &= \sum_{i=1}^n \ln(1) - \ln[\pi(1 + (x - \theta)^2)] \\
 &= \sum_{i=1}^n -\ln[\pi(1 + (x - \theta)^2)]
 \end{aligned}$$

**Note:** please find the code use for this question in the appendix

When we start at 150:  $\theta = 162.0665$ ,  $se = 0.5057734$

When we start at 200:  $\theta = 188.9784$ ,  $se = 0.5027846$

**Reason:** The log likelihood function is non-convex/concave (so multi-modal) from curve below, there are multiple extremum and the numerical approximation method used by R (Newton's approximation for example) will converge to different peaks depending on your starting point



**Does this invalidate the calibration of the confidence:** Yes, it does since Theorem 4.1 in the book states that the consistency of the ML estimator

asymptotic normality of the ML estimator  $\left( \lim_{n \rightarrow \infty} \frac{\tilde{\theta} - \theta}{1/\sqrt{n\mathcal{I}_1(\theta)}} \rightarrow N(0, 1) \right)$  relies on the "regularity condition" which we no longer have - thus we can no longer say the confidence interval is calibrated.

## Question 2

1. Using the ML method, we have

$$\begin{aligned}f_{X_1, \dots, X_n}(x_1, \dots, x_n) &= \prod_{i=1}^n ax^{a-1}e^{-x^a} \\ \ln f_{X_1, \dots, X_n}(x_1, \dots, x_n) &= \sum_{i=1}^n \ln[ax^{a-1}e^{-x^a}] \\ &= \sum_{i=1}^n \ln(a) + (a-1)[\ln(x)] + (-x^a) \\ &= n \ln(a) + (a-1) \sum_{i=1}^n \ln x - \sum_{i=1}^n x^a\end{aligned}$$

**Note:** please find the code in the appendix

Using the mle function in R, we were able to find:

$$\hat{a} = 1.194001, \quad se(\hat{a}) = 0.1762217$$

2. Thus we can find the confidence interval by using

$$\begin{aligned}CI &= \hat{a} \pm z_{0.95} \cdot se(\hat{a}) \\ &= 1.194001 \pm 1.644854 \cdot 0.1762217 \\ &= [0.9041421, 1.4838599]\end{aligned}$$

# Appendix

Question 1.3:

```
negativeLogLikelihood <- function(theta) {  
  xMinusThetaSquared = (data - theta)^2  
  sumTing <- sum(-log(1 + (data - theta)^2))  
  return (-sumTing)  
}
```

Question 2.1

```
voltage_data <- read_csv("voltage_spikes.csv")  
n <- nrow(voltage_data)  
x <- voltage_data$overvoltage  
head(x)  
  
negativeLogLikelihood <- function(alpha) {  
  sumTing <- n*log(alpha) + (alpha-1)*sum(log(x)) - sum(x^alpha)  
  return(-sumTing)  
}  
  
curve(negativeLogLikelihood, 0, 10)  
  
fit <- mle((negativeLogLikelihood), start = list(alpha = 1),  
           method = "L-BFGS-B")  
print(summary(fit))  
  
sd = 0.1762217  
alpha_hat = 1.194001  
  
lower = alpha_hat - qnorm(0.95) * sd  
upper = alpha_hat + qnorm(0.95) * sd  
  
print(lower)  
print(upper)
```