STAT 305 Assignment 3

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Question 1

1. We have $Y_i \sim Pois(\mu)$, therefore we have

$$\begin{aligned}
f_{Y_i|M}(y_i \mid \mu) &= \frac{e^{-\mu}\mu^{y_i}}{y_i!} \\
\therefore f_{Y_1,\ldots,Y_n|M}(y_1,\ldots,y_n \mid \mu) &= \prod_{i=1}^{n} \frac{e^{-\mu}\mu^{y_i}}{y_i!} \\
&= \frac{e^{-n\mu}\mu^{\sum_{i=1}^n y_i}}{\prod_{i=1}^n y_i}
\end{aligned}$$

We also have the μ takes on the prior distribution of an exponential with prior mean 10 (so $\lambda = 1/10$), therefore

$$\begin{aligned}
p_M(\mu) &= \frac{1}{10}e^{-\frac{1}{10}\cdot\mu} \\
\therefore p_{M|Y_1,\ldots,Y_n}(\mu \mid y_1,\ldots,y_n) &= \frac{\frac{1}{10}e^{-\frac{1}{10}\cdot\mu} \cdot \frac{e^{-n\mu}\mu^{\sum_{i=1}^n y_i}}{\prod_{i=1}^n y_i!}}{f_{Y_1,\ldots,Y_n}(y_1,\ldots,y_n)} \\
&\propto \frac{1}{10}e^{-\frac{1}{10}\cdot\mu} \cdot \frac{e^{-n\mu}\mu^{\sum_{i=1}^n y_i}}{\prod_{i=1}^n y_i!} \\
&\propto e^{-\frac{1}{10}\cdot\mu} \cdot e^{-n\mu}\mu^{\sum_{i=1}^n y_i} \\
&= e^{-(\frac{1}{10}+n)\mu} \cdot \mu^{\sum_{i=1}^n y_i} \\
&\sim Gamma(\sum_{i=1}^{n} y_i + 1, \frac{1}{10} + n)
\end{aligned}$$

2. Following the formula given, we have

$$
\begin{aligned}
P(Y_{1897} = 0 \mid Y) &= E[P(Y_{1987} = 0 \mid \mu) \mid Y] \\
&= E\left[\frac{e^{-\mu}\mu^0}{0!} \mid Y\right] \\
&= E[e^{-\mu} \mid Y] \\
&= \int_{-\infty}^{\infty} e^{-\mu} \cdot p_{M|Y}(\mu \mid y)\, d\mu \\
&= \int e^{-\mu} \cdot \frac{\beta^\alpha}{T(\alpha)}\mu^{\alpha-1}e^{-\beta\mu} \\
&\qquad \text{where } \alpha = \sum_{i=1}^{n} y_i + 1 \text{ and } \beta = \frac{1}{10} + n \\
&\text{notice this is } E(e^{tX}) \text{ or MGF of a Gamma when } t = -1 \\
&= \left(\frac{\beta}{\beta - (-1)}\right)^{\alpha} \\
&= \left(\frac{\frac{1}{10} + 16}{\frac{1}{10} + 16 + 1}\right)^{57+1} \\
&= \left(\frac{16.1}{17.1}\right)^{58} \\
&= 0.03034774367 \\
&\approx 0.03
\end{aligned}
$$

Question 2

1. We have the posterior density

$$
\begin{aligned}
f_{\Pi|Y}(\pi \mid y) &= \frac{f_{\Pi}(\pi) \cdot f_{Y|\Pi}(y \mid \pi)}{f_Y(y)} \\
&= \frac{\left[\sum_{i=1}^{3} \frac{1}{3} Beta(\pi, \alpha_i, \beta_i)\right] \cdot \binom{8197}{y} \pi^y (1-\pi)^{8197-y}}{\int \sum_{i=1}^{3} \frac{1}{3} Beta(\pi, \alpha_i, \beta_i) \cdot \binom{8197}{y} \pi^y (1-\pi)^{8197-y}} \\
&= \frac{\left[\sum_{i=1}^{3} \frac{1}{3} Beta(\pi, \alpha_i, \beta_i) \cdot \pi^y (1-\pi)^{8197-y}\right] \cdot \binom{8197}{y}}{\sum_{i=1}^{3} \int \frac{1}{3} Beta(\pi, \alpha_i, \beta_i) \cdot \binom{8197}{y} \pi^y (1-\pi)^{8197-y}} \\
&= \frac{\left[\sum_{i=1}^{3} \frac{1}{3} \frac{1}{B(\alpha_i,\beta_i)} \pi^{\alpha_i+y-1} (1-\pi)^{\beta_i+8197-y-1}\right] \cdot \binom{8197}{y}}{\sum_{i=1}^{3} \int \frac{1}{3} \frac{1}{B(\alpha_i,\beta_i)} \pi^{\alpha_i+y-1} (1-\pi)^{\beta_i+8197-y-1} \cdot \binom{8197}{y}} \\
&= \frac{\sum_{i=1}^{3} \left[\frac{1}{3} Beta(\pi, \alpha_i+y, \beta_i+8197-y)\right] \cdot \binom{8197}{y}}{\frac{1}{3} \frac{1}{B(\alpha_i,\beta_i)} \binom{8197}{y} \sum_{i=1}^{3} \int \pi^{\alpha_i+y-1} (1-\pi)^{\beta_i+8197-y-1}} \\
&= \frac{\sum_{i=1}^{3} Beta(\pi, \alpha_i+y, \beta_i+8197-y)}{\sum_{i=1}^{3} \frac{1}{B(\alpha_i,\beta_i)} B(\alpha_i+y, \beta_i+8197-y)} \\
&= \sum_{i=1}^{3} \pi_i^{'} \cdot Beta(\pi, \alpha_i^{'}, \beta_i^{'}) \\
\text{where } \pi_i^{'} &= \sum_{i=1}^{3} \frac{1}{B(\alpha_i,\beta_i)} B(\alpha_i+y, \beta_i+8197-y), \\
\alpha_i^{'} &= a_i + y \text{ and} \\
\beta_i^{'} &= \beta_i + 8197 - y \quad \square
\end{aligned}
$$

2. Using the posterior from the previous question,

$$
\begin{aligned}
f_{\Pi|Y}(\pi \mid y=51) &= \frac{\sum_{i=1}^{3} Beta(\pi, \alpha_i + 51, \beta_i + 8146)}{\sum_{i=1}^{3} \frac{1}{B(\alpha_i, \beta_i)} B(a_i + 51, \beta_i + 8146)} \\
&= \frac{Beta(\pi, 52, 8156) + Beta(\pi, 52, 8147) + Beta(\pi, 61, 8147)}{\frac{B(52,8156)}{B(1,10)} + \frac{B(52,8147)}{B(1,1)} + \frac{B(61,8147)}{B(10,1)}} \\
&= \frac{Beta(\pi, 52, 8156) + Beta(\pi, 52, 8147) + Beta(\pi, 61, 8147)}{3\left[\frac{1}{3}\frac{B(52,8156)}{B(1,10)} + \frac{1}{3}\frac{B(52,8147)}{B(1,1)} + \frac{1}{3}\frac{B(61,8147)}{B(10,1)}\right]} \\
&= \frac{Beta(\pi, 52, 8156) + Beta(\pi, 52, 8147) + Beta(\pi, 61, 8147)}{3 \cdot Z} \\
&= \frac{\sum_{i=1}^{3} Beta(\pi, \alpha_i + 51, \beta_i + 8146)}{3Z}
\end{aligned}
$$

Question 3

1. Using the law of total probability, we have

$$
\begin{aligned}
P(L(Y) \leq \Pi \leq R(Y)) &= E\big[P[L(Y) \leq \Pi \leq R(y) \mid Y]\big] \\
&= E[0.95] && \text{by definition of Credible Interval} \\
&= 0.95
\end{aligned}
$$

2. Since π is randomly generated from the prior and y is also randomly generated based on π so both Π and Y are random variables, and thus, as showed above, the probability of the random variable Π falling within the left and right bound ($L(Y)$ and $R(Y)$, respectively) is 95%. Therefore, it's calibrated