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Question 1

Confirmed

Question 2

1.

$$
\begin{aligned}
f_X(x) &= c \cdot \exp\left(-\frac{|x-3|}{2}\right) \\
F_X(x) &= \int_{-\infty}^{\infty} c \cdot \exp\left(-\frac{|x-3|}{2}\right) dx = 1 \\
&= c \cdot \left[\int_{-\infty}^{3} \exp\left(\frac{x-3}{2}\right)\, dx + \int_{3}^{\infty} \exp\left(\frac{-x+3}{2}\right)\, dx\right] \\
&= c(2+2) \\
\Rightarrow 4c &= 1 \\
\therefore c &= \frac{1}{4}
\end{aligned}
$$

2.

$$
\begin{aligned}
f_X(x) &= \frac{1}{4}\exp(-\frac{|x-3|}{2}) \\
M_X(t) &= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx \\
&= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{4}\exp(-\frac{|x-3|}{2})\, dx \\
&= \frac{1}{4}\int_{-\infty}^{\infty} \exp(tx + -\frac{|x-3|}{2})\, dx \\
&= \frac{1}{4}\left[\int_{-\infty}^{3} \exp\left(tx + \frac{x-3}{2}\right)\, dx + \int_{3}^{\infty} \exp\left(tx + \frac{-x+3}{2}\right)\, dx\right] \\
&= \frac{1}{4}\left[\frac{2}{1+2t}\exp\left(\frac{-3+x}{2} + tx\right)\Big|_{\infty}^{3} + \frac{2}{-1+2t}\exp\left(\frac{3-x}{2} + tx\right)\Big|_{3}^{\infty}\right] \\
&= \frac{1}{4}\left[\frac{2e^{3t}}{2t+1} - \frac{2e^{3t}}{2t-1}\right] \\
&= -\frac{e^{3t}}{(2t+1)(2t-1)}
\end{aligned}
$$

3. where is the intergral $\int e^{tx} f_X(x)\ dx$ is finite??

$$\begin{aligned}
\lim_{x\to\infty} e^{tx+\frac{-x+3}{2}} &= \lim_{x\to\infty} e^{tx-\frac{x}{2}+\frac{3}{2}} \\
&= \lim_{x\to\infty} e^{tx-\frac{x}{2}} \cdot e^{\frac{3}{2}} \\
&= \lim_{x\to\infty} e^{tx-\frac{x}{2}} \cdot \lim_{x\to\infty} e^{\frac{3}{2}} \\
&= \lim_{x\to\infty} e^{x\left(t-\frac{1}{2}\right)} \cdot \lim_{x\to\infty} e^{\frac{3}{2}}
\end{aligned}$$

in order for $\lim_{x\to\infty} e^{tx-\frac{x}{2}}$ to tend to 0, we need $\left(t-\frac{1}{2}\right) < 0$ since $\lim_{x\to\infty} e^{-x} = 0$

$$\therefore \lim_{x\to\infty} e^{tx+\frac{-x+3}{2}} = 0 \cdot e^{\frac{3}{2}} = 0 \qquad \text{for } t < \frac{1}{2}$$

$$\text{similarly, } \lim_{x\to-\infty} e^{tx+\frac{x-3}{2}} = 0 \qquad \text{for } t > -\frac{1}{2}$$

$\therefore$ **The MGF is defined on the neighbourhood of** $-\frac{1}{2} < t < \frac{1}{2}$

4.

$$\begin{aligned}
M_X'(t) &= \frac{d}{dt}\left(-\frac{e^{3t}}{(2t+1)(2t-1)}\right) \\
&= -\frac{3e^{3t}(2t+1)(2t-1) - 8e^{3t}t}{(2t+1)^2(2t-1)^2}
\end{aligned}$$

5.

$$\begin{aligned}
M_X''(t) &= \frac{d}{dt}\left[-\frac{3e^{3t}(2t+1)(2t-1) - 8e^{3t}t}{(2t+1)^2(2t-1)^2}\right] \\
&= -\frac{e^{3t}(144t^4 - 192t^3 + 24t^2 + 48t + 17)}{(2t+1)^3(2t-1)^3}
\end{aligned}$$

6.

$$\begin{aligned}
E[X] &= M_X'(t) \text{ at } t = 0 \\
&= -\frac{3e^{3t}(2t+1)(2t-1) - 8e^{3t}t}{(2t+1)^2(2t-1)^2}\Big|t = 0 \\
&= 3
\end{aligned}$$

7.

$$
\begin{aligned}
E[X^2] &= M_X''(t) \text{ at } t = 0 \\
&= -\frac{e^{3t}(144t^4 - 192t^3 + 24t^2 + 48t + 17)}{(2t+1)^3(2t-1)^3}\Big| t = 0 \\
&= 17
\end{aligned}
$$

$$
\begin{aligned}
V[X] &= E[X^2] - (E[X])^2 \\
&= 17 - (3)^2 \\
&= 17 - 9 \\
&= 8
\end{aligned}
$$

Question 3

1.

$$
\begin{aligned}
M_{Y_0 Z_0}(t) &= E\left[e^{tY_0 Z_0}\right] \\
&= E\left[E\left[e^{(tZ_0)Y_0} \mid Z_0\right]\right] \\
&= E\left[M_{Y_0}(tZ_0)\right] \\
&= E\left[\frac{\lambda}{\lambda - tZ_0}\right] && \text{since } Y \sim Exp(\lambda) \\
&= \sum_{i=1}^{\infty} \frac{\lambda}{\lambda - tz_0} \cdot f_{Z_0}(z_0) \\
&= \frac{\lambda}{\lambda - 0} \cdot 0.5 + \frac{\lambda}{\lambda - t} \cdot 0.5 && \text{since } X \sim Bern(0.5) \\
&= \frac{1}{2} + \frac{\lambda}{2(\lambda - t)} \\
&= \frac{(\lambda - t) + \lambda}{2(\lambda - t)} \\
&= \frac{2\lambda - t}{2(\lambda - t)}
\end{aligned}
$$

2.

$$\text{let } L = \frac{1}{2}X_1 \qquad \therefore M_L(t) = M_{X_1}\left(\frac{1}{2}t\right)$$

$$M_{Y_1Z_1}(t) = M_{Y_0Z_0}(t) = M_{X_1}(t) \qquad \text{since their distribution doesn't depend on } n$$

$$\begin{aligned} M_{X_2}(t) &= M_L(t) \cdot M_{Y_1Z_1}(t) && \text{via independence} \\ &= M_{X_1}\left(\frac{1}{2}t\right) \cdot M_{X_1}(t) \\ &= \frac{2\lambda - \frac{t}{2}}{2(\lambda - \frac{t}{2})} \cdot \frac{2\lambda - t}{2(\lambda - t)} \\ &= \frac{4\lambda - t}{4(\lambda - t)} \end{aligned}$$

3. Through extrapolation from the results of M_{X_3} and M_{X_4}

$$M_{X_n}(t) = \frac{2^n(\lambda) - t}{2^n(\lambda - t)}$$

4. We can prove this by taking the limit of the MGF of X_n as n approaches infinity

$$\begin{aligned} \lim_{n\to\infty} M_{X_n}(t) &= \lim_{n\to\infty} \frac{2^n(\lambda) - t}{2^n(\lambda - t)} \\ &= \frac{1}{\lambda - t} \cdot \lim_{n\to\infty} \left(\frac{2^n\lambda - t}{2^n}\right) \\ &= \frac{1}{\lambda - t} \cdot \lim_{n\to\infty} \left(\frac{2^n\lambda\left(1 - \frac{t}{2^n\lambda}\right)}{2^n}\right) \\ &= \frac{1}{\lambda - t} \cdot \lim_{n\to\infty} \left(-\frac{t}{2^n} + \lambda\right) \\ &= \frac{1}{\lambda - t} \cdot \left[\lim_{n\to\infty}(\lambda) - \lim_{n\to\infty}\left(\frac{t}{2^n}\right)\right] \\ &= \frac{1}{\lambda - t}(\lambda - 0) \\ &= \frac{\lambda}{\lambda - t} \\ &= \text{ MGF of the Exponential} \\ &\therefore X_n \xrightarrow{d} X \text{ where the X is exponentially distributed} \end{aligned}$$

Question 4

1.

$$\begin{aligned}
&\mu_{\text{op}} \pm z_{0.975}\frac{\sigma}{\sqrt{n}}\\
&\mu_{\text{op}} \pm 1.96\frac{4.2}{\sqrt{27}}\\
\Rightarrow\ &93.5 \pm 1.96\frac{4.2}{\sqrt{27}}\\
=&[91.9, 95.1]
\end{aligned}$$

2. (a) $\exists\mu = \mu_{\text{non-op}} = \mu_{\text{op}}$, and because of this we can say $\mu \in I_{\text{non-op}}$ and $\mu \in I_{\text{op}}$. And since there's a value that's in both sets (interval), by definition, $I_{\text{non-op}} \cap I_{\text{o}} \neq \varnothing$

   (b)

$$\begin{aligned}
P(R) &\leq P\left[(\mu_{\text{non-op}} \notin I_{\text{non-op}}) \vee (\mu_{\text{op}} \notin I_{\text{op}})\right]\\
&\leq P(\mu_{\text{non-op}} \notin I_{\text{non-op}}) + P(\mu_{\text{op}} \notin I_{\text{op}}) - P\left[(\mu_{\text{non-op}} \notin I_{\text{non-op}}) \wedge (\mu_{\text{op}} \notin I_{\text{op}})\right]\\
&\leq P(\mu_{\text{non-op}} \notin I_{\text{non-op}}) + P(\mu_{\text{op}} \notin I_{\text{op}}) - P(\mu_{\text{non-op}} \notin I_{\text{non-op}})P(\mu_{\text{op}} \notin I_{\text{op}})\\
&\leq [1 - P(\mu_{\text{non-op}} \in I_{\text{non-op}})] + [1 - P(\mu_{\text{op}} \in I_{\text{op}})]\\
&\qquad - [1 - P(\mu_{\text{non-op}} \in I_{\text{non-op}})][1 - P(\mu_{\text{op}} \in I_{\text{op}})]\\
&\leq 0.05 + 0.05 - 0.05(0.05)\\
&\leq 0.0975
\end{aligned}$$

3. Theoretically we can since we just need to actually compute the intersection of the two, perhaps through conditional probability, but with the currently given info, it is not possible. However, since we know that $P(A \cap B) \geq 0$, we can say

$$\begin{aligned}
P(R) &\leq P(A) + P(B) - P(A \cap B)\\
&\leq P(A) + P(B) && \text{since } P(A + B) \geq 0\\
&\leq P(A) + P(B)\\
&\leq 0.05 + 0.05\\
&= 0.1
\end{aligned}$$

   So thus, we are still able to get an upper bound of $P(R) \leq 0.1$ without independence. But exact calculations is not possible without independence