L01 23. 11. 2024

THM: $V = \mathbb{R}^n$ (over the field $\mathbb{R}$). Then a function

$$f : \mathbb{R}^n \to \mathbb{R}$$

is <u>linear</u> (so $\forall \alpha, \beta \in \mathbb{R}, \forall x, y \in \mathbb{R}^n;\ f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$)

if only if

$$\exists a \in \mathbb{R}^n \quad \forall x \in \mathbb{R}^n \qquad f(x) = \boxed{a \cdot x} = \sum_{i=1}^{n} a_i x_i .$$

In addition, in such case, a is unique.

(CAUCHY SCHWARTZ INEQUALITY $\quad |a \cdot x| \leq \|a\| \cdot \|x\|$.)

DEF: In V is a vector space over $\mathbb{R}$ then we define its:

<u>ALGEBRAIC DUAL</u> $\quad V^* := \{ f : V \to \mathbb{R} \text{ LINEAR} \}$.

<u>TOPOLOGICAL DUAL</u> $\quad V' := \{ f : V \to \mathbb{R} \text{ LINEAR } \underline{\text{AND CONTINUOUS}} \}$

SO BY THM ABOVE CAN BE STATED AS: $(\mathbb{R}^n)^* \simeq \mathbb{R}^n$

COROLLARY: $(\mathbb{R}^n)' \simeq \mathbb{R}^n$ (for every integer $n \geq 1$).

MORE EXPLICITLY, <u>ALL</u> LINEAR FUNCTIONS ARE ALSO CONTINUOUS.

WE ARE GOING TO SHOW THAT A LINEAR FUNCTION

$$f(x) = a \cdot x$$

IS LIPSCHITZ CONTINUOUS, i.e.,

$$\overset{?}{\exists} C > 0 \quad \forall x, y \in \mathbb{R}^n \quad |f(x) - f(y)| \leq C \|x - y\|.$$

INDEED.

$$|f(x) - f(y)| \overset{def}{=} |a \cdot x - a \cdot y|$$

$$\overset{\text{LINEARITY}}{=} |a \cdot (x - y)|$$

$$\underset{\text{INEQ.}}{\overset{\text{C.S.}}{\leq}} \underbrace{\boxed{\|a\|}}_{= C.} \cdot \|x - y\|.$$

MAIN PROBLEM: FIND EXPLICITLY THE TOPOLOGICAL DUAL (CERTAIN VECTOR SUBSPACES ?) OF

$$B(S) := \{ f : S \to \mathbb{R} \text{ BOUNDED} \},$$
$$= \{ f : S \to \mathbb{R}, \sup_{x \in S} |f(x)| < \infty \}$$

WHERE S IS A GIVEN SET.

$B(S)$ IS A NORMED SPACE, WHERE THE NORM IS DEFINED BY

$$\forall f \in B(S) \quad \|f\|_\infty := \sup_{x \in S} |f(x)|.$$

$$\left(\begin{array}{l}\|f\| \geq 0 \quad \forall f \\ \|f\|=0 \Leftrightarrow f=0 \\ \|\alpha f\|=|\alpha| \cdot \|f\| \\ \|f+g\| \leq \|f\|+\|g\|\end{array}\right)$$

THM: $(B(S), \|\cdot\|)$ IS A BANACH SPACE (THAT IS, IT IS A NORMED VECTOR SPACE WHICH IS COMPLETE).

DEF: FIX A SET S. A FAMILY OF SUBSETS $\Sigma \subseteq P(S)$ IS A CALLED ALGEBRA (ON S) IF:

(i) $\Sigma \neq \emptyset$.

(ii) $\forall A, B \in \Sigma \Longrightarrow A \cup B \in \Sigma$

(iii) $\forall A \in \Sigma \longrightarrow A^c := S \setminus A \in \Sigma$.

(IF (ii) IS REPLACED BY $\forall (A_n) \in \Sigma$, $\bigcup_{n=1}^{\infty} A_n \in \Sigma$, THEN Σ IS CALLED σ-ALGEBRA)

EXAMPLES

1. $\Sigma = P(S)$ IS AN ALGEBRA, GIVEN AN ARBITRARY SET S.

2. $\Sigma = \{\emptyset, S\}$ IS AN ALGEBRA.

3. $S := \mathbb{N} = \{0, 1, 2, \ldots\}$

$\Sigma := \{A \subseteq \mathbb{N} : A$ is finite or A^c is finite$\}$.

(all three properties are verified)

4. $S := \mathbb{N} = \{0, 1, 2, \dots\}$

$$\Sigma := \{A \subseteq \mathbb{N} : A \cap 2\mathbb{N} \text{ is finite or } 2\mathbb{N} \setminus A \text{ is finite}\}$$

5. $S := \{1, 2, 3\}$

$$\Sigma := \{\emptyset, \boxed{\{1\}}, \boxed{\{2,3\}}, \{1, 2, 3\}\}.$$

? 6. $S = \mathbb{N} = \{0, 1, 2, \dots\}$

$$\Sigma = \{\{0, n\} : n \geq 1\}.$$

IS NOT ALGEBRA, BUT IT SATISFIES THIS WEAKER PROPERTY:

(ii') $\forall A, B \in \Sigma, \quad A \cap B = \emptyset \Longrightarrow A \cup B \in \Sigma$.

DEF: FIX S AND AN ALGEBRA $\Sigma \subseteq P(S)$.
A FUNCTION $f : S \to \mathbb{R}$ IS SAID TO BE

Σ- MEASURABLE:

IF:

$\forall$ interval $I \subseteq \mathbb{R} \qquad f^{-1}[I] \in \Sigma$.

EXAMPLES: • $S = \mathbb{R}$ $\quad \Sigma' = \{\emptyset, (-\infty, 1), [1, +\infty), \mathbb{R}\}$

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto x.$$

f IS NOT Σ'-MEASURABLE: FOR EXAMPLE, IF $I = (0,1)$ THEN

$$f^{-1}[I] = (0,1) \notin \Sigma'.$$

• $S = \mathbb{R}$ $\quad \Sigma' = \{\emptyset, (-\infty, 1), [1, +\infty), \mathbb{R}\}$

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto \sqrt{2}$$

PICK $I \subseteq \mathbb{R}$ INTERVAL. THEN $f^{-1}[I] = \begin{cases} \mathbb{R} & \text{if } \sqrt{2} \in I \\ \emptyset & \text{if } \sqrt{2} \notin I \end{cases}$

(LEMMA: IF $f : S \to \mathbb{R}$ IS CONSTANT, THEN f IS Σ-MEASURABLE FOR <u>EVERY</u> CHOICE OF Σ').

• $S = \mathbb{R}$ $\quad \Sigma' = \{\emptyset, (-\infty, 1), [1, +\infty), \mathbb{R}\}$

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto \begin{cases} \sqrt{2} & \text{if } x < 1 \\ \sqrt{3} & \text{if } x \geq 1. \end{cases}$$

$\forall I \subseteq \mathbb{R}$ INTERVAL,

$$f^{-1}[I] = \begin{cases} \emptyset & \text{if } \sqrt{2}, \sqrt{3} \notin I \\ (-\infty, 1) & \text{if } \sqrt{2} \in I, \sqrt{3} \notin I \\ [1, \infty) & \text{if } \sqrt{2} \notin I, \sqrt{3} \in I \\ \mathbb{R} & \text{if } \sqrt{2}, \sqrt{3} \in I. \end{cases}$$

So f is Σ-MEASURABLE

"IN GENERAL, IF IS CONSTANT ON MEMBERS OF Σ (FINITELY MANY TIMES), THEN f IS Σ-MEASURABLE".

DEF: FIX S AND AN ALGEBRA $\Sigma \subseteq P(S)$.

THEN A FUNCTION $f: S \to \mathbb{R}$ IS SAID TO BE **SIMPLE**

IF:

(i) f IS Σ-MEASURABLE

(ii) $\operatorname{Im}(f) (:= f[S])$ IS FINITE.

THE SET OF SIMPLE FUNCTIONS IS DEFINED $B_0(\Sigma)$

EXERCISE: ① A FUNCTION $f: S \to \mathbb{R}$ IS SIMPLE IF AND ONLY IF:

(i') $\forall x \in \mathbb{R}$, $f^{-1}(x) \in \Sigma$.

(ii) $\operatorname{Im}(f)$ IS FINITE.

② PROVE THAT $B_0(\Sigma)$ IS A VECTOR SUBSPACE OF THE BANACH SPACE $(B(S), \|\cdot\|)$.

DEF: FIX S AND ALGEBRA $\Sigma \subseteq P(S)$. DEFINE

$$B(\Sigma) := \overline{B_0(\Sigma)}$$

closure of $B_0(\Sigma)$ in the bigger

space $(B(S), \|\cdot\|)$

"SMALLEST CLOSED SET"
$=$ CONTAINING $B_0(\Sigma)$

$$= \left\{ \lim_{n\to\infty} f_n : \forall n,\ f_n \text{ is SIMPLE} \right\}.$$

def.

$$\text{"}\lim_{n\to\infty} f_n = f\text{"} \iff \text{"}\lim_{n\to\infty} d(f_n, f) = 0$$

$$\iff \lim_{n\to\infty} \|f_n - f\| = 0.$$

$$\iff \lim_{n\to\infty} \sup_{x\in S} |(f_n - f)(x)| = 0$$

EXERCISE: 3 PROVE THAT $B(\Sigma)$ $(= \overline{B_0(\Sigma)})$ IS ALWAYS A VECTOR SPACE.

4 PROVE THAT $B(\Sigma)$ IS A BANACH SPACE, THAT IS, A NORMED COMPLETE VECTOR SPACE.

MAIN PROBLEM:

FIND THE TOPOLOGICAL DUAL OF

$B(\Sigma')$, THAT IS.

$$(B(\Sigma'))' := \{T : B(\Sigma') \to \mathbb{R} \text{ LINEAR AND CONTINUOUS}\}.$$

$$\text{SO: } (B(\Sigma'))' = \ ?.$$

REPRESENTATIONS OF SIMPLE FUNCTIONS.

$$B_0(\Sigma') = \{f : S \to \mathbb{R} \ \ \Sigma\text{-MEASURABLE \& FINITE IMAGE}\}$$

$$= \left\{ f(x) = \begin{cases} \alpha_1 & \text{if } x \in A_1 \ (\in \Sigma') \\ \alpha_2 & \text{if } x \in A_2 \ (\in \Sigma') \\ \vdots & \vdots \\ \alpha_n & \text{if } x \in A_n \ (\in \Sigma') \end{cases} \right\}$$

$$= \left\{ f(x) = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{A_i}(x) \text{ with } \begin{array}{l} \alpha_1, \dots, \alpha_n \in \mathbb{R} \\ A_1, \dots, A_n \in \Sigma' \end{array} \right\}.$$

$$\mathbb{1}_{A_i}(x) = \begin{cases} 1 & \text{if } x \in A_i \\ 0 & \text{if } x \notin A_i \end{cases}$$

- THIS REPRESENTATION IS <u>NOT</u> UNIQUE:

FOR INSTANCE, CONSIDER

$$S = \mathbb{R} \qquad \Sigma = \{\emptyset, (-\infty, 1), [1, \infty), \mathbb{R}\}$$

CONSIDER THE FUNCTION $\quad f : S \longrightarrow \mathbb{R}$

$$x \longmapsto \begin{cases} 1 & \text{if } x < 1 \\ 2 & \text{if } x \geq 1. \end{cases}$$

f is SIMPLE.

$$f(x) = \mathbb{1}_{(-\infty, 1)}(x) + 2 \cdot \mathbb{1}_{[1, \infty)}(x)$$

$$= \mathbb{1}_{\mathbb{R}}(x) + \mathbb{1}_{[1, \infty)}(x).$$

"CANONICAL REPRESENTATION OF $f \in \mathcal{B}_0(\Sigma)$"

$$f = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{A_i}, \quad \text{where}$$

$\alpha_1 < \alpha_2 < \alpha_3 < \dots < \alpha_n \; (\in \mathbb{R})$ and

$A_1, A_2, \ldots, A_n$ IS PARTITION OF S,
$A_1, \ldots, A_n \in \Sigma$.

ABOUT $B(\Sigma)$: $B(\Sigma) := \overline{B_0(\Sigma)}$

$\stackrel{?}{=} \cancel{\{f \in B(S), f \text{ MEASURABLE}\}}$.

- ① $\{f \in B(S) : f \text{ MEASURABLE}\} \subseteq B(\Sigma)$
- ② $\exists (S, \Sigma)$ $\{f \in B(S) : f \text{ MEASURABLE}\} \neq B(\Sigma)$
- ③ IF Σ IS σ-ALGEBRA, THEN

$$\{f \in B(S) : f \text{ MEASURABLE}\} = B(\Sigma)$$

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MEASURES

FIX A SET S AND AN ALGEBRA $\Sigma \subseteq P(S)$:

DEF: A MAP $\mu : \Sigma \longrightarrow \mathbb{R}$ IS A CHARGE IF:

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

FOR EVERY $A, B \in \Sigma$ SUCH THAT $A \cap B = \phi$.

EXAMPLES: • $S = \mathbb{N}$ $\quad \Sigma = P(\mathbb{N})$

DEFINE
$$\mu : \Sigma \to \mathbb{R}$$
$$A \longmapsto \underbrace{\sum_{n\in A} \frac{1}{2^n}}_{\mu(A)}$$

FOR EXAMPLE
$$\mu(\mathbb{N}) = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2.$$
$$\mu(2\mathbb{N}) = \sum_{n=0}^{\infty} \frac{1}{2^{2n}} = \sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}.$$
$$\mu(\{0\}) = \sum_{n\in\{0\}} \frac{1}{2^n} = \frac{1}{2^0} = 1.$$

FIX $A, B \subseteq \mathbb{N}$

$A \cap B = \phi$
$$\mu(A\cup B) \overset{?}{=} \mu(A) + \mu(B)$$
$$\sum_{n\in A\cup B} \frac{1}{2^n} = \sum_{n\in A} \frac{1}{2^n} + \sum_{n\in B} \frac{1}{2^n}$$

• $\mu : \Sigma \to \mathbb{R}$
$$A \longmapsto \underbrace{0}_{\mu(A)}$$
$$\mu(A\cup B) \overset{?}{=} \mu(A) + \mu(B)$$

CHARGE. $\boxed{\checkmark}$

• $S = \mathbb{N}$ $\quad \Sigma' = \{ A \subseteq \mathbb{N} : A \text{ finite or } A^c \text{ finite} \}$.

$$\mu : \Sigma' \longrightarrow \mathbb{R}$$

$$A \longmapsto \begin{cases} 1 & \text{if } A^c \text{ finite.} \\ 0 & \text{if } A \text{ finite} \end{cases}$$

FIX $A, B \in \Sigma'$ WITH $\underline{A \cap B = \emptyset}$. THEN $\mu(A \cup B) \overset{?}{=} \mu(A) + \mu(B)$

CASE 1: A fin, B fin $\qquad 0 = 0 + 0$

CASE 2: A fin, B^c fin $\qquad 1 = 0 + 1$

CASE 3: A^c fin, B fin $\qquad 1 = 1 + 0$.

CASE 4: ~~A^c finite & B^c finite~~

$\Downarrow$

$A^c \cap B^c \neq \emptyset$.

EXERCISE: ⑤ PROVE THAT THE FAMILY

$$\{\mu : \Sigma \to \mathbb{R} \text{ CHARGE}\}$$

IS A VECTOR SUBSPACE OF $\mathbb{R}^{\Sigma}$.

<u>DEF</u>: IF $\mu : \Sigma \to \mathbb{R}$ IS A CHARGE, DEFINE

$$\|\mu\| := \sup_{A \in \Sigma} |\mu(A)|.$$

WE SAY "<u>μ IS BOUNDED</u>" IF $\|\mu\| < \infty$. DEFINE

$$ba(\Sigma) := \{\mu : \Sigma \to \mathbb{R} \ \underline{\text{CHARGE}} \ \underline{\text{BOUNDED}}\}$$

BOUNDED ADDITIVE

(THERE IS AN EXAMPLE OF $\mu : \Sigma \to \mathbb{R}$ WITH $\|\mu\| = +\infty$).

(<u>EXAMPLE</u>: $S = \mathbb{N}$ $\Sigma = \{A \subseteq \mathbb{N} : A$ finite or)

A^c finite$\}$

$$\left(\forall A \in \Sigma', \quad \mu(A) := \sum_{n \in A} \frac{(-1)^n}{n+1}. \right)$$

EXERCISE 6: PROVE THAT

$ba(\Sigma')$ IS A NORMED VECTOR SPACE.

LEMMA: FIX S AND ALGEBRA $\Sigma' \subseteq P(S)$.

If $\mu : \Sigma' \to \mathbb{R}$ CHARGE THEN

$$\mu(\emptyset) = 0.$$

Proof: If $A = B = \emptyset$ then

$$\mu(\emptyset \cup \emptyset) = \mu(\emptyset) + \mu(\emptyset)$$

$$\Downarrow$$

$$\cancel{\mu(\emptyset)} = \cancel{\mu(\emptyset)} + \mu(\emptyset)$$

$$\Downarrow$$

$$\mu(\emptyset) = 0. \quad \blacksquare$$

LEMMA: FIX S AND ALGEBRA $\Sigma' \subseteq P(S)$.

If $\mu : \Sigma' \to \mathbb{R}$ CHARGE BOUNDED $(\mu \in ba(\Sigma'))$

THEN:

$\forall A_1, A_2, \ldots, A_n$ DISJOINT $(A_1, \ldots, A_n \in \Sigma)$ $\quad \sum_{i=1}^{n} |\mu(A_i)| \leq 2\|\mu\|.$

FAIL STRATEGY: $\sum_{i=1}^{n} |\mu(A_i)| \leq \sum_{i=1}^{n} \|\mu\| = n \|\mu\|.$ TOO WEAK!

2???

PROOF:

$$\sum_{i=1}^{n} |\mu(A_i)| = \sum_{i:\mu(A_i)>0} |\mu(A_i)| + \sum_{i:\mu(A_i)\leq 0} |\mu(A_i)|$$

$$= \Big|\mu\Big(\underbrace{\bigcup_{i:\mu(A_i)>0} A_i}_{:=A}\Big)\Big| + \Big|\mu\Big(\underbrace{\bigcup_{i:\mu(A_i)\leq 0} A_i}_{:=B}\Big)\Big|$$

$$= |\mu(A)| + |\mu(B)|$$

$$\leq \|\mu\| + \|\mu\|$$

$$= 2\|\mu\|.$$