Homework 2

Kevin Shuman

January 2020

1

To show what's asked, I first want to write my function $f(x)$ as a Taylor expansion, where I'm going to leave the coefficients as just letters to save writing:

$$f(x) = a + bx + cx^2 + dx^3 + ex^4 + gx^5 + ... \tag{1}$$

The next step is to integrate this function to find the exact form of what $f(x)$ is so we have something to compare our approximation with.

$$\int_{-h/2}^{h/2} f(x)dx = [ax + \frac{b}{2}x^2 + \frac{c}{3}x^3 + \frac{d}{4}x^4 + \frac{e}{5}x^5 + \frac{g}{6}x^6 + ...]\Big|_{-\frac{h}{2}}^{\frac{h}{2}} . \tag{2}$$

Due to the symmetry in the integration limits, the even terms cancel and we're left with

$$\int_{-h/2}^{h/2} f(x)dx = ah + \frac{ch^3}{12} + \frac{eh^5}{80} + ... \tag{3}$$

Assuming the same form for $f(x)$ and knowing $f_1 = f(\eta = -1)$, $f_2 = f(\eta = 0)$, $f_3 = f(\eta = 1)$, where $\eta = \frac{x}{h}$, we can determine what the integral approximates to. The reason $\eta = \frac{x}{h}$ is because we are looking at one interval the size of h, this also tells us we are evaluating $f(x)$ at $x = -h$, $x = 0$, $x = h$ for f_1, f_2, f_3, respectively. The result follows,

$$\begin{aligned}\int_{-h/2}^{h/2} f(x)dx &\simeq h * [\frac{1}{24}f_1 + \frac{11}{12}f_2 + \frac{1}{12}f_3] \\ &= ah + \frac{ch^3}{12} + \frac{eh^5}{12}.\end{aligned} \tag{4}$$

Subtracting out results to get the error, we get, $e = \frac{f(0)^{(4)}}{4!}$,

$$error \simeq \frac{17}{240}(\frac{f(0)^{(4)}}{4!})h^5. \tag{5}$$

To show the error for the extended method goes as $\frac{1}{N}$, N the number of steps in the sum, we need only divide our interval, h, by N and multiply out error by N. We get,

$$
\begin{aligned}
error &\simeq \frac{17}{240}(\frac{f(0)^{(4)}}{4!})(\frac{h}{N})^5 N \\
&\sim \frac{1}{N^4}.
\end{aligned} \tag{6}
$$

2

We want to formulate a function that interpolates three points by a quadratic polynomial, call it $y(x)$. To do this, I follow the example done in class.

Say I know the three points and they are spaced evenly by h. Due to the symmetry, I'm going to say the points are evaluated at $x = -h$, $x = 0$, and $x = h$ for points f_1, f_2, and f_3, respectively. To find the quadratic, I have at $x = -h$ that f_1, similarly for the other points, so what gives me that?

$$
y(x) = (\frac{x^2 - xh}{2h^2})f_1 - ((\frac{x}{h})^2 - 1)f_2 + (\frac{x^2 + xh}{2h^2})f_3. \tag{7}
$$

Now, we want to integrate this quadratic over the three domains surrounding each known f. Call these integrals 1, 2, and 3 in the following order,

$$
\begin{aligned}
\int_{-h/2}^{-3h/2} y(x)dx &= h * [\frac{25}{24}f_1 - \frac{1}{12}f_2 + \frac{1}{24}f_3] \\
\int_{-h/2}^{h/2} y(x)dx &= h * [\frac{1}{24}f_1 + \frac{11}{12}f_2 + \frac{1}{12}f_3] \\
\int_{h/2}^{3h/2} y(x)dx &= h * [\frac{1}{24}f_1 - \frac{1}{12}f_2 + \frac{25}{24}f_3].
\end{aligned} \tag{8}
$$

To create the extended Tres Hermanos rule, we follow the by adding these integrals together for different points in the way followed on page four in the notes: `http://solar.physics.montana.edu/kankel/ph567/LectureNotes/02.1.HigherOrder.pdf`.

The final result is an approximation to our integral, given N points described by

$$
f_n = f(-\frac{h}{2} + (n - \frac{1}{2})H), H = \frac{h}{N}, \tag{9}
$$

is

$$
\int_{-h/2}^{h/2} f(x)dx \simeq h[\frac{26}{25}f_1 + \frac{21}{24}f_2 + \frac{25}{24}f_3 + \sum_{n=4}^{N-3} f_n + \frac{25}{24}f_{N-2} + \frac{21}{24}f_{N-1} + \frac{26}{24}f_N]. \tag{10}
$$

The integral given isn't in this form, but a simply substitution, $x = y + \frac{1}{2}$ can get us there,

$$
\int_0^1 \frac{(1-x)^{\frac{1}{3}}}{\ln(x)}dx = \int_{-1/2}^{1/2} \frac{(\frac{1}{2} - y)^{\frac{1}{3}}}{\ln(\frac{1}{2} + y)}dy. \tag{11}
$$

But wait, there's more! Clearly there is a singularity in the integrand, which will screw up the numerical integration. To avoid this, we learned a trick in class to handle it, add zero! The process is given here: `http://solar.physics.montana.edu/kankel/ph567/LectureNotes/02.0a.IntegratingSingularities.pdf`.

The result gives us the following integral to numerically evaluate:

$$\int_{-1/2}^{1/2}[\frac{(\frac{1}{2}-y)^{\frac{1}{3}}}{\ln{(\frac{1}{2}+y)}}-(\frac{1}{2}-y)^{-\frac{2}{3}}]dy-3. \tag{12}$$

3

I calculated the integral using Mathematica to 15 significant figures to have a value to compare my results with and to get an estimate of the errors. The value I got is:

$$\int_{-1/2}^{1/2}[\frac{(\frac{1}{2}-y)^{\frac{1}{3}}}{\ln{(\frac{1}{2}+y)}}-(\frac{1}{2}-y)^{-\frac{2}{3}}]dy-3=-2.5562482319856224262. \tag{13}$$

Due to computational limitations, I was only able to reach 9 significant figures with my code. For every increase in order in steps, the time it took to run the code increase by that same order until the number of steps reached 10 billions, where it was significantly shorter than expected, 30 minutes as opposed to 3 and a half hours. This leads me to believe I wasn't able to create a variable with enough memory and the code decided to stop.

The value I reached with 1 billion steps was:

$$\int_{-1/2}^{1/2}[\frac{(\frac{1}{2}-y)^{\frac{1}{3}}}{\ln{(\frac{1}{2}+y)}}-(\frac{1}{2}-y)^{-\frac{2}{3}}]dy-3=-2.556248232027416. \tag{14}$$

Below shows the log of the errors verse the log of the step size. As can be seen, the error isn't going as the expected $\frac{1}{N^4}$, but more like $\frac{1}{N}$.