
Discrete Logarithm Based Cryptography with Abelian Varieties

A COMPARISON OF TWO MODERN TECHNIQUES:
THE GENERAL NUMBER FIELD SIEVE AND
SHOR'S QUANTUM ALGORITHM

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Abstract

WE DO SOME STUFF.

Contents

1	The Discrete Logarithm Problem	3
1.1	The Diffie-Hellman Key Exchange	3
1.2	A Brief History of the Groups \mathbb{F}_p^* and $\mathbb{F}_{2^n}^*$	3
2	Abelian Varieties	4
2.1	Dimension	5
2.2	Genus	5
3	Elliptic Curves	5
3.1	The Group Operation	5
3.2	Scalar Multiplication of a Point	6
3.3	Finding Points	6
3.4	Counting Points	8
3.5	Generating Provably Random Elliptic Curves	9
3.6	Parameter Selection	9
4	Hyperelliptic Curves	9
4.1	The Jacobian of a Hyperelliptic Curve	9
4.2	Representation of Divisors	10
4.3	The Group Operation	10
4.4	Scalar Multiplication	11
4.5	Counting Points	11
4.6	Parameter Selection	11
5	Algorithms	11
6	References	13

1 The Discrete Logarithm Problem

Given an arbitrary finite cyclic group G with group operation \cdot and generator $g \in G$, discrete exponentiation by a in G is defined by

$$g^a = \overbrace{g \cdot g \cdots g}^{a \text{ times}}$$

If $y = g^a$ is known, computing a is called finding the discrete logarithm of y . With the method of fast exponentiation, y can be computed quickly, only $O(\log a)$ group operations. On the other hand, computing a can be much harder. In fact, in [2] it was shown that in an arbitrary group for which only the group operation and discrete exponentiation can be applied to group elements, computing discrete logarithms will take at least $O(\sqrt{|G|})$ operations. In most cases though, more structure is known about the group in use.

1.1 The Diffie-Hellman Key Exchange

This one-way property of discrete exponentiation has proven to be very useful for cryptographic purposes. The most notable of these is in the Diffie-Hellman Key Exchange Protocol in which two parties A and B wish to share a secret key k .

Algorithm 1 Diffie-Hellman Key Exchange Protocol

- 1: A and B share a publicly known group G and generator g .
 - 2: A chooses a random private exponent a and computes g^a .
 - 3: B chooses a random private exponent b and computes g^b .
 - 4: A sends g^a to B and B sends g^b to A .
 - 5: A raises g^b to their own private exponent a to obtain $k = (g^b)^a = g^{ab}$.
 - 6: B raises g^a to their own private exponent b to obtain $k = (g^a)^b = g^{ba}$.
-

The two parties may now use k to communicate with any cryptographically secure communication protocol needing a symmetric key. The described protocol relies on the hardness of computing g^{ab} given g^a and g^b , which is conjectured in [3] to be equivalent to computing discrete logarithms. This protocol has the nice property that it's applicable to any finite cyclic group. We look at some simple examples of groups which have already had significant use in cryptography.

1.2 A Brief History of the Groups \mathbb{F}_p^* and $\mathbb{F}_{2^n}^*$

Given a finite field \mathbb{F} , the multiplicative units $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$ form a finite cyclic group and thus may be used for discrete logarithm based cryptography. It is a standard result from algebra that every finite field has order p^n , where p is a prime and $n \in \mathbb{Z}^+$. We divide the discussion of the cryptographic properties of the group $\mathbb{F}_{p^n}^*$ into two cases; when $n = 1$ and when $n > 1$.

In the latter case, when working with finite fields of order p^n , the arithmetic is only really tractable when $p = 2$. In the 1980's, researchers at the University of Waterloo made attempts to construct discrete logarithm cryptosystems based on $\mathbb{F}_{2^{127}}^*$ which initially paralleled RSA in terms of bits of security. But in 1986, Don Coppersmith devised an astonishing algorithm in [7] which could compute

discrete logarithms in the group $\mathbb{F}_{2^{127}}^*$ in about 5 minutes. Further attempts were made to increase the size to $n = 593$ but similar adaptations of Copper-smith's algorithm made researchers abandon public key cryptosystems based on the discrete logarithm problem in $\mathbb{F}_{2^n}^*$.

When $n = 1$, we have a group which is essentially just the non-zero integers mod a prime. As one might expect, when p is small, computing discrete logs in \mathbb{F}_p^* can be done quickly with just trial exponentiation. When p is large though, say $p = 2^{1000}$, this method becomes completely intractable, even on today's fastest computers. That being said, there is an attack described in [6] which adapts the Number Field Sieve to solve discrete logs in \mathbb{F}_p^* . This attack has running time very similar to factoring

$$O(p) = e^{1.923(\log p)^{1/3}(\log \log p)^{2/3}}$$

This basically means that the bit length required for p in \mathbb{F}_p^* based cryptosystems is the same as the bit length required for the modulus in RSA. In today's standards that means $p = 2^{2048}$. Although this is cryptographically viable, in practice using such large values of p has its limitations, such as bandwidth in network communications or memory in a hand-held device. These two cases made researchers search for alternative groups with cryptographically strong properties.

2 Abelian Varieties

In this report we focus on groups which arise from the solution set of polynomial equations over finite fields. The most famous of these groups is the set of points on an elliptic curve characterized by the equation $y^2 = x^3 + ax + b$ where $a, b \in \mathbb{F}_p$ satisfy $4a^3 + 27b^2 \not\equiv 0 \pmod{p}$. In recent years, groups arising from these equations have found significant success in public key cryptography even though it is still an open question whether these kind of groups are actually cryptographically secure! One of the benefits of elliptic curves is that the fastest known attack on them has $O(\sqrt{N})$ complexity, where N is the order of the group. This means significantly smaller keys can be used in comparison to the key length of RSA. A natural question is

What about other polynomial equations?

It turns out that there are infinitely many groups which arise from the solution sets of polynomial equations. Which of these groups are cryptographically viable is a vast active area of research. First we develop some notations required to describe these groups.

Let K be a field. Let $A = K[x_1, \dots, x_n]$ represent the polynomial ring in n variables over K . Given a subset $T \subseteq A$, we may define

$$Z(T) = \{P \in K^n \mid f(P) = 0 \text{ for all } f \in T\}$$

to be the set of all common zeros of polynomials in T . We call $Y \subseteq K^n$ an *algebraic subs* if $Y = Z(T)$ for some subset $T \subseteq A$. A not so obvious but usual

fact to keep in mind is that if $T \subseteq A$ and $J = \langle T \rangle$ is the ideal generated by elements of T , then $Z(T) = Z(J)$. We say an algebraic set Y is reducible if it can be written as the union of two smaller algebraic sets. For example, let $Y = Z(x^2 - yz, xz - x)$ as a subset of \mathbb{C}^3 . That is, Y is the set of complex valued points in \mathbb{C}^3 which satisfy each of the equations $x^2 - yz$ and $xz - x$. Notice that

$$\begin{aligned} Y &= Z(x^2 - yz, xz - x) \\ &= Z(x^2 - yz, x(z - 1)) \\ &= Z(x^2 - yz, x) \cup Z(x^2 - yz, z - 1) \\ &= Z(yz, x) \cup Z(x^2 - yz, z - 1) \\ &= Z(y, x) \cup Z(z, x) \cup Z(x^2 - yz, z - 1) \end{aligned}$$

So Y is reducible in \mathbb{C}^3 . If an algebraic set is not reducible, it is called irreducible.

2.1 Dimension

2.2 Genus

3 Elliptic Curves

Let p be an odd prime and let $\mathbb{E} = Z(y^2 - x^3 - ax - b)$. We call \mathbb{E} and *elliptic curve* defined over $K = \mathbb{F}_p$ if $4a^3 + 27b^2 \not\equiv 0 \pmod{p}$. It was first realized by [Need reference](#) Abel and Jacobi in the 1700's that, remarkably, the \mathbb{F}_p -rational points of \mathbb{E} can be transformed into a group using a very specific group operation. In this section we describe this group operation along with other algorithms needed for cryptographic purposes.

3.1 The Group Operation

Given two \mathbb{F}_p -rational points $P_1 = (x_1, y_1), P_2 = (x_2, y_2)$ which we want to add together, we first define the value

$$s = \begin{cases} \frac{y_1 - y_2}{x_1 - x_2} \pmod{p} & \text{if } P_1 \neq P_2 \\ \frac{3x_1^2 - a}{2y_1} \pmod{p} & \text{if } P_1 = P_2 \end{cases}$$

Then we define the coordinates of the point $P_3 = P_1 + P_2$ to be

$$\begin{aligned} x_3 &= s^2 - x_1 - x_2 \\ y_3 &= y_1 + s(x_3 - x_1) \end{aligned}$$

It's not immediately obvious that $P_3 = (x_3, y_3)$ is even a point on \mathbb{E} and less obvious that this operation satisfies the axioms of a group.

Note that in steps 7 and 10, modular inverses must be calculated. Also note that if $x_1 = x_2$ but $P_1 \neq P_2$, the second coordinates satisfy $y_1 = -y_2$. So the line from P_1 to P_2 is just a vertical line at x_1 . This is taken to be the point at infinity \mathcal{O} . Also, when implementing this algorithm, the point at infinity \mathcal{O} may be instantiated with any variable because no arithmetic is ever performed on it.

Algorithm 2 The addition of two points $P_1 = (x_1, y_1), P_2 = (x_2, y_2)$ on an elliptic curve $\mathbb{E} : y^2 = x^3 + ax + b$

```

1: function ADD( $\mathbb{E}, P_1, P_2$ )
2:   if  $P_1 = \mathcal{O}$  then
3:     return  $P_2$ 
4:   else if  $P_2 = \mathcal{O}$  then
5:     return  $P_1$ 
6:   else if  $P_1 = P_2$  then
7:      $s \leftarrow (3x_1^2 + a)(2y_1)^{-1} \bmod p$ 
8:   else
9:     if  $x_1 \neq x_2$  then
10:       $s \leftarrow (y_1 - y_2)(x_1 - x_2)^{-1} \bmod p$ 
11:    else
12:      return  $\mathcal{O}$ 
13:    end if
14:  end if
15:   $x_3 \leftarrow s^2 - x_1 - x_2 \bmod p$ 
16:   $y_3 \leftarrow y_1 + s(x_3 - x_1) \bmod p$ 
17:  return  $P_3 = (x_3, y_3)$ 
18: end function

```

3.2 Scalar Multiplication of a Point

For many discrete logarithm protocols (such as Diffie-Hellman or DSA), we require to add point P to itself many times in order to perform discrete exponentiation. That is, given an integer m we need to calculate

$$mP = P + \overset{m \text{ times}}{P} + \dots + P$$

fast in order to be cryptographically reasonable. The following algorithm does this.

Algorithm 3 Scalar multiplication of a point P by an integer m

```

1: function SCALARMULT( $m, P$ )
2:   if  $m == 0$  then
3:     return  $\mathcal{O}$ 
4:   else if  $m == 1$  then
5:     return  $P$ 
6:   else if  $m == 0 \bmod 2$  then
7:     return SCALARMULT( $m/2, P + P$ )
8:   else
9:     return  $P + \text{SCALARMULT}(m - 1, P)$ 
10:  end if
11: end function

```

3.3 Finding Points

Now that we have developed arithmetic on an elliptic curve \mathbb{E} , then next step is to figure out how to find points on \mathbb{E} . If $y^2 = x^3 + ax + b$ for $a, b \in \mathbb{F}_p$,

then finding \mathbb{F}_p -rational points on \mathbb{E} is equivalent to determining if $x^3 + ax + b$ is a square mod p . This is a classical problem in number theory which can be reformulated to determining the value of the *Legendre Symbol* of a mod p , which is defined as

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a square mod } p \\ -1 & \text{if } a \text{ is not a square mod } p \\ 0 & \text{if } p \text{ divides } a \end{cases}$$

where (in our case) $a = x^3 + ax + b$. The following five properties let us determine whether a is a square mod p in polynomial time. Let $a, b \in \mathbb{Z}$ and p, q be odds primes.

(i) If $a \equiv b \pmod{p}$, then $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$

(ii) $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right) \left(\frac{b}{p}\right)$

(iii) $\left(\frac{-1}{p}\right) = 1$ if $p \equiv 1 \pmod{4}$, and $\left(\frac{-1}{p}\right) = -1$ if $p \equiv 3 \pmod{4}$

(iv) $\left(\frac{2}{p}\right) = 1$ if $p \equiv \pm 1 \pmod{8}$, and $\left(\frac{2}{p}\right) = -1$ if $p \equiv \pm 3 \pmod{8}$

(v) If p, q are distinct, then

$$\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right) \text{ if } p \text{ or } q \equiv 1 \pmod{4}$$

$$\left(\frac{p}{q}\right) = -\left(\frac{q}{p}\right) \text{ if } p \equiv q \equiv 3 \pmod{4}$$

For example, to determine if 105 is a square mod 227, we simply compute

$$\begin{aligned} \left(\frac{105}{227}\right) &\stackrel{(ii)}{=} \left(\frac{3}{227}\right) \left(\frac{5}{227}\right) \left(\frac{7}{227}\right) \\ &\stackrel{(v)}{=} (-1) \left(\frac{227}{3}\right) \left(\frac{227}{5}\right) (-1) \left(\frac{227}{7}\right) \\ &\stackrel{(i)}{=} \left(\frac{2}{3}\right) \left(\frac{2}{5}\right) \left(\frac{3}{7}\right) \\ &\stackrel{(iv)}{=} (-1)(-1)(-1) \left(\frac{7}{3}\right) \\ &\stackrel{(i)}{=} (-1) \left(\frac{1}{3}\right) \\ &= -1 \end{aligned}$$

So 105 is not a square mod 227. A full description of how to compute Legendre symbols is provided in algorithm 5. This process is incredibly quick but doesn't actually tell us the squareroot of a , if a is indeed a square mod p . If $\left(\frac{a}{p}\right) = 1$, the following method finds x such that $x^2 = a \pmod{p}$. We break the

calculations into two cases.

If $p \equiv 3 \pmod{4}$, then $x = a^{(p+1)/4}$ satisfies $x^2 \equiv a \pmod{p}$. The second case where $p \equiv 1 \pmod{4}$ is more involved.

1. Pick random r such that $\left(\frac{r^2-4a}{p}\right) = -1$ and write $d = r^2 - 4a$.
2. let $\alpha = \frac{r+\sqrt{d}}{2}$ and $\alpha^k = \frac{V_k+U_k\sqrt{d}}{2}$ where V_k, U_k are the coefficients of $1, \sqrt{d}$ in the k -th power of α .
3. $x = 2^{-1}V_{(p+1)/2}$ satisfies $x^2 \equiv a \pmod{p}$.

For this case, the proof that x does satisfy $x^2 \equiv a \pmod{p}$ is quite involved but can be found in [GarrWALSH NOTES](#). Putting all this together we obtain the following algorithm which finds random points on an elliptic curve \mathbb{E} .

Algorithm 4 Find random points on elliptic curve $\mathbb{E} : y^2 - x^3 - ax - b$ modulo an odd prime p

```

1: function RANDOMPOINTS( $\mathbb{E}, p$ )
2:   pick random  $x \in \{1, \dots, p-1\}$ 
3:   while not LEGENDRE( $x^3 + ax + b, p$ ) do
4:     pick another  $x \in \{1, \dots, p-1\}$ 
5:   end while
6: end function

```

3.4 Counting Points

When using an elliptic curve \mathbb{E} for discrete logarithm based cryptosystems, it is importance to know the number of points on \mathbb{E} . This is because in 1978 Stephan Pohlig and Martin Hellman came up with an attack, described in [5], which uses the order of the group to solve discrete logs. The attack proceeds as follows: Given G , a finite cyclic group with order N , we may factor $N = p_1^{e_1} p_2^{e_2} \dots p_s^{e_s}$ where p_1, p_2, \dots, p_s are primes and $e_1, e_2, \dots, e_s \in \mathbb{Z}^+$. All the subgroups G_1, G_2, \dots, G_s of G will have order $p_1^{e_1}, p_2^{e_2}, \dots, p_s^{e_s}$ respectively. Given an element $h = g^a \in G$, the Pohlig-Hellman attack solves for a in the smaller subgroups G_1, G_2, \dots, G_s and uses the Chinese remainder theorem to piece these solutions back together to solve for a in the bigger group G . What they noticed is that when using this method, the complexity of solving for a in G (which can be done in $O(\sqrt{N})$ bit operations) gets reduced to $O(p_1^{e_1/2}) + O(p_2^{e_2/2}) + O(p_s^{e_s/2})$, i.e. the complexity the sum of the complexities of solving in the smaller groups. This means a group's bits of security is only as high as the bits of security of its largest subgroup. Therefore we want the order of the groups that we use to be prime, or at least a very small multiple of a prime, to avoid this attack.

With this in mind, we need an algorithm which calculates the number of points on an elliptic curve \mathbb{E} and thus the order of the group \mathbb{E} . Intuitively, the number of points must be much less than p^2 , for this would correspond to a curve

that zeros every point in \mathbb{Z}_p^2 . In fact, a much better estimate is given by Hasse in [??]. Denote the number of \mathbb{F}_p -points on \mathbb{E} by $\#E(\mathbb{F}_p)$, then

$$\#E(\mathbb{F}_p) = q + 1 + t \quad (1)$$

where $|t| \leq 2\sqrt{q}$. So essentially finding $\#E(\mathbb{F}_p)$ is equivalent to finding t . The first tractable algorithm was designed by Rene Schoof in 1985 and was inspired by a very special map called *the map of frobenious*

$$\begin{aligned} \phi : E &\rightarrow E \\ (x, y) &\mapsto (x^p, y^p) \end{aligned}$$

which satisfies the equation

$$\phi^2 + p = -t\phi \quad (2)$$

in the endomorphism ring $\text{End}(E)$. The idea is to find t in the interval $[-2\sqrt{p}, 2\sqrt{p}]$ which gives equivalence in (2). The problem is when p is large (300 bits), this interval is too big to brute force. Schoof's idea was to reduce this guess work by computing t which satisfies (2) in the l -torsion subgroups $\mathbb{E}[l]$ for a sufficient number of small primes l , then use the chinese remainder theorem to recover t . A detailed explanation of Schoof's algorithm can be found in algorithm 5

3.5 Generating Provably Random Elliptic Curves

3.6 Parameter Selection

4 Hyperelliptic Curves

Unlike elliptic curves, when the genus g of a curve \mathcal{C} is greater than 1, the set of points on \mathcal{C} will not always form a group.

4.1 The Jacobian of a Hyperelliptic Curve

Luckily, there is another way to form an abelian group with hyperelliptic curves. Indeed, let \mathfrak{D} be the set of all formal finite sums

$$\sum_i m_i P_i$$

where $m_i \in \mathbb{Z}$ and P_i are points on the curve \mathcal{C} . We call elements of \mathfrak{D} divisors of \mathcal{C} . Given a rational function f in $\mathbb{Z}_p[\mathcal{C}]$, we can define the corresponding divisor to f as

$$(f) = \sum_i m_i P_i$$

where P_i are the zeros and poles of f with multiplicities m_i .

Divisors of this form are called principal divisors and we let \mathfrak{P} denote the subset of all of them in \mathfrak{D} . If we define the operation on \mathfrak{D} by

$$\sum_i m_i P_i + \sum_i m'_i P_i = \sum_i (m_i + m'_i) P_i$$

then \mathfrak{D} becomes an abelian group. Unfortunately, this group is far too large and unstructured for cryptographic purposes. So we consider the subgroup \mathfrak{D}^0 of all divisors of \mathfrak{D} whose coefficients sum to 0. That is, divisors $\sum_i m_i P_i$ such that $\sum_i m_i = 0$.

This subgroup is still infinite, but that can be remedied by defining two divisors D_1, D_2 of \mathfrak{D}^0 to be equal if $D_1 - D_2$ is equal to the divisor of a rational function on \mathfrak{C} . That is, $D_1 - D_2 = (f)$ for $f \in \mathbb{Z}_p[\mathfrak{C}]$. This new quotient group, denoted

$$\mathfrak{J} = \mathfrak{D}^0 / \mathfrak{P}$$

is called the jacobian of the curve \mathfrak{C} and is a finite cyclic group. This will be the group used to build hyperelliptic cryptosystems.

4.2 Representation of Divisors

Although the Jacobian \mathfrak{J} of an hyperelliptic curve \mathfrak{C} is a finite abelian group, elements of \mathfrak{J} are very hard to represent.

To make the group operation in \mathfrak{J} tractable, we utilize the mumford representation of a divisor which is described as follows. Let D be a semi-reduced with points $P_i = (x_i, y_i)$. We associate to D polynomials $a, b \in \mathbb{Z}_p[x]$ such that

$$a(x) = \prod_i^r (x - x_i)$$

$$b(x_i) = y_i \quad 1 \leq i \leq r$$

where $\deg b < \deg a$ and $(x - x_i)^{k_i} \mid b - y_i$, if k_i is the multiplicity of P_i . Denote this representation $D \stackrel{\text{def}}{=} \text{div}(a, b)$.

4.3 The Group Operation

The group operation can be divided into two parts - *Composition* and *reduction* as described in [4].

Given two divisors represented as $D_1 = \text{div}(a_1, b_1), D_2 = \text{div}(a_2, b_2)$

1. compute $d_0 = \gcd(a_1, a_2)$ and find the unique $c_1, e_1 \in \mathbb{Z}_p[x]$ such that $d_0 = c_1 a_1 + e_1 a_2$
2. compute $d = \gcd(d_1, b_1 + b_2)$ and find the unique $c_2, e_2 \in \mathbb{Z}_p[x]$ such that $d = c_2 d_1 + e_2 (b_1 + b_2)$
3. compute $a_3 = \frac{a_1, a_1}{d^2}$
4. compute $b_3 = \frac{c_2 c_1 a_1 + c_2 e_1 a_2 + e_2 (b_1 b_2 + f)}{d} \bmod \frac{a_1, a_1}{d^2}$
5. compute $a'_3 = \frac{f - b_3^2}{a_3}$ and $b'_3 = -b_3 \bmod a'_3$
6. while $\deg(a'_3) > g$, reassign $a_3 = a'_3, b_3 = b'_3$ and repeat step 5
7. divide a'_3 by its leading coefficient so that a'_3 becomes monic
8. the output $\text{div}(a'_3, b'_3) = D_1 + D_2$

4.4 Scalar Multiplication

4.5 Counting Points

4.6 Parameter Selection

5 Algorithms

Algorithm 5 The Legendre symbol of an integer a modulo a prime p

```
1: function LEGENDRE( $a, p$ )
2:    $s \leftarrow 1$ 
3:   for  $q \in \text{FACTORS}(a)$  do
4:      $s \leftarrow s * \text{LEGENDREFORPRIME}(q, p)$ 
5:   end for
6:   function LEGENDREFORPRIME( $q, p$ )
7:     if  $q = 1$  then
8:       return 1
9:     else if  $q \equiv 0 \pmod p$  then
10:      return 0
11:     else if  $q = -1$  then
12:       if  $p \equiv 1 \pmod 4$  then
13:         return 1
14:       else
15:         return -1
16:       end if
17:     else if  $q = 2$  then
18:       if  $p \equiv \pm 1 \pmod 8$  then
19:         return 1
20:       else
21:         return -1
22:       end if
23:     else if  $q > p$  then
24:       return LEGENDREFORPRIME( $q \pmod p, p$ )
25:     else if  $q \equiv 1 \pmod 4$  or  $p \equiv 1 \pmod 4$  then
26:       return LEGENDREFORPRIME( $p, q$ )
27:     else
28:       return -LEGENDREFORPRIME( $p, q$ )
29:     end if
30:   end function
31:   return  $s$ 
32: end function
```

Algorithm 6 Schoofs algorithm to calculate the number of points on elliptic curve $\mathbb{E} : y^2 = x^3 - ax - b$ over \mathbb{F}_p

```

1: function ORDER( $\mathbb{E}, p$ )
2:    $S \leftarrow$  set of odd primes such that  $\prod_{l \in S} l > 4\sqrt{p}$ 
3:    $C \leftarrow \emptyset$ 
4:   if  $\gcd(x^p - x, x^3 + ax + b) == 1$  then
5:      $C \leftarrow C \cup \{(2, 1)\}$ 
6:   else
7:      $C \leftarrow C \cup \{(2, 0)\}$ 
8:   end if
9:   for  $l \in S$  do
10:     $\psi_l \leftarrow \text{DIVISIONPOLYNOMIAL}(l)$ 
11:     $p_l = p \bmod l$ 
12:     $(x', y') \leftarrow (x^{p^2}, y^{p^2}) \oplus [p_l](x, y)$ 
13:    for  $t \in \{1, 2, \dots, \frac{l-1}{2}\}$  do
14:       $(x_t, y_t) = [t](x, y)$ 
15:      if  $x' == x_t \bmod \psi_l$  then
16:        if  $\frac{(y' - y_t)}{y} == 0 \bmod \psi_l$  then
17:           $C \leftarrow C \cup \{(l, t)\}$ 
18:        else
19:           $C \leftarrow C \cup \{(l, -t)\}$ 
20:        end if
21:      Next  $t$ 
22:    end if
23:  end for
24:  if  $p == w^2 \bmod l$  for some  $w \in Z$  then
25:    if  $\gcd(\text{numerator}(x^p - x_w), \psi_l) == 1$  then
26:       $C \leftarrow C \cup \{(l, 0)\}$ 
27:    else
28:      if  $\gcd(\text{numerator}(\frac{y^p - y_w}{y}), \psi_l) == 1$  then
29:         $C \leftarrow C \cup \{(l, -2w)\}$ 
30:      else
31:         $C \leftarrow C \cup \{(l, 2w)\}$ 
32:      end if
33:    end if
34:  else
35:     $C \leftarrow C \cup \{(l, 0)\}$ 
36:  end if
37: end for
38: return CHINESEREMAINDERTHEOREM( $C$ )
39: end function

```

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