CS 6041
Theory of Computation

Review 2

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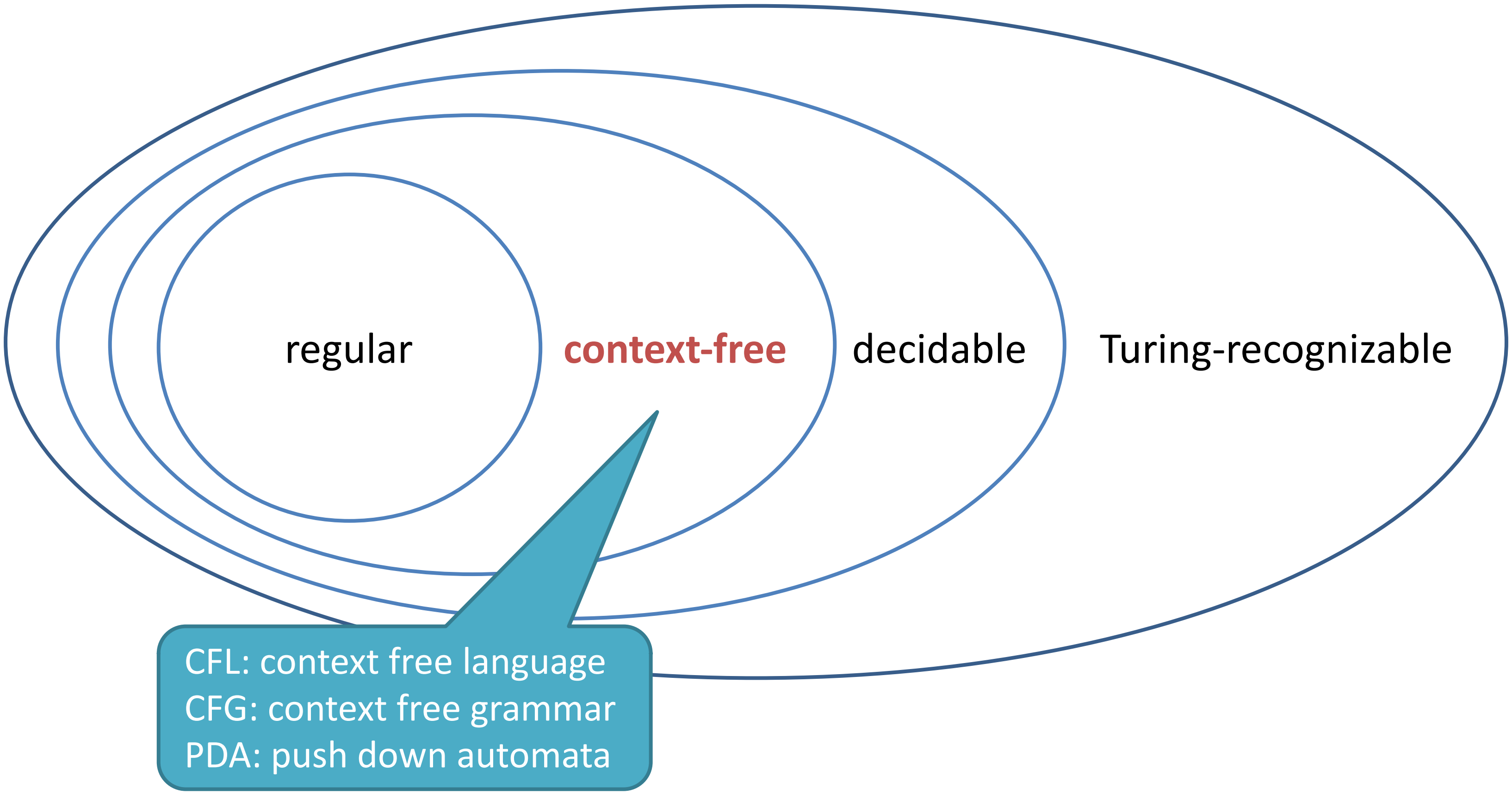
Exam 2 – similar as Exam 1

- 10 True/False question
  - 2 points each

- 4 short answer question
  - 20 points each

- 100 = 2*10 + 4*20

Context-free language



Context Free Grammar

- Example, G_1

Variable:
A, B

Start variable:
A

3 substitution rules
(productions)

$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#$$

Terminals:
0, 1, #

$$\begin{aligned} A &\Rightarrow 0A1 \\ &\Rightarrow 00A11 \\ &\Rightarrow 000A111 \\ &\Rightarrow 000B111 \\ &\Rightarrow 000\#111 \end{aligned}$$

The language of grammar

- Grammar G_1:

$$A \rightarrow 0A1$$

$$A \rightarrow B$$

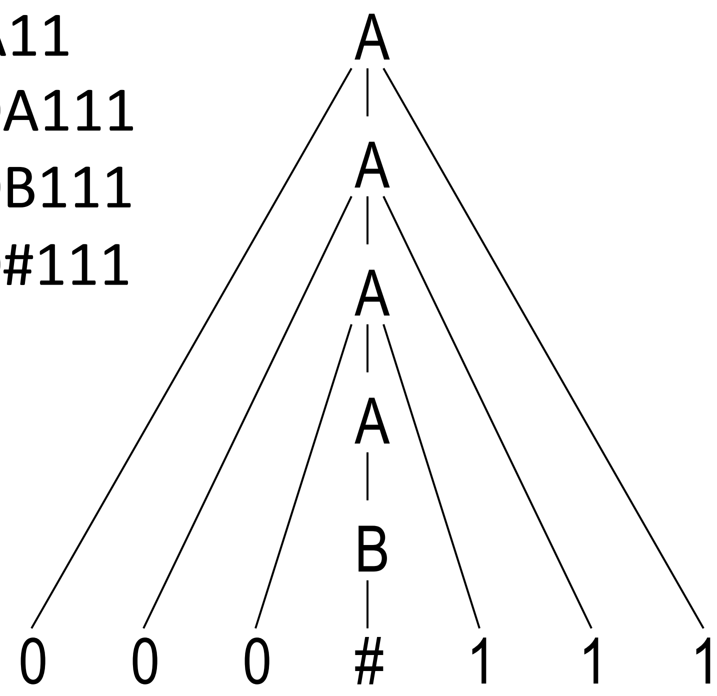
$$B \rightarrow \#$$

- The language of G_1:

$L(G_1)=\{\ 0^n\#1^n \mid n \geq 0\ \}$

- Context-free language
  - Languages generated by context-free grammars

$$\begin{aligned} A &\Rightarrow 0A1 \\ &\Rightarrow 00A11 \\ &\Rightarrow 000A111 \\ &\Rightarrow 000B111 \\ &\Rightarrow 000\#111 \end{aligned}$$



000#111

Definition of context-free grammar

- Context-free grammar is a 4-tuple $G=(V,\Sigma,R,S)$,

  1) V: finite variable set

  2) Σ: finite terminal set

  3) R: finite rule set

     $(A \to w,\ w \in (V \cup \Sigma)^*)$

  4) $S \in V$: start variable

Design CFG for languages

- Design CFG is much difficult than designing an automata for language

- Basic idea:

  1. divide CFL into small parts
  2. design CFG for each small part
  3. combine them together

Design context-free grammar

- Design CFG for $\{w | w=0^n1^n \text{ or } w=1^n0^n, n\geq 0\}$

Design context-free grammar

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Design context-free grammar

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Generating same number of 0 and 1
Generating 0 before 1

01 0011 000111 00..011..1

Design context-free grammar

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01   0011   000111   00..011..1

Design context-free grammar

- Design CFG for $\{w \mid w=0^n1^n \text{ or } w=1^n0^n, n \geq 0\}$
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Generating same number of 0 and 1
Generating 0 before 1

01 0011 000111 00..01..11

$S \rightarrow 0S1$

Design context-free grammar

- Design CFG for $\{w \mid w=0^n1^n \text{ or } w=1^n0^n, n\geq 0\}$
  - Design CFG for $\{w \mid w=0^n1^n, n\geq 0\}$
    - $G_1=(\{S\},\{0,1\}, \{S\rightarrow 0S1, S\rightarrow \varepsilon\}, S)$
  - Design CFG for $\{w \mid w=1^n0^n, n\geq 0\}$
    - $G_2=(\{S\},\{0,1\}, \{S\rightarrow 1S0, S\rightarrow \varepsilon\}, S)$

Design context-free grammar

- Design CFG for $\{w|w=0^n1^n \text{ or } w=1^n0^n, n\geq 0\}$
  - Design CFG for $\{w|w=0^n1^n, n\geq 0\}$
    - $G_1=(\{S_1\},\{0,1\},\{S_1\rightarrow 0S_11, S_1\rightarrow\varepsilon\},S_1)$
  - Design CFG for $\{w|w=1^n0^n, n\geq 0\}$
    - $G_2=(\{S_2\},\{0,1\},\{S_2\rightarrow 1S_20, S_2\rightarrow\varepsilon\},S_2)$

Design context-free grammar

- Design CFG for $\{w|w=0^n1^n \text{ or } w=1^n0^n, n\geq 0\}$
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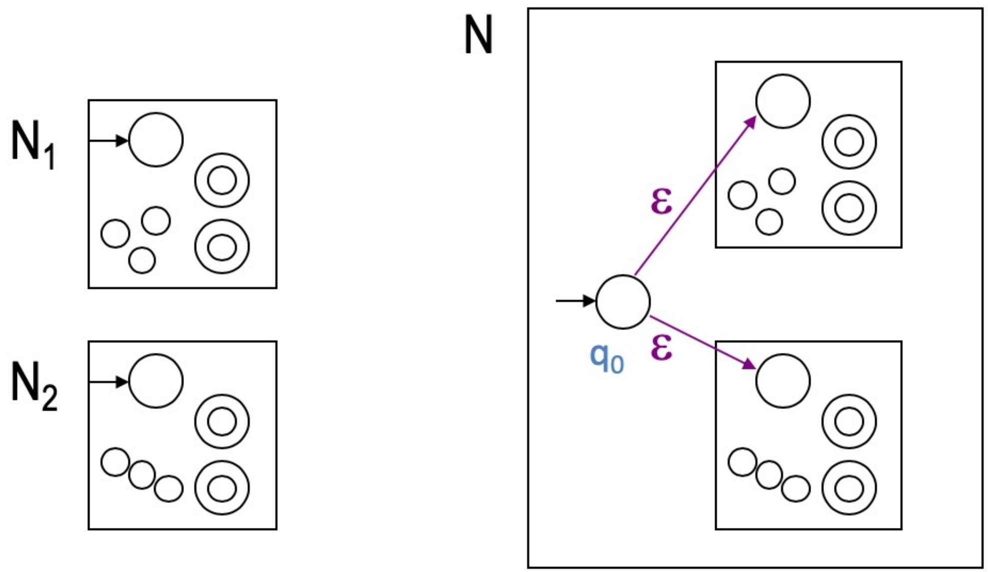

Design context-free grammar

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  - Design CFG for $\{w|w=1^n0^n, n\geq 0\}$
    - $G_2= (\{S_2\},\{0,1\},\{S_2\to 1S_20, S_2\to\varepsilon\},S_2)$
  - $G=(\{S,S_1,S_2\},\{0,1\},$

    $\{ S\to S_1,\ S\to S_2,\ S_1\to 0S_11,\ S_1\to\varepsilon,\ S_2\to 1S_20,\ S_2\to\varepsilon\},$

    $S)$

Design CFG for languages

- Design CFG is much difficult than designing an automata for language

- Other ideas:

  1. Simulate the regular expressions
  2. Look for a pattern from example strings
  3. ...

Design CFG for languages

- L={w| w has at least three 1s}, $\Sigma = \{0,1\}$

$$\Sigma^* 1\ \Sigma^*\ 1\ \Sigma^*\ 1\ \Sigma^*$$

Design CFG for languages

- L={w| w has at least three 1s}, $\Sigma = \{0,1\}$

$\Sigma^{*} 1 \Sigma^{*} 1 \Sigma^{*} 1 \Sigma^{*}$

$S \rightarrow R1R1R1R$

Design CFG for languages

- L={w| w has at least three 1s}, $\Sigma = \{0,1\}$

$\Sigma^*1\ \Sigma^*1\ \Sigma^*1\ \Sigma^*$

S → R1R1R1R

R → 0R

R → 1R

R → ε

Design CFG for languages

- L={w| w has odd length}, Σ = {0,1}

$$\Sigma(\Sigma\,\Sigma)^*$$

Design CFG for languages

- L={w| w has odd length}, Σ = {0,1}

$\Sigma(\Sigma\,\Sigma)^*$

$$\left.\begin{array}{l} S \rightarrow 0 \\ S \rightarrow 1 \end{array}\right\}$$

Design CFG for languages

- L={w| w has odd length}, Σ = {0,1}

$\Sigma(\Sigma\,\Sigma)^{*}$

S → 0

S → 1

S → S00

S → S01

S → S10

S → S11

Design CFG for languages

- L={w| w has odd length}, Σ = {0,1}

$\Sigma(\Sigma\,\Sigma)^*$

S → 0

S → 1

S → S00

S → S01

S → S10

S → S11

Design CFG for languages

- L={w| w has odd length and the middle symbol is 0}, $\Sigma = \{0,1\}$

0
000
001
100
101
00011
…

Design CFG for languages

- L={w| w has odd length and the middle symbol is 0}, $\Sigma = \{0,1\}$

$S \rightarrow 0$
$S \rightarrow 0S0$
$S \rightarrow 0S1$
$S \rightarrow 1S0$
$S \rightarrow 1S1$

0
000
001
100
101
00011
…

Design CFG for languages

- $L = \{0^n 1^n \mid n \geq 0\}$. $\Sigma = \{0,1\}$

ε, 01, 0011, …

$S \rightarrow 0S1 \mid \varepsilon$

Design CFG for languages

- $L = \{0^n 1^{2n} \mid n \geq 0\}$. $\Sigma = \{0,1\}$

ε, 011, 001111, …

$S \rightarrow 0S11 \mid \varepsilon$

Design CFG for languages

- $L = \{00^*11^*\}$. $\Sigma = \{0,1\}$

01, 011, 0011, …

How to design 00^*

How to design 11^*

Design CFG for languages

- $L = \{00^*11^*\}$. $\Sigma = \{0,1\}$

How to design 00^*

$C \rightarrow 0$

$C \rightarrow 0C$

Design CFG for languages

- $L = \{00^*11^*\}$. $\Sigma = \{0,1\}$

How to design 11^*

$D \rightarrow 1$

$D \rightarrow 1D$

Design CFG for languages

- $L = \{00^*11^*\}$. $\Sigma = \{0,1\}$

How to design 00^*

$C \rightarrow 0$

$C \rightarrow 0C$

How to design 11^*

$D \rightarrow 1$

$D \rightarrow 1D$

How to design 00^*11^*

$S \rightarrow CD$

$C \rightarrow 0C \mid 0$

$D \rightarrow 1D \mid 1$

Ambiguity

- If a grammar generates the ***same*** string in several ***different*** ways, we say that the string is derived *ambiguously* in that grammar.

- If a grammar generates some string ambiguously, we say that the grammar is ***ambiguous***.

- G_5: E $\rightarrow$

  E+E |

  E$\times$E |

  (E) | a

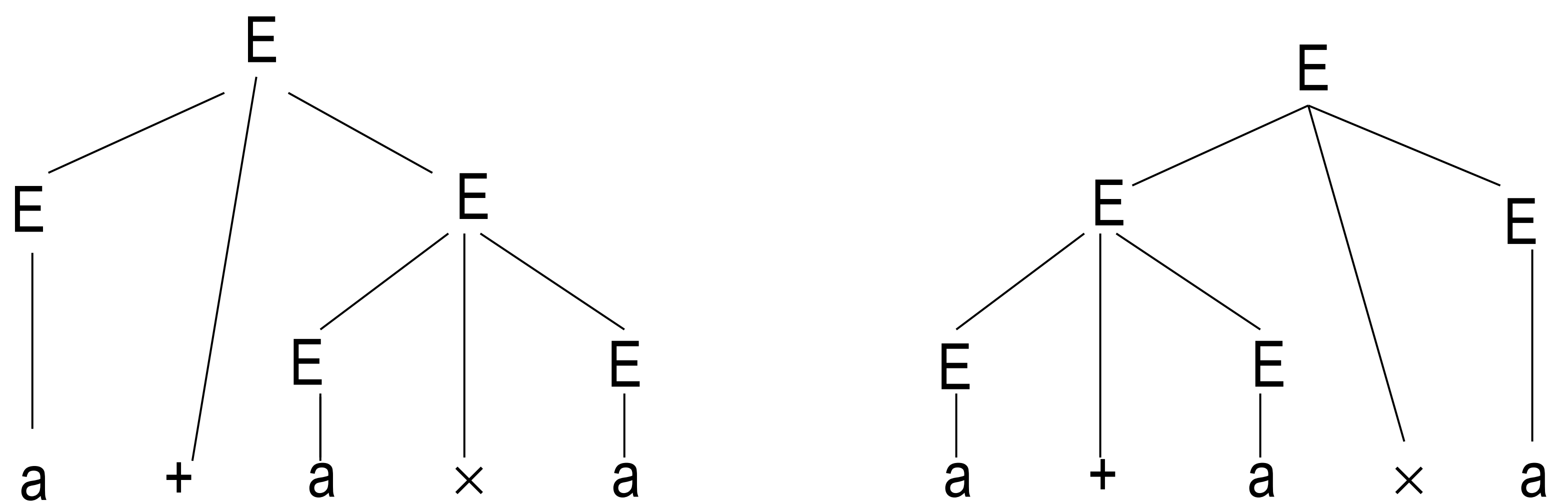
Ambiguity

- G_5: $E \rightarrow$

  $E+E \mid$

  $E \times E \mid$

  $(E) \mid a$



Leftmost derivation

- A derivation of a string w in a grammar G is a ***leftmost derivation*** if at every step the ***leftmost*** remaining variable is the one replaced

- $E \Rightarrow E+E$

  $\Rightarrow a+E$

  $\Rightarrow a+E\times E$

  $\Rightarrow a+a\times E \Rightarrow a+a\times a$

- G_5: $E \rightarrow$

  $E+E \mid$

  $E\times E \mid$

  $(E) \mid a$

Two different leftmost derivation

- E
  $\Rightarrow$ E+E
  $\Rightarrow$ a+E
  $\Rightarrow$ a+E$\times$E
  $\Rightarrow$ a+a$\times$E
  $\Rightarrow$ a+a$\times$a

- E
  $\Rightarrow$ E$\times$E
  $\Rightarrow$ E+E $\times$E
  $\Rightarrow$ a+E$\times$E
  $\Rightarrow$ a+a$\times$E
  $\Rightarrow$ a+a$\times$a

- G_5: E $\rightarrow$
  E+E |
  E$\times$E |
  (E) | a

Chomsky normal form (CNF)

- CNF: only allow CFG in the following forms
  - $S \to \varepsilon$
  - $A \to BC$
  - $A \to a$

Only start variable S can generate ε

Variables can only generate:
1, two variables
2, single terminal

✓ $S \to \varepsilon$	✗ $A \to \varepsilon$
✓ $A \to BC$	✗ $A \to B$
✓ $A \to a$	✗ $A \to abcd$
	✗ $A \to aB$

Techniques for CNF

- Add new start variable if needed
- $A \rightarrow \varepsilon$, merge above rules with A
- $A \rightarrow B$, replace B with terminals or other rules
- $A \rightarrow aB$, replace with $U \rightarrow a$, $A \rightarrow UB$
- $A \rightarrow abcd$, replace with $A \rightarrow aU_1$, $U_1 \rightarrow bU_2$, $U_2 \rightarrow cd$
- $A \rightarrow BCD$, similar as the above

✓ $S \rightarrow \varepsilon$
✓ $A \rightarrow BC$
✓ $A \rightarrow a$
✗ $A \rightarrow \varepsilon$
✗ $A \rightarrow B$
✗ $A \rightarrow abcd$
✗ $A \rightarrow aB$

Chomsky normal form example

G_6: $S \rightarrow ASA \mid aB,$

$A \rightarrow B \mid S$

$B \rightarrow b \mid \varepsilon$

Get the CNF for G_6

✓ $S \rightarrow \varepsilon$	✗ $A \rightarrow \varepsilon$
✓ $A \rightarrow BC$	✗ $A \rightarrow B$
✓ $A \rightarrow a$	✗ $A \rightarrow abcd$
	✗ $A \rightarrow aB$

Step 1

G_6: $S \rightarrow ASA \mid aB,$

$A \rightarrow B \mid S$

$B \rightarrow b \mid \varepsilon$

$S \rightarrow \varepsilon$

$A \rightarrow BC$

$A \rightarrow a$

$A \rightarrow \varepsilon$

$A \rightarrow B$

$A \rightarrow abcd$

$A \rightarrow aB$

(1) $S_0 \rightarrow S$

$S \rightarrow ASA \mid aB$

$A \rightarrow B \mid S$

$B \rightarrow b \mid \varepsilon$

To simply the start variable

Step 2

(2a) $S_0 \rightarrow S$

$S \rightarrow ASA \mid aB$

$A \rightarrow B \mid S$

$B \rightarrow b \mid \varepsilon$

Only start variable S can generate ε

$S \rightarrow \varepsilon$

$A \rightarrow BC$

$A \rightarrow a$

$A \rightarrow \varepsilon$

$A \rightarrow B$

$A \rightarrow abcd$

$A \rightarrow aB$

$S_0 \rightarrow S$

$S \rightarrow ASA \mid aB \mid a$

$A \rightarrow B \mid S \mid \varepsilon$

$B \rightarrow b$

Step 2

(2b) $S_0 \rightarrow S$

$S \rightarrow ASA \mid aB \mid a$

$A \rightarrow B \mid S \mid \varepsilon$

$B \rightarrow b$

Only start variable S can generate ε

$S \rightarrow \varepsilon$

$A \rightarrow BC$

$A \rightarrow a$

$A \rightarrow \varepsilon$

$A \rightarrow B$

$A \rightarrow abcd$

$A \rightarrow aB$

$S_0 \rightarrow S$

$S \rightarrow ASA \mid aB \mid a \mid SA \mid AS \mid S$

$A \rightarrow B \mid S$

$B \rightarrow b$

No A→ε anymore

Step 3

(3a) $S_0 \rightarrow S$

$S \rightarrow ASA \mid aB \mid a \mid SA \mid AS \mid S$

$A \rightarrow B \mid S$

$B \rightarrow b$

$S \rightarrow \varepsilon$

$A \rightarrow BC$

$A \rightarrow a$

$A \rightarrow \varepsilon$

$A \rightarrow B$

$A \rightarrow abcd$

$A \rightarrow aB$

$S_0 \rightarrow S$

$S \rightarrow ASA \mid aB \mid a \mid SA \mid AS$

$A \rightarrow B \mid S$

$B \rightarrow b$

Step 3

(3b) $S_0 \rightarrow S$

$S \rightarrow ASA \mid aB \mid a \mid SA \mid AS$

$A \rightarrow B \mid S$

$B \rightarrow b$

$S \rightarrow \varepsilon$

$A \rightarrow BC$

$A \rightarrow a$

$A \rightarrow \varepsilon$

$A \rightarrow B$

$A \rightarrow abcd$

$A \rightarrow aB$

$S_0 \rightarrow ASA \mid aB \mid a \mid SA \mid AS$

$S \rightarrow ASA \mid aB \mid a \mid SA \mid AS$

$A \rightarrow B \mid S$

$B \rightarrow b$

Step 3

(3c) $S_0 \rightarrow ASA \mid aB \mid a \mid SA \mid AS$

$S \rightarrow ASA \mid aB \mid a \mid SA \mid AS$

$A \rightarrow B \mid S$

$B \rightarrow b$

$S_0 \rightarrow ASA \mid aB \mid a \mid SA \mid AS$

$S \rightarrow ASA \mid aB \mid a \mid SA \mid AS$

$A \rightarrow b \mid S$

$B \rightarrow b$

✓ $S \rightarrow \varepsilon$

✓ $A \rightarrow BC$

✓ $A \rightarrow a$

✗ $A \rightarrow \varepsilon$

✗ $A \rightarrow B$

✗ $A \rightarrow abcd$

✗ $A \rightarrow aB$

Step 3

(3d) $S_0 \rightarrow ASA \mid aB \mid a \mid SA \mid AS$

$S \rightarrow ASA \mid aB \mid a \mid SA \mid AS$

$A \rightarrow b \mid S$

$B \rightarrow b$

$S_0 \rightarrow ASA \mid aB \mid a \mid SA \mid AS$

$S \rightarrow ASA \mid aB \mid a \mid SA \mid AS$

$A \rightarrow b \mid ASA \mid aB \mid a \mid SA \mid AS$

$B \rightarrow b$

$S \rightarrow \varepsilon$

$A \rightarrow BC$

$A \rightarrow a$

$A \rightarrow \varepsilon$

$A \rightarrow B$

$A \rightarrow abcd$

$A \rightarrow aB$

Step 4

(4) $S_0 \rightarrow ASA \mid aB \mid a \mid SA \mid AS$

$S \rightarrow ASA \mid aB \mid a \mid SA \mid AS$

$A \rightarrow b \mid ASA \mid aB \mid a \mid SA \mid AS$

$B \rightarrow b$

$S_0 \rightarrow AA_1 \mid aB \mid a \mid SA \mid AS$

$S \rightarrow AA_1 \mid aB \mid a \mid SA \mid AS$

$A \rightarrow b \mid AA_1 \mid aB \mid a \mid SA \mid AS$

$B \rightarrow b$

$A_1 \rightarrow SA$

$S \rightarrow \varepsilon$

$A \rightarrow BC$

$A \rightarrow a$

$A \rightarrow \varepsilon$

$A \rightarrow B$

$A \rightarrow abcd$

$A \rightarrow aB$

Step 5

(5) $S_0 \rightarrow AA_1 \mid aB \mid a \mid SA \mid AS$

$S \rightarrow AA_1 \mid aB \mid a \mid SA \mid AS$

$A \rightarrow b \mid AA_1 \mid aB \mid a \mid SA \mid AS$

$B \rightarrow b$

$A_1 \rightarrow SA$

$S_0 \rightarrow AA_1 \mid UB \mid a \mid SA \mid AS$

$S \rightarrow AA_1 \mid UB \mid a \mid SA \mid AS$

$A \rightarrow b \mid AA_1 \mid UB \mid a \mid SA \mid AS$

$B \rightarrow b$

$A_1 \rightarrow SA$

$U \rightarrow a$

$S \rightarrow \varepsilon$

$A \rightarrow BC$

$A \rightarrow a$

$A \rightarrow \varepsilon$

$A \rightarrow B$

$A \rightarrow abcd$

$A \rightarrow aB$

Step 5

(5) $S_0 \rightarrow AA_1 \mid UB \mid a \mid$

$SA \mid AS$

$S \rightarrow AA_1 \mid UB \mid a \mid$

$SA \mid AS$

$A \rightarrow b \mid AA_1 \mid UB \mid a \mid$

$SA \mid AS$

$B \rightarrow b$

$A_1 \rightarrow SA$

$U \rightarrow a$

G6 in CNF

G_6: $S \rightarrow ASA \mid aB,$

$A \rightarrow B \mid S$

$B \rightarrow b \mid \varepsilon$

✓ $S \rightarrow \varepsilon$

✓ $A \rightarrow BC$

✓ $A \rightarrow a$

✗ $A \rightarrow \varepsilon$

✗ $A \rightarrow B$

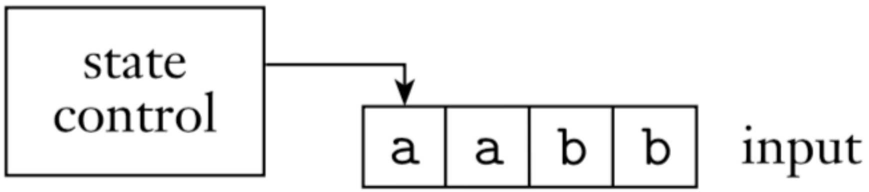
✗ $A \rightarrow abcd$

✗ $A \rightarrow aB$

What does PDA look like?

finite automaton

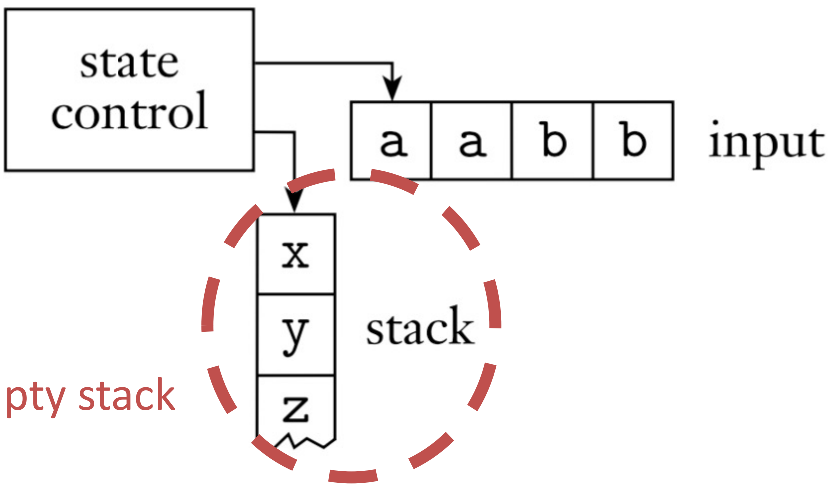
Memory = 1



pushdown automaton

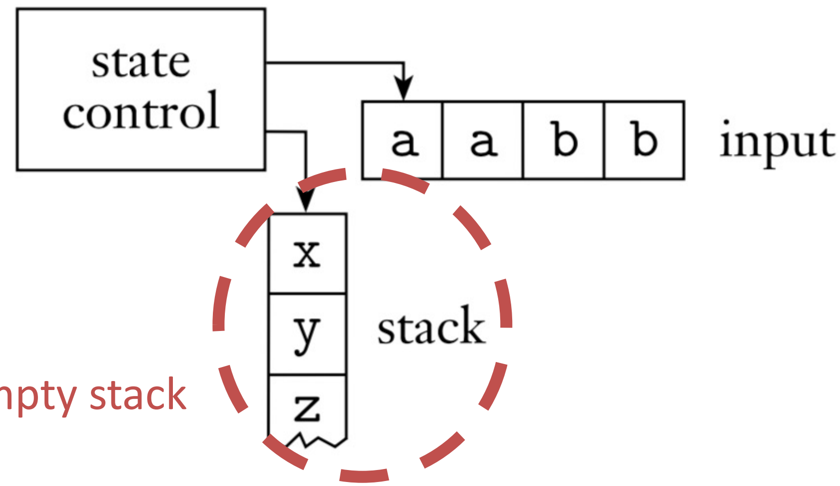
Memory = N

$ means empty stack



What does PDA looks like?

pushdown automaton

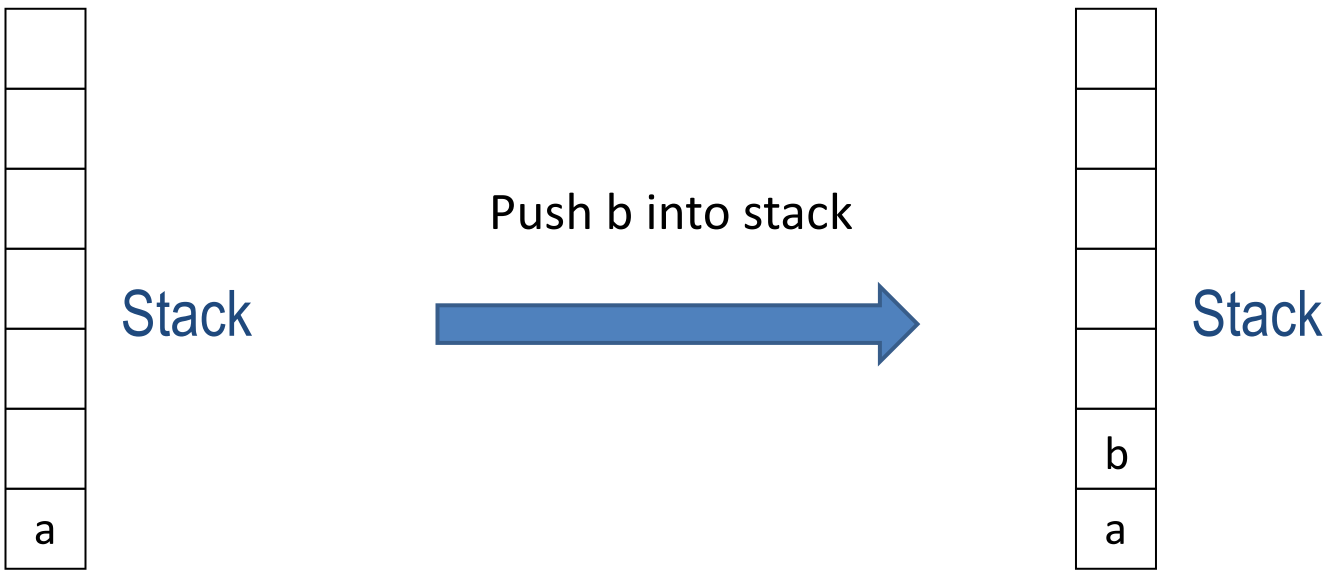


$ means empty stack

- Pushdown automata has more memories than finite automata
- PDA = finite automata + **<u>A stack (unlimited size)</u>**

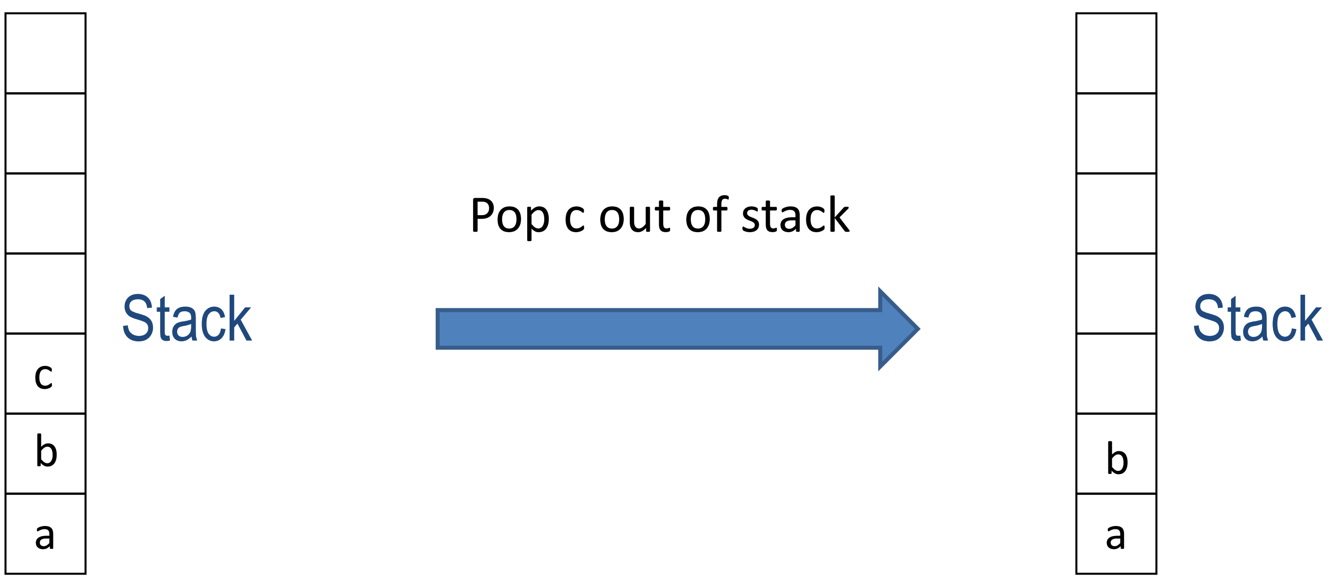
Stack operation

- Push: add to the top of stack



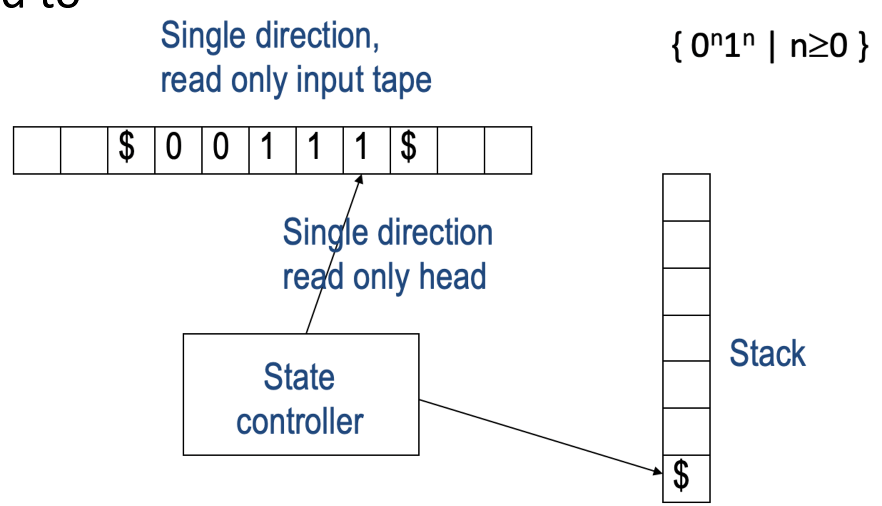
Stack operation

- Pop: remove from the top of stack



Informal description for PDA to recognize some languages

- L = {w | w has some features}
- Read symbols from input
  - **STEP1:** regular?
    - If the language is regular, do not need to use stack; if not regular, define operations on stack
  - **STEP2:** define operations:
    - When to push
    - When to pop
  - **STEP 3:** determine accept/reject:
    - Under which cases, accept
    - Under which cases, reject



Definition of PDA (non-deterministic)

- PDA $M=(Q,\Sigma,\Gamma,\delta,q_0,F)$, where

1) Q: set of states

2) Σ: input alphabet, $\Sigma_\varepsilon=\Sigma\cup\{\varepsilon\}$

3) Γ: stack alphabet, $\Gamma_\varepsilon=\Gamma\cup\{\varepsilon\}$

4) δ: $Q\times\Sigma_\varepsilon\times\Gamma_\varepsilon\rightarrow P(Q\times\Gamma_\varepsilon)$,

   transition function

5) $q_0\in Q$: start state

6) $F\subseteq Q$: accept state set

Design PDA

$	aaa…a	bbb…b	ccc…c
q_1	q_2	q_3	q_4

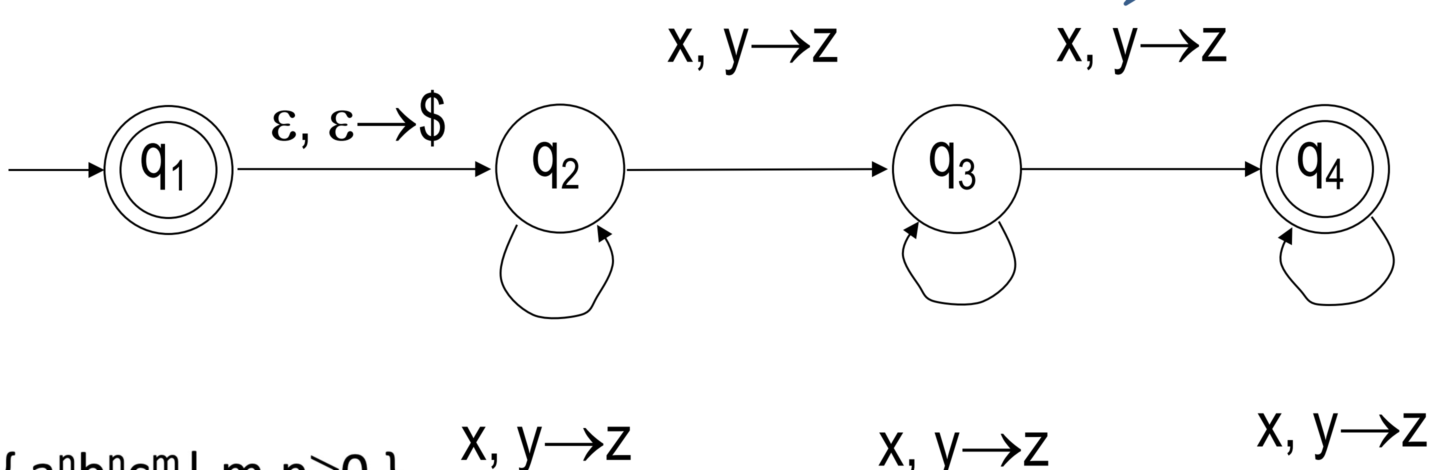
- $L(M_2)=\{ a^n b^n c^m \mid m,n \geq 0 \}$
  - Operation:
    - For an input a, and push a into stack
    - For an input b, pop one a from the stack
  - Determine accept/reject
    - If the stack is empty when finish reading b, then after reading all the cs, accept;
    - Otherwise, reject;

Design PDA

$	aaa…a	bbb…b	ccc…c
q_1	q_2	q_3	q_4

- $L(M_2)=\{ a^nb^nc^m \mid m,n\geq 0 \}$

x,y->z
x: input
y->z: the top of
stack changes

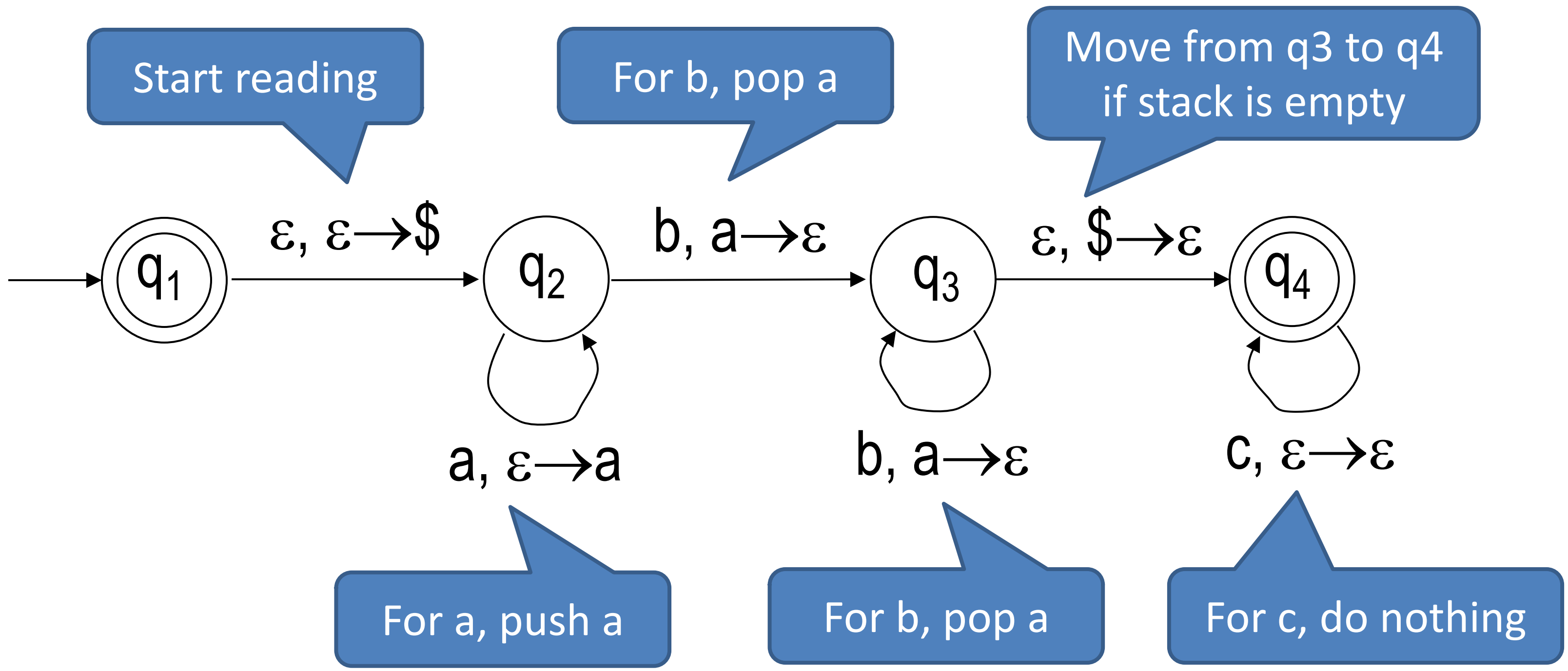


- $L(M_2)=\{ a^nb^nc^m \mid m,n\geq 0 \}$
  - Operation:
    - For an input a, and push a into stack
    - For an input b, pop one a from the stack
  - Determine accept/reject
    - If the stack is empty when finish reading b, then after reading all the cs, accept;
    - Otherwise, reject;

Design PDA

\$	aaa…a	bbb…b	ccc…c
q_1	q_2	q_3	q_4

- $L(M_2)=\{\ a^n b^n c^m \mid m,n \geq 0\ \}$



Pumping lemma

Suppose A is CFL,

then there exist a number p(the pumping length) where,

if $s \in A$ and $|s| \geq p$, then s = **uvxyz**,

Satisfying the following

1) $\forall i \geq 0$, $uv^i xy^i z \in A$;

2) $|vy|>0$;

3) $|vxy| \leq p$.

1) $\forall i \geq 0, uv^i xy^i z \in A$;

2) $|vy|>0$;

3) $|vxy| \leq p$.

Example: $B=\{ a^n b^n c^n \mid n \geq 0 \}$

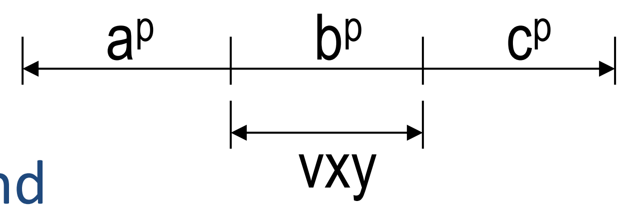
- Proof:

Suppose B is CFL, p is the pumping length,

let $s=a^p b^p c^p$

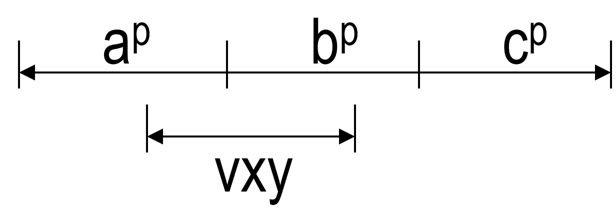
Then s = uvxyz, that

$\forall i \geq 0, uv^i xy^i z \in B$;



$|vy|>0$, v and y have at least one kind of symbol;

$|vxy| \leq p$, v and y have at most two kinds of symbol;



1) $\forall i \geq 0$, $uv^ixy^iz \in A$;

2) $|vy|>0$;

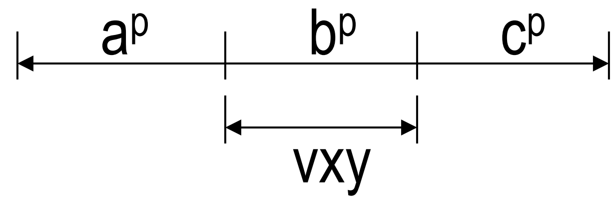
3) $|vxy| \leq p$.

Example: $B=\{ a^nb^nc^n \mid n \geq 0 \}$

- Proof:

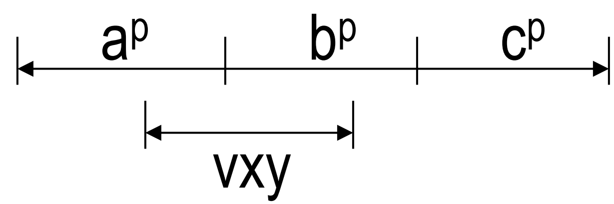
If v and y have one kind of symbol,

then in uv^ixy^iz ($i>1$), a/b/c has different numbers;



If v and y have two kinds of symbol,

then in uv^ixy^iz ($i>1$), a/b/c has different numbers;



Contradiction.

Exam 2

- 10 True/False question
  - 2 points each

- 4 short answer question
  - 20 points each

- Time: check the time @D2L