CS 6041
Theory of Computation

Reducibility

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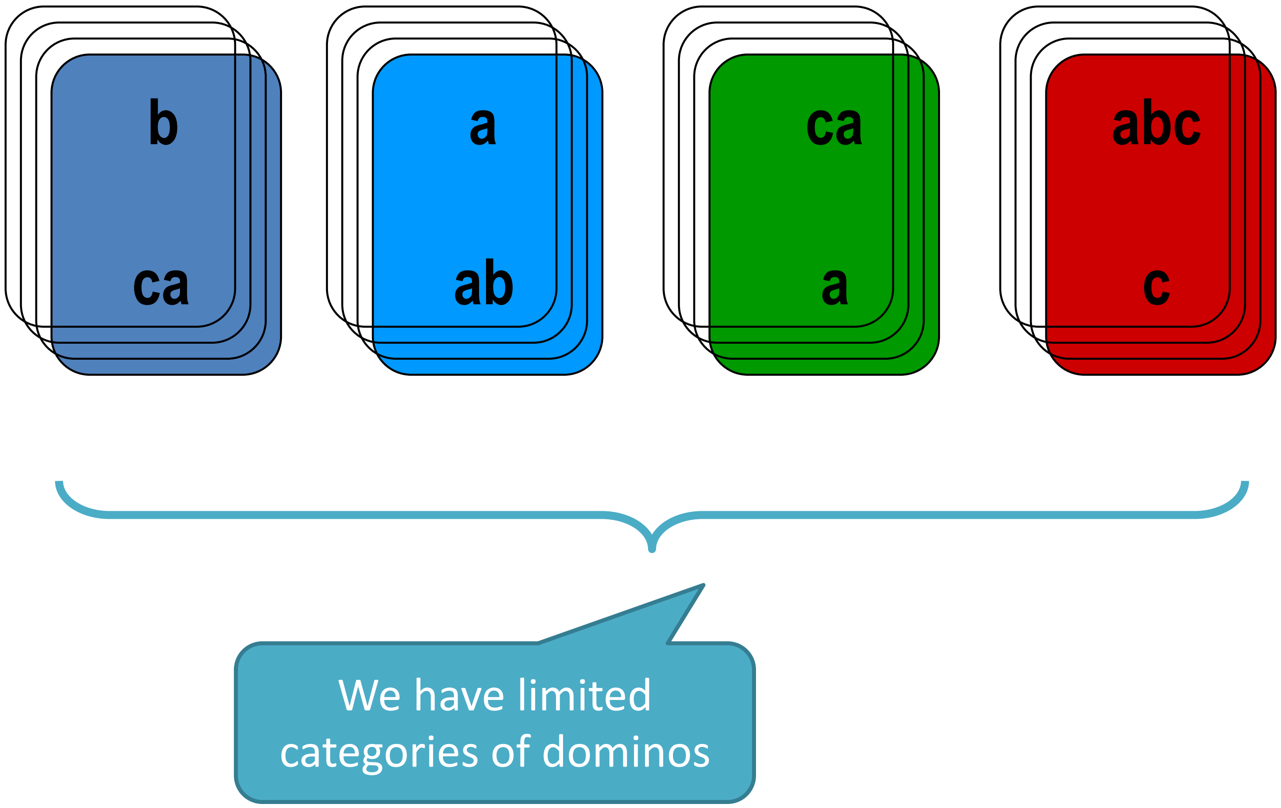
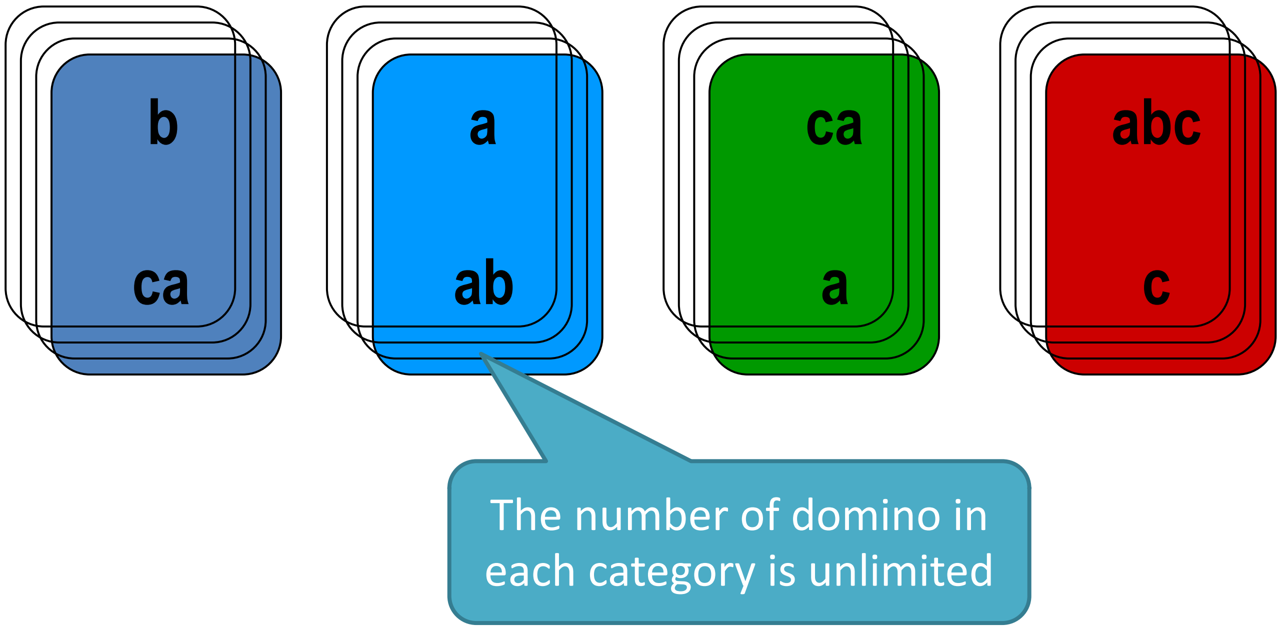
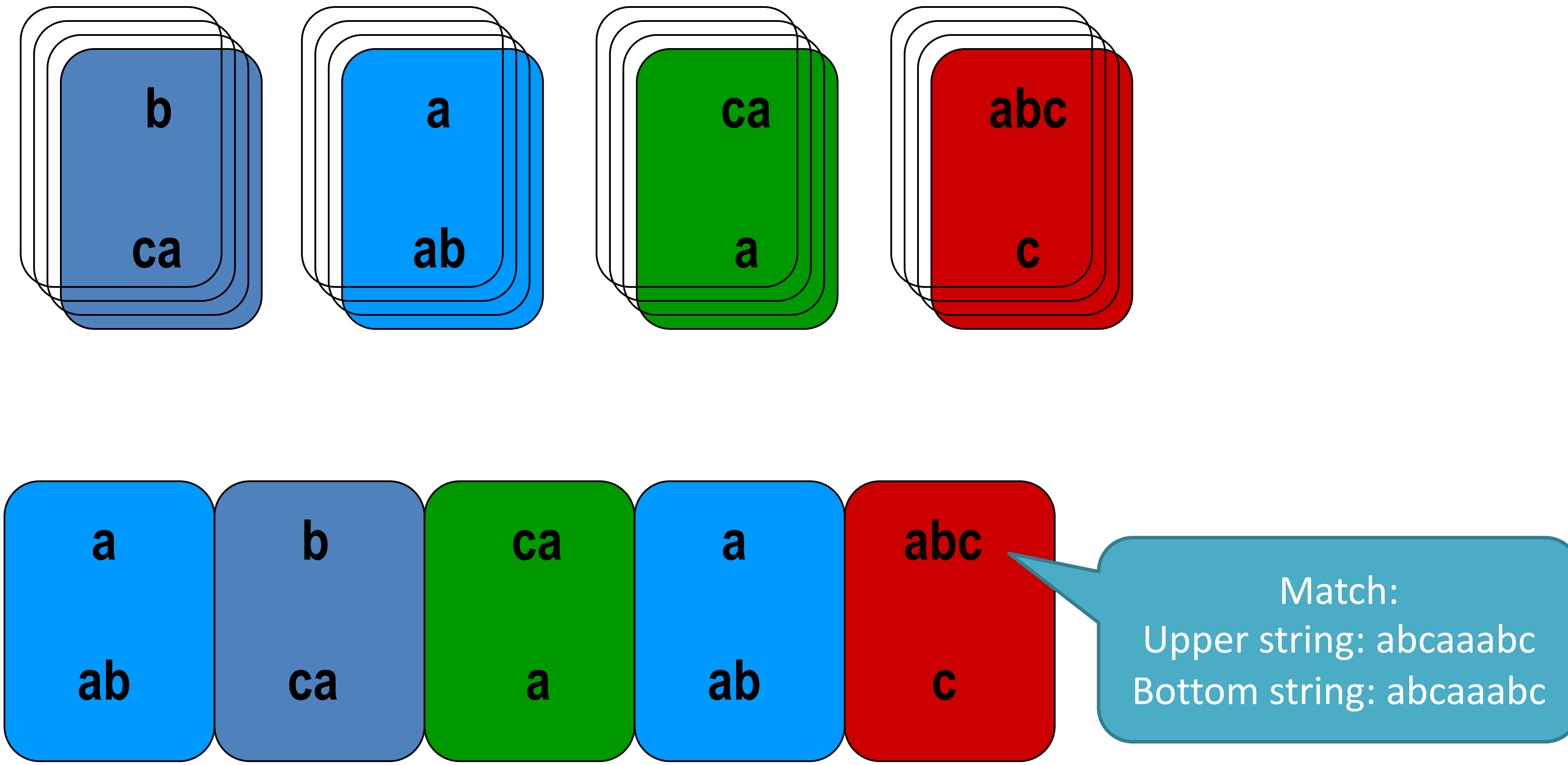
Post Correspondence Problem (PCP)



Post Correspondence Problem (PCP)

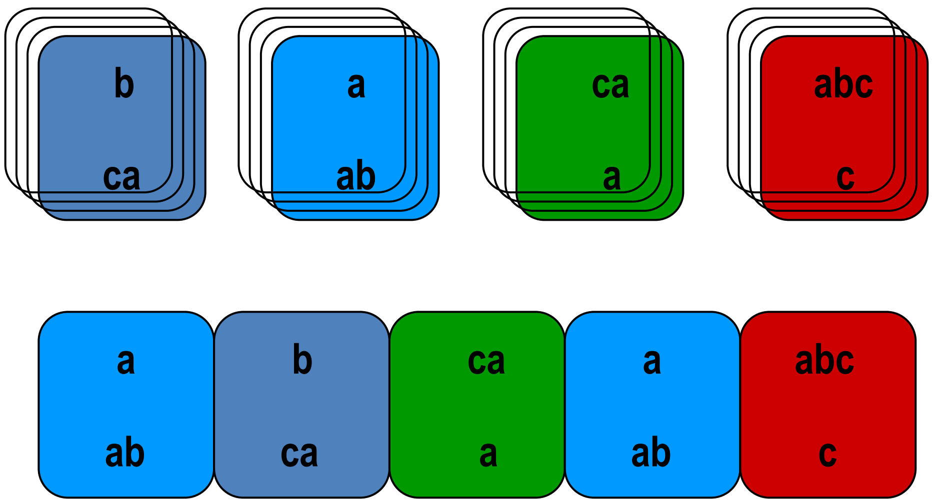


Post Correspondence Problem (PCP)



Post Correspondence Problem (PCP)

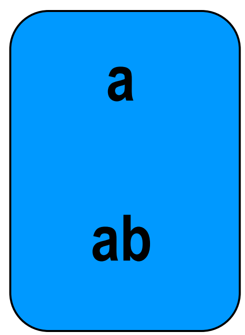
- Whether a collection of dominos has a match



- PCP = {⟨P⟩ | P is an instance of the Post Correspondence Problem with a match}.

Description of PCP

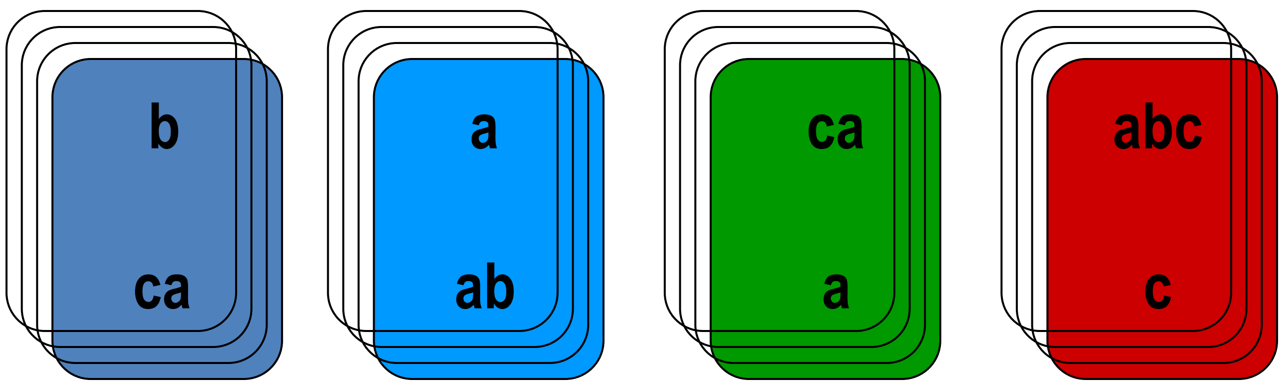
An individual domino $\left[\frac{a}{ab}\right]$



Description of PCP

A collection of dominos

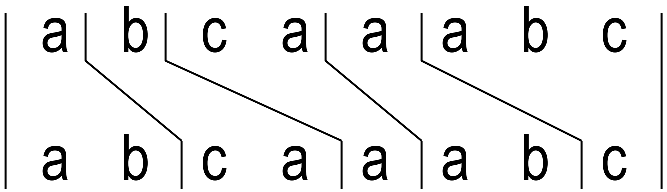
$$\left\{\left[\frac{b}{ca}\right], \left[\frac{a}{ab}\right], \left[\frac{ca}{a}\right], \left[\frac{abc}{c}\right]\right\}$$

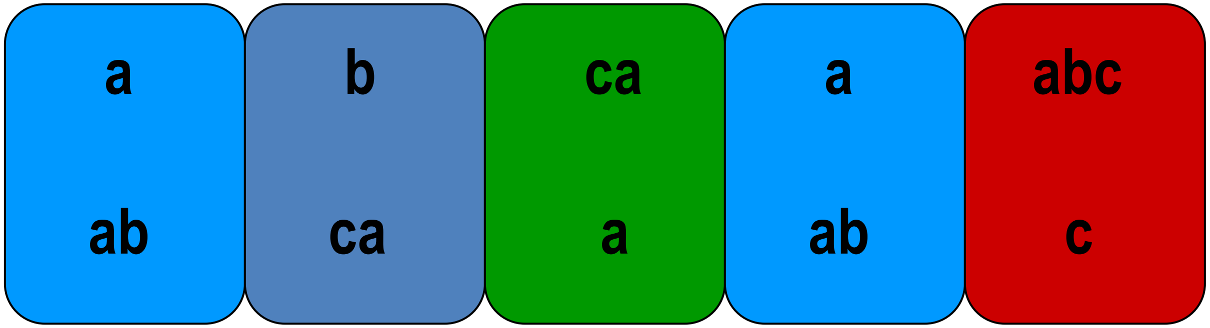


Description of PCP

A match

$$\left[\frac{a}{ab}\right]\left[\frac{b}{ca}\right]\left[\frac{ca}{a}\right]\left[\frac{a}{ab}\right]\left[\frac{abc}{c}\right]$$





A collection without a match

- For a given collection

$$\left\{\left[\frac{abc}{ab}\right], \left[\frac{ca}{a}\right], \left[\frac{acc}{ba}\right]\right\}$$

- It cannot contain a match because every top string is longer than the corresponding bottom string

Theorem 5.15

- PCP is undecidable

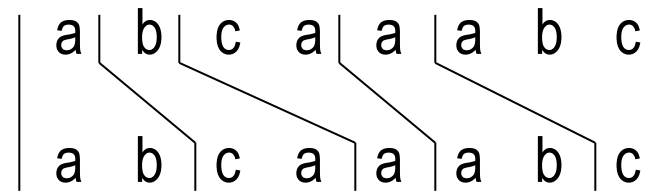
- Proof:

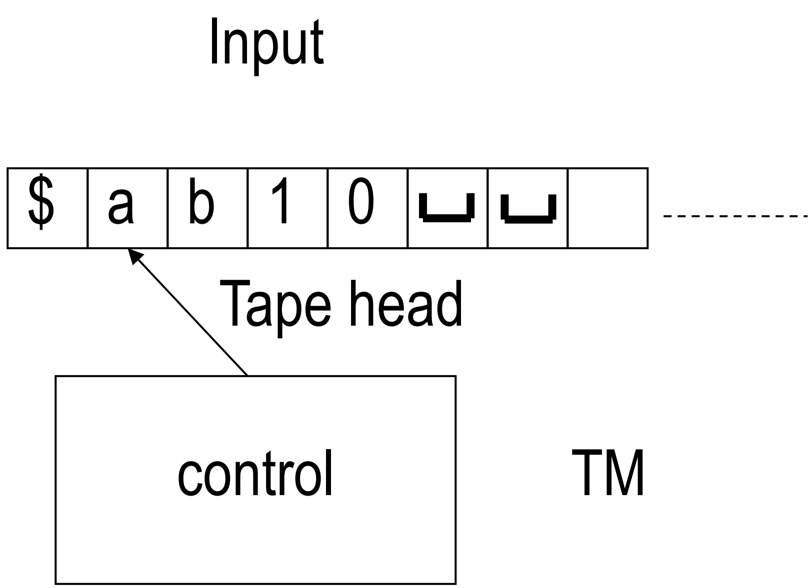
  To simplify the problem, we create ***Modified Post Correspondence Problem (MPCP)*** ,

  MPCP = {⟨P⟩| P is an instance of the PCP with a match that starts with the first domino}.

Theorem 5.15

$$\left[\frac{a}{ab}\right]\left[\frac{b}{ca}\right]\left[\frac{ca}{a}\right]\left[\frac{a}{ab}\right]\left[\frac{abc}{c}\right]$$





- The PCP program is very similar with the A_{TM} program

Theorem 5.15 proof

- Proof:

  Suppose PCP is decidable

  We construct TM S to decide A_{TM} (Theorem 4.11: A_{TM} is undecidable)

  PCP

  $\mathbf{M = (Q,\Sigma,\Gamma,\delta,q_0,q_{accept},q_{reject})}$

  S constructs an instance of the PCP P that has a match iff M accepts w:

  A_{TM}

  - (1) S first constructs an instance P' of the MPCP;
  - (2) Transfer P' into P;

Theorem 5.15 proof

- Proof:

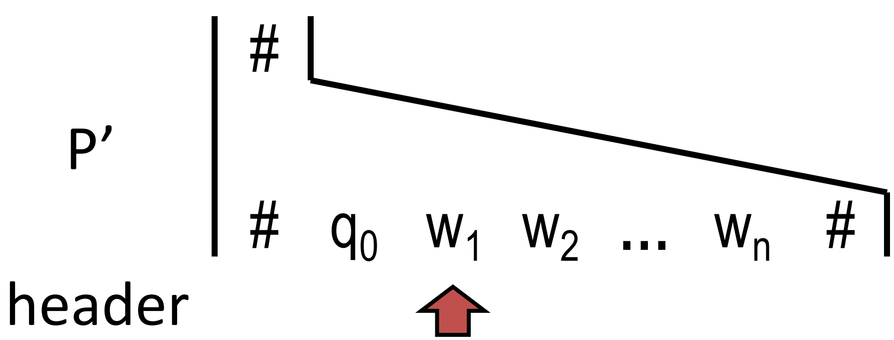
  (1) S first constructs an instance P’ of the MPCP;

  (1.1) Generating beginning configuration

  Put $\left[\frac{\#}{\# q_0 w_1 w_2 \ldots w_n \#}\right]$ into P’ as $\left[\frac{t_1}{b_1}\right]$

TM M = (Q, Σ, Γ, δ, q_0, q_{accept}, q_{reject})

Input w



Theorem 5.15 proof

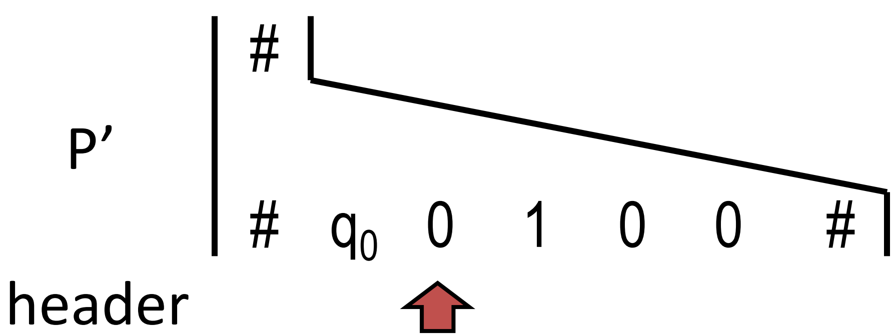
- Proof:

  (1) S first constructs an instance P' of the MPCP;

  (1.1) Generating beginning configuration

  Suppose Γ={0,1,2, ␣}, w=0100,

  Put the following into P' : $\left[\frac{\#}{\# q_0 0100\#}\right] = \left[\frac{t_1}{b_1}\right]$



Theorem 5.15 proof

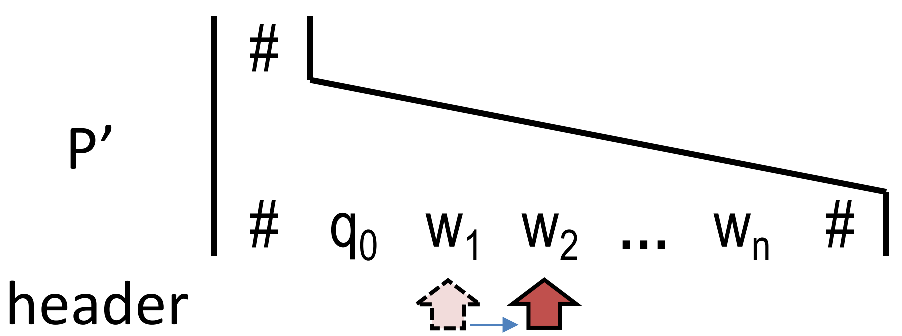
- Proof:

  (1) S first constructs an instance P’ of the MPCP;

  (1.2) The head move to the right

  For each a, b$\in\Gamma$ and q, r$\in$Q, where q $\neq$ q_{reject},

  if **δ(q, a) = (r, b, R),** then put $\left[\frac{qa}{br}\right]$ into P’



Theorem 5.15 proof

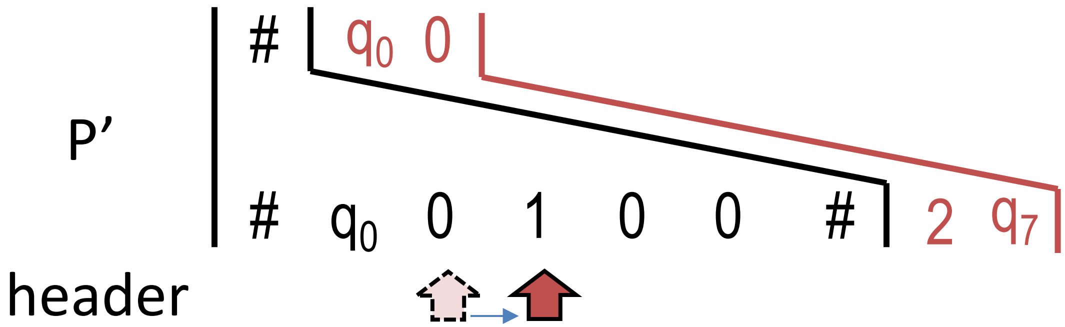
- Proof:

  (1) S first constructs an instance P' of the MPCP;

  (1.2) The head moves to the right

  For each a, b $\in \Gamma$ and q, r $\in$ Q, where q $\neq q_{reject}$,

  if $\delta(q_0, 0) = (q_7, 2, R)$, then put $\left[\frac{q_0 0}{2 q_7}\right]$ into P'



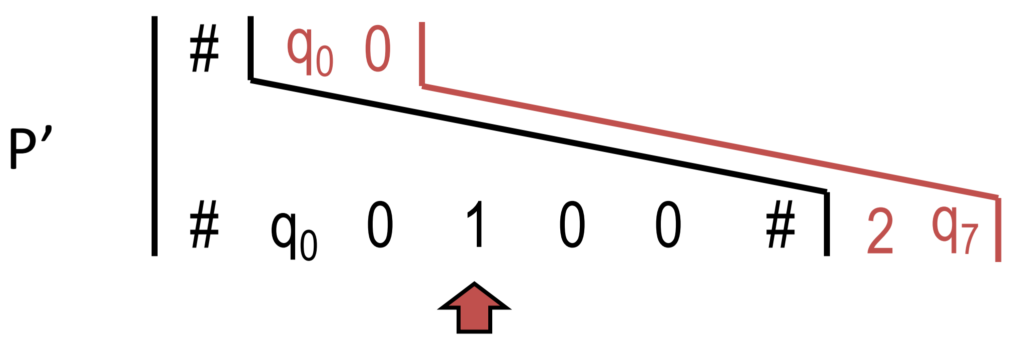
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P' of the MPCP;

  (1.3) The head moves to the left

  Discuss it later on.



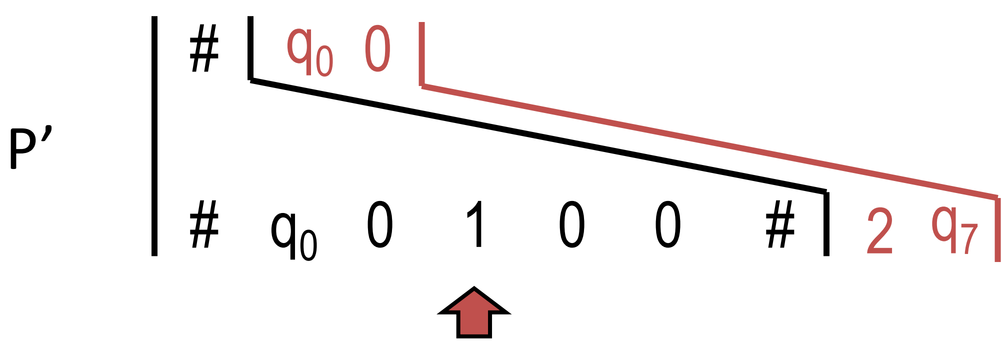
Theorem 5.15 proof

- Proof:

(1) S first constructs an instance P' of the MPCP;

(1.4) For each a∈Γ,

put $\left[\frac{a}{a}\right]$ into P'



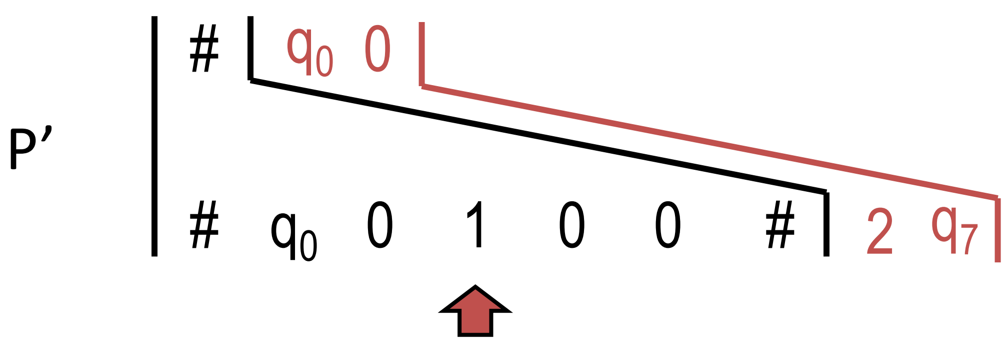
Theorem 5.15 proof

- Proof:

(1) S first constructs an instance P′ of the MPCP;

(1.4) $\Gamma = \{0, 1, 2, \sqcup\}$,

put $\left[\frac{0}{0}\right]$, $\left[\frac{1}{1}\right]$, $\left[\frac{2}{2}\right]$ and $\left[\frac{\sqcup}{\sqcup}\right]$ into P′



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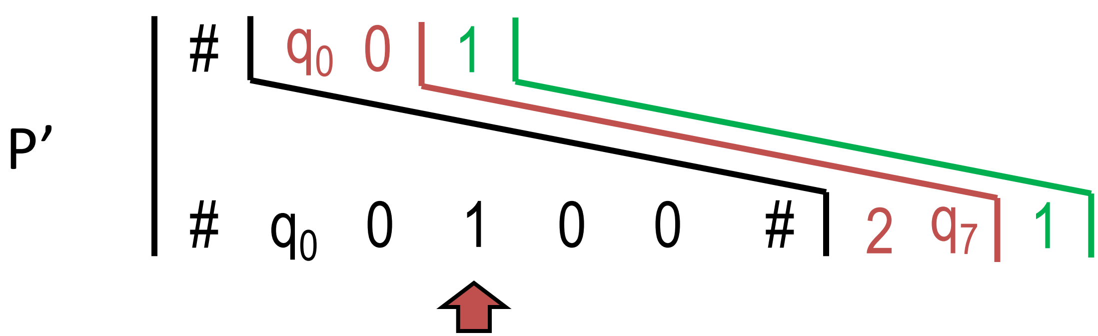


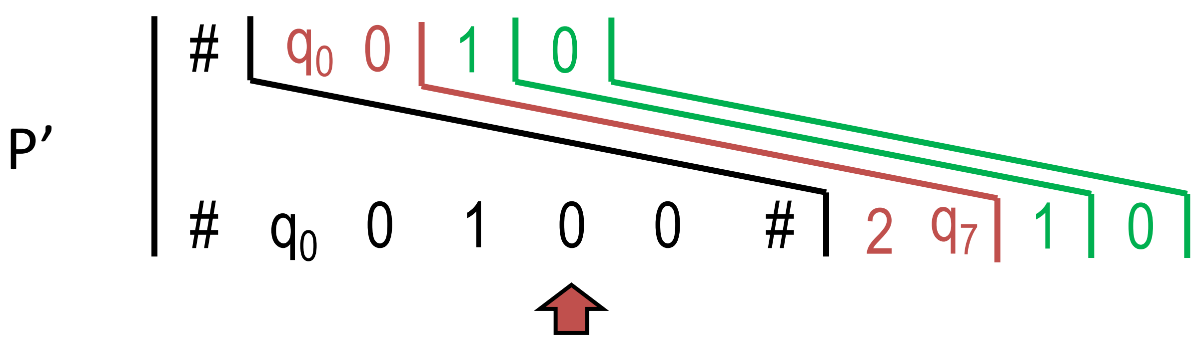
Theorem 5.15 proof

- Proof:

(1) S first constructs an instance P’ of the MPCP;

(1.4) Γ = {0, 1, 2, ␣},

put $\left[\frac{0}{0}\right]$, $\left[\frac{1}{1}\right]$, $\left[\frac{2}{2}\right]$ and $\left[\frac{␣}{␣}\right]$ into P’



Theorem 5.15 proof

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(1) S first constructs an instance P’ of the MPCP;

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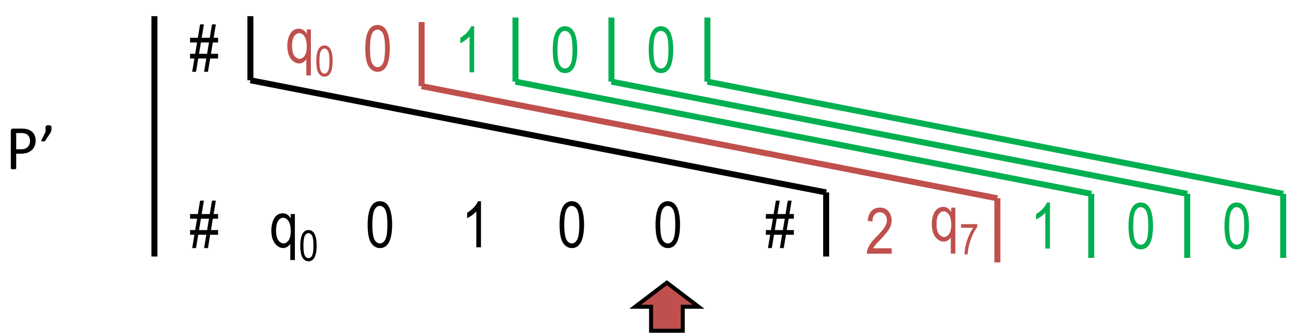
put $\left[\frac{0}{0}\right]$, $\left[\frac{1}{1}\right]$, $\left[\frac{2}{2}\right]$ and $\left[\frac{\sqcup}{\sqcup}\right]$ into P’



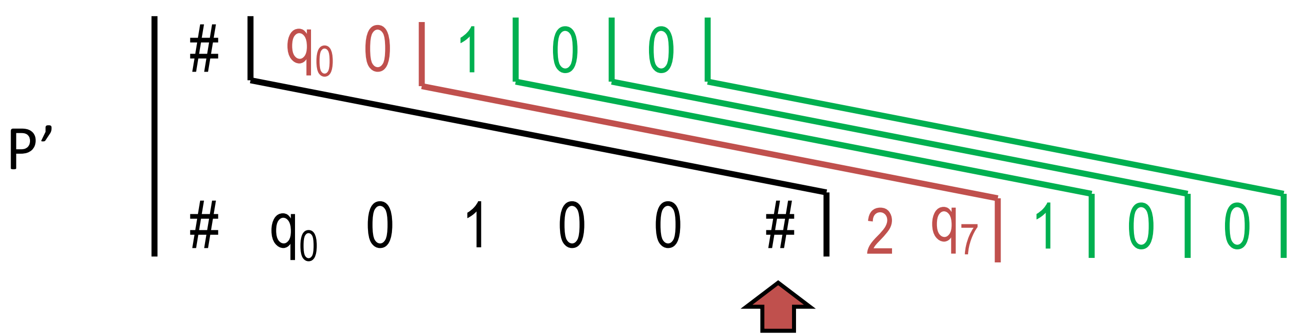
Theorem 5.15 proof

- Proof:

(1) S first constructs an instance P’ of the MPCP;

(1.5) Copy # and Put ␣ at the end of configuration

put $\left[\frac{\#}{\#}\right]$ and $\left[\frac{\#}{\sqcup\#}\right]$ into P’



Theorem 5.15 proof

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  (1) S first constructs an instance P' of the MPCP;

  (1.5) Copy # and Put ␣ at the end of configuration

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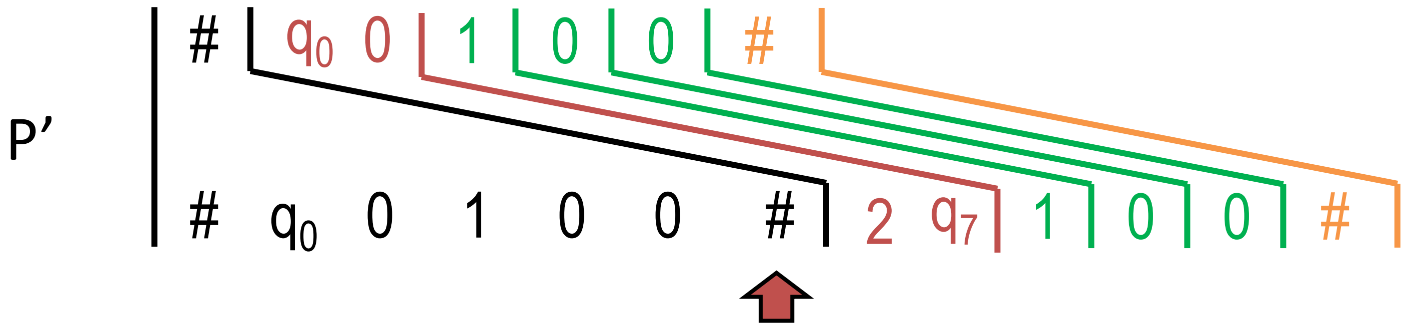

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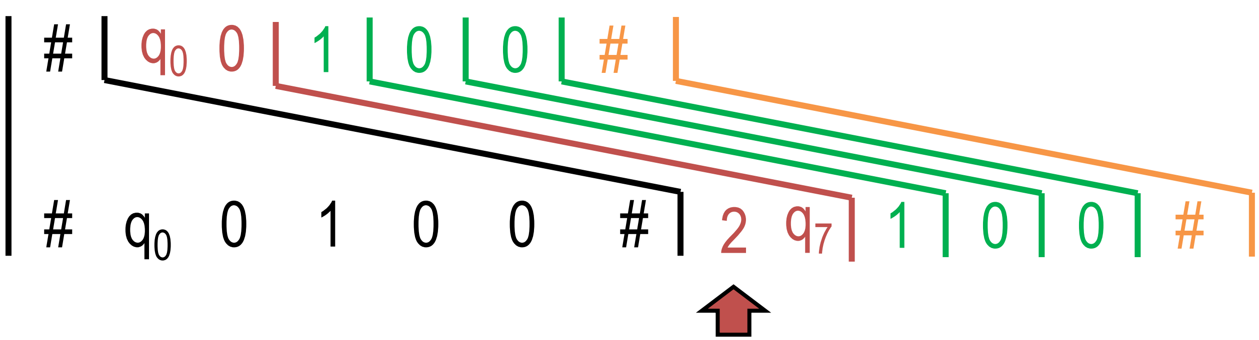
  (1) S first constructs an instance P’ of the MPCP;

  (1.5) Copy # and Put ␣ at the end of configuration

  put $\left[\frac{\#}{\#}\right]$ and $\left[\frac{\#}{\text{␣}\#}\right]$ into P’

P’



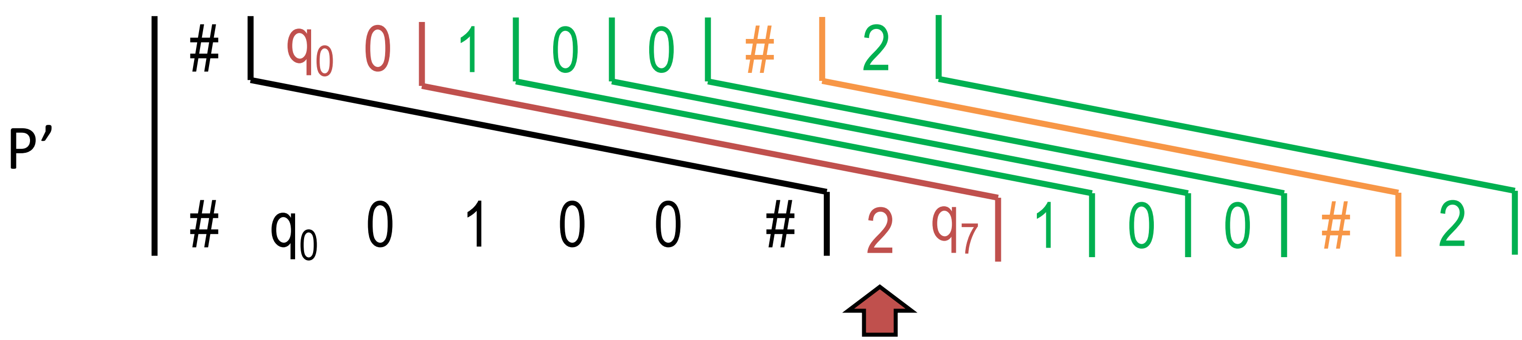
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P’ of the MPCP;

  (1.5) Copy # and Put ␣ at the end of configuration

  put $\left[\frac{\#}{\#}\right]$ and $\left[\frac{\#}{\textvisiblespace\#}\right]$ into P’



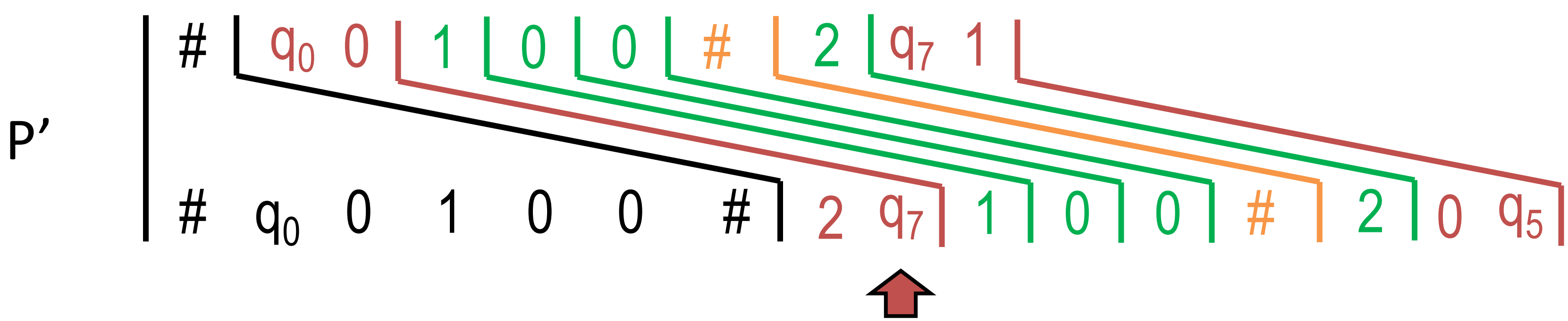
Theorem 5.15 proof

- Proof:

(1) S first constructs an instance P’ of the MPCP;

(1.5) Copy # and Put ␣ at the end of configuration

suppose $\delta(q_7, 1) = (q_5, 0, R)$, then put $\left[\frac{q_7 1}{0 q_5}\right]$ into P’



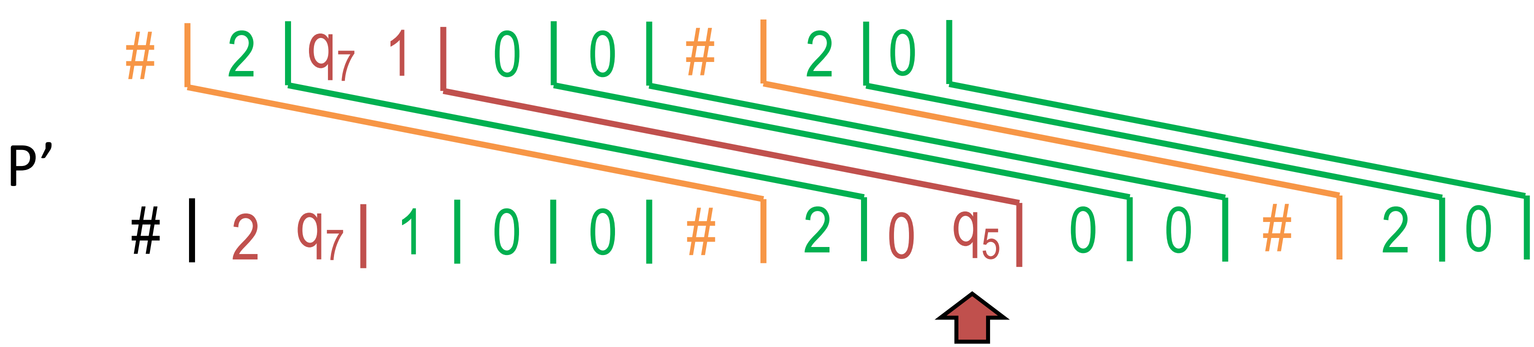
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P’ of the MPCP;

  (1.5) Copy # and Put ␣ at the end of configuration

  keep putting $\left[\frac{0}{0}\right]$, $\left[\frac{1}{1}\right]$, $\left[\frac{2}{2}\right]$, $\left[\frac{\sqcup}{\sqcup}\right]$, $\left[\frac{\#}{\#}\right]$ and $\left[\frac{\#}{\sqcup\#}\right]$ into P’



Theorem 5.15 proof

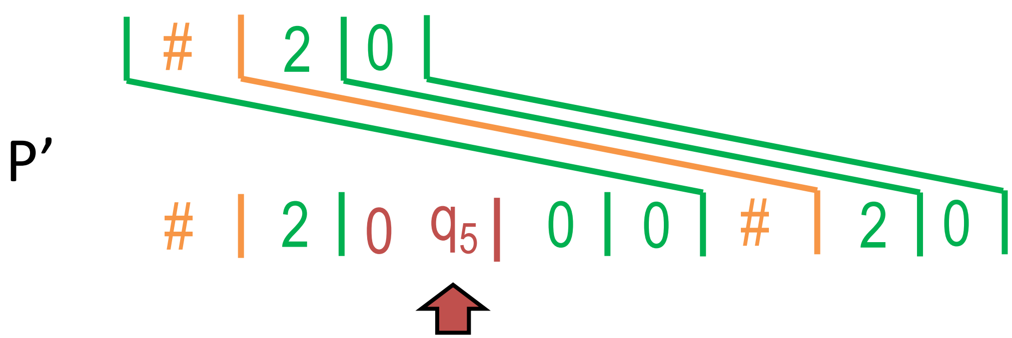
- Proof:

(1) S first constructs an instance P' of the MPCP;

(1.3) The head moves to the left

For each a, b, c$\in\Gamma$ and q, r$\in$Q, where q $\neq$ q_{reject},

if δ**(q, a) = (r, b, L),** then put $\left[\frac{cqa}{rcb}\right]$ into P' , c is the element on the left of q



Theorem 5.15 proof

- Proof:

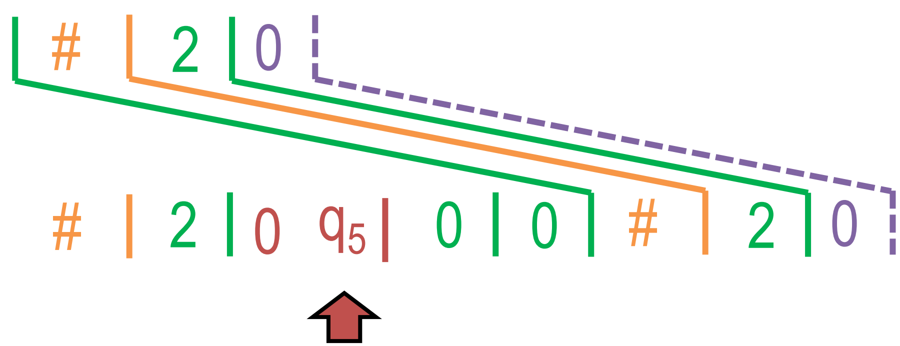
  (1) S first constructs an instance P' of the MPCP;

  (1.3) The head moves to the left

  Suppose $\delta(q_5, 0) = (q_9, 2, L)$, remove the old 0

  if $\delta(q, a) = (r, b, L)$, then put $\left[\frac{cqa}{rcb}\right]$ into P' , c is the element on the left of q

P'



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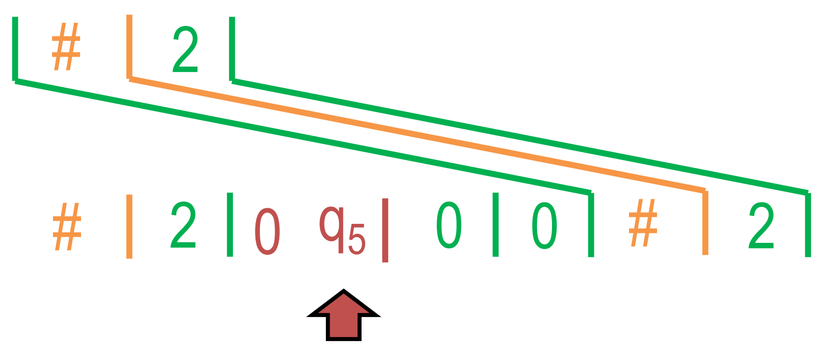


Theorem 5.15 proof

- Proof:

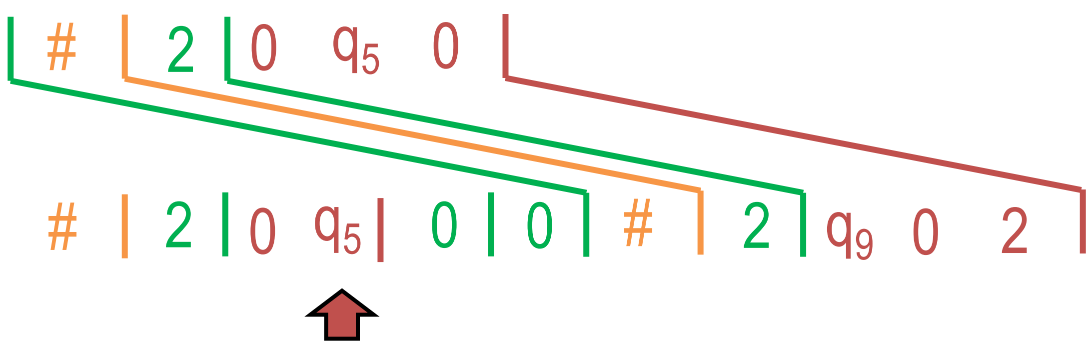
(1) S first constructs an instance P' of the MPCP;

(1.3) The head moves to the left

Suppose $\delta(q_5, 0) = (q_9, 2, L)$, put $\left[\frac{0q_50}{q_902}\right]$ into P'

if $\delta(q, a) = (r, b, L)$, then put $\left[\frac{cqa}{rcb}\right]$ into P' , c is the element on the left of q

P'



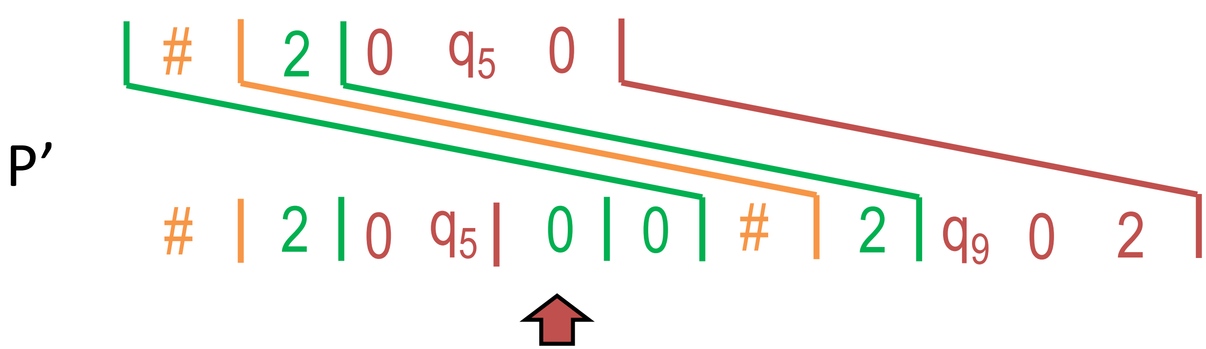
Theorem 5.15 proof

- Proof:

(1) S first constructs an instance P' of the MPCP;

(1.3) The head moves to the left

keep putting $\left[\frac{0}{0}\right]$, $\left[\frac{1}{1}\right]$, $\left[\frac{2}{2}\right]$, $\left[\frac{\sqcup}{\sqcup}\right]$, $\left[\frac{\#}{\#}\right]$ and $\left[\frac{\#}{\sqcup\#}\right]$ into P'



Theorem 5.15 proof

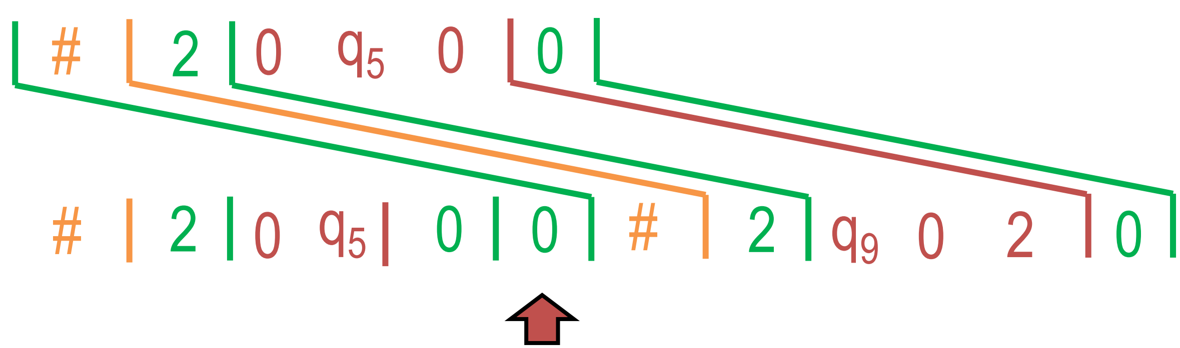
- Proof:

  (1) S first constructs an instance P’ of the MPCP;

  (1.3) The head moves to the left

  keep putting $\left[\frac{0}{0}\right]$, $\left[\frac{1}{1}\right]$, $\left[\frac{2}{2}\right]$, $\left[\frac{\sqcup}{\sqcup}\right]$, $\left[\frac{\#}{\#}\right]$ and $\left[\frac{\#}{\sqcup\#}\right]$ into P’

P’



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  (1) S first constructs an instance P’ of the MPCP;

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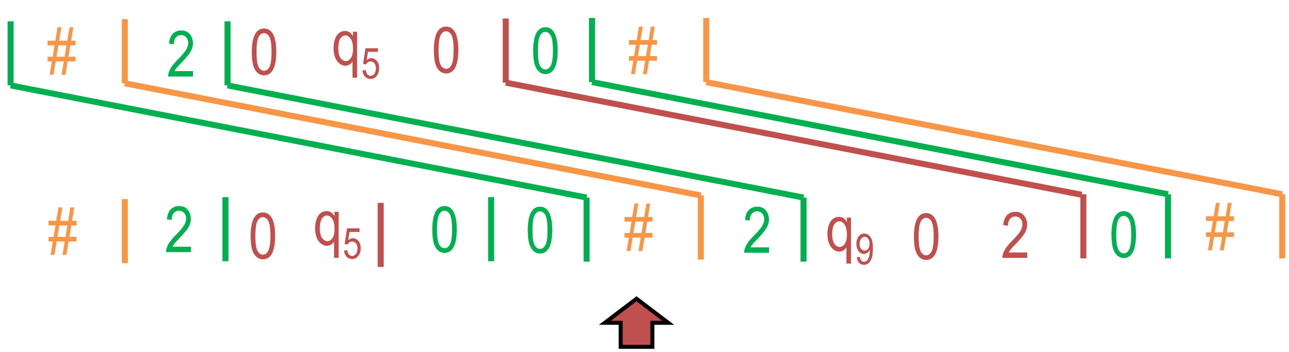
P’



Theorem 5.15 proof

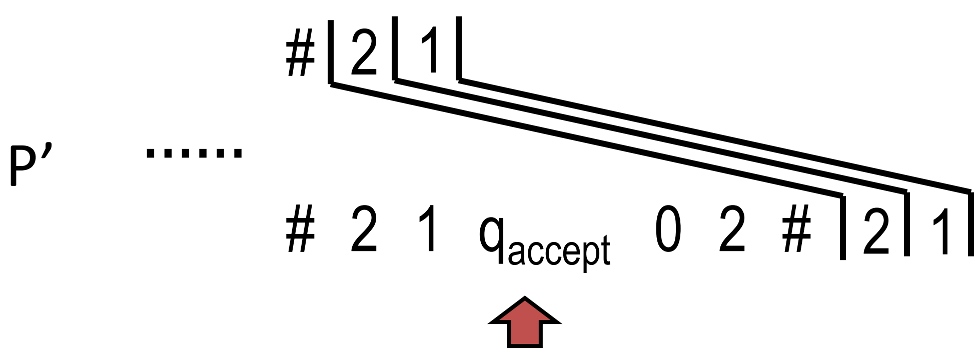
- Proof:

  (1) S first constructs an instance P’ of the MPCP;

  keep putting something into P’ until M halts:

  - if M rejects, S also rejects, means no match in PCP;
  - if M accepts, add something to the top to match the bottom.

Connect PCP match program with whether M accepts w



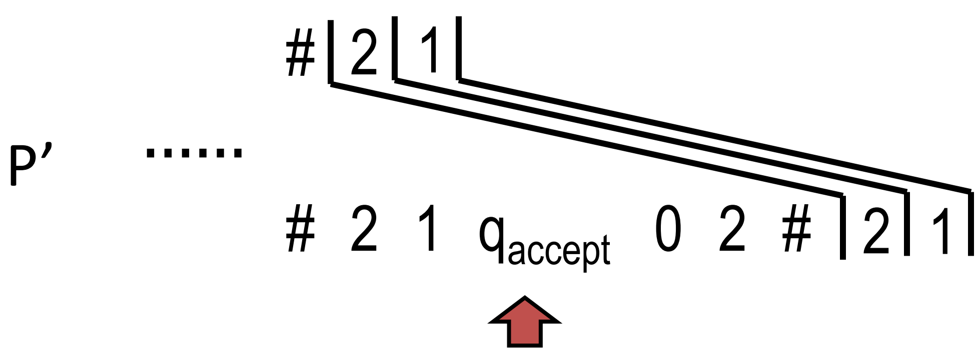
Theorem 5.15 proof

- Proof:

(1) S first constructs an instance P' of the MPCP;

(1.6) For a $\in \Gamma$,

put $\left[\frac{aq_{accept}}{q_{accept}}\right]$ and $\left[\frac{q_{accept}a}{q_{accept}}\right]$ into P'



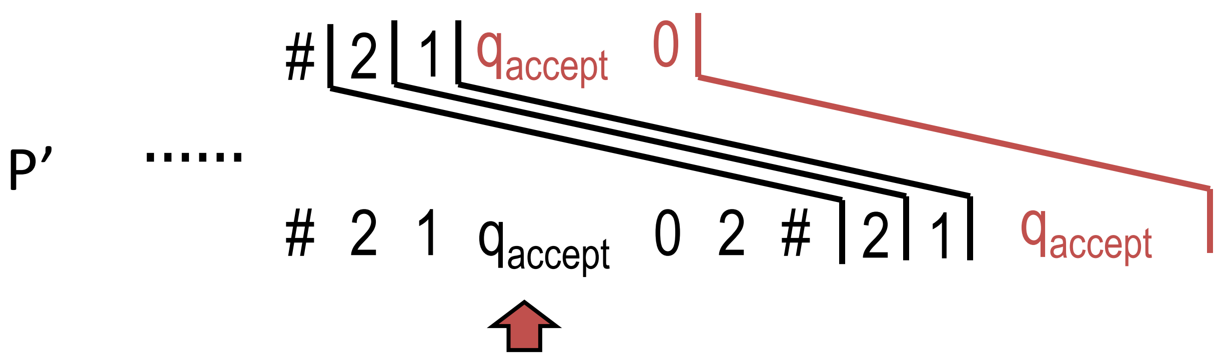
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P’ of the MPCP;

  (1.6) For a $\in \Gamma$,

$$\text{put } \left[\frac{aq_{accept}}{q_{accept}}\right] \text{ and } \left[\frac{q_{accept}a}{q_{accept}}\right] \text{ into P’}$$



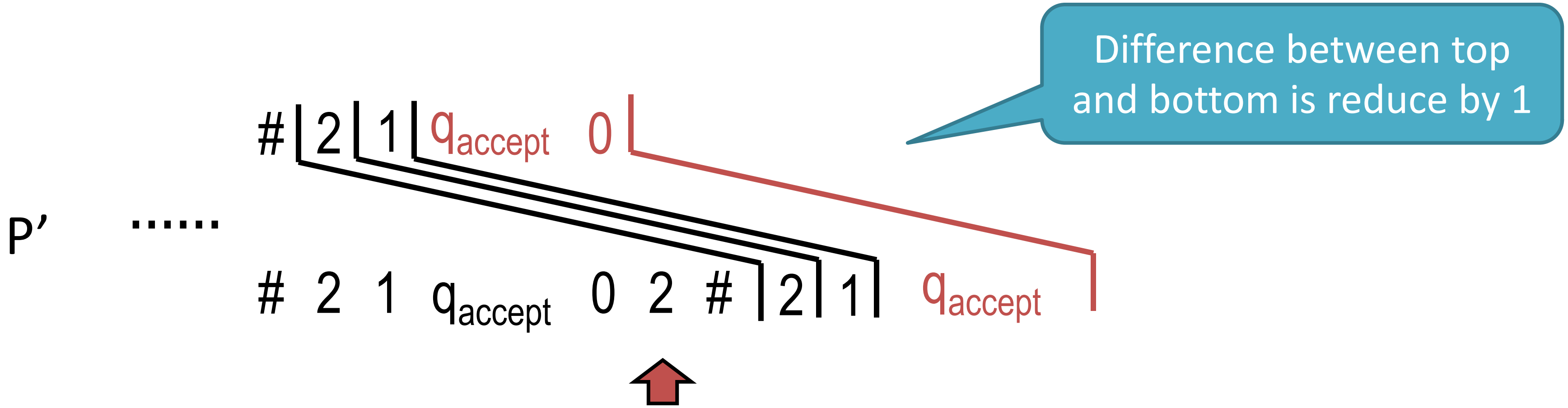
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.6) For a$\in\Gamma$,

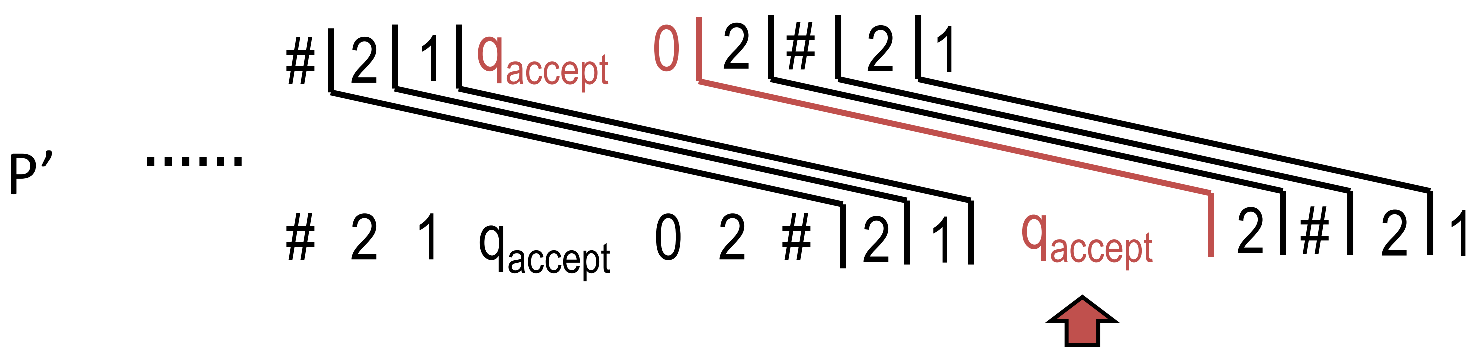
  $$\text{put } \left[\frac{aq_{accept}}{q_{accept}}\right] \text{ and } \left[\frac{q_{accept}a}{q_{accept}}\right] \text{ into P'}$$



Theorem 5.15 proof

- Proof:
  - (1) S first constructs an instance P' of the MPCP;
  - (1.6) For $a \in \Gamma$,

    put $\left[\frac{a q_{accept}}{q_{accept}}\right]$ and $\left[\frac{q_{accept} a}{q_{accept}}\right]$ into P'



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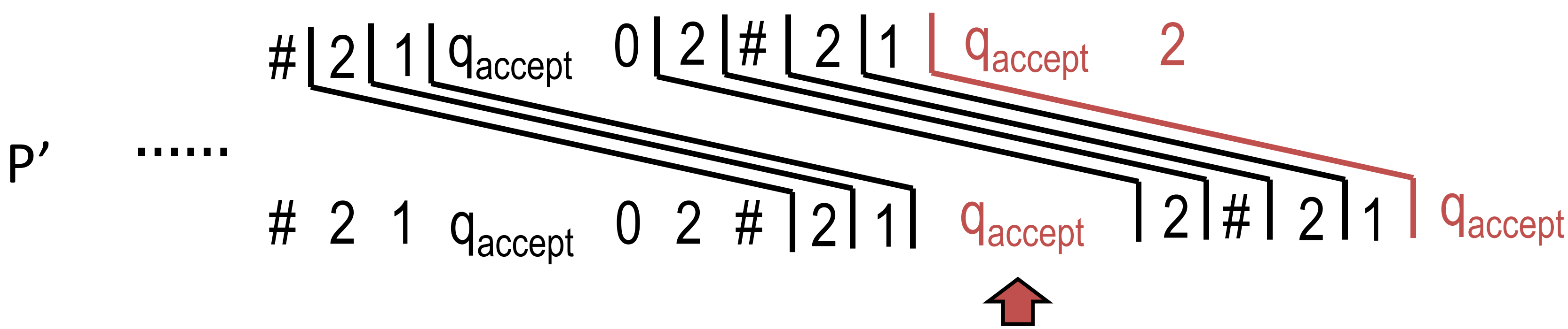
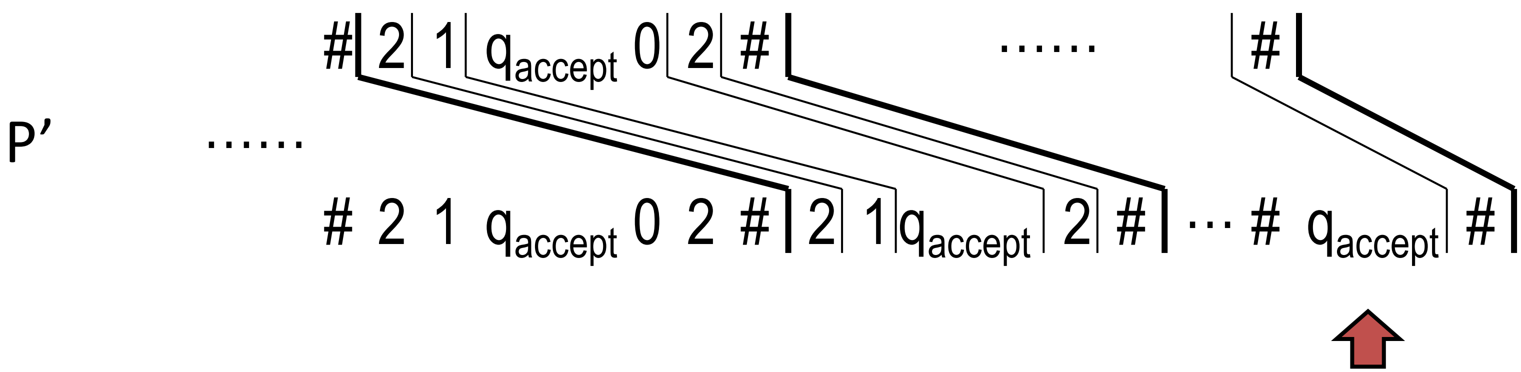


Theorem 5.15 proof

- Proof:
  - (1) S first constructs an instance P′ of the MPCP;
  - (1.6) For $a \in \Gamma$,

    Keep doing until the top and bottom difference is 1.



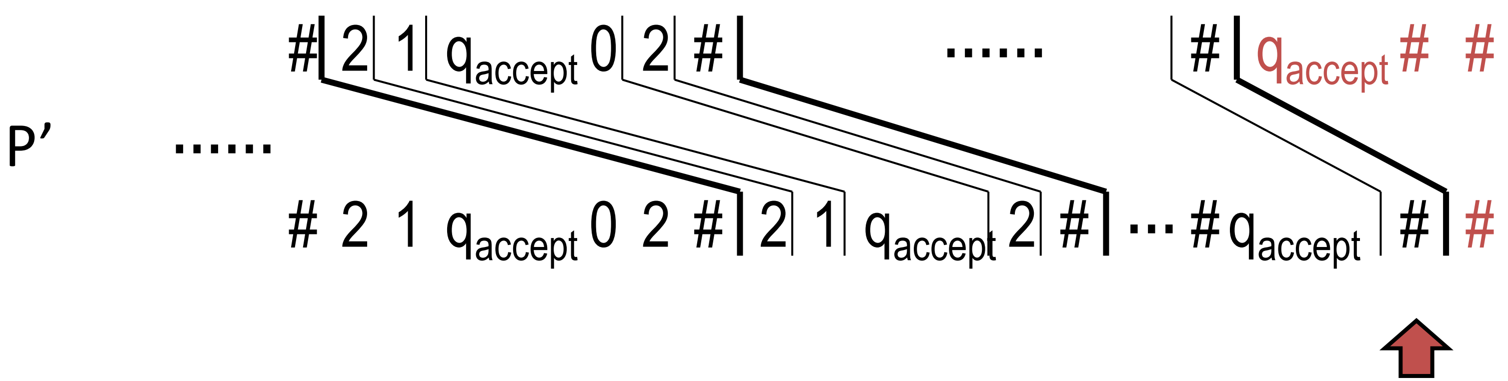
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P' of the MPCP;

  (1.7) Finish the match

  put $\left[\frac{q_{accept}\#\#}{\#}\right]$ into P'



Theorem 5.15 proof

- Proof:

(2) Transfer P' into P

P' is MPCP = {⟨P⟩| P is an instance of the PCP with a match that starts with the first domino}.

Suppose $u=u_1u_2\ldots u_n$ to be any string of length n, define

- ⋆u = ∗u1∗u2 ∗…∗u3
- u⋆ = $u_1 * u_2 * \ldots * u_3 *$
- ⋆u⋆ = $* u_1 * u_2 * \ldots * u_3 *$

Theorem 5.15 proof

$\star u \; = *u1*u2 *\ldots*u3$

$u\star \; = u_1*u_2 *\ldots*u_3*$

$\star u\star = *u_1*u_2 *\ldots*u_3*$

- Proof:

(2) Transfer P' into P

If P' = $\left\{\left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \ldots, \left[\frac{t_k}{b_k}\right]\right\}$, then let

P = $\left\{\left[\frac{\star t_1}{\star b_1 \star}\right], \left[\frac{\star t_2}{b_2 \star}\right], \left[\frac{\star t_3}{b_3 \star}\right], \ldots, \left[\frac{\star t_k}{b_k \star}\right], \left[\frac{*\Delta}{\Delta}\right]\right\}$

First domino: The only element as beginning

The match in P must be in shape as $\left[\frac{\star t_1}{\star b_1 \star}\right] \ldots \left[\frac{*\Delta}{\Delta}\right]$

Theorem 5.15 proof

- Proof:

Suppose PCP is decidable

We construct TM S using PCP match or not to decide A_{TM}

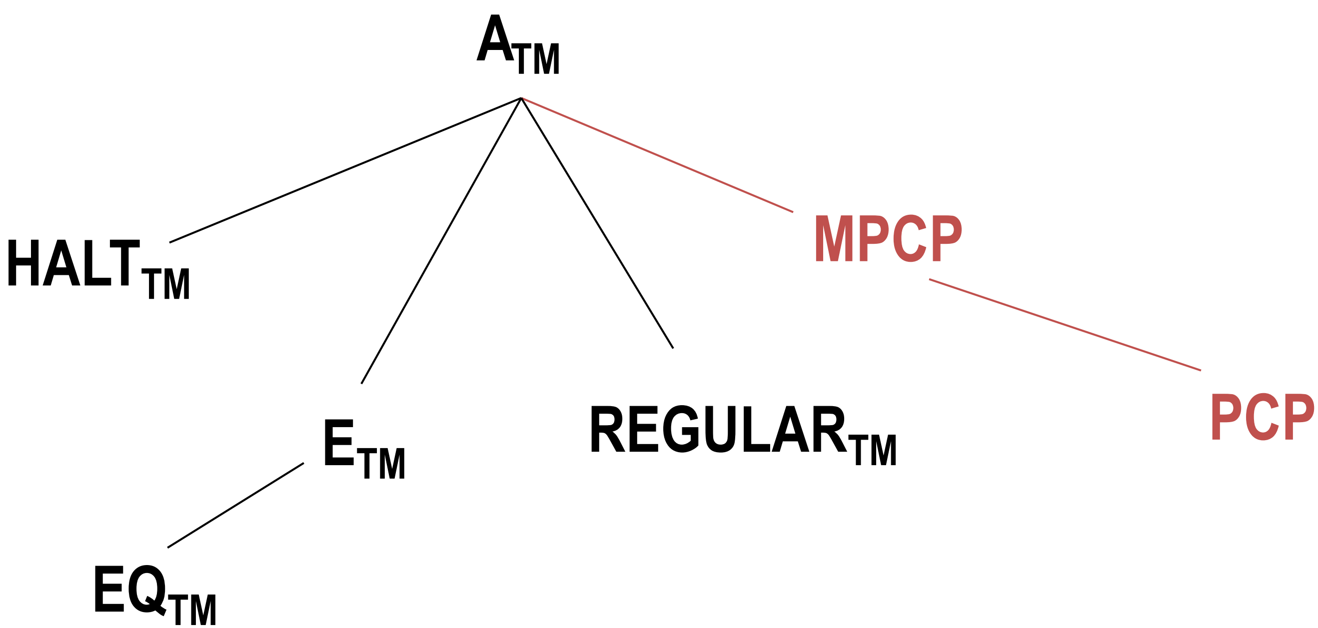
- (1) S first constructs an instance P' of the MPCP;
- (2) Transfer P' into P;

Theorem 4.11: A_{TM} is undecidable. Contradiction!

The suppose is wrong. Thus PCP is undecidable.

Conclusion

- Relationship of languages on reducibility



Conclusion

- Closure on operations

	Complement $\overline{A}$	Intersection $\cap$	Union $\cup$	Star A^*
Regular/DFA/ NFA	√	√	√	√
CFL/ PDA	×	×	√	√
Turing-decidable TM	√	√	√	√
Turing-recognizable TM	×	√	√	√