CS 6041
Theory of Computation

Context-free language

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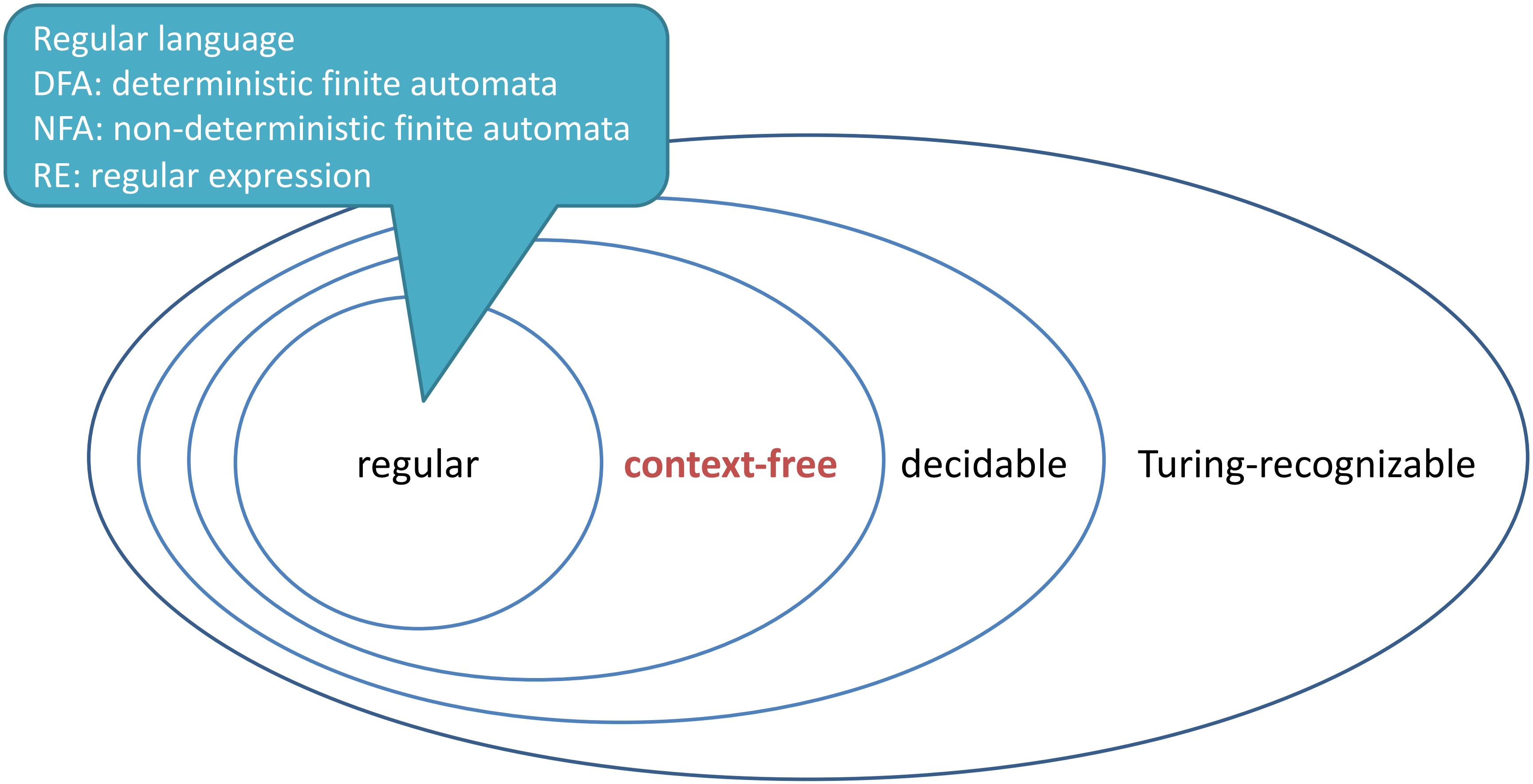
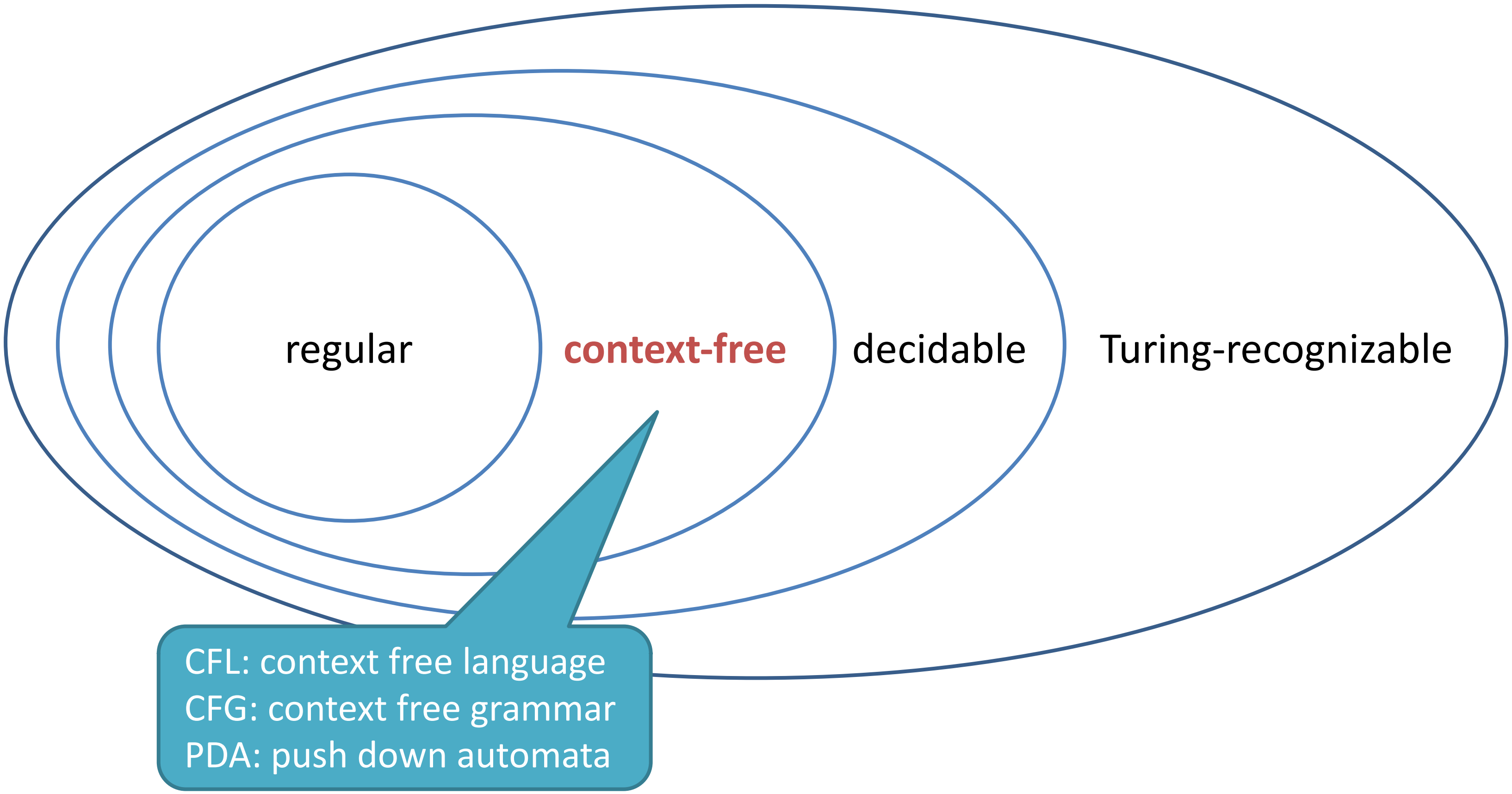
Outline

- Context-free language
  - Context-free language and grammar
  - Parse tree
  - Definition of CFG
- Design CFG
  - Example
  - Ambiguity
  - Leftmost derivation

Context-free language



Context-free language



Context Free Grammar

- Example, G_1

3 substitution rules (productions)

$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#$$

Context Free Grammar

- Example, G_1

Variable:
A, B

$A \rightarrow 0A1$
$A \rightarrow B$
$B \rightarrow \#$

Context Free Grammar

- Example, G_1

Start variable: A

$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#$$

Context Free Grammar

- Example, G_1

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Terminals:
0, 1, #

Context Free Grammar

- Example, G_1

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A, B

Start variable:
A

3 substitution rules (productions)

$$A \rightarrow 0A1$$
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$$B \rightarrow \#$$

Terminals:
0, 1, #

$$\begin{aligned} A &\Rightarrow 0A1 \\ &\Rightarrow 00A11 \\ &\Rightarrow 000A111 \\ &\Rightarrow 000B111 \\ &\Rightarrow 000\#111 \end{aligned}$$

Context Free Grammar

- The sequence of substitutions to obtain a string is called a ***derivation***

Grammar G_1: $A \rightarrow 0A1$

$A \rightarrow B$

$B \rightarrow \#$

Rule: $\rightarrow$

The language of G_1:

$L(G_1)=\{ 0^n\#1^n \mid n \geq 0 \}$

$$
\begin{aligned}
A &\Rightarrow 0A1 \\
&\Rightarrow 00A11 \\
&\Rightarrow 000A111 \\
&\Rightarrow 000B111 \\
&\Rightarrow 000\#111
\end{aligned}
$$

derivation : $\Rightarrow$

$\Leftrightarrow$ $A \Rightarrow^* 000\#111$

Abbreviating the CFGs

- Grammar G_1:

$$A \rightarrow 0A1$$

$$A \rightarrow B$$

$$B \rightarrow \#$$

- Abbreviation of G_1 :

$$G_1: \; A \rightarrow 0A1 \mid B$$

$$B \rightarrow \#$$

Parse tree

A

- Grammar G_1:

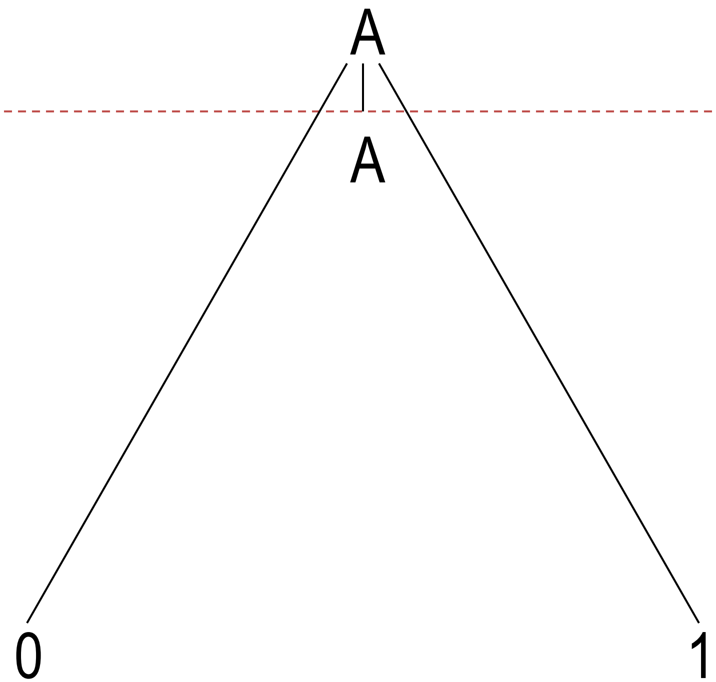
  $A \rightarrow 0A1$

  $A \rightarrow B$

  $B \rightarrow \#$

- Derivation: A
- Parse tree

Parse tree



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- Derivation: $A \Rightarrow 0A1$
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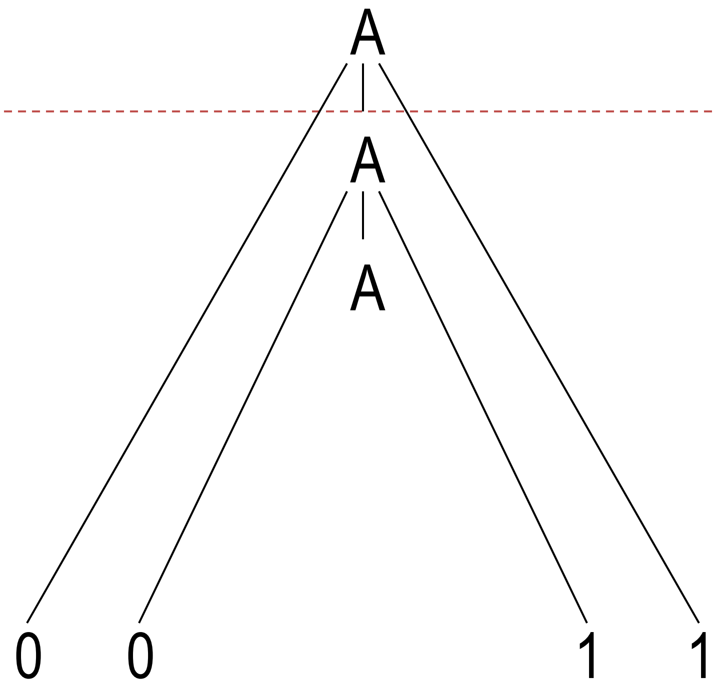

- Grammar G_1:

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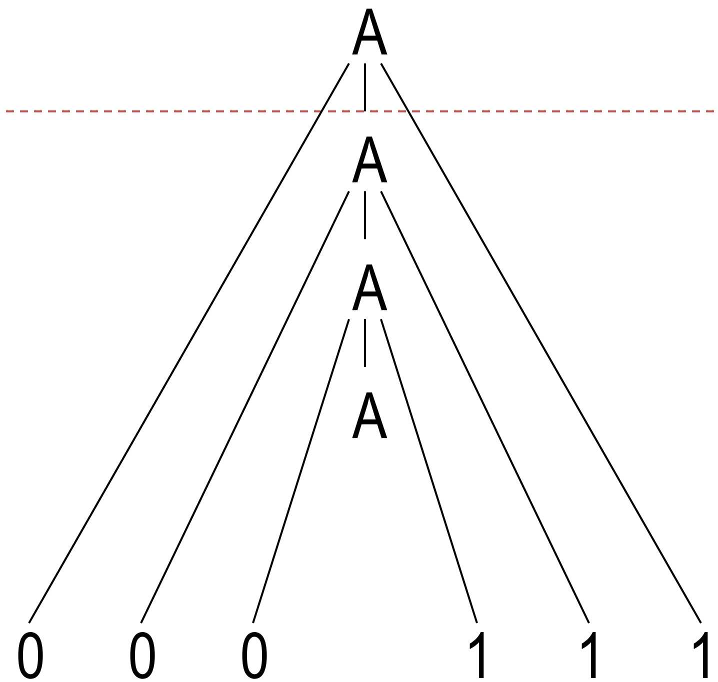
Parse tree



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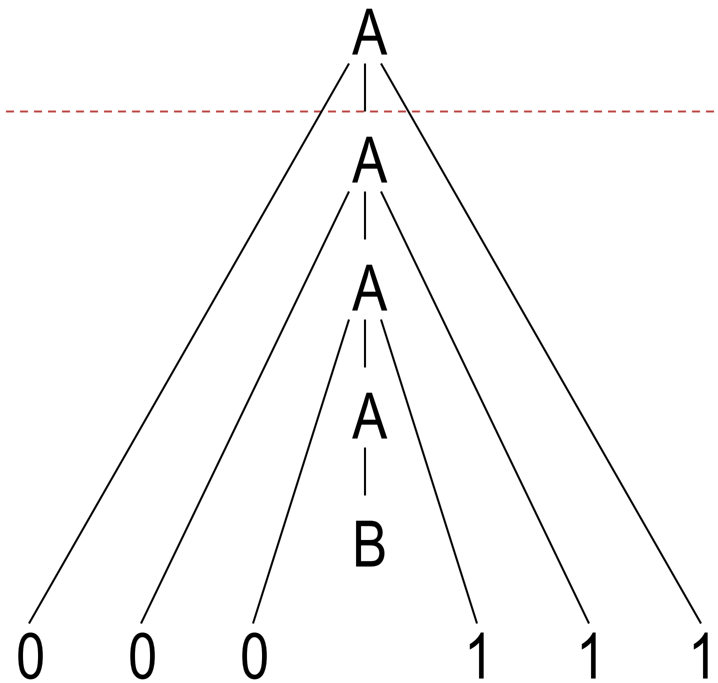

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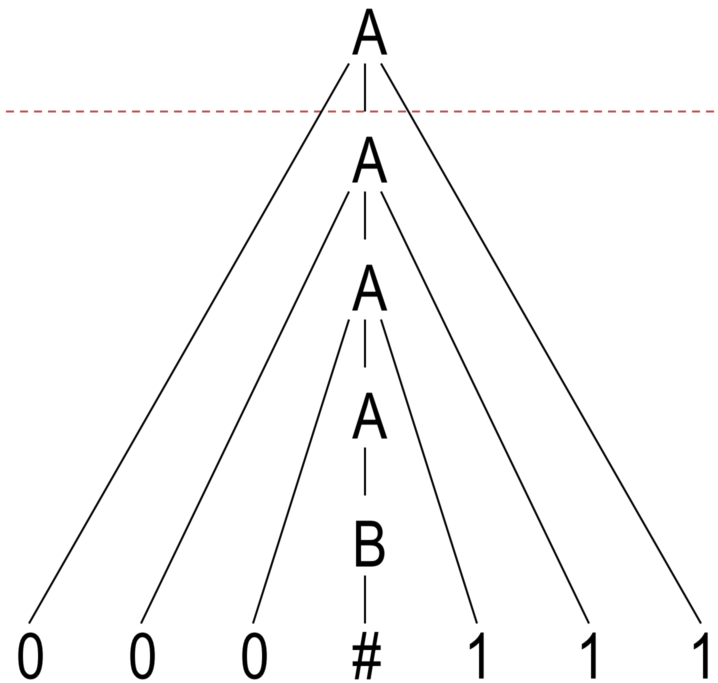


- Derivation: $A \Rightarrow 0A1 \Rightarrow 00A11$

  $\Rightarrow 000A111 \Rightarrow 000B111 \Rightarrow 000\#111$

- Parse tree

The language of grammar

- Grammar G_1:

  $A \rightarrow 0A1$

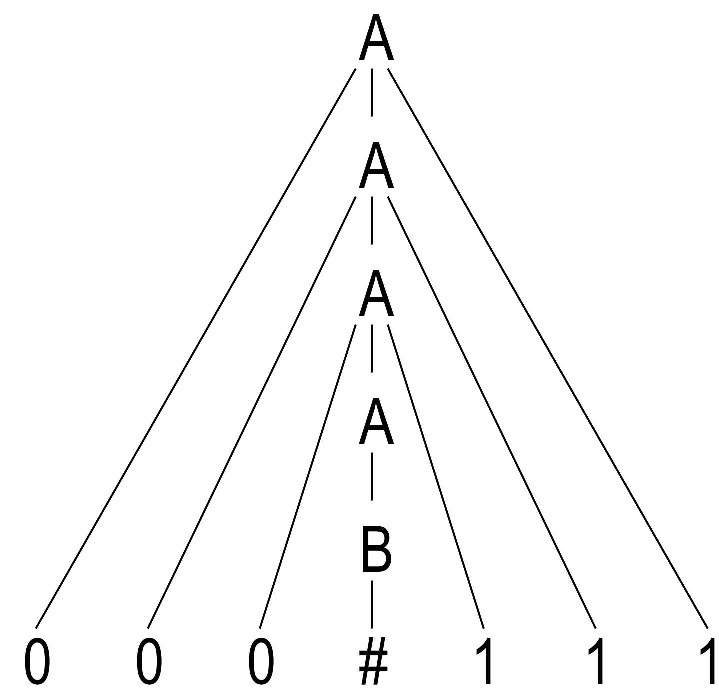
  $A \rightarrow B$

  $B \rightarrow \#$

- The language of G_1:

  $L(G_1)=\{ 0^n\#1^n \mid n>0 \}$

- Context-free language
  - Languages generated by context-free grammars



000#111

Definition of context-free grammar

- Context-free grammar is a 4-tuple $G=(V,\Sigma,R,S)$,

  1) V: finite variable set

  2) Σ: finite terminal set

  3) R: finite rule set

     ($A \rightarrow w$, $w \in (V \cup \Sigma)^*$)

  4) $S \in V$: start variable

Definition of context-free grammar

- Yield
  - If $A \rightarrow w$ is a rule of the grammar, we say that uAv ***yields*** uwv
- Derive
  - u ***derives*** v ($u \Rightarrow v$), if $u \Rightarrow u_1 \Rightarrow u_2 \Rightarrow \ldots \Rightarrow u_k \Rightarrow v$
- The language of grammar
  - $L(G)=\{ w \in \Sigma^* \mid S \Rightarrow^* w \}$
- Context-free language (CFL)
  - The language of CFG

Example

- Grammar G_1:

  $A \to 0A1$

  $A \to B$

  $B \to \#$

- $G_1 = ($

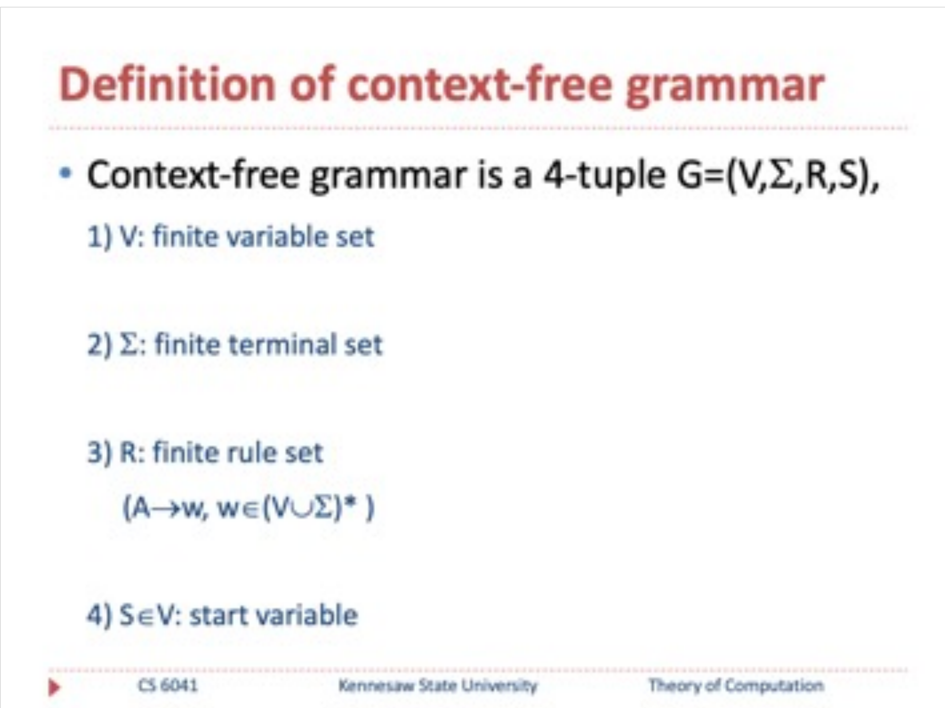
  $\{A,B\},$

  $\{0,1,\#\},$

  $\{A \to 0A1, A \to B, B \to \#\},$

  A

  $)$



Example

- Grammar G_1:

  S -> S+S | S*S | a

- G_1 = (

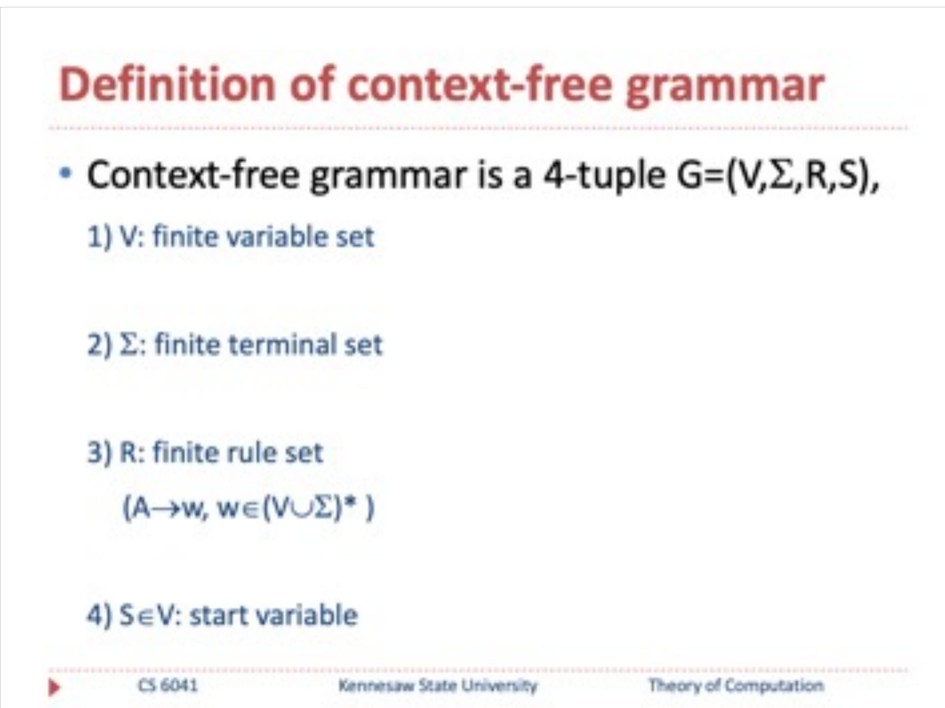
  {S},

  {a,+,*},

  {S -> S+S | S*S | a},

  S

)



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  4) $S \in V$: start variable

Question: how to derive it?

- $G_3=(\{S\},\{a,b\},R,S)$, R is

  $\{ S \rightarrow aSb \mid SS \mid \varepsilon \}$

$S \Rightarrow abab$?

S

$\Rightarrow SS$

$\Rightarrow aSbS$

$\Rightarrow abS$

$\Rightarrow abaSb$

$\Rightarrow abab$

$S \Rightarrow aaabbb$?

S

$\Rightarrow aSb$

$\Rightarrow aaSbb$

$\Rightarrow aaaSbbb$

$\Rightarrow aaabbb$

$S \Rightarrow aababb$?

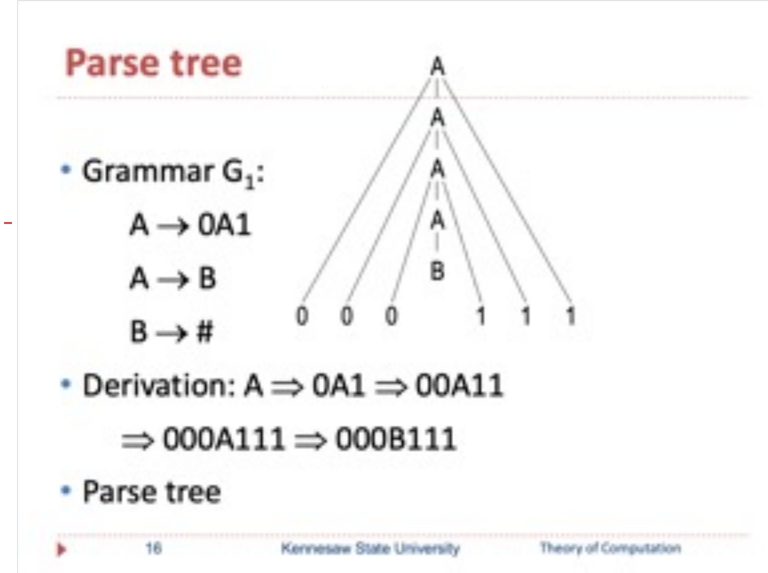
S

$\Rightarrow aSb$

…. //follow by $S \Rightarrow abab$

$\Rightarrow aababb$

Example of Parse tree



- $G_4=(V,\Sigma,R,E)$,

  $V=\{E, T, F\}$,

  $\Sigma=\{ a, +, \times, (,) \}$,

  $R=\{$

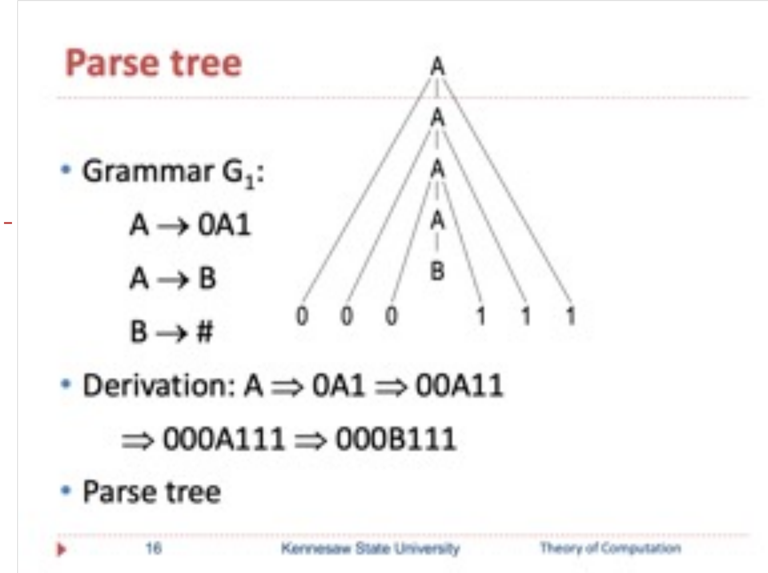
  $$E \rightarrow E + T \mid T$$

  $$T \rightarrow T \times F \mid F$$

  $$F \rightarrow (E) \mid a$$

  $\}$

Parse tree of a+a×a



- $G_4=(V,\Sigma,R,E)$,

  V={E, T, F},

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  R={

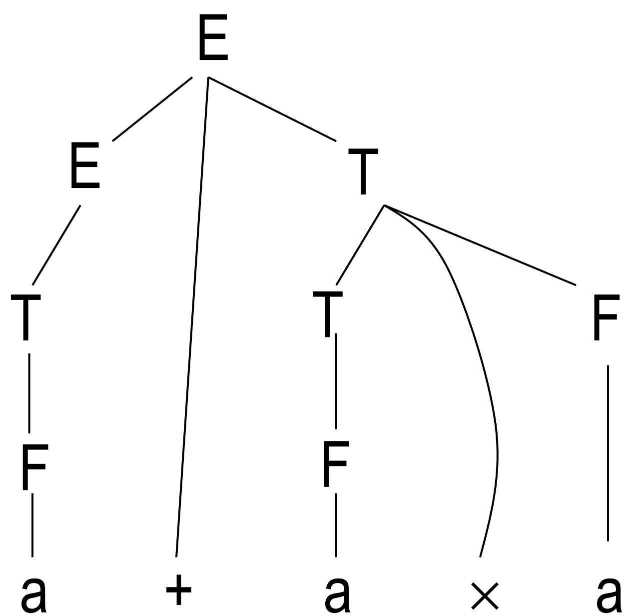
  $E \to E + T \mid T$

  $T \to T \times F \mid F$

  $F \to (E) \mid a$

  }

  E



Parse tree of (a+a)×a

- $G_4=(V,\Sigma,R,E)$,

  V={E, T, F},

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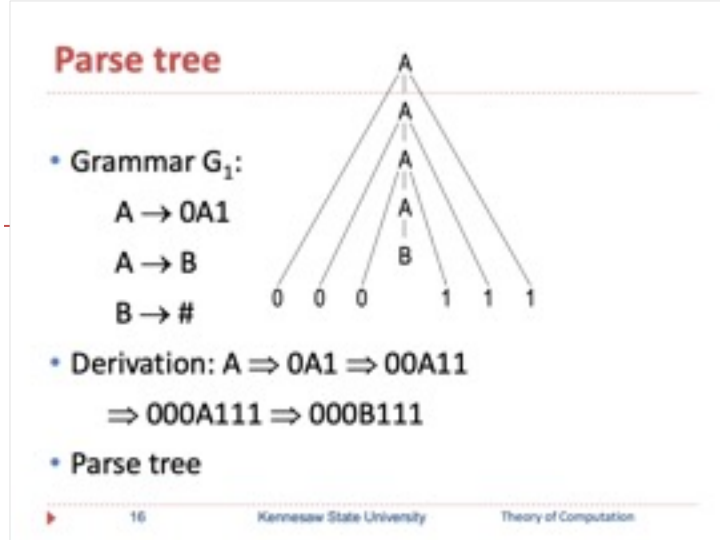

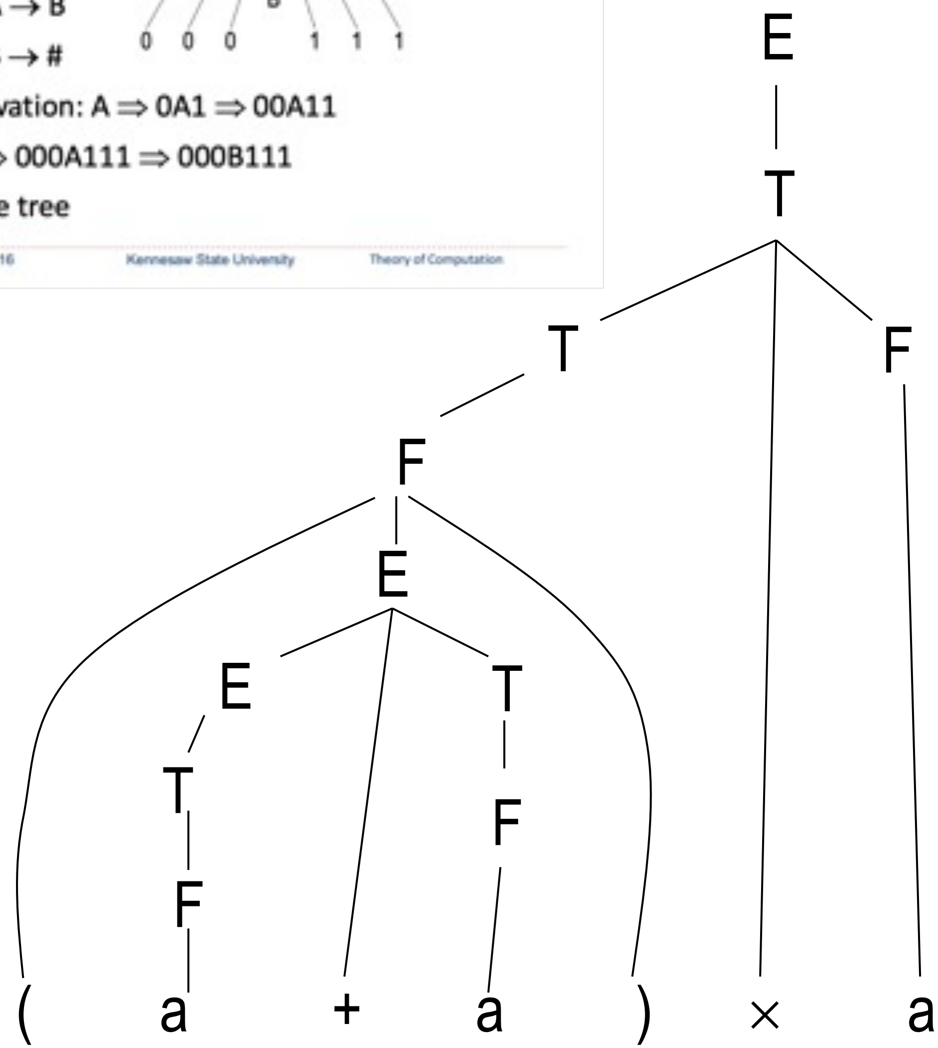

Outline

- Context-free language
  - Context-free language and grammar
  - Parse tree
  - Definition of CFG
- Design CFG
  - Example
  - Ambiguity
  - Leftmost derivation

Design context-free grammar

- Design CFG for $\{w|w=0^n1^n \text{ or } w=1^n0^n, n\geq 0\}$
  - Design CFG for $\{w|w=0^n1^n, n\geq 0\}$
    - $G_1=(\{S\},\{0,1\}, \{S\rightarrow 0S1, S\rightarrow \varepsilon\}, S)$
  - Design CFG for $\{w|w=1^n0^n, n\geq 0\}$
    - $G_2=(\{S\},\{0,1\}, \{S\rightarrow 1S0, S\rightarrow \varepsilon\}, S)$

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    - $G_1=(\{S_1\},\{0,1\},\{S_1\rightarrow 0S_11, S_1\rightarrow \varepsilon\},S_1)$
  - Design CFG for $\{w|w=1^n0^n, n\geq 0\}$
    - $G_2= (\{S_2\},\{0,1\},\{S_2\rightarrow 1S_20, S_2\rightarrow \varepsilon\},S_2)$

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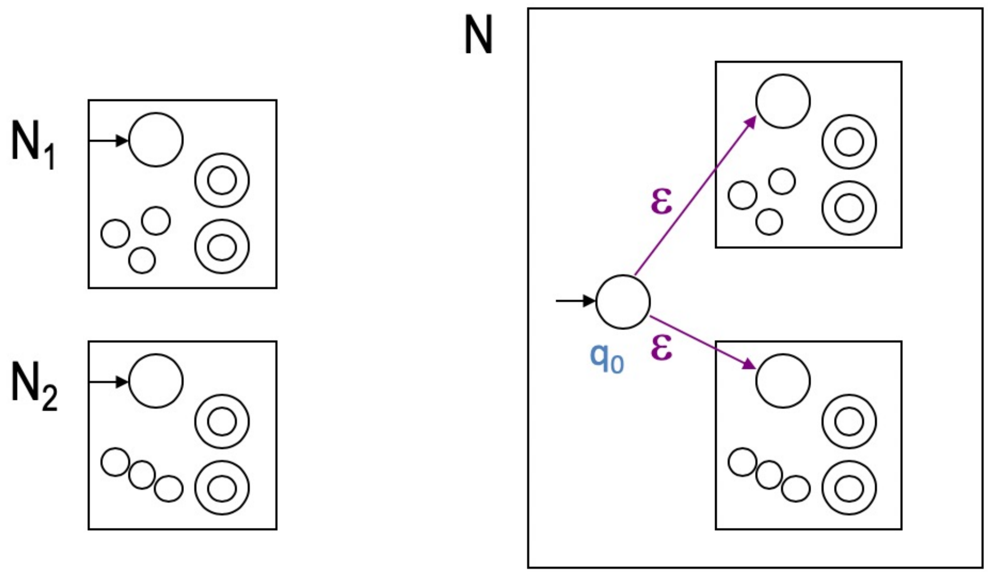

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    - $G_2= (\{S_2\},\{0,1\},\{S_2\rightarrow 1S_20, S_2\rightarrow \varepsilon\},S_2)$
  - $G=(\{S,S_1,S_2\},\{0,1\},$

    $\{ S\rightarrow S_1,\ S\rightarrow S_2,\ S_1\rightarrow 0S_11,\ S_1\rightarrow \varepsilon,\ S_2\rightarrow 1S_20,\ S_2\rightarrow \varepsilon\},$

    $S)$

Combine CFG into one

- General case:

  Add $S \rightarrow S_1 \mid S_2 \mid \ldots \mid S_k$

  - S is the new start variable
  - $S_1, S_2, \ldots, S_k$ are original start variables

- CFL is closure on the Union operation

Operation on languages

	RL: DFA/NFA/RE	CFL: CFG/PDA	TM
Union	close	close	?
Concatenation	close	?	?
Star	close	?	?
Complement	close	?	?
Boolean operation	close	?	?

Design CFG for languages

- Design CFG is much difficult than designing an automata for language

- Basic idea:

  1. divide CFL into small parts
  2. design CFG for each small part
  3. combine them together

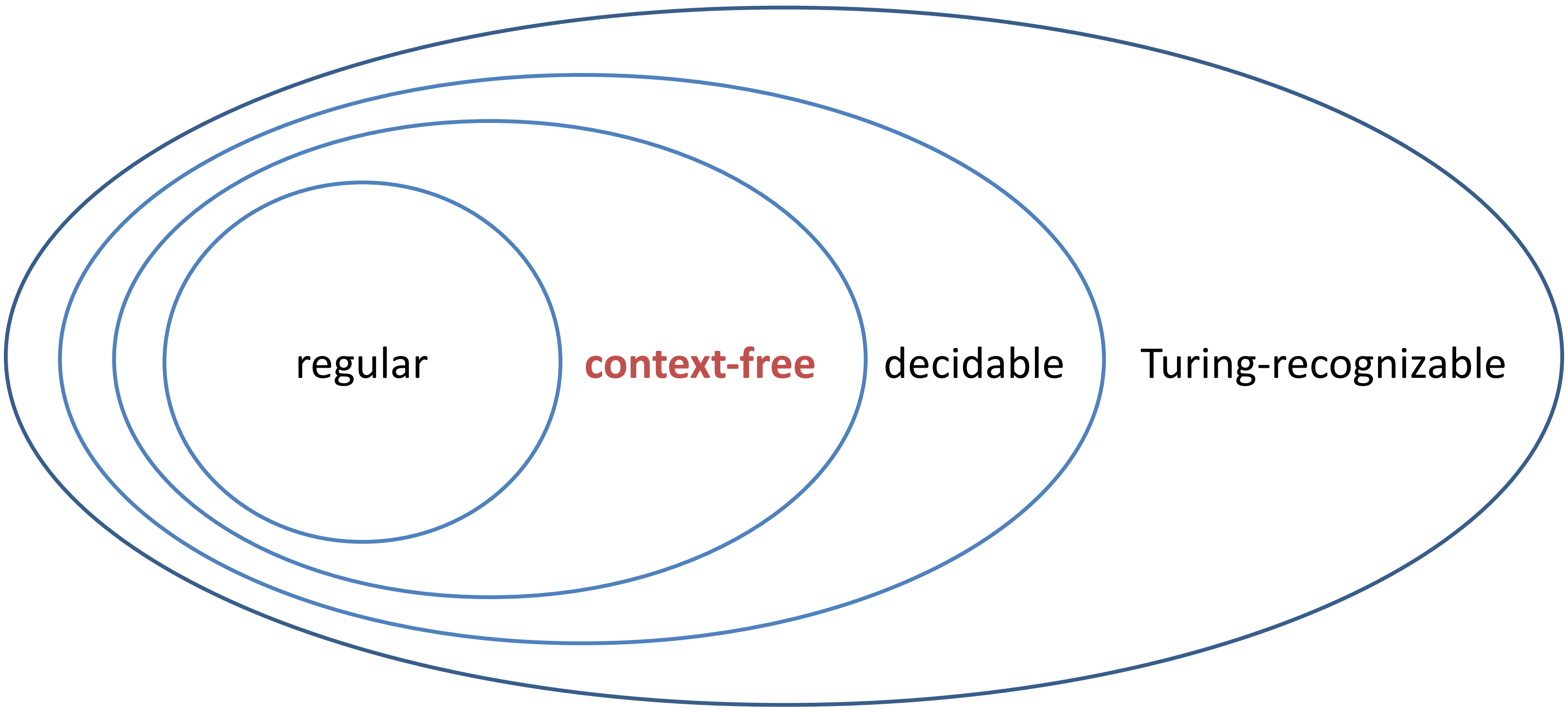
Design CFG for languages

- Design CFG is much difficult than designing an automata for language

- Other ideas:

  1. Simulate the regular expressions
  2. Look for a pattern from example strings
  3. ...

Design CFG for regular languages



Design CFG for regular languages

- Transfer DFA into equivalent CFG

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- Let DFA $M=(Q,\Sigma,\delta,q_0,F)$

  then CFG $G=(V,\Sigma,R,R_0)$

Design CFG for regular languages

- Transfer DFA into equivalent CFG
- Let DFA $M=(Q,\Sigma,\delta,q_0,F)$
  - $Q=\{q_0,q_1,\ldots,q_k\}$,

  then CFG $G=(V,\Sigma,R,R_0)$
  - $V=\{R_0,R_1,\ldots,R_k\}$,

Design CFG for regular languages

- Transfer DFA into equivalent CFG
- Let DFA $M=(Q,\Sigma,\delta,q_0,F)$
  - $Q=\{q_0,q_1,\ldots,q_k\}$,
  - $\delta(q_i,a)=q_j$,

  then CFG $G=(V,\Sigma,R,R_0)$
  - $V=\{R_0,R_1,\ldots,R_k\}$,
  - $R_i \rightarrow aR_j$,

Design CFG for regular languages

- Transfer DFA into equivalent CFG
- Let DFA $M = (Q,\Sigma,\delta,q_0,F)$
  - $Q=\{q_0,q_1,\ldots,q_k\}$,
  - $\delta(q_i,a)=q_j$,
  - $q_i \in F$

  then CFG $G=(V,\Sigma,R,R_0)$
  - $V=\{R_0,R_1,\ldots,R_k\}$,
  - $R_i \to aR_j$,
  - $R_i \to \varepsilon$

Grammar G_1:

$A \to 0A1$

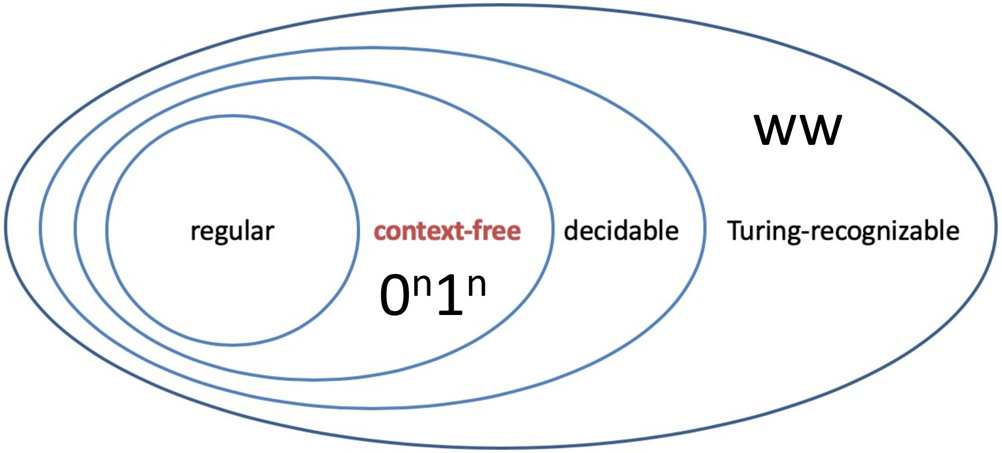
$A \to B$

$B \to \varepsilon$

More languages

- 0^n1^n
  - is not regular language, proved by pumping lemma
  - is a context-free language built by CFG

    $R \rightarrow 0R1$, $R \rightarrow \varepsilon$



- ww
  - is not regular language
  - Is not context-free language

Ambiguity

- If a grammar generates the ***same*** string in several ***different*** ways, we say that the string is derived *ambiguously* in that grammar.

- If a grammar generates some string ambiguously, we say that the grammar is ***ambiguous***.

- G_5: $E \rightarrow$

  $E+E$ |

  $E \times E$ |

  (E) | a

Ambiguity

- G_5: $E \rightarrow$

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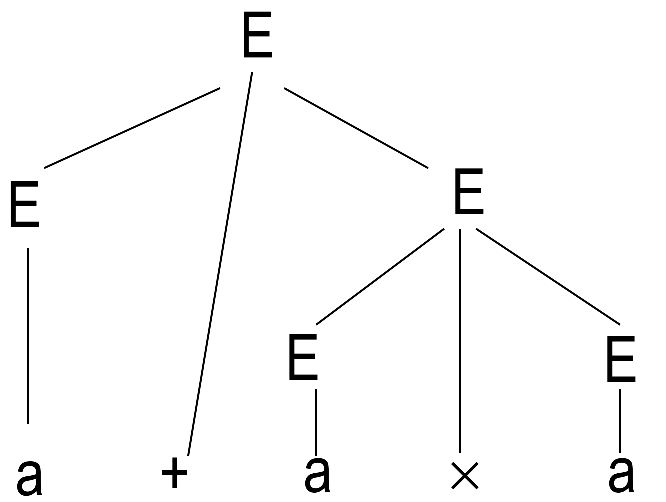
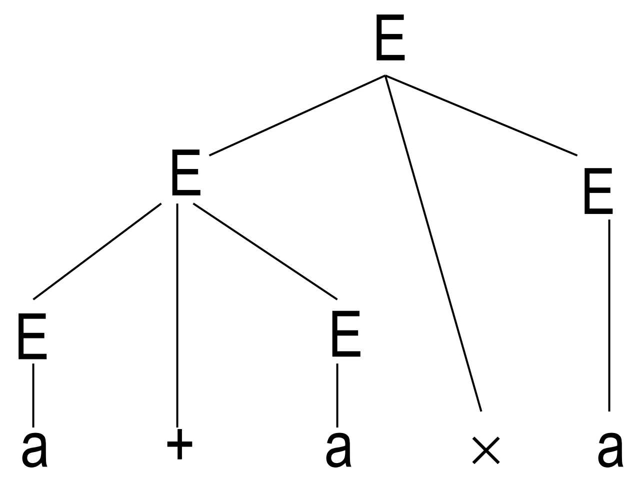
Is the grammar ambiguous

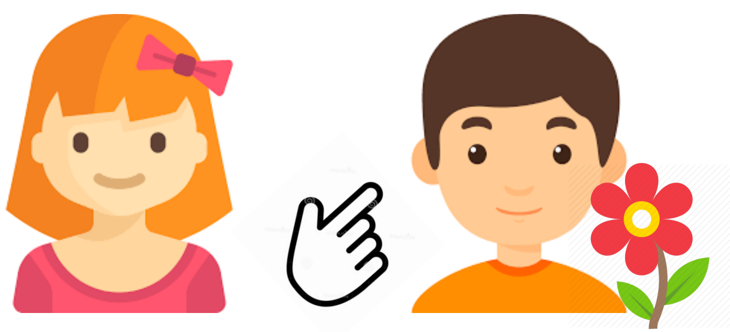


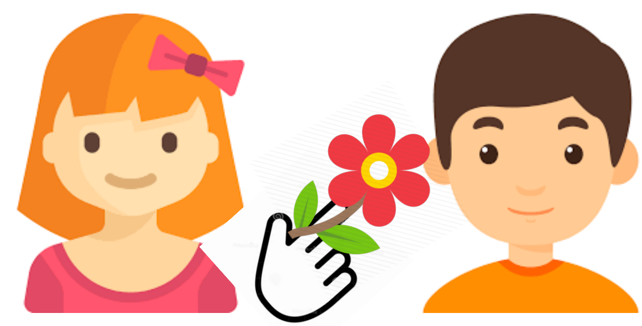


Ambiguity in real life

Is the grammar ambiguous

- G_2:
- the_girl_touches_the_boy_with_flower





Leftmost derivation

- A derivation of a string w in a grammar G is a *leftmost derivation* if at every step the *leftmost* remaining variable is the one replaced

- $E \Rightarrow E+E$

  $\Rightarrow a+E$

  $\Rightarrow a+E\times E$

  $\Rightarrow a+a\times E$

  $\Rightarrow a+a\times a$

- G_5: $E \rightarrow$

  $E+E \mid$

  $E\times E \mid$

  $(E) \mid a$

Two different leftmost derivation

- E
  $\Rightarrow E+E$
  $\Rightarrow a+E$
  $\Rightarrow a+E\times E$
  $\Rightarrow a+a\times E$
  $\Rightarrow a+a\times a$

- E
  $\Rightarrow E\times E$
  $\Rightarrow E+E\times E$
  $\Rightarrow a+E\times E$
  $\Rightarrow a+a\times E$
  $\Rightarrow a+a\times a$

- G_5: $E \rightarrow$
  $E+E \mid$
  $E\times E \mid$
  $(E) \mid a$

Ambiguity

- A string w is derived ***ambiguously*** in context-free grammar G if it has two or more different ***leftmost derivations***.

- Grammar G is ***ambiguous*** if it generates some string ambiguously.

- Some context-free languages can be generated only by ambiguous grammars. (***inherently ambiguous***)

Inherently ambiguous example

- $\{ 0^i 1^j 2^k \mid i=j \text{ or } j=k \}$
  - $\{ 0^n 1^n 2^m \mid n,m \geq 0 \} \cup \{ 0^m 1^n 2^n \mid n,m \geq 0 \}$
  - $0^n 1^n 2^n$ can only be generated by ambiguous grammars (due to the language definition)
  - Human languages like English/French/Spanish/Chinese/Japanese/Hindi … are inherently ambiguous

Conclusion

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  - Leftmost derivation