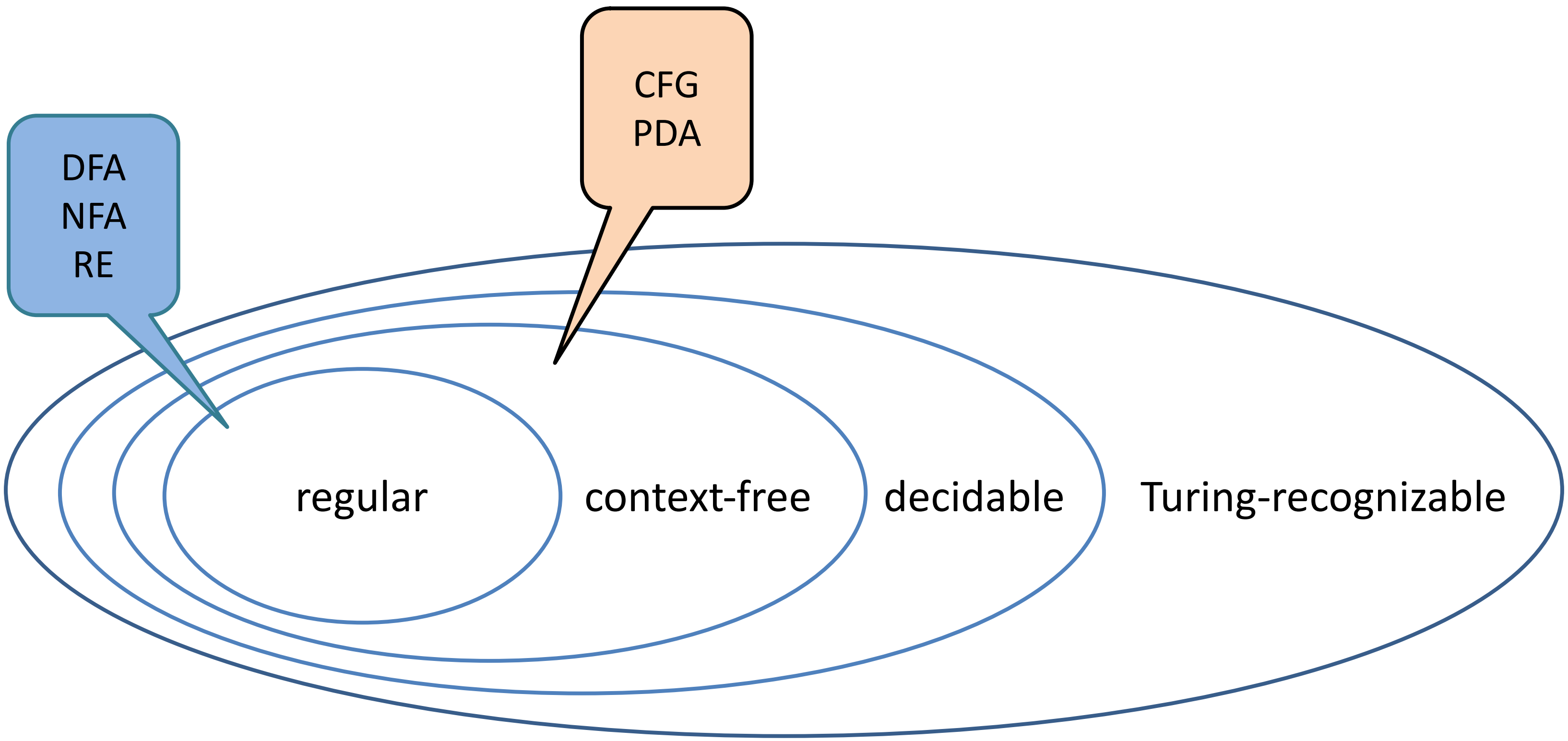
CS 6041
Theory of Computation

Pushdown Automata

Kun Suo
Computer Science, Kennesaw State University
https://kevinsuo.github.io/

Pushdown Automata (PDA)



Pushdown Automata (PDA)

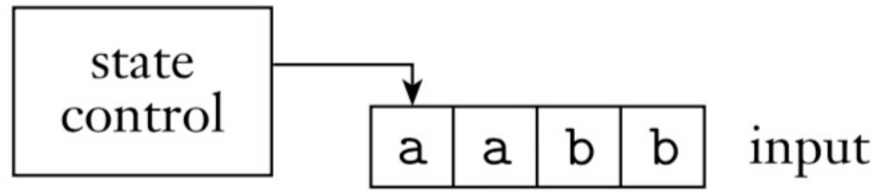
- Pushdown automatas are equivalent in power to context-free grammars (PDA=CFG)

- PDA can recognize some nonregular languages

What does PDA look like?

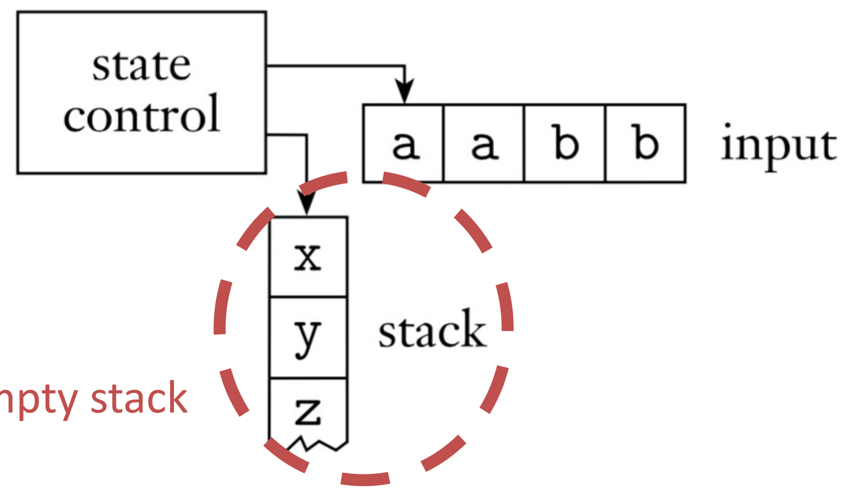
finite automaton

Memory = 1

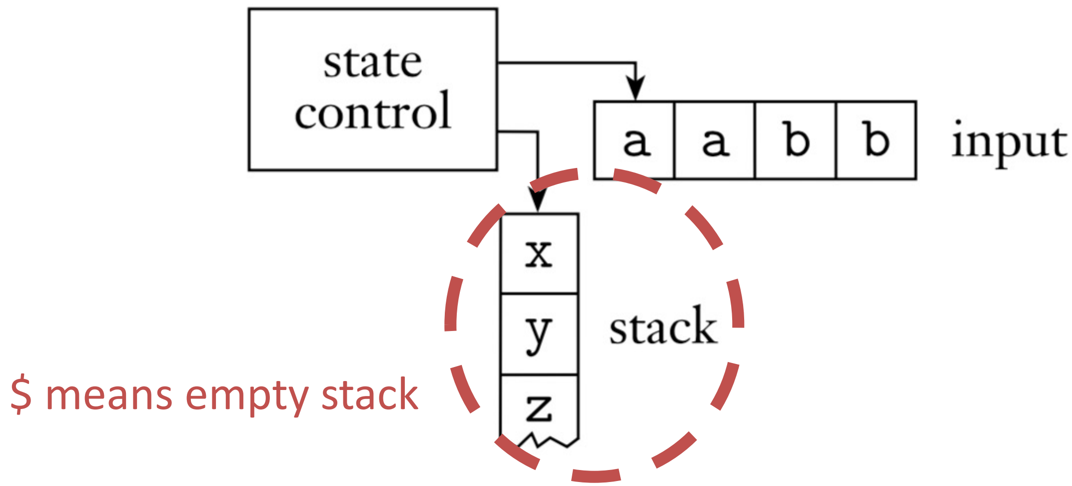


pushdown automaton

Memory = N



What does PDA look like?

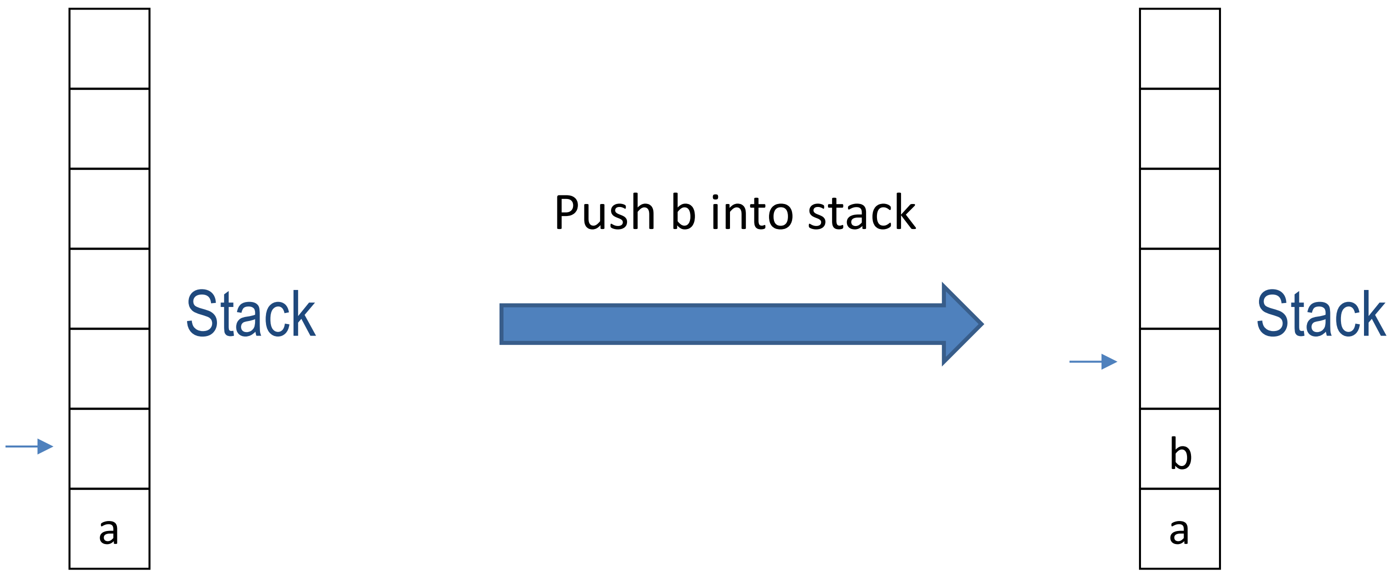


pushdown automaton

- Pushdown automata has more memories than finite automata
- PDA = finite automata + **<u>A stack (unlimited size)</u>**

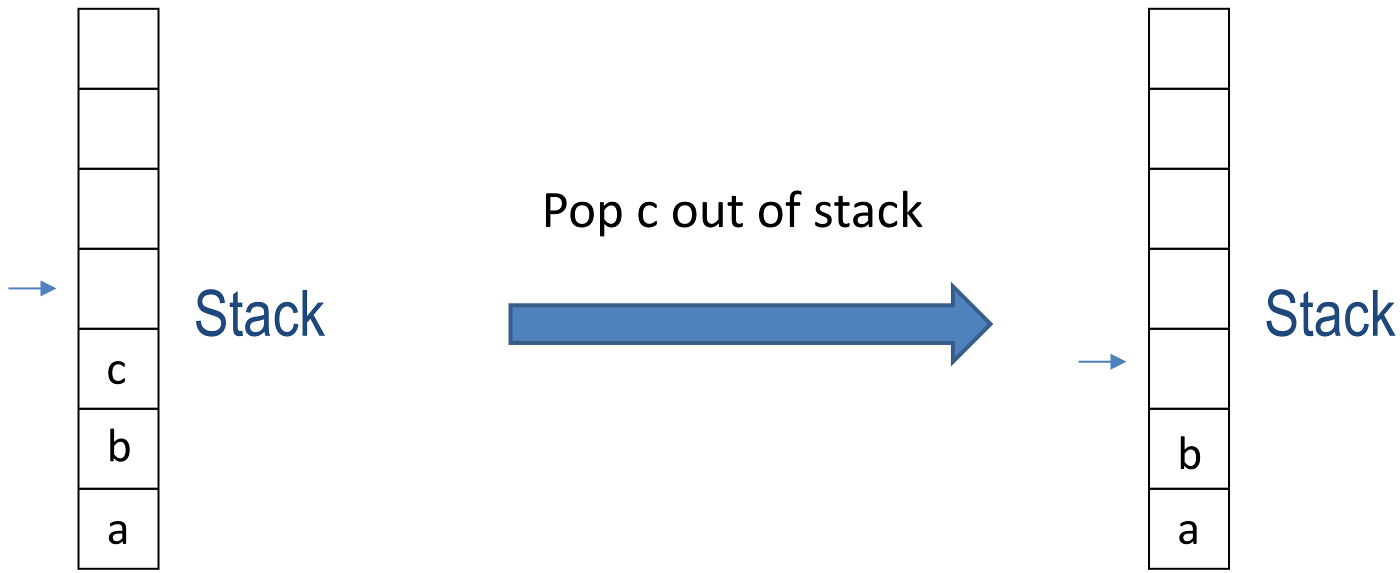
Stack operation

- Push: add to the top of stack

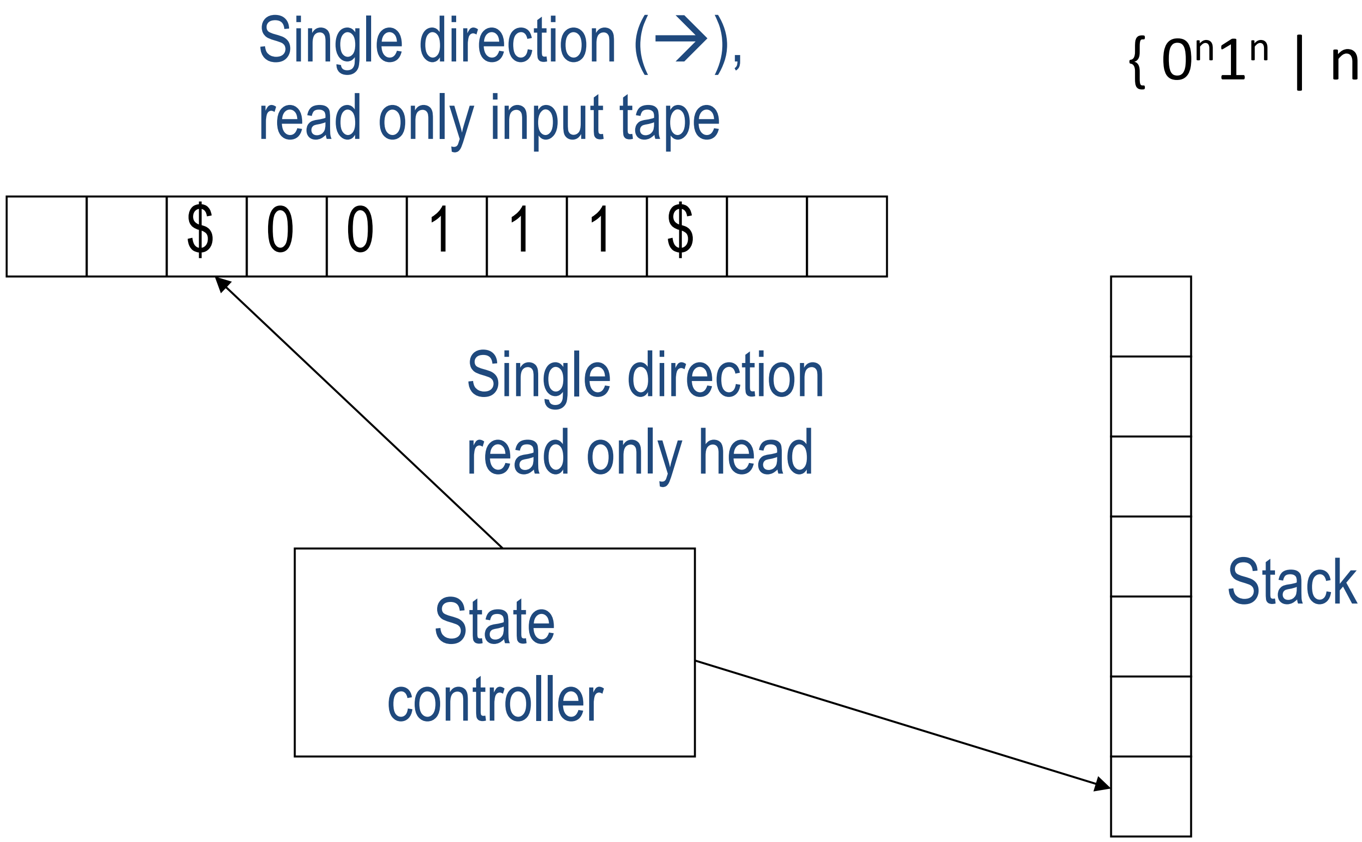


Stack operation

- Pop: remove from the top of stack



To test whether 00111 is in 0^n1^n



$\{ 0^n1^n \mid n \geq 0 \}$

To test whether 00111 is in 0^n1^n

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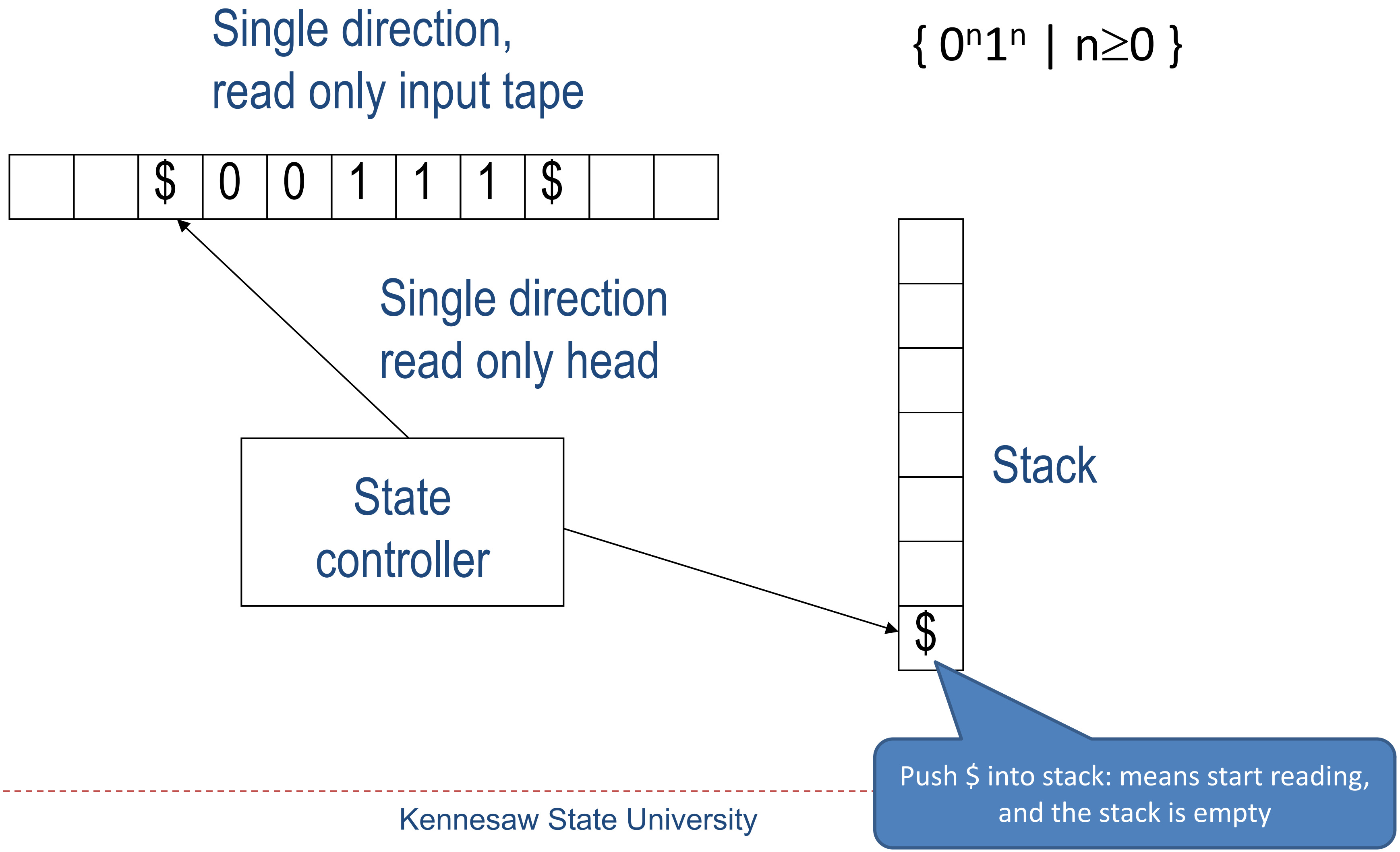

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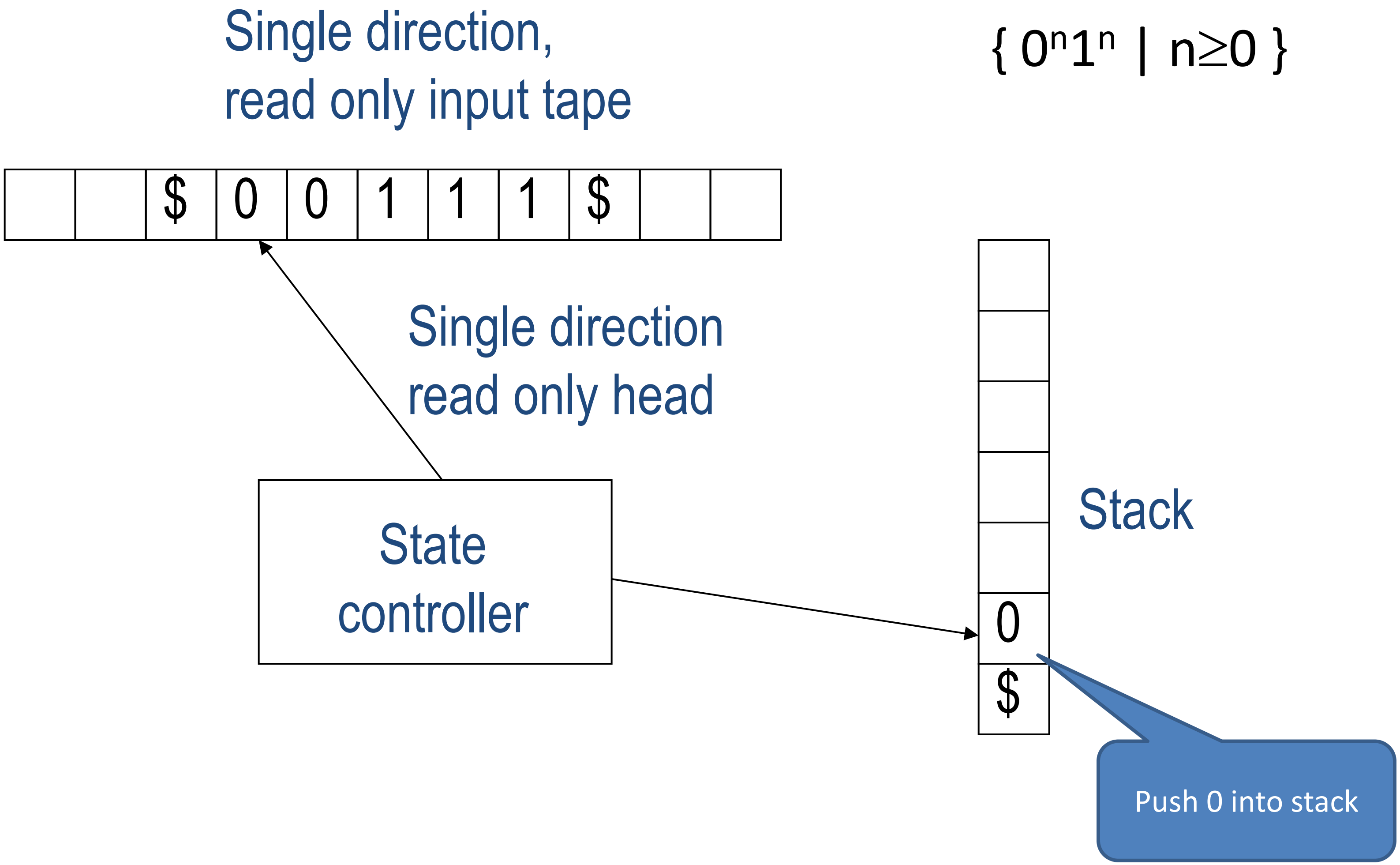

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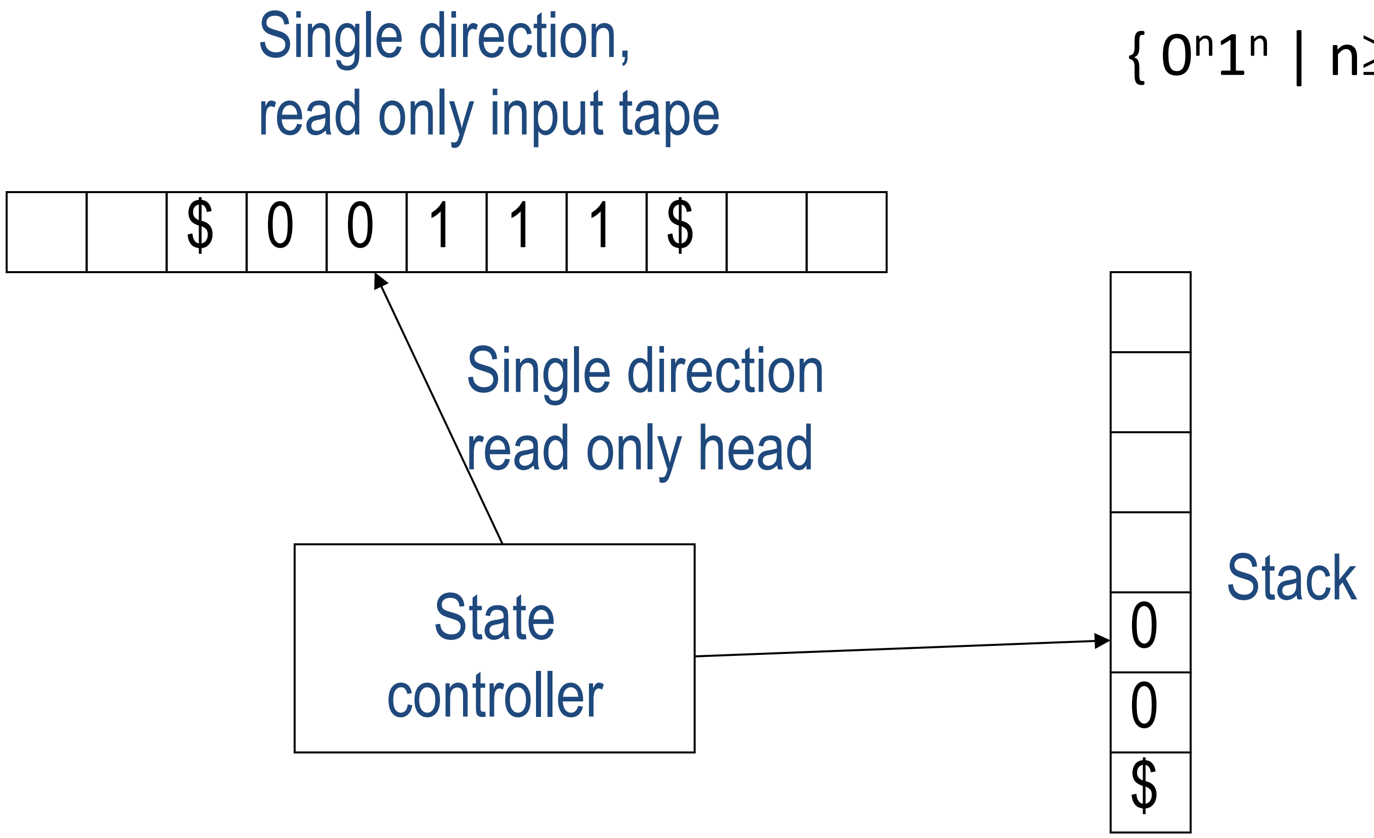

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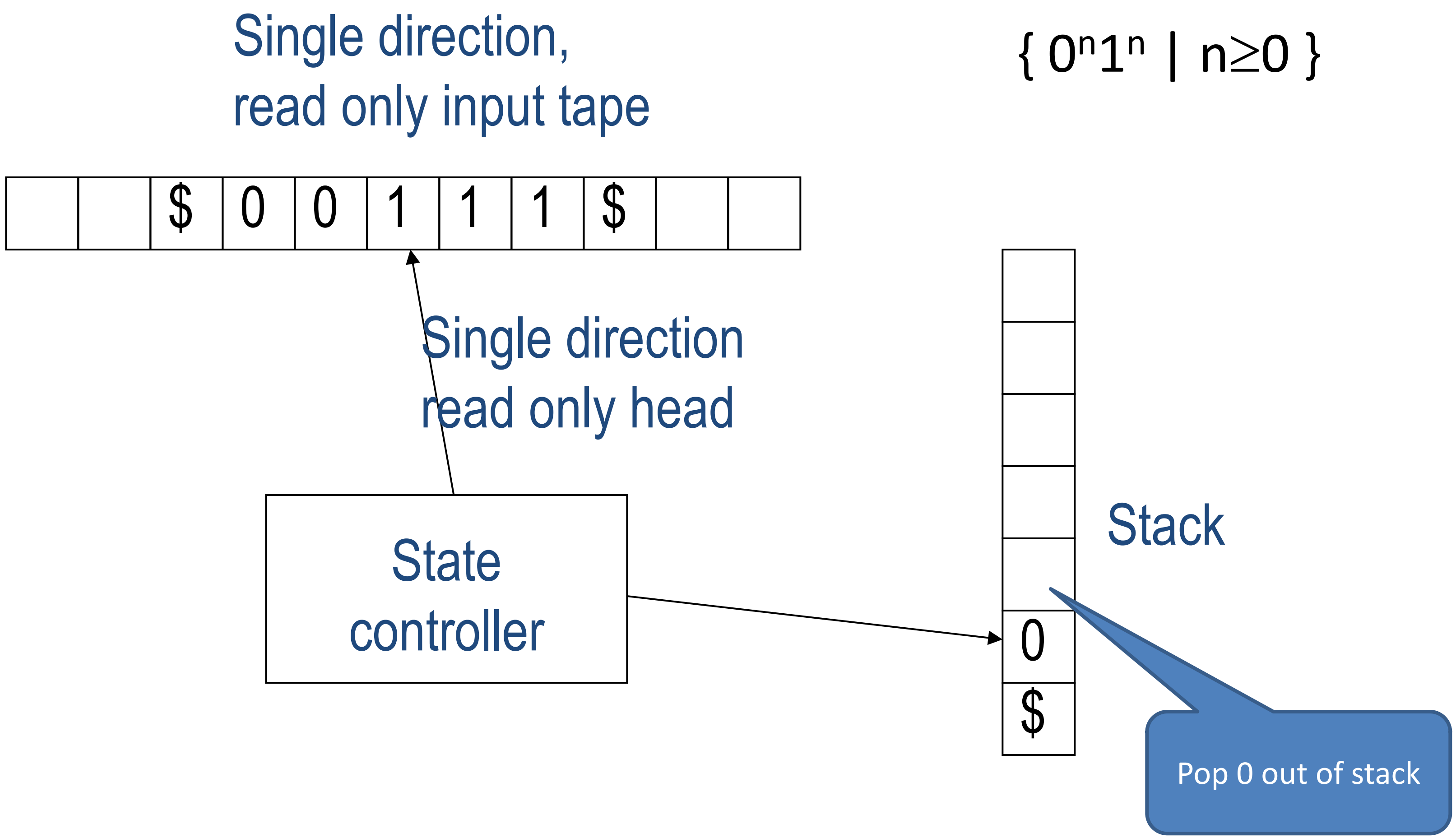

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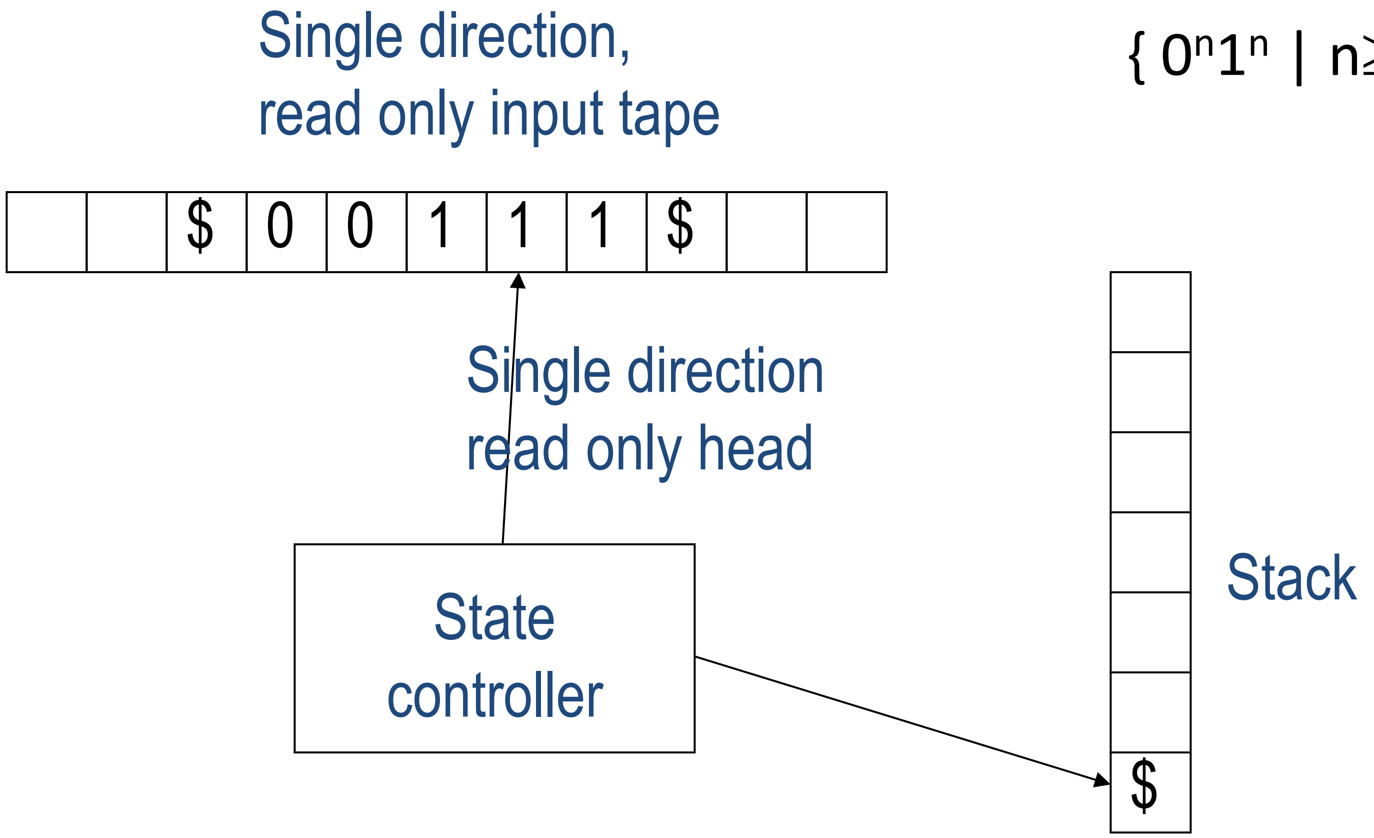

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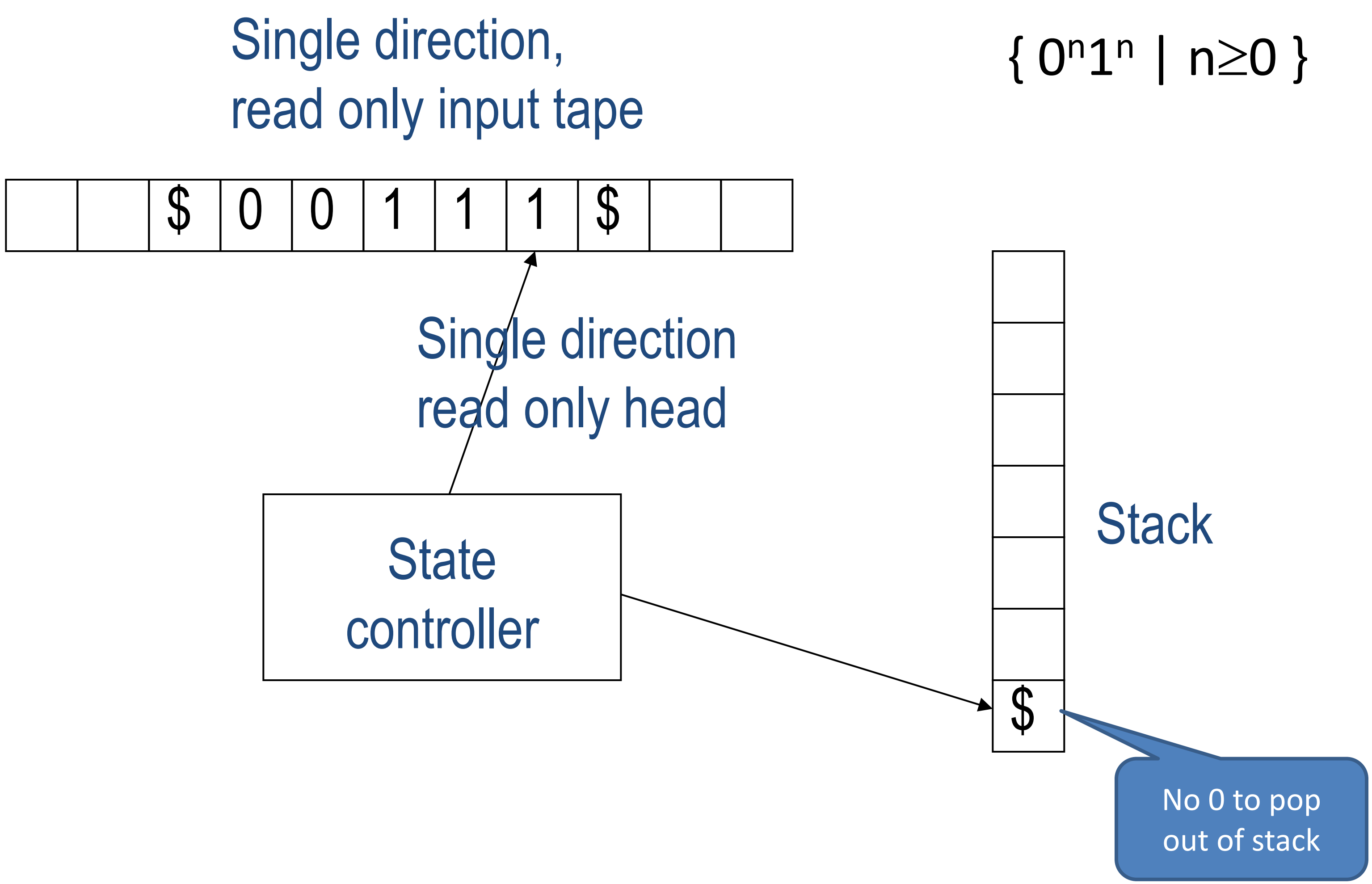
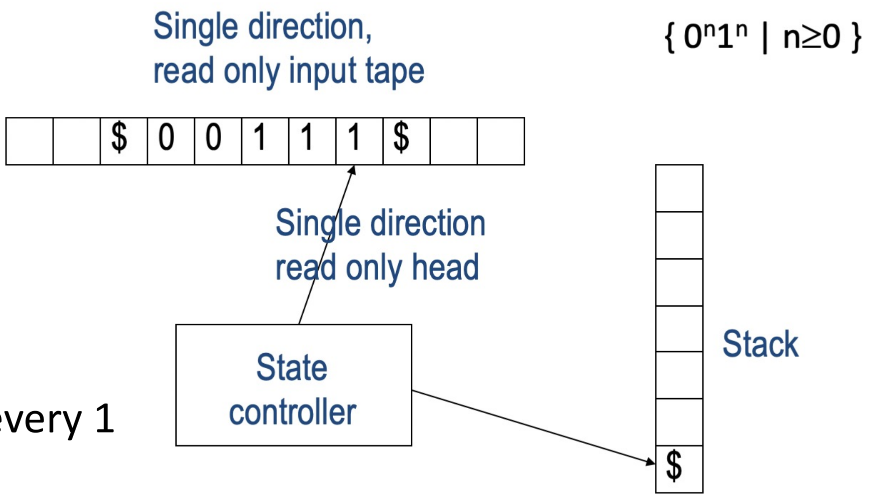


Informal description for PDA to recognize some languages

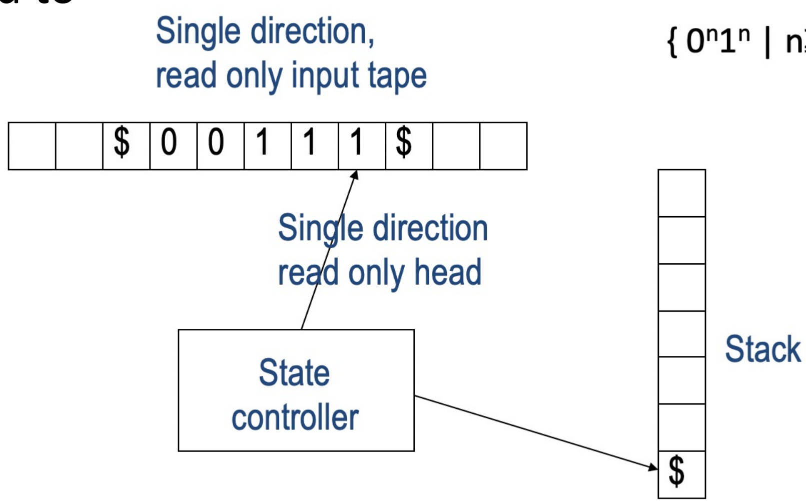
- $A = \{0^n1^n \mid n \geq 0\}$
- Read symbols from input
  - Operation
    - For every 0s, push 0 into stack
    - When read 1s, pop one 0 from stack for every 1
  - Determine accept/reject:
    - When finish reading string and there is no 0s in stack, accept;
    - When there exist 0s after 1s (not in 0^n1^n), reject.
    - When tape is not finished while the stack is empty (1>0), reject;
    - When tape finished while the stack is non-empty (1<0) reject;



Informal description for PDA to recognize some languages

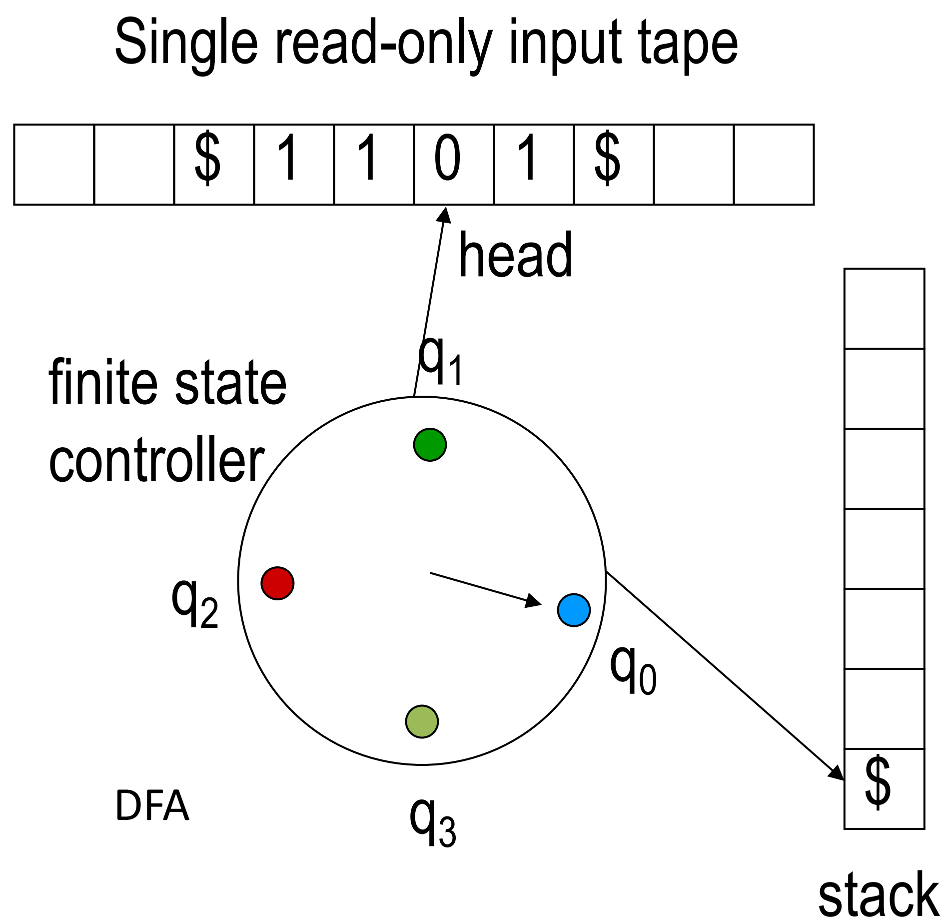
- L = {w | w has some features}
- Read symbols from input
  - **STEP1:** regular?
    - If the language is regular, do not need to use stack; if not regular, define operations on stack
  - **STEP2:** define operations:
    - When to push
    - When to pop
  - **STEP 3:** determine accept/reject:
    - Under which cases, accept
    - Under which cases, reject

{ 0^n1^n | $n \geq 0$ }



Question: Informal description

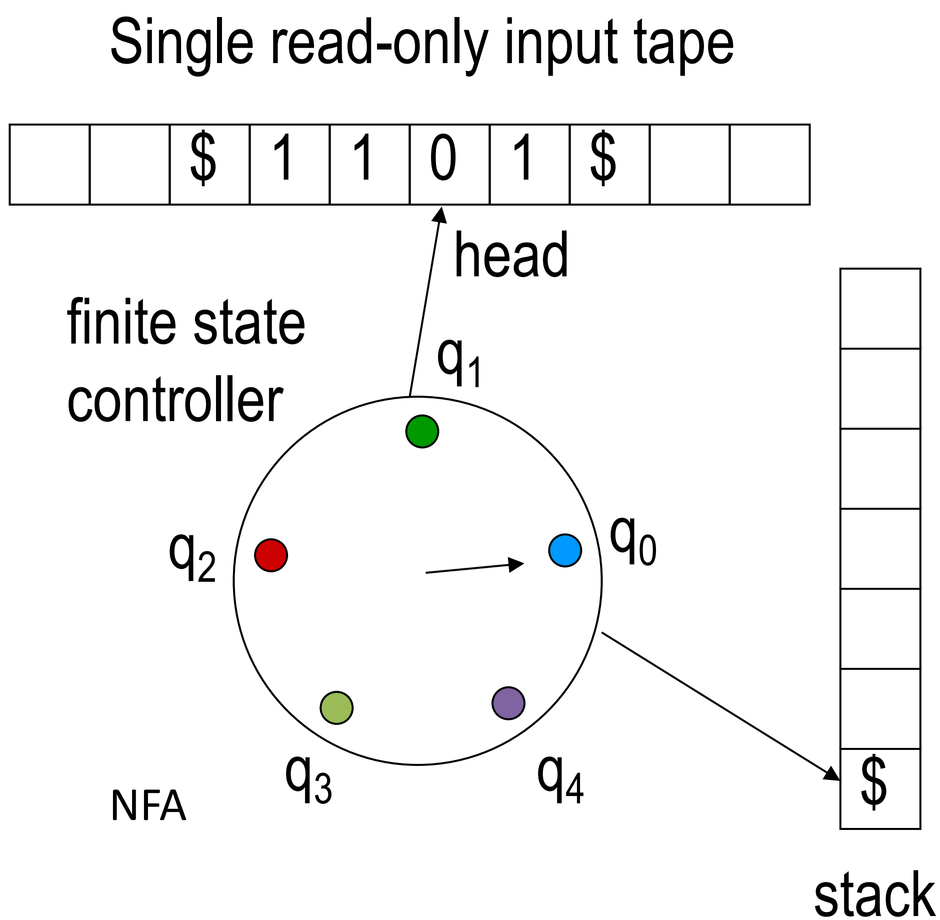
- L1={w| w has at least three 1s}
  - This set is regular ($\Sigma^*1\Sigma^*1\Sigma^*1\Sigma^*$), so the PDA doesn't even need to use its stack.
  - The PDA scans the string and uses its finite control to maintain a counter which counts up to 3. The PDA accepts the moment it sees three ones.



q_i:
No 1s; one 1; two 1s; three and more 1s;

Question: Informal description

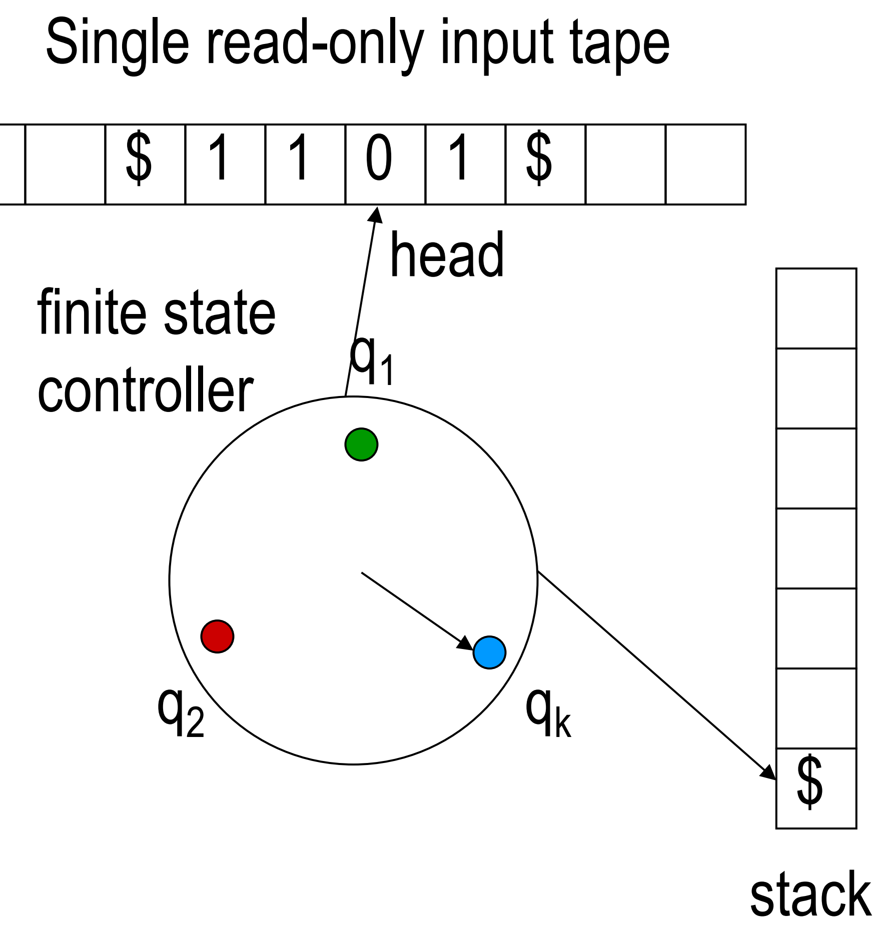
- L2={w| w starts and ends with the same symbol}
  - This set is regular (0(Σ)*0 U 1(Σ)*1), so the PDA doesn't even need to use its stack.
  - The PDA scans the string and keep track of the first and last symbol in its finite control. If they are the same, accepts.



q_i:
ε; 1***0; 0***1; 1***1; 0***0

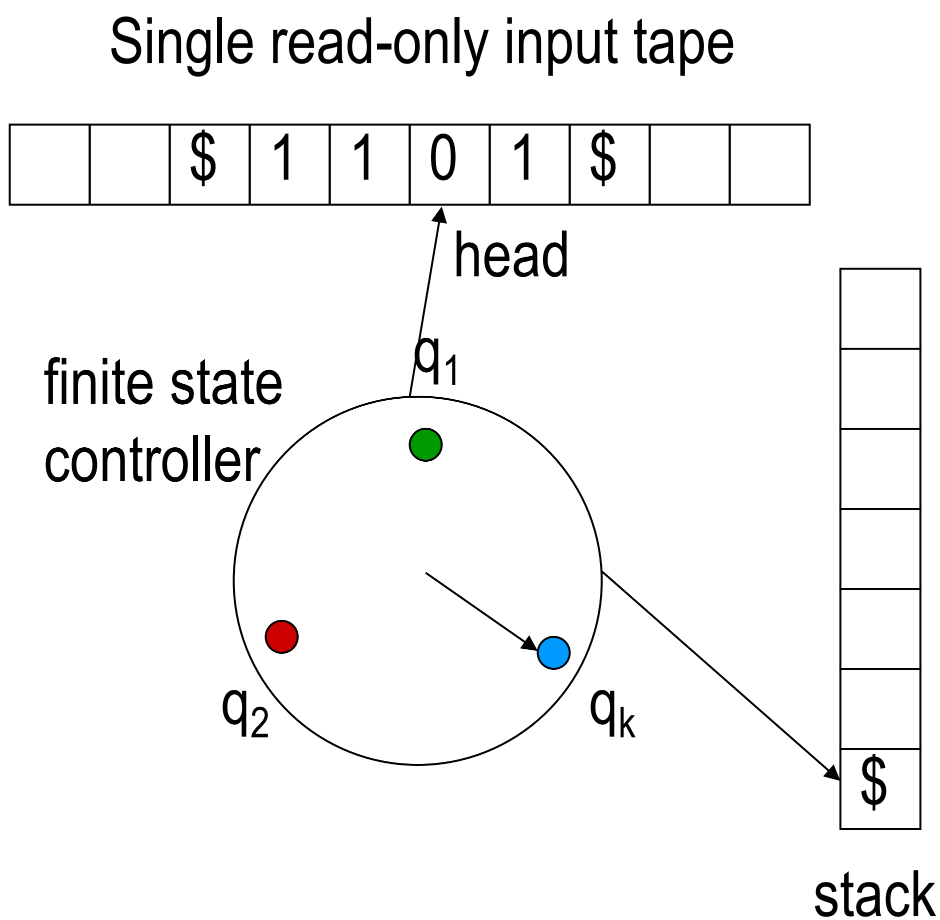
Question: Informal description

- L3={w| w has more 1s than 0s}
  - This set is not regular.
  - The PDA scans across the input.
    - POP: If it sees a 1 and its top stack symbol is a 0, it pops the stack. Similarly, if it scans a 0 and its top stack symbol is a 1, it pops the stack.
    - PUSH: In all other cases, it pushes the input symbol onto the stack.
  - After it scans the input, if there is a 1 on top of the stack, it accepts. Otherwise it rejects.



Question: Informal description

- $L4=\emptyset$
  - Just reject.



Definition of PDA (non-deterministic)

- PDA $M=(Q,\Sigma,\Gamma,\delta,q_0,F)$, where

  1) Q: set of states

  2) Σ: input alphabet, $\Sigma_\varepsilon=\Sigma\cup\{\varepsilon\}$

  3) Γ: stack alphabet, $\Gamma_\varepsilon=\Gamma\cup\{\varepsilon\}$

  4) $\delta: Q\times\Sigma_\varepsilon\times\Gamma_\varepsilon\rightarrow P(Q\times\Gamma_\varepsilon)$,

     transition function

  5) $q_0\in Q$: start state

  6) $F\subseteq Q$: accept state set

PDA vs. NFA

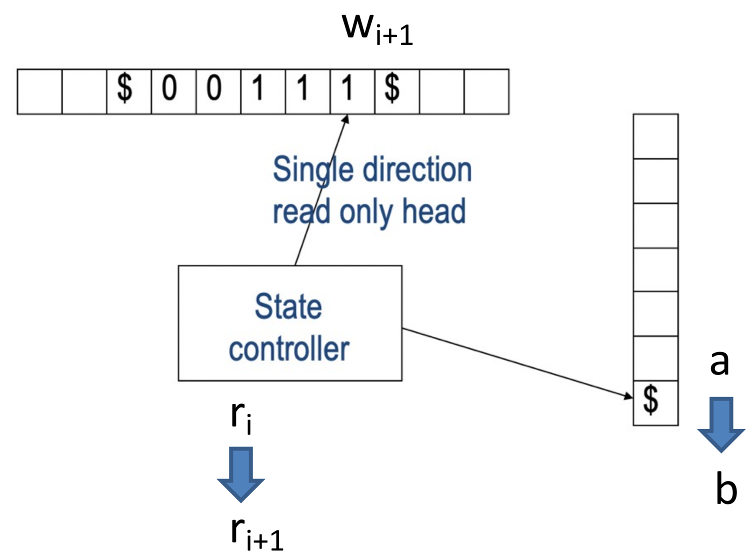
A ***pushdown automaton*** is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$, where Q, Σ, Γ, and F are all finite sets, and

1. Q is the set of states,
2. Σ is the input alphabet,
3. Γ is the stack alphabet,
4. $\delta\colon Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \longrightarrow \mathcal{P}(Q \times \Gamma_\varepsilon)$ is the transition function,
5. $q_0 \in Q$ is the start state, and
6. $F \subseteq Q$ is the set of accept states.

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta\colon Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

Computation on PDA



- $M=(Q,\Sigma,\Gamma,\delta,q_0,F)$;

  input $w=w_1w_2\ldots w_m$,

  $w_i \in \Sigma_\varepsilon$

- Computation: (state, stack)

  $(r_0,s_0), (r_1,s_1), \ldots, (r_m,s_m)$,

  Where $r_i \in Q$, $s_i \in \Gamma^*$, satisfying

  1) $(r_0,s_0)=(q_0,\varepsilon)$;

  At first, the first state is q_0 and stack is empty

  2) $(r_{i+1},b) \in \delta(r_i,w_{i+1},a)$;

  where $s_i=at$; $s_{i+1}=bt$,

  $a,b \in \Sigma_\varepsilon$,

  $t \in \Gamma^*$ (t are other elements in stack)

  After input w_{i+1}, state changes from r_i to r_{i+1} and the top element in stack changes from a to b

Computation on PDA

- Accept of computation:

  3) $r_m \in F$;

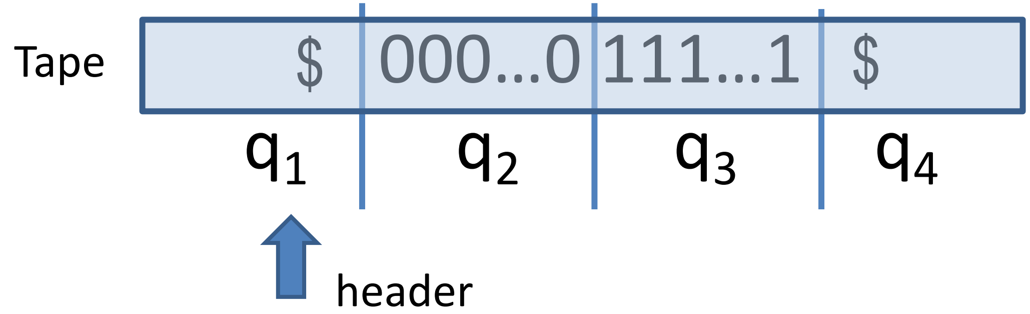
- M accepts w:

  M is at accept states after input of w

- The language that M accepts:

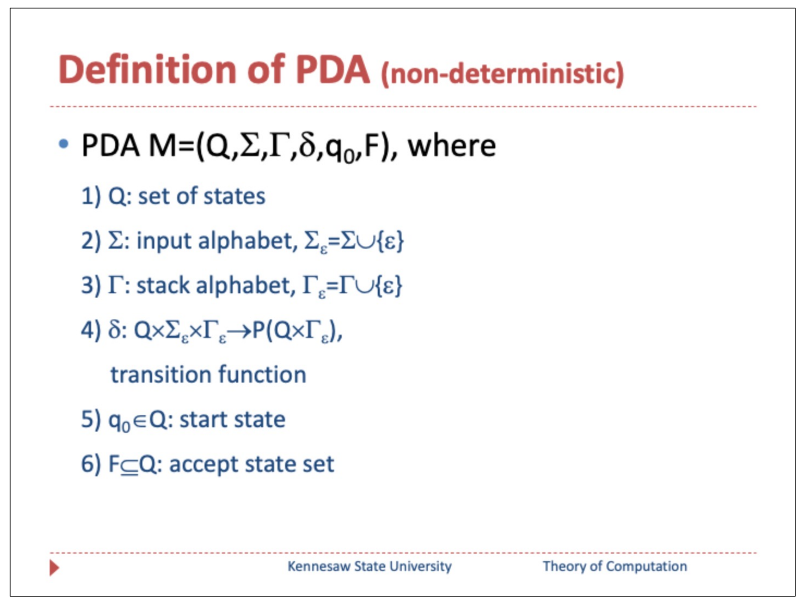
  $L(M) = \{ x \mid M \text{ accepts } x \}$

PDA example



- $L = \{0^n1^n \mid n \geq 0\}$
- $M_1=(\{q_1,q_2,q_3,q_4\},\{0,1\},\{0,\$\},\delta,q_1,\{q_1,q_4\})$

Can you explain what this PDA means?



Definition of PDA (non-deterministic)

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  2) Σ: input alphabet, $\Sigma_\varepsilon=\Sigma\cup\{\varepsilon\}$
  3) Γ: stack alphabet, $\Gamma_\varepsilon=\Gamma\cup\{\varepsilon\}$
  4) δ: $Q\times\Sigma_\varepsilon\times\Gamma_\varepsilon\rightarrow P(Q\times\Gamma_\varepsilon)$, transition function
  5) $q_0\in Q$: start state
  6) $F\subseteq Q$: accept state set

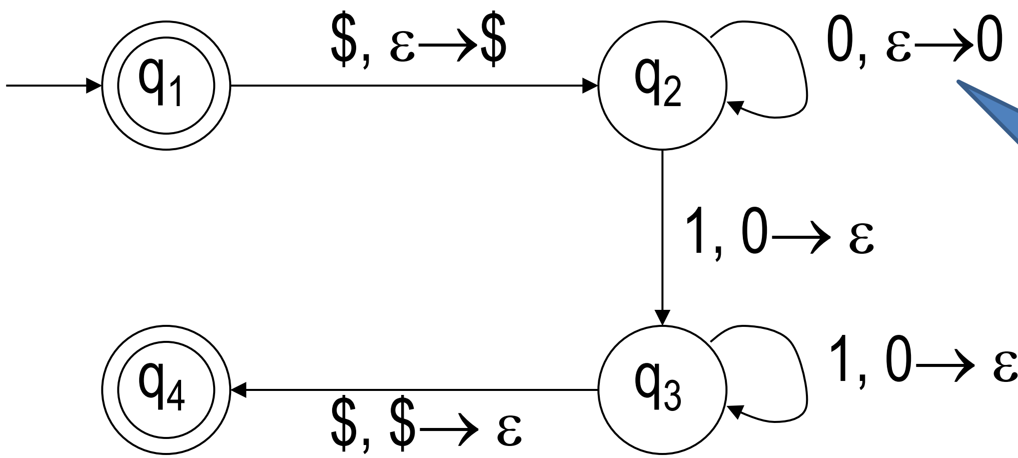
Kennesaw State University Theory of Computation

PDA example

\$	000…0	111…1	\$
q_1	q_2	q_3	q_4

We only put 0 or \$ into stack

- $L = \{0^n 1^n \mid n \geq 0\}$
- $M_1 = (\{q_1, q_2, q_3, q_4\}, \{0,1\}, \{0,\$\}, \delta, q_1, \{q_1, q_4\})$



a,b->c
a: input
b->c: the top of stack changes

M_1

PDA example

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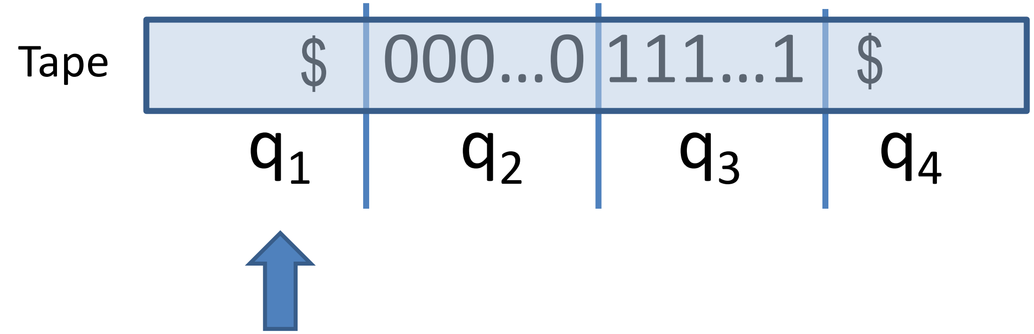
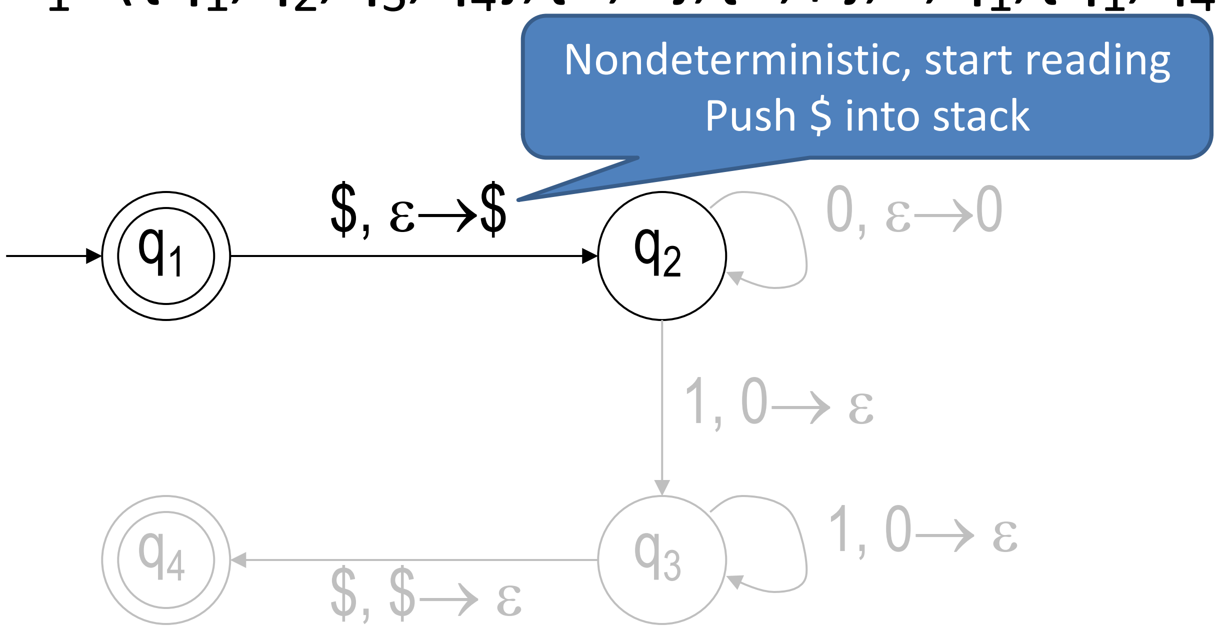
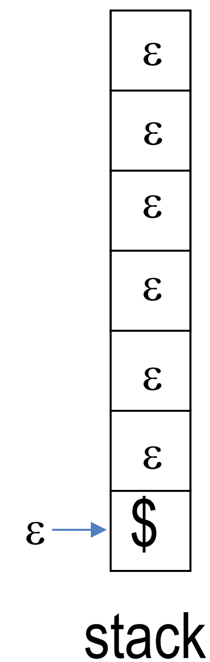
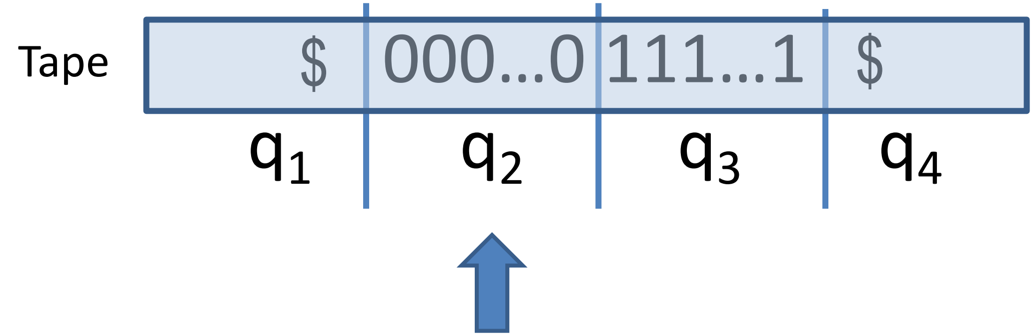




M_1



PDA example



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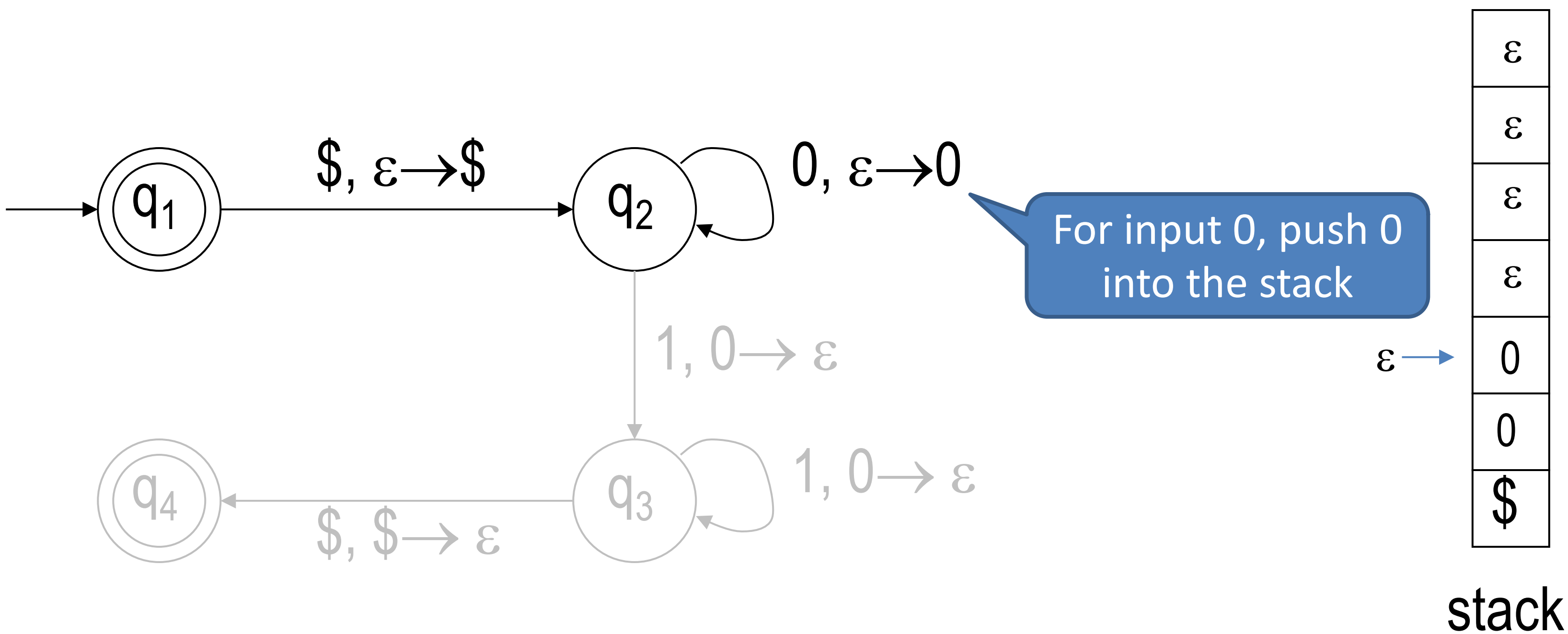
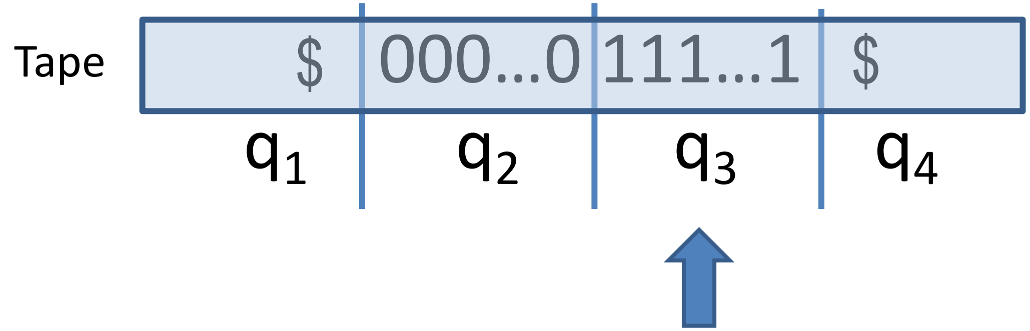


M_1

PDA example



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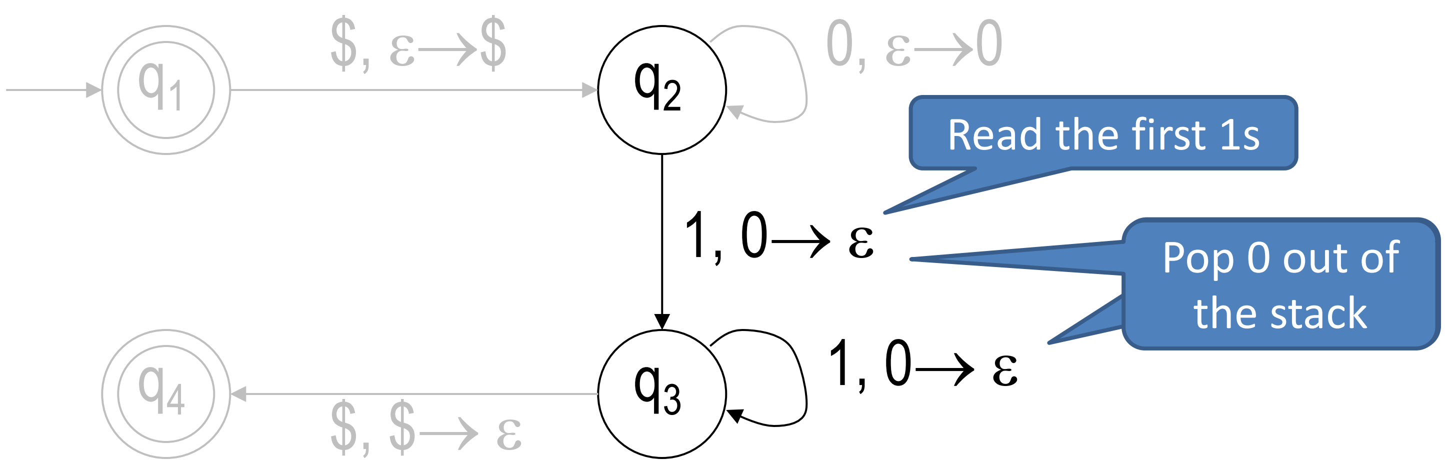
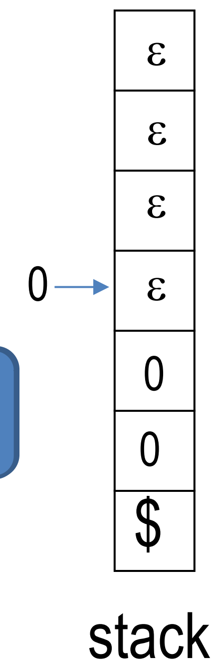
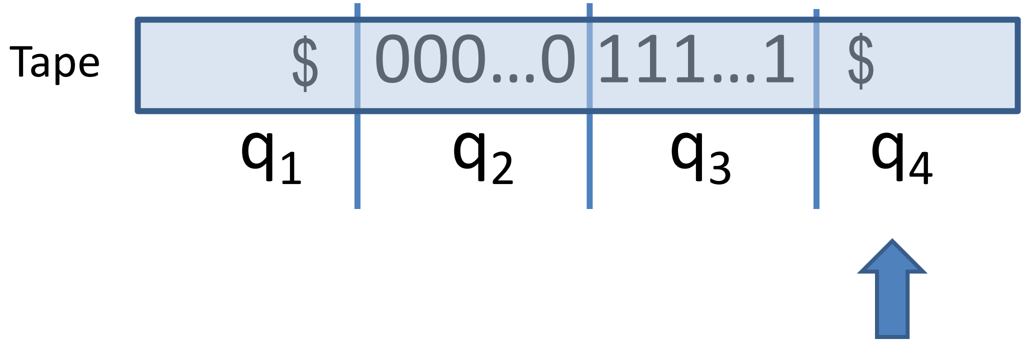


M_1



PDA example



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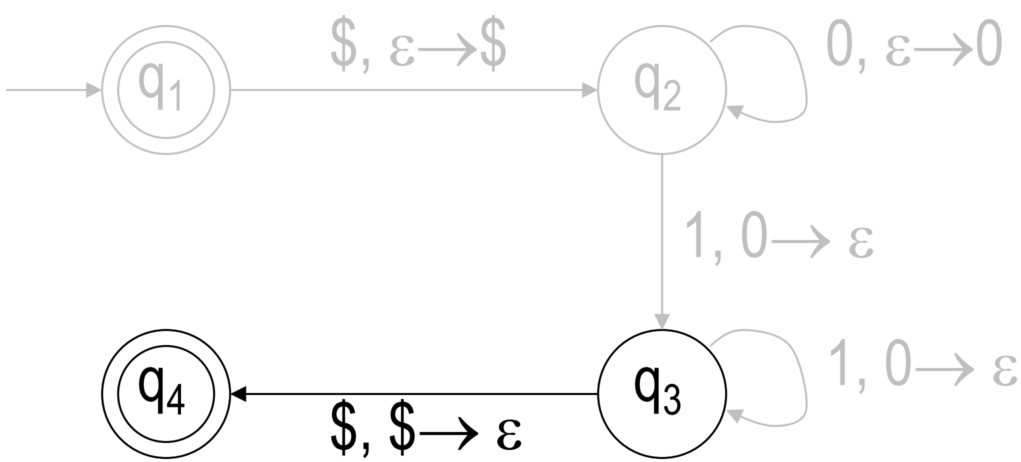
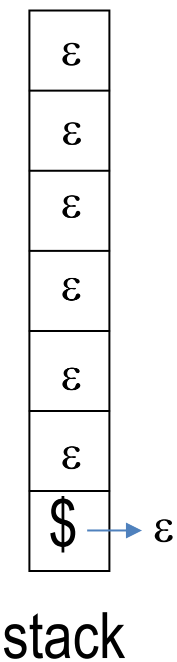
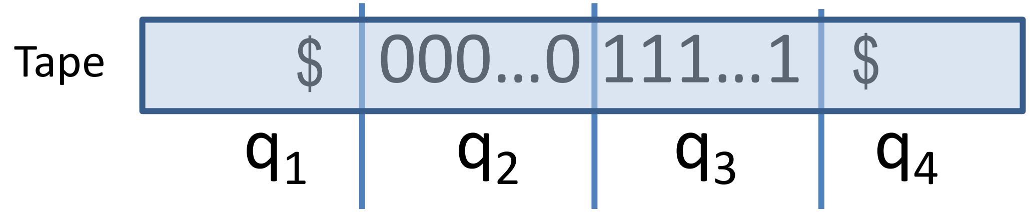
M_1





If $ is popped out of stack, means stack is empty;
If input is $, means string finishes. Accept

PDA example



- $L = \{0^n 1^n \mid n \geq 0\}$
- $M_1 = (\{q_1, q_2, q_3, q_4\}, \{0,1\}, \{0,\$\}, \delta, q_1, \{q_1, q_4\})$

If input is 0,
(q_2, ε) changes to $(q_2, 0)$
ε in stack change to 0 (PUSH 0)

$\delta: Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \rightarrow P(Q \times \Gamma_\varepsilon)$

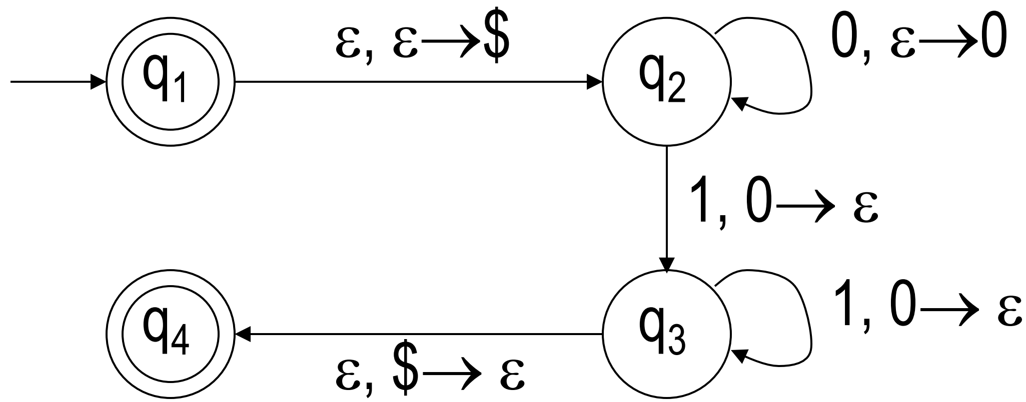
δ table

Σ_ε	Input		0			1			ε		
Γ_ε	stack		0	\$	ε	0	\$	ε	0	\$	ε
	state	q_1	∅	∅	∅	∅	∅	∅	∅	∅	$\{(q_2,\$)\}$
Q		q_2	∅	∅	$\{(q_2,0)\}$	$\{(q_3,\varepsilon)\}$	∅	∅	∅	∅	∅
		q_3	∅	∅	∅	$\{(q_3,\varepsilon)\}$	∅	∅	∅	$\{(q_4,\varepsilon)\}$	∅
		q_4	∅	∅	∅	∅	∅	∅	∅	∅	∅

PDA example

ε, ε→$
=
$, ε→$

δ graph



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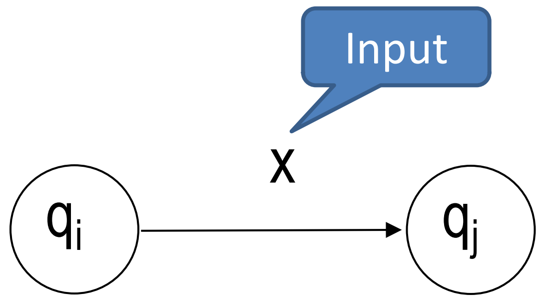
δ table

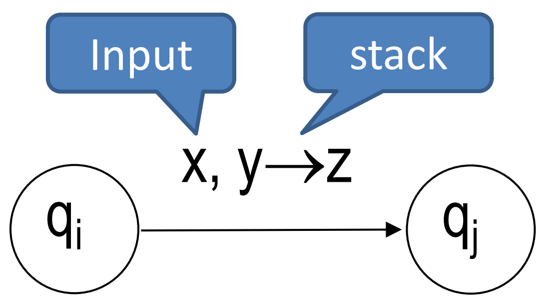
Input		0			1			ε		
stack		0	$	ε	0	$	ε	0	$	ε
state	q_1	∅	∅	∅	∅	∅	∅	∅	∅	{(q_2,$)}
	q_2	∅	∅	{(q_2,0)}	{(q_3,ε)}	∅	∅	∅	∅	∅
	q_3	∅	∅	∅	{(q_3,ε)}	∅	∅	∅	{(q_4,ε)}	∅
	q_4	∅	∅	∅	∅	∅	∅	∅	∅	∅

Design PDA

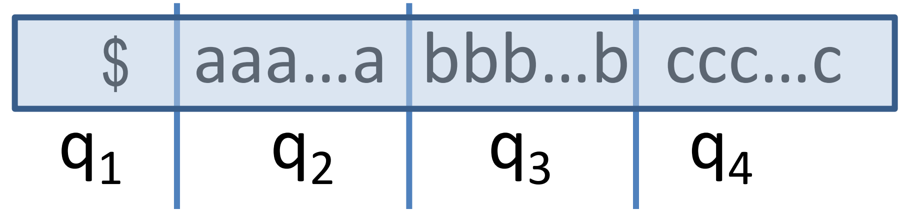
DFA/NFA:



PDA:

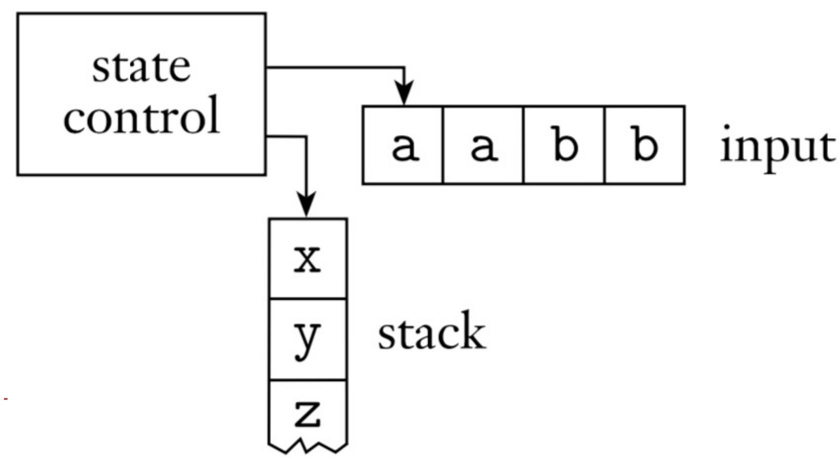


Design PDA

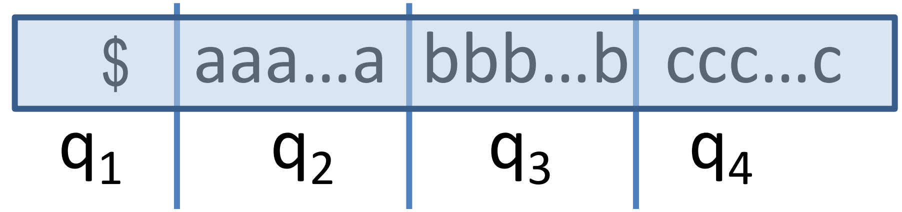


Can you explain the states?

- $L(M_2)=\{ a^n b^n c^m \mid m,n \geq 0 \}$
  - Operation:
    - For an input a, and push a into stack
    - For an input b, pop one a from the stack
  - Determine accept/reject
    - If the stack is empty when finish reading b, then after reading all the cs, accept;
    - Otherwise, reject;



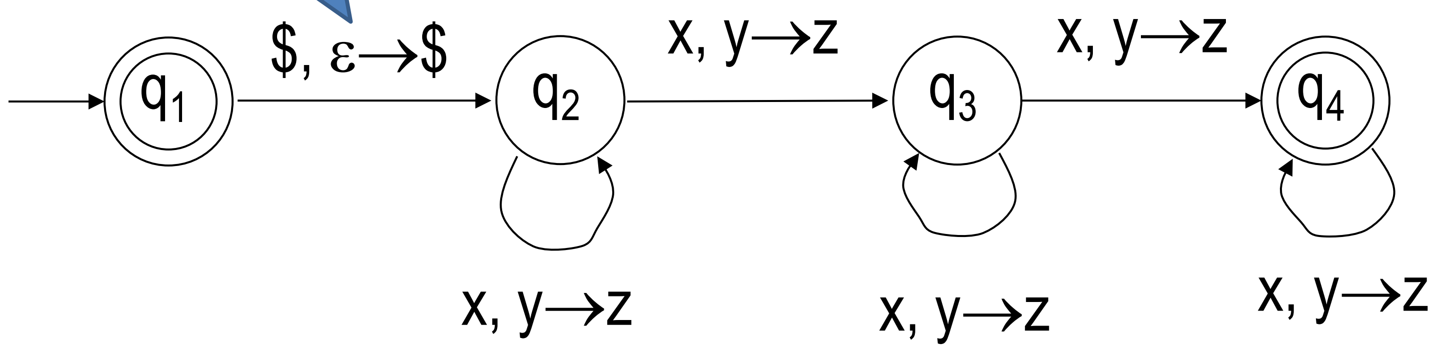
Design PDA

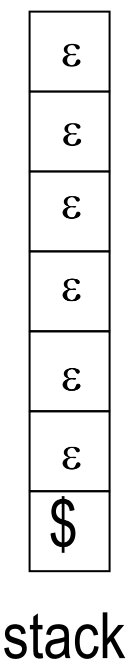


- $L(M_2)=\{ a^n b^n c^m \mid m,n \geq 0 \}$

x,y->z
x: input
y->z: the top of stack changes

Start reading
Push $ into stack





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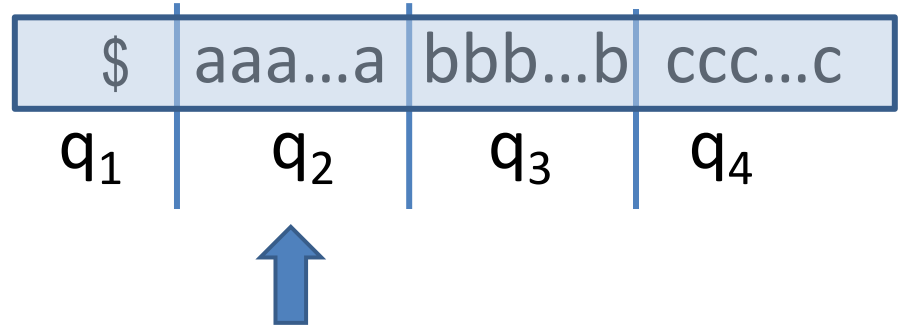
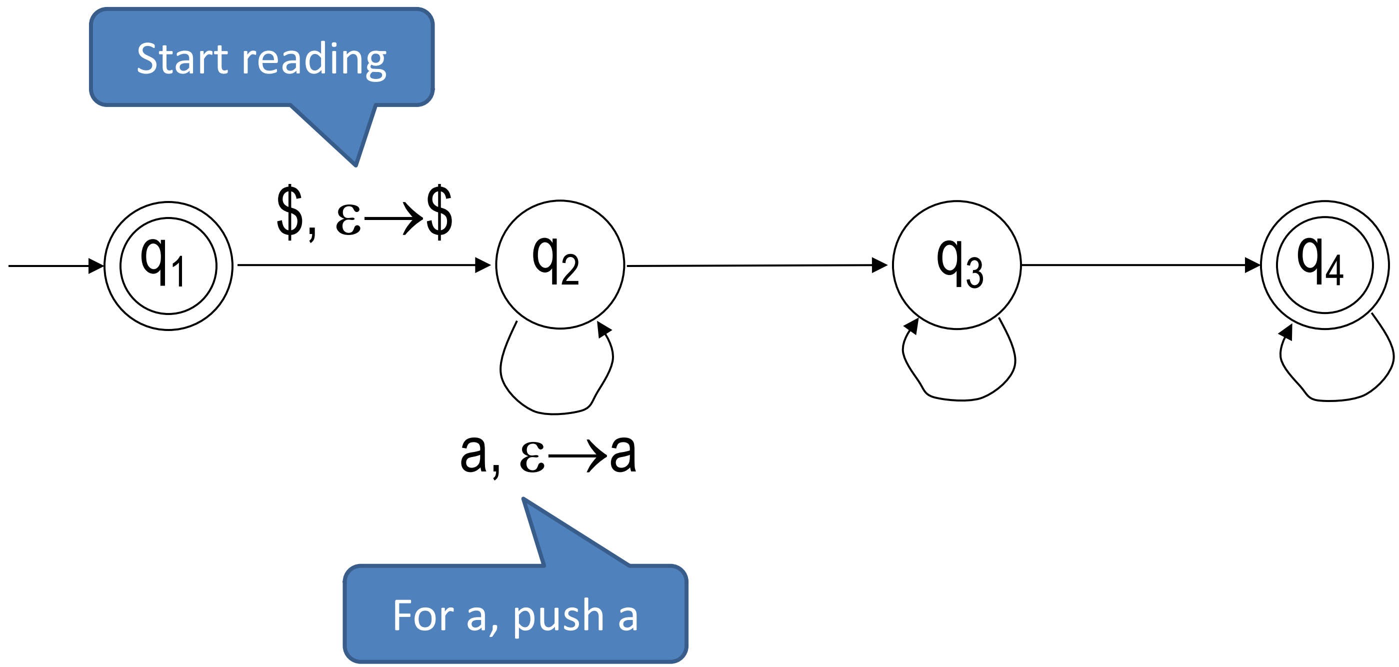
    - Otherwise, reject;

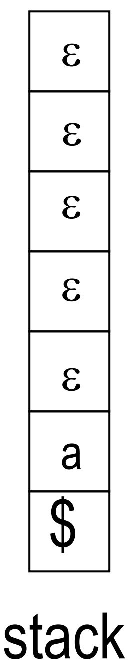
Can you define the transitions x,y->z?

Design PDA

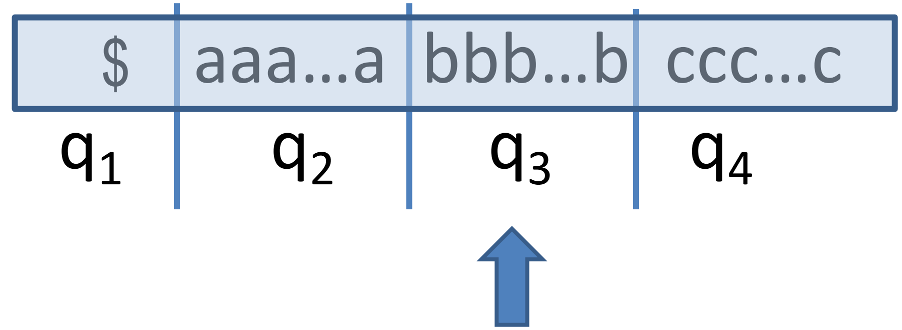


- $L(M_2)=\{ a^n b^n c^m \mid m,n \geq 0 \}$





Design PDA



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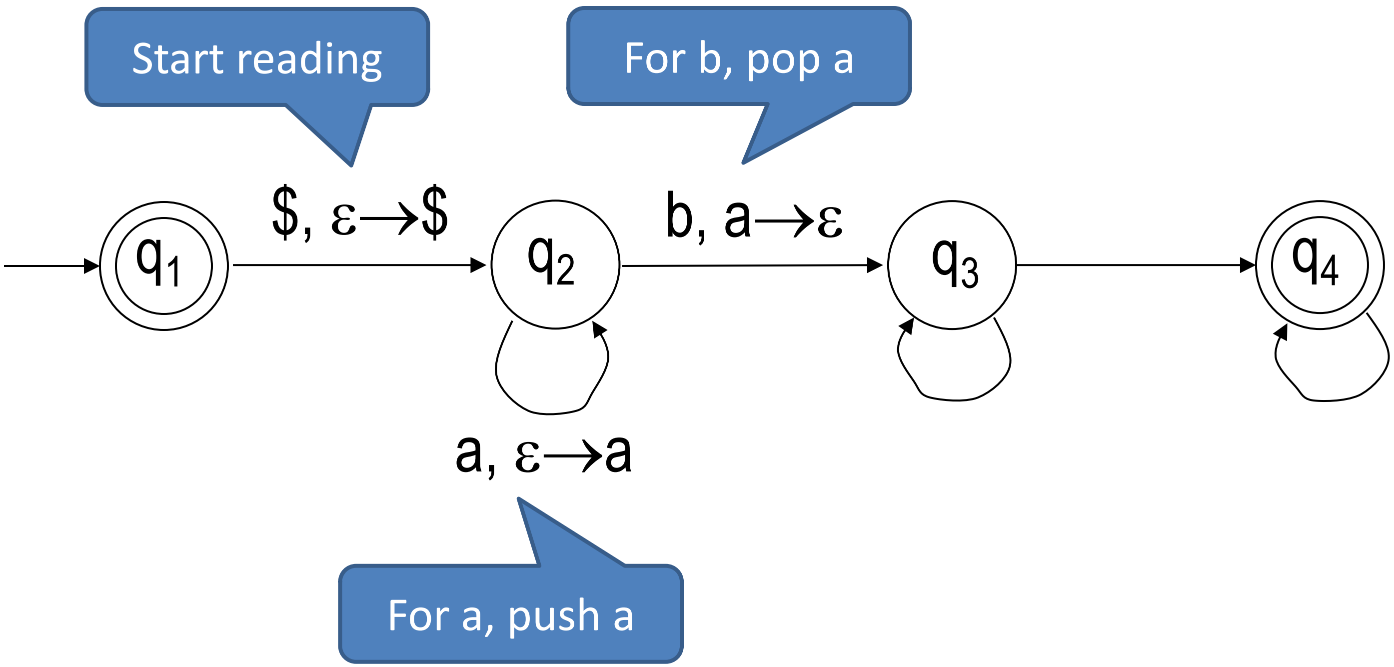
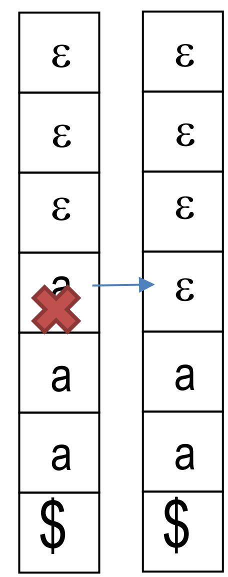
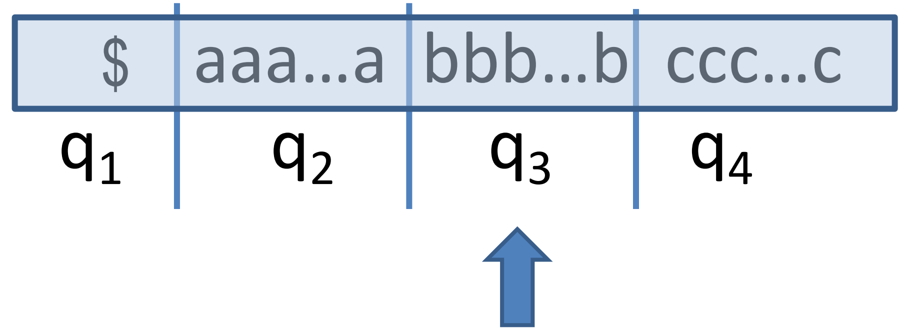


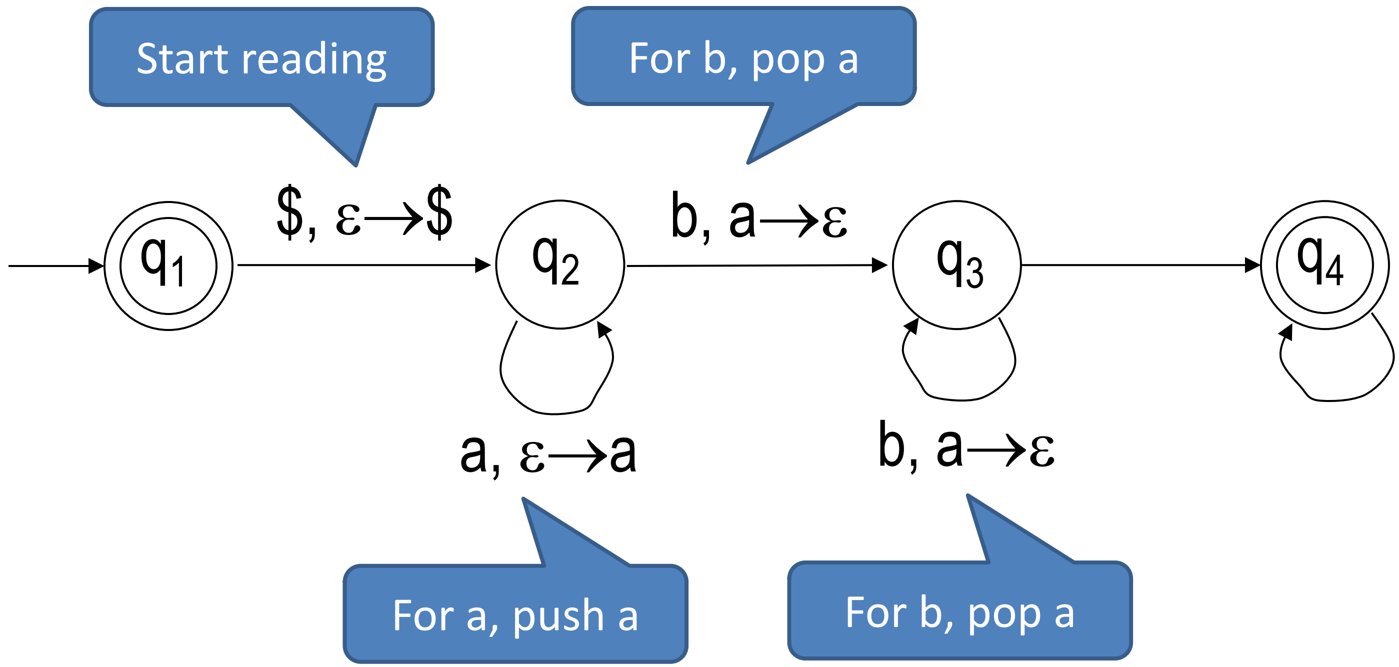


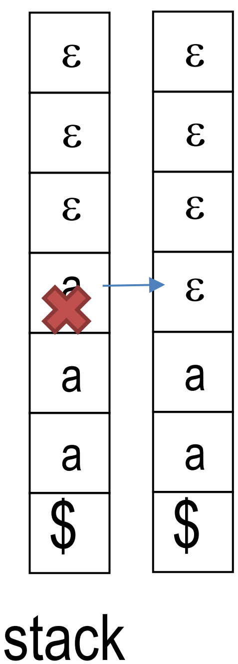
stack

Design PDA

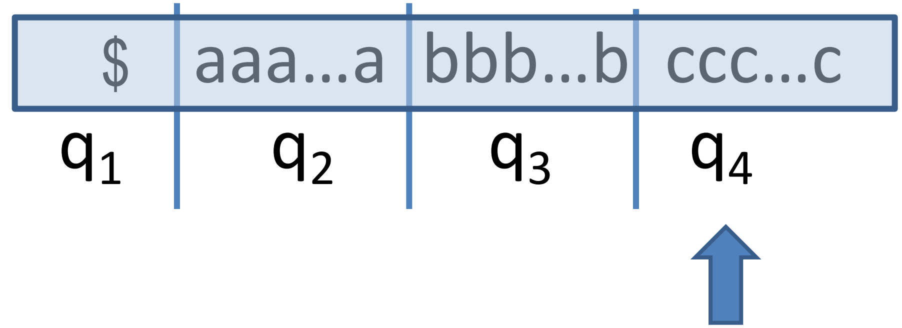


- $L(M_2)=\{\ a^n b^n c^m \mid m,n \geq 0\ \}$

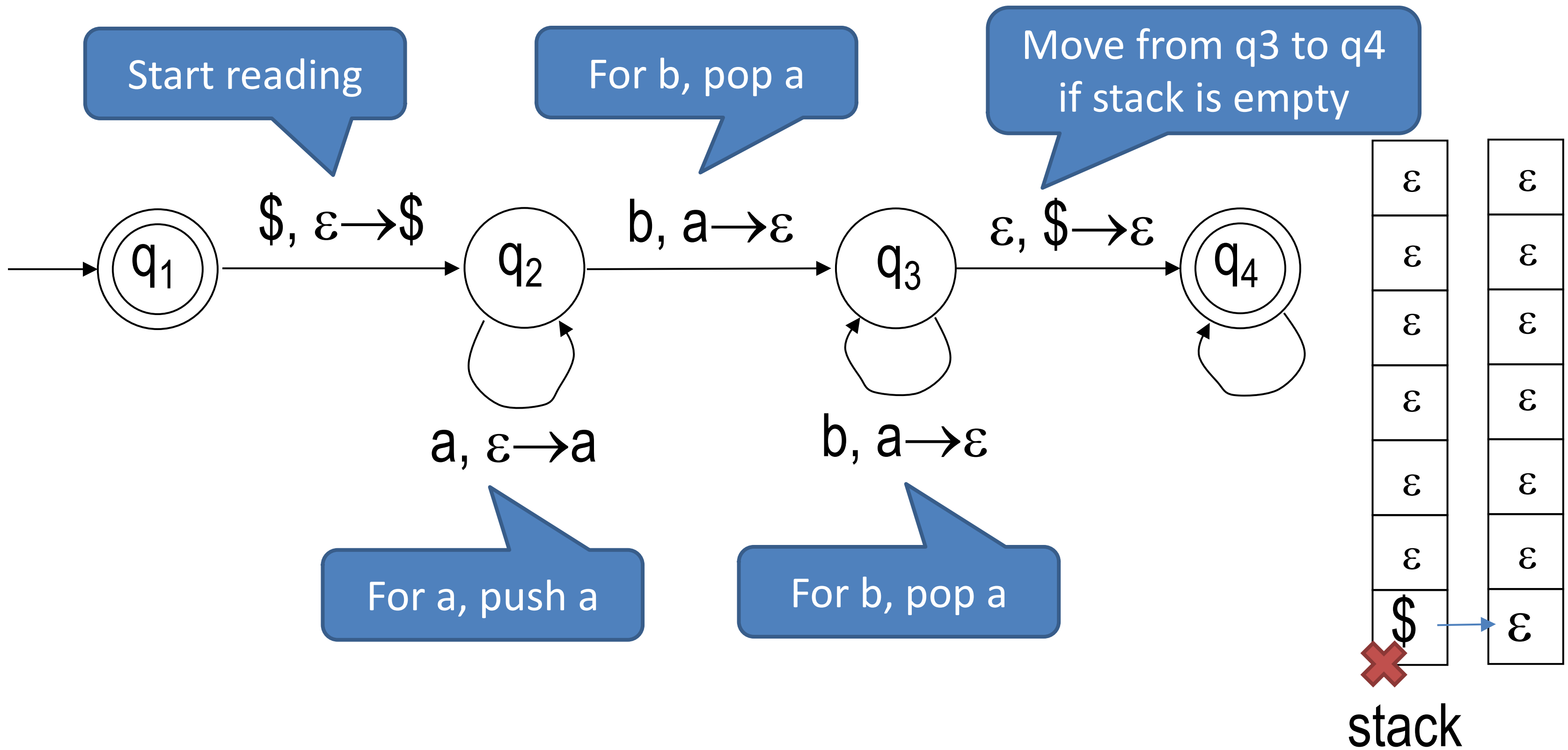




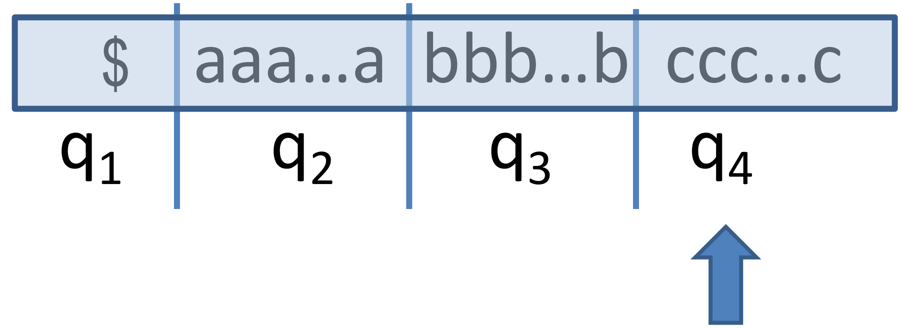
Design PDA



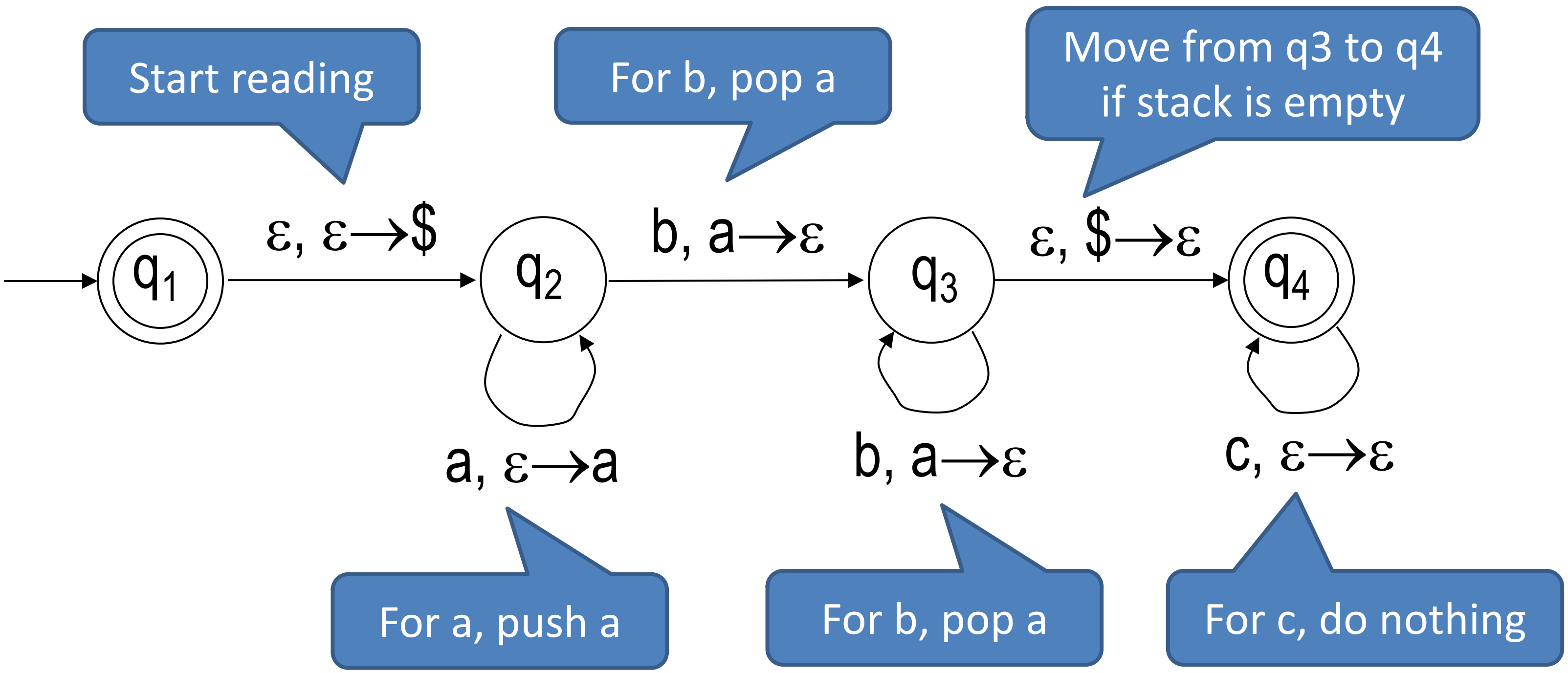
- $L(M_2)=\{\, a^n b^n c^m \mid m,n \geq 0 \,\}$



Design PDA



- $L(M_2)=\{\ a^n b^n c^m \mid m,n \geq 0\ \}$



Design PDA

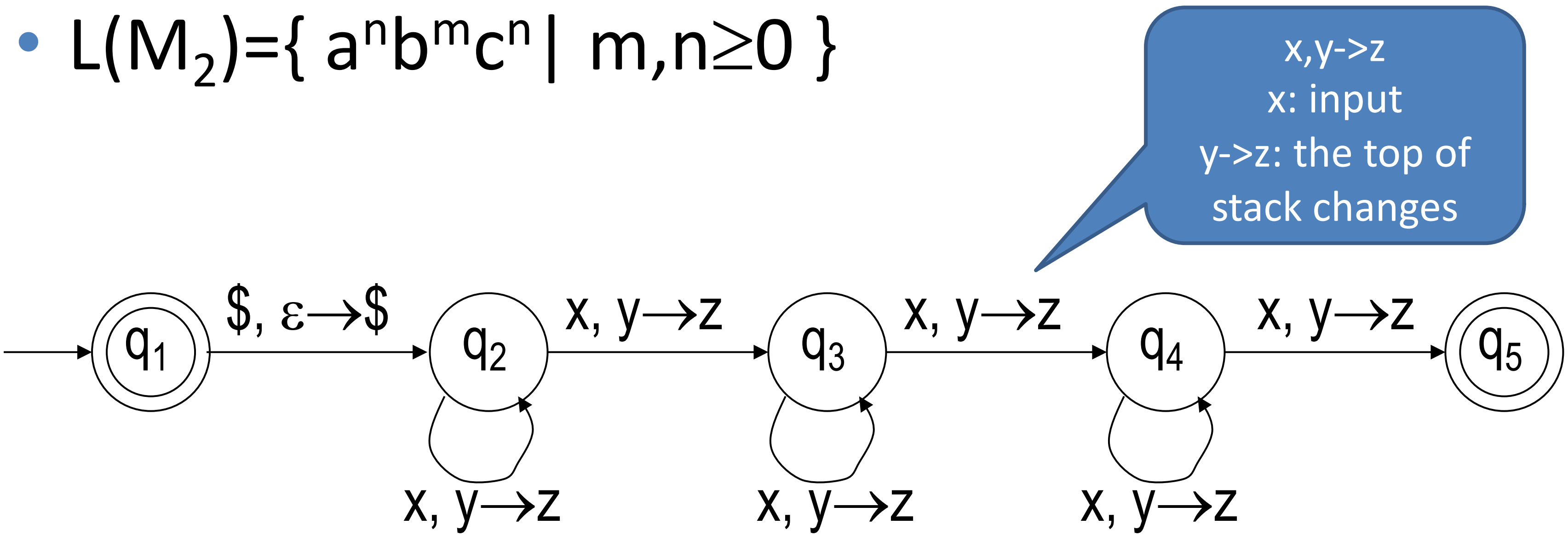
$\$$	aaa…a	bbb…b	ccc…c	$\$$
q_1	q_2	q_3	q_4	q_5

- $L(M_2)=\{ a^n b^m c^n \mid m,n \geq 0 \}$
  - Operation:
    - For an input a, and push a into stack
    - After reading some bs, every time, for an input c, pop one a from the stack
  - Determine accept/reject
    - If the stack is empty when input is done, accept;
    - Otherwise, reject.

Design PDA

\$	aaa…a	bbb…b	ccc…c	\$
q_1	q_2	q_3	q_4	q_5

- $L(M_2)=\{ a^n b^m c^n \mid m,n \geq 0 \}$



- $L(M_2)=\{ a^n b^m c^n \mid m,n \geq 0 \}$
  - Operation:
    - For an input a, and push a into stack
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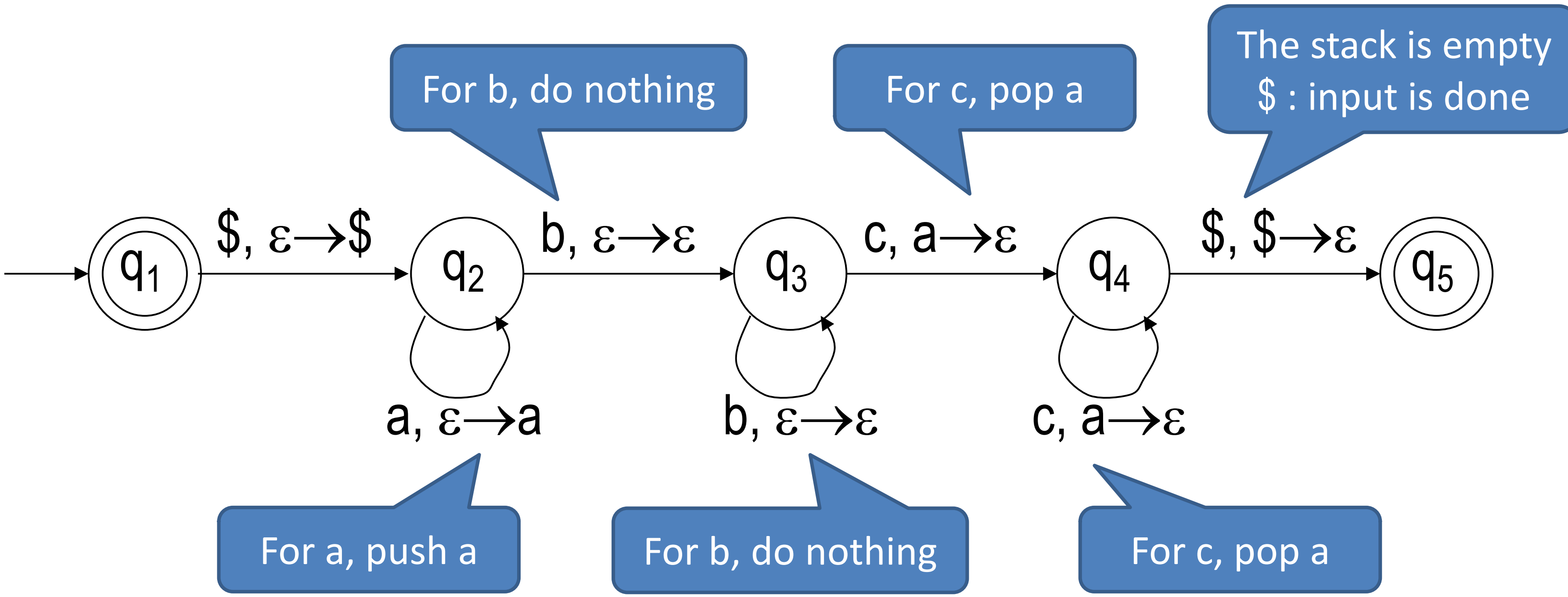
Design PDA

$	aaa...a	bbb...b	ccc...c	$
q_1	q_2	q_3	q_4	q_5

- $L(M_2)=\{\ a^n b^m c^n \mid m,n>0\ \}$



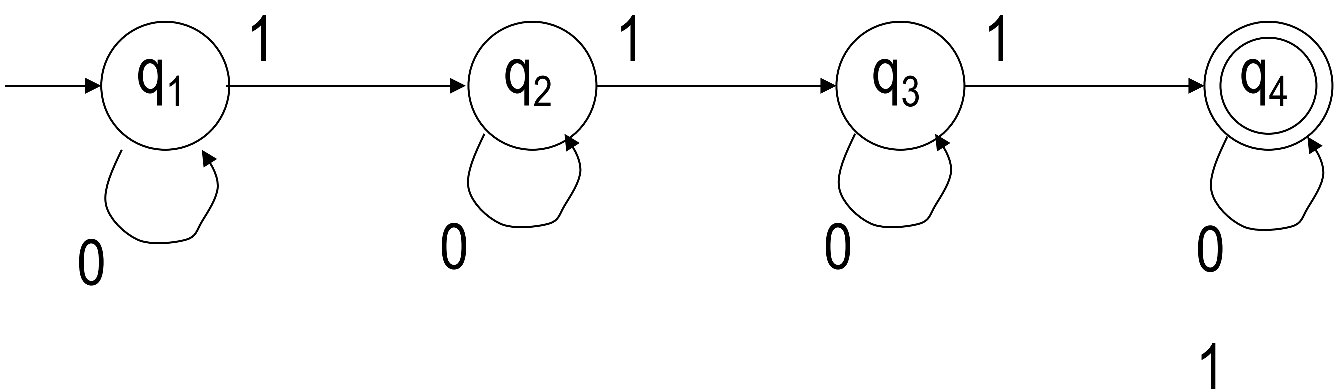
Design PDA

- $L(M_3)$={ w| w contains at least three 1s}, input = {0, 1}
  - Input : 001101
    - Output : Accepted
  - Input : 100010
    - Output : Not Accepted
  - Regular language
    - Does not need the stack, $\varepsilon \rightarrow \varepsilon$

Design PDA

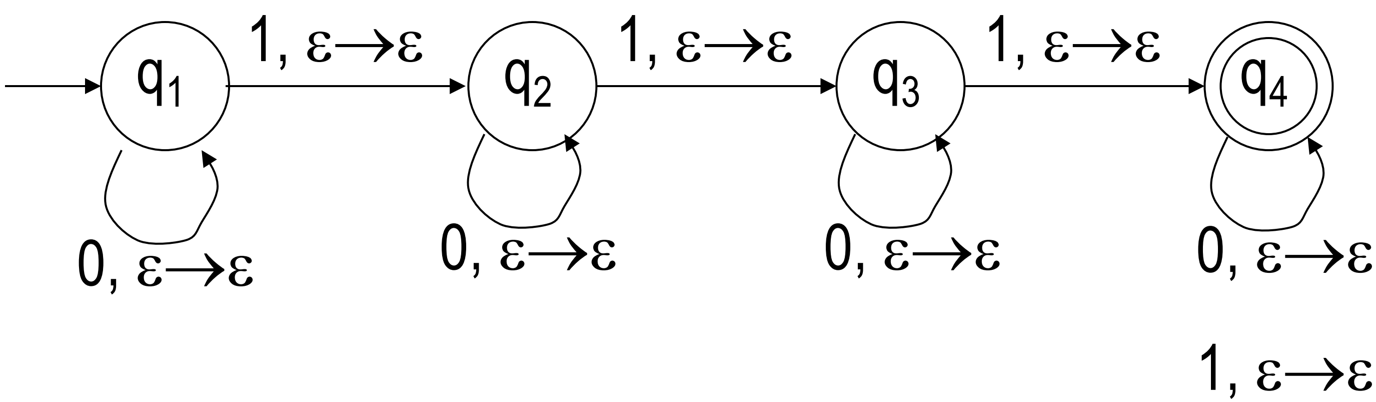
- $L(M_3)$={ w| w contains at least three 1s}, input = {0, 1}

What are the states?



Design PDA

- $L(M_3)$={ w| w contains at least three 1s}, input = {0, 1}



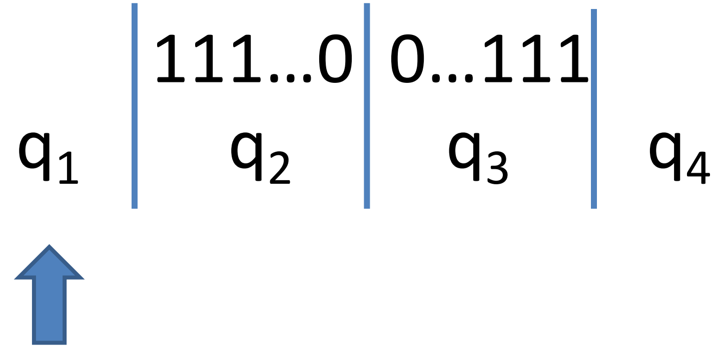
Design PDA: { ww^R }

	111…0	0…111	
q_1	q_2	q_3	q_4

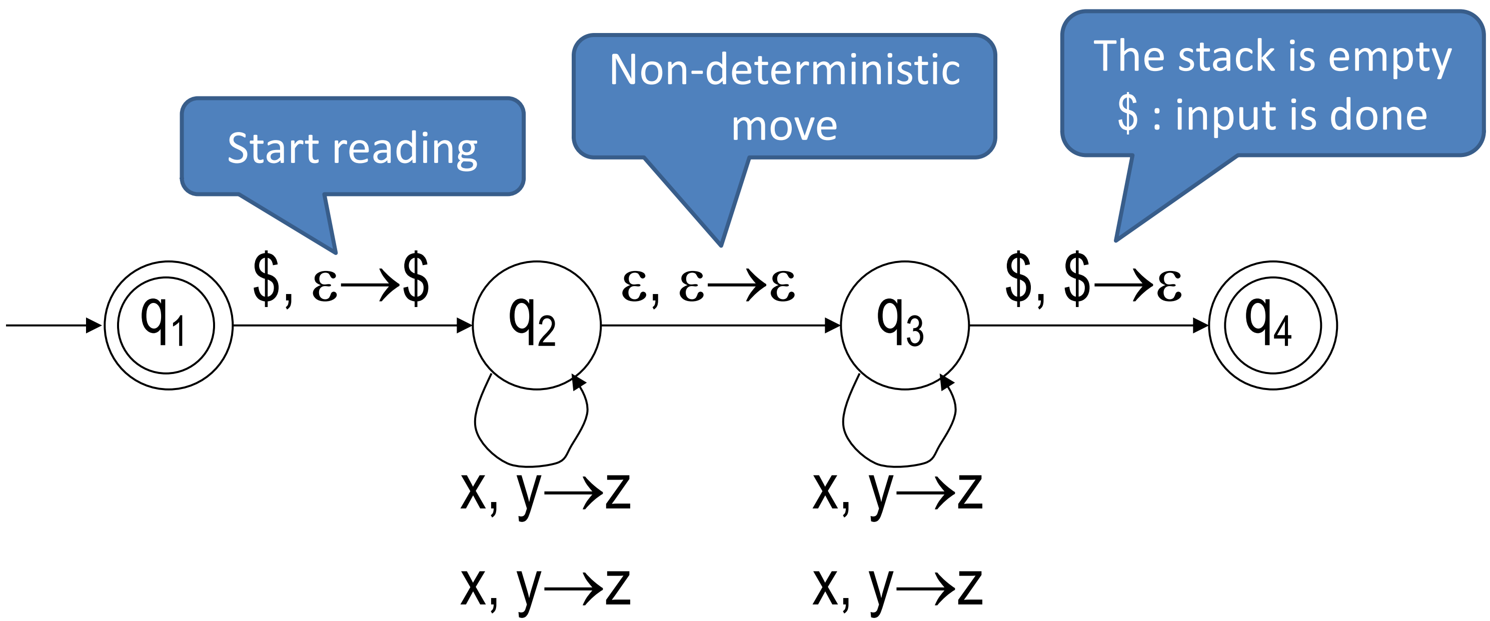
- Palindromes:
- Examples:
  - NOON
  - 123321
  - abba

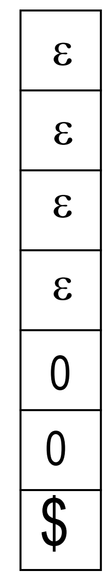
What are the states?

Design PDA: { ww^R }



- Palindromes:



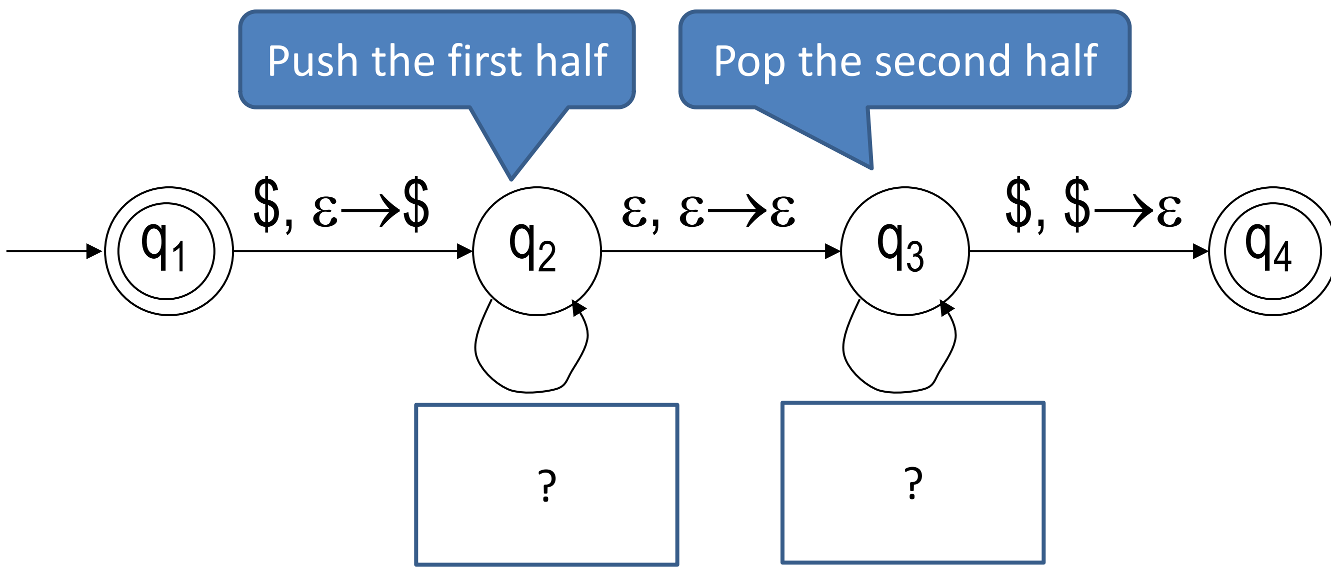


stack

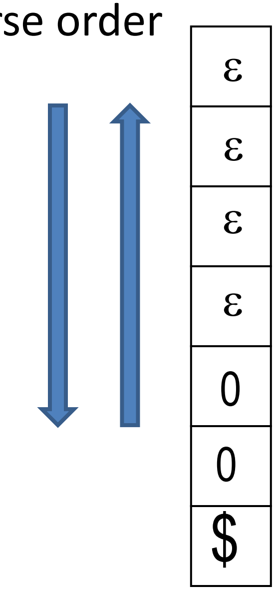
Design PDA: { ww^R }

	abc…z	z…cba	
q_1	q_2	q_3	q_4

- Palindromes:

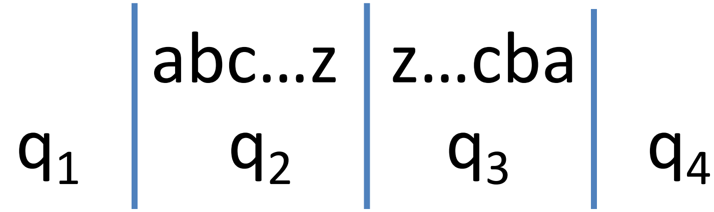


The stack itself already guarantees reverse order

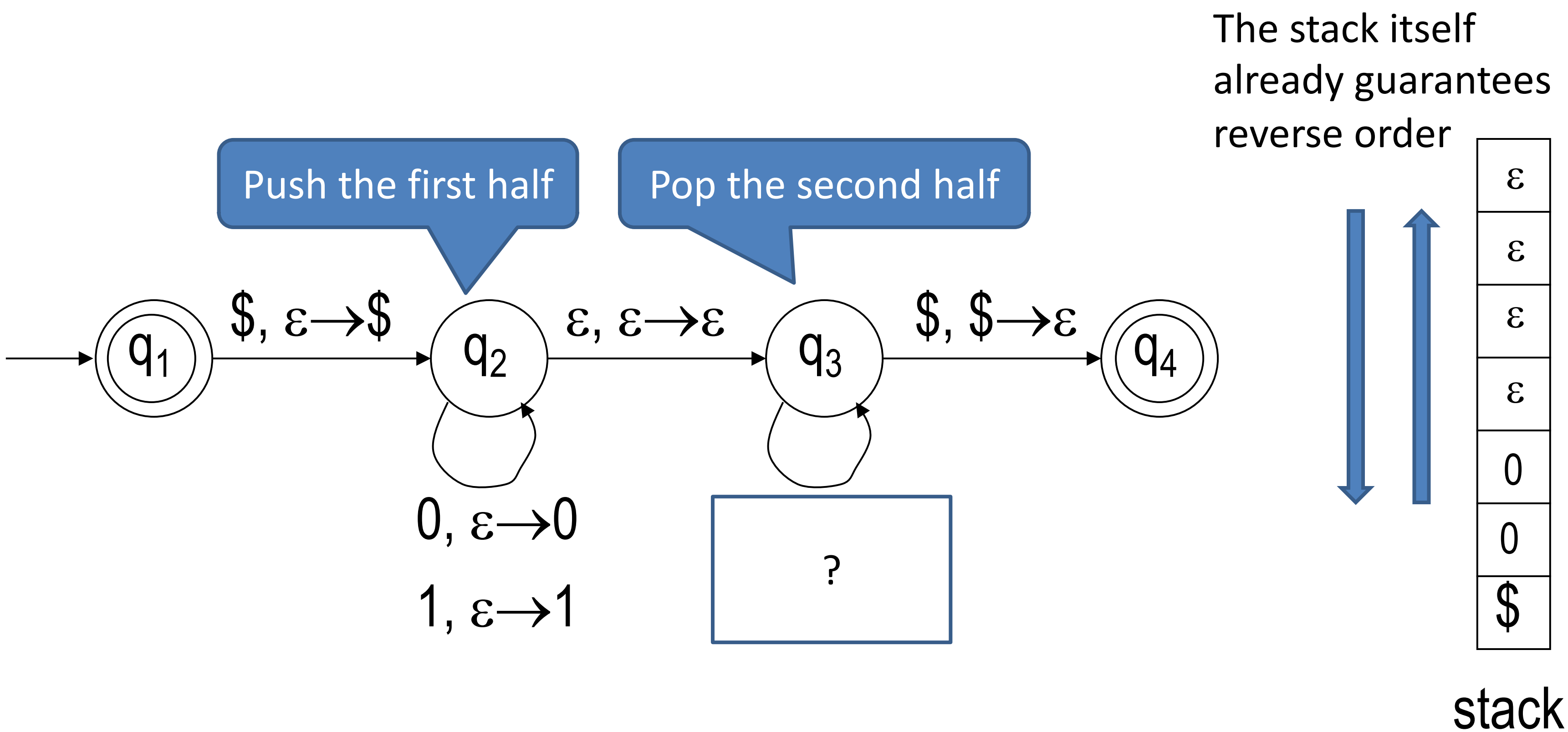


stack

Design PDA: { ww^R }



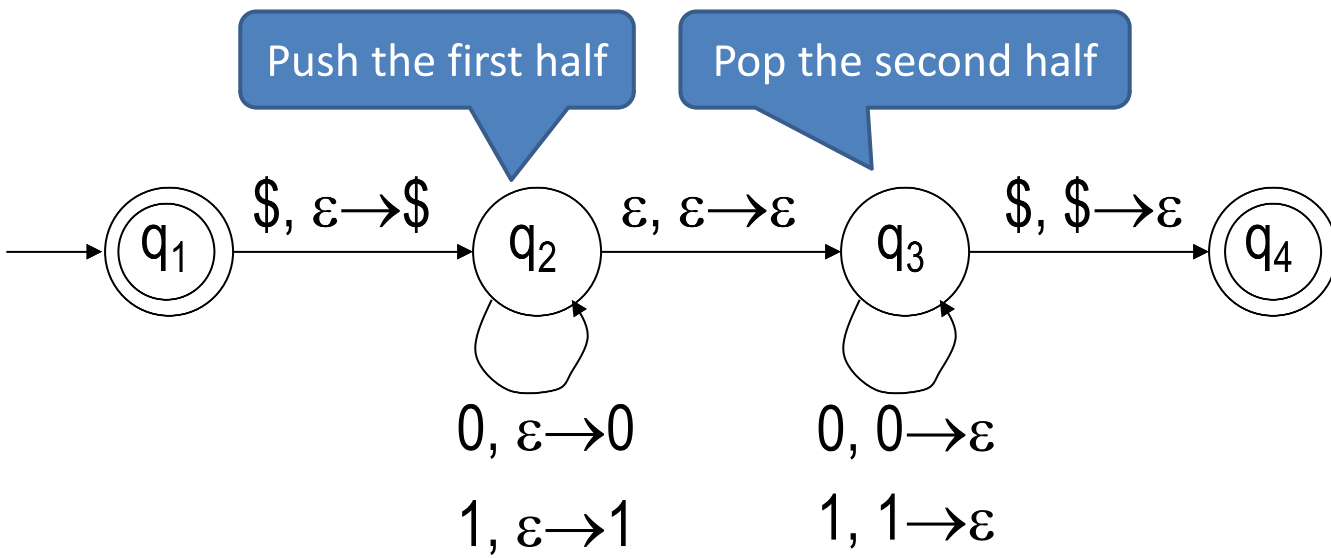
- Palindromes:



Design PDA: { ww^R }

	abc…z	z…cba	
q_1	q_2	q_3	q_4

- Palindromes:



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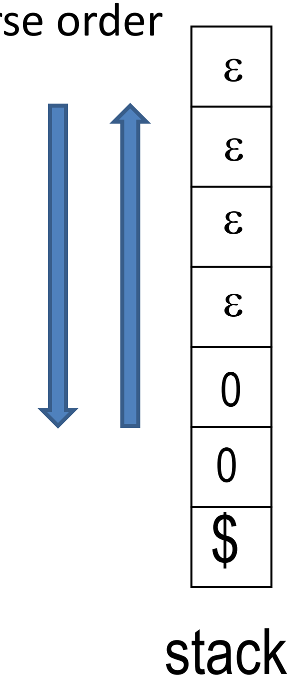

Conclusion

- What is pushdown automata (PDA)?
- How to use PDA to recognize some CFL? Informal description
- Definition of PDA $M=(Q,\Sigma,\Gamma,\delta,q_0,F)$
- PDA examples, δ: $x, y \rightarrow z$

PUSH z: $x, \varepsilon \rightarrow z$

POP z. : $x, z \rightarrow \varepsilon$