CS 6041
Theory of Computation

Reducibility

Kun Suo

Computer Science, Kennesaw State University

https://kevinsuo.github.io/

Reducibility

- If A reduces to B, we can use a solution to B to solve A
- Example:
  - Look for a place - - > Get a map
  - Go to a place - -> Take a car
- If A is reduced to B:
  - If we can do B, then we can also do A
  - If we cannot do A, then we cannot do B

Counter-proposition

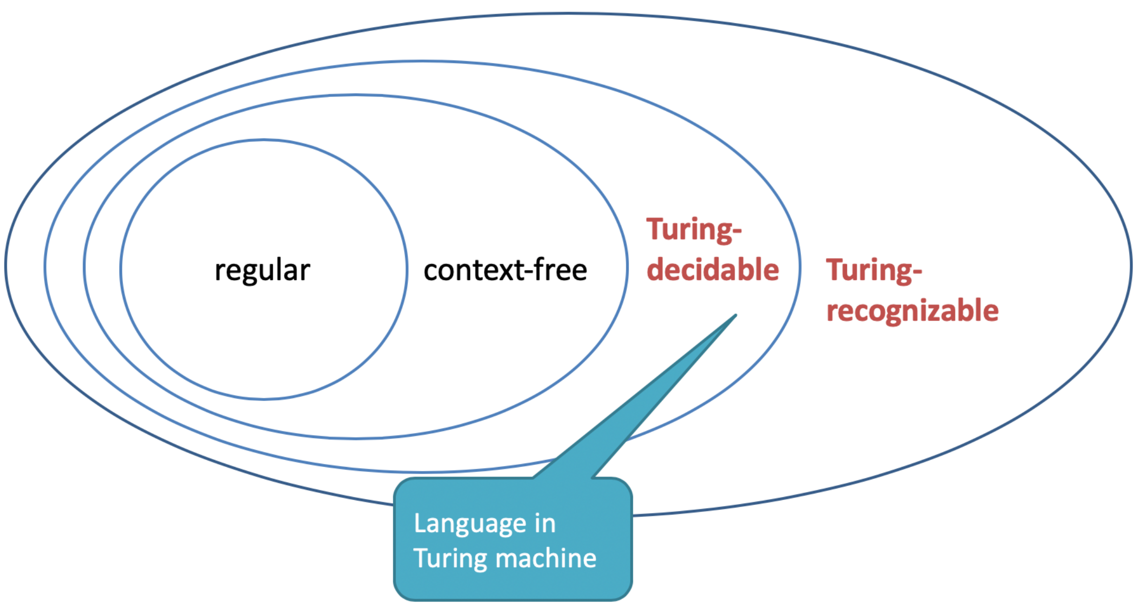
Revisit: The output of Turing Machine

- Accept
- Reject
- Loop

Accept, Reject } Halt -> Decidable

Loop = Never Halt

Halt -> Decidable, Never Halt } Recognizable



Revisit: Decidability

- Decidable?

	DFA/NFA/RE	CFG	TM
Acceptance (A)	√	√	×
Emptiness (E)	√	√	
Equivalence (EQ)	√	×	

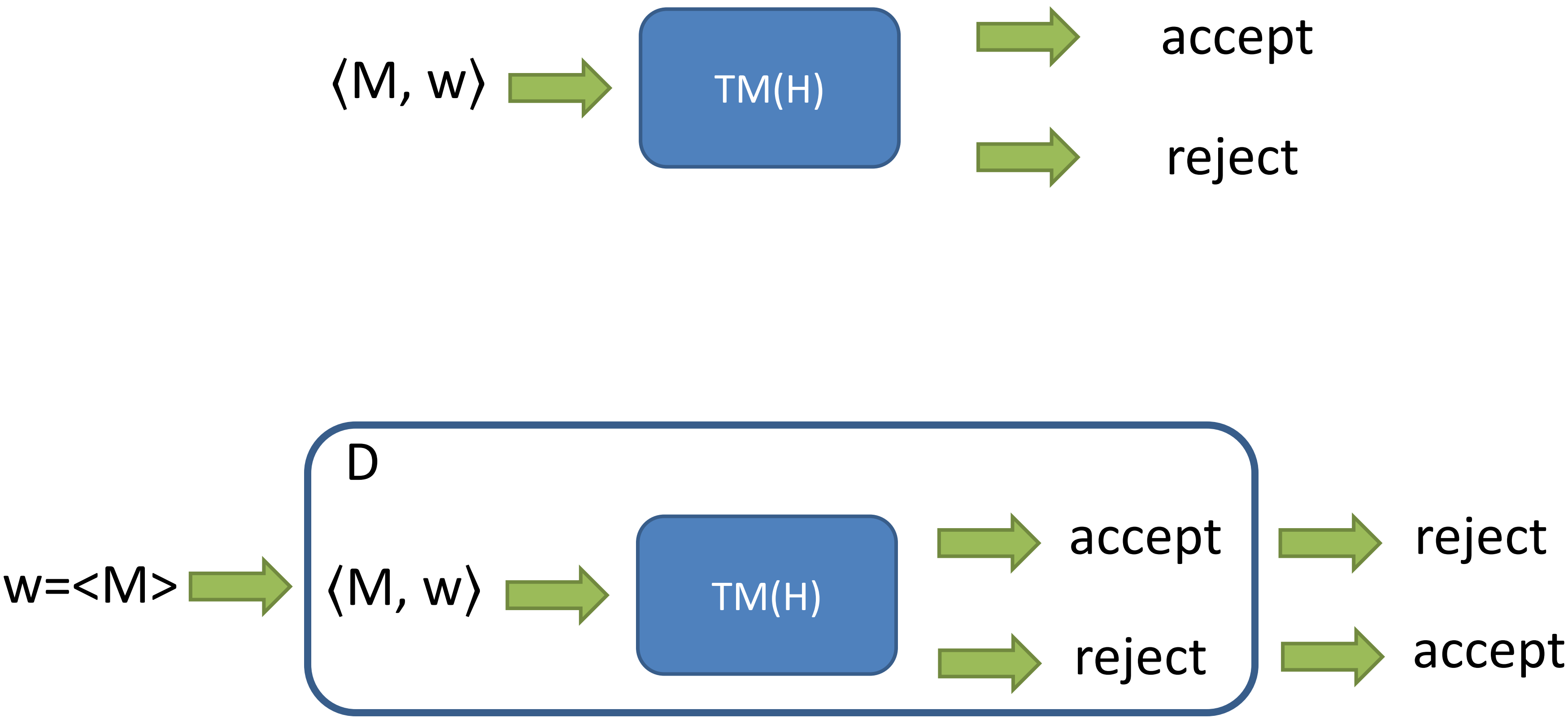
Revisit: Decidable problems for Turing Machine

- Acceptance problem for Turing Machine
  - Whether a Turing machine accepts a given input string

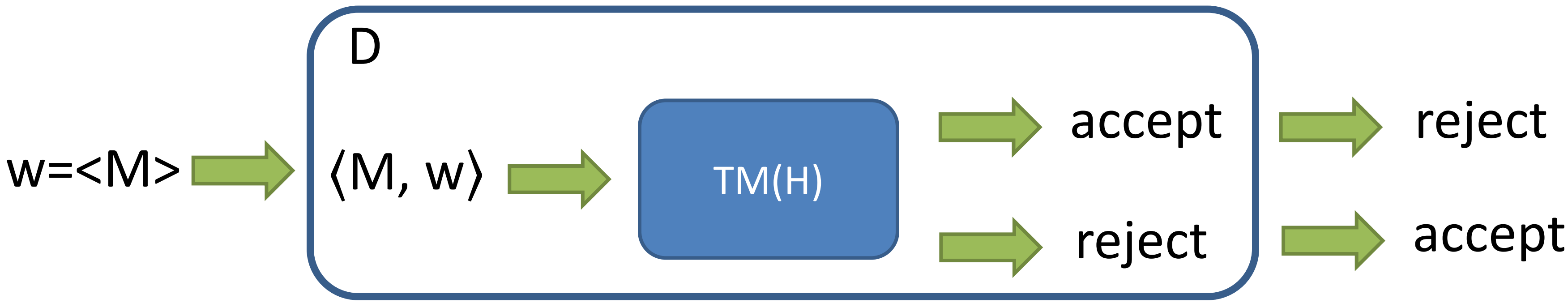
We do not know whether M accepts w or not

- Language:
  - $A_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$

Revisit: Decidable problems for Turing Machine



Revisit: Decidable problems for Turing Machine



$$D(< M >) = \begin{cases} \text{accept,} & \text{if M does not accept } < M > \\ \text{reject,} & \text{if M accepts } < M > \end{cases}$$

$$D(< D >) = \begin{cases} \text{accept,} & \text{if D does not accept } < D > \\ \text{reject,} & \text{if D accepts } < D > \end{cases}$$

Contradiction!

Revisit: Decidability

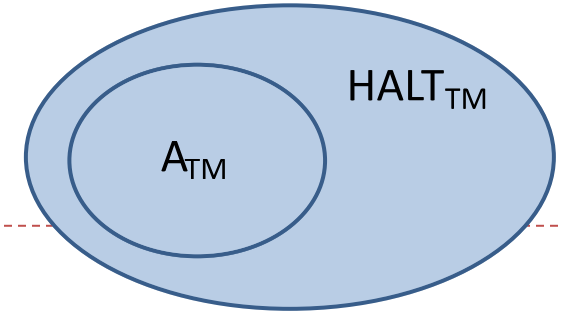
- Decidable?

	DFA/NFA/RE	CFG	TM
Acceptance (A)	√	√	×
Emptiness (E)	√	√	
Equivalence (EQ)	√	×	
Halt			?

1. Halting problem

- TM halting problem:
  - whether a Turing machine M halts (by accepting or rejecting) on a given input w

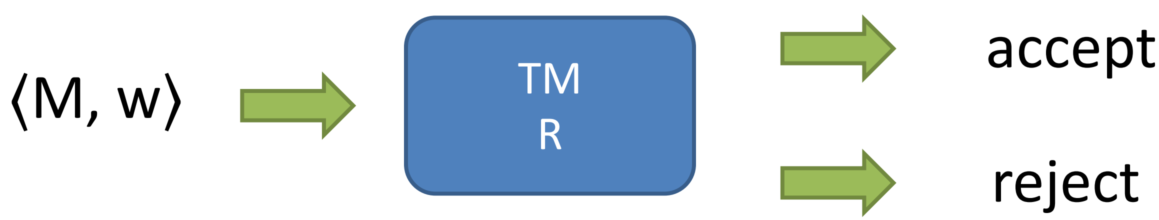
1. Halting problem



- Language
  - $HALT_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ halts on input } w\}$.

    vs.

    $A_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts input } w\}$.



Theorem 5.1

- $HALT_{TM}$ is undecidable

- Proof (prove by contradiction):

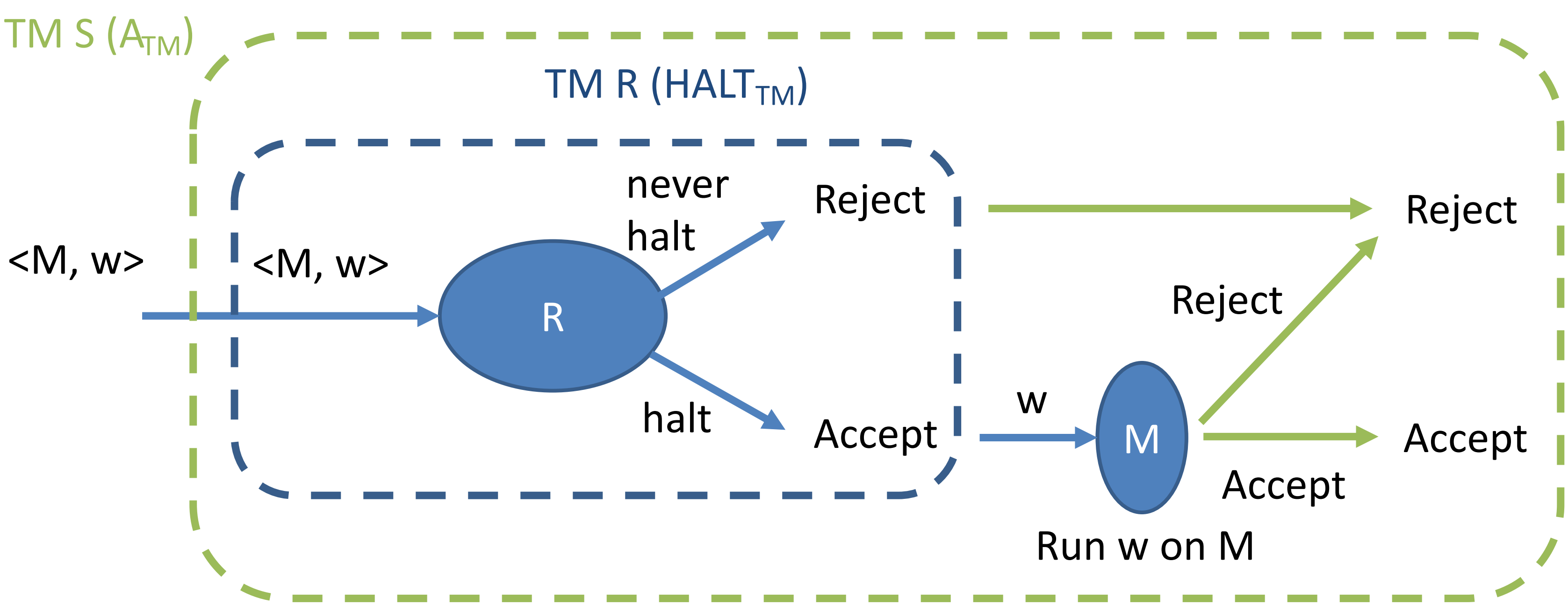
  Suppose TM R decides $HALT_{TM}$

  Then we create a TM S to decide A_{TM}

S = “On input ⟨M, w⟩, M is a TM and w is a string:

1. Run TM R on input ⟨M, w⟩.
2. If R rejects, which means never halt. Then S rejects.
3. If R accepts, which means R will halt (accept or reject)
   we simulate M on w until it halts.
4. If M has accepted, accept;
   if M has rejected, reject.”

It means A_{TM} is decidable. Contradiction!



Theorem 5.1

- $HALT_{TM}$ is undecidable
- If the $HALT_{TM}$ is decidable, then we can get A_{TM} is also decidable. However, we already proved A_{TM} is undecidable.
- A_{TM} is reduced to $HALT_{TM}$

Rethink A_{TM}

- Acceptance problem for Turing Machine
  - Whether a Turing machine accepts a given input string

The output of Turing Machine

- Accept
- Reject
- Loop

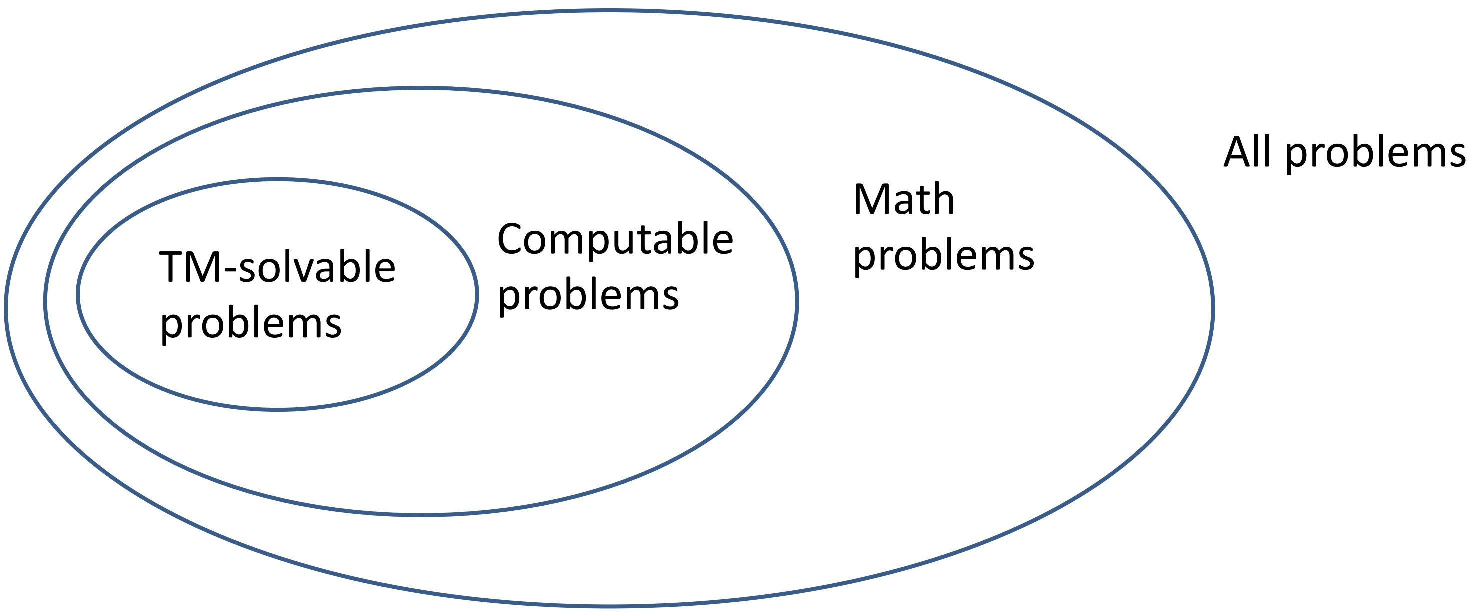
$\left.\begin{array}{l}\text{Accept}\\ \text{Reject}\end{array}\right\}$ Halt

Loop = Never Halt

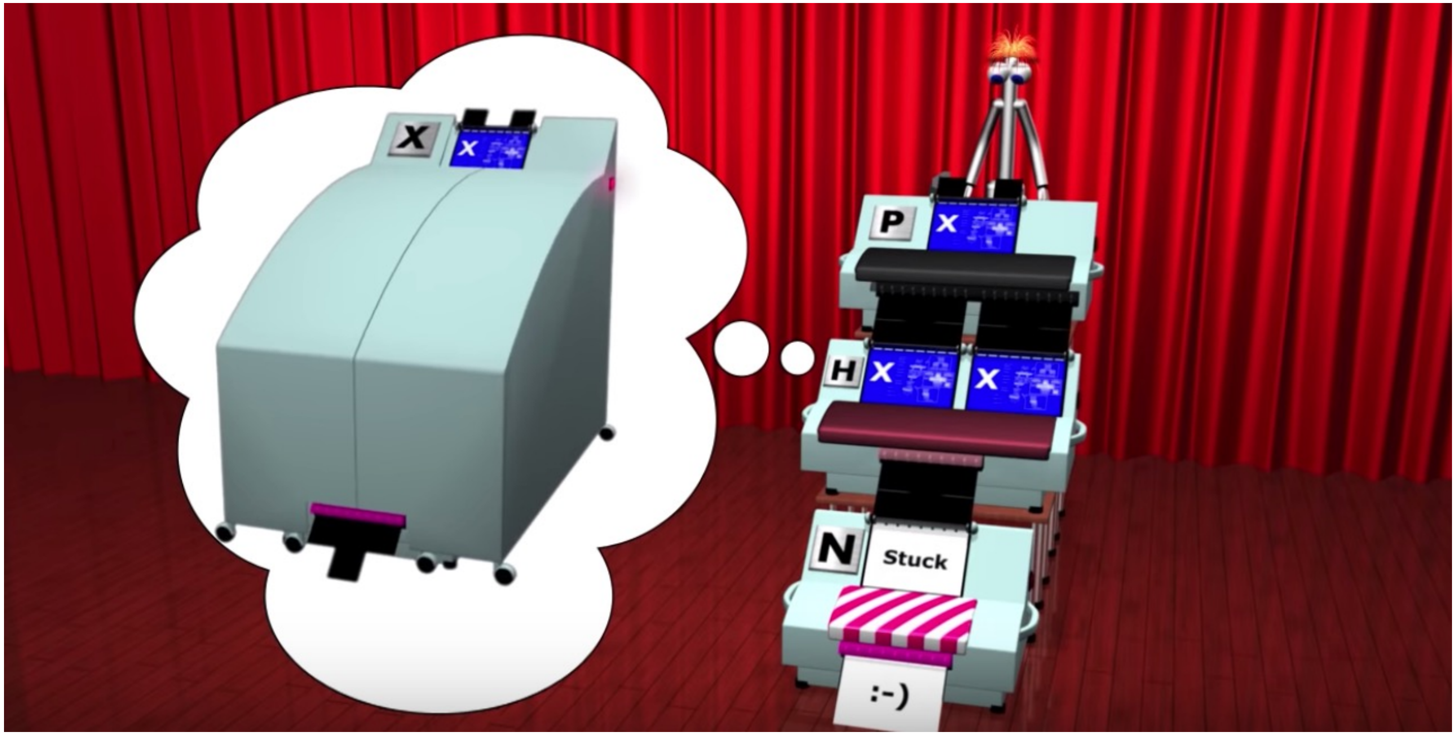
We do not know whether M accepts w or not is because TM could never halt

$HALT_{TM}$

- The $HALT_{TM}$ problem just proves that the Turing machine (or computers) is not omnipotent



$HALT_{TM}$



https://youtu.be/92WHN-pAFCs

2. Emptiness of Turing machine

- Decidable?

	DFA/NFA/RE	CFG	TM
Acceptance (A)	√	√	×
Emptiness (E)	√	√	?
Equivalence (EQ)	√	×	

2. Emptiness of Turing machine

- Emptiness of Turing machine
  - Whether or not a TM never accept any string w

- Language
  - $E_{TM} = \{ \langle M \rangle \mid M \text{ is a TM and } L(M)=\emptyset \}$

Theorem 5.2

- E_{TM} is undecidable
  - $E_{TM} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

- Proof:

Input is M
Nothing related to w

We need create contradiction between

$$E_{TM} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$$

and

$$A_{TM} = \{\langle M,w \rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$$

Theorem 5.2 proof

- Proof:

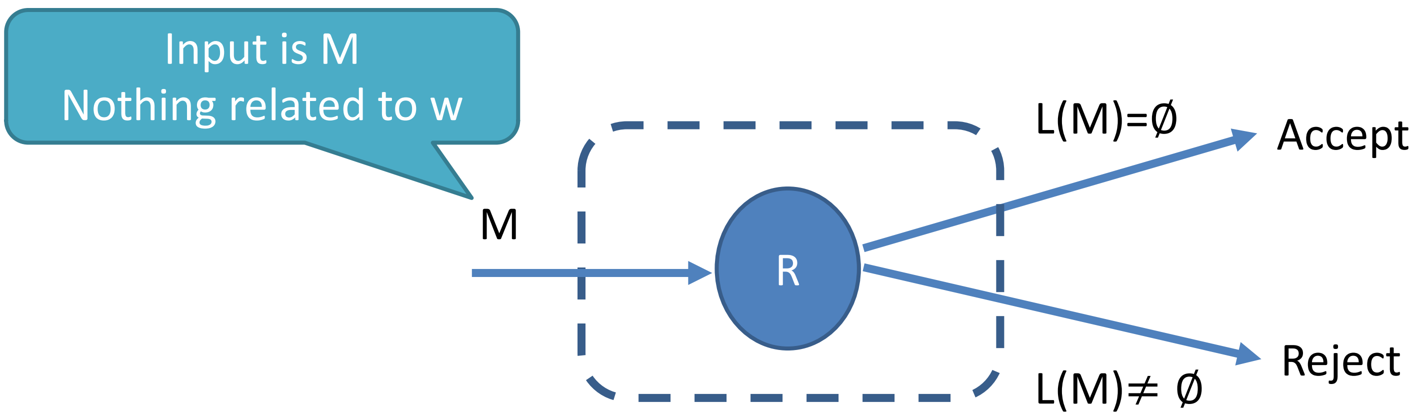
  Suppose E_{TM} is decidable, then TM R decides E_{TM}

  R = "On input M,

  if M does not accept anything, then L(M)=∅, R accept;

  if M accept something, then L(M)≠ ∅, R reject;

  "



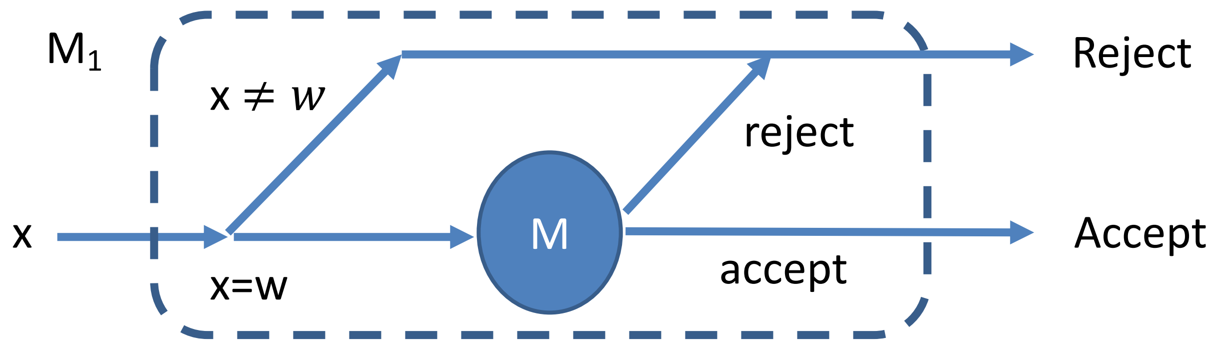
Theorem 5.2 proof

- Proof

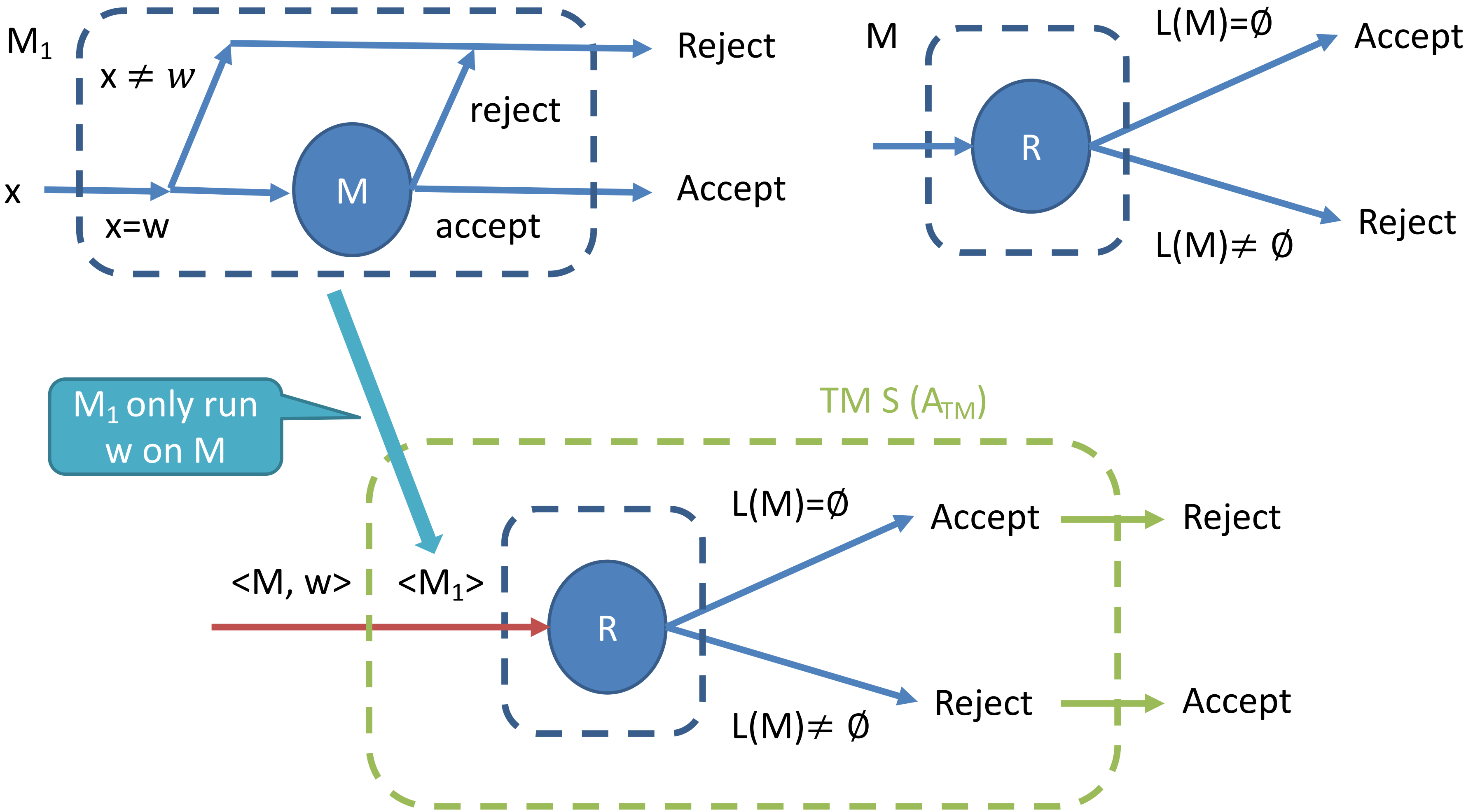
Create a TM M_1 involving both M and w

Create a TM M_1, that

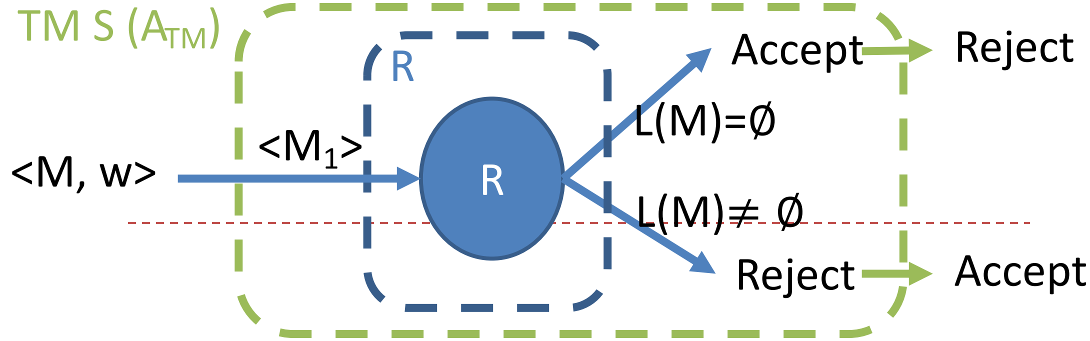
- If input $\neq w$, M_1 reject (Only string M_1 can accept is w);
- If input is w, test w on M
  - If M accepts w, M_1 accepts
  - If M rejects w, M_1 rejects



Theorem 5.2 proof



- Proof



Integrate the R and M_1

R = "On input M_1,

if M does not accept w, then L(M)=∅, R accept;

if M accept w, then L(M)≠ ∅, R reject;

"

Based on R, we can create TM S to decide A_{TM}

S = "On input <M, w>

Run R on input M_1 (M_1 = <M, w>)

If R accepts, S rejects;

M rejects w ⇒ R accepts ⇒ S rejects

If R rejects, S accepts.

M accepts w ⇒ R rejects ⇒ S accepts

",

Contradiction! A_{TM} is not decidable

3. Regular issue of TM

- Decidable?

	DFA	CFG	TM
Acceptance (A)	√	√	×
Emptiness (E)	√	√	×
Equivalence (EQ)	√	×	
Halt			×
Regular			?

3. Regular issue of TM

- Regular issue of TM
  - Whether a given Turing machine has an equivalent finite automaton or recognizes a regular language

- Language
  - $REGULAR_{TM} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is a regular language}\}.$

Theorem 5.3

- $REGULAR_{TM}$ is undecidable.
  - $REGULAR_{TM} = \{\langle M \rangle \mid M$ is a TM and $L(M)$ is a regular language$\}$.

Input is M
Nothing related to w

- Proof:

We need create contradiction between

$REGULAR_{TM} = \{\langle M \rangle \mid M$ is a TM and $L(M)$ is a regular language$\}$.

and

$A_{TM} = \{\langle M, w \rangle \mid M$ is a TM and M accepts $w\}$

Theorem 5.3 proof

Input is M
Nothing related to w

- Proof:

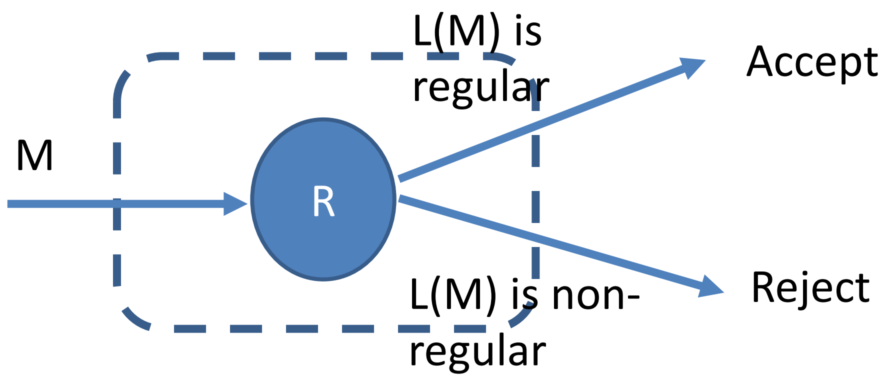
  Suppose $REGULAR_{TM}$ is decidable, then TM R decides $REGULAR_{TM}$

  R = "On input M,

  if L(M) is not a regular language, R rejects;

  if L(M) is a regular language, R accepts;

  "



Theorem 5.3 proof

Create a TM M_2 involving both M and w

- Proof

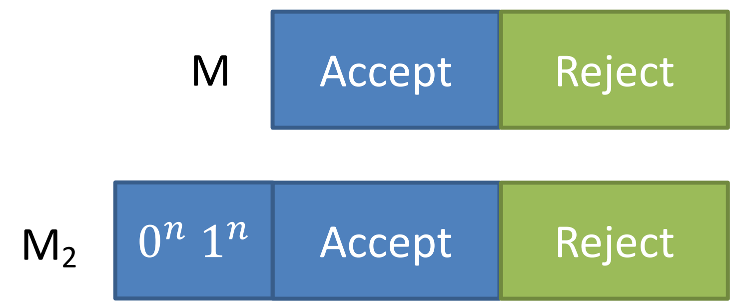
Create a TM M_2, for input w:

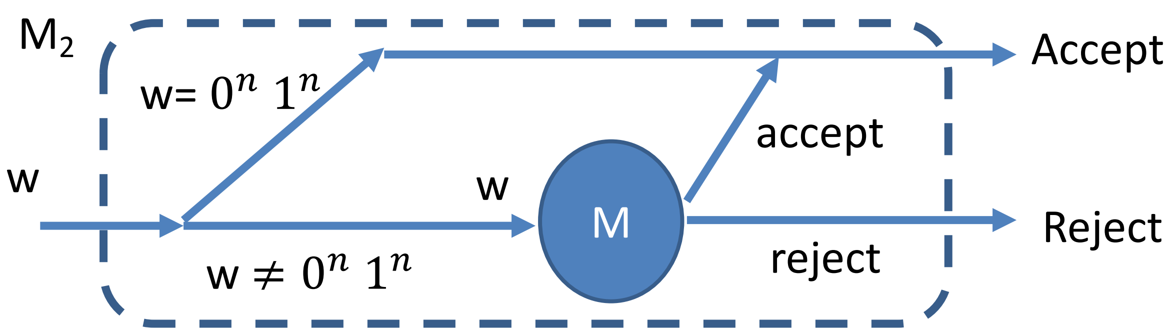
If w is in format of $0^n 1^n$ (not regular language), accept;

If w is not in that format, run w on M:

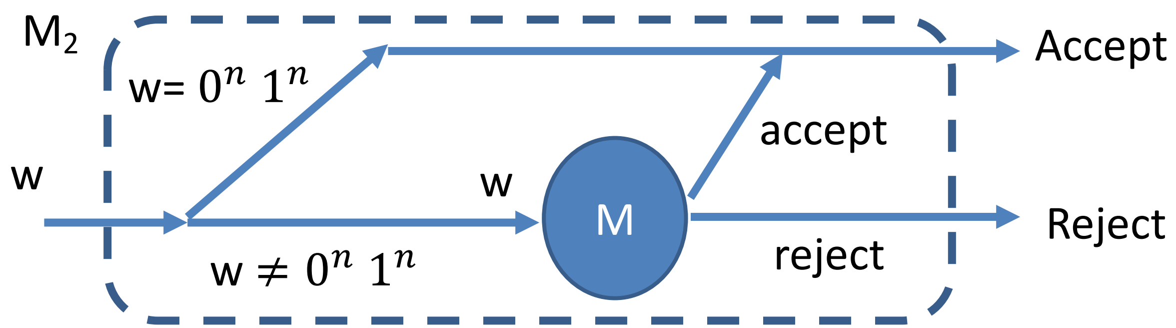
If M accepts w, M_2 accepts;

otherwise, rejects.



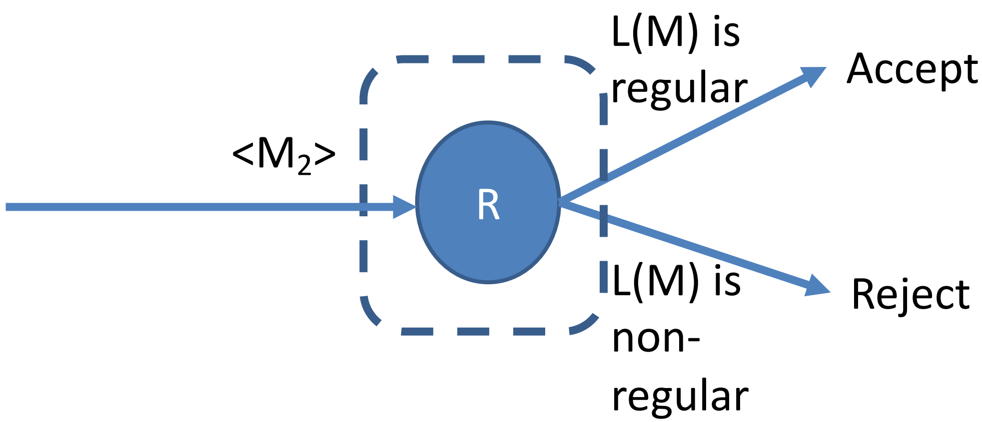


Theorem 5.3 proof



$$L(M_2) = \begin{cases} regular, & if\ M\ accept\ w \\ 0^n\ 1^n, & if\ M\ does\ not\ accept\ w \end{cases}$$

Theorem 5.3 proof



- Proof

$$L(M_2) = \begin{cases} regular, & if\ M\ accept\ w \\ 0^n\ 1^n, & if\ M\ does\ not\ accept\ w \end{cases}$$

Integrate the R and M_2

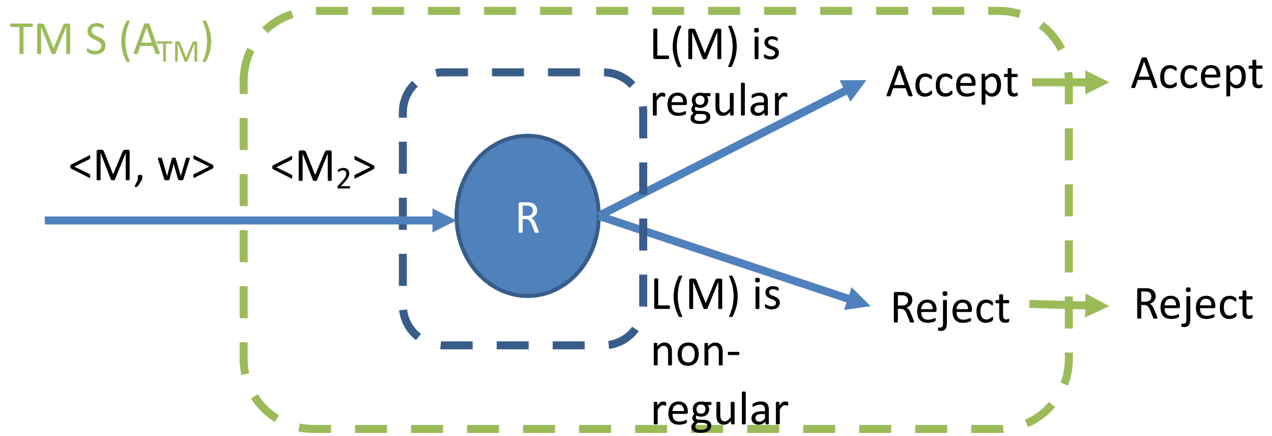
R = "On input M_2,

if M does not accept w, then $L(M_2)$ is non-regular language ($0^n\ 1^n$), R rejects;

if M accepts w, then $L(M_2)$ is a regular language, R accepts;

"

Theorem 5.3 proof



Based on R, we can create TM S to decide A_{TM}

S = "On input <M, w>, convert <M, w> into M_2

Run R on input M_2

If R accepts, S accepts;

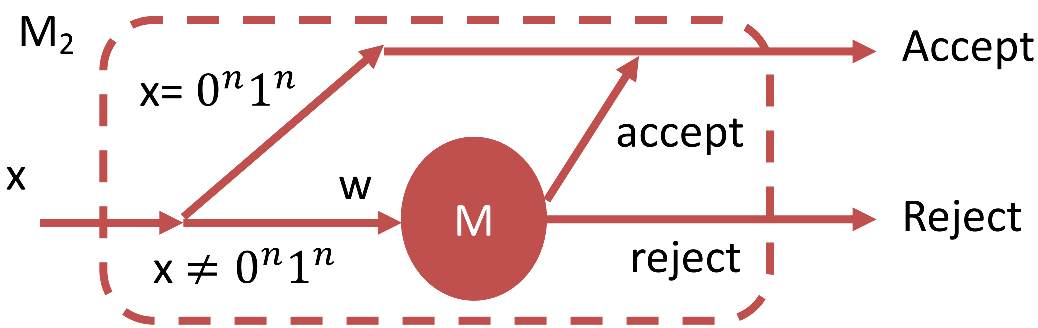
If R rejects, S rejects. ",

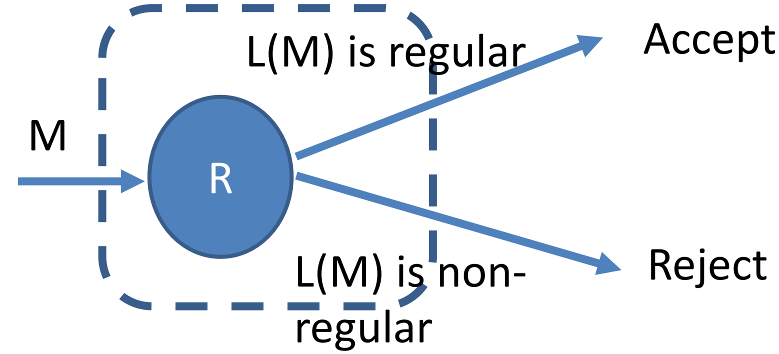
Contradiction!

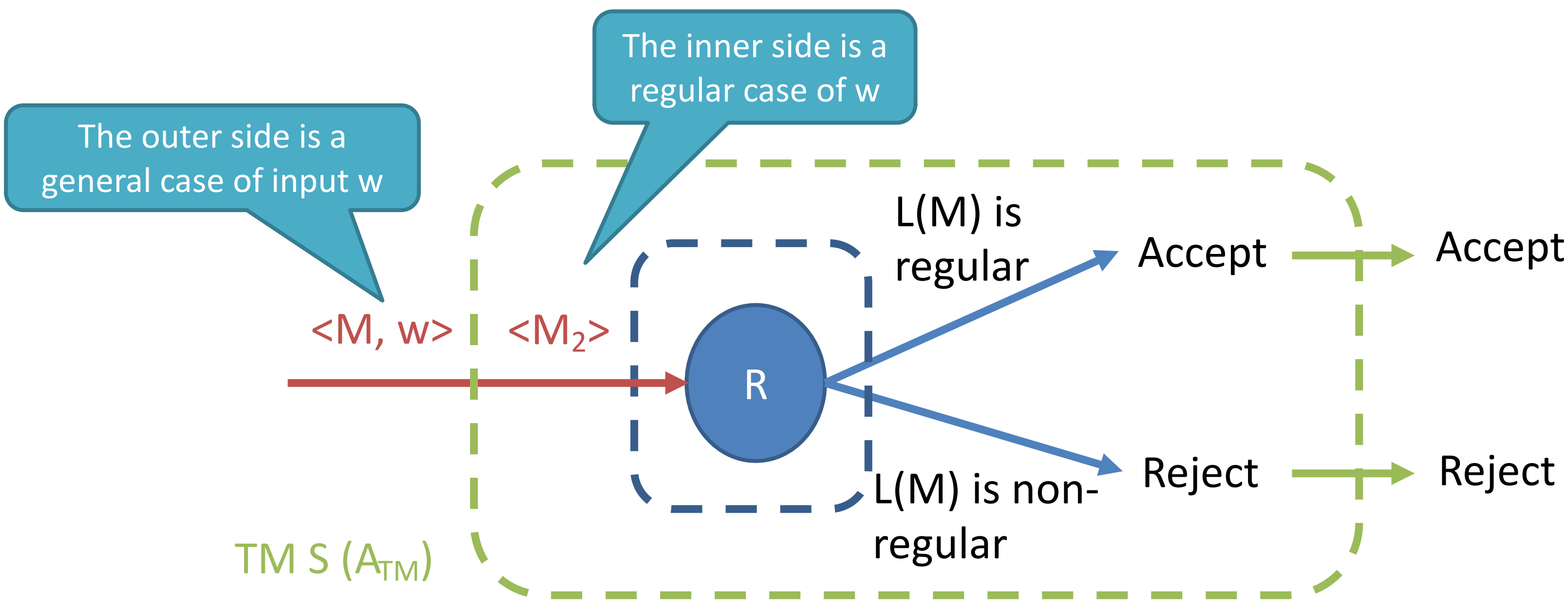
M accepts w ⇒ R accepts ⇒ S accepts

M rejects w ⇒ R rejects ⇒ S rejects

Theorem 5.3 proof







4. Equivalence of TM

- Decidable?

	DFA	CFG	TM
Acceptance (A)	√	√	×
Emptiness (E)	√	√	×
Equivalence (EQ)	√	×	?
Halt			×
Regular			×

4. Equivalence of TM

- Definition
  - Whether two TMs can recognize the same language

- Language
  - $EQ_{TM} = \{\langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are TMs}$
    and
    $L(M_1) = L(M_2)\}$

Theorem 5.4

- EQ_{TM} is undecidable.
  - $EQ_{TM} = \{\langle M_1,M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

- Proof:

We need create contradiction between

$$EQ_{TM} = \{\langle M_1,M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$$

and

$$E_{TM} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset \}$$

Theorem 5.4 proof

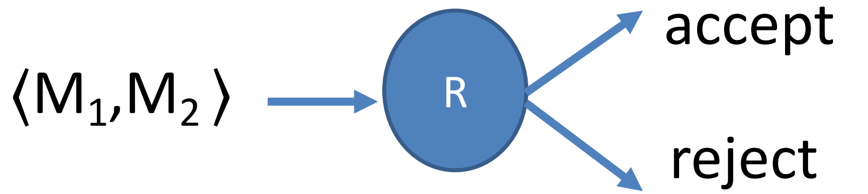
- Proof:

Suppose EQ_{TM} is decidable, TM R decide EQ_{TM}

R = "On input $<M_1, M_2>$,

if $L(M_1) = L(M_2)$, accept;

if $L(M_1) \neq L(M_2)$, reject."



Create another TM S for E_{TM}

$L(M_a) = \emptyset$

S = "On input $<M>$,

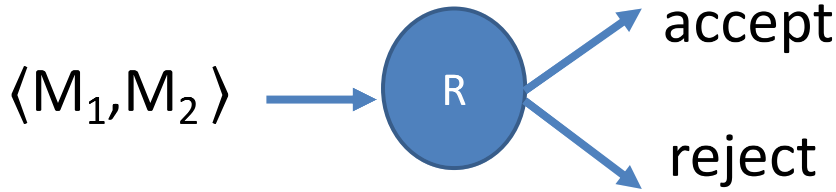
Execute R on input $<M, M_a>$, M_a is a TM rejects all inputs,

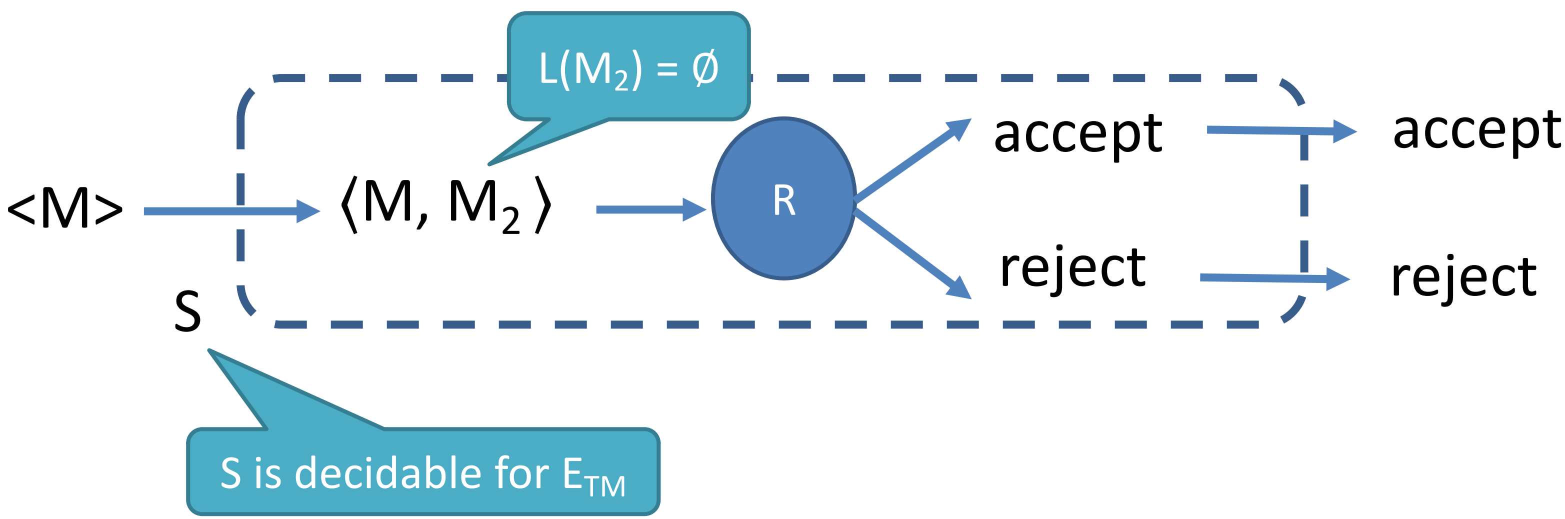
if R accepts, S accepts;

if R rejects, S rejects."

Theorem 5.4 proof

- Proof:





Theorem 5.4 proof

- Proof:

S = “On input <M>,

Execute R on input <M, M_a>, M_a is a TM rejects all inputs,

if $L(M) = \emptyset$, S accepts;

if $L(M) \neq \emptyset$, S rejects.”

Because R id decidable, then S is also decidable.

However, for E_{TM} is not decidable (based on theorem 5.2).

Contradiction!

4. Equivalence of TM

- Decidable?

	DFA	CFG	TM
Acceptance (A)	√	√	×
Emptiness (E)	√	√	×
Equivalence (EQ)	√	×	×
Halt			×
Regular			×

Conclusion

- $HALT_{TM}$ is undecidable
  - We do not know whether a TM will halt on a given input
- E_{TM} is undecidable
  - We do not know whether a TM never accept any strings
- $REGULAR_{TM}$ is undecidable
  - We do not know whether a TM has an equivalent DFA/NFA/RE
- EQ_{TM} is undecidable
  - We do not know whether two VMs recognize the same language

Conclusion

- Relationship of languages on reducibility