CS 6041
Theory of Computation

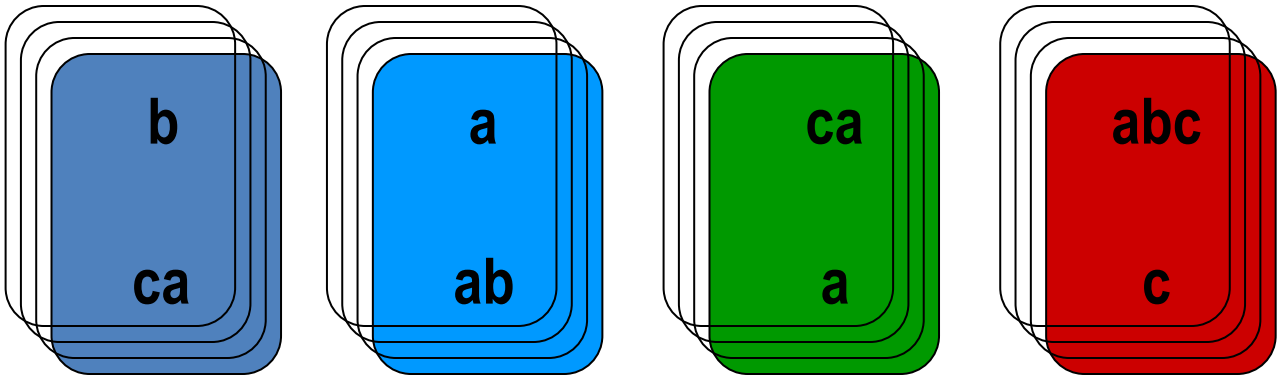
Reducibility

Kun Suo

Computer Science, Kennesaw State University

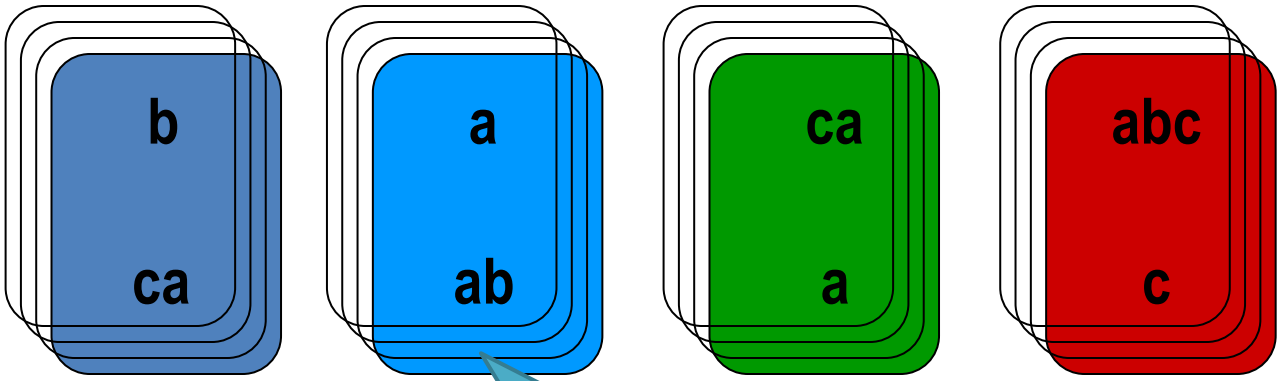
https://kevinsuo.github.io/

Post Correspondence Problem (PCP)

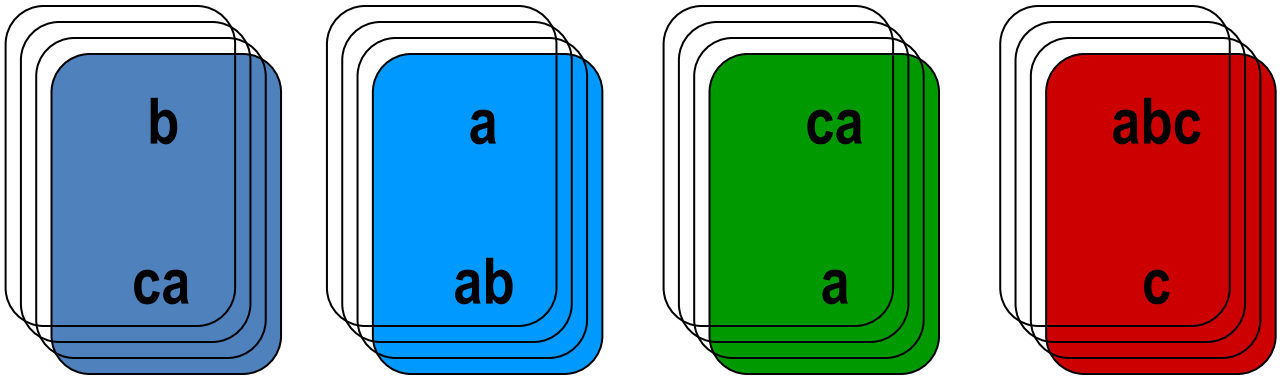


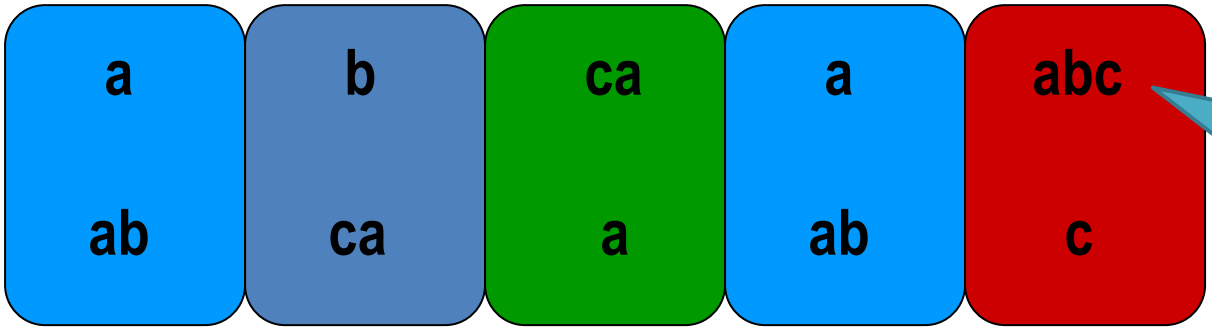
We have limited categories of dominos

Post Correspondence Problem (PCP)



Post Correspondence Problem (PCP)

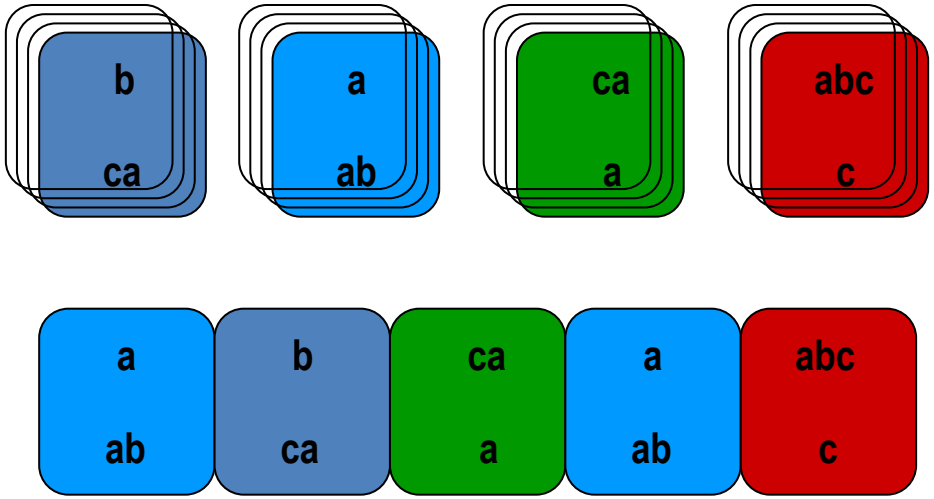




Match:
Upper string: abcaaabc
Bottom string: abcaaabc

Post Correspondence Problem (PCP)

- Whether a collection of dominos has a match

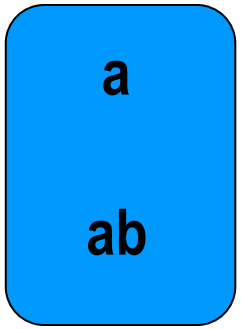


- PCP = {⟨P⟩ | P is an instance of the Post Correspondence Problem with a match}.

Description of PCP

An individual domino

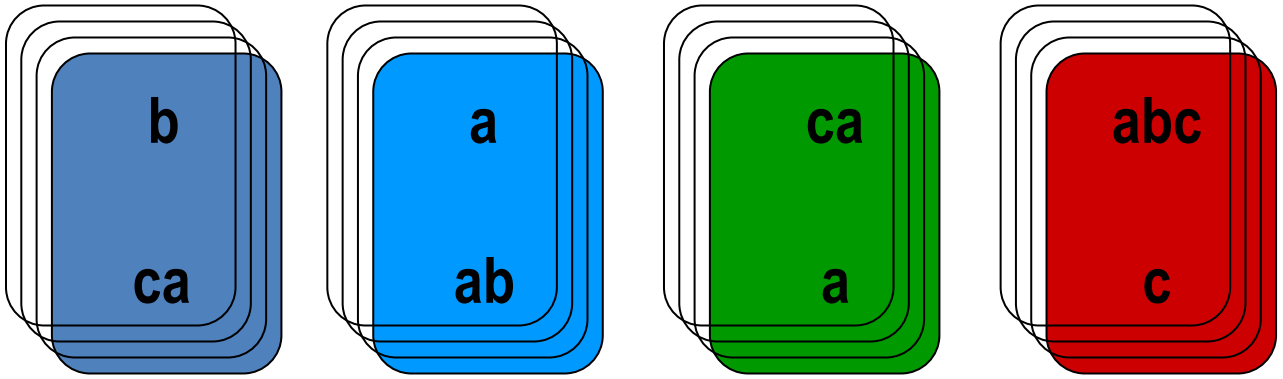
$$\left[\frac{a}{ab}\right]$$



Description of PCP

A collection of dominos

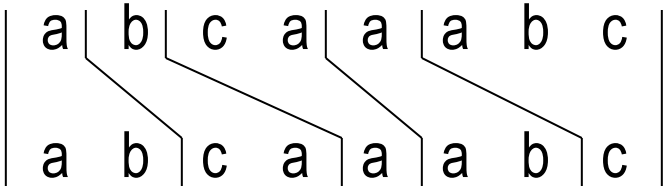
$$\left\{\left[\frac{b}{ca}\right], \left[\frac{a}{ab}\right], \left[\frac{ca}{a}\right], \left[\frac{abc}{c}\right]\right\}$$

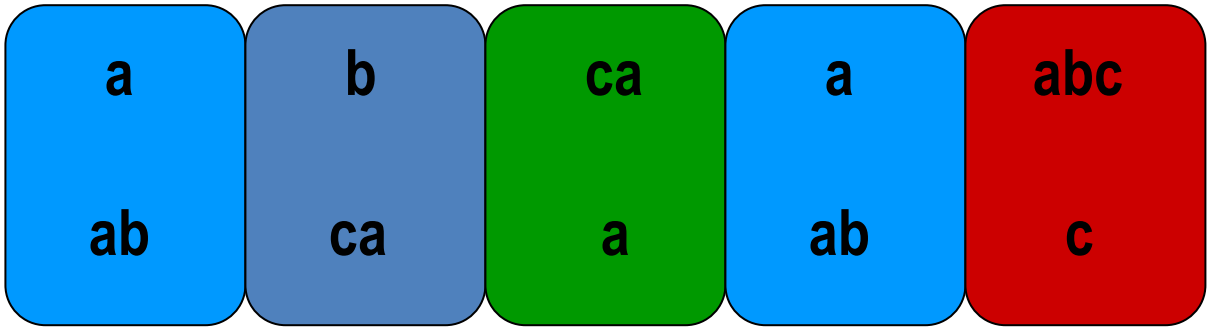


Description of PCP

A match

$$\left[\frac{a}{ab}\right]\left[\frac{b}{ca}\right]\left[\frac{ca}{a}\right]\left[\frac{a}{ab}\right]\left[\frac{abc}{c}\right]$$





A collection without a match

- For a given collection

$$\left\{\left[\frac{abc}{ab}\right], \left[\frac{ca}{a}\right], \left[\frac{acc}{ba}\right]\right\}$$

- It cannot contain a match because every top string is longer than the corresponding bottom string

Theorem 5.15

- PCP is undecidable

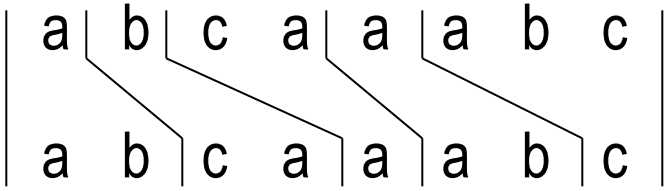
- Proof:

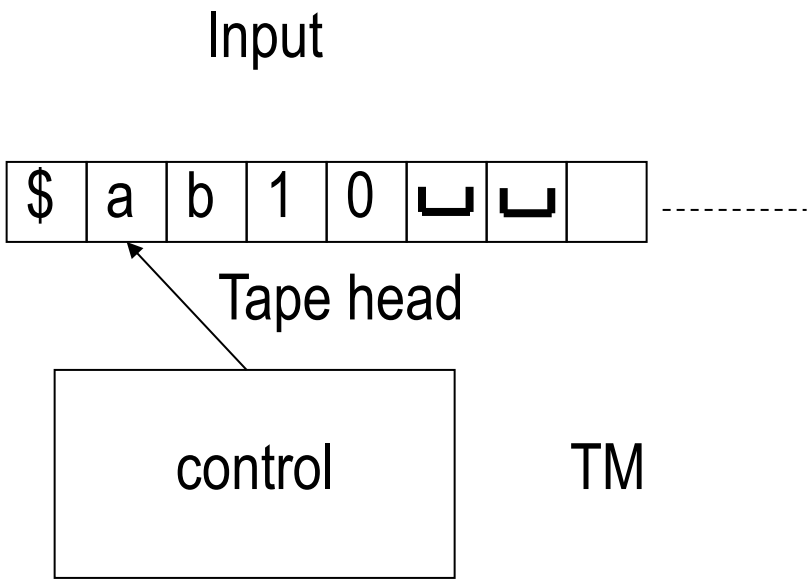
  To simplify the problem, we create ***Modified Post Correspondence Problem (MPCP)*** ,

  MPCP = {⟨P⟩| P is an instance of the PCP with a match that starts with the first domino}.

Theorem 5.15

$$\left[\frac{a}{ab}\right]\left[\frac{b}{ca}\right]\left[\frac{ca}{a}\right]\left[\frac{a}{ab}\right]\left[\frac{abc}{c}\right]$$





- The PCP program is very similar with the A_{TM} program

Theorem 5.15 proof

- Proof:

  Suppose PCP is decidable

  We construct TM S to decide A_{TM} (Theorem 4.11: A_{TM} is undecidable)

  PCP

  $M = (Q,\Sigma,\Gamma,\delta,q_0,q_{accept},q_{reject})$

  S constructs an instance of the PCP P that has a match iff

  M accepts w: ← A_{TM}

  - (1) S first constructs an instance P' of the MPCP;
  - (2) Transfer P' into P;

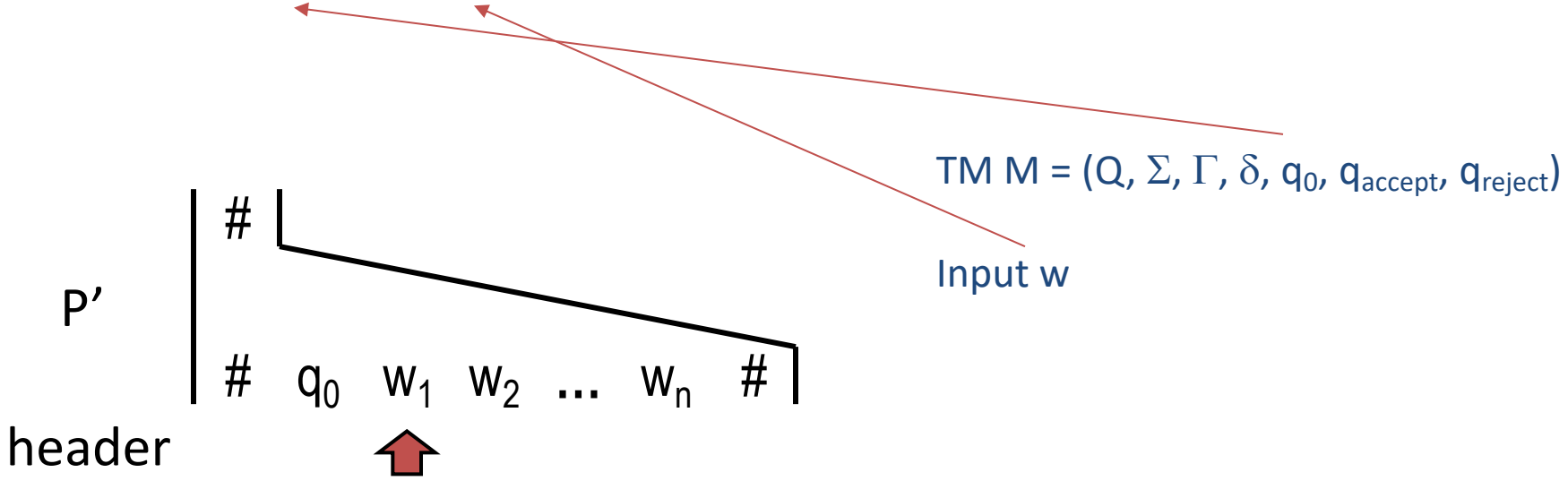
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.1) Generating beginning configuration

  Put $\left[\frac{\#}{\#q_0 w_1 w_2 \ldots w_n \#}\right]$ into P′ as $\left[\frac{t_1}{b_1}\right]$



Theorem 5.15 proof

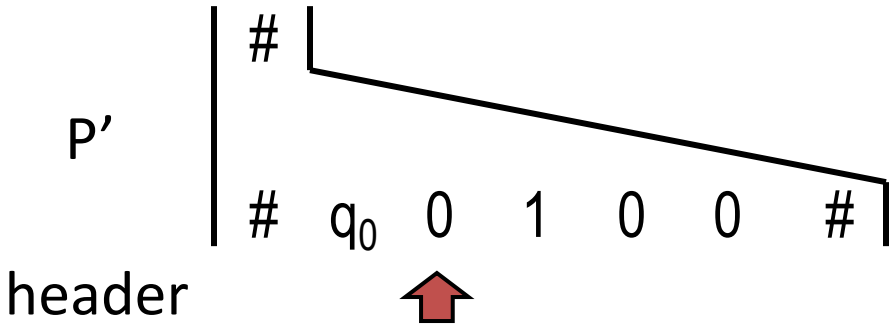
- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.1) Generating beginning configuration

  Suppose Γ={0,1,2, ␣}, w=0100,

  Put the following into P′ : $\left[\frac{\#}{\#q_0 0100\#}\right] = \left[\frac{t_1}{b_1}\right]$



Theorem 5.15 proof

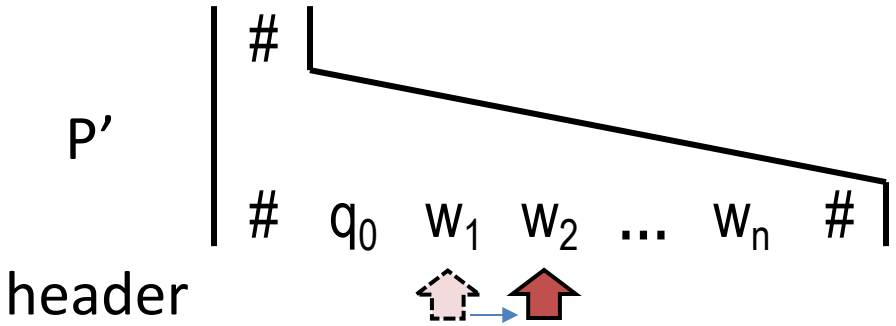
- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.2) The head move to the right

  For each a, b$\in\Gamma$ and q, r$\in$Q, where q $\neq$ q_{reject},

  if **δ(q, a) = (r, b, R),** then put $\left[\frac{qa}{br}\right]$ into P′



Theorem 5.15 proof

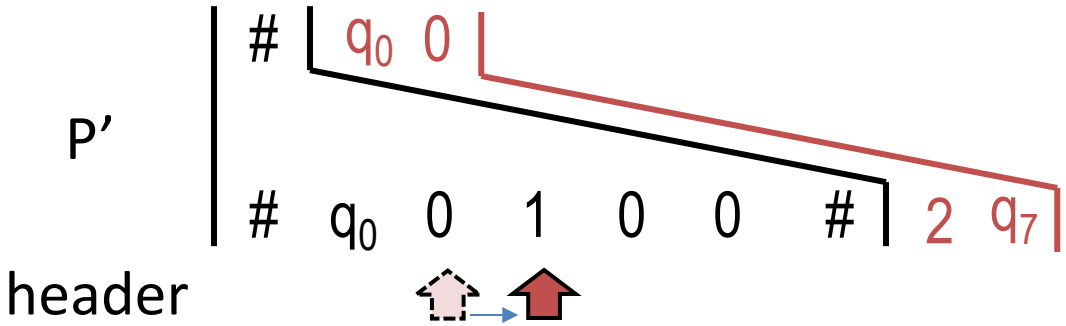
- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.2) The head moves to the right

  For each a, b$\in\Gamma$ and q, r$\in$Q, where q $\neq$ q_{reject},

  if $\delta(q_0, 0) = (q_7, 2, R)$, then put $\left[\frac{q_0 0}{2 q_7}\right]$ into P′



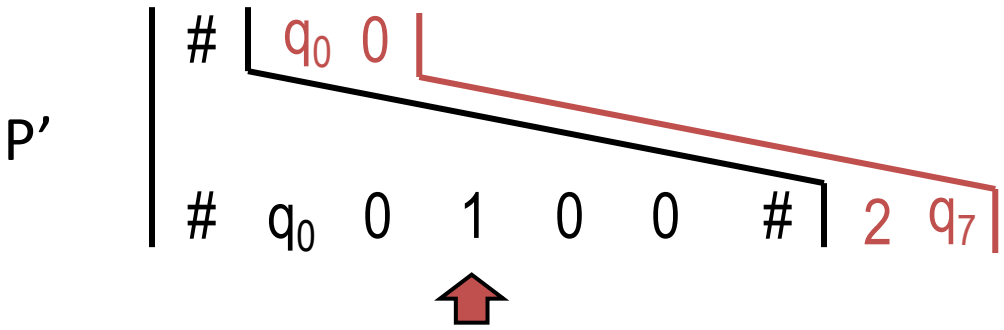
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.3) The head moves to the left

  Discuss it later on.



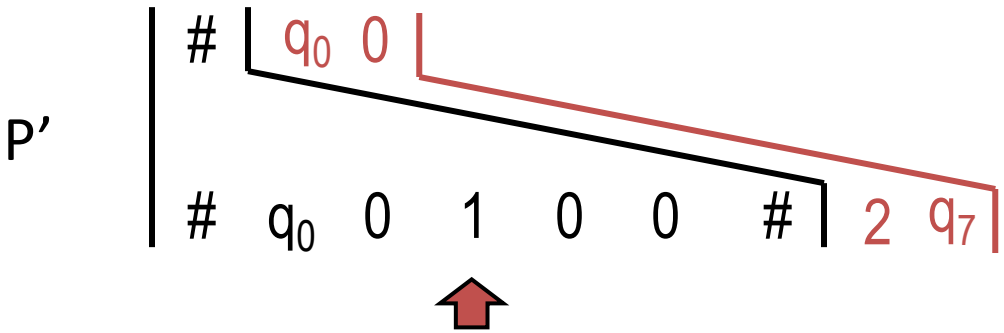
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.4) For each a∈Γ,

  put $\left[\frac{a}{a}\right]$ into P′



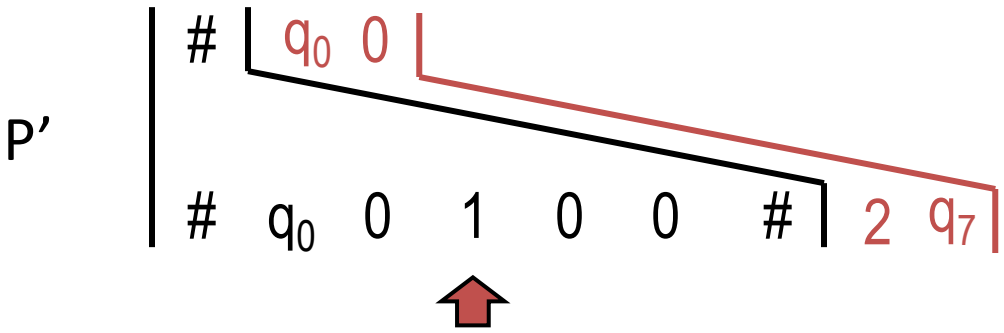
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.4) $\Gamma = \{0, 1, 2, \sqcup\}$,

  put $\left[\frac{0}{0}\right]$, $\left[\frac{1}{1}\right]$, $\left[\frac{2}{2}\right]$ and $\left[\frac{\sqcup}{\sqcup}\right]$ into P′



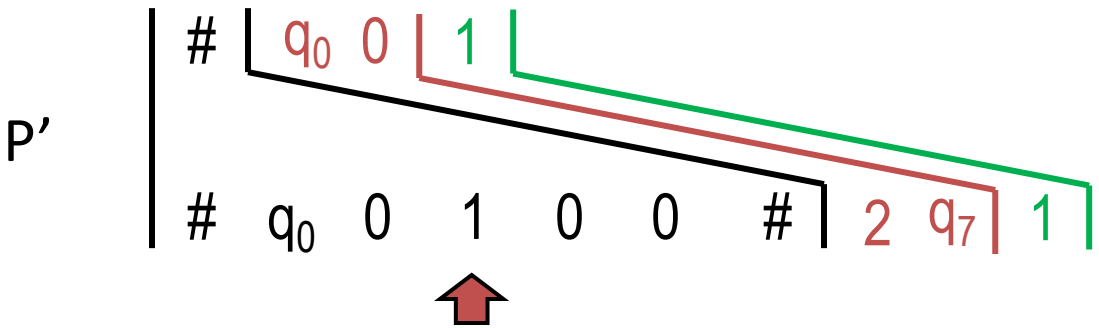
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.4) $\Gamma = \{0, 1, 2, \sqcup\}$,

  put $\left[\frac{0}{0}\right]$, $\left[\frac{1}{1}\right]$, $\left[\frac{2}{2}\right]$ and $\left[\frac{\sqcup}{\sqcup}\right]$ into P′



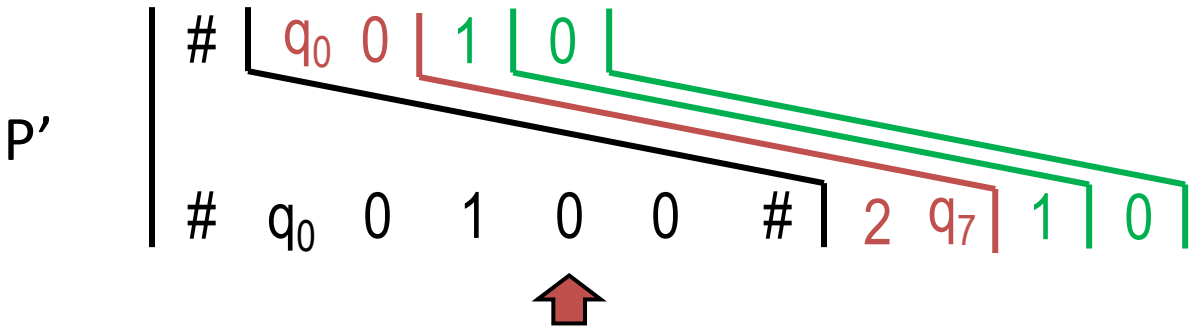
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.4) $\Gamma = \{0, 1, 2, \sqcup\}$,

  put $\left[\frac{0}{0}\right]$, $\left[\frac{1}{1}\right]$, $\left[\frac{2}{2}\right]$ and $\left[\frac{\sqcup}{\sqcup}\right]$ into P′



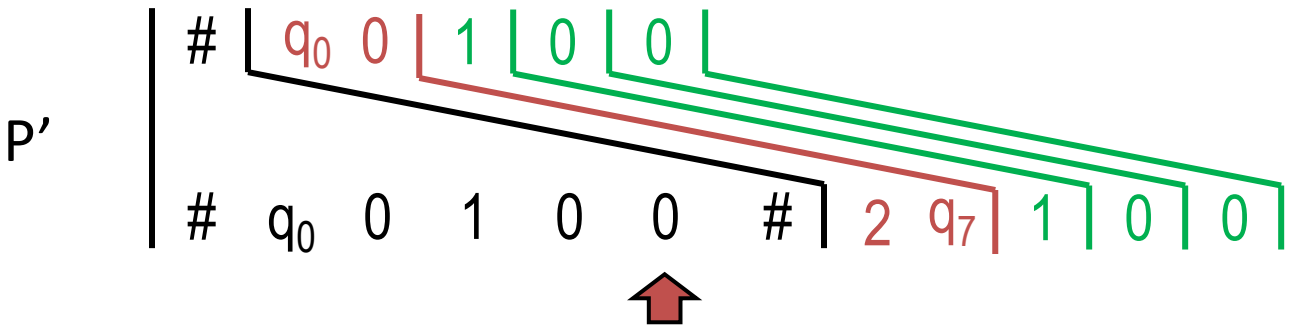
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.4) $\Gamma = \{0, 1, 2, \sqcup\}$,

  put $\left[\frac{0}{0}\right]$, $\left[\frac{1}{1}\right]$, $\left[\frac{2}{2}\right]$ and $\left[\frac{\sqcup}{\sqcup}\right]$ into P′



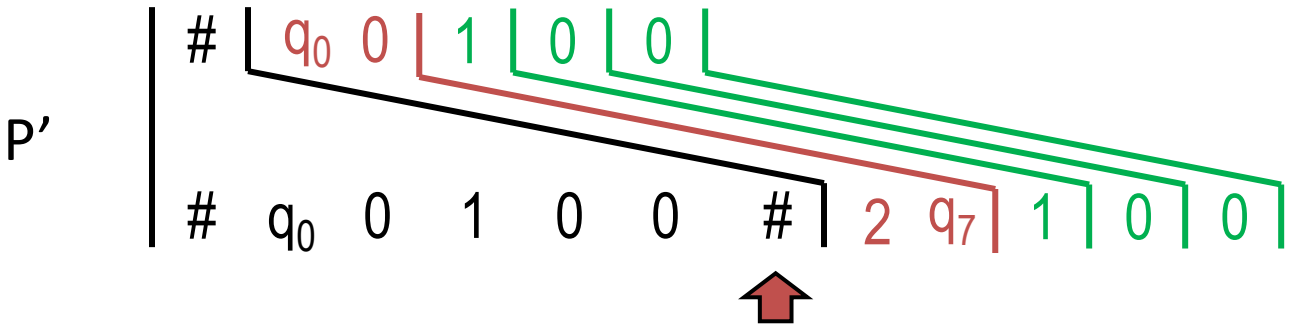
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.5) Copy # and Put ␣ at the end of configuration

  put $\left[\frac{\#}{\#}\right]$ and $\left[\frac{\#}{\sqcup\#}\right]$ into P′



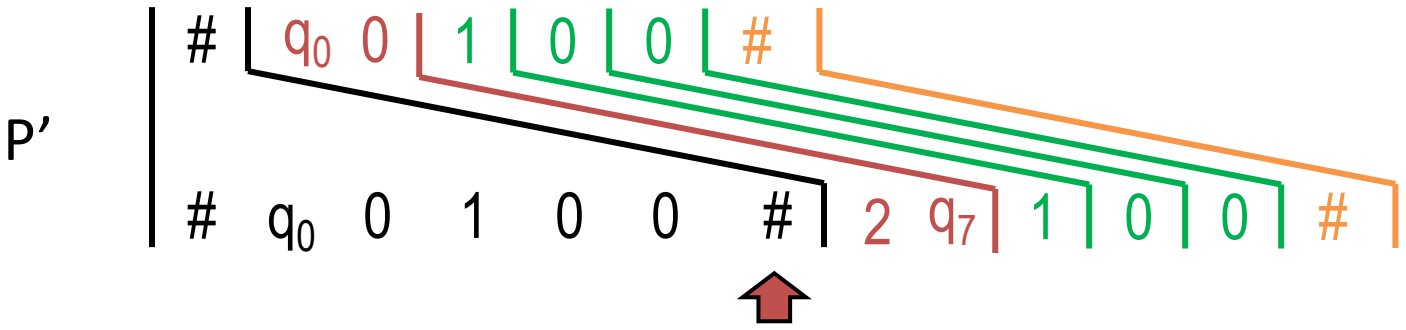
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.5) Copy # and Put ␣ at the end of configuration

  put $\left[\frac{\#}{\#}\right]$ and $\left[\frac{\#}{\text{␣}\#}\right]$ into P′



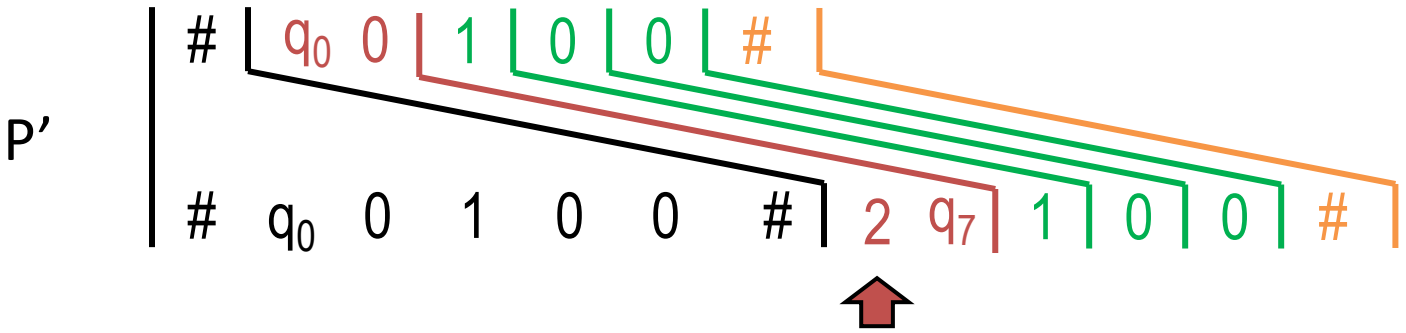
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.5) Copy # and Put ␣ at the end of configuration

  put $\left[\frac{\#}{\#}\right]$ and $\left[\frac{\#}{\text{␣}\#}\right]$ into P′



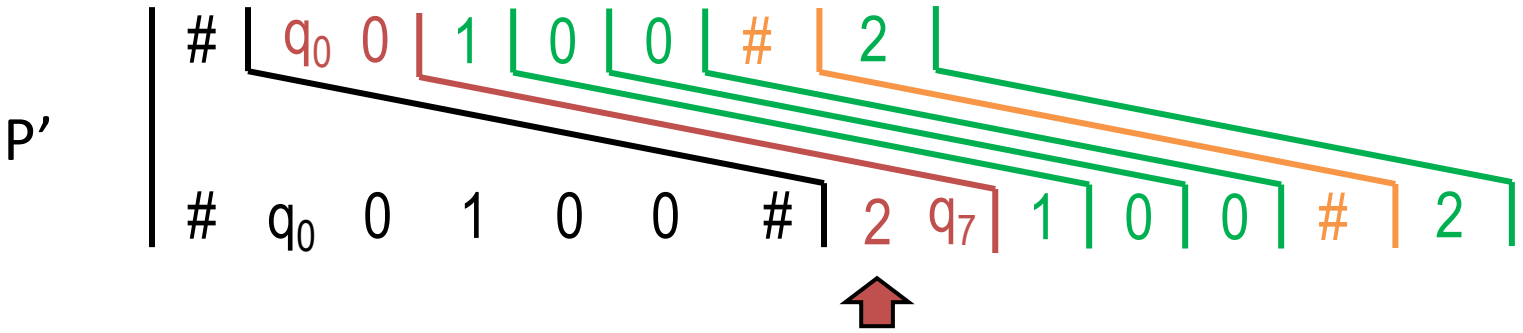
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.5) Copy # and Put ␣ at the end of configuration

  put $\left[\frac{\#}{\#}\right]$ and $\left[\frac{\#}{\sqcup\#}\right]$ into P′



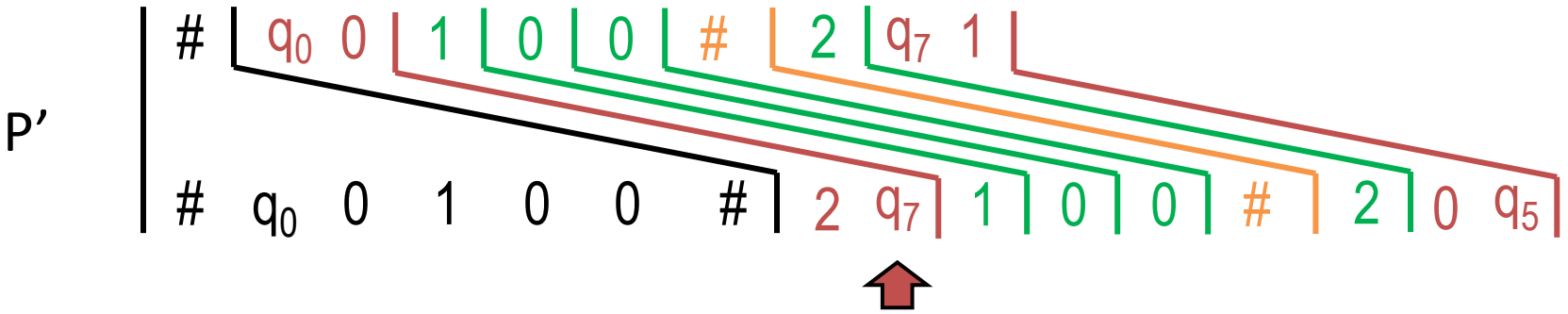
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.5) Copy # and Put ␣ at the end of configuration

  suppose $\delta(q_7, 1) = (q_5, 0, R)$, then put $\left[\frac{q_7 1}{0 q_5}\right]$ into P′



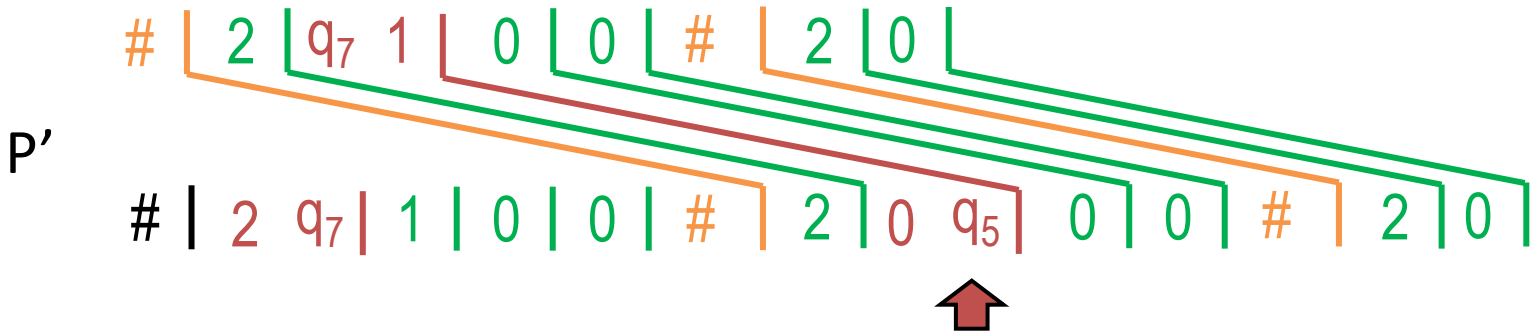
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.5) Copy # and Put ␣ at the end of configuration

  keep putting $\left[\frac{0}{0}\right]$, $\left[\frac{1}{1}\right]$, $\left[\frac{2}{2}\right]$, $\left[\frac{␣}{␣}\right]$, $\left[\frac{\#}{\#}\right]$ and $\left[\frac{\#}{␣\#}\right]$ into P′



Theorem 5.15 proof

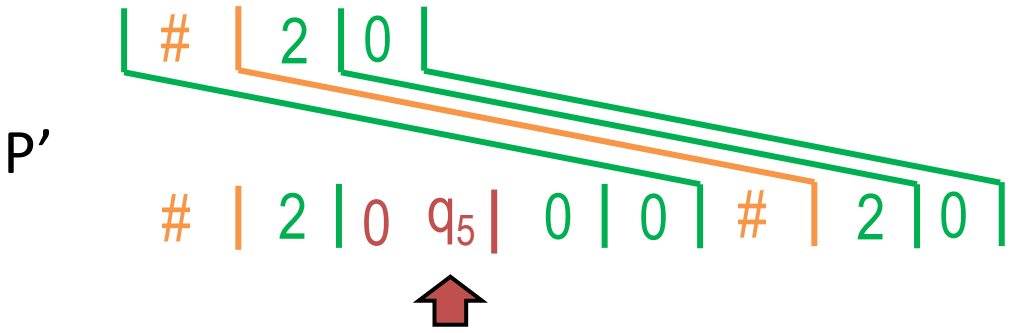
- Proof:

  (1) S first constructs an instance P' of the MPCP;

  (1.3) The head moves to the left

  For each a, b, c$\in\Gamma$ and q, r$\in$Q, where q $\neq$ q_{reject},

  if **δ(q, a) = (r, b, L),** then put $\left[\frac{cqa}{rcb}\right]$ into P' , c is the element on the left of q



Theorem 5.15 proof

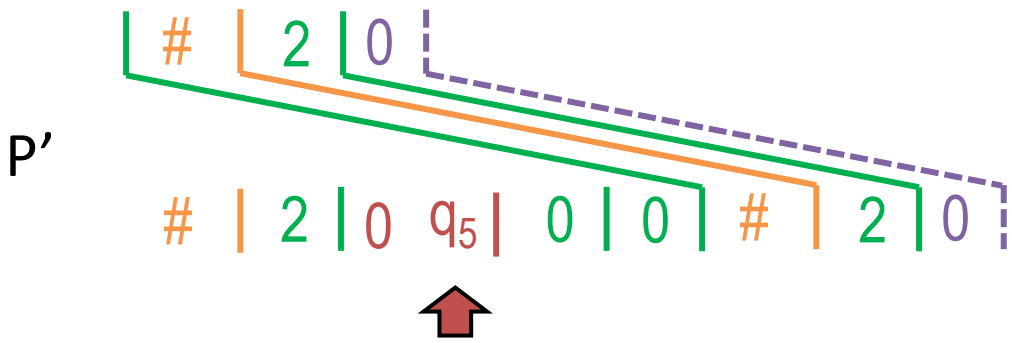
- Proof:

(1) S first constructs an instance P′ of the MPCP;

(1.3) The head moves to the left

Suppose $\delta(q_5, 0) = (q_9, 2, L)$, remove the old 0

if $\delta(q, a) = (r, b, L)$, then put $\left[\frac{cqa}{rcb}\right]$ into P′ , c is the element on the left of q



Theorem 5.15 proof

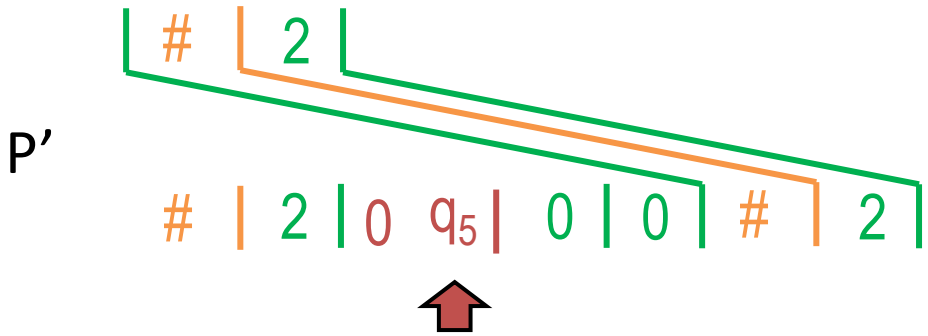
- Proof:

(1) S first constructs an instance P′ of the MPCP;

(1.3) The head moves to the left

Suppose $\delta(q_5, 0) = (q_9, 2, L)$, remove the old 0

if $\delta(q, a) = (r, b, L)$, then put $\left[\frac{cqa}{rcb}\right]$ into P′ , c is the element on the left of q



Theorem 5.15 proof

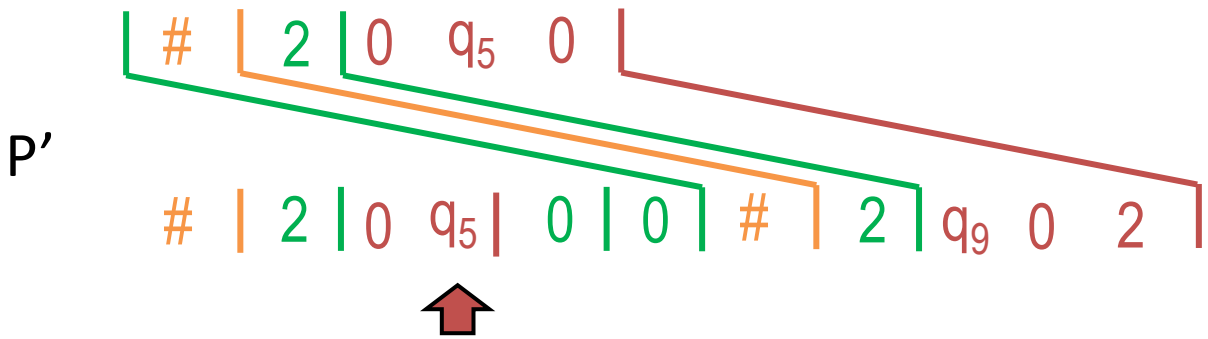
- Proof:

(1) S first constructs an instance P′ of the MPCP;

(1.3) The head moves to the left

Suppose $\delta(q_5, 0) = (q_9, 2, L)$, put $\left[\frac{0q_50}{q_902}\right]$ into P′

if $\delta(q, a) = (r, b, L)$, then put $\left[\frac{cqa}{rcb}\right]$ into P′ , c is the element on the left of q



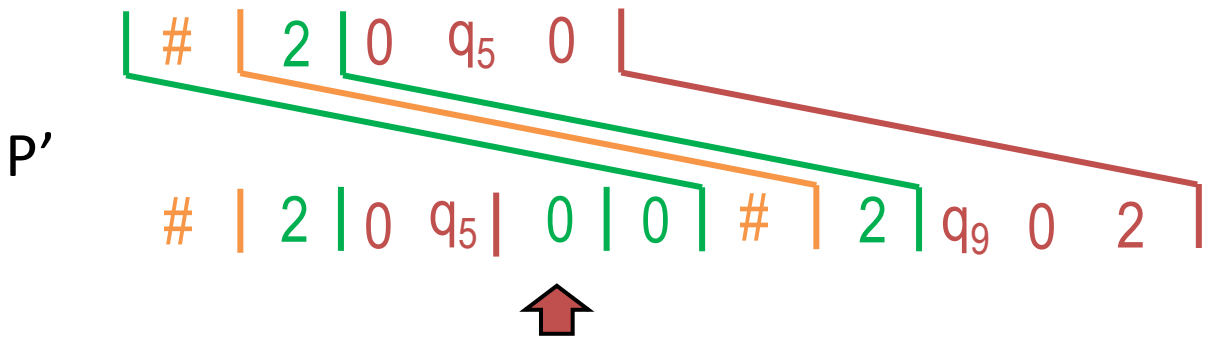
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.3) The head moves to the left

  keep putting $\left[\frac{0}{0}\right]$, $\left[\frac{1}{1}\right]$, $\left[\frac{2}{2}\right]$, $\left[\frac{\sqcup}{\sqcup}\right]$, $\left[\frac{\#}{\#}\right]$ and $\left[\frac{\#}{\sqcup\#}\right]$ into P′



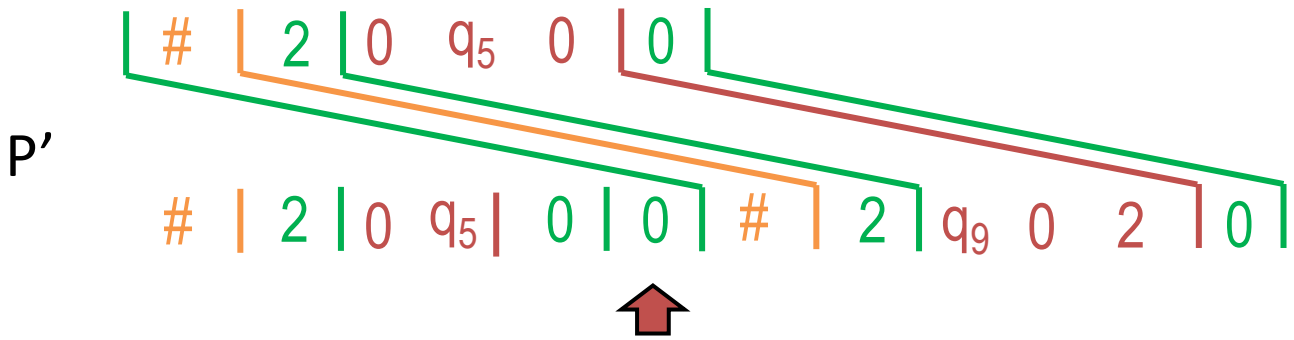
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.3) The head moves to the left

  keep putting $\left[\frac{0}{0}\right]$, $\left[\frac{1}{1}\right]$, $\left[\frac{2}{2}\right]$, $\left[\frac{\sqcup}{\sqcup}\right]$, $\left[\frac{\#}{\#}\right]$ and $\left[\frac{\#}{\sqcup\#}\right]$ into P′



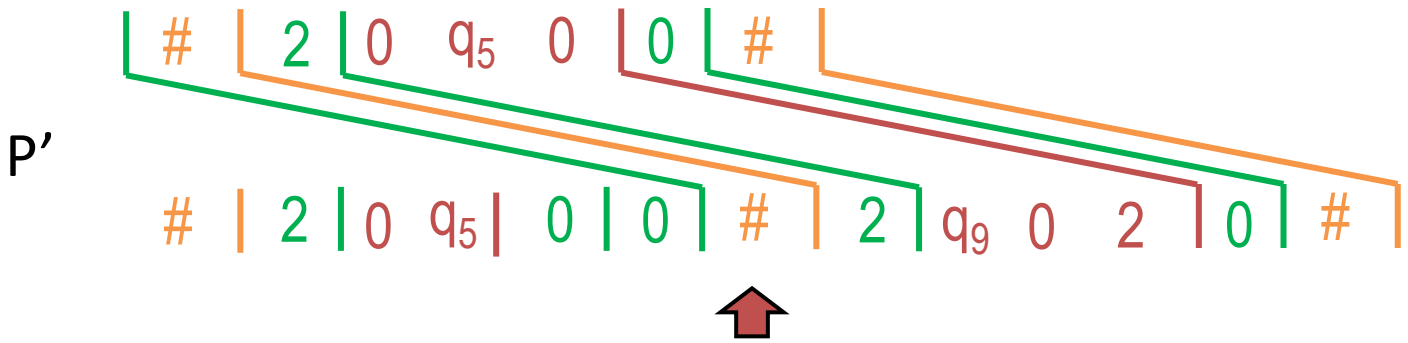
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.3) The head moves to the left

  keep putting $\left[\frac{0}{0}\right]$, $\left[\frac{1}{1}\right]$, $\left[\frac{2}{2}\right]$, $\left[\frac{\sqcup}{\sqcup}\right]$, $\left[\frac{\#}{\#}\right]$ and $\left[\frac{\#}{\sqcup\#}\right]$ into P′



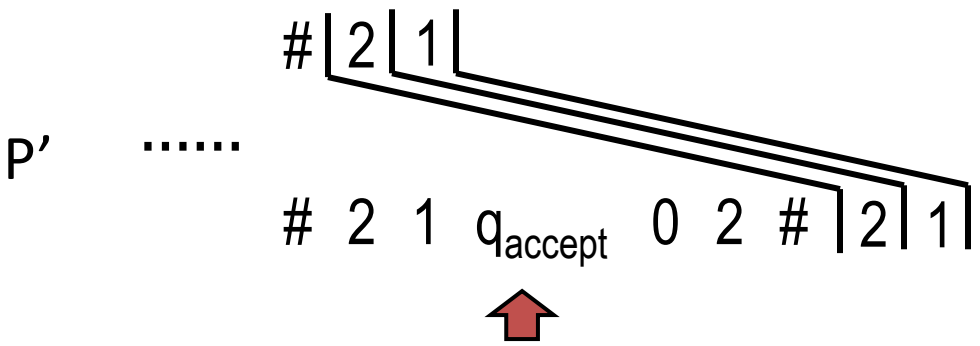
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P' of the MPCP;

  keep putting something into P' until M halts:

  - if M rejects, S also rejects, means no match in PCP;
  - if M accepts, add something to the top to match the bottom.



Connect PCP match program with whether M accepts w

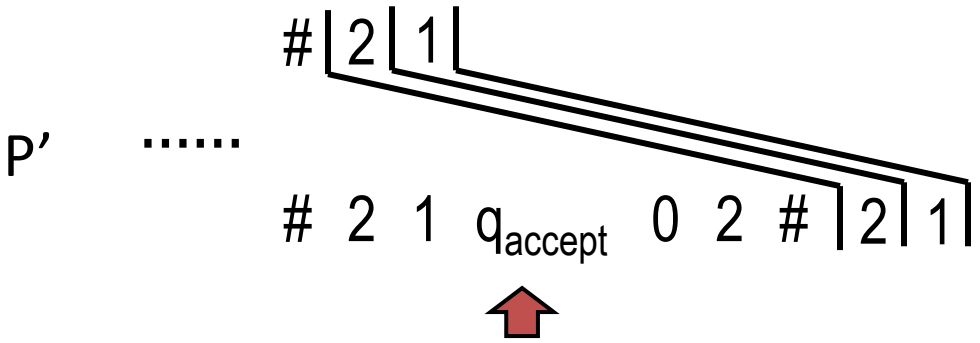
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.6) For a$\in\Gamma$,

  put $\left[\frac{aq_{accept}}{q_{accept}}\right]$ and $\left[\frac{q_{accept}a}{q_{accept}}\right]$ into P′



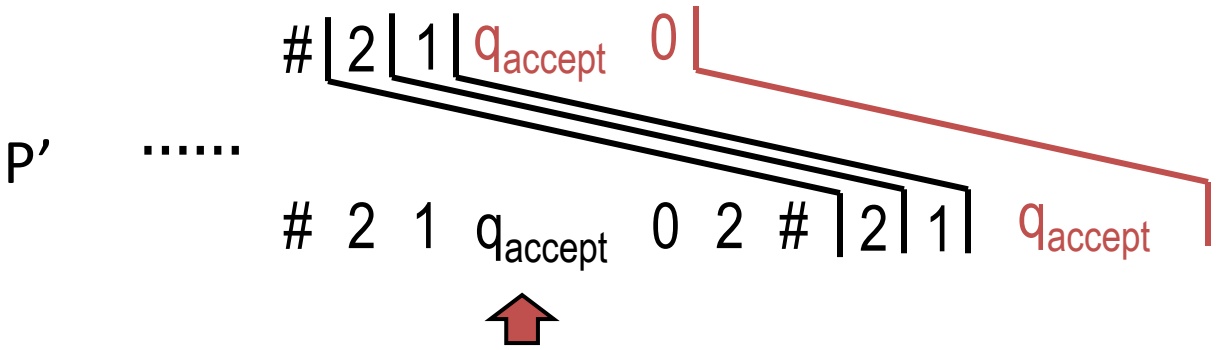
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.6) For a$\in\Gamma$,

  put $\left[\frac{aq_{accept}}{q_{accept}}\right]$ and $\left[\frac{q_{accept}a}{q_{accept}}\right]$ into P′



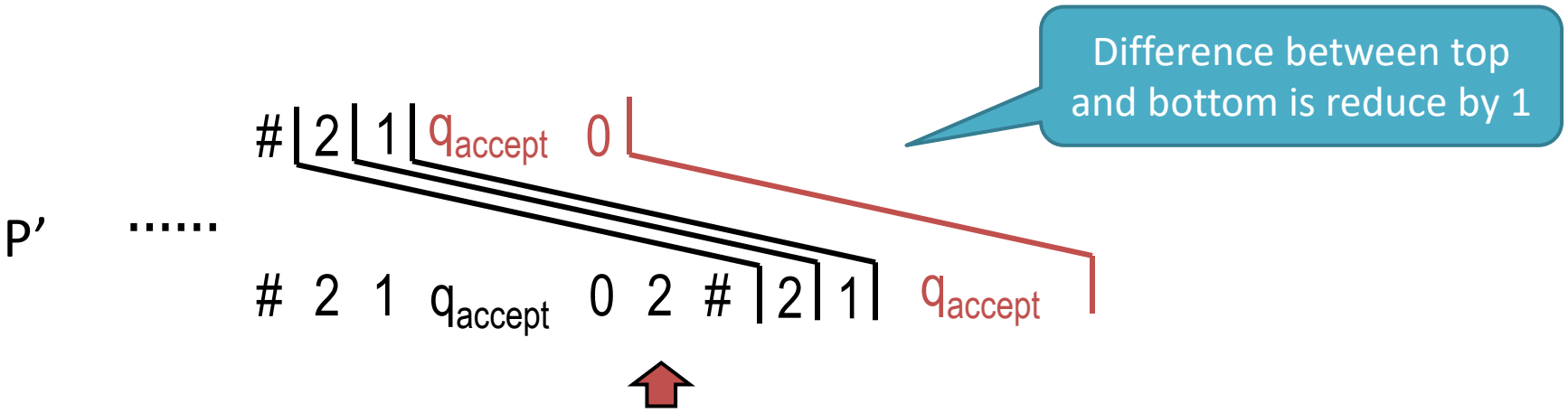
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.6) For a$\in\Gamma$,

  put $\left[\frac{aq_{accept}}{q_{accept}}\right]$ and $\left[\frac{q_{accept}a}{q_{accept}}\right]$ into P′



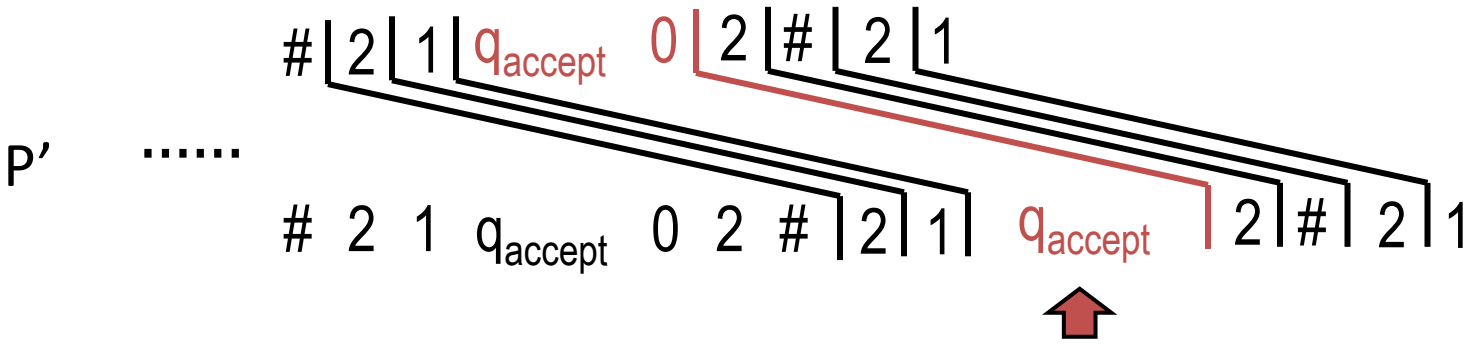
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.6) For a$\in\Gamma$,

  put $\left[\frac{aq_{accept}}{q_{accept}}\right]$ and $\left[\frac{q_{accept}a}{q_{accept}}\right]$ into P′



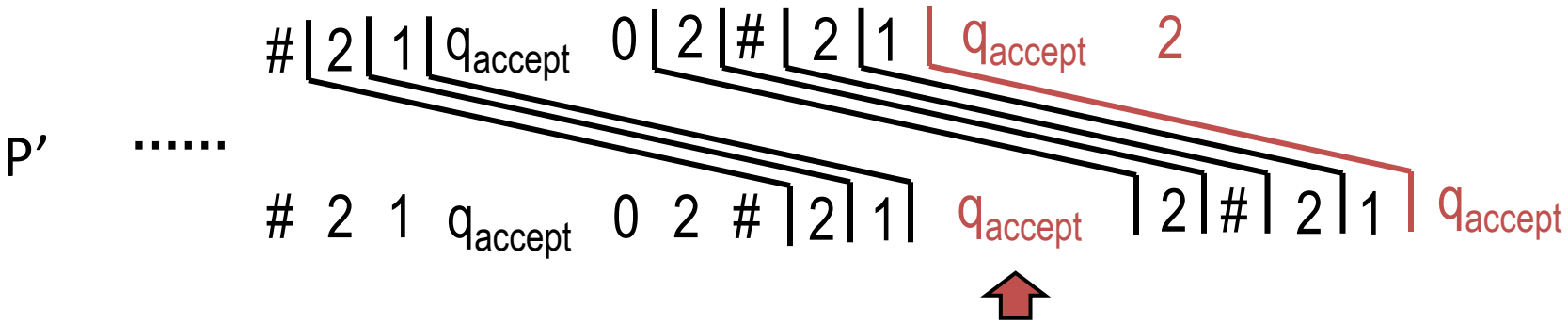
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.6) For $a \in \Gamma$,

  put $\left[\frac{aq_{accept}}{q_{accept}}\right]$ and $\left[\frac{q_{accept}a}{q_{accept}}\right]$ into P′



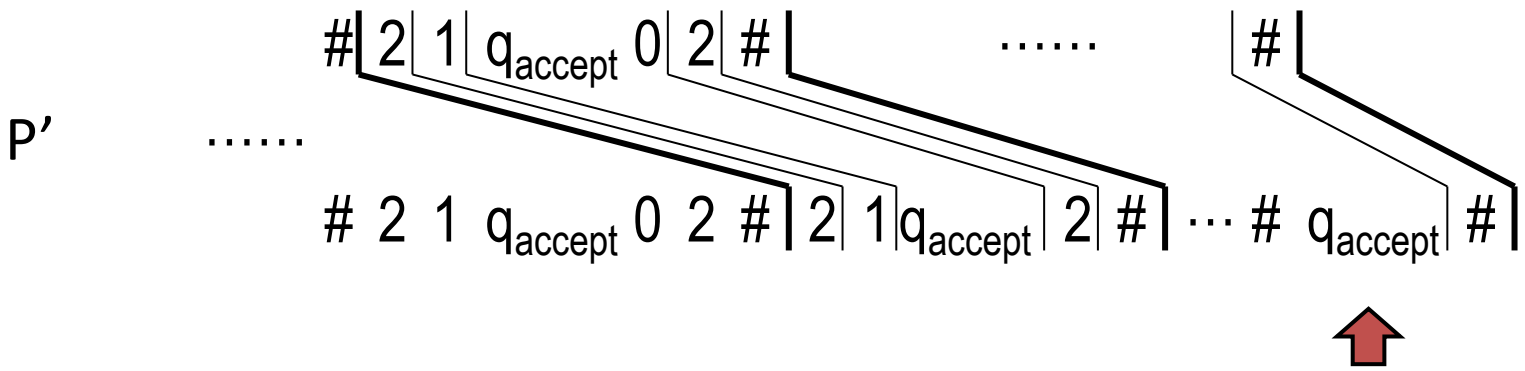
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.6) For $a \in \Gamma$,

  Keep doing until the top and bottom difference is 1.



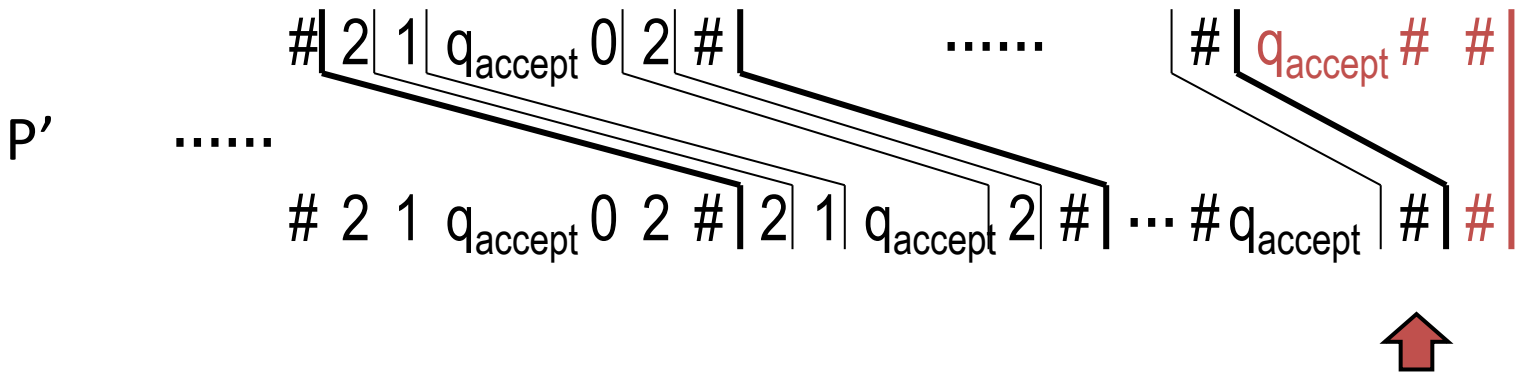
Theorem 5.15 proof

- Proof:

  (1) S first constructs an instance P′ of the MPCP;

  (1.7) Finish the match

  put $\left[\frac{q_{accept}\#\#}{\#}\right]$ into P′



Theorem 5.15 proof

- Proof:

  (2) Transfer P' into P

  P' is MPCP = {⟨P⟩| P is an instance of the PCP with a match that starts with the first domino}.

  Suppose $u=u_1u_2...u_n$ to be any string of length n, define

  - $\star u$ = $*u1*u2*...*u3$
  - $u\star$ = $u_1*u_2*...*u_3*$
  - $\star u\star$ = $*u_1*u_2*...*u_3*$

Theorem 5.15 proof

$\star u = *u1*u2*\ldots*u3$

$u\star = u_1*u_2*\ldots*u_3*$

$\star u\star = *u_1*u_2*\ldots*u_3*$

- Proof:

  (2) Transfer P' into P

  If P' = $\left\{\left[\frac{t_1}{b_1}\right], \left[\frac{t_2}{b_2}\right], \ldots, \left[\frac{t_k}{b_k}\right]\right\}$, then let

  P = $\left\{\left[\frac{\star t_1}{\star b_1 \star}\right], \left[\frac{\star t_2}{b_2 \star}\right], \left[\frac{\star t_3}{b_3 \star}\right], \ldots, \left[\frac{\star t_k}{b_k \star}\right], \left[\frac{*\Delta}{\Delta}\right]\right\}$

  First domino: The only element as beginning

  The match in P must be in shape as $\left[\frac{\star t_1}{\star b_1 \star}\right] \ldots \left[\frac{*\Delta}{\Delta}\right]$

Theorem 5.15 proof

- Proof:

  Suppose PCP is decidable

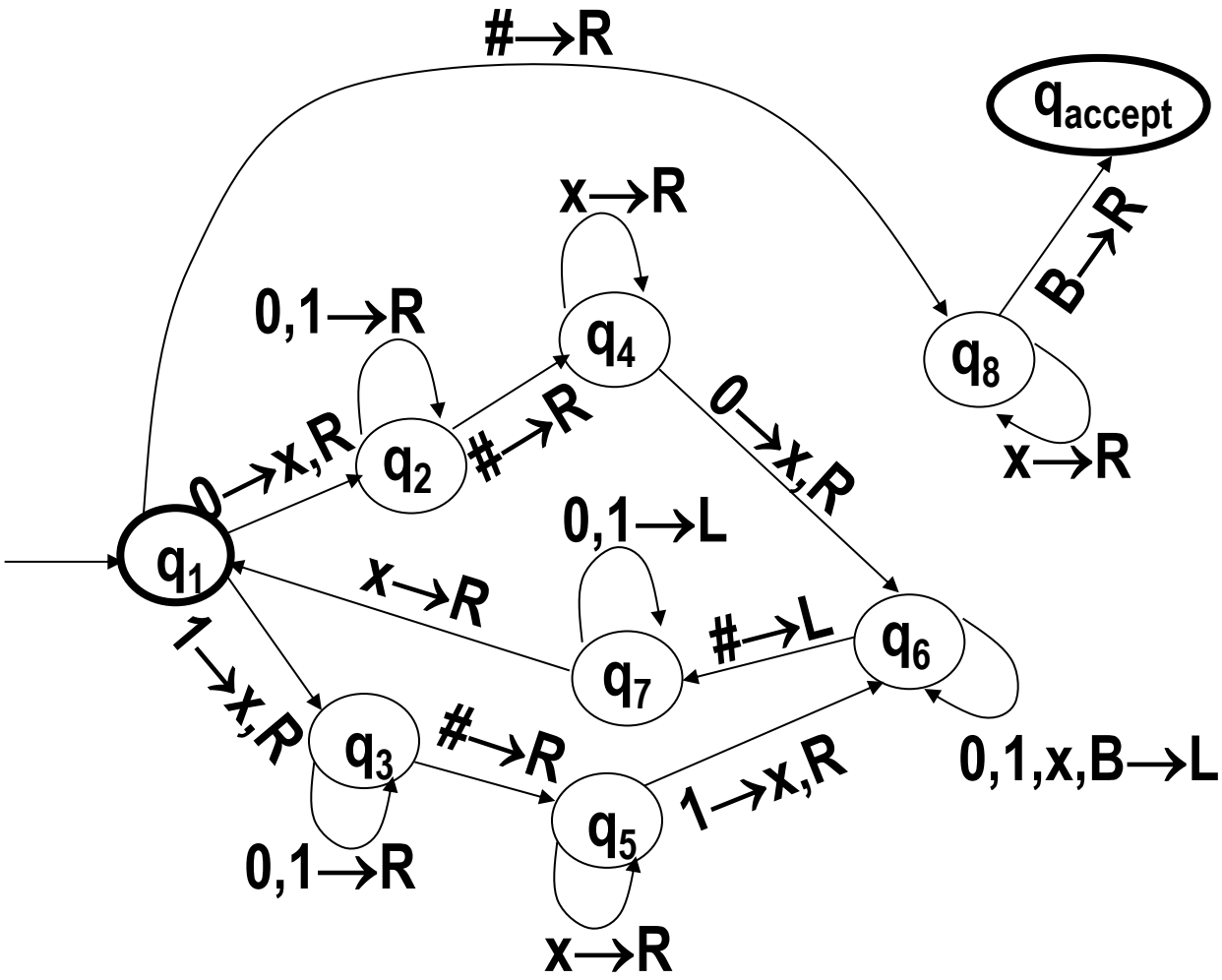
  We construct TM S using PCP match or not to decide A_{TM}

  - (1) S first constructs an instance P' of the MPCP;
  - (2) Transfer P' into P;

  Theorem 4.11: A_{TM} is undecidable. Contradiction!

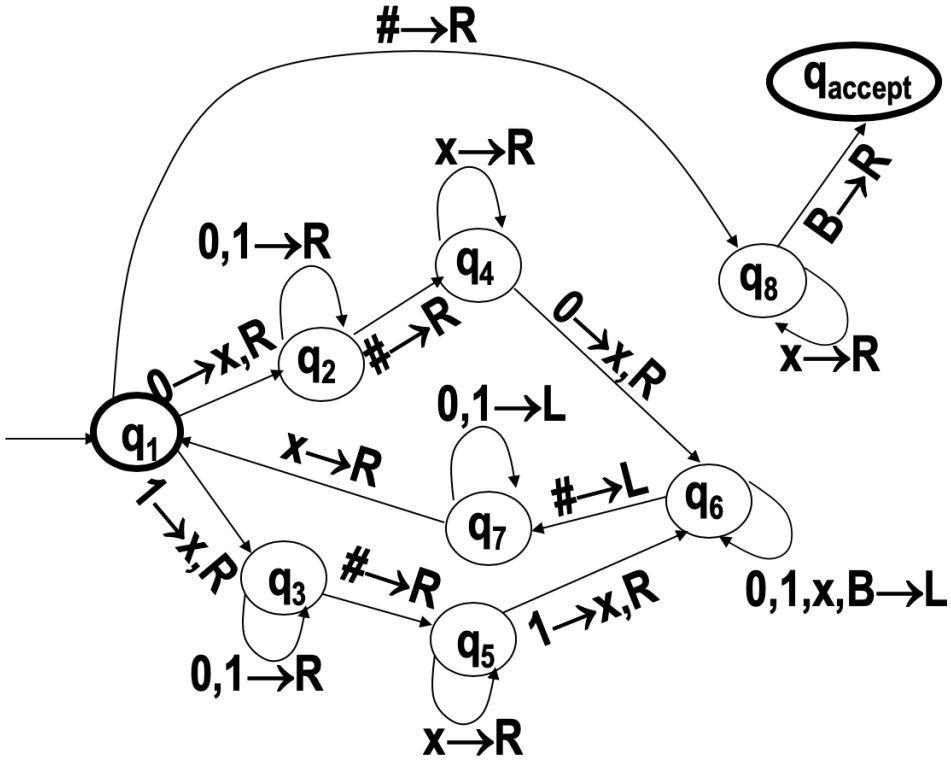
  The suppose is wrong. Thus PCP is undecidable.

Example of TM: B={ w#w | w∈{0,1}* }

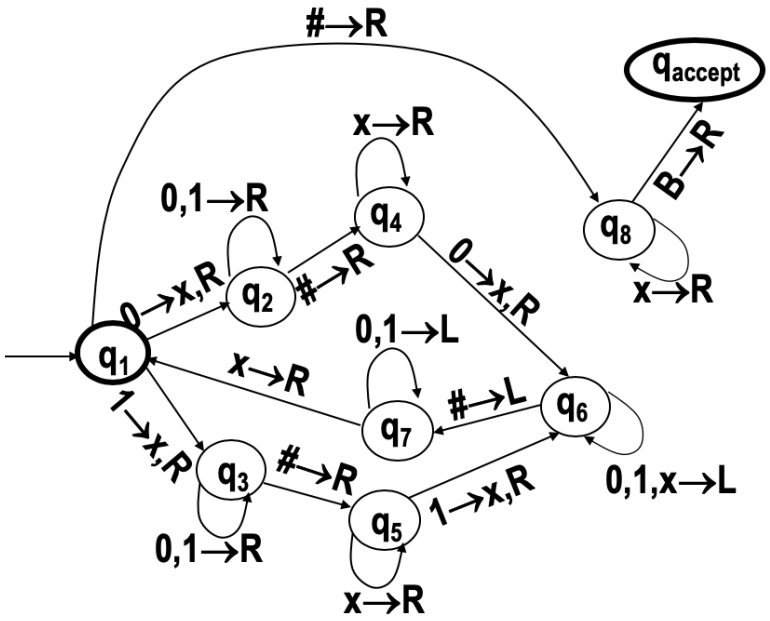


Example of PCP to A_{TM}

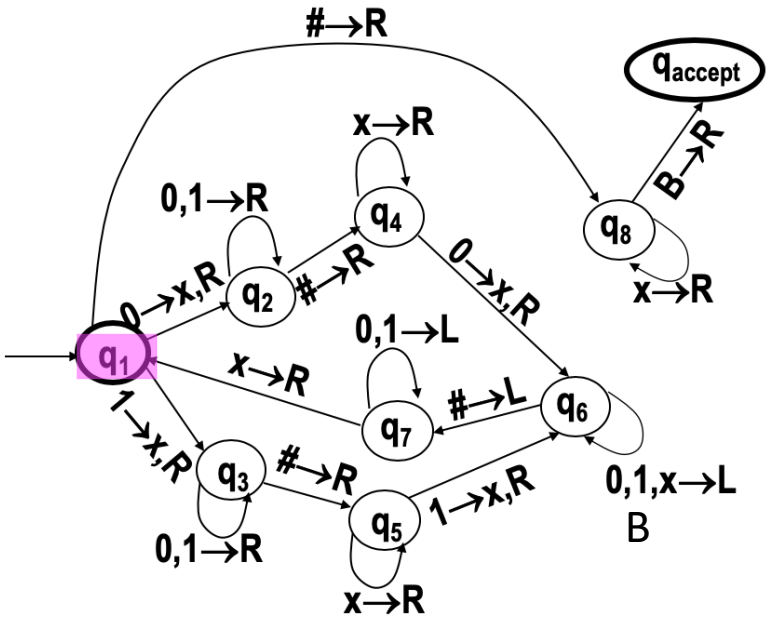
- $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$
  - Q= {q0, q1, q2}
  - Σ = {0, 1, x, #},
  - Γ = {0, 1, x, #, B}
  - q0={q1}
  - $q_{accept} = q_{accept}$
  - q_{reject} = all not q_{accept}



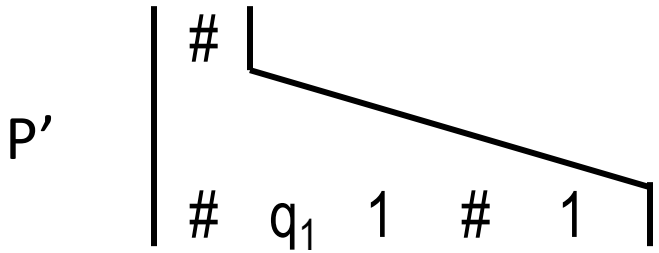
Example of PCP to A_{TM} , input is 1#1



Example of PCP to A_{TM} , input is 1#1

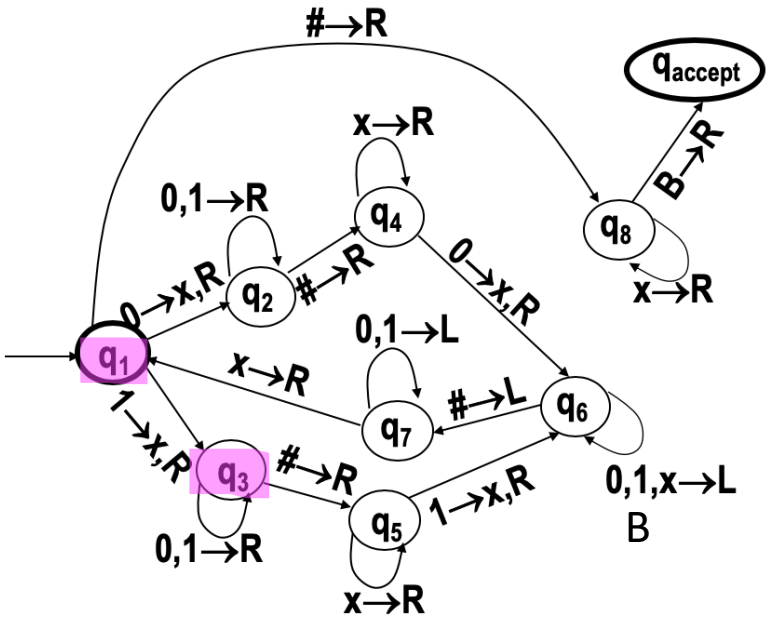


Put $\left[\frac{\#}{\#q_0 w_1 w_2 \ldots w_n \#}\right]$ into P' as $\left[\frac{t_1}{b_1}\right]$



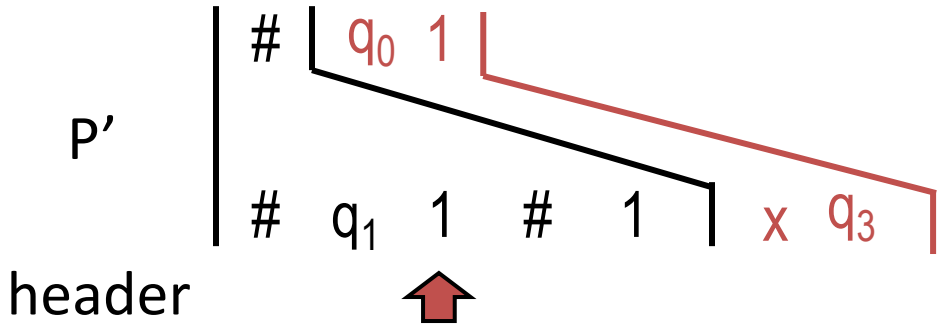
header

Example of PCP to A_{TM} , input is x#1

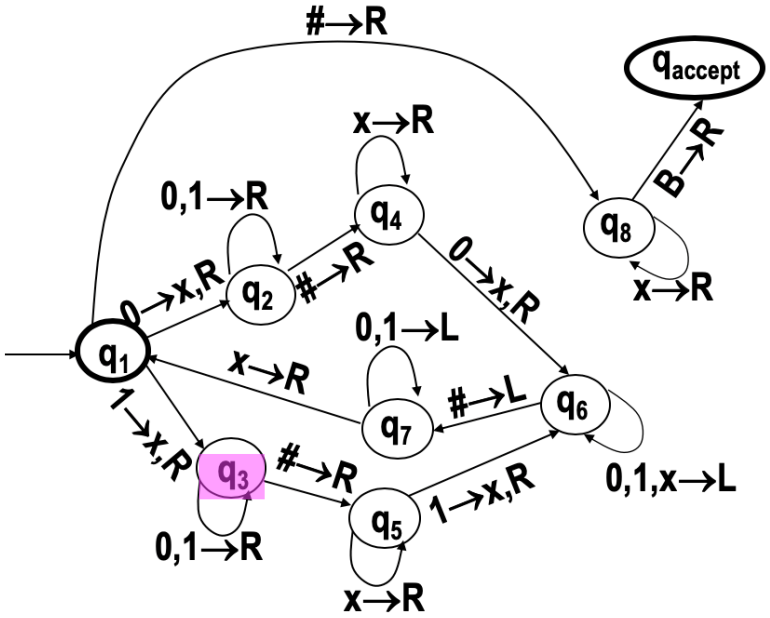


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

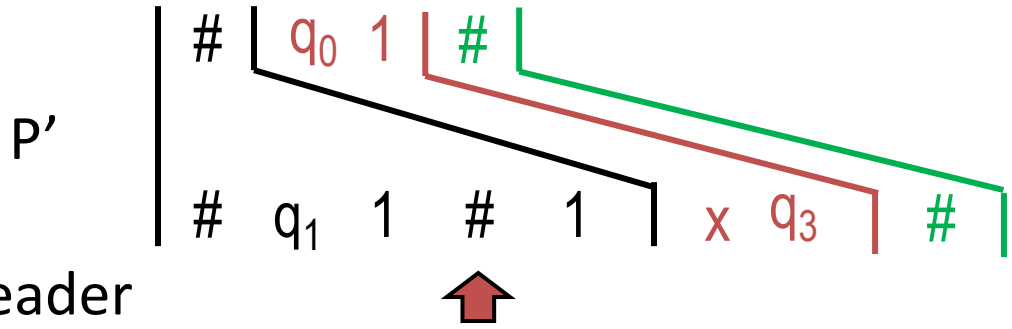


Example of PCP to A_{TM} , input is x#1

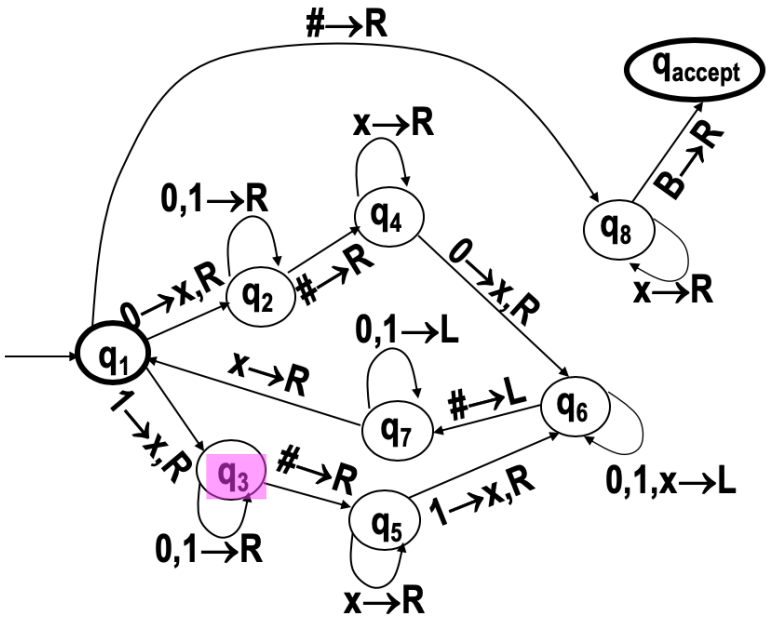


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

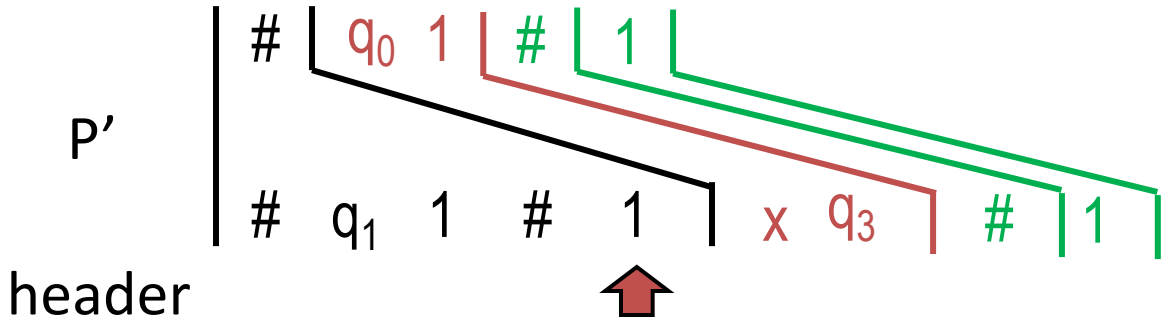


Example of PCP to A_{TM} , input is x#1

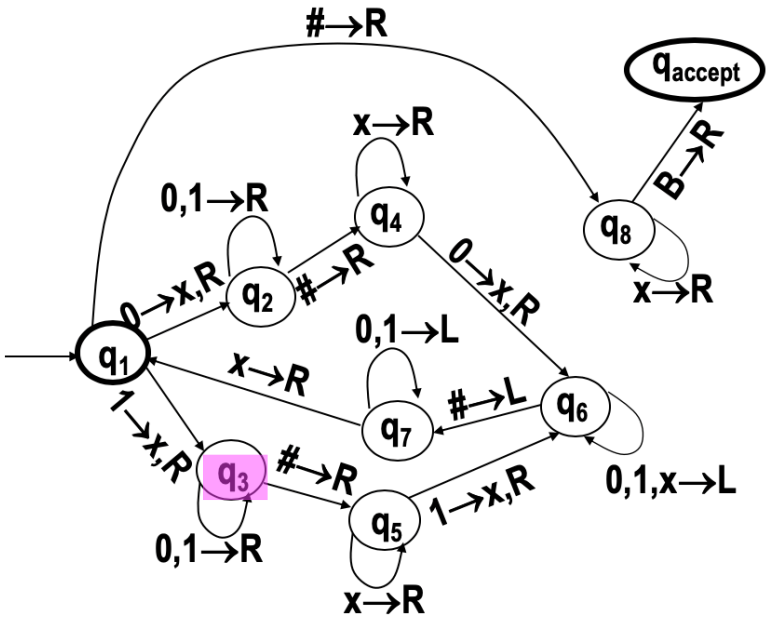


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

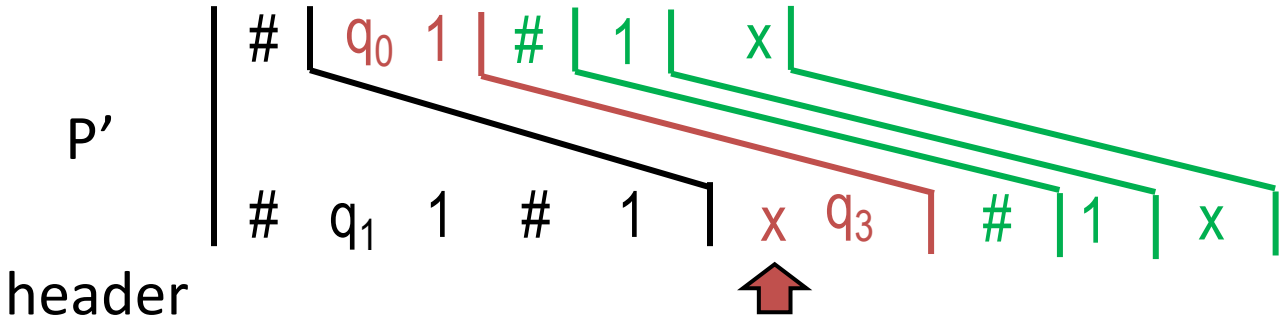


Example of PCP to A_{TM} , input is x#1

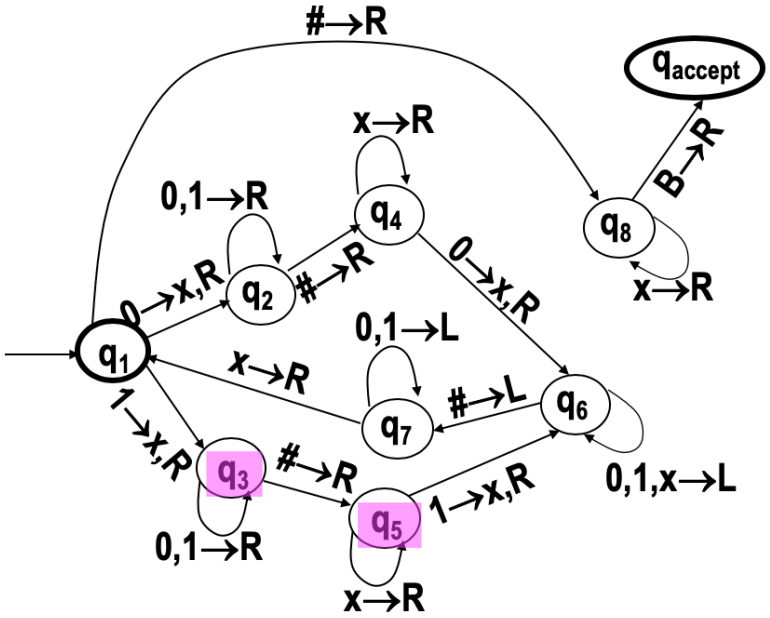


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

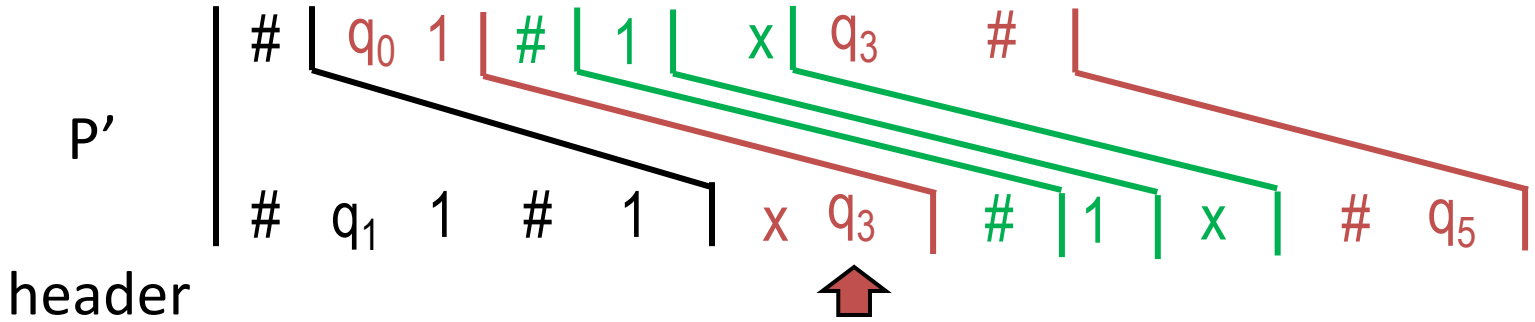


Example of PCP to A_{TM} , input is x#1

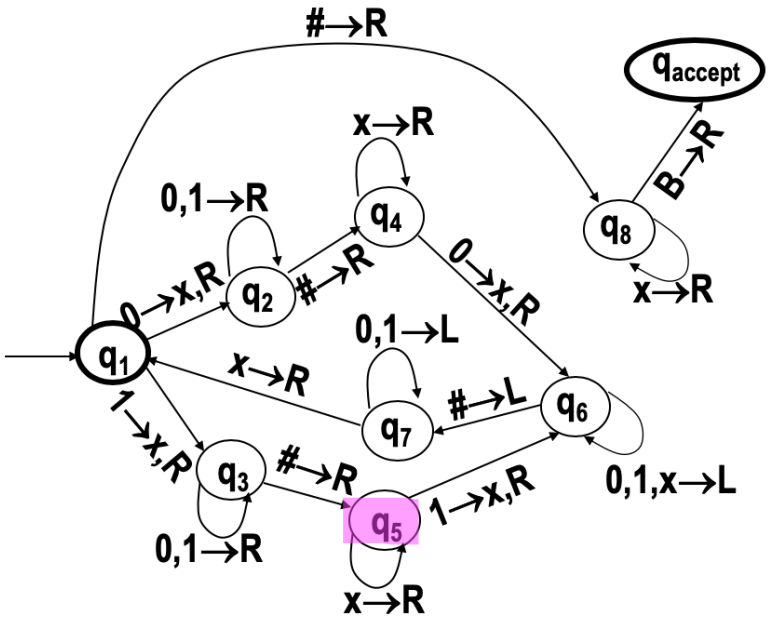


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

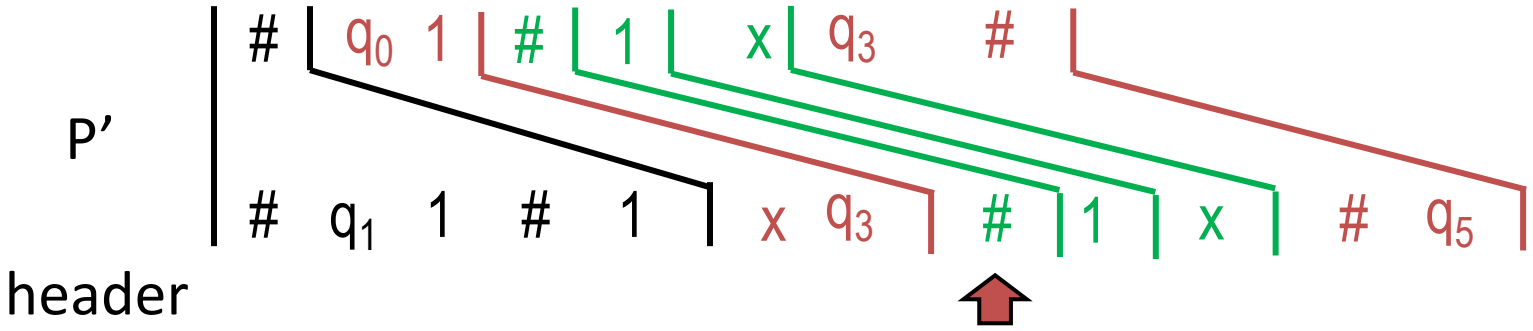


Example of PCP to A_{TM} , input is x#1

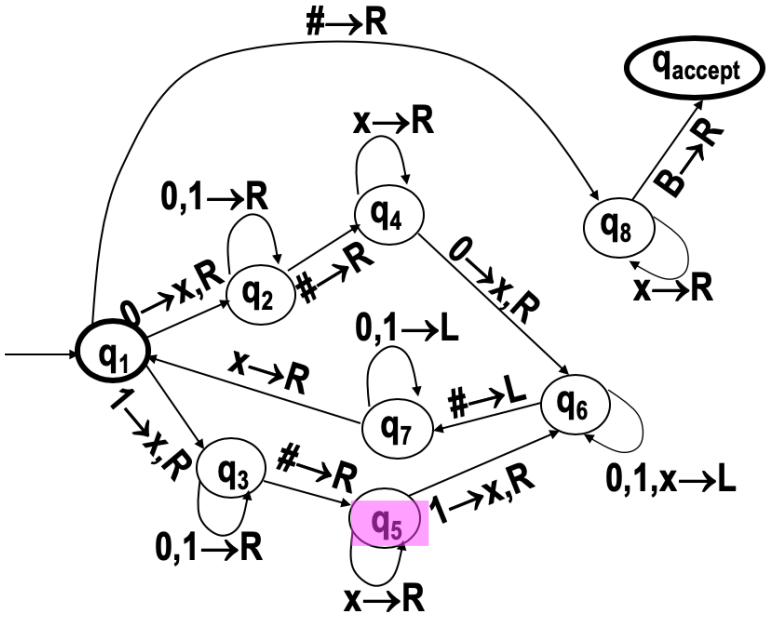


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'



Example of PCP to A_{TM} , input is x#1



if $\delta(q, a) = (r, b, R)$,

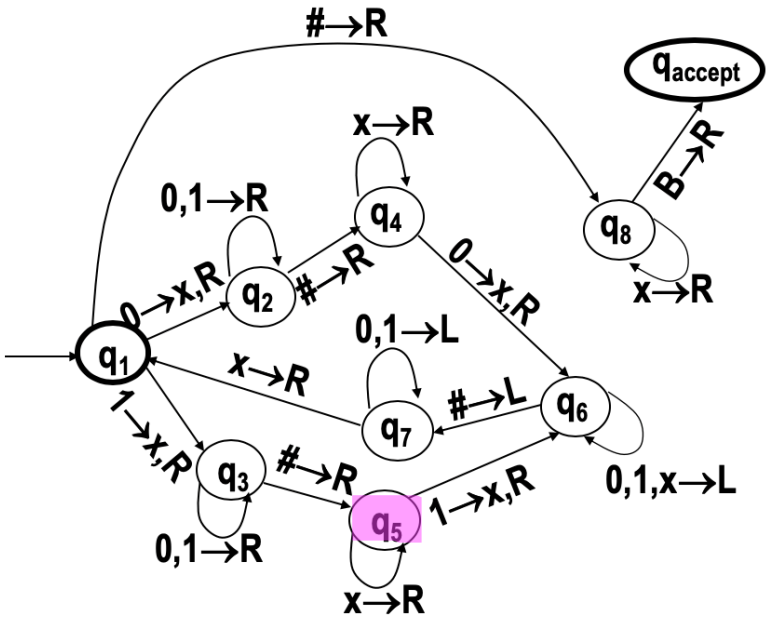
then put $\left[\frac{qa}{br}\right]$ into P'

P'

#	q_0	1	#	1	x	q_3	#	1						
#	q_1	1	#	1	x	q_3	#	1	x	#	q_5	1		

header

Example of PCP to A_{TM} , input is x#1



if $\delta(q, a) = (r, b, R)$,

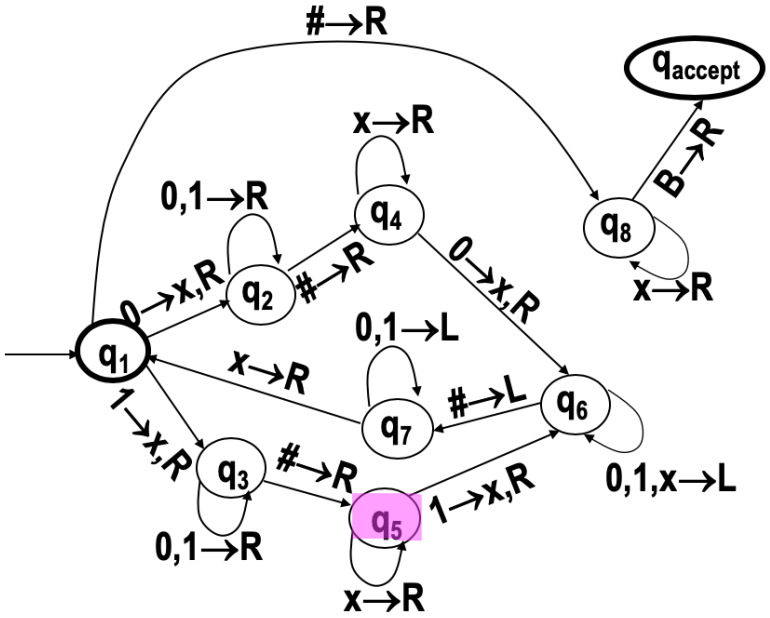
then put $\left[\frac{qa}{br}\right]$ into P'

P'

#	q_0	1	#	1	x	q_3	#	1						
#	q_1	1	#	1	x	q_3	#	1	x	#	q_5	1		

header

Example of PCP to A_{TM} , input is x#1



if $\delta(q, a) = (r, b, R)$,

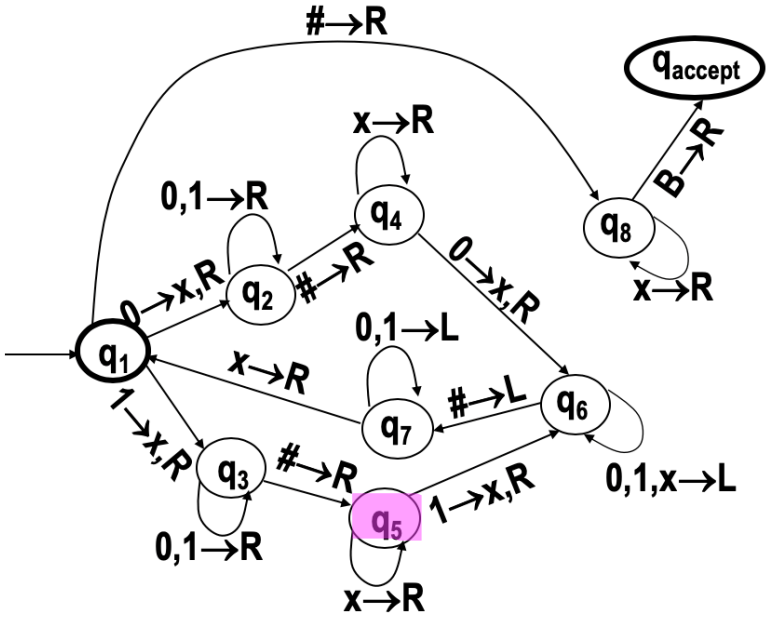
then put $\left[\frac{qa}{br}\right]$ into P'

P'

#	q_0	1	#	1	x	q_3	#	1	x					
#	q_1	1	#	1	x	q_3	#	1	x	#	q_5	1	x	

header

Example of PCP to A_{TM} , input is x#1



if $\delta(q, a) = (r, b, R)$,

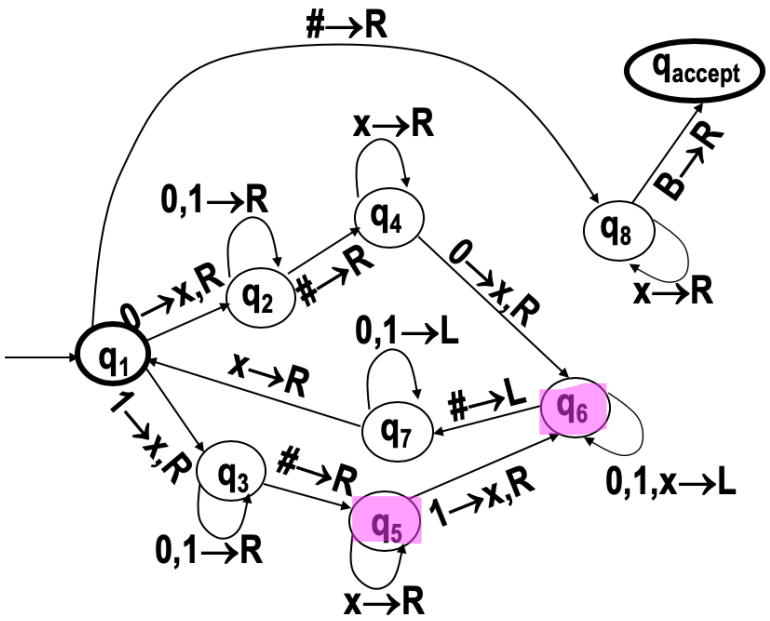
then put $\left[\frac{qa}{br}\right]$ into P'

P'

header

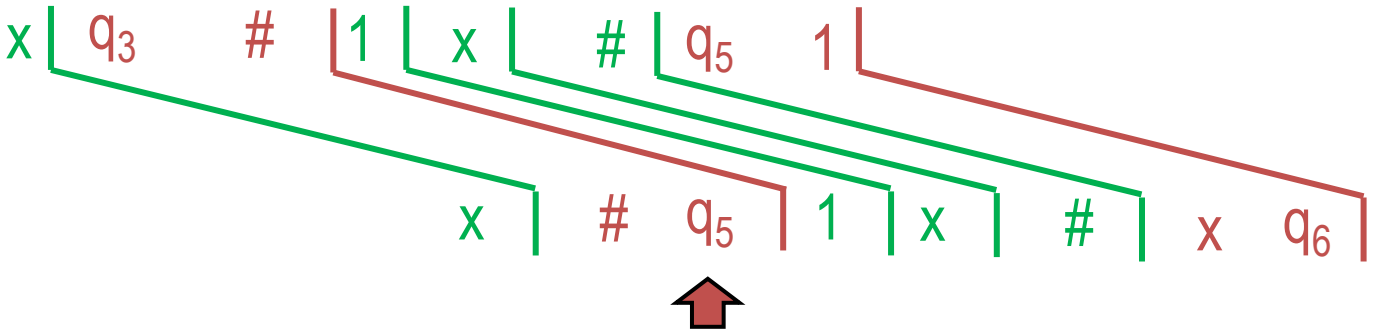
#	q_0	1	#	1	x	q_3	#	1	x	#						
#	q_1	1	#	1	x	q_3	#	1	x	#	q_5	1	x	#		

Example of PCP to A_{TM} , input is x#x

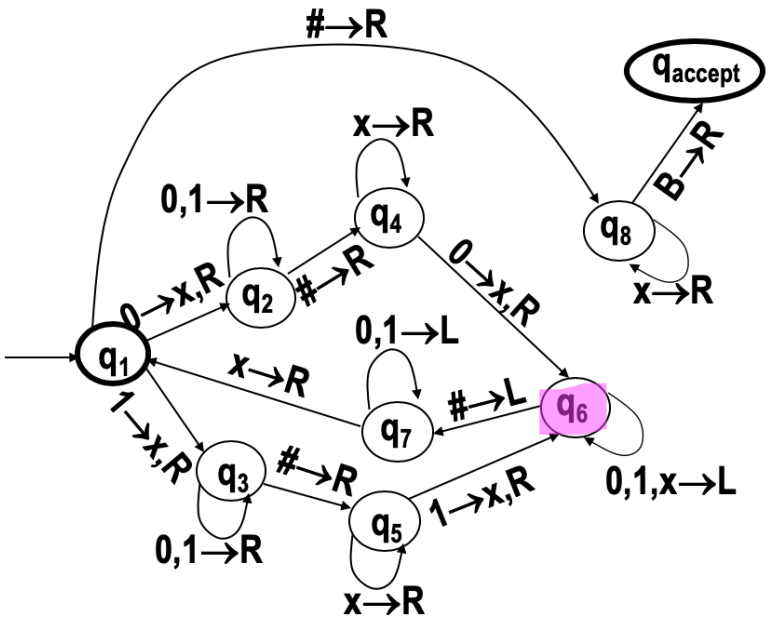


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

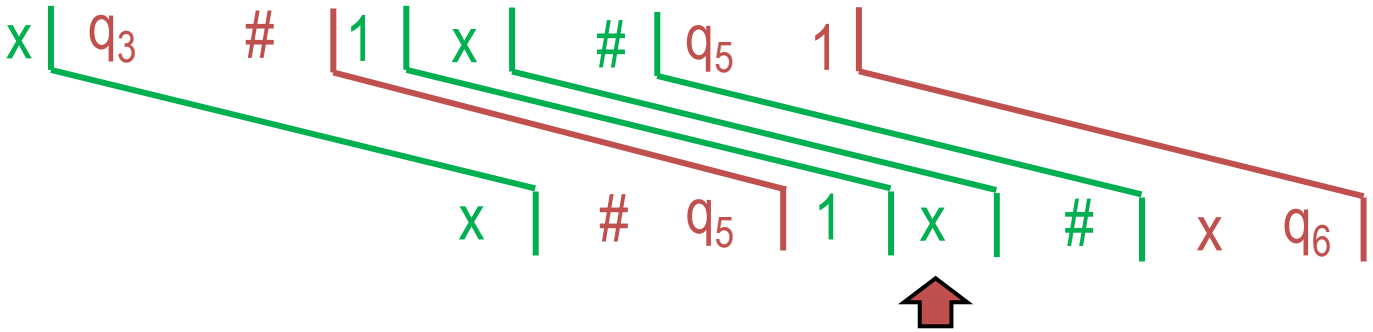


Example of PCP to A_{TM} , input is x#x

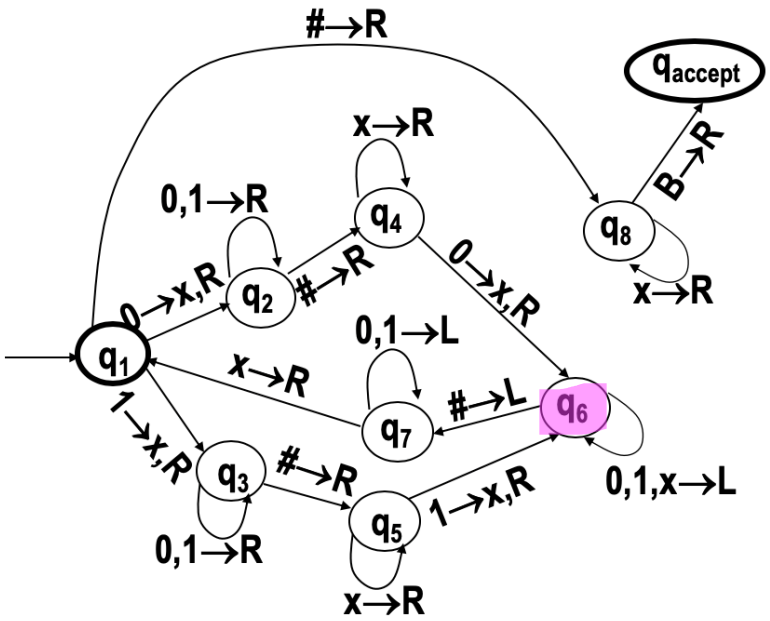


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

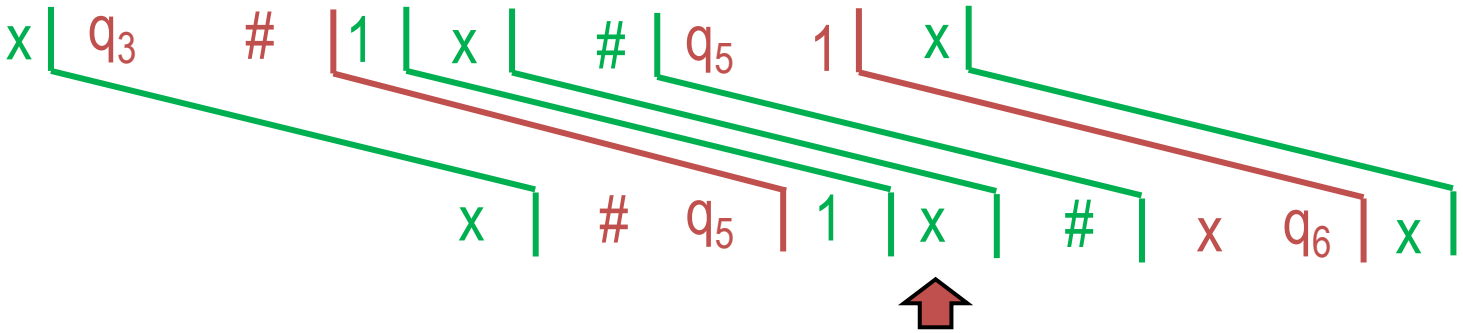


Example of PCP to A_{TM} , input is x#x

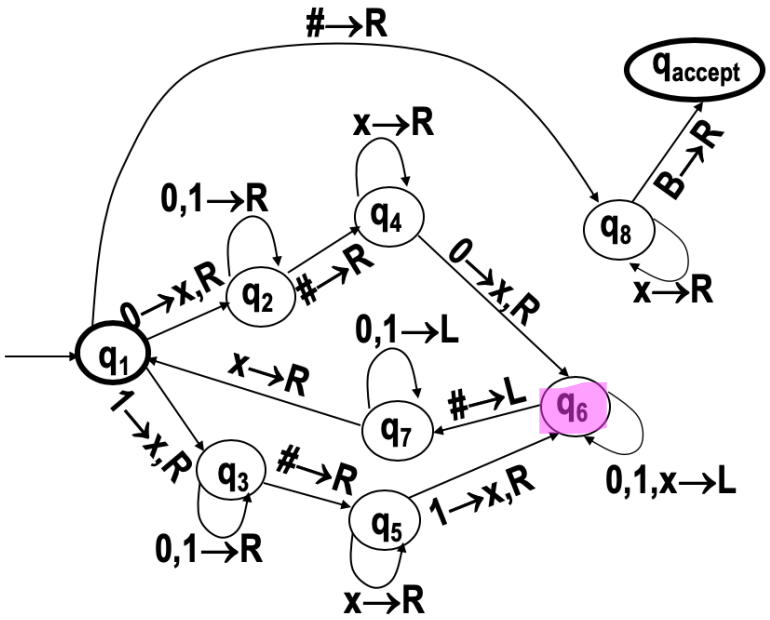


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

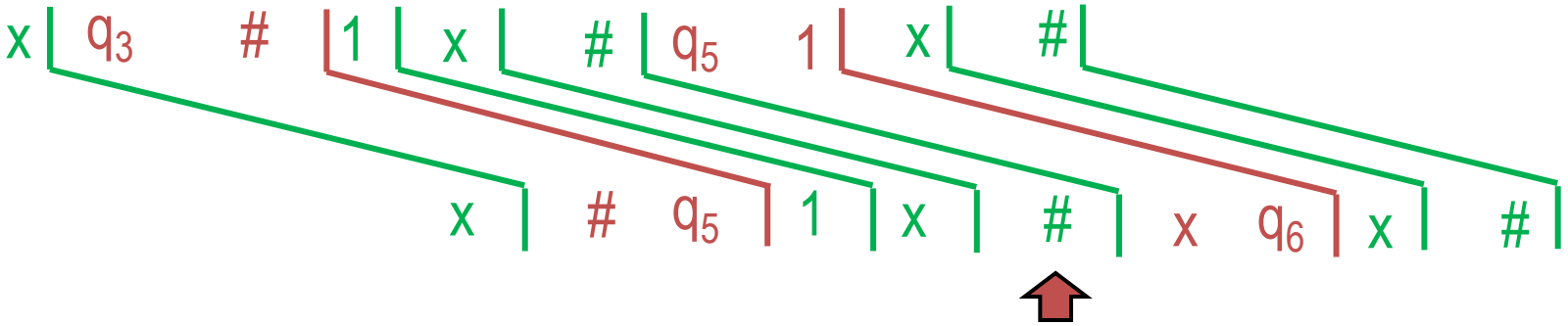


Example of PCP to A_{TM} , input is x#x

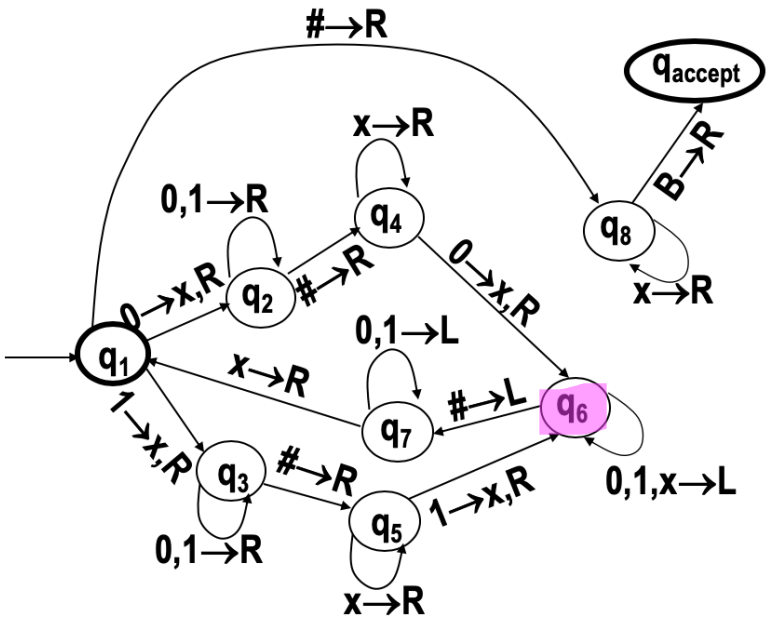


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

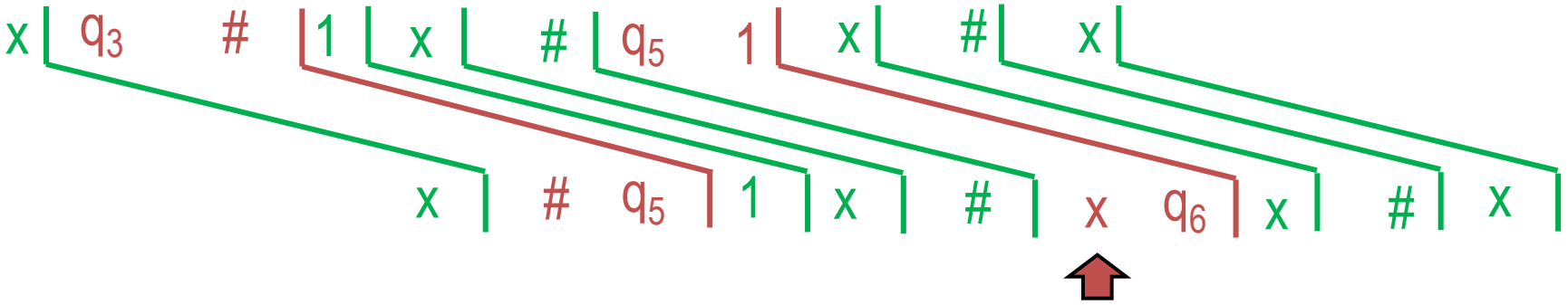


Example of PCP to A_{TM} , input is x#x

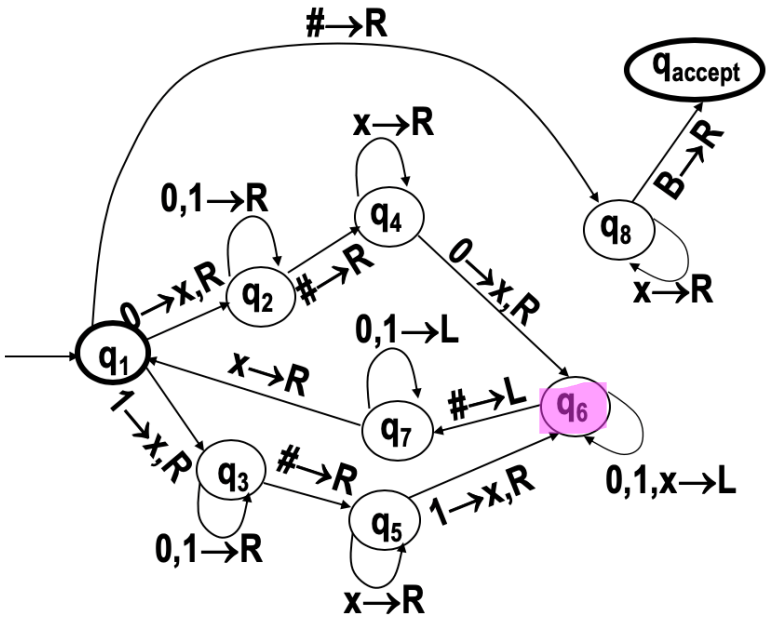


if $\delta(q, a) = (r, b, R)$,

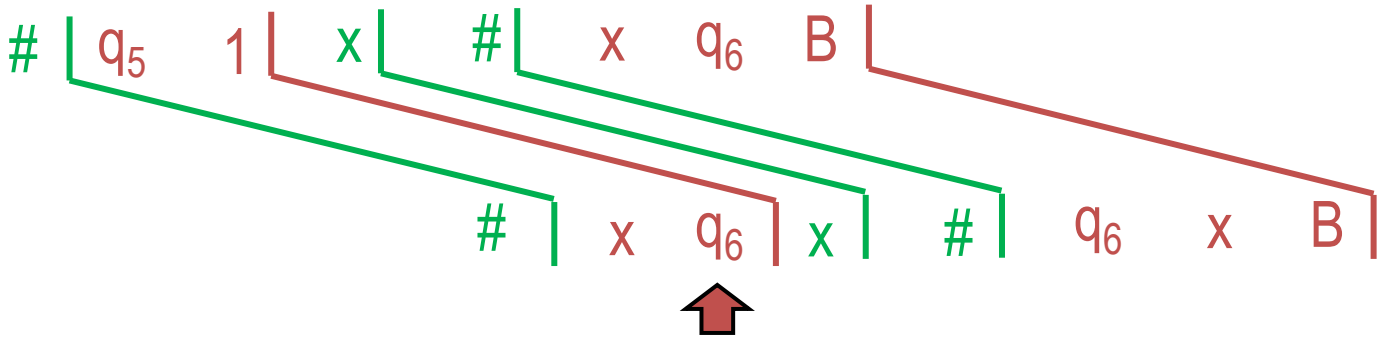
then put $\left[\frac{qa}{br}\right]$ into P'



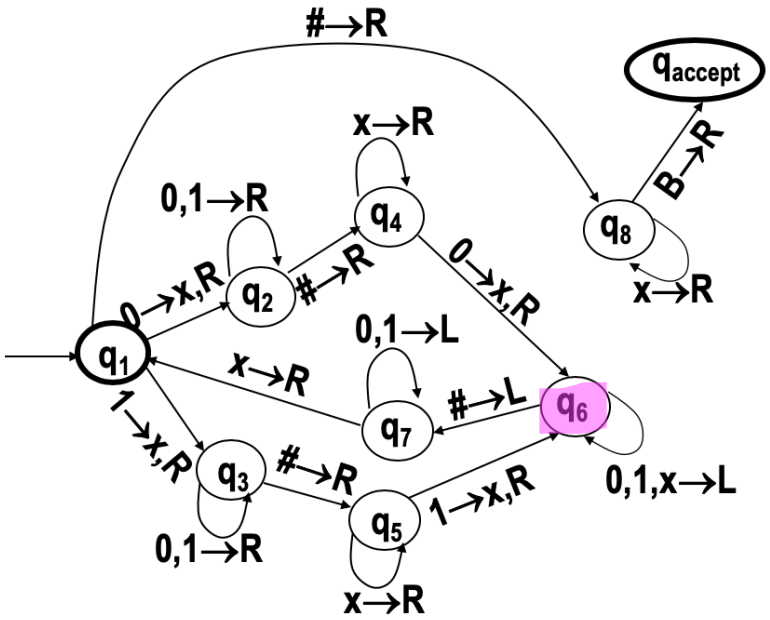
Example of PCP to A_{TM} , input is x#x



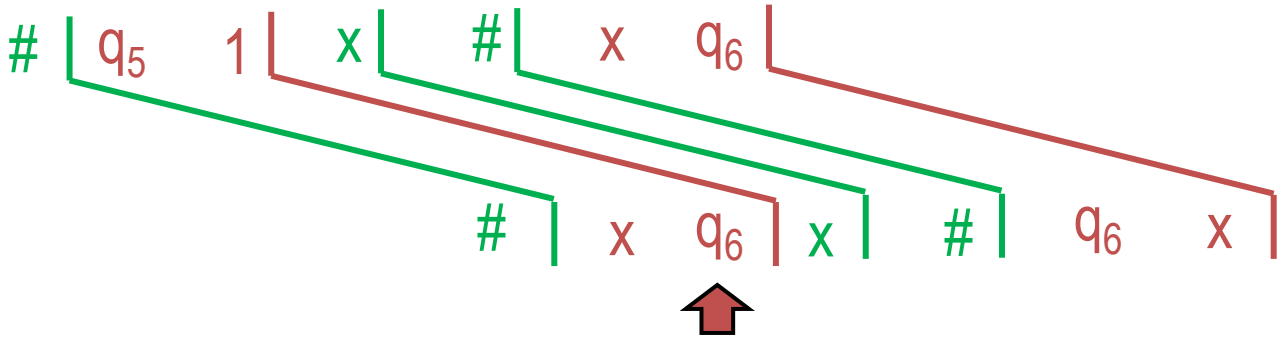
if $\delta(q, a) = (r, b, L)$, then put $\left[\frac{cqa}{rcb}\right]$ into P' , c is the element on the left of q



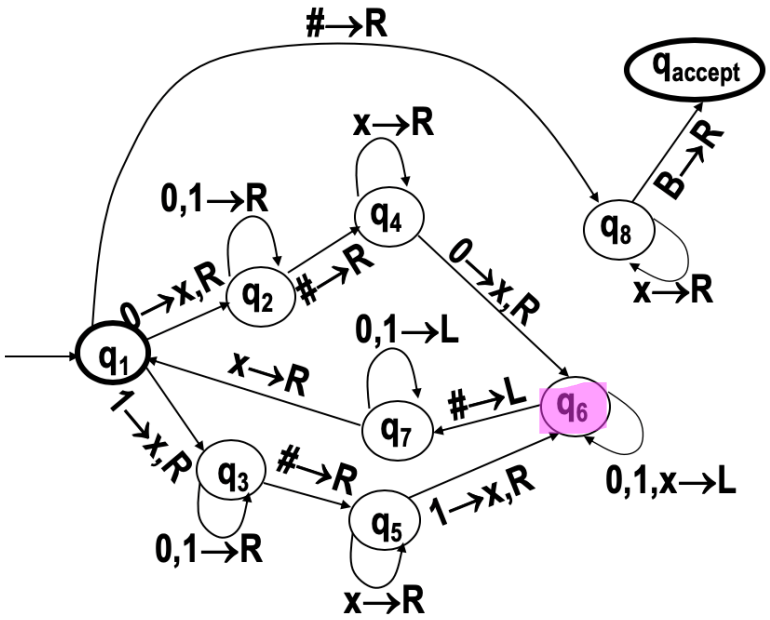
Example of PCP to A_{TM} , input is x#x



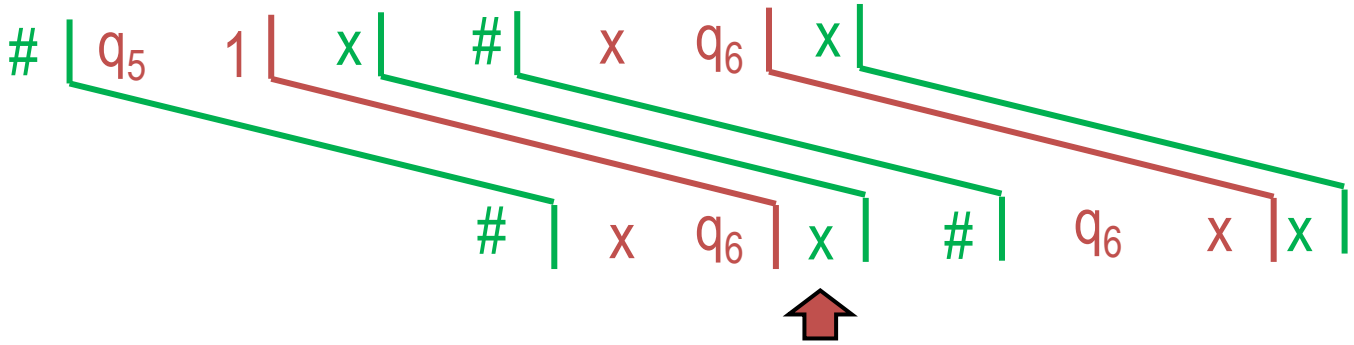
if $\delta(q, a) = (r, b, L)$, then put $\left[\frac{cqa}{rcb}\right]$ into P' , c is the element on the left of q



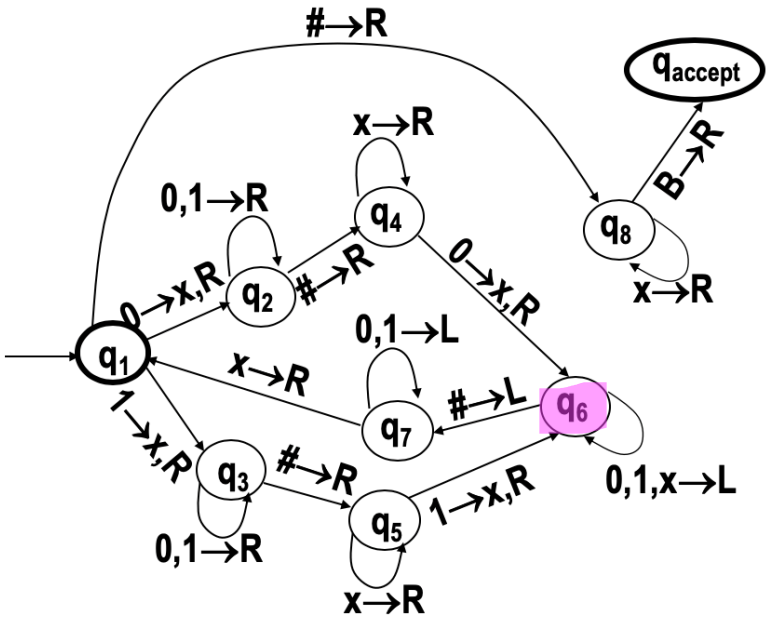
Example of PCP to A_{TM} , input is x#x



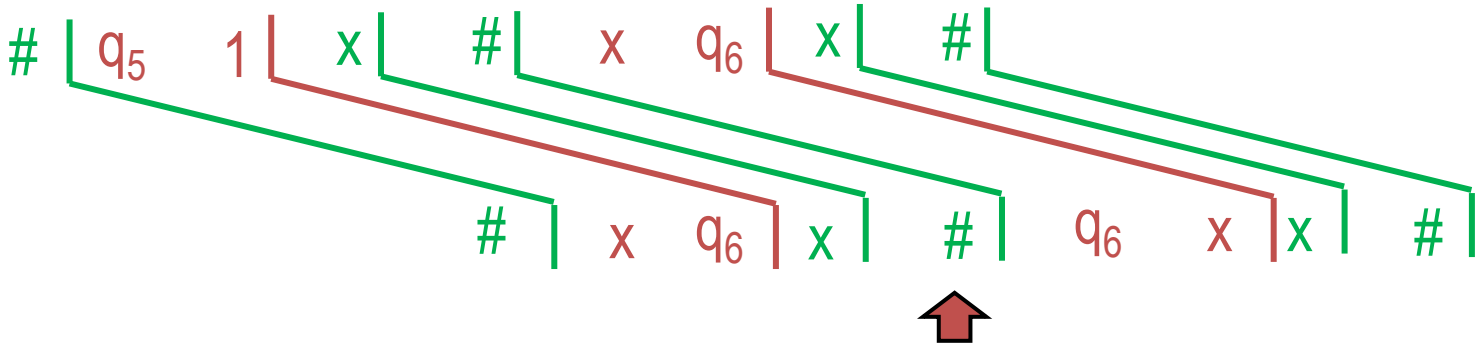
if $\delta(q, a) = (r, b, L)$, then put $\left[\frac{cqa}{rcb}\right]$ into P' , c is the element on the left of q



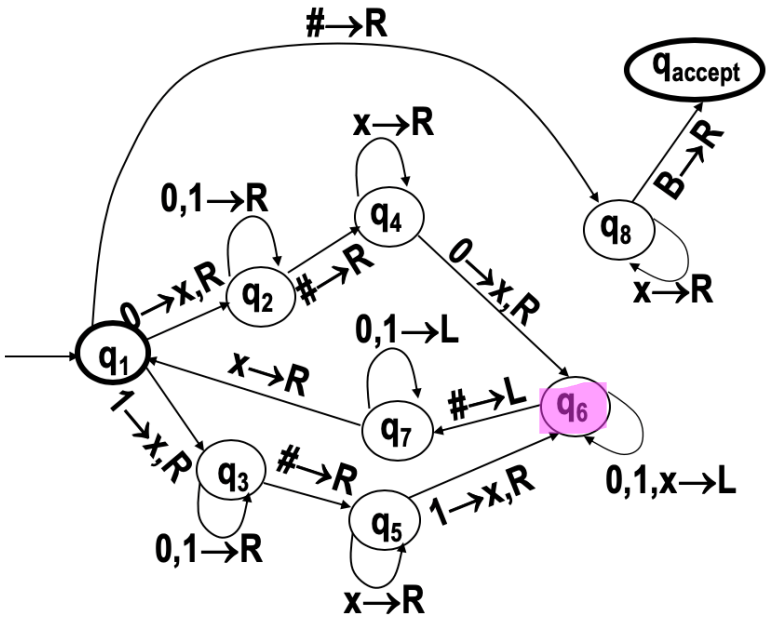
Example of PCP to A_{TM} , input is x#x



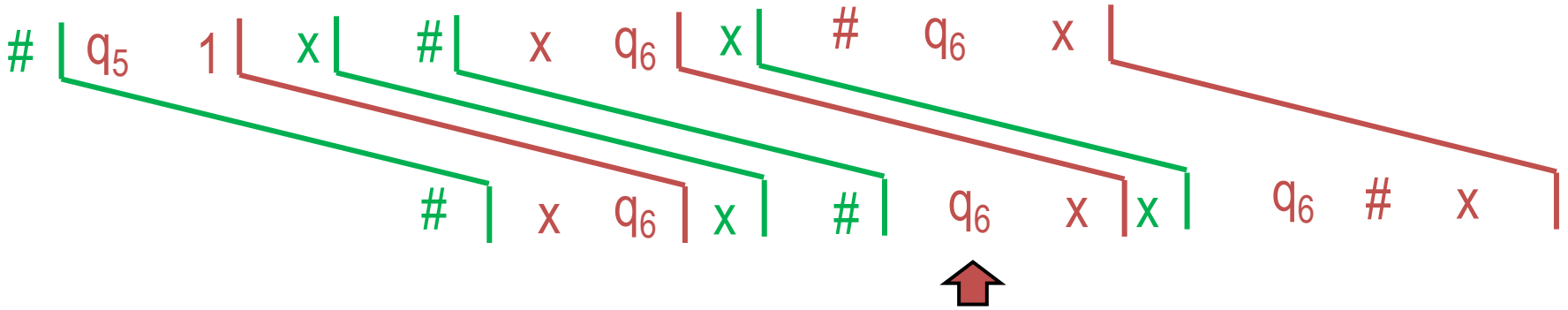
if $\delta(q, a) = (r, b, L)$, then put $\left[\frac{cqa}{rcb}\right]$ into P' , c is the element on the left of q



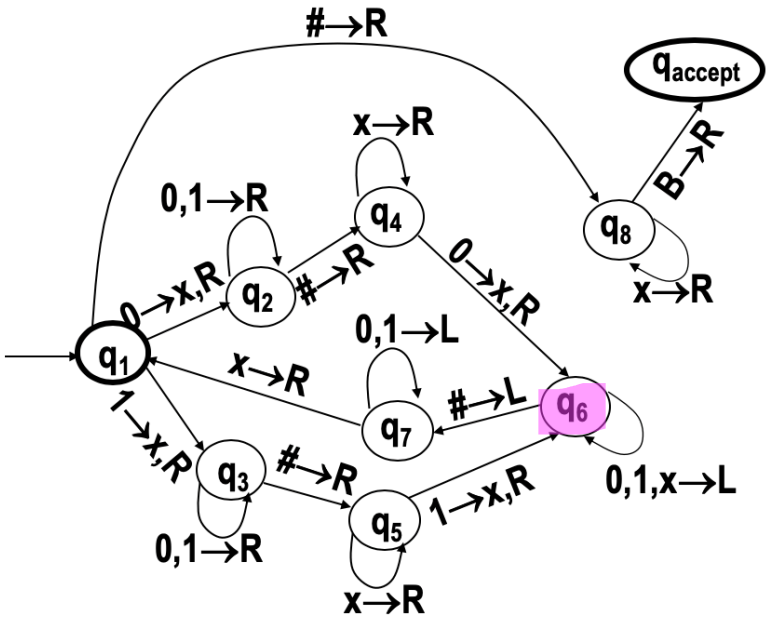
Example of PCP to A_{TM} , input is x#x



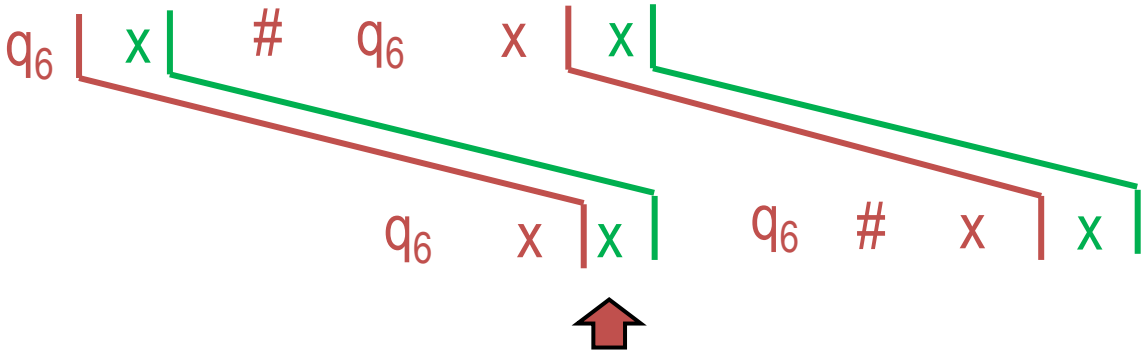
if $\delta(q, a) = (r, b, L)$, then put $\left[\frac{cqa}{rcb}\right]$ into P' , c is the element on the left of q



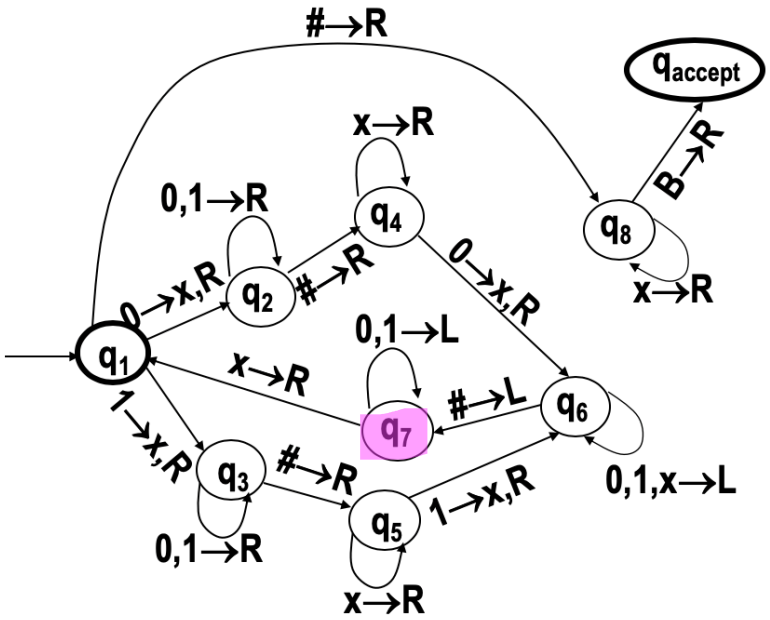
Example of PCP to A_{TM} , input is x#x



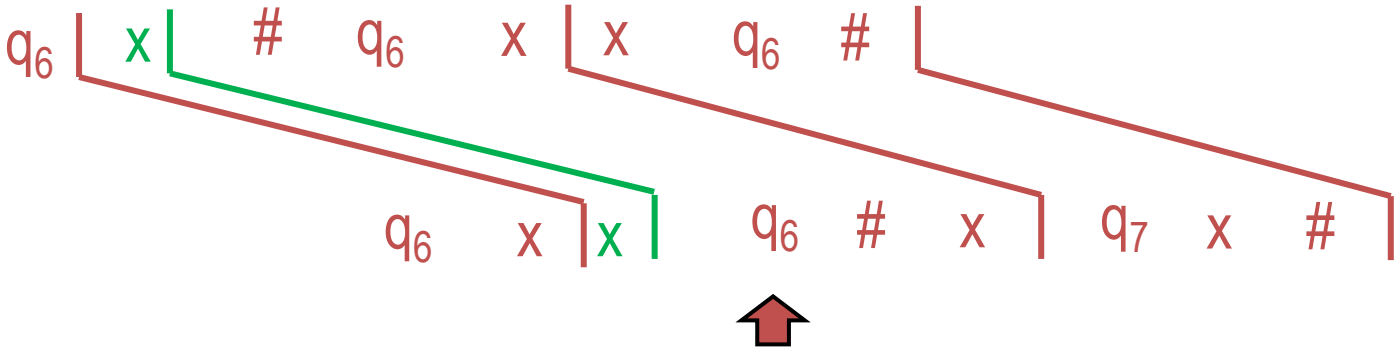
if $\delta(q, a) = (r, b, L)$, then put $\left[\frac{cqa}{rcb}\right]$ into P' , c is the element on the left of q



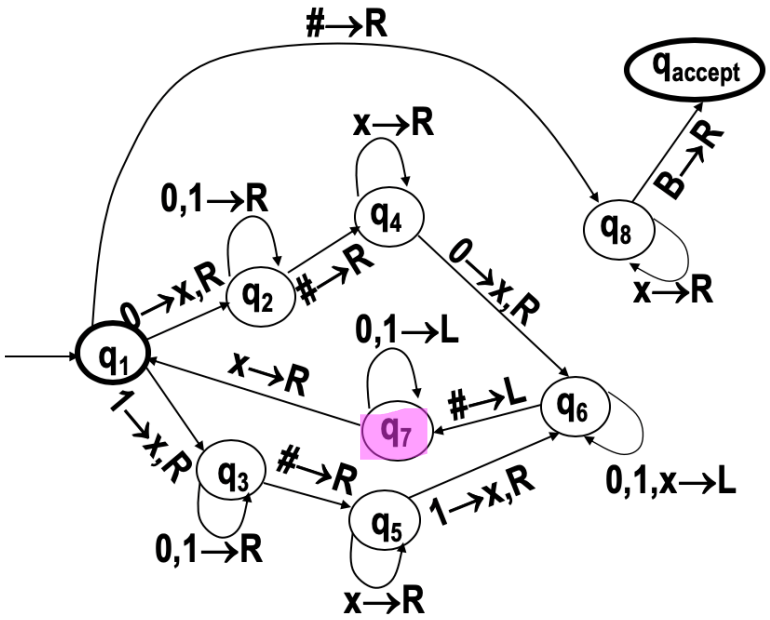
Example of PCP to A_{TM} , input is x#x



if $\delta(q, a) = (r, b, L)$, then put $\left[\frac{cqa}{rcb}\right]$ into P' , c is the element on the left of q

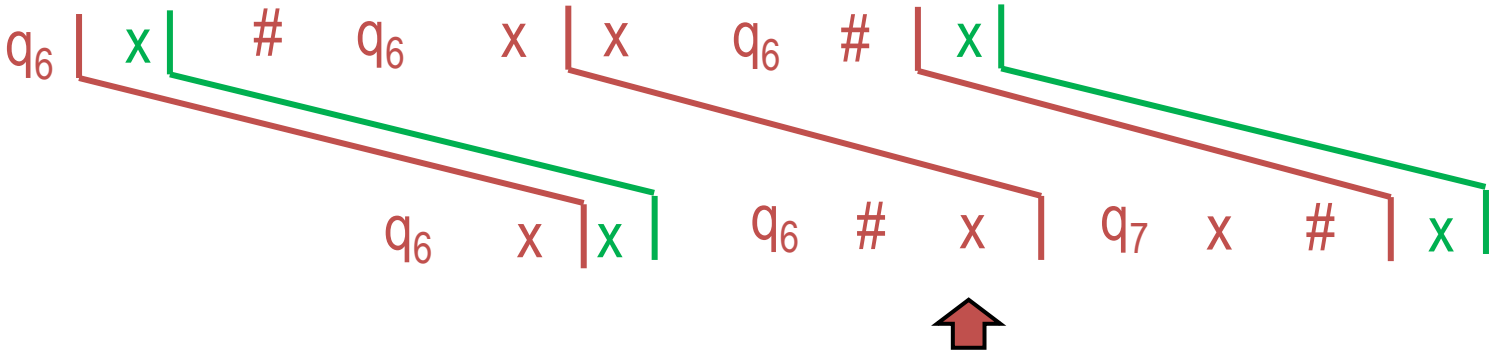


Example of PCP to A_{TM} , input is x#x

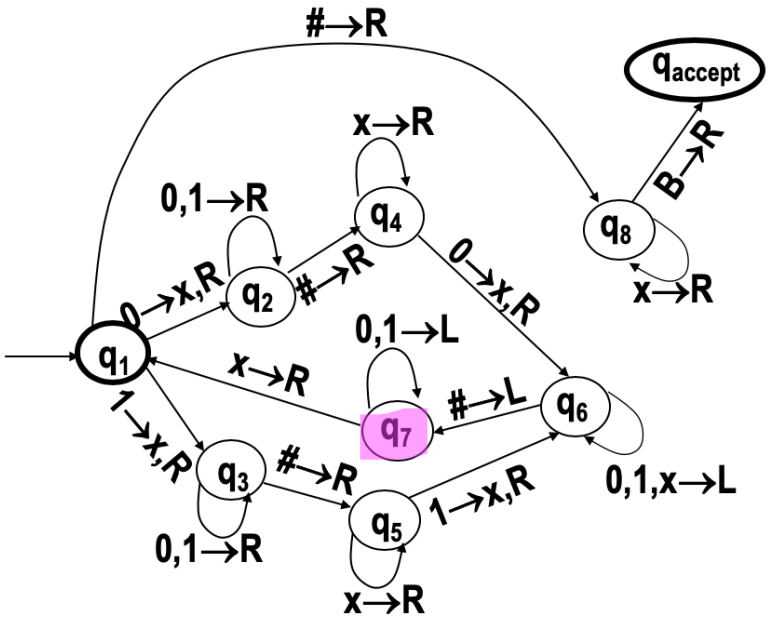


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

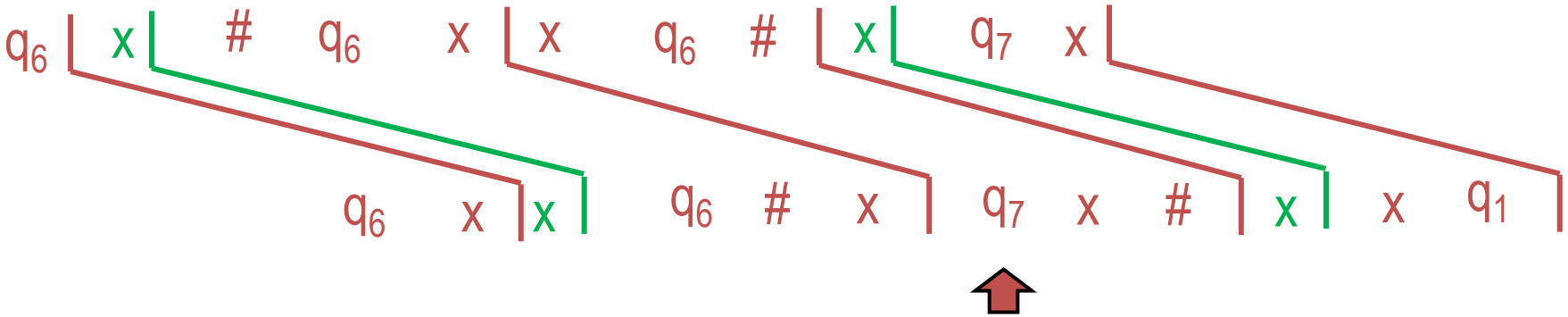


Example of PCP to A_{TM} , input is x#x

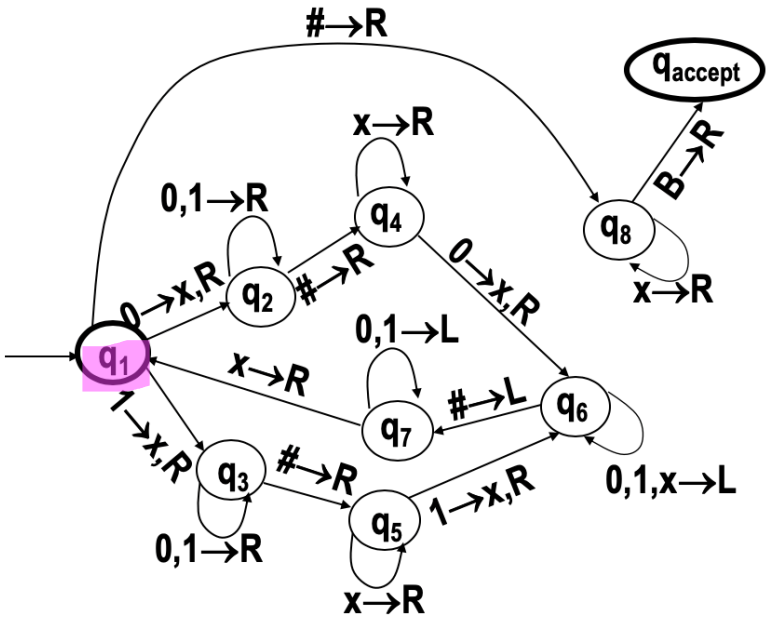


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

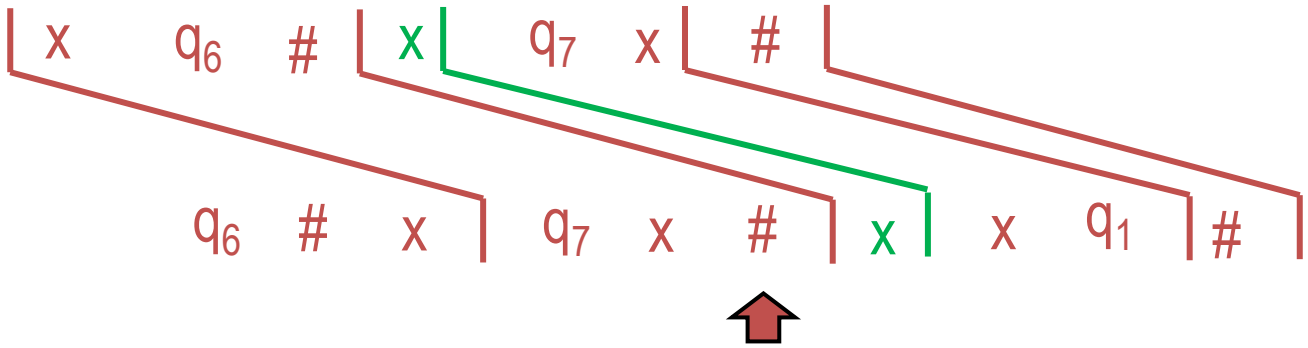


Example of PCP to A_{TM} , input is x#x

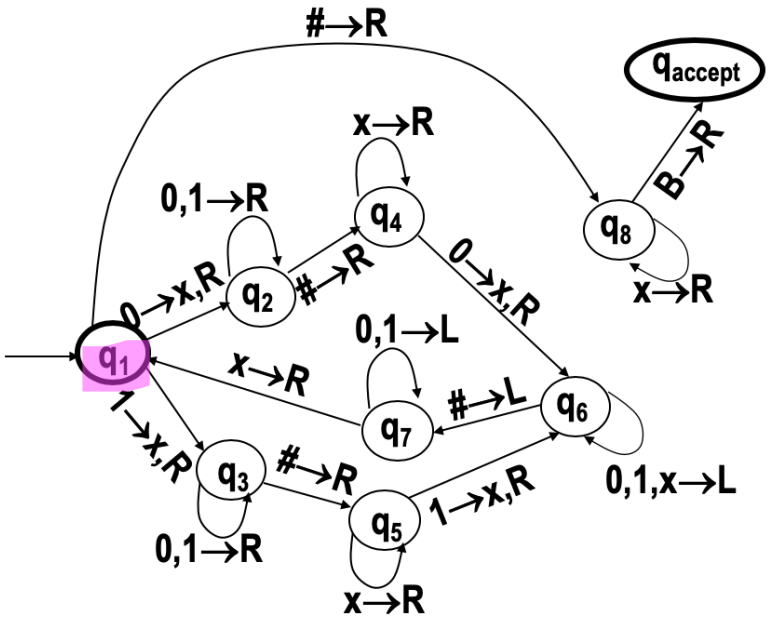


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

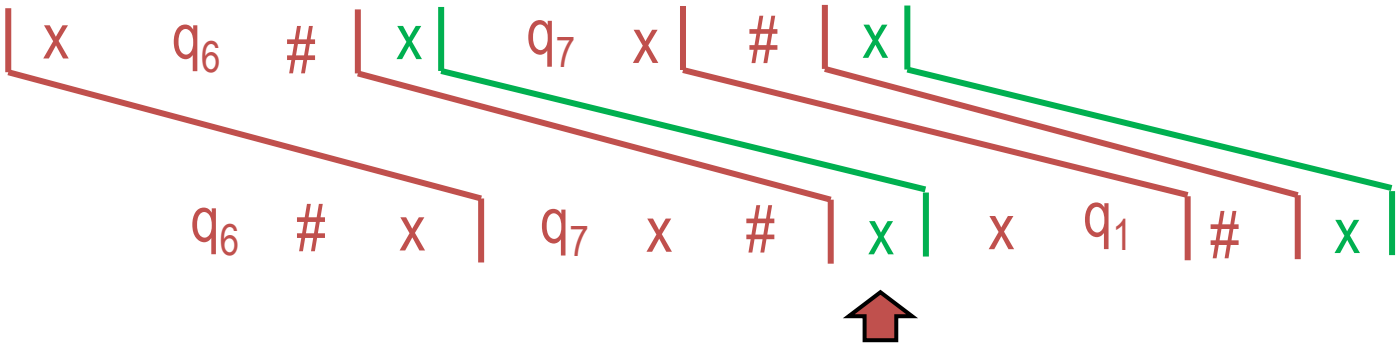


Example of PCP to A_{TM} , input is x#x

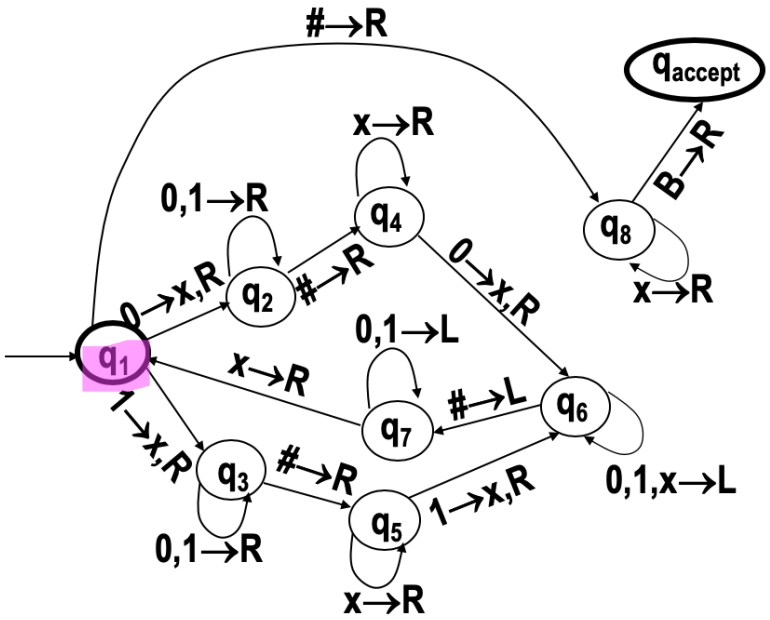


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

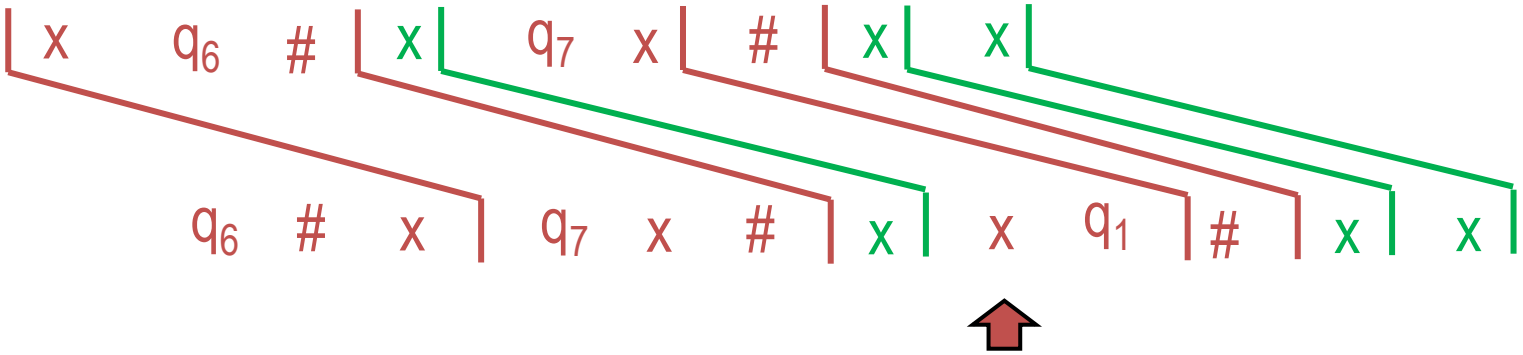


Example of PCP to A_{TM} , input is x#x

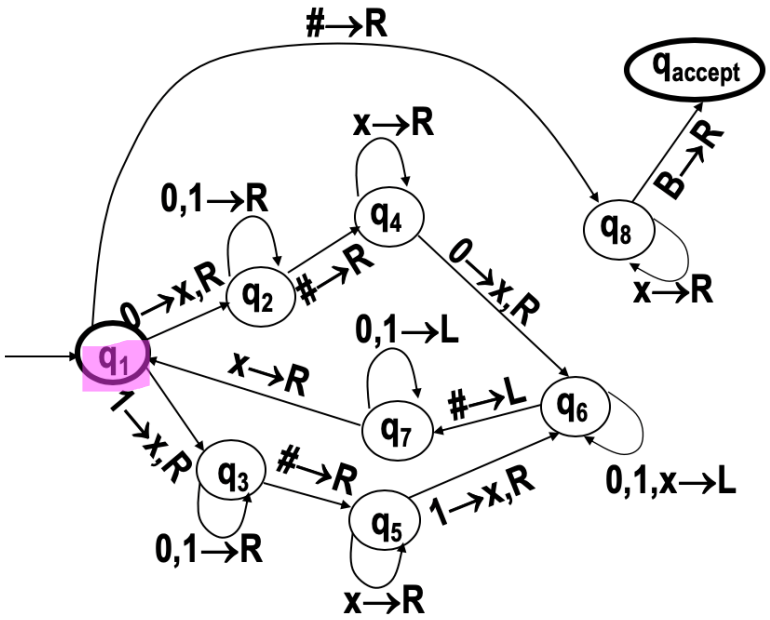


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

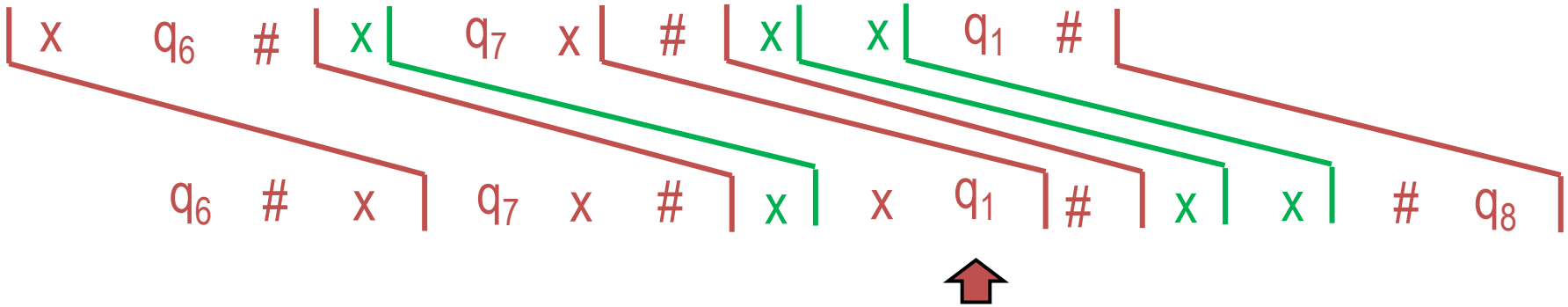


Example of PCP to A_{TM} , input is x#x

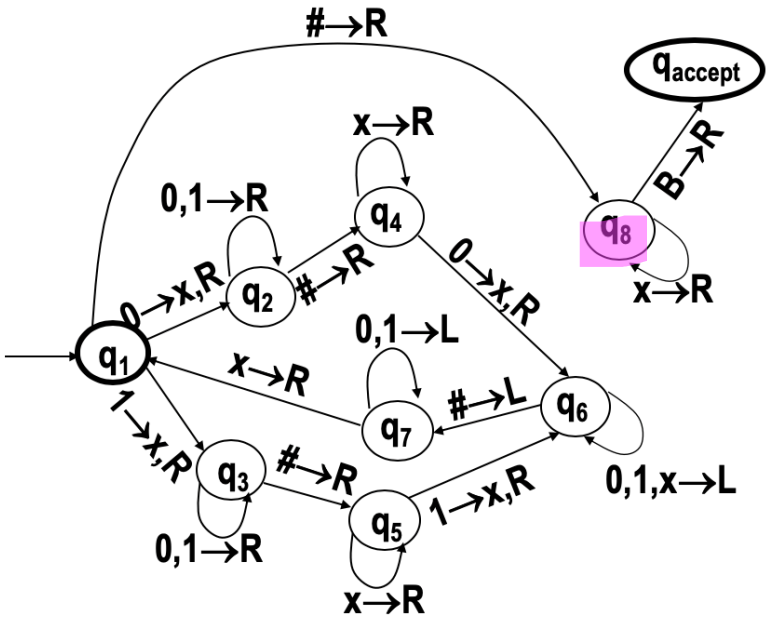


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

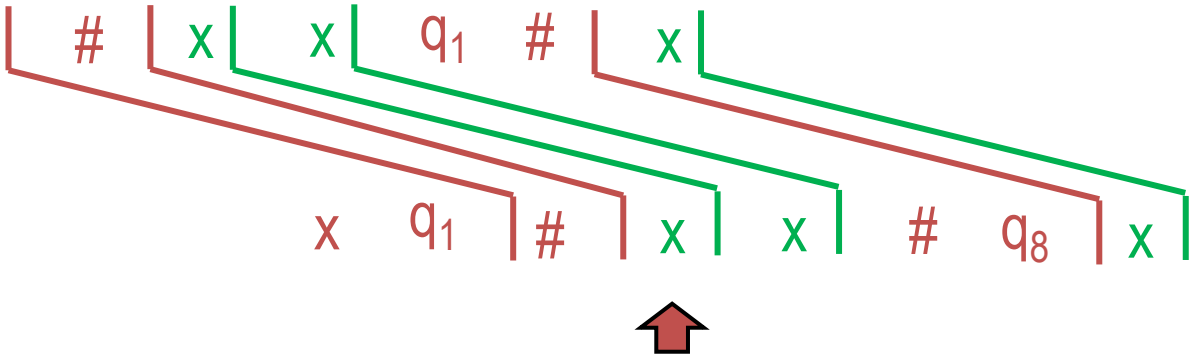


Example of PCP to A_{TM} , input is x#x

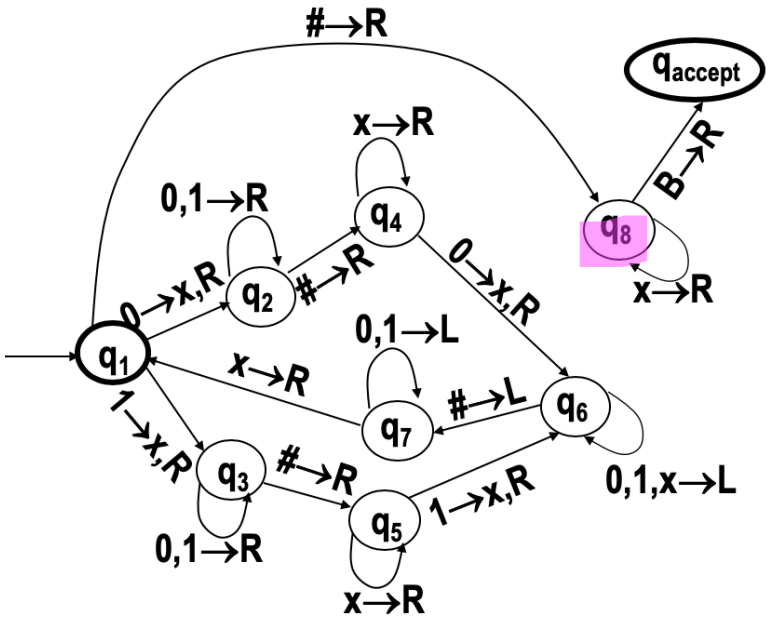


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

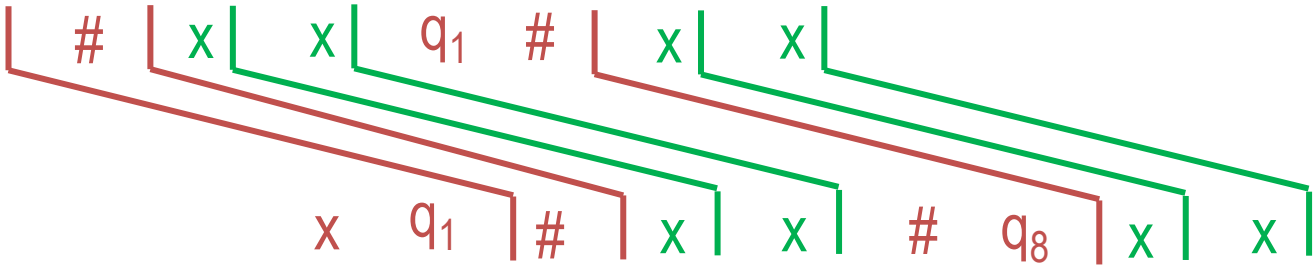


Example of PCP to A_{TM} , input is x#x

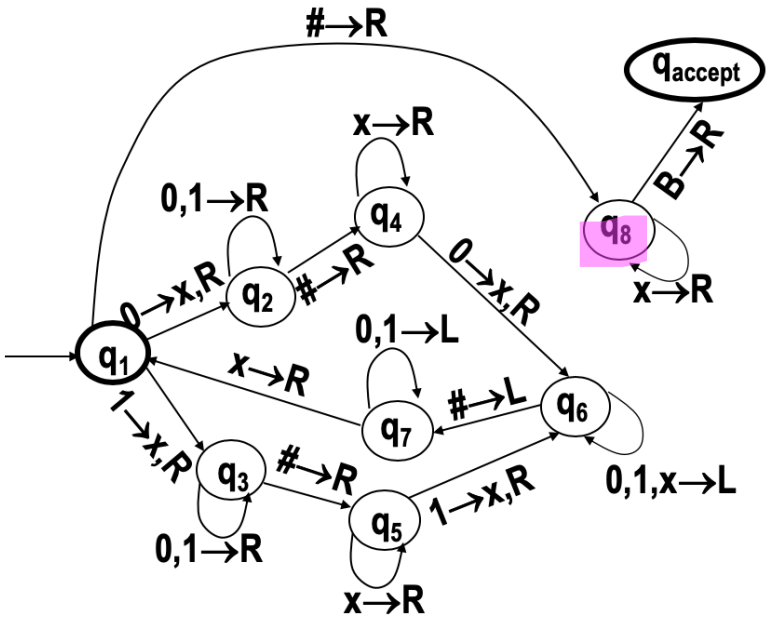


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

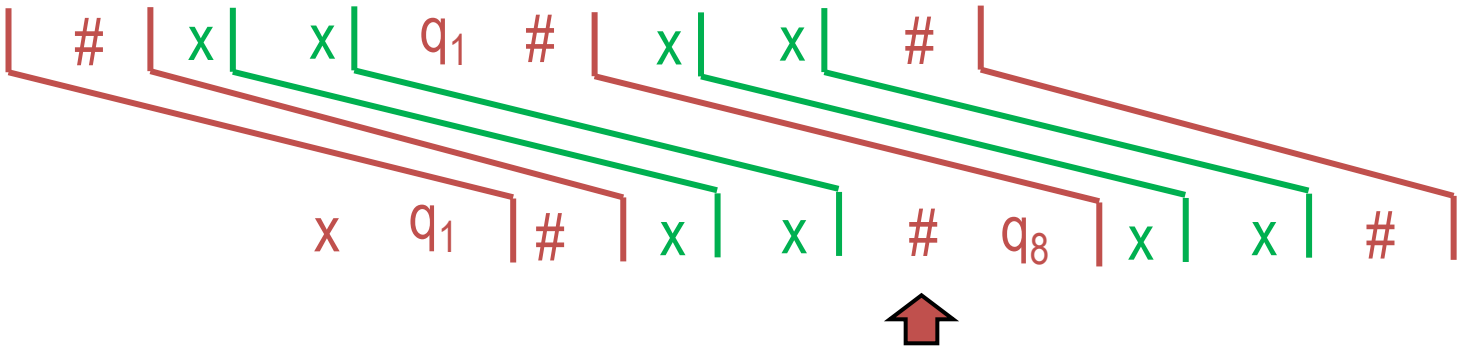


Example of PCP to A_{TM} , input is x#x

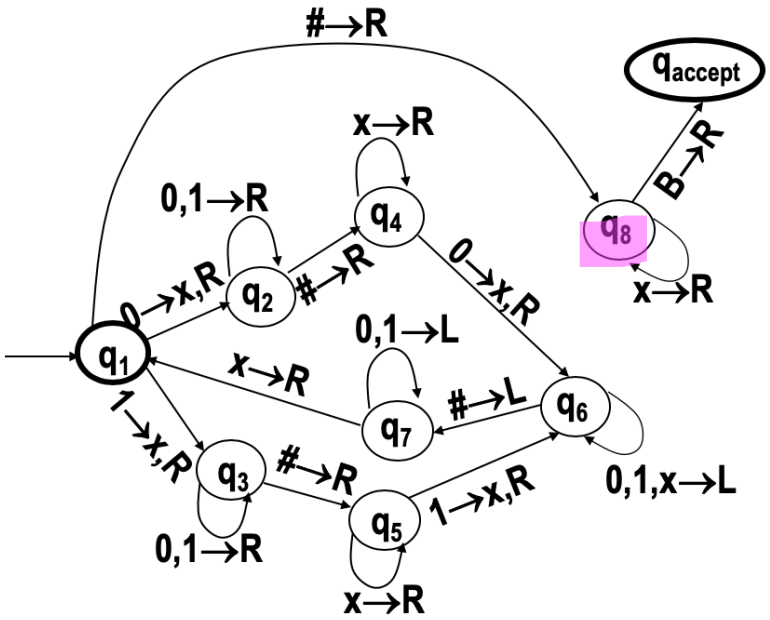


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

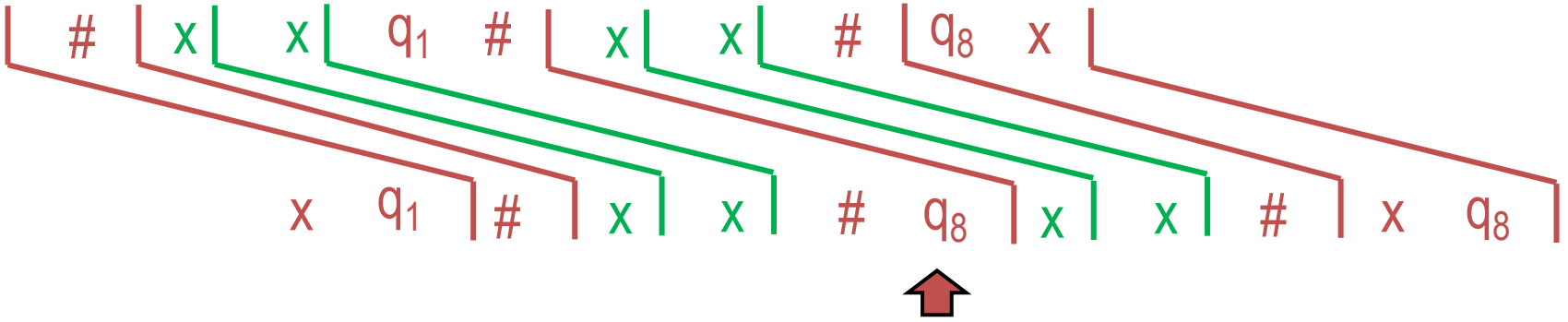


Example of PCP to A_{TM} , input is x#x

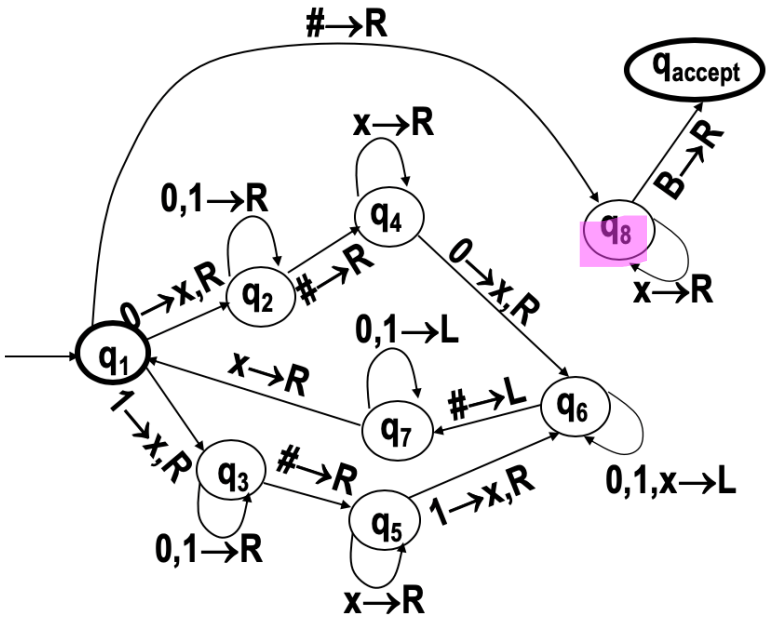


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

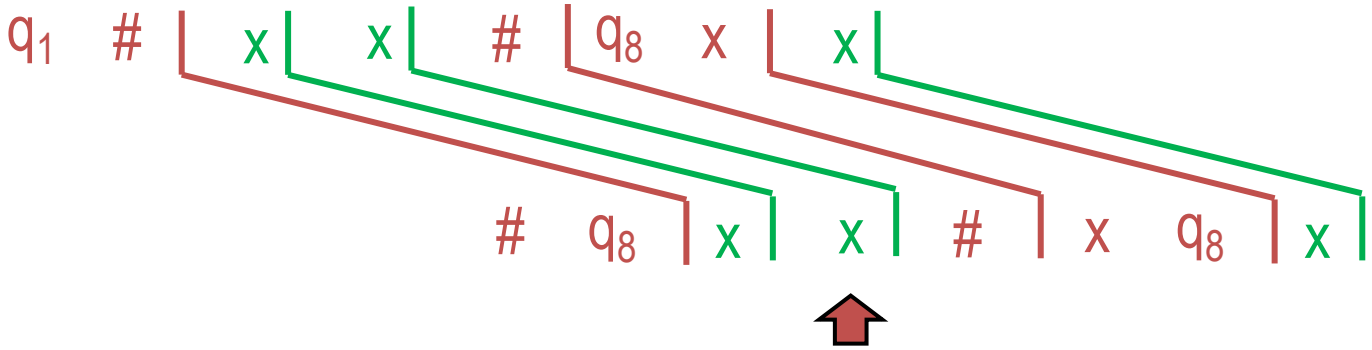


Example of PCP to A_{TM} , input is x#x

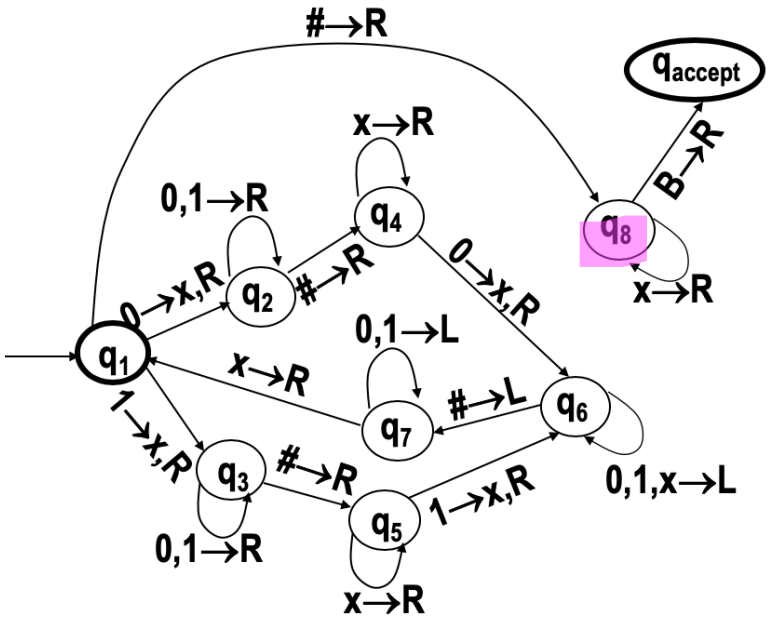


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

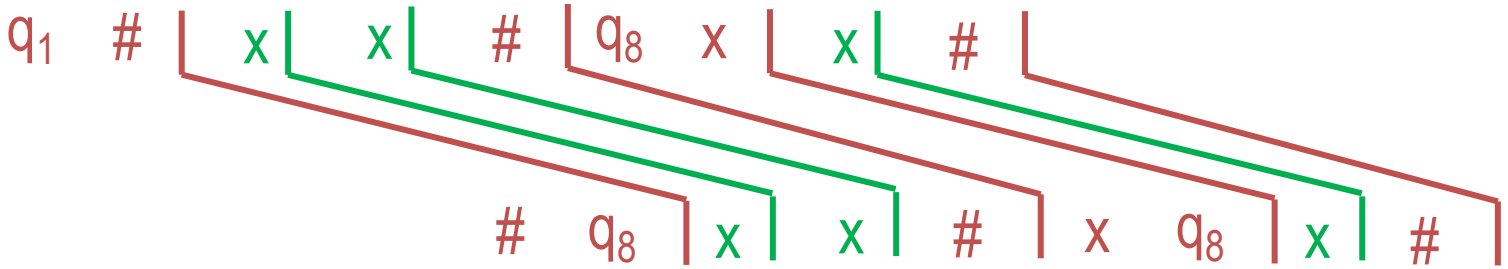


Example of PCP to A_{TM} , input is x#x

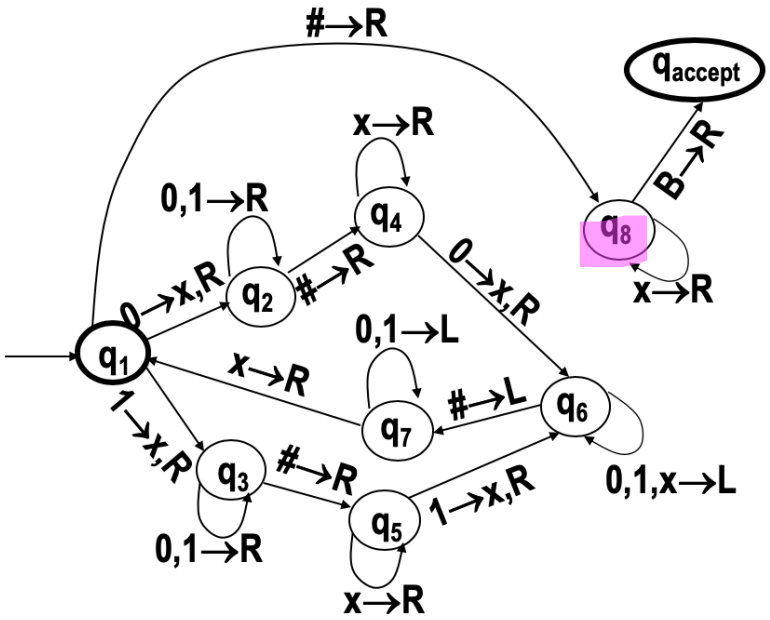


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

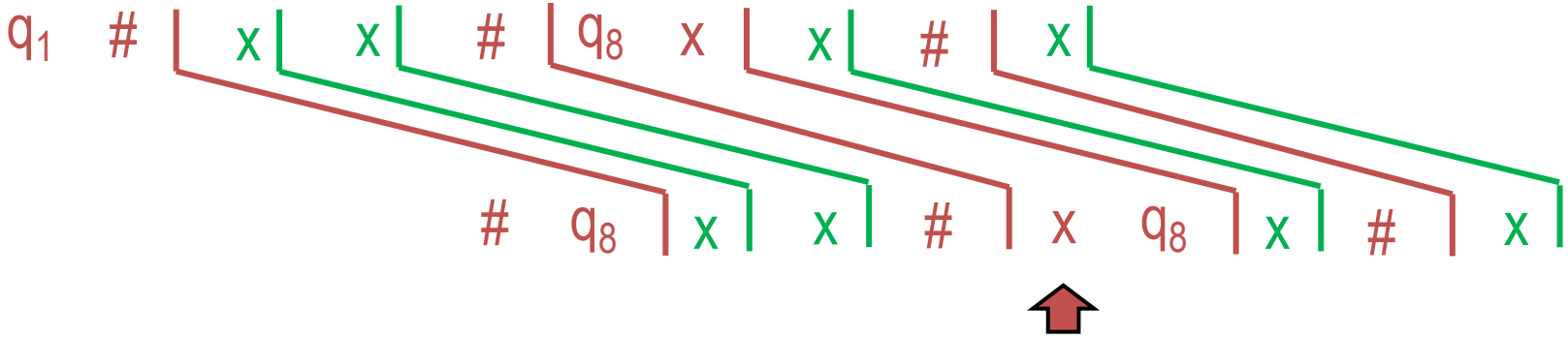


Example of PCP to A_{TM} , input is x#x

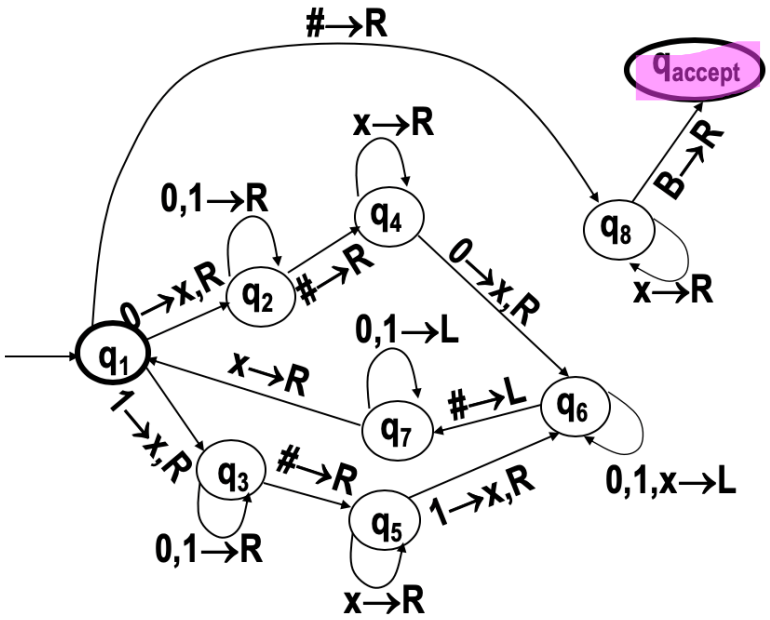


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

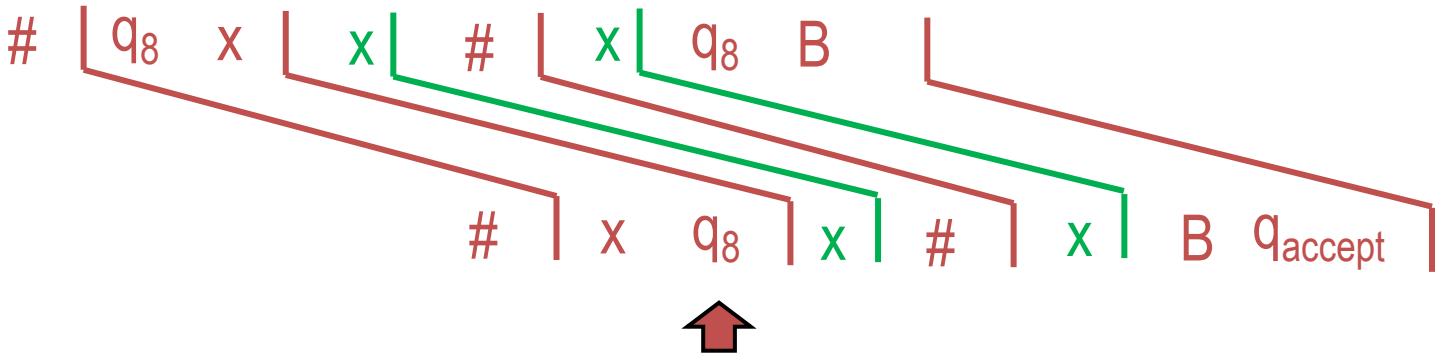


Example of PCP to A_{TM} , input is x#x

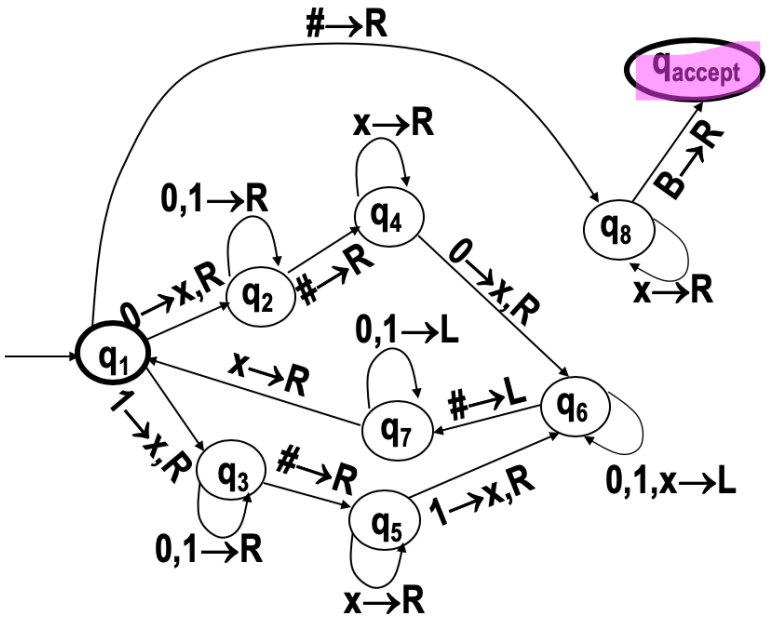


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

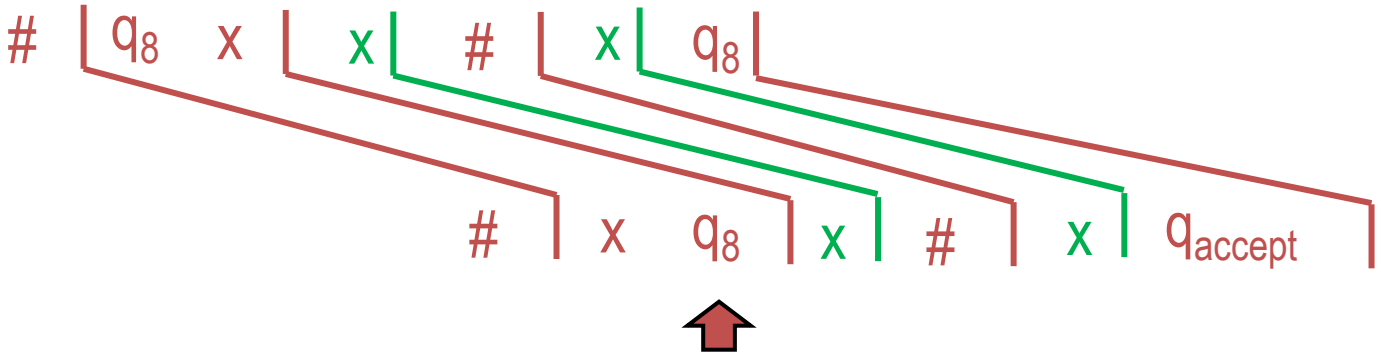


Example of PCP to A_{TM} , input is x#x

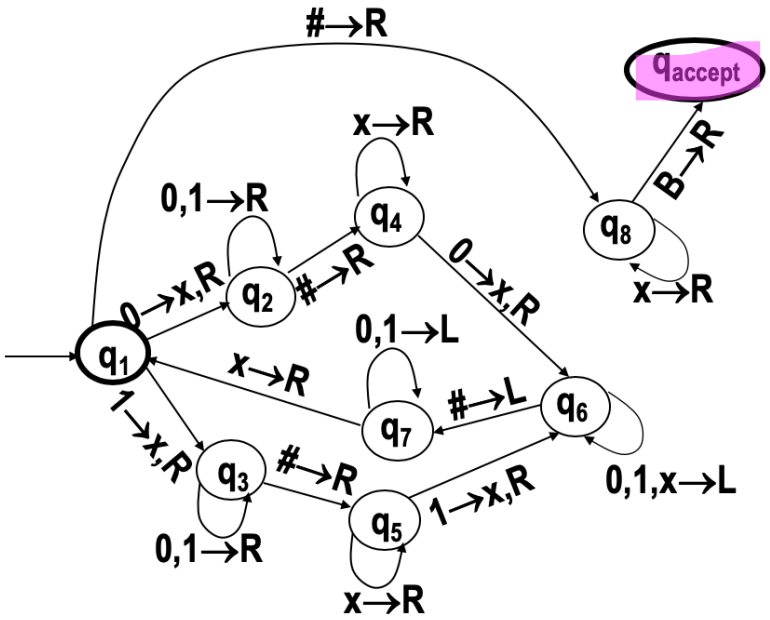


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

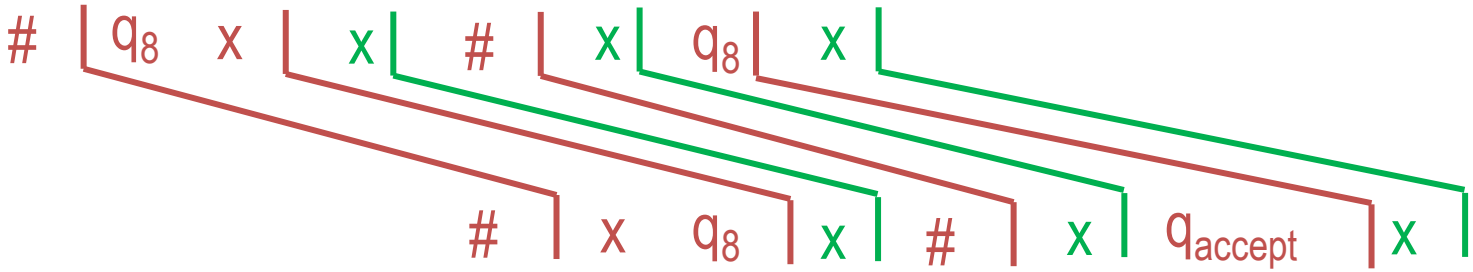


Example of PCP to A_{TM} , input is x#x

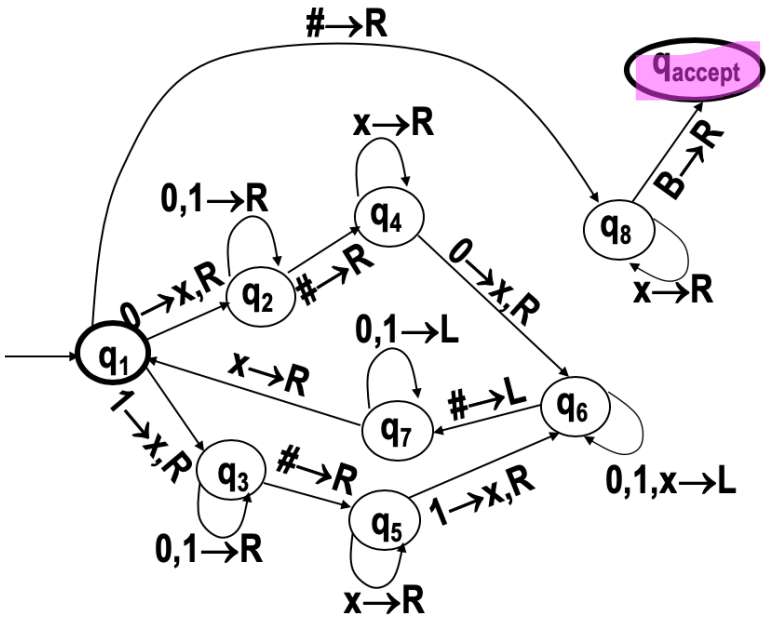


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

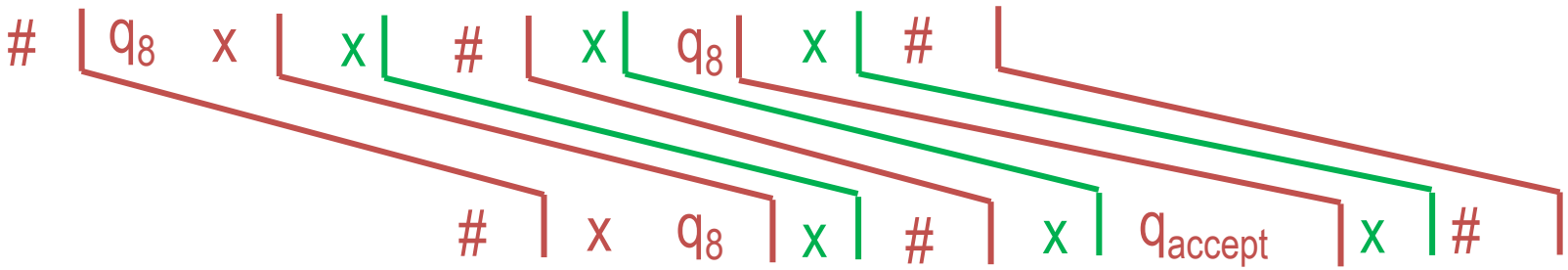


Example of PCP to A_{TM} , input is x#x

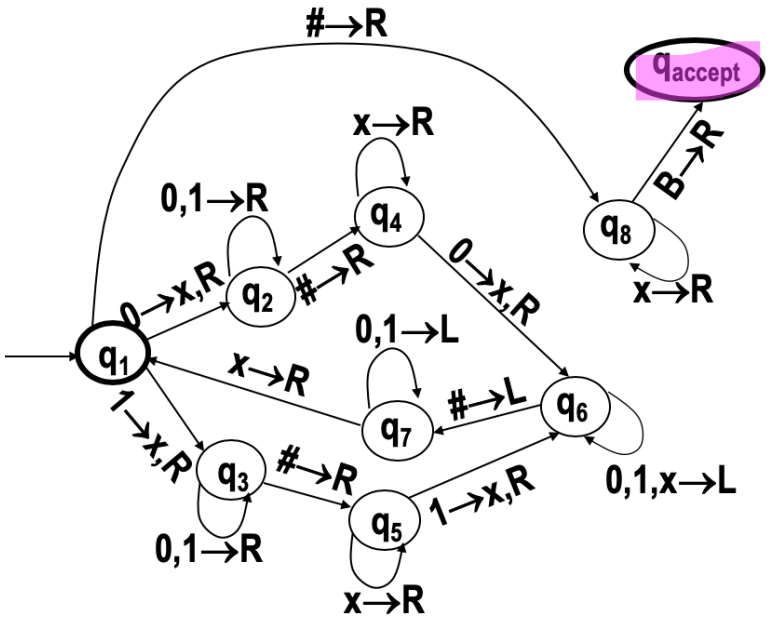


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

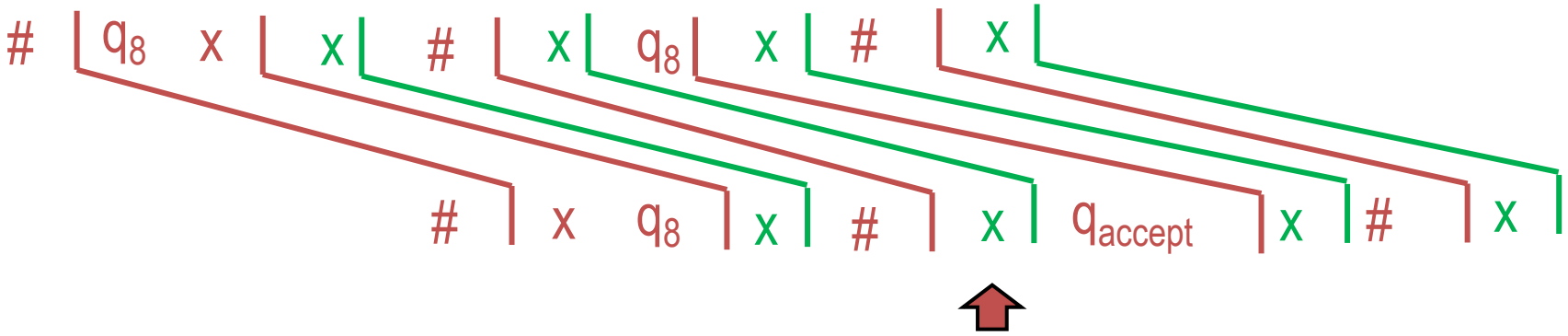


Example of PCP to A_{TM} , input is x#x

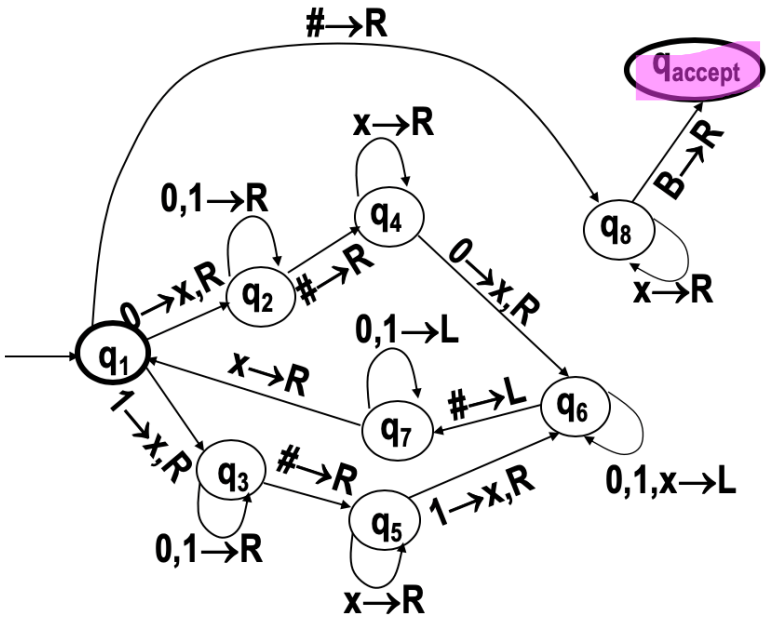


if $\delta(q, a) = (r, b, R)$,

then put $\left[\frac{qa}{br}\right]$ into P'

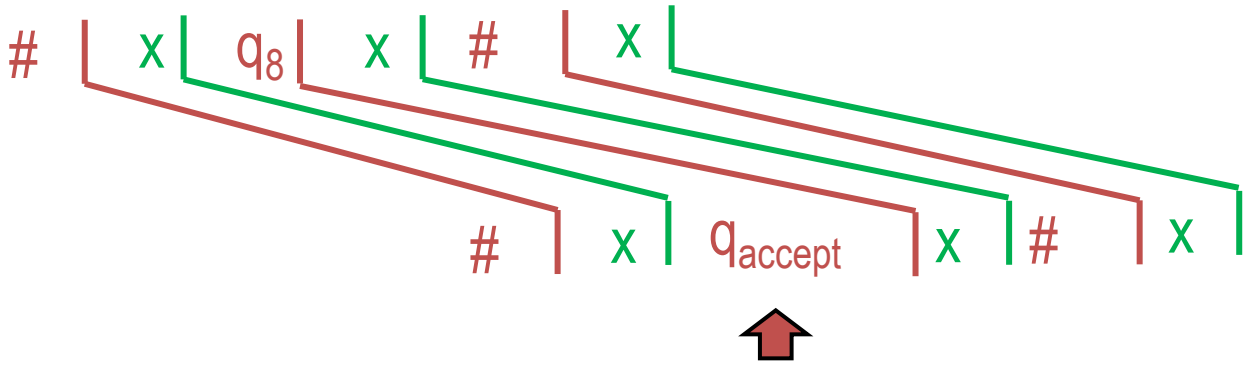


Example of PCP to A_{TM} , input is x#x

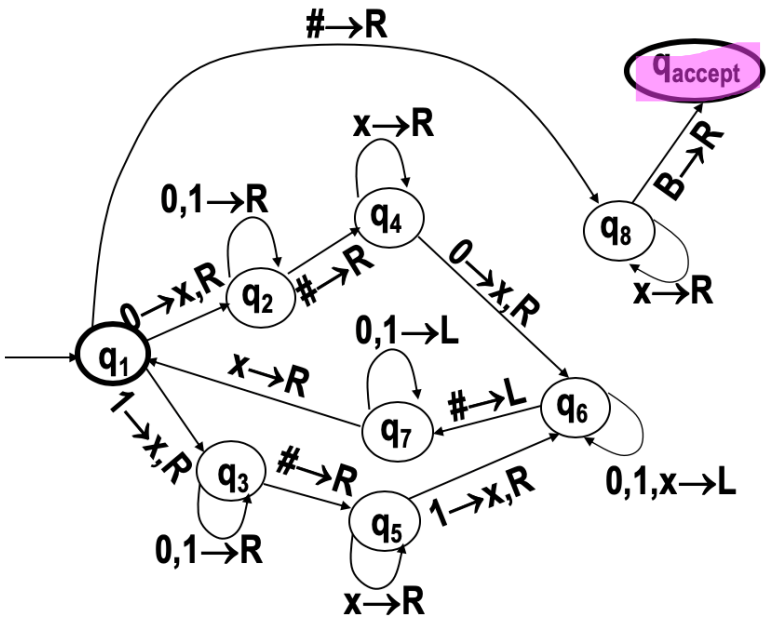


if $\delta(q, a) = (r, b, R)$,

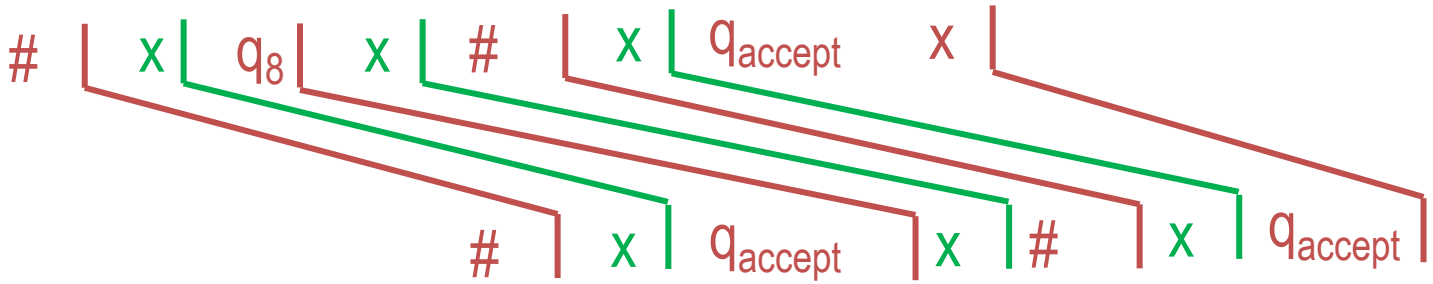
then put $\left[\frac{qa}{br}\right]$ into P'



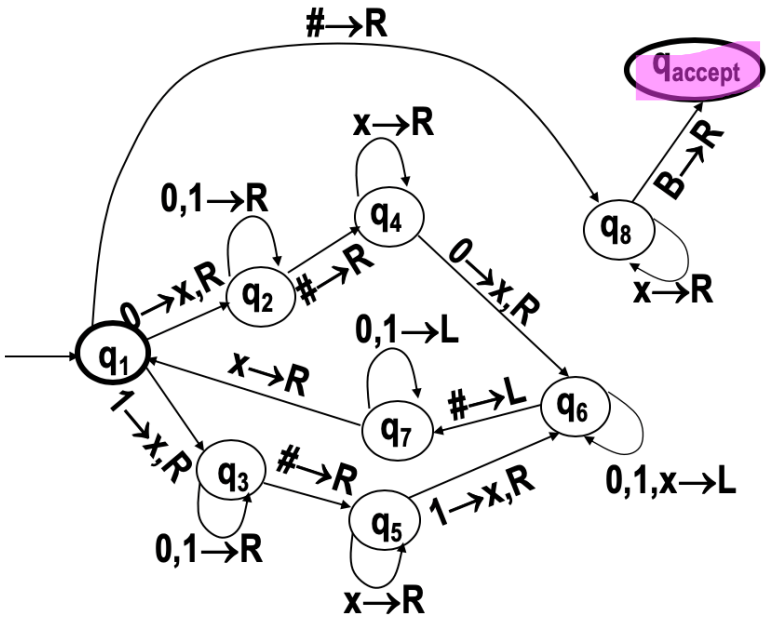
Example of PCP to A_{TM} , input is x#x



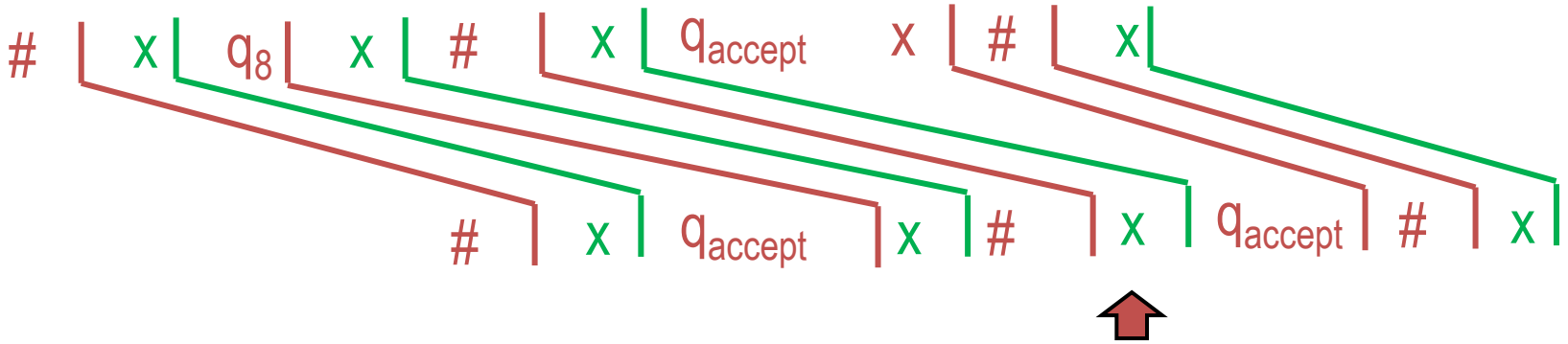
put $\left[\frac{aq_{accept}}{q_{accept}}\right]$ and $\left[\frac{q_{accept}a}{q_{accept}}\right]$ into P'



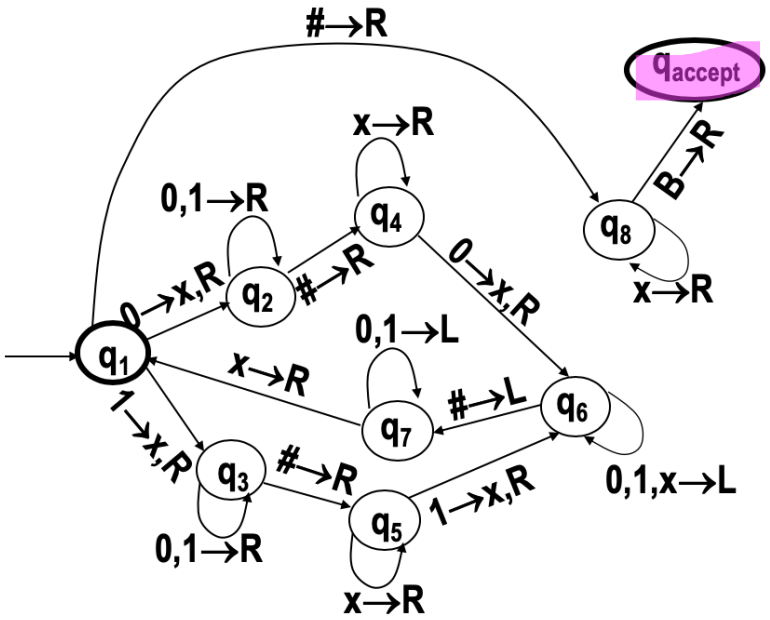
Example of PCP to A_{TM} , input is x#x



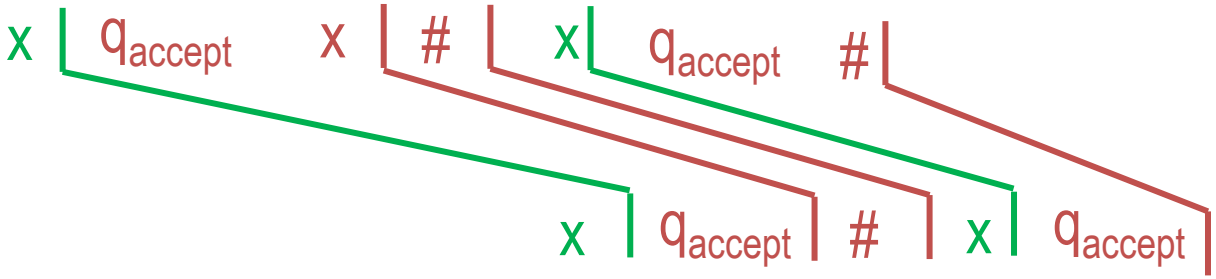
put $\left[\frac{aq_{accept}}{q_{accept}}\right]$ and $\left[\frac{q_{accept}a}{q_{accept}}\right]$ into P'



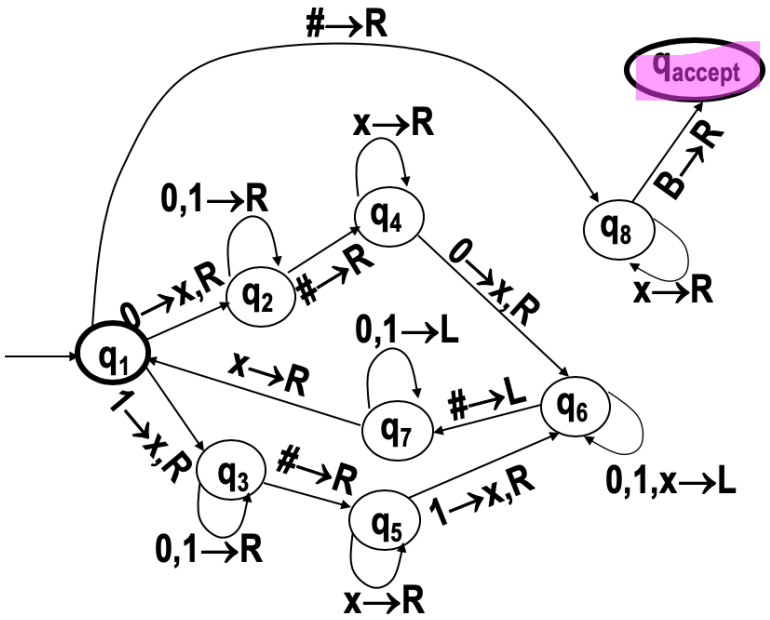
Example of PCP to A_{TM} , input is x#x



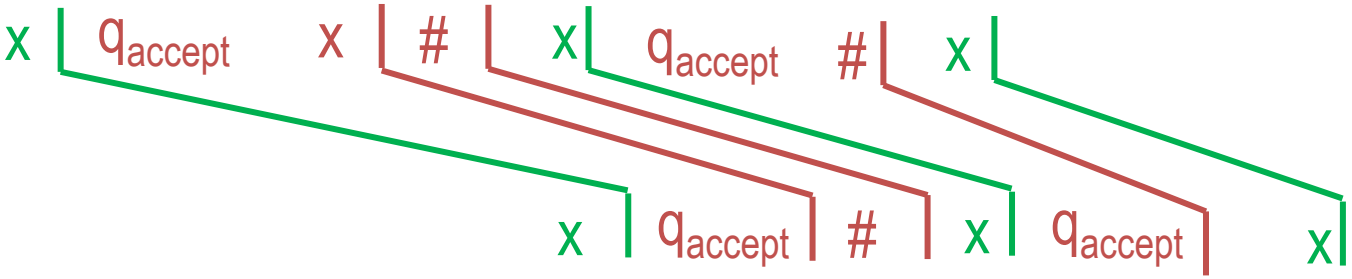
put $\left[\frac{aq_{accept}}{q_{accept}}\right]$ and $\left[\frac{q_{accept}a}{q_{accept}}\right]$ into P'



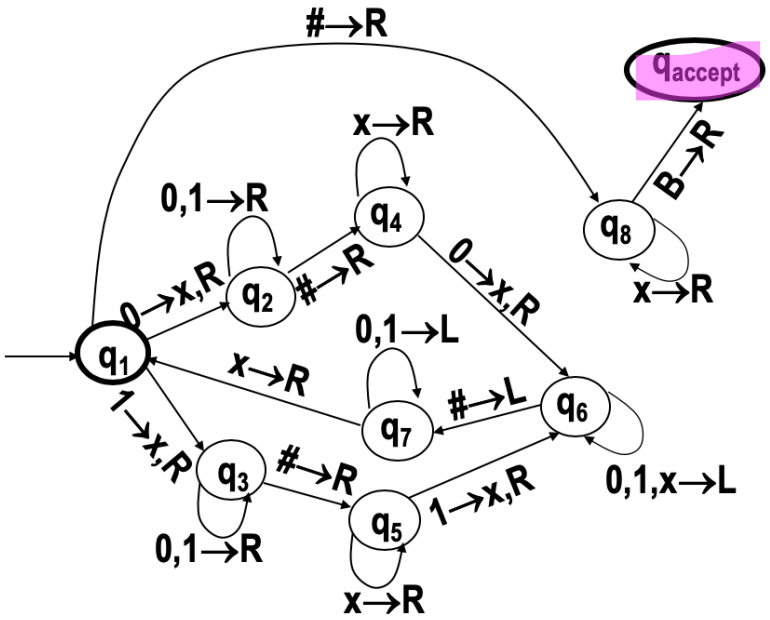
Example of PCP to A_{TM} , input is x#x



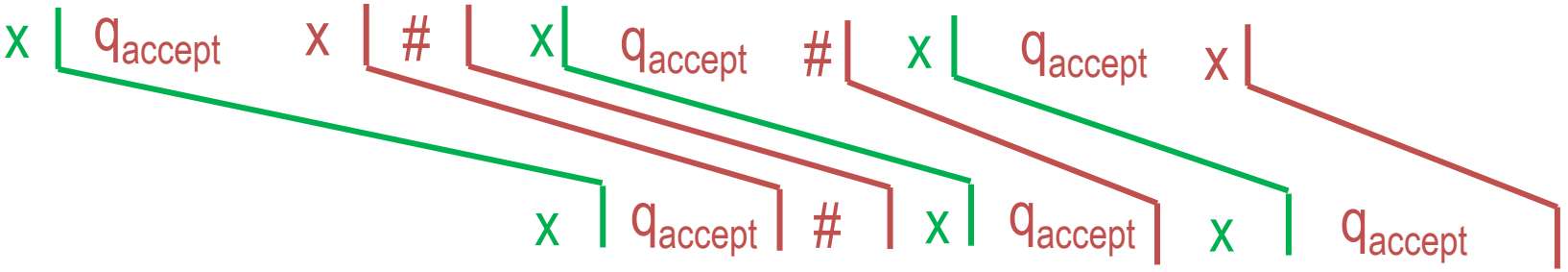
put $\left[\frac{aq_{accept}}{q_{accept}}\right]$ and $\left[\frac{q_{accept}a}{q_{accept}}\right]$ into P'



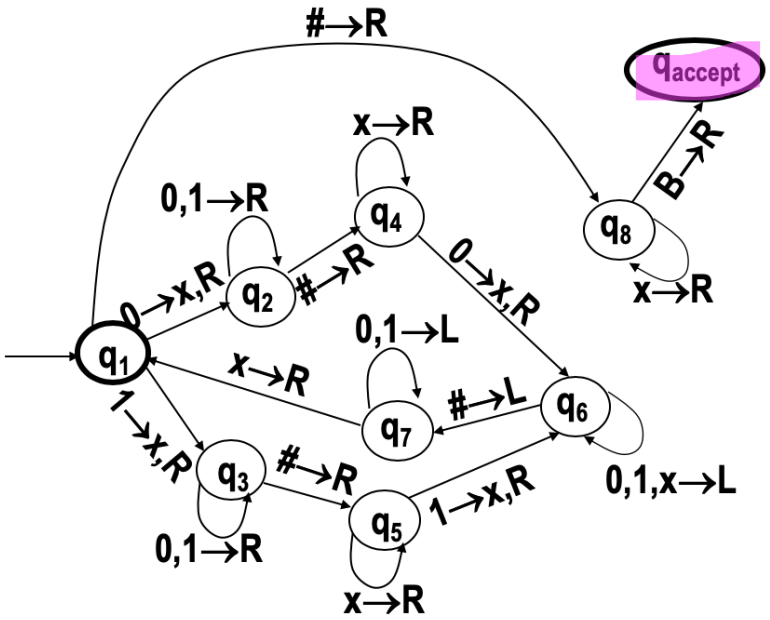
Example of PCP to A_{TM} , input is x#x



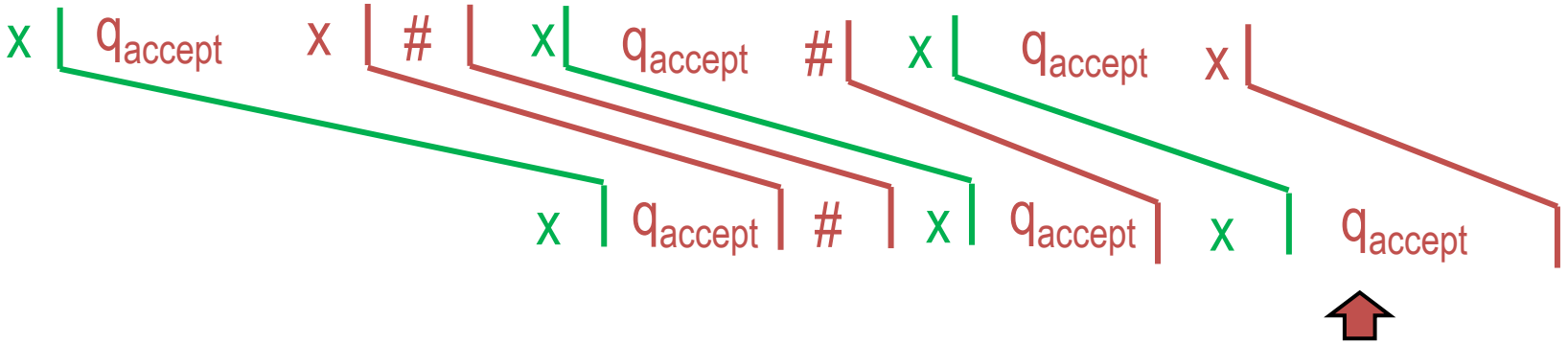
put $\left[\frac{aq_{accept}}{q_{accept}}\right]$ and $\left[\frac{q_{accept}a}{q_{accept}}\right]$ into P'



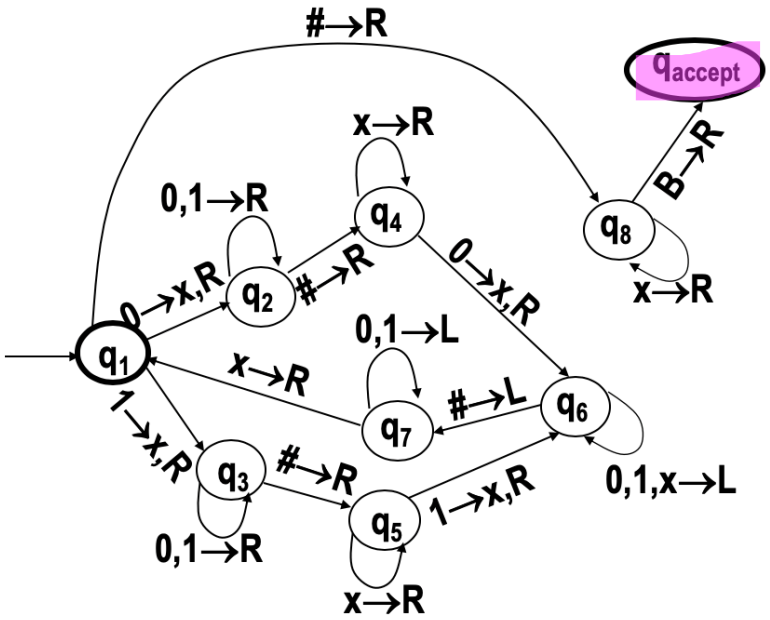
Example of PCP to A_{TM} , input is x#x



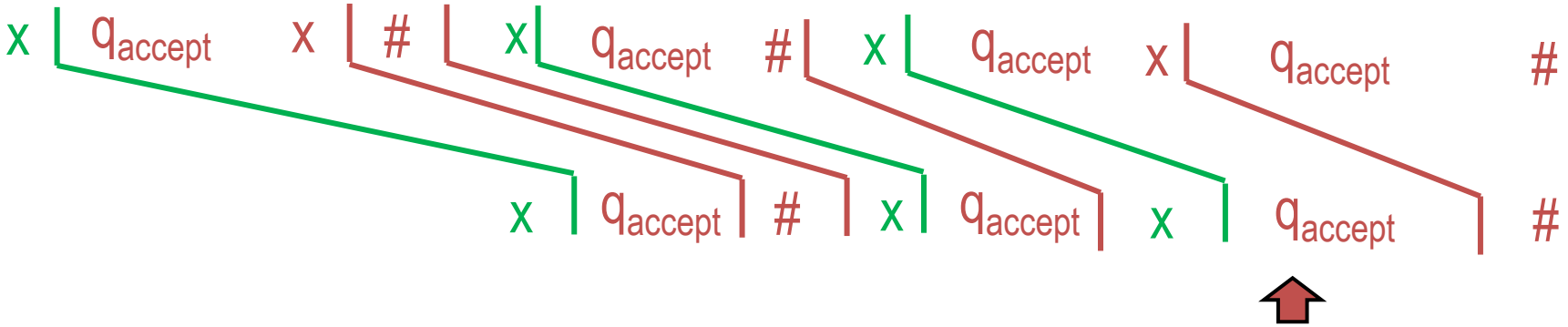
Finish the match, put $\left[\frac{q_{accept}\#\#}{\#}\right]$ into P'



Example of PCP to A_{TM} , input is x#x

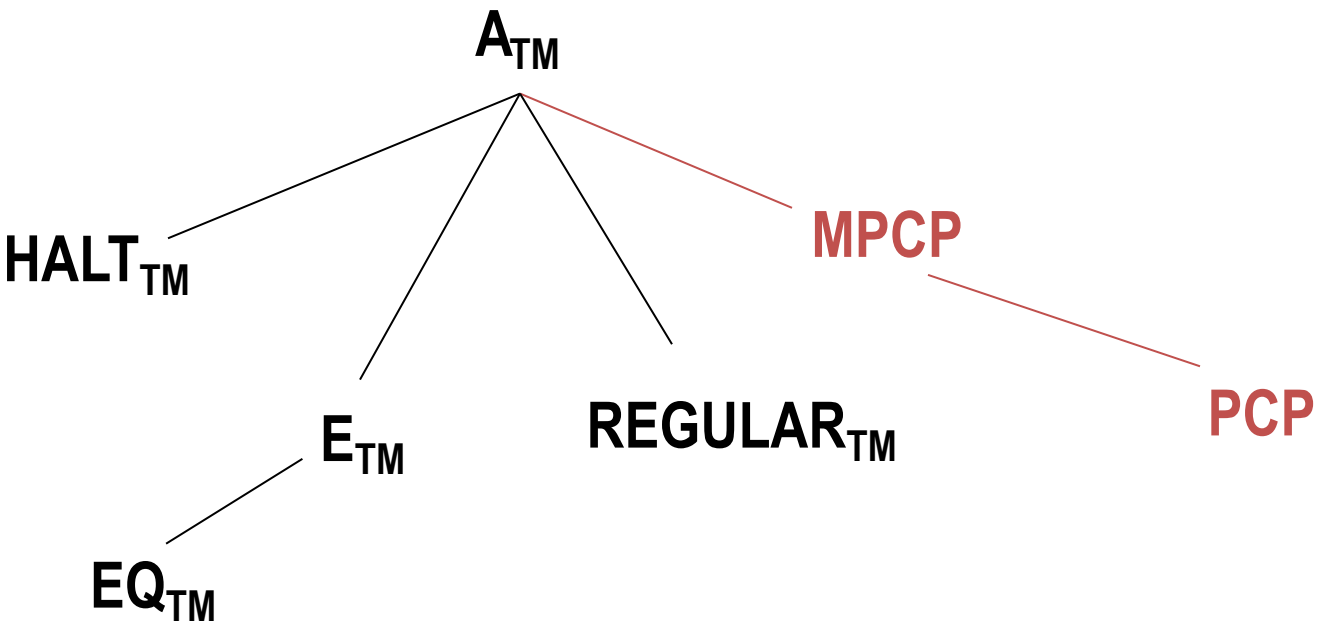


Finish the match, put $\left[\frac{q_{accept}\#}{\#}\right]$ into P'



Conclusion

- Relationship of languages on reducibility



Conclusion

- Closure on operations

	Complement $\overline{A}$	Intersection ∩	Union ∪	Star A^*
Regular/DFA/ NFA	√	√	√	√
CFL/ PDA	×	×	√	√
Turing-decidable TM	√	√	√	√
Turing-recognizable TM	×	√	√	√