Methods of Knowledge Representation Using Type-1 Fuzzy Sets

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Outline

- Introduction
- Basic terms
- Operations on fuzzy sets
- The extension principle
- Fuzzy numbers
- Triangular norms and negations
- Fuzzy relations and their properties
- Approximate reasoning
- Fuzzy inference systems
- Application of fuzzy sets

Introduction

- Real world phenomena is ambiguous and imprecise
  - “high temperature”
  - “young man”
  - “average height”
  - “large city”
- Describe using classical theory of sets and bivalent logic
  - Unable to formally describe
- Fuzzy sets theory to help

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Universe of Discourse

- Case 1: a lot of money
  - If we limit to USD [0, 1000], we get a different sum
  - If we limit to USD [0, 1000000], we get another different sum
  - What we limit is the universe of discourse

Definition: Fuzzy Set

The fuzzy set A in a given (non-empty) space $\mathbf{X}$, which is denoted as $A \subseteq \mathbf{X}$, is the set of pairs

$$A = \{(x, \mu_A(x))\,; x \in \mathbf{X}\},$$

in which

$$\mu_A : \mathbf{X} \to [0,\ 1]$$

is the membership function of a fuzzy set A.

- Three cases:

1) $\mu_A(x) = 1$ means the full membership of element x to the fuzzy set A, i.e. $x \in A$,

2) $\mu_A(x) = 0$ means the lack of membership of element x to the fuzzy set A, i.e. $x \notin A$,

3) $0 < \mu_A(x) < 1$ means a partial membership of element x to the fuzzy set A.

Definition: Fuzzy Set (2)

- $\mathbf{X} = \{x_1, ..., x_n\}$ is a finite set
- Fuzzy set $A \subseteq \mathbf{X}$

$$A = \frac{\mu_A(x_1)}{x_1} + \frac{\mu_A(x_2)}{x_2} + \ldots + \frac{\mu_A(x_n)}{x_n} = \sum_{i=1}^{n} \frac{\mu_A(x_i)}{x_i}$$

- $\frac{\mu_A(x_i)}{x_i}$ $i = 1, ..., n$ means $(x_i, \mu_A(x_i))$ $i = 1, ..., n$
- "+" means set union
- If X is infinite

$$A = \int_{\mathbf{X}} \frac{\mu_A(x)}{x}$$

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Example: Natural Numbers

- X = N, a set of natural numbers
- Define the term: set of natural number “close to 7”

$$A = \frac{0.2}{4} + \frac{0.5}{5} + \frac{0.8}{6} + \frac{1}{7} + \frac{0.8}{8} + \frac{0.5}{9} + \frac{0.2}{10}$$

Example: Real Numbers

- X = R, a set of real numbers
- Define the term: set of real number “close to 7”
  - How?

Example: Real Numbers

- X = R, a set of real numbers
- Define the term: set of real number “close to 7”
  - Membership function: $\mu_A(x) = \dfrac{1}{1+(x-7)^2}$
  - The fuzzy set

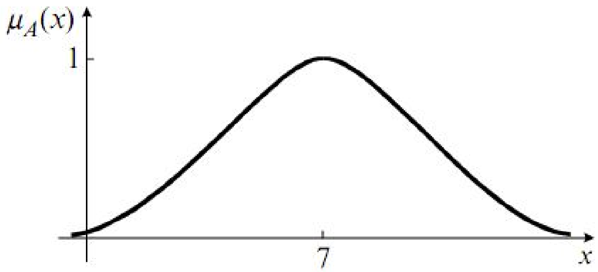
$$A = \int_{\mathbf{X}} \frac{\left[1+(x-7)^2\right]^{-1}}{x}$$

  - Many other ways…

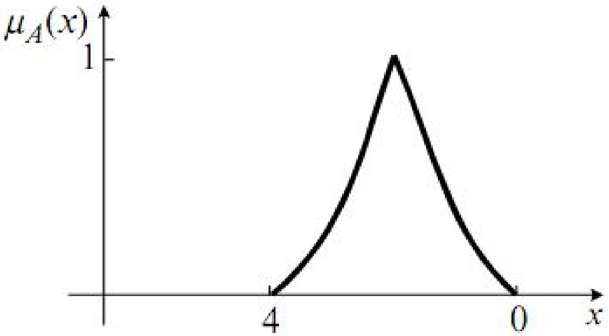
$$\mu_A(x) = \begin{cases} 1-\sqrt{\dfrac{|x-7|}{3}}, & \text{if} \quad 4 \le x \le 10, \\ 0, & \text{otherwise.} \end{cases}$$

Example: Real Numbers (2)

$$\mu_A(x) = \frac{1}{1+(x-7)^2}$$



$$\mu_A(x) = \begin{cases} 1-\sqrt{\dfrac{|x-7|}{3}}, & \text{if} \quad 4 \le x \le 10, \\ 0, & \text{otherwise.} \end{cases}$$



Example: Proper Temp.

- Appropriate temperature of water for swimming
  - Universe of discourse $X = [15°, ..., 25°]$
  - Vacationer A:

$$A = \frac{0.1}{16} + \frac{0.3}{17} + \frac{0.5}{18} + \frac{0.8}{19} + \frac{0.95}{20} + \frac{1}{21} + \frac{0.9}{22} + \frac{0.8}{23} + \frac{0.75}{24} + \frac{0.7}{25}.$$

  - Vacationer B:

$$B = \frac{0.1}{15} + \frac{0.2}{16} + \frac{0.4}{17} + \frac{0.7}{18} + \frac{0.9}{19} + \frac{1}{20} + \frac{0.9}{21} + \frac{0.85}{22} + \frac{0.8}{23} + \frac{0.75}{24} + \frac{0.7}{25}.$$

Remark

- The fuzzy sets theory describes the uncertainty in a different sense than the probability theory
- Probability: the probability of casting 4, 5 or 6 while tossing a dice
- Fuzzy: the imprecise notion "casting a large number of pips"

$$A = \frac{0.6}{4} + \frac{0.8}{5} + \frac{1}{6}$$

$$A = \frac{0.1}{3} + \frac{0.5}{4} + \frac{0.85}{5} + \frac{1}{6}$$

Singleton Function

- The singleton is a specific function

$$\mu_A(x) = \begin{cases} 1, & \text{if} \quad x = \overline{x}, \\ 0, & \text{if} \quad x \neq \overline{x}. \end{cases}$$

- This membership function characterizes a single-element fuzzy set

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Gaussian Membership Function

- Gaussian membership function

$$\mu_A(x) = \exp\left(-\left(\frac{x-\overline{x}}{\sigma}\right)^2\right)$$

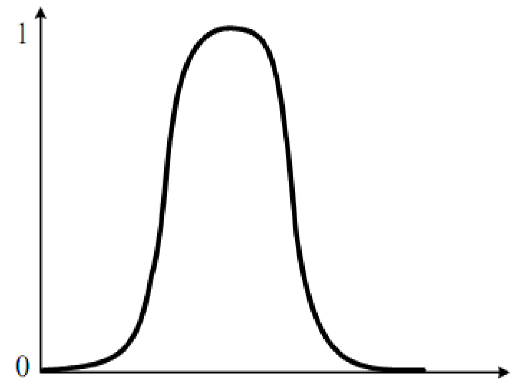
  – x is the middle
  – σ defines the width of the Gaussian curve

Bell Membership Function

- Bell membership function

$$\mu(x; a, b, c) = \frac{1}{1 + \left| \frac{x - c}{a} \right|^{2b}}$$

  - Parameter a defines width
  - Parameter b defines slopes
  - Parameter c defines center



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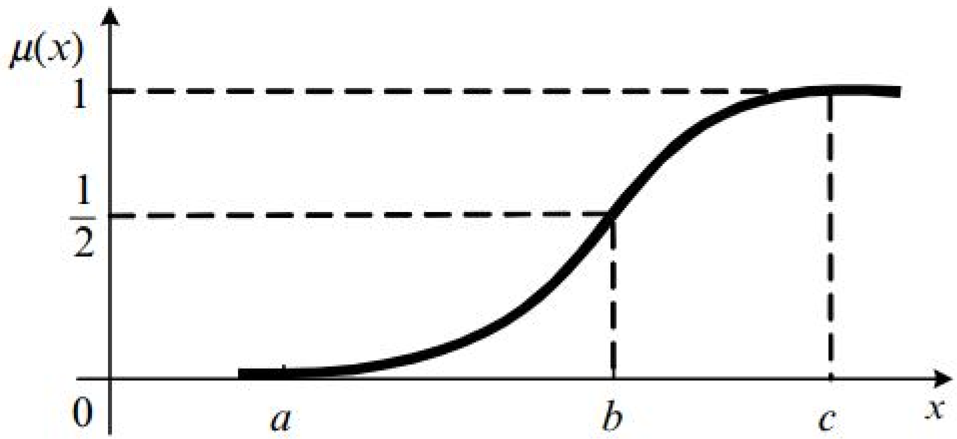
Membership Function of Class s

- Membership function of class s

$$
s(x; a, b, c) = \begin{cases} 0 & \text{for} \quad x \le a, \\ 2\left(\dfrac{x-a}{c-a}\right)^2 & \text{for} \quad a < x \le b, \\ 1 - 2\left(\dfrac{x-c}{c-a}\right)^2 & \text{for} \quad b < x \le c, \\ 1 & \text{for} \quad x > c. \end{cases}
$$

$$
b = (a + c)/2
$$



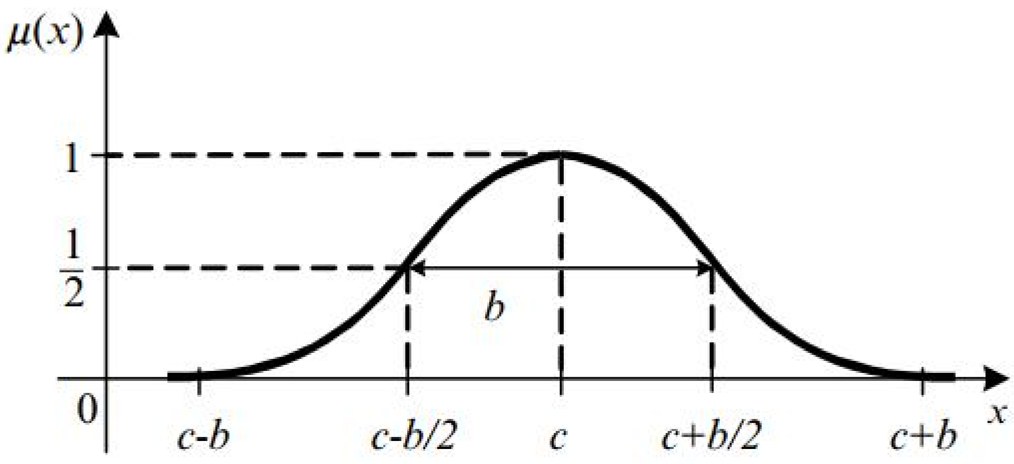
Membership Function of Class π

- Membership function of class π

$$\pi(x; b, c) = \begin{cases} s(x; c-b,\ c-b/2, c) & \text{for} \quad x \le c, \\ 1 - s(x; c, c+b/2, c+b) & \text{for} \quad x > c. \end{cases}$$

zero values for $x \ge c + b$ and $x \le c - b$

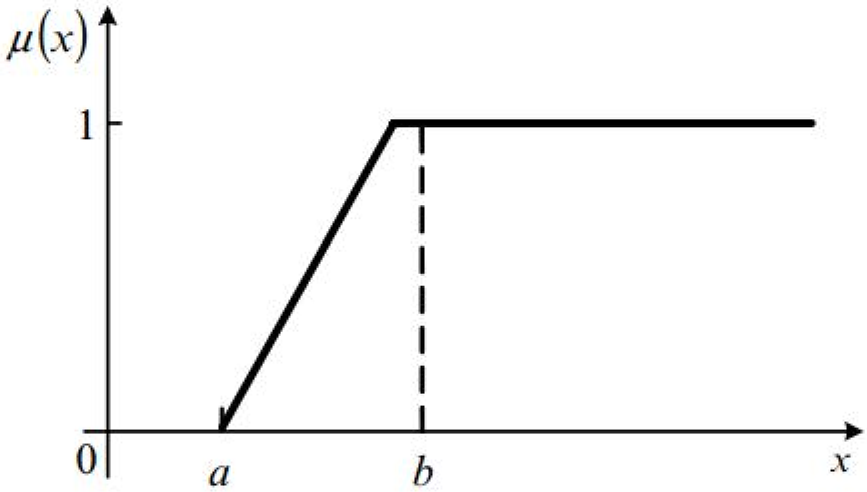
$x = c \pm b/2$ its value is 0.5



Membership Function of Class γ

- Membership function of class γ

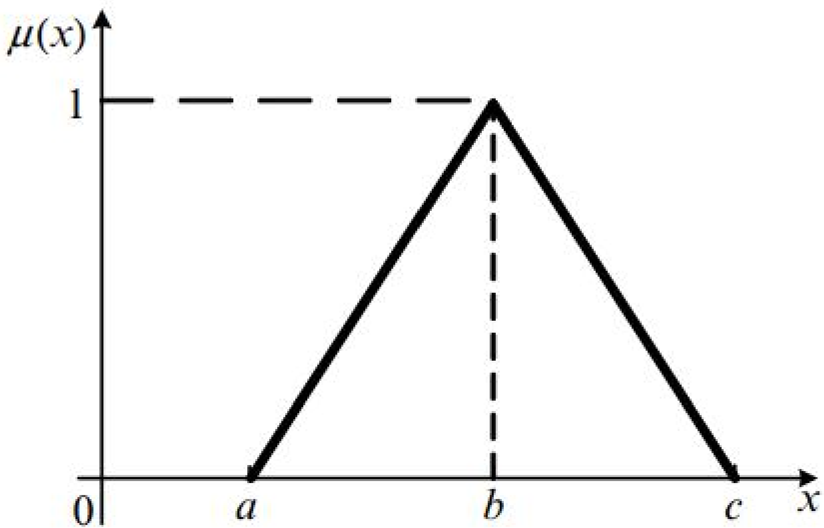
$$\gamma(x; a, b) = \begin{cases} 0 & \text{for} \quad x \le a, \\ \dfrac{x-a}{b-a} & \text{for} \quad a < x \le b, \\ 1 & \text{for} \quad a > b. \end{cases}$$



Membership Function of Class t

- Membership function of class t

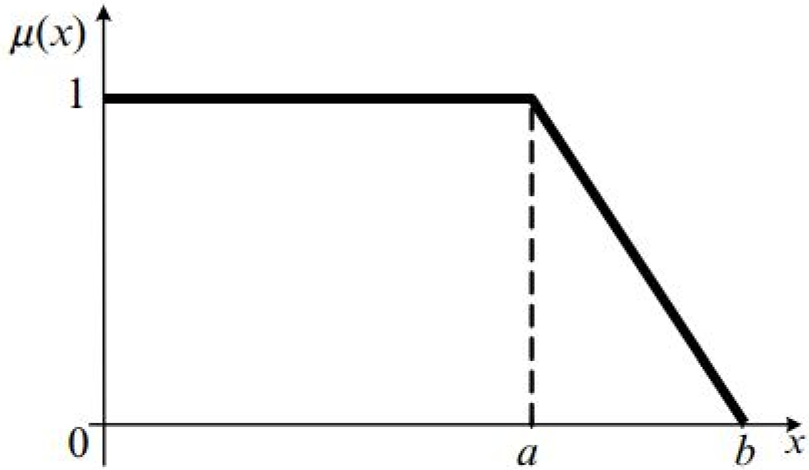
$$t\,(x;a,b,c)=\begin{cases}0 & \text{for} \quad x\le a,\\ \dfrac{x-a}{b-a} & \text{for} \quad a<x\le b,\\ \dfrac{c-x}{c-b} & \text{for} \quad b<x\le c,\\ 0 & \text{for} \quad x>c.\end{cases}$$



Membership Function of Class L

- Membership function of class L

$$L(x; a, b) = \begin{cases} 1 & \text{for} \quad x \le a, \\ \dfrac{b - x}{b - a} & \text{for} \quad a < x \le b, \\ 0 & \text{for} \quad a > b. \end{cases}$$



Multidimensional Membership Functions

- Assume case: independence of variables
- Multidimensional membership functions: Cartesian product of fuzzy sets

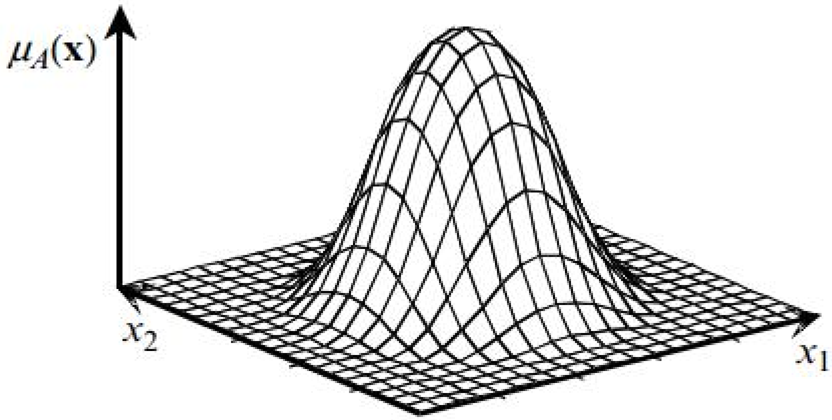
Membership Function of Class Π

- Membership function of class Π

$$
\mu_A(\mathbf{x}) = \begin{cases} 1 - 2 \cdot \left(\dfrac{\|\mathbf{x} - \bar{\mathbf{x}}\|}{\alpha} \right)^2 & \text{for} \quad \|\mathbf{x} - \bar{\mathbf{x}}\| \leq \dfrac{1}{2}\alpha, \\ 2 \cdot \left(1 - \dfrac{\|\mathbf{x} - \bar{x}\|}{\alpha} \right)^2 & \text{for} \quad \dfrac{1}{2}\alpha < \|\mathbf{x} - \bar{\mathbf{x}}\| \leq \alpha, \\ 0 & \text{for} \quad \|\mathbf{x} - \bar{\mathbf{x}}\| > \alpha, \end{cases}
$$

$\bar{\mathbf{x}}$ is the center of the membership function

$\alpha > 0$ is the parameter defining its spread



Radial Membership Function

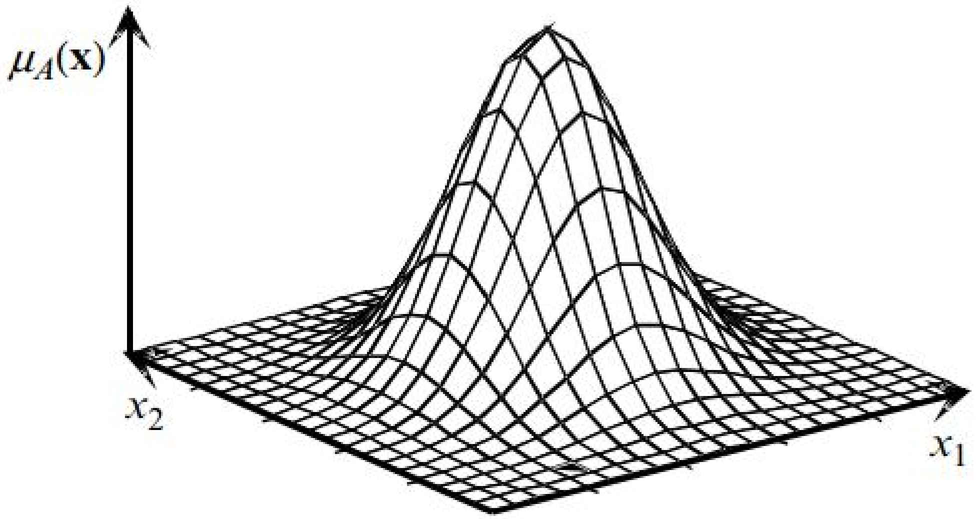
- Radial membership function

$$\mu_A(x) = \exp\left(-\left(\frac{x-\overline{x}}{\sigma}\right)^2\right)$$

$$\mu_A(\mathbf{x}) = \mathrm{e}^{\frac{\|\mathbf{x}-\overline{\mathbf{x}}\|^2}{2\cdot\sigma^2}}$$

$\overline{x}$ is the center

parameter σ influences the shape



Ellipsoidal Membership Function

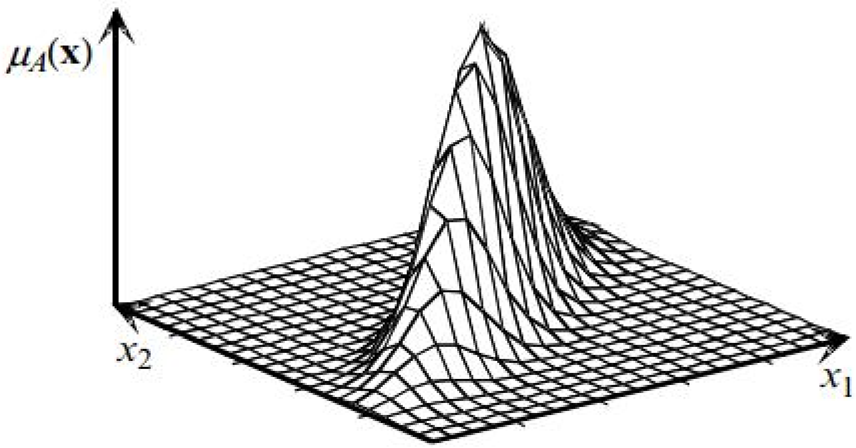
- Ellipsoidal membership function

$$\mu_A(\mathbf{x}) = \exp\left(-\frac{(\mathbf{x}-\overline{\mathbf{x}})^T \mathbf{Q}^{-1} (\mathbf{x}-\overline{\mathbf{x}})}{\alpha}\right)$$

$\overline{\mathbf{x}}$ is the center

$\alpha > 0$ is the parameter defining the spread

$\mathbf{Q}$ is the so-called covariance matrix

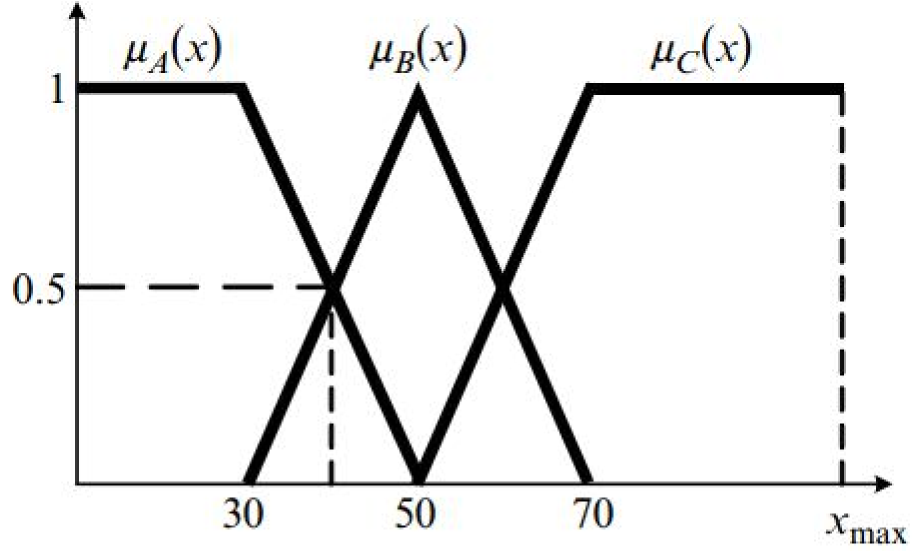


Example: Car Speed

- Three imprecise statements
  - “low speed of the car“, ”medium speed of the car“, "high speed of the car“
  - Assume set A is of the L type, B: t type, C: class γ

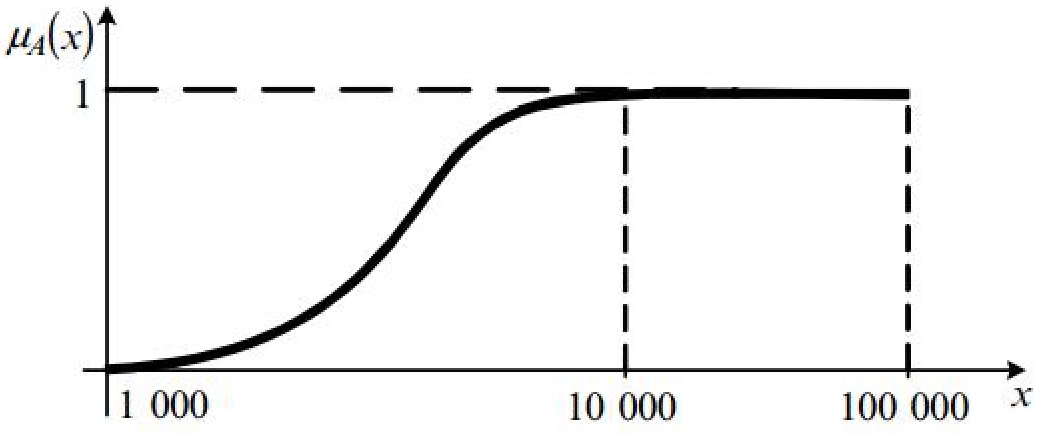
interval $[0, x_{\max}]$ as the universe of discourse $\mathbf{X}$

$x_{\max}$ is the maximum speed



Example: Amount of Money

- Class s function, X= [0; 100000], a= 1000, c= 10000



Definition: Support, Height, Normal

- Support of a fuzzy set

$$\text{supp } A = \{x \in \mathbf{X};\ \mu_A(x) > 0\}$$

If $X = \{1, 2, 3, 4, 5\}$ and

$$A = \frac{0.2}{1} + \frac{0.4}{2} + \frac{0.7}{4},$$

then $\text{supp } A = \{1, 2, 4\}$.

Definition: Support, Height, Normal

- Height of a fuzzy set

$$h(A) = \sup_{x \in \mathbf{X}} \mu_A(x)$$

If $\mathbf{X} = \{1, 2, 3, 4\}$ and

$$A = \frac{0.3}{2} + \frac{0.8}{3} + \frac{0.5}{4},$$

then $h(A) = 0.8.$

Definition: Support, Height, Normal

- Normal of a fuzzy set

$$\mu_{A_{nor}}(x) = \frac{\mu_A(x)}{h(A)}$$

$$A = \frac{0.1}{2} + \frac{0.5}{4} + \frac{0.3}{6}$$

$$A_{nor} = \frac{0.2}{2} + \frac{1}{4} + \frac{0.6}{6}$$

Definition: Empty, Included, Equal

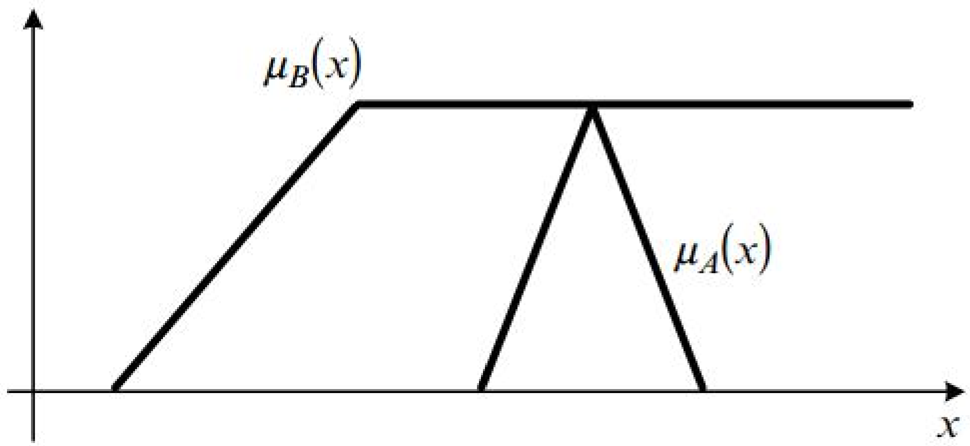
- Empty fuzzy set

$$\mu_A(x) = 0 \text{ for each } x \in \mathbf{X}$$

- Included fuzzy set

The fuzzy set A is *included* in the fuzzy set B, which shall be notated $A \subset B$, if and only if

$$\mu_A(x) \le \mu_B(x)$$



Definition: Empty, Included, Equal

- Equal fuzzy set

$$\mu_A(x) = \mu_B(x)$$

- Equality degree of fuzzy sets

$$E(A = B) = 1 - \max_{x \in T} \left| \mu_A(x) - \mu_B(x) \right|,$$

where $T = \{x \in X : \mu_A(x) \neq \mu_B(x)\}$

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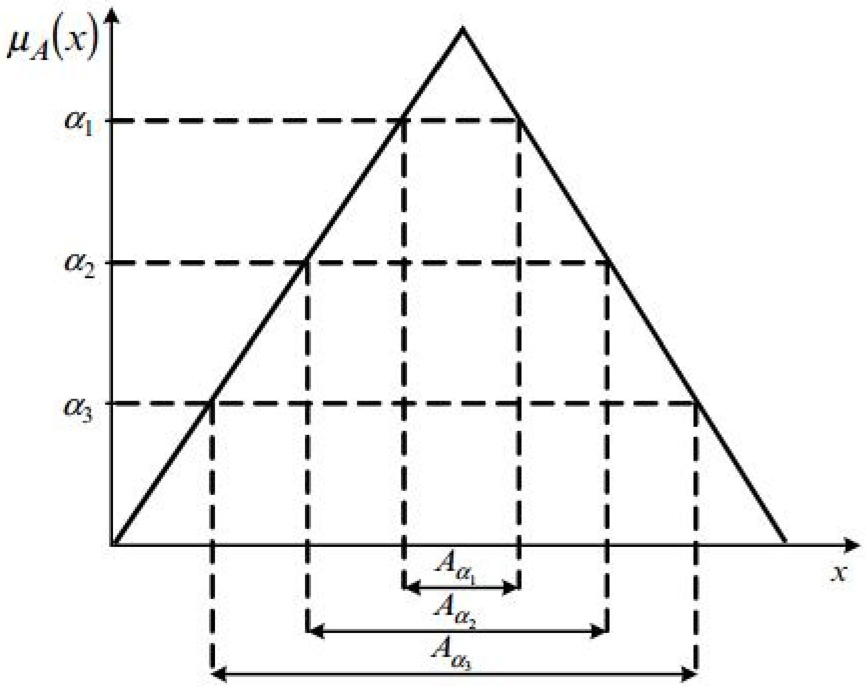
Definition: α-cut

α-cut of the fuzzy set $A \subseteq \mathbf{X}$, notated as A_α is called the following non-fuzzy set:

$$A_\alpha = \{x \in \mathbf{X} : \mu_A(x) \geq \alpha\}, \quad \forall_{\alpha \in [0,1]},$$

or the set defined by the characteristic function

$$\chi_{A_\alpha}(x) = \begin{cases} 1 & \text{for} \quad \mu_A(x) \geq \alpha, \\ 0 & \text{for} \quad \mu_A(x) < \alpha. \end{cases}$$



Example: α-cut

Let us consider the fuzzy set $A \subseteq \mathbf{X}$

$$A = \frac{0.1}{2} + \frac{0.3}{4} + \frac{0.7}{5} + \frac{0.8}{8} + \frac{1}{10},$$

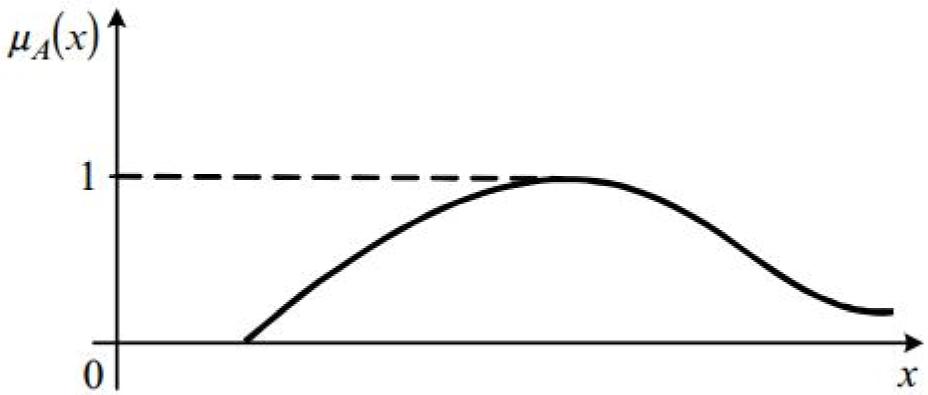
while $\mathbf{X} = \{1, ..., 10\}$. According to Definition 4.8 particular α-cuts are defined as follows:

$$\begin{aligned}
A_0 &= \mathbf{X} = \{1, ..., 10\}, \\
A_{0.1} &= \{2,\ 4,\ 5,\ 8,\ 10\}, \\
A_{0.3} &= \{4,\ 5,\ 8,\ 10\}, \\
A_{0.7} &= \{5,\ 8,\ 10\}, \\
A_{0.8} &= \{8,\ 10\}, \\
A_1 &= \{10\}.
\end{aligned}$$

Definition: Convex

The fuzzy set $A \subseteq \mathbf{R}$ is *convex* if and only if for any $x_1, x_2 \in \mathbf{R}$ and $\lambda \in [0,1]$ the following occurs

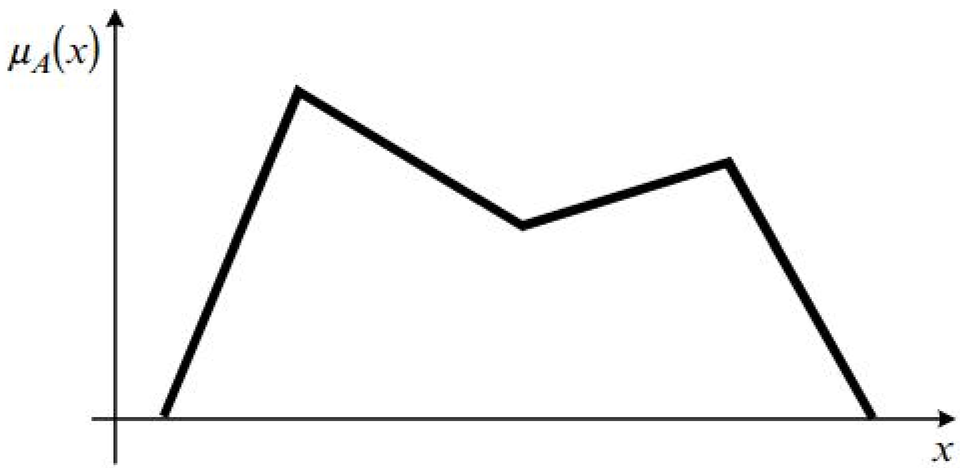
$$\mu_A[\lambda x_1 + (1-\lambda)x_2] \geq \min\{\mu_A(x_1), \mu_A(x_2)\}.$$



Definition: Concave

The fuzzy set $A \subseteq \mathbf{R}$ is *concave* if and only if there are such points x_1, $x_2 \in \mathbf{R}$ and $\lambda \in [0,1]$, that the following inequality holds

$$\mu_A\left[\lambda x_1 + (1-\lambda)\,x_2\right] < \min\left\{\mu_A\left(x_1\right), \mu_A\left(x_2\right)\right\}.$$



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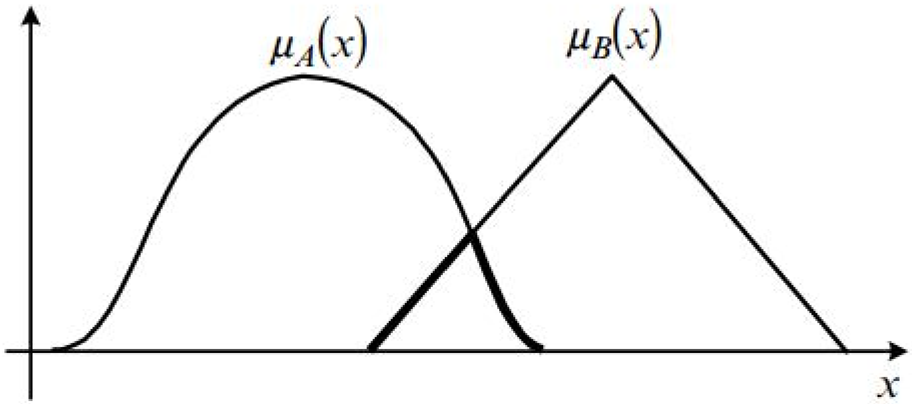
Intersection of Fuzzy Sets

The intersection of fuzzy sets A, $B \subseteq \mathbf{X}$ is the fuzzy set $A \cap B$ with the membership function

$$\mu_{A \cap B}(x) = \min\left(\mu_A(x), \mu_B(x)\right)$$

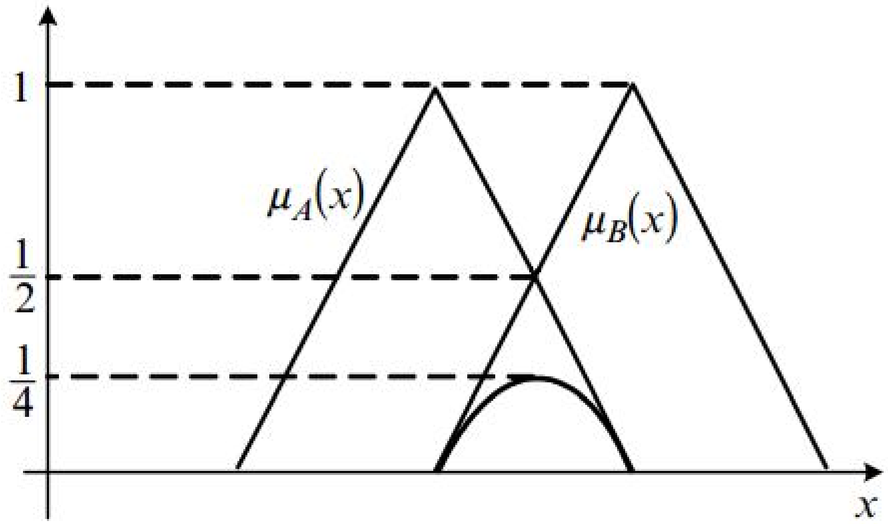
$$\mu_{A_1 \cap A_2 \cdots \cap A_n}(x) = \min\left[\mu_{A_1}(x), \mu_{A_2}(x), \ldots, \mu_{A_n}(x)\right]$$



Algebraic Product of Fuzzy Sets

The algebraic product of fuzzy sets A and B is the fuzzy set $C = A \cdot B$ defined as follows:

$$C = \{(x, \mu_A(x) \cdot \mu_B(x)) \mid x \in \mathbf{X}\}.$$

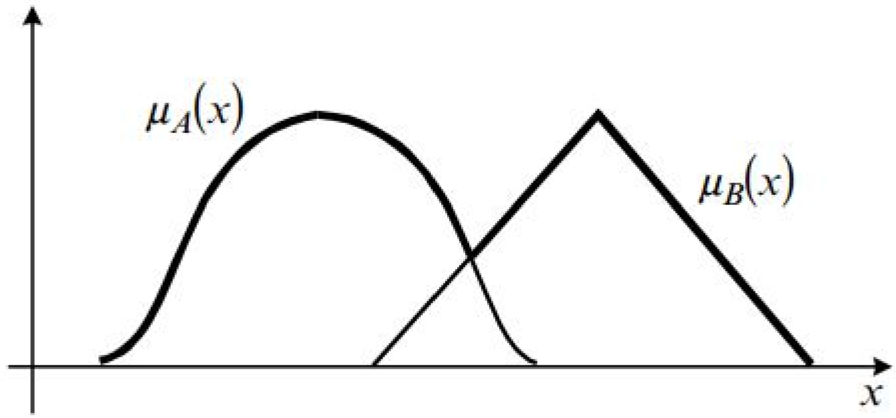


Union of Fuzzy Sets

The union of fuzzy sets A and B is the fuzzy set $A \cup B$ defined by the membership function

$$\mu_{A\cup B}(x) = \max\left(\mu_A(x), \mu_B(x)\right)$$

$$\mu_{A_1\cup A_2\cup\cdots\cup A_n}(x) = \max\left[\mu_{A_1}(x), \mu_{A_2}(x), ..., \mu_{A_n}(x)\right]$$



Example: Intersection and Union

$$\mathbf{X} = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7\}$$

$$A = \frac{0.9}{3} + \frac{1}{4} + \frac{0.6}{6},$$

$$B = \frac{0.7}{3} + \frac{1}{5} + \frac{0.4}{6}.$$

$$A \cap B = \frac{0.7}{3} + \frac{0.4}{6}.$$

$$A \cup B = \frac{0.9}{3} + \frac{1}{4} + \frac{1}{5} + \frac{0.6}{6}$$

$$A \cdot B = \frac{0.63}{3} + \frac{0.24}{6}.$$

Decomposition Theorem

Any fuzzy set $A \subseteq \mathbf{X}$ may be presented in the form

$$A = \bigcup_{\alpha \in [0,1]} \alpha A_\alpha,$$

where αA_α means a fuzzy set, to the elements of which the following membership degrees have been assigned:

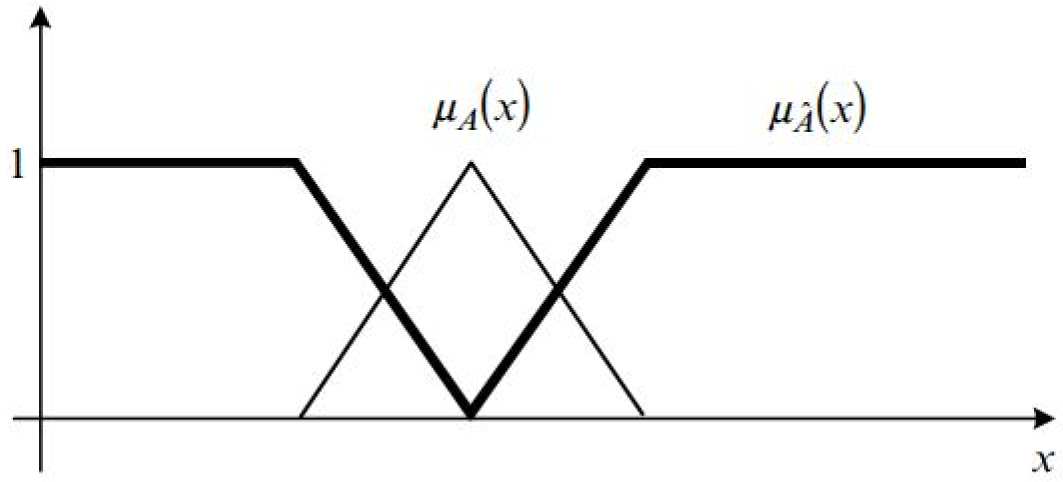
$$\mu_{\alpha A_\alpha}(x) = \begin{cases} \alpha & \text{for} \quad x \in A_\alpha, \\ 0 & \text{for} \quad x \notin A_\alpha. \end{cases}$$

Complement of Fuzzy Set

The complement of a fuzzy set $A \subseteq \mathbf{X}$ is the fuzzy set $\widehat{A}$ with the membership function

$$\mu_{\widehat{A}}(x) = 1 - \mu_A(x)$$

for each $x \in \mathbf{X}$.



Example: Complement

$$\mathbf{X} = \{1, 2, 3, 4, 5, 6\}$$

$$A = \frac{0.3}{2} + \frac{1}{3} + \frac{0.7}{5} + \frac{0.9}{6}.$$

$$\widehat{A} = \frac{1}{1} + \frac{0.7}{2} + \frac{1}{4} + \frac{0.3}{5} + \frac{0.1}{6}.$$

$$A \cap \widehat{A} = \frac{0.3}{2} + \frac{0.3}{5} + \frac{0.1}{6} \neq \varnothing$$

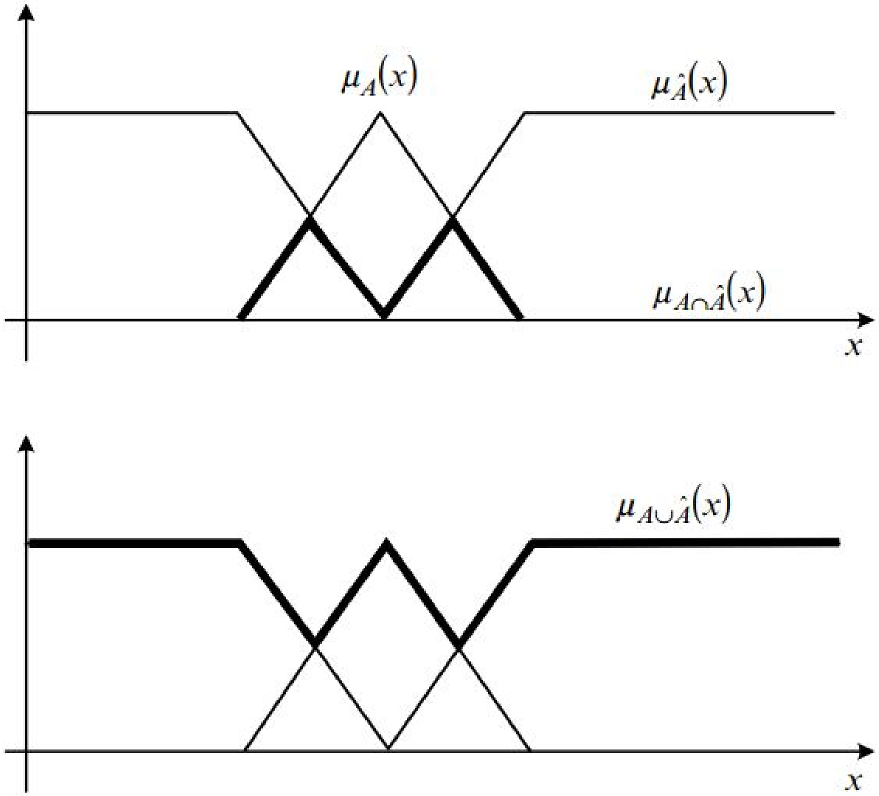
$$A \cup \widehat{A} = \frac{1}{1} + \frac{0.7}{2} + \frac{1}{3} + \frac{1}{4} + \frac{0.7}{5} + \frac{0.9}{6} \neq \mathbf{X}.$$

Remark: Notice the Inequality

- Why?

$$\mu_{A \cap \hat{A}}(x) = \min\left(\mu_A(x), \mu_{\hat{A}}(x)\right) \leq \frac{1}{2}$$

$$\mu_{A \cup \hat{A}}(x) = \max\left(\mu_A(x), \mu_{\hat{A}}(x)\right) \geq \frac{1}{2}$$



Definition: Cartesian Product

The Cartesian product of fuzzy sets $A \subseteq \mathbf{X}$ and $B \subseteq \mathbf{Y}$ is notated as $A \times B$ and defined as

$$\mu_{A\times B}(x,y) = \min\left(\mu_A(x), \mu_B(y)\right)$$

or

$$\mu_{A\times B}(x,y) = \mu_A(x)\,\mu_B(y)$$

$$\mu_{A_1\times A_2\times\ldots\times A_n}(x_1,x_2,\ldots,x_n) = \min(\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_n}(x_n))$$

or

$$\mu_{A_1\times A_2\times\ldots\times A_n}(x_1,x_2,\ldots,x_n) = \mu_{A_1}(x_1)\,\mu_{A_2}(x_2), \ldots, \mu_{A_n}(x_n)$$

Example: Cartesian Product

$$\mathbf{X} = \{2,\ 4\}\ ,\ \mathbf{Y} = \{2,\ 4,\ 6\} \text{ and}$$

$$A = \frac{0.5}{2} + \frac{0.9}{4},$$

$$B = \frac{0.3}{2} + \frac{0.7}{4} + \frac{0.1}{6}.$$

$$A \times B = \frac{0.3}{(2,2)} + \frac{0.5}{(2,4)} + \frac{0.1}{(2,6)} + \frac{0.3}{(4,2)} + \frac{0.7}{(4,4)} + \frac{0.1}{(4,6)}.$$

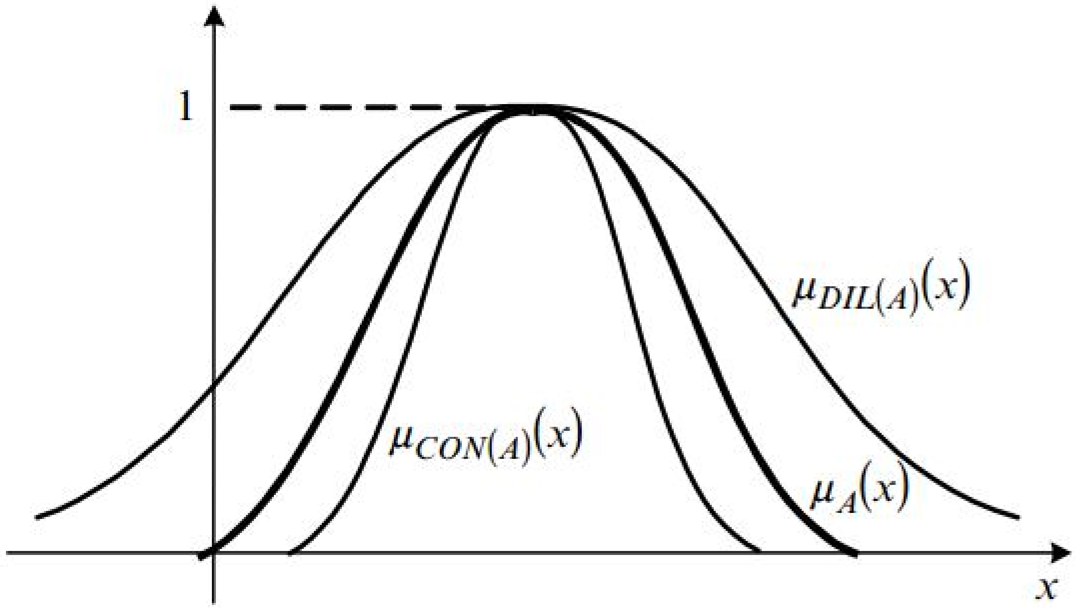
Definition: Concentration and Dilation

The concentration of a fuzzy set $A \subseteq \mathbf{X}$ shall be notated as $CON(A)$ and defined as

$$\mu_{CON(A)}(x) = (\mu_A(x))^2$$

The dilation of a fuzzy set $A \subseteq \mathbf{X}$ shall be notated as $DIL(A)$ and defined as

$$\mu_{DIL(A)}(x) = (\mu_A(x))^{0.5}$$



Example: Concentration and Dilation

$$\mathbf{X} = \{1,\ 2,\ 3,\ 4\}$$

$$A = \frac{0.4}{2} + \frac{0.7}{3} + \frac{1}{4},$$

$$CON\,(A) = \frac{0.16}{2} + \frac{0.49}{3} + \frac{1}{4},$$

$$DIL\,(A) = \frac{0.63}{2} + \frac{0.84}{3} + \frac{1}{4}.$$

Outline

- Introduction
- Basic terms
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Extension Principle

- Extend different mathematical operations from non-fuzzy to fuzzy sets

Consider a non-fuzzy mapping f of the space X in the space Y

$$f : \mathbf{X} \rightarrow \mathbf{Y}$$

given fuzzy set $A \in \mathbf{X}$

$$A = \frac{\mu_A(x_1)}{x_1} + \frac{\mu_A(x_2)}{x_2} + \cdots + \frac{\mu_A(x_n)}{x_n}$$

then

$$B = f(A) = \frac{\mu_A(x_1)}{f(x_1)} + \frac{\mu_A(x_2)}{f(x_2)} + \cdots + \frac{\mu_A(x_n)}{f(x_n)}$$

Example: Extension Principle

$$A = \frac{0.1}{3} + \frac{0.4}{2} + \frac{0.7}{5}$$

$$\text{and } f(x) = 2x + 1$$

$$B = f(A) = \frac{0.1}{7} + \frac{0.4}{5} + \frac{0.7}{11}$$

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Example: Extension Principle

$$A = \frac{0.3}{-2} + \frac{0.5}{3} + \frac{0.7}{2}$$

and $f(x) = x^2$

$$B = f(A) = \frac{0.5}{9} + \frac{0.7}{4}$$

Extension Principle I

$$B = f(A) = \{(y, \mu_B(y)) \mid y = f(x),\ x \in \mathbf{X}\},$$

where

$$\mu_B(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_A(x), & \text{if} \quad f^{-1}(y) \neq \varnothing, \\ 0, & \text{if} \quad f^{-1}(y) = \varnothing. \end{cases}$$

Extension Principle II

$$f : \mathbf{X}_1 \times \mathbf{X}_2 \times \ldots \times \mathbf{X}_n \rightarrow \mathbf{Y}$$

and given fuzzy sets $A_1 \subseteq \mathbf{X}_1$, $A_2 \subseteq \mathbf{X}_2, ..., A_n \subseteq \mathbf{X}_n$, then

$$B = f(A_1, ..., A_n) = \{(y, \mu_B(y)) \mid y = f(x_1, ..., x_n), (x_1, ..., x_n) \in \mathbf{X}\},$$

while

$$\mu_B(y) = \begin{cases} \sup\limits_{\substack{(x_1,...,x_n \\ \in f^{-1}(y))}} \min\{\mu_{A_1}(x_1), ..., \mu_{A_n}(x_n)\}, & \text{if } f^{-1}(y) \neq \varnothing, \\ 0, & \text{if } f^{-1}(y) = \varnothing. \end{cases}$$

Example: Extension Principle II

$$f(x_1, x_2) = \frac{x_1 x_2}{(x_1 + x_2)}$$

determine a fuzzy set $B = f(A_1, A_2)$

$$B = f(A_1, A_2) = \int_{x_1 \in \mathbf{X}_1} \int_{x_2 \in \mathbf{X}_2} \sup_{\substack{(x_1, \ldots, x_n \\ \in f^{-1}(y))}} \min\left(\mu_{A_i}(x_1), \mu_{A_2}(x_2)\right) \Big| \frac{x_1 x_2}{x_1 + x_2}$$

Example: Extension Principle II

$$A_1 = \frac{0.7}{1} + \frac{1}{2} + \frac{0.8}{3} \qquad A_2 = \frac{0.8}{3} + \frac{1}{4} + \frac{0.9}{5} \qquad y = f(x_1, x_2) = x_1 x_2$$

$$B = f(A_1, A_2) \qquad B \subseteq \mathbf{Y} = \{1, 2, \ldots, 36\}$$

$$B = f(A_1, A_2) = \sum_{i,j=1}^{3} \left[\min\left(\mu_{A_1}\left(x_1^{(i)}\right), \mu_{A_2}\left(x_2^{(j)}\right)\right)\right] / x_1^{(i)} x_2^{(j)}$$

$$= \frac{\min(0.7;0.8)}{3} + \frac{\min(0.7;1)}{4} + \frac{\min(0.7;0.9)}{5}$$

$$+ \frac{\min(1;0.8)}{6} + \frac{\min(1;1)}{8} + \frac{\min(1;0.9)}{10} + \frac{\min(0.8;0.8)}{9}$$

$$+ \frac{\min(0.8;1)}{12} + \frac{\min(0.8;0.9)}{15}$$

$$= \frac{0.7}{3} + \frac{0.7}{4} + \frac{0.7}{5} + \frac{0.8}{6} + \frac{1}{8} + \frac{0.8}{9} + \frac{0.9}{10} + \frac{0.8}{12} + \frac{0.8}{15}$$

Example: Extension Principle II

$\mathbf{X}$ is the Cartesian product of sets $\mathbf{X}_1 = \mathbf{X}_2 = \{1, 2, 3, 4\}$

$$A_1 = \frac{0.7}{1} + \frac{1}{2} + \frac{0.8}{3} \qquad A_2 = \frac{0.8}{2} + \frac{1}{3} + \frac{0.6}{4}$$

$$B = f(A_1, A_2) \qquad B \subseteq \mathbf{Y} = \{1, 2, \ldots, 16\}$$

$$B = f(A_1, A_2) = ?$$

Example: Extension Principle II

$\mathbf{X}$ is the Cartesian product of sets $\mathbf{X}_1 = \mathbf{X}_2 = \{1, 2, 3, 4\}$

$$A_1 = \frac{0.7}{1} + \frac{1}{2} + \frac{0.8}{3} \qquad A_2 = \frac{0.8}{2} + \frac{1}{3} + \frac{0.6}{4}$$

$$B = f(A_1, A_2) \qquad B \subseteq \mathbf{Y} = \{1, 2, \ldots, 16\}$$

$$\begin{aligned} B = f(A_1, A_2) = {} & \frac{\min(0.7; 0.8)}{2} + \frac{\min(0.7; 1)}{3} \\ & + \frac{\max[\min(0.7; 0.6); \min(1; 0.8)]}{4} \\ & + \frac{\max[\min(1; 1); \min(0.8; 0.8)]}{6} \\ & + \frac{\min(1; 0.6)}{8} + \frac{\min(0.8; 1)}{9} + \frac{\min(0.8; 0.6)}{12} \\ = {} & \frac{0.7}{2} + \frac{0.7}{3} + \frac{0.8}{4} + \frac{1}{6} + \frac{0.6}{8} + \frac{0.8}{9} + \frac{0.6}{12}. \end{aligned}$$

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Fuzzy Number

A fuzzy set A defined on the set of real numbers, $A \subseteq \mathbf{R}$, the membership function of which

$$\mu_A : \mathbf{R} \to [0,1]$$

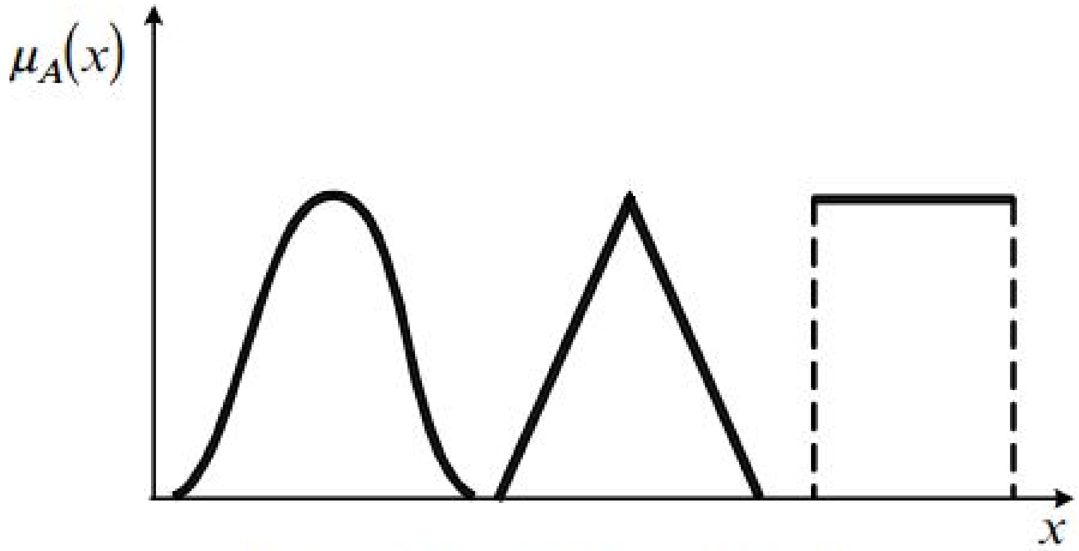
meets the conditions:

1) $\sup_{x \in \mathbf{R}} \mu_A(x) = 1$, i.e. the fuzzy set A is normal,

2) $\mu_A[\lambda x_1 + (1-\lambda)x_2] \geq \min\{\mu_A(x_1), \mu_A(x_2)\}$, i.e. the set A is convex,

3) $\mu_A(x)$ is a continuous function by intervals, is called a *fuzzy number*.

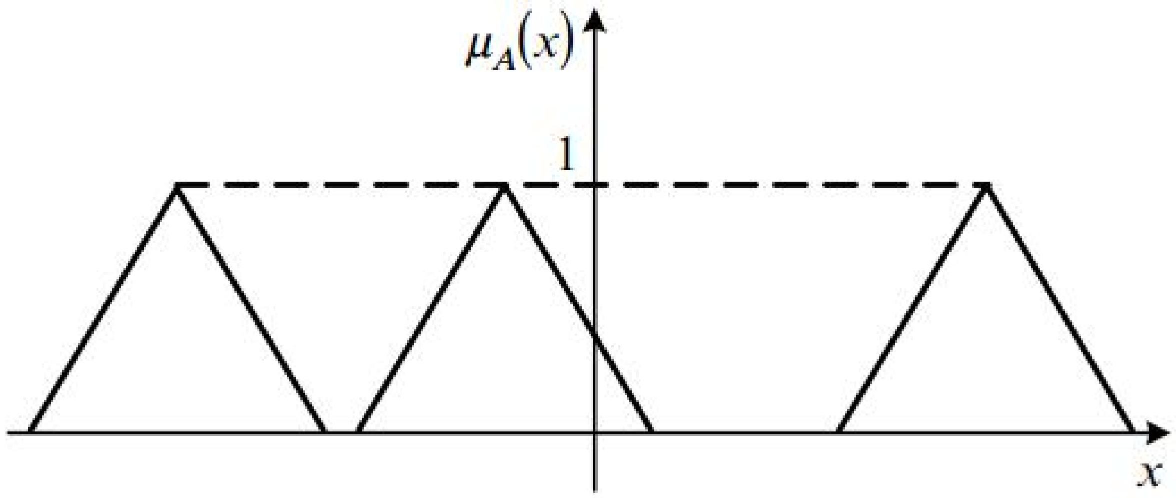
Fuzzy Numbers



Examples of fuzzy numbers

Positive/Negative Fuzzy Numbers

The fuzzy number $A \subseteq \mathbf{R}$ is *positive*, if $\mu_A(x) = 0$ for all $x < 0$.
The fuzzy number $A \subseteq \mathbf{R}$ is *negative*, if $\mu_A(x) = 0$ for all $x > 0$.



Basic Arithmetic Operations

- Adding two fuzzy numbers

$$A_1 \oplus A_2 \stackrel{\text{def}}{=} B,$$

$$\mu_B(y) = \sup_{\substack{x_1, x_2 \\ y = x_1 + x_2}} \min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2)\}$$

- Subtracting two fuzzy numbers

$$A_1 \ominus A_2 \stackrel{\text{def}}{=} B,$$

$$\mu_B(y) = \sup_{\substack{x_1, x_2 \\ y = x_1 - x_2}} \min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2)\}$$

Basic Arithmetic Operations

- Multiplying two fuzzy numbers

$$A_1 \odot A_2 \stackrel{\text{def}}{=} B,$$

$$\mu_B(y) = \sup_{\substack{x_1, x_2 \\ y = x_1 \cdot x_2}} \min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2)\}$$

- Dividing two fuzzy numbers

$$A_1 \oplus\!\!\!\!\!\div\, A_2 \stackrel{\text{def}}{=} B,$$

$$\mu_B(y) = \sup_{\substack{x_1, x_2 \\ y = x_1 : x_2}} \min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2)\}$$

Example: Arithmetic Operations

$$A_1 = \frac{0.7}{2} + \frac{1}{3} + \frac{0.6}{4} \qquad A_2 = \frac{0.8}{3} + \frac{1}{4} + \frac{0.5}{6}$$

$$\begin{aligned} A_1 \oplus A_2 = & \frac{\min(0.7;\ 0.8)}{5} + \frac{\max\{\min(0.7;\ 1),\min(1;\ 0.8)\}}{6} \\ & + \frac{\max\{\min(1;\ 1),\min(0.6;\ 0.8)\}}{7} \\ & + \frac{\max\{\min(0.7;\ 0.5),\min(0.6;\ 1)\}}{8} \\ & + \frac{\min(1;\ 0.5)}{9} + \frac{\min(0.6;\ 0.5)}{10} \\ = & \frac{0.7}{5} + \frac{0.8}{6} + \frac{1}{7} + \frac{0.6}{8} + \frac{0.5}{9} + \frac{0.5}{10}. \end{aligned}$$

- How about subtraction, multiplication, division?

Example: Arithmetic Operations

$$
\begin{aligned}
A_1 \odot A_2 = & \frac{\min(0.7;\ 0.8)}{6} + \frac{\min(0.7;\ 1)}{8} + \frac{\min(1;\ 0.8)}{9} \\
& + \frac{\max\{\min(0.7;\ 0.5), \min(1;\ 1), \min(0.6;\ 0.8)\}}{12} \\
& + \frac{\min(0.6;\ 1)}{16} + \frac{\min(1;\ 0.5)}{18} + \frac{\min(0.6;\ 0.5)}{24} \\
= & \frac{0.7}{6} + \frac{0.7}{8} + \frac{0.8}{9} + \frac{1}{12} + \frac{0.6}{16} + \frac{0.5}{18} + \frac{0.5}{24}.
\end{aligned}
$$

- Do the rest parts by yourself!

Unary Operations

- Unary operations on fuzzy numbers
  - Reversal of sign operation
  - Inverse operation
  - Scaling operation
  - Exponent operation
  - Absolute value operation

Unary Operations

- Reversal of sign operation

$$f(x) = -x \quad A \subseteq \mathbf{R} \quad -A \subseteq \mathbf{R}$$
$$\mu_{-A}(x) = \mu_A(-x)$$

- Inverse operation

$$f(x) = x^{-1}, x \neq 0$$
$$\mu_{A^{-1}}(x) = \mu_A\left(x^{-1}\right)$$

- Scaling operation

$$f(x) = \lambda x, \lambda \neq 0$$
$$\mu_{\lambda A}(x) = \mu_A\left(x\lambda^{-1}\right)$$

Unary Operations

- Exponent operation

$$f(x) = \mathrm{e}^{x}, x > 0$$

$$\mu_{\mathrm{e}^A}(x) = \begin{cases} \mu_A(\log\ x) & \text{for} \quad x > 0, \\ 0 & \text{for} \quad x < 0, \end{cases}$$

- Absolute value operation

$$\mu_{|A|}(x) = \begin{cases} \max(\mu_A(x), \mu_A(-x)) & \text{for} \quad x \geq 0, \\ 0 & \text{for} \quad x < 0. \end{cases}$$

Example: Unary Operations

$$A = \frac{0.7}{1} + \frac{1}{2} + \frac{0.6}{5},$$

$$-A = \frac{0.6}{-5} + \frac{1}{-2} + \frac{0.7}{-1},$$

$$A^{-1} = \frac{0.6}{0.2} + \frac{1}{0.5} + \frac{0.7}{1}.$$

- Note that

$$A + (-A) \neq \frac{1}{0}$$

$$A \cdot A^{-1} \neq \frac{1}{1}.$$

Remark

- The fuzzy numbers are characterized by a lack of opposite and inverse fuzzy number with relation to adding and multiplication
  - impossible to use, for instance, the elimination method to solve equations with fuzzy numbers

$$A + (-A) \neq \frac{1}{0}$$

$$A \cdot A^{-1} \neq \frac{1}{1}.$$

Fuzzy Numbers of L-P Type

- Let L, P be the functions mapping $(-\infty,\infty) \to [0,1]$
  - Meet conditions:

    1) $L(-x) = L(x)$ and $P(-x) = P(x)$,
    2) $L(0) = 1$ and $P(0) = 1$,
    3) L and P are nonincreasing functions in the interval $[0,+\infty)$
  - Examples of function L:

$$L(x) = P(x) = \mathrm{e}^{-|x|^p} \quad p > 0$$

$$L(x) = P(x) = \frac{1}{1+|x|^p} \quad p > 0$$

$$L(x) = P(x) = \max(0, 1-|x|^p) \quad p > 0$$

$$L(x) = P(x) = \begin{cases} 1 & \text{for} \quad x \in [-1,1], \\ 0 & \text{for} \quad x \notin [-1,1]. \end{cases}$$

Fuzzy Numbers of L-P Type

The fuzzy number $A \subseteq \mathbf{R}$ is *a fuzzy number of the* $L - P$ *type* if and only if

$$\mu_A(x) = \begin{cases} L\left(\dfrac{m-x}{\alpha}\right), & \text{if} \quad x \le m, \\ P\left(\dfrac{x-m}{\beta}\right), & \text{if} \quad x \ge m, \end{cases}$$

$A(\mu_A(m) = 1)$

α-positive real number, called the left-sided spread

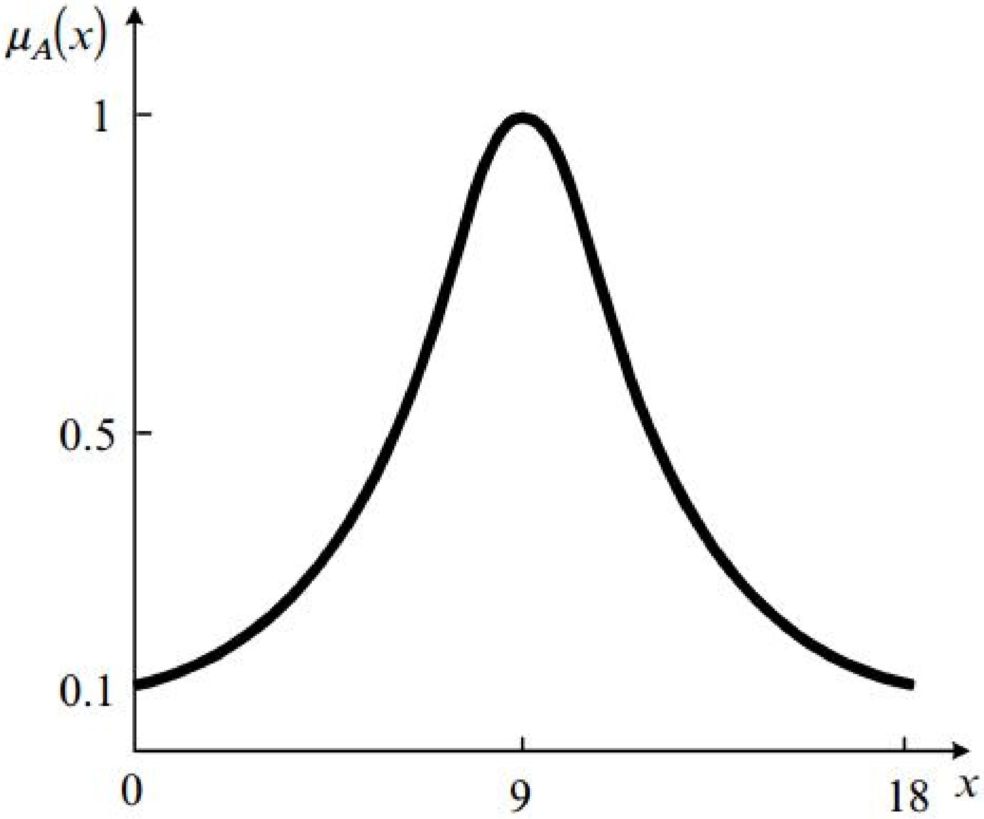
β-positive real number, called the right-sided spread

$$A = (m_A, \alpha_A, \beta_A)_{LP}$$

Example: L-P Type Fuzzy Number

$$A = (9, 3, 3)_{LP}.$$

$$L(x) = P(x) = \frac{1}{1+x^2}.$$



Flat Fuzzy Number

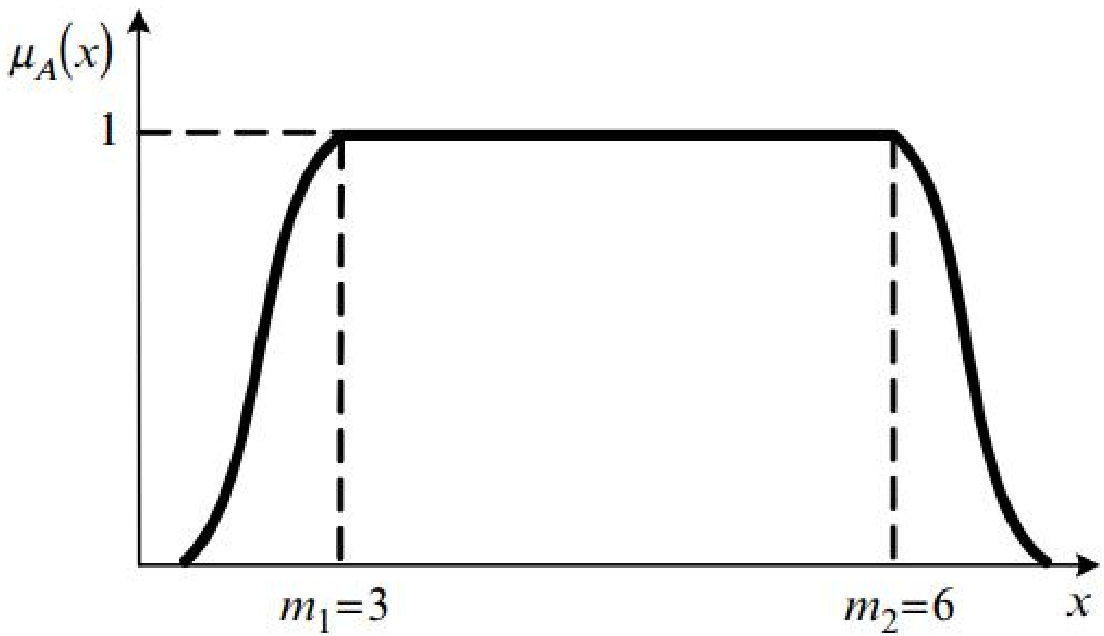
$$\mu_A(x) = \begin{cases} L\left(\dfrac{m_1 - x}{\alpha}\right), & \text{if} \quad x \le m_1, \\ 1, & \text{if} \quad m_1 \le x \le m_2, \\ P\left(\dfrac{x - m_2}{\beta}\right), & \text{if} \quad x \ge m_2. \end{cases}$$

$$A = (m_1, m_2, \alpha, \beta)_{LP}.$$

Example: Flat Fuzzy Number

- "the price of the motorbike in this store varies from approx. 3,000 USD to 6,000 USD"

$$A = (3, 6, \alpha, \beta)_{LP}$$



Triangular Fuzzy Number

- A triangular fuzzy number A is defined on the interval $[a_1,a_2]$, the membership function takes the value equal to 1 in the point a_M

$$A=(a_1, a_M, a_2)$$

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Generalized Intersection/Union

- Originally:

$$\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x)),$$
$$\mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x)).$$

- Generalized:

$$\mu_{A\cap B}(x) = T(\mu_A(x), \mu_B(x))$$
$$\mu_{A\cup B}(x) = S(\mu_A(x),\ \mu_B(x))$$

Function T is called t-norm

Function S is called t-conorm

t-norm and t-conorm belong to the so-called triangular norms

t-norm

- Function T:[0,1]×[0,1]→[0,1] is called a t-norm if:

(i) function T is nondecreasing with relation to both arguments

$$T(a,c) \leq T(b,d) \quad \text{for} \quad a \leq b,\ c \leq d$$

(ii) function T satisfies the condition of commutativity

$$T(a,b) = T(b,a)$$

(iii) function T satisfies the condition of associativity

$$T(T(a,b),c) = T(a,T(b,c))$$

(iv) function T satisfies the boundary condition

$$T(a,1) = a,$$

where $a,b,c,d \in [0,1]$.

$$T(a,b) = a \overset{T}{*} b.$$

Weighted t-norm

$$T^* \{a_1, \ldots, a_n;\ w_1, \ldots, w_n\} = \mathop{T}_{i=1}^{n} \{1 - w_i (1 - a_i)\},$$

$$0 \leq w_i \leq 1, \quad i = 1, \ldots, n.$$

t-conorm

- t-conorm function S:[0,1]×[0,1]→[0,1]
  - Also meets the condition of commutativity and associativity, and boundary condition
- Weighted t-conorm

$$S^*\{a_1, \ldots, a_n;\ w_1, \ldots, w_n\} = \mathop{S}_{i=1}^{n} \{w_i a_i\}$$

- Dual triangular norms

$$\mathop{S}_{i=1}^{n} \{a_i\} = 1 - \mathop{T}_{i=1}^{n} \{1 - a_i\}$$

$$\mathop{T}_{i=1}^{n} \{a_i\} = 1 - \mathop{S}_{i=1}^{n} \{1 - a_i\}$$

Archimedean Property

$$T\{a,a\} < a < S\{a,a\}$$

for each $a \in (0,1)$

np/st Type Dual Triangular Norm

- np (nilpotent) type triangular norm

$$T\{a_1, a_2, ..., a_n\} = 0,$$

$$S\{a_1, a_2, ..., a_n\} = 1.$$

- st (strict) type triangular norm

$$T\{a_1, a_2, \ldots, a_n\} > 0,$$

$$S\{a_1, a_2, \ldots, a_n\} < 1$$

for $0 < a_i < 1$, $i = 1, ..., n$, $n \geq 2$ and $a_1 = a_2 = ... = a_n$

Min/Max Type Dual Triangular Norm

$$T_M\{a_1, a_2\} = \min\{a_1, a_2\},$$

$$S_M\{a_1, a_2\} = \max\{a_1, a_2\},$$

$$T_M\{a_1, a_2, \dots, a_n\} = \min_{i=1,\dots,n}\{a_i\},$$

$$S_M\{a_1, a_2, \dots, a_n\} = \max_{i=1,\dots,n}\{a_i\}.$$

- Min/max triangular norms are dual but are not Archimedean

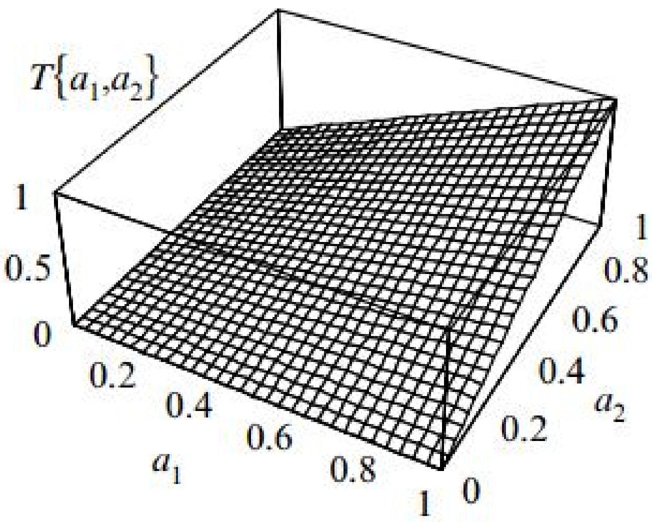
Algebraic Triangular Norms

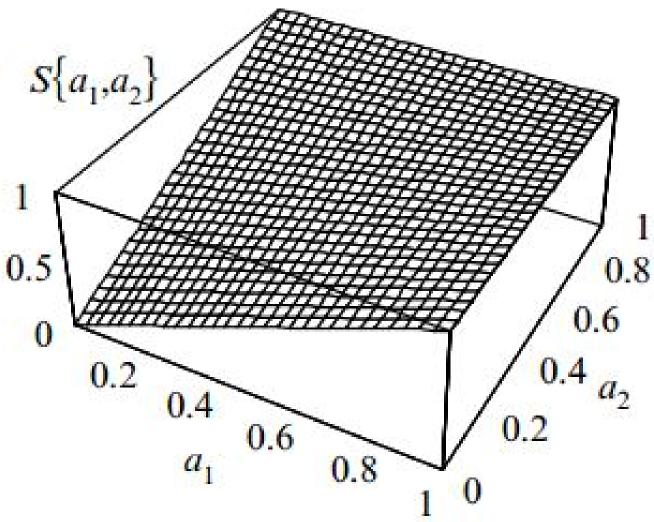
$$T_P\{a_1, a_2\} = a_1 a_2,$$

$$S_P\{a_1, a_2\} = a_1 + a_2 - a_1 a_2,$$

$$T_P\{a_1, a_2, ..., a_n\} = \prod_{i=1}^{n} a_i,$$

$$S_P\{a_1, a_2, ..., a_n\} = 1 - \prod_{i=1}^{n} (1 - a_i)$$





Algebraic Triangular Norms

- Algebraic triangular norms are dual triangular norms of the strict type

Remarks

- Try to find the definitions:
  - Łukasiewicz triangular norms
  - Boundary triangular norms
  - Multiplicative generator of Archimedean t-norm/t-conorm
  - Additive generator of Archimedean t-norm/t-conorm

Negations

- Negations are extensions of a logical contradiction

(i) A nonincreasing function $N : [0,1] \to [0,1]$ is called a *negation*, if $N(0) = 1$ and $N(1) = 0$.

(ii) Negation $N : [0,1] \to [0,1]$ is called a *st* (*strict*) *type* negation, if it is continuous and decreasing.

(iii) A negation of *st* (*strict*) type is called *strong type* negation, if it is involution, i.e. $N(N(a)) = a$.

Negations

- Zadeh negation: strong type

$$N(a) = 1 - a$$

- Yager negation: strict type

$$N(a) = (1 - a^p)^{\frac{1}{p}}, \quad p > 0$$

- Sugeno negation: strong type

$$N(a) = \frac{1 - a}{1 + pa}, \quad p > -1$$

De Morgan Triple

- T: t-norm, S: t-conorm, N: strong negation
  - Then

$$\mathop{S}_{i=1}^{n}\{a_i\} = N^{-1}\left(\mathop{T}_{i=1}^{n}\{N(a_i)\}\right)$$

$$\mathop{T}_{i=1}^{n}\{a_i\} = N^{-1}\left(\mathop{S}_{i=1}^{n}\{N(a_i)\}\right)$$

  - t-norm T and t-conorm S are called N-dual

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Fuzzy Relation

- The fuzzy relation **R** between two non-empty (non-fuzzy) sets **X** and **Y** is called the fuzzy set determined on the Cartesian product **X** × **Y**

$$R \subseteq \mathbf{X} \times \mathbf{Y} = \{(x, y) : x \in \mathbf{X},\ y \in \mathbf{Y}\}.$$

$$R = \{((x, y), \mu_R(x, y))\}, \quad \forall_{x \in \mathbf{X}} \forall_{y \in \mathbf{Y}},$$

$$\mu_R : \mathbf{X} \times \mathbf{Y} \to [0, 1]$$

$$R = \sum_{\mathbf{X} \times \mathbf{Y}} \frac{\mu_R(x, y)}{(x, y)}$$

$$R = \int_{\mathbf{X} \times \mathbf{Y}} \frac{\mu_R(x, y)}{(x, y)}$$

Example: Fuzzy Relation

- "y is more or less equal to x"
  - X={3, 4, 5}, Y={4, 5, 6}

$$R = \frac{1}{(4,4)} + \frac{1}{(5,5)} + \frac{0.8}{(3,4)} + \frac{0.8}{(4,5)} + \frac{0.8}{(5,4)} + \frac{0.8}{(5,6)} + \frac{0.6}{(3,5)} + \frac{0.6}{(4,6)} + \frac{0.4}{(3,6)}.$$

$$\mu_R(x,y) = \begin{cases} 1 & \text{if} \quad x = y, \\ 0.8 & \text{if} \quad |x-y| = 1, \\ 0.6 & \text{if} \quad |x-y| = 2, \\ 0.4 & \text{if} \quad |x-y| = 3. \end{cases}$$

Example: Fuzzy Relation

- Relation R notated using matrix

$$\begin{array}{c c} & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.8 & 0.6 & 0.4 \\ 1 & 0.8 & 0.6 \\ 0.8 & 1 & 0.8 \end{bmatrix} \end{array}$$

where $x_1 = 3$, $x_2 = 4$, $x_3 = 5$, and $y_1 = 4$, $y_2 = 5$, $y_3 = 6$

Example: Fuzzy Relation

- “a person of agex is much older than a person of age y”

  How?

Example: Fuzzy Relation

- “a person of age x is much older than a person of age y”

  Let X=Y=[0,120] be the human lifespan

$$\mu_R(x,y) = \begin{cases} 0 & \text{if} \quad x-y \le 0, \\ \dfrac{x-y}{30} & \text{if} \quad 0 < x-y < 30, \\ 1 & \text{if} \quad x-y \ge 30 \end{cases}$$

Composition of sup-Ttype of Fuzzy Relations

- R⊆X×Y, S⊆Y×Z
- Composition R∘S⊆X×Z, membership function

$$\mu_{R\circ S}(x,z) = \sup_{y\in \mathbf{Y}} \left\{ \mu_R(x,y) \overset{T}{*} \mu_S(y,z) \right\}$$

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Basic Properties of Fuzzy Relations

1	$R \circ I = I \circ R = R$
2	$R \circ O = O \circ R = O$
3	$(R \circ S) \circ T = R \circ (S \circ T)$
4	$R^m \circ R^n = R^{m+n}$
5	$(R^m)^n = R^{mn}$
6	$R \circ (S \cup T) = (R \circ S) \cup (R \circ T)$
7	$R \circ (S \cap T) \subseteq (R \circ S) \cap (R \circ T)$
8	$S \subset T \rightarrow R \circ S \subset R \circ T$

Remarks

- Try to find the definitions:
  - Cylindrical extension
  - Projectionof fuzzy set

Outline

- Introduction
- Basic terms
- Operations on fuzzy sets
- The extension principle
- Fuzzy numbers
- Triangular norms and negations
- Fuzzy relations and their properties
- Approximate reasoning
- Fuzzy inference systems

Inference in Binary Logic

Premise Implication	A $A \rightarrow B$
Inference	B

Premise Implication	$\overline{B}$ $A \rightarrow B$
Inference	$\overline{A}$

Generalized Modus Ponens Inference Rule

Premise	x is A'
Implication	**IF** x is A **THEN** y is B
Inference	y is B'

- A, A’ ⊆X and B,B’ ⊆Yare fuzzy sets, and x, y are the linguistic variables (“low speed”, “young person”,…)

Example: Generalized Modus Ponens

Premise	The car speed is high
Implication	If the car speed is very high, then the noise level is high
Inference	The noise level in the car is medium-high

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Example: Generalized Modus Ponens

- Linguistic variable T_1 and T_2
  - T_1={"low", "medium", "high", "very high"}
  - T_2={"low", "medium", "medium-high", "high"}
- Fuzzy sets
  - A="very high speed of the car"
  - A'="high speed of the car"
  - B="high noise level"
  - B'="medium-high noise level"

Example: Generalized Modus Ponens

- Fuzzy set A="very high speed of the car" is not equal to the fuzzy set A' ="high speed of the car"
- The inference of the fuzzy rule relates to a certain fuzzy set B', which is defined by the composition of the fuzzy set A' and a fuzzy implication A→B

$$B' = A' \circ (A \rightarrow B)$$

- The fuzzy implication A→B is equivalent to a certain fuzzy relation $R \in X \times Y$ with the membership function $\mu_R(x, y)$

$$\mu_{B'}(y) = \sup_{x \in \mathbf{X}} \left\{ \mu_{A'}(x) \overset{T}{*} \mu_{A \rightarrow B}(x, y) \right\}$$

Example: Generalized Modus Ponens

$$\mu_{B'}(y) = \sup_{x \in \mathbf{X}} \{\min[\mu_{A'}(x), \mu_{A \to B}(x, y)]\}$$

- Assume

1) $A' = A$,
2) $A' =$ "very A", while $\mu_{A'}(x) = \mu_A^2(x)$,
3) $A' =$ "more or less A", while $\mu_{A'}(x) = \mu_A^{1/2}(x)$,
4) $A' =$ "not A", while $\mu_{A'}(x) = 1 - \mu_A(x)$.

Intuitive Relations

Relation	Premise x is A'	Inference y is B'
1	x is A	y is B
2a	x is "very A"	y is "very B"
2b	x is "very A"	y is B
3a	x is "more or less A"	y is "more or less B"
3b	x is "more or less A"	y is B
4a	x is "not A"	y is undefined
4b	x is "not A"	y is "not B"

Other Issues

- Generalized fuzzy modus tollens inference rule
- Inference rules for the Mamdani model
- Inference rules for the logical model
  - Fuzzy implication

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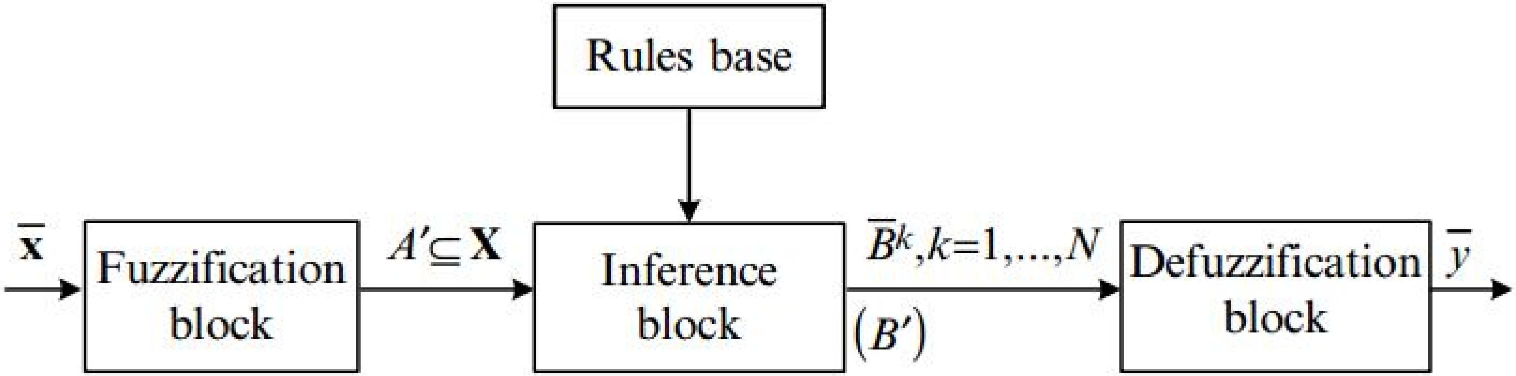
Fuzzy Inference System

- Elements
  - Rules base
  - Fuzzification block
  - Inference block
  - Defuzzification block



Rules Base

- Rules base (linguistic model): a set of fuzzy rules

$$R^{(k)}: \textbf{IF } x_1 \text{ is } A_1^k \textbf{ AND } x_2 \text{ is } A_2^k \textbf{ AND...AND}$$

$$x_n \text{ is } A_n^k \textbf{ THEN } y_1 \text{ is } B_1^k \textbf{ AND } y_2 \text{ is } B_2^k \textbf{ AND...AND } y_m \text{ is } B_m^k$$

$$A_i^k \subseteq \mathbf{X}_i \subset \mathbf{R}, \quad i = 1, ..., n,$$

$$B_j^k \subseteq \mathbf{Y}_j \subset \mathbf{R}, \quad j = 1, ..., m,$$

$$[x_1, x_2, \ldots, x_n]^T = \mathbf{x} \in \mathbf{X}_1 \times \mathbf{X}_2 \times \ldots \times \mathbf{X}_n,$$

$$[y_1, y_2, \ldots, y_m]^T = \mathbf{y} \in \mathbf{Y}_1 \times \mathbf{Y}_2 \times \ldots \times \mathbf{Y}_m.$$

Rules Base

- Assume outputs are independent

$$R^{(k)}: \textbf{IF} \quad x_1 \quad \text{is} \quad A_1^k \quad \textbf{AND} \quad x_2 \quad \text{is} \quad A_2^k \quad \textbf{AND}$$
$$\ldots \textbf{AND} \quad x_n \quad \text{is} \quad A_n^k \quad \textbf{THEN} \quad y \quad \text{is} \quad B^k;$$

- $R^{(k)} \subseteq X \times Y$ is a fuzzy set with membership function

$$\mu_{R^{(k)}}(\mathbf{x}, y) = \mu_{A^k \to B^k}(\mathbf{x}, y)$$

Fuzzification Block

- Fuzzification block is a control system with fuzzy logic operates on fuzzy sets

$$\overline{\mathbf{x}} = [\overline{x}_1, \overline{x}_2, ..., \overline{x}_n]^T$$

$$A' \subseteq \mathbf{X} = \mathbf{X}_1 \times \mathbf{X}_2 \times ... \times \mathbf{X}_n$$

- The fuzzy set A' is an input of the inference block

$$\mu_{A'}(\mathbf{x}) = \delta(\mathbf{x} - \overline{\mathbf{x}}) = \begin{cases} 1, & \text{if} \quad \mathbf{x} = \overline{\mathbf{x}}, \\ 0, & \text{if} \quad \mathbf{x} \neq \overline{\mathbf{x}}. \end{cases}$$

- If the input considers interference:

$$\mu_{A'}(\mathbf{x}) = \exp\left[-\frac{(\mathbf{x} - \overline{\mathbf{x}})^T (\mathbf{x} - \overline{\mathbf{x}})}{\delta}\right], \quad \delta > 0.$$

Inference Block

- Find an appropriate fuzzy set at the output of this block
- Two cases
  - Obtain N fuzzy sets $B_k \subseteq Y$ according to the generalized fuzzy modus ponens inference rule

$$\overline{B}^{(k)} = A' \circ (A^k \to B^k), \quad k = 1, \ldots, N$$

$$\mu_{\overline{B}^k}(y) = \sup_{\mathbf{x} \in \mathbf{X}} \left[\mu_{A'}(\mathbf{x}) \overset{T}{*} \mu_{A^k \to B^k}(\mathbf{x}, y) \right]$$

Defuzzifigification Block

The output value of the inference block is either N fuzzy sets $\overline{B}^k$ with membership functions $\mu_{\overline{B}^k}(y)$, $k = 1, 2, \ldots, N$, or a single fuzzy set B' with membership function $\mu_{B'}(y)$.

- Output method
  - Center average defuzzification method
  - Center of sums defuzzification method
  - Center of gravity method(or center of area method)
  - Maximum membership function method

Center Average Defuzzification Method

$$\overline{y} = \frac{\sum_{k=1}^{N} \mu_{\overline{B}^k}\left(\overline{y}^k\right)\overline{y}^k}{\sum_{k=1}^{N} \mu_{\overline{B}^k}\left(\overline{y}^k\right)},$$

where $\overline{y}^k$ is the point in which the function $\mu_{B^k}(y)$ takes the maximum value, i.e.

$$\mu_{B^k}\left(\overline{y}^k\right) = \max_{y} \mu_{B^k}(y).$$

Center of Sums Defuzzification Method

$$\overline{y} = \frac{\int_{\mathbf{Y}} y \sum_{k=1}^{N} \mu_{\overline{B}^k}(y)\, \mathrm{d}y}{\int_{\mathbf{Y}} \sum_{k=1}^{N} \mu_{\overline{B}^k}(y)\, \mathrm{d}y}.$$

Center of Gravity Method

$$\bar{y} = \frac{\displaystyle\int_{\mathbf{Y}} y\mu_{B'}(y)\,\mathrm{d}y}{\displaystyle\int_{\mathbf{Y}} \mu_{B'}(y)\,\mathrm{d}y} = \frac{\displaystyle\int_{\mathbf{Y}} y\ S_{k=1}^{N}\ \mu_{\overline{B}^k}(y)}{\displaystyle\int_{\mathbf{Y}} S_{k=1}^{N}\ \mu_{\overline{B}^k}(y)},$$

In a discrete case,

$$\bar{y} = \frac{\displaystyle\sum_{k=1}^{N} \mu_{B'}\left(\bar{y}^k\right)\bar{y}^k}{\displaystyle\sum_{k=1}^{N} \mu_{B'}\left(\bar{y}^k\right)}.$$

Maximum Membership Function Method

$$\mu_{B'}(\overline{y}) = \sup_{y \in \mathbf{Y}} \mu_{B'}(y)$$

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