M115 Quiz#3 Spring 2022

Rubric:
show up to the quiz meeting: 4pts
questions 1,2,3,4 : 1,1,2,2 pts, respectively.
total possible: 10pts

1. Claim: If $n, a, b \in \mathbf{N}$ and $n = a \cdot b$, then either $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.
a) Prove this claim using the contrapositive.
b) Prove the claim using a proof by contradiction.
c) Is there any difference to the proofs in parts (a) and (b)? If so, what is it?

a) Proof (Contrapositive).

Suppose you have natural numbers $n, a, b \in N$.
You want to show $P \rightarrow Q$ "If P, then Q" whereby
P := "$n = a \cdot b$" and Q := "$a \leq \sqrt{n}$ or $b \leq \sqrt{n}$".
To prove this by the contrapositive we should give a direct argument to $\neg Q \rightarrow \neg P$.
We start by assuming $\neg Q$. So we're assuming $a > \sqrt{n}$ and $b > \sqrt{n}$.
Our goal here is to conclude $\neg P$. So we want to argue that $n \neq a \cdot b$.

Using the stuff we're assuming, multiply that stuff together. What happens?

$$a \cdot b > \sqrt{n} \cdot \sqrt{n} = n.$$

Using the assumption we've got that $a \cdot b > n$, which means that $a \cdot b \neq n$ which is what we wanted to show.

1b) Proof (Contradiction).

Suppose we have natural numbers $n, a, b \in \mathbf{N}$. We still want to show "If P, then Q," where P and Q are the same as above.

There are many ways to structure this, even in the limits of a proof by contradiction, but you could say that since we want to show $P \rightarrow Q$. Let's assume $\neg(P \rightarrow Q)$. This is equivalent to assuming $P \wedge \neg Q$. So we are assuming that $n = a \cdot b$, and that $a > \sqrt{n}$ and $b > \sqrt{n}$. The goal is to conclude a contradiction. The contradiction comes using the same logic as in part (a): multiplying the a and b together give you something larger than n, which is a contradiction since we assumed in this part that $n = a \cdot b$.

Another way to reason this goes a little more directly. We want to show $P \rightarrow Q$. So we start by assuming P. After assuming P, we want to somehow conclude Q. For the sake of contradiction, assuming Q isn't true. In other words, we assume $\neg Q$ is true. All together we have assume both P and $\neg Q$, which is the same beginning as in the above paragraph. The contradiction that you get is the same.

1c) What is the difference between the two approaches? The biggest difference is in what you assume. When proving $P \to Q$ by contraposition you assume only $\neg Q$, but when you try to prove it by contradiction you assume both P and $\neg Q$.

Another difference in the two methods is in the thing you want to conclude. In the proof by contraposition you want to conclude with $\neg P$, while in the proof by contradiction you want to reach a contradiction. In the problem at hand that contradiction comes in the form of $P \wedge \neg P$, but in general proving $P \to Q$ by contradiction may entail reaching all sorts of contradictions, it doesn't have to involve P or Q.

2. a) Using the definitions of even and odd integers, prove that for all integers n, $n + 9$ is even iff $n^2 - 4$ is odd.

a) There are two parts to this 'iff' proof. You need to show that "if $n + 9$ is even, then $n^2 - 4$ is odd." And you need to show that "if $n^2 - 4$ is odd, then $n + 9$ is even."

Here's the first part.
Assume n is an integer (this often goes without saying, but it's here for the record.)
Assume $n + 9$ is even.
This means $n + 9 = 2k$ for some integer k. (… what does this say about n?)
Then $n = 2k - 9 = 2(k - 5) + 1$, so n is odd. (… so what about $n^2 - 4$?)
Then $n^2 - 4 = 4(k-5)^2 + 4(k-5) + 1 - 4 = 2\big(2(k-5)^2 + 2(k-5) - 2\big) + 1$
This means $n^2 - 4$ is odd.
(Once they had an odd n, most students said something like "if n is odd, then n^2 is also odd, and $n^2 - 4$ is odd too." This was acceptable.)
We have shown that for all integers n, if $n + 9$ is even, then $n^2 - 4$ is odd.

Here's the second part.
Assume n is an integer and $n^2 - 4$ is odd.
This means $n^2 - 4 = 2k + 1$ for some integer k.
Then $n^2 = 2k + 5 = (2k + 2) + 1$, so n^2 is odd.
Then n itself must be odd, because if it were even then n^2 would be even, contrary to what we have deduced.
Then $n = 2j + 1$ for some integer j.
Then $n + 9 = 2j + 10 = 2(j + 5)$,
so $n + 9$ is even.
We have shown that for all integers n if $n^2 - 4$ is odd, then $n + 9$ is even.

All together we have shown that for all integers n, $n + 9$ is even iff $n^2 - 4$ is odd. QED

2b) Given any n integers $a_1, a_2, \ldots, a_n$ prove that at least one of the given integers must be less than or equal to their average.

Proof (contradiction):

Suppose n is positive integer, and $a_1, a_2, \ldots a_n$ are integers. We want to show that one of them must me less than or equal to $A = \dfrac{a_1 + a_2 + \ldots + a_n}{n}$. So let's suppose for the sake of contradiction that this is not the case.

So we're assuming $a_i > A$ for $i = 1,2,3,...n$. Is there a contradiction to be had somewhere?!

Yep. Add up all the a_is. Since their all greater than A, their sum is greater than $n \cdot A$

$a_1 + a_2 + \ldots + a_n > n \cdot A = a_1 + a_2 + \ldots + a_n.$

So this sum is greater than itself, which is kinda weird. That's a contradiction (no number is ever greater than itself.)

Thus not all the assumptions we made can be true. So if n is a positive integer and $a_1, a_2, \ldots a_n$ are integers, then at least one of them is less than or equal to A.

2c) Here's a Boolean function of 3 variables: $F(x, y, z) = x + y + z$. Express this function as a full sum of products. The 'full' means that each product should have all three of x,y and z, whereby some of them may be complemented. For example $xy\bar{z}$ is one product in the sum.

This boolean function only has one 3-tuple where it's not equal to 1, namely (x,y,z)=(0,0,0). So $\bar{x}\bar{y}\bar{z}$ is the only product that won't be in its sum-of-products expansion.

$F(x, y, z) = xyz + \bar{x}yz + x\bar{y}z + xy\bar{z} + \bar{x}\bar{y}z + \bar{x}y\bar{z} + x\bar{y}\bar{z}.$

3.

a) Prove or disprove that if x and y are rational numbers, then x^y is rational.

We can disprove the claim that "If x and y are rational numbers, then x^y is rational" with a counterexample. $x = 2,\ \ y = 1/2$ works. This x and y are both rational, but $x^y = \sqrt{2}$ is not.

Formally we have shown that $\forall x \in \mathbf{Q}\ \ \forall y \in \mathbf{Q}\ \ x^y \in \mathbf{Q}$ is false by showing that its negation

$\exists x \in \mathbf{Q}\ \ \exists y \in \mathbf{Q}\ \ x^y \notin \mathbf{Q}$ is true.

3b) Prove or disprove that there exist a rational x and an irrational y so that x^y is rational.

This one is true. There does exists a rational x and an irrational y so that x^y is rational. $x = 0$ pairs well with any $y \in \mathbf{R} - \mathbf{Q}$ here. For example $x = 0, \quad y = \sqrt{2}$ works since $x^y = 0 \in \mathbf{Q}$.

3c) Prove or disprove that for any rational number x and any irrational number y there exists an irrational number, z, between x and y.

This is true. We can prove that $\forall x \in \mathbf{Q}\ \forall y \in \mathbf{R} - \mathbf{Q}\ \exists z \in \mathbf{R} - \mathbf{Q}\ \ z$ is between x and y.

Proof: You could translate that statement into English with “If x is rational, and y is irrational, then there exists an irrational z between x and y”

We’ll start a proof with the assumption that we have a rational number x and an irrational number y. Taking the average of these two numbers is a good idea for this problem because

1) $z = (x + y)/2$ is definitely irrational. If it were rational, then $2z - x = y$ would be rational too, contrary to the assumption that y is irrational.

2) $z = (x + y)/2$ is between x and y. Geometrically, it’s the number halfway between the numbers x and y on the number line.

Thus, no matter what $x \in \mathbf{Q}$ and $y \in \mathbf{R} - \mathbf{Q}$ are, we have an irrational number between them. QED.

d) Prove or disprove that for any rational number x and any irrational number y there exists a rational number, z, between x and y.

This one is not easy, but it’s true.

Proof. Suppose $x \in \mathbf{Q}$ and $y \in \mathbf{R} - \mathbf{Q}$.

These two numbers can’t be equal, since a number that is irrational can’t also be rational, so one of them is less than the other. Without loss of generality we’ll assume $x < y$.

x is rational. So $x + \frac{1}{n}$ is rational for all $n \geq 1$. Furthermore, $\left\{ x + \frac{1}{n} \right\}_{n \geq 1}$ is a sequence which converges to x. This means that it gets arbitrarily close to x as n gets larger. So we can pick n large enough so that $x + \frac{1}{n} < y$. But then $x + \frac{1}{n}$ is a rational number that is between x and y, as desired.

(In the case when $x > y$, the same argument works with the sequence $x - \frac{1}{n}$.)

4. Claim: If B is a Boolean Algebra and $x, y \in B$, then $x + xy = x$.

4a) Here is an incorrect proof of the claim. Why is this proof incorrect?

Proof: Suppose B is a Boolean Algebra, $x \in B$ and $y \in B$.

Case 1: if $x = 0$, then using the facts that $0 \cdot y = 0$ for all y in B, and that $0 + 0 = 0$, we get $x + xy = 0 + 0 \cdot y = 0 = x$.

Case 2: if $x = 1$, then using the facts that $1 \cdot y = y$ for all y in B, and that $1 + y = 1$ for all y in B, we get that $x + xy = 1 + y = 1 = x$.

In all possible cases we have $x + xy = x$. QED.

The problem with this proof is that a case argument doesn't cover all cases. For example, here is a Boolean Algebra:

x_1	x_2	F_1	F_2	F_4	F_5	...	F_{15}	F_{16}
1	1	0	1	0	0		0	1
1	0	0	0	1	0		1	1
0	1	0	0	0	1		1	1
0	0	0	0	0	0		1	1

This is the set of Boolean Functions on two variables. This is the tricky point. The elements of the Boolean Algebra are not just bit variables (like the x_1 and x_2 in this table). The elements of this BA are the functions! Amongst all those functions there is a 0 element (namely F_1) and a 1 element (namely F_{16}), but there are also lots of other elements in this Boolean Algebra, many of which are not the special 0 element or 1 element.

If you wanted to do a proof by cases, then you'd have 16 cases to consider here - and that's just for this specific Boolean Algebra! To give a general argument, a proof by cases doesn't appear to be a very good approach.

4b) Either give a correct proof of the claim, or disprove it.

It's true. Here's a proof: $x + xy = x \cdot 1 + xy = x \cdot (1 + y) = x \cdot 1 = x$.

This proof uses property of the BA's 1 element (namely $1 \cdot x = x$ and $1 + x = 1$ for all $x \in B$), and the distributive property,