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filename: Math115 homework5
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desc: https://courses.csail.mit.edu/6.042/spring18/mcs.pdf (Links to an external site.) please do these problems: 4.8, 4.9, 4.15, 4.19, 4.20

4.8

Let A, B and C be sets. Prove that

$A \bigcup B \bigcup C = (A - B) \bigcup (B - C) \bigcup (C - A) \bigcup (A \bigcap B \bigcap C))$

using a chain of IFF's as Section 4.1.5.

$A \bigcup B \bigcup C = (A - B) \bigcup (B - C) \bigcup (C - A) \bigcup (A \bigcap B \bigcap C))$

converting minus signs to equivalent set expressions

iff $\underline{(A \bigcap \bar{B})} \cup \underline{(B \bigcap \bar{C})} \cup \underline{(C \bigcap \bar{A})} \cup (A \bigcap B \bigcap C))$

manipulating the right side, we get

iff $(A \bigcap \bar{B}) \cup (B \bigcap \bar{C}) \cup (C \bigcap \bar{A}) \cup \underline{((A \bigcap B) \bigcup (B \bigcap C) \bigcup (C \bigcap A))}$

iff $(A \bigcap \bar{B}) \cup (B \bigcap \bar{C}) \cup (C \bigcap \bar{A}) \cup (A \bigcap B) \cup (B \bigcap C) \cup (C \bigcap A)$

Associative property to shuffle things around

iff $(A \bigcap \bar{B}) \cup (A \bigcap B) \cup (B \bigcap \bar{C}) \cup (B \bigcap C) \cup (C \bigcap \bar{A}) \cup (C \bigcap A)$

Reverse distribute for A, B, C

iff $(A \bigcap (\bar{B} \bigcup B)) \cup (B \bigcap (\bar{C} \bigcup C)) \cup (C \bigcap (\bar{A} \bigcup A))$

Universe $= set \bigcup \bar{set}$

iff $(A \bigcap U) \cup (B \bigcap U) \cup (C \bigcap U)$

Identity: $A \bigcap U = A$

$= A \bigcup B \bigcup C$

4.9

Union distributes over the intersection of two sets:

$A \bigcup (B \bigcap C) = (A \bigcup B) \bigcap (A \bigcup C)$

Use (4.11) and the Well Ordering Principle to prove the Distributive Law of union over the intersection of n sets:

$A \bigcup (B_1 \bigcap ... \bigcap B_{n-1} \bigcap B_n) = (A \bigcup B_1) \bigcap ... \bigcap (A \bigcup B_{n-1}) \bigcap (A \bigcap B_n)$

Extending formulas to an arbitrary number of terms is a common (if mundane) application of the WOP.

Prove by contradiction/induction.

Let's say $P(n) \leftrightarrow Q(n)$

and let

$P(n) = A \bigcup (B_1 \bigcap ... \bigcap B_n)$

and

$Q(n) = (A \bigcup B_1) \bigcap ... \bigcap (A \bigcap B_n)$

So the setup would be to assume $(B_1 \bigcap ... \bigcap B_n) \in P(n)$ and test $Q(n)$ to find a set (or sets) where $(B_1 \bigcap ... \bigcap B_n \bigcap ... \notin Q(n)$.

- Case 1: $P(0)$ case. Let a single set be represented as B. We'll say set $B \in P(n)$.

  Since $Q(n) = A \bigcup B$

  I would have to say $B \in Q(n)$

  so $B \in P(n) \bigcap Q(n)$

- Case 2: Lets 1 to n sets be represented by $B_1...B_n$ and $B_1...B_n \in P(n)$.

  Since $Q(n) = (A \bigcup B_1) \bigcap ... \bigcap (A \bigcap B_n)$.

  I would have to say $B_1...B_n \in Q(n)$

  so $B_1...B_n \in P(n) \bigcap Q(n)$

- Similar arguments follow if we assume $(B_1 \bigcap ... \bigcap B_n) \in Q(n)$ and tested $P(n)$ if the set(s) exist(s).

- Since I was unable to find a set that exist in $P(n)$ and not in $Q(n)$, I couldn't prove the contradiction. So the distibutive law is valid.

4.15

a) Give a simple example where the following result fails, and briefly explain why:

False Theorem. For sets $A, B, CandD$, let

$L ::= (A \bigcup B) \times (C \bigcup D)$.

$R ::= (A \times C) \bigcup (B \times D)$.

Then $L = R$.

b) Identify the mistake in the following proof of the False Theorem.

Bogus proof. Since L and R are both sets of pairs, it's sufficient to prove that

$(x, y) \in L \leftrightarrow (x, y) \in R$ for all x, y.

The proof will be a chain of iff implications:

$(x, y) \in R$

iff $(x, y) \in (A \times C) \bigcup (B \times D)$

iff $(x, y) \in A \times C, or(x, y) \in B \times D$

iff ($x \in A$ and $y \in C$) or else ($x \in B$ and $y \in D$)

iff either $x \in A$ or $x \in B$, and either $y \in C$ or $y \in D$

iff $x \in A \bigcup B$ and $y \in C \bigcup D$

iff $(x, y) \in L$.

c) Fix the proof to show that $R \subseteq L$.

- a) this proof is false because L would have two times more combinations or products than R.

  For example if A, B, C and D have 2 elements each. $L ::= (A \bigcup B) \times (C \bigcup D)$ would have 16 combinations total. $R ::= (A \times C) \bigcup (B \times D)$ would have 8 combinations total.

- b) The step with "*iff either* $x \in A$ *or* $x \in B$, *and either* $y \in C$ *or* $y \in D$" is incorrect. Because this statement implies for R, x can belong to A or B and y can belong to either C or D. But for R, if x belongs to A it has to belong to C as well. So a combination such as $x \in A \bigcup B$ wouldn't be valid for R.

- c) $(x, y) \in R$

  iff $(x, y) \in (A \times C) \bigcup (B \times D)$

  iff $(x, y) \in A \times C, or(x, y) \in B \times D$

  iff ($x \in A$ and $y \in C$) or else ($x \in B$ and $y \in D$)

  iff ($x \notin A$ and $y \notin C$) or ($y \notin B$ and $y \notin D$)

  iff ($x \in A$ and $y \in D$) or ($y \in B$ and $y \in C$)

  iff $x \in A \bigcup B$ and $y \in C \bigcup D$

  iff $(x, y) \subseteq L$

  $\therefore R \subseteq L$.

4.19

For each of the following real-valued functions on the real numbers, indicate whether it is a bijection, a surjection but not a bijection, an injection but not a bijection, or neither an injection nor a surjection.

a) $x \rightarrow x + 2$

b) $x \rightarrow 2x$

c) $x \rightarrow x^2$

d) $x \rightarrow x^3$

e) $x \rightarrow sin(x)$

f) $x \rightarrow xsin(x)$

g) $x \to e^x$

- a) bijective.

  $\forall x_1, x_2 \in \mathbb{R} : x \to x + 2$

  let x_1 and $x_2 \in \mathbb{R}$ and $f(x_1) = f(x_2)$

  $x_1 + 2 = x_2 + 2$

  subtract 2 from both sides

  $x_1 = x_2$

  x_1 has to equal x_2, this implies this function is injective. The function can continuously map an x in its domain to the codomain through out $\mathbb{R}$, so by the intermediate value theorum the function is surjective as well.

- b) bijective.

  $\forall x_1, x_2 \in \mathbb{R} : x \to 2x$

  let x_1 and $x_2 \in \mathbb{R}$ and $f(x_1) = f(x_2)$

  $2x_1 = 2x_2$

  dividing 2 on both sides

  $x_1 = x_2$

  x_1 has to equal x_2, this implies this function is injective. The function can continuously map an x in its domain to the codomain through out $\mathbb{R}$, so by the intermediate value theorum the function is surjective as well.

- c) neither an injection nor a surjection.

  $\forall x_1, x_2 \in \mathbb{R} : x \to x^2$

  let x_1 and $x_2 \in \mathbb{R}$ and $f(x_1) = f(x_2)$

  $(x_1)^2 = (x_2)^2$

  If $x_1 \neq x_2$ they can still map to the same element in the codomain, example -2 and 2 would both map to 4, so not injective. There also isn't mapping of an $f(x)$ to every $\mathbb{R}$ because x^2 is always positive, so not surjective.

- d) bijection.

  $\forall x_1, x_2 \in \mathbb{R} : x \to x^3$

  let x_1 and $x_2 \in \mathbb{R}$ and $f(x_1) = f(x_2)$

  if $x_1 \neq x_2$

  $(x_1)^3$ and $(x_2)^3$ would only be equal if $x_1 = x_2$, so this is injective.

  The function can continuously map an x in its domain to the codomain through out $\mathbb{R}$, so by the intermediate value theorum the function is surjective as well.

- e) neither an injection nor a surjection.

  $\forall x_1, x_2 \in \mathbb{R} : x \to sin(x)$

let x_1 and $x_2 \in \mathbb{R}$ and $f(x_1) = f(x_2)$

$sin(x_1) = sin(x_2)$

This is not injective because multiple x values can be mapped to $f(x)$. There isn't mapping of a $f(x)$ to every $\mathbb{R}$, so it is not surjective.

- f) surjection but not a bijection.

  the explanation is similar to e) but because of the multiplied by x, the function can span continuously to ∞ or -∞, so by the intermediate value theorum, this function is surjective.

- g) injection but not a bijection.

  $\forall x_1, x_2 \in \mathbb{R} : x \to e^x$

  let x_1 and $x_2 \in \mathbb{R}$ and $f(x_1) = f(x_2)$

  if $x_1 \neq x_2$, there is no combination x_1 and x_2 that can make $e^{x_1} = e^{x_2}$, therefore injective.

  if we take e^x to ∞ or -∞ there would not be a e^x to map from $-\infty$ to 0. So I would say it's not surjective.

4.20

Let $f : A \to B$ and $g : B \to C$ be functions and $h : A \to C$ be their composition, namely, $h(a) ::= g(f(a))$ for all $a \in A$.

a) Prove that if f and g are surjections, then so is h.

b) Prove that if f and g are bijections, then so is h.

c) If f is a bijection, then so is f^{-1}.

- a) proof by deduction

  Assuming f and g are surjective, then C is a subset of B and B is a subset of A. This implies C would be a subset of A.

  so then $f \supseteq g \supseteq h$, which means h has to be surjective as well.

- b) proof by deduction

  Assuming f and g are injective, a similar argument can be said compared to a).

  let $x...x_n \in \mathbb{R}$ be in the set of $f : A \to B$

  and let $y...y_n \in \mathbb{R}$ be in the set of $g : B \to C$

  and let $z...z_n \in \mathbb{R}$ be in the set of $h : A \to C$

  if f is in $\mathbb{R}$ and g is in $\mathbb{R}$. Then h would also have to be in $\mathbb{R}$.

- c) If f is bijective $\to f^{-1}$ is bijective.

  let $\forall x, y \in \mathbb{R} : x \to f(x)$.

  $f(x) = y$ and $y \in \mathbb{R}$

$f^{-1}(y) = x$ and $x \in \mathbb{R}$

So the inverse function would just remap all y's to x's.

example: $x \in \mathbb{R} : f(x) = x^3$

let $x = 2n$, $n \in \mathbb{R}$

$f(x) = (2n)^3$

So $f^{-1}(x) = x^{\frac{1}{3}}$

$((2n)^3)^{\frac{1}{3}} = (2n)^{\frac{3}{3}} = 2n$

so we can say $(f^{-1}(f(x))) = x$, or a bijective functions inverse is bijective since it just undoes what the function did.