M115 Homework2 Solutions, Spring 2022.

1. Prove $\neg P \vee (R \to \neg Q) \equiv \neg(P \wedge Q \wedge R)$ without using a truth table.

Proof by cases:
Case 1: P,Q and R are all true; Case 2: At least one of P,Q,R are false.

In case 1, the right side is $\neg(T \wedge T \wedge T) = \neg T = F$. The right side is also false because both $\neg P = \neg T$ is false, and $R \to \neg Q = T \to \neg T = T \to F$ is false.

In case 2, since at least one of P,Q and R is false, $P \wedge Q \wedge R$ will be false, so the right side will be true. As for the left side, consider sub-cases 2.1: P is false; 2.2: Q is false; 2.3: R is false. Note that in case 2, at least one of these sub-cases must hold (otherwise we would be in case 1). In all of these sub-cases we want to show that the left side is true, which means that either $\neg P$ should be true, or $R \vee \neg Q$ should be true, or both.

sub-case 2.1. In this case $\neg P$ is true, so the left side is true.
sub-case 2.2. In this case $\neg Q$ is true, so $R \to \neg Q$ is true (regardless of R), so the left side is true.
sub-case 2.3. In this case $R \to \neg Q$ is true (regardless of Q), so the left side is true.

In all sub-cases we have the left side is true, which is what we wanted.
Thus in both cases 1 and 2 the two sides have the same truth value, so the propositions are equivalent.

Note: the equivalence can also be proven using some formulaic manipulations, e.g. disjunctive normal form, then deMorgan's Law.

1.3
a) Multiplying an inequality by a negative number, such as $\log_{10}(1/2)$, should flip the inequality. The multiplication happened, but not the flip.

b) The issue must be with the units. Trade $1\$$ for $100c$ and see if it fits.

$$1c = 0.01 \cdot 1\$ = 0.01 \cdot (100c) =^{???} (0.1 \cdot 100c)^2$$

The square quantity is $0.1 \cdot 100c \cdot 0.1 \cdot 100c$, which amounts to one cent times a dollar - not the same as what we started with. Squaring units is problematic.
A more hands-on version of this problem may be had if you apply the same logic to conclude that $1cm = 1m$. The argument there has issues with quantities of length like centimeters (cm) and meters (m) being equal to quantities of area (cm^2). Such quantities can't be compared.

c) There's cancellation of 0 here. going from $(a-b)(a+b) = (a-b)b$ to $a+b = b$ entails cancelling $a-b$. In order to cancel something like that, it must be not 0, which is not the case in this problem since it starts with $a = b$, so $a - b = 0$.

1.7. Prove max(r,s)+min(r,s)=r+s for all real numbers r and s.

(You could also say: If r and s are real numbers, then max(r,s)+min(r,s)=r+s.)

Proof.

Suppose you have two real numbers, r and s.

Consider cases:
Case 1. r=s
Case 2. r<s.
Case 3. r>s.

Case 1. If r=s, then max(r,s)=min(r,s)=r=s. So max(r,s)+min(r,s) = 2r = r+s.

Case 2. If r<s, then max(r,s) = s and min(r,s) = r. So max(r,s)+min(r,s) = s+r = r+s.

Case 3. If r>s, then max(r,s)=r and min(r,s)=s. So max(r,s)+min(r,s)=r+s.

Note: There's some symmetry here between r and s, since max(r,s)=max(s,r) and min(r,s)=min(s,r), and since r+s=s+r. You could say that in the case r≠s you may assume *without loss of generality* that $r < s$, since the alternative case $r > s$ appeals to the same argument with r and s switched.

1.8 This one was done in the week 2 lessons, darn it. I didn't see the likeness until it was too late. But here it is again in case you missed it in the lesson.

Case 1. $\sqrt{2}^{\sqrt{2}}$ is rational. In this case, we're done. We have shown that there exists an irrational base $x = \sqrt{2}$ and an irrational exponent $y = \sqrt{2}$ so that x^y is irrational.

Case 2. $\sqrt{2}^{\sqrt{2}}$ is irrational. In this case, take this irrational number to be the base x, and take $y = \sqrt{2}$ to be the irrational exponent. Then $x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = (\sqrt{2})^2 = 2$, which is rational.

In either case we have irrational numbers x and y so that x^y is rational.

1.10

a) If $a+b+c=d$ then d is even iff either one of a,b,c are even, or all of a,b,c are even.

Proof.

Assume $a+b+c=d$.

We want to show d is even iff either one of a,b,c are even, or all of a,b,c are even.
This entails proving two implications:
1) If d is even, then either one of a,b,c are even, or all of a,b,c are even.
2) If either one of a,b,c are even, or all of a,b,c are even, then d is even.

We can prove (1) using the contrapositive of the statement: we'll assume that it's not the case that either one of a,b,c or all of a,b,c are even, and show that d is not even (i.e. it's odd).

(In other words using the P, W and T as given in the problem statement, we're arguing this: if not W and not T, then not P.)

Assuming not W and not W, then there are two cases:
case 1.1: none of a,b,c are even
case 1.2: two of a,b,c are.

In case 1.1 a,b,c are all odd, so there are some integers j,k,l so that $a=2j+1$, $b=2k+1$ and $c=2l+1$. Adding these up yields $d=a+b+c = 2(j+k+l)+3 = 2(j+k+l+1)+1$, which is odd.

In case 1.2 because of symmetry in the expression $a+b+c$ we'll assume without loss of generality (w.l.o.g.) that a is odd, and b and c are both even. In this case there are integers j,k and l so that $a=2k+1$, $b=2j$ and $c=2l$. Then $d=a+b+c = 2(k+j+l)+1$, which is still odd.

In both cases 1.1 and 1.2 we have that d is not even. So we've proven the contrapositive of the implication (1), so (1) is true.

We can prove 2) directly using cases.
Case 2.1: one of a,b,c are even.
Case 2.2: all of a,b,c are even.

In both cases we want to show that d is even.

In case 2.1 because of symmetry in the expression $a+b+c$ we may assume w.l.o.g. that a is even, while b and c are both odd. So $a=2j$, $b=2k+1$ and $c=2l+1$ for some integers j,k,l. Adding up a, b and c we get $d=a+b+c=2j+2k+1+2l+1=2(j+k+l+1)$, which is even since it's twice an integer.

In case 2.2, we have integers j,k,l so that $a=2j$, $b=2k$ and $c=2l$. In this case $d=a+b+c=2(j+k+l)$, which is still even.
In both cases 2.1 and 2.2 d is even, so (2) is proven.

We've proven both implications (1) and (2), so the 'iff' statement is true. Yay!

1.10 b). If $w^2 + x^2 + y^2 = z^2$ then z is even iff $w, x,$ and y are even.

Proof. Assume $w, x, y, z \in \mathbf{N}$ and $w^2 + x^2 + y^2 = z^2$. We want to show that z is even iff w, x and y all are. As in part (a), this entails two implications.

1) If z is even, then so are w,x, and y.
2) If w,x and y are even, then so is z.

I'll do 2) first since I think that direction is a little easier, using a direct argument.

Suppose that w,x and y are all even. This means that there exists some integers j,k,l so that w=2j, x=2k, and y=2l. Then squaring these things and adding them up is gonna result in something even, like this:

$$z^2 = w^2 + x^2 + y^2 = 4j^2 + 4k^2 + 4l^2 = 2(2j^2 + 2k^2 + 2l^2).$$

This means that z^2 is even. We saw in the lessons (week2day2, pg 18) that if a product mn is even, then at least one of m or n is even. Applying this here with the even number $z^2 = z \cdot z$, it must be that z is even or z is even (using the proposition from class with $m = n = z$). This means z must be even no matter what. This is what we wanted to show in the implication 2).

Now regarding the implication 1) we can prove this using the contrapositive. Suppose it's not the case that all of w,x and y are even. There are cases to consider.

Case 1.1 one of w,x,y are odd. (w.l.o.g. w is odd, x and y are even.)
Case 1.2 two of w,x,y are odd. (w.l.o.g. w and x are odd, y is even.)
Case 1.3 all three of w,x,y are odd.

In each of these cases we're going to show that z^2 is not divisible by 4, which will eventually imply that z is not even, which is what the contrapositive argument requires here in 1).

Case 1.1: There exists integers j,k,l so that w=2j+1, x=2k and y=2l. In this case we get

$$z^2 = w^2 + x^2 + y^2 = 4(j^2 + j) + 1 + 4k^2 + 4l^2 = 4(j^2 + j + k^2 + l^2) + 1$$

this means z^2 is one more than a multiple of 4, so it's not divisible by 4.

Case 1.2: There exists integers j,k,l so that w=2j+1, x=2k+1 and y=2l. In this case we get

$$z^2 = w^2 + x^2 + y^2 = 4(j^2 + j) + 1 + 4(k^2 + k) + 1 + 4l^2 = 4(j^2 + j + k^2 + k + l^2) + 2$$

z^2 is even in this case, sure, but it's not divisible by 4, because it's 2 more than a multiple of 4.

Case 1.3: There exists integers j,k,l so that w=2j+1, x=2k+1 and y=2l+1. In this case we get

$$z^2 = w^2 + x^2 + y^2$$
$$= 4(j^2 + j) + 1 + 4(k^2 + k) + 1 + 4(l^2 + l) + 1 = 4(j^2 + j + k^2 + k + l^2 + 1) + 3$$

This z^2 still isn't a multiple of 4, since it's 3 more than a multiple of 4.

So what? Who cares? We've only shown that if w,x and y are not all even, then in all possible cases z^2 isn't a multiple of 4. But what we want to conclude is that z isn't even.

We need to fill a little gap here with a 'helping theorem,' also called a 'Lemma'

Lemma: if z is even, then z^2 is divisible by 4.

(Proof of lemma: if z is even, then z=2k for some k, and squaring this yields 4 times k squared, which is 4 times another integer, so z squared is a multiple of 4.)

Using this lemma's contrapositive, we have that if z^2 is not divisible by 4, then z is odd.

Applying this to each of the cases we looked at above, we can conclude that z is odd in each case, and that's what we wanted to show to complete the contrapositive of (1).

Having shown implications 1) and 2), we have shown that z is even iff w,x and y all are. QED.

optional x/c problems. 1.1, 1.4, 1.5

1.1. I didn't get this problem completely. I managed to rearrange the shapes into the cxc square with the yellow square in the middle. The yellow square has side length b-a, so equating areas this gives

$(b-a)^2 + 4\left(\frac{ab}{2}\right) = c^2$, and this simplifies to the desired equation. How to do the alleged second arrangement of the shapes into two squares eludes me for the moment.

c) I'd say the square happens regardless of a and b because of the angles of the triangle. Regardless of the values of a and b, the angles of the corners of the triangle will add up to 90 degrees, which results in the square of side-length c.

d) One assumption that is being made is the the angles of a triangle add up to 180 degrees. Without this fact, arranging the shapes into a square (or many squares) might not be possible. This makes me wonder if the pythagorean theorem is true on the surface of a sphere?

1.4. Actually I thought this line of argument looks pretty good. The only thing that I find that isn't perfect is the direction of the argument. Technically, you should start from something that you know to be true, such as $(a-b)^2 \geq 0$, and work towards something that you want to conclude, like $\dfrac{a+b}{2} \geq \sqrt{ab}$. What this proof is arguing is $Q \to P$ when it should be arguing $P \to Q$. Alternatively, if the author replaced each of the 'so' with something like 'iff,' then the logic would be stronger. The proof would be that $P \leftrightarrow Q$, which I believe is actually the case in this problem.

1.5. I'm not sure about this one. I think Monday is left up in the air. Students may rule out Friday based on what happens before Friday, but that same argument can't be applied to Monday, since nothing in the week comes before Monday.