M115 Exam Directions, Spring 2022.

This exam is a take-home exam, followed by an interview via Zoom. During the interview I will ask you to walk me through four or five of the problems on the following pages. I may only choose a part of a problem (e.g. part (a)), and I might change a problem a little so that it's very similar to, but not exactly the same as one of the exam problems.

How do you ace the exam?
Do the problems as best you can *before* you meet with me for the exam. During the meeting, pretend like you are the teacher, and I am a very inquisitive student. I will ask you questions including but not limited to: "What is the answer to problem number (x)?" "how do I do the next step?" "what are the assumptions that I make in this part of the proof?" and "what do I want to conclude in this part of the proof?"

What is allowed?
1. While preparing for the interview exam, you can use anything you want: notes, calculators, internet, friends and family, tutors, and fellow students. If you want to share your solutions to these problems on Canvas or elsewhere on the internet, please be my guest.

2. During the interview you aren't allowed to use the internet, reference class slides, or speak to anyone but me. You may refer to any amount of hand-written notes (on paper or a tablet). I might ask you to send me a picture of the notes that you refer to for the exam.

3. You may use a calculator to check your answers if you want, but you should be able to justify the steps that you take to solve a problem without using a calculator.

When will this interview happen?
I have set up appointment times throughout the day, on Tuesday 3/22 and Thursday 3/24

1. Click on Calendar (it's above the Inbox on the left side of your Canvas account page.)
2. Click on "Find Appointment" (on the right side of the Calendar page.)
3. Select our Course: "Discrete Mathematics 31761-002"
4. You should see several slots on the days Tuesday 3/22 and Thursday 3/24. Pick an available slot at a time that best suits you. Click that slot!
5. A window will pop up with a 'reserve' button in the bottom corner. Click that to reserve your meeting with me.
6. When the time of your appointment has arrived, meet me on zoom via the link given in the exam announcement in our Canvas homepage.

Signup for the meeting sooner rather than later. Appointments are first-come first-serve.

(If you prefer to meet with me on Friday 3/25, this can be arranged. But the accommodation will cost you a point on the exam. This includes the cases where you don't sign up until the last minute, only to find all available appointments have been taken.)

(exam problems are on the next pages —->)

M115 Exam#1 Spring2022

1. State the definitions of the following concepts.
a) proposition
b) rational number
c) even integer
d) odd integer
e) injective function
f) surjective function
g) the union of two sets

2. Two of the following three propositions are equivalent. Which two? Prove that they are equivalent using a truth table. $\neg(P \to Q), \quad P \wedge \neg Q, \quad (\neg P) \to \neg Q.$

3. Prove or disprove the following equivalences.
a) $(P \to Q) \to R \equiv P \to (Q \to R)$
b) $(\neg P) \leftrightarrow (\neg Q) \equiv P \leftrightarrow Q$

4. Say D_1 is the domain of all students at CCSF, D_2 is the domain of all classes at CCSF. Say T is a propositional function with domain $D_1 \times D_2$, $T(x, y) :=$ x has taken y.

Express the negation of the following propositions in English.

a) $\forall x \in D_1 \quad T(x, M115)$
b) $\exists y \in D_2 \quad \neg T(\text{Felipe}, y)$
c) $\forall x \in D_1\ \exists y \in D_2 \quad T(x, y)$

5. Prove or disprove the following claims.
a) Given any two distinct integers there exists a rational number strictly between them.
b) Given any two distinct integers there exists an irrational number strictly between them.
c) Given any two distinct irrational numbers there exists an integer strictly between them.

6. Here are some sets contained in the universe $U = \{0,1,2,3,4,5\}$
$A = \{0,5\}, \quad B = \{2,3,4\}, \quad C = \{1,3,5\}$. Compute the following sets.

a) $\overline{A \cup B}$ b) $(C - A) \cup (B - C)$ c) $\overline{A} \cap B \cap \overline{C}$

7. Prove or disprove the following claims. In your arguments please rely on the definitions of ∪, ∩, ⊆ and/or equality of sets.
a) For all sets A, B, C, if $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.
b) For all sets A, B, C, if $A \cup C \subseteq B \cup C$, then $A \subseteq B$.
c) If you have two sets A and B, then $A \cup B = B$ iff $A \cap B = A$.

Directions: #8-#10 Will require some proofs by induction. When you do induction, please state the base case(s) clearly, as well as the induction assumption and where it is used in the proof.

8.Use induction to prove that n^2 equals the sum of the first n positive odd integers.
In other words $1 + 3 + \ldots + (2n - 1) = n^2$ for all integers $n \geq 1$.

9. Suppose $a_0 = 1, \quad a_1 = 2, \quad a_{n+2} = 5a_{n+1} - 6a_n$ for $n \in \mathbf{N}$.
a) Compute a_2, a_3, a_4, a_5.
b) Do you see a pattern in part (a)? Make a hypothesis about a_n, then prove it using induction.

10. (This is question 7.14 in the online textbook)
The *2-3-averaged numbers* N23 are a subset of the real-number interval $[0,1]$ defined as follows:

Base cases: $0 \in N23, \quad 1 \in N23$.

Recursive Case: If $x \in N23$ and $y \in N23$, then $\dfrac{2x + 3y}{5} \in N23$.

10a) Use induction to show that $\left(\dfrac{3}{5}\right)^n \in N23$ for all $n \in \mathbf{N}$.

10b) Use structural induction to show that
$\forall x, y \in [0,1]$ if $x \in N23$ and $y \in N23$, then $xy \in N23$.

Hint: Start a proof with the assumption that $y \in N23$. Show that $\forall x \in N23 \quad xy \in N23$.

11. One of the following functions is not a bijection. Which one and why not?

i) $f : \mathbf{R} \to [1,\infty), \quad f(x) = e^x + e^{-x}$
ii) $g : [-2,0] \to [0,16], \quad g(x) = x^4$
iii) $h : [0,\pi] \to [-1,1], \quad h(x) = \cos(x)$

12. $p : \mathbf{R} \to \mathbf{R} \quad p(x) = 2x + 3$.
Prove that p is a bijection. Be sure to use definitions of injection and surjection in your proof.

13. Prove or disprove the following claims.
a) If $f : \mathbf{R} \to \mathbf{R}$ and $f(x) > x^2$ for all $x \in \mathbf{R}$, then f is injective.
b) For all functions $f : A \to B$ and $g : B \to C$, if the composition $h = g \circ f : A \to C$ is injective, then f is injective.
c) For all functions $f : A \to B$ and $g : B \to C$, if the composition $h = g \circ f : A \to C$ is injective, then g is injective.

14. What is wrong with this proof.

Claim: $f : \mathbf{R} \to \mathbf{R} \quad f(x) = x^2$ is bijective.

Proof:
Using the function $g(x) = \sqrt{x}$ we see that if $f(x_1) = f(x_2)$, then $x_1^2 = x_2^2$ then $x_1 = \sqrt{x_1^2} = \sqrt{x_2^2} = x_2$, so f is injective.

Also, if $y \in \mathbf{R}$, then using $x = \sqrt{y}$ gives that $f(x) = \left(\sqrt{y}\right)^2 = y$, so f is surjective.

(Note on 14. It is clear that the claim is not true. However, showing that it's not true doesn't answer the question. The question asks for reason(s) why the proof is not correct.)

15. What, if anything, is wrong with this proof?

Claim: $A \cup (B_1 \cap B_2 \cap \ldots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \ldots \cap (A \cup B_n)$
for all sets $A, B_1, B_2, \ldots, B_n$ for all $n \in \mathbf{N}$.

Proof (induction on n):

Base Case: $n = 0$.
In this case there aren't any sets B_i, both sides of the equation are equal to A, so the equation is true.

Inductive Step: Suppose the claim is true when $n = k$ for some integer $k \geq 0$.

Then
$A \cup (B_1 \cap B_2 \cap \ldots \cap B_{k+1})$
$= A \cup ((B_1 \cap B_2 \cap \ldots \cap B_k) \cap B_{k+1})$
$= A \cup (B_1 \cap B_2 \cap \ldots \cap B_k) \cap (A \cup B_{k+1})$
$= ((A \cup B_1) \cap (A \cup B_2) \cap \ldots \cap (A \cup B_k)) \cap (A \cup B_{k+1})$
$= (A \cup B_1) \cap (A \cup B_2) \cap \ldots \cap (A \cup B_k) \cap (A \cup B_{k+1})$

By induction the claim is true for all $n \in \mathbf{N}$.

The page will include some potential extra credit problems. This page might be updated in the coming days.

xc1 (1pt).
The function $f : \mathbf{R} \to \mathbf{R}$ with $f(x) = x^2$ is not surjective, but what about $f : \mathbf{C} \to \mathbf{C}$ with $f(x) = x^2$. Here $\mathbf{C} = \{a + bi : a, b \in \mathbf{R}\}$ is the set of complex numbers, where $i := \sqrt{-1}$.

xc2. (pending).