name: kevin wong
filename: Math115 homework15
date: 05/16/2022
desc: https://courses.csail.mit.edu/6.042/spring18/mcs.pdf (Links to an external site.) please do these problems: 15.16, 15.37, 15.58, 15.73, 15.79

15.16

Problem 15.16.
Your class tutorial has 12 students, who are supposed to break up into 4 groups of 3 students each. Your Teaching Assistant (TA) has observed that the students waste too much time trying to form balanced groups, so he decided to pre-assign students to groups and email the group assignments to his students.

(a) Your TA has a list of the 12 students in front of him, so he divides the list into consecutive groups of 3. For example, if the list is ABCDEFGHIJKL, the TA would define a sequence of four groups to be $(\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\})$. This way of forming groups defines a mapping from a list of twelve students to a sequence of four groups. This is a k-to-1 mapping for what k?

(b) A group assignment specifies which students are in the same group, but not any order in which the groups should be listed. If we map a sequence of 4 groups,

$$(\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\}),$$

into a group assignment

$$\{\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\}\},$$

this mapping is j-to-1 for what j?

(c) How many group assignments are possible?

(d) In how many ways can $3n$ students be broken up into n groups of 3?

- a) $k = (3!)^4$, a $(3!)^4$-to-1 mapping
- b) $j = 4!$, a $4!$-to-1 mapping
- c) $\frac{12!}{4! * (3!)^4} = 15400$ possible group assignments
- d) 3n students can be broken up into $\frac{3n!}{n! * (3!)^n}$ possible group assignments

---

15.37

Problem 15.37.

Let C_{41} be the graph with vertices $\{0, 1, \ldots, 40\}$ and edges

$$\langle 0\text{—}1\rangle, \langle 1\text{—}2\rangle, \ldots, \langle 39\text{—}40\rangle, \langle 40\text{—}0\rangle,$$

and let K_{41} be the *complete graph* on the same set of 41 vertices.

You may answer the following questions with formulas involving exponents, binomial coefficients, and factorials.

(a) How many edges are there in K_{41}?

(b) How many isomorphisms are there from K_{41} to K_{41}?

(c) How many isomorphisms are there from C_{41} to C_{41}?

(d) What is the chromatic number $\chi(K_{41})$?

(e) What is the chromatic number $\chi(C_{41})$?

(f) How many edges are there in a spanning tree of K_{41}?

(g) A graph is created by adding a single edge between nonadjacent vertices of a tree with 41 vertices. What is the largest number of cycles the graph might have?

(h) What is the smallest number of leaves possible in a spanning tree of K_{41}?

(i) What is the largest number of leaves possible in a in a spanning tree of K_{41}?

(j) How many spanning trees does C_{41} have?

(k) How many spanning trees does K_{41} have?

(l) How many length-10 paths are there in K_{41}?

(m) How many length-10 cycles are there in K_{41}?

- a) There are 41 edges in C_{41} and $\frac{41(41-1)}{2} = 820$ edges in K_{41}
- b) There are 41! isomorphisms from K_{41} to K_{41}. Basically all permutations of the vertices in the graph. Any of the vertices in this graph can be interchanged and the adjacencies will be preserved.
- c) There are 41! isomorphisms from C_{41} to C_{41}, same reason as part b.
- d) $\chi(K_{41}) = 41$ because every vertex has an edge to each vertex in the graph.
- e) $\chi(C_{41}) = 3$ because this graph has an old number of vertices with each vertex having degree 2.
- f) There are 40 edges in a spanning tree of K_{41}

- g) If a graph is created by adding a single edge between nonadjacent vertices of a tree with 41 vertices, the largest number of cycles the graph would have would be 1 because a tree is acyclic.
- h) The smallest number of leafs possible in a spanning tree of K_{41} is 2. Basically C_{41} with any one edge removed.
- i) The largest number of leafs possible in a spanning tree of K_{41} is 40. 40 leafs branching off a vertex.
- j) C_{41} can have a maximum of 41 spanning trees, by removing anyone of the 41 edges.
- k) A complete graph can have a maximum of n^{n-2} spanning trees, so K_{41} has 41^{41-2} spanning trees.
- l) There are 40 edges for each vertex in K_{41}, so $40 * 40 * 40 * 40 * 40 * 40 * 40 * 40 * 40 * 40$ length-10 paths in K_{41}
- m) There are 40 edges for each vertex in K_{41}, so $40 * 39 * 38 * 37 * 36 * 35 * 34 * 33 * 32 * 31$ length 10 cycles in K_{41}

15.58

Problem 15.58.
We want to count step-by-step paths between points in the plane with integer coordinates. Only two kinds of step are allowed: a right-step which increments the x coordinate, and an up-step which increments the y coordinate.

(a) How many paths are there from $(0, 0)$ to $(20, 30)$?

(b) How many paths are there from $(0, 0)$ to $(20, 30)$ that go through the point $(10, 10)$?

(c) How many paths are there from $(0, 0)$ to $(20, 30)$ that do *not* go through either of the points $(10, 10)$ and $(15, 20)$?

Hint: Let P be the set of paths from $(0, 0)$ to $(20, 30)$, N_1 be the paths in P that go through $(10, 10)$ and N_2 be the paths in P that go through $(15, 20)$.

- a) $\frac{50!}{20!*(50-20)!} = P$. The same as 50 choose 20. The order of going right and up don't matter in this problem.
- b) | paths that go to (10,10) | + | paths the go from (10, 10) to (20, 30) |

  $\frac{20!}{10!*(20-10)!} + \frac{30!}{10!*(30-10)!} = N_1$
- c) $P - N_1 - N_2$

  The paths that go through $(15, 20)$ are

  | paths that go to (15,20) | + | paths the go from (15, 20) to (20, 30) |

  are $\frac{35!}{15!*(35-15)!} + \frac{15!}{5!*(15-5)!} = N_2$

So the paths from (0,0) to (20, 30) that do not go through either (10,10) and (15,20) are calculated using the expression below.

$\frac{50!}{20!*(50-20)!} - (\frac{20!}{10!*(20-10)!} + \frac{30!}{10!*(30-10)!}) - (\frac{35!}{15!*(35-15)!} + \frac{15!}{5!*(15-5)!})$

15.73

Problem 15.73.
Give a combinatorial proof for this identity:

$$\sum_{r=0}^{n} \binom{n}{r}\binom{m}{k-r} = \binom{n+m}{k}$$

- Let n and m be the number of decisions in two different sets. In the time you have you can only accomplish k decisions.
- RHS, only allowed to make k decisions from either the n pool or the m pool.
- LHS, if r is increased by one, that takes away a decision from the m. Summing 0 to n counts all possible decisions or cases from r equal to 0 to r equal to n.
- So the LHS and the RHS are both counting k decisions possible out of $n+m$ number of decisions.

15.79

Problem 15.79.
Give a combinatorial proof of

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (n-1) \cdot n = 2\binom{n+1}{3}$$

Hint: Classify sets of three numbers from the integer interval $[0..n]$ by their maximum element.

- The below is just a guess, I don't know, but I believe the math works out with what I have below.
- The LHS is equal to $\sum_{i=0}^{n} i^2 - i = RHS$. This counts all the sets of $i^2 - i$ from 0 to n.
- The RHS is counts all the elements for $n+1$ choose 3 elements two times. The elements chosen have to be in a specific order to satisfy the elements of the rows in the table below.
- The both seem to count the sum of the elements in row 1 in the image below.

```
0*1 + 1*2 + 2*3 + 3*4 + 4*5 + 5*6 + 6*7 + 7*8+...+ (n-1)n

difference between each number forms this triangle shape with 3 rows:

0   2   6   12  20  30  42  56 ... "row 1"
  2   4   6   8   10  12  14 ...   "row 2"
    2   2   2   2   2   2 ...      "row 3"
      0   0   0   0   0...
```

END