M115 Homework3 Solutions, Spring 2022.

1.12. $\forall n > 0$ if a^n is even, then a is even.

Proof: Suppose n is a positive integer, and that a^n is even. (A tacit assumption here also is that a is an integer - I'm going to try to prove this in the case when the conclusion 'a is even' actually makes sense.)

We want to conclude that a is even, so for the sake of contradiction let's assume that this doesn't happen, and try to reach a contradiction. So we assume that a is odd.

This means that $a = 2k + 1$ for some k. Then

$$a^n = (2k+1)^n = (2k+1) \cdot (2k+1) \cdot \ldots \cdot (2k+1)$$

with n factors of $2k + 1$ in that product. There might be some fancy theorems you could appeal to here (e.g. something called The Binomial Theorem might come in handy), but even without those we can see that this is going to be an odd number because upon distributing all the 2k's and the 1's you'll have a single $1 \cdot 1 \cdot 1 ... \cdot 1 = 1$ from the product of all the 1's, and then every other product will have at least one factor of 2k. This means a^n could be expressed as 1 more than 2 times some integer, so a^n is odd. This contradicts the original assumption that a^n is even. Therefore if a^n is even, it must also be that a is even too. QED.

1.13. This question is part of the first problem of quiz#3. I'm not going to write anything here just yet.

1.15. Find some integers m, n so that m goes into n^2, but not n itself. Here are some: $(m, n) = (9,6)$. If you want $m < n$ then $(m, n) = (4,6)$ would work. There are lots of examples, but they would all need for m to be not prime, because if m were prime it'll divide n iff it divides n^2. Prime numbers are strict in that way.

1.17. Prove that $\log_4(6)$ is irrational.

Proof (by contradiction): suppose not!

Assume $\log_4(6)$ is rational and try to find a contradiction.

$\log_4(6)$ being rational means that it's a fraction of integers:

$\log_4(6) = p/q$ with $q \neq 0$. Notice that $1 = \log_4(4) < \log_4(6)$ since the log is an increasing function. So with this fraction p/q we need to have $p > q$.

Multiply both sides by q and use some log rules to get $p = \log_4(6^q)$.

Exponentiate both sides by 4 to get $4^p = 6^q$.

Use some algebra rules to express this like $2^{2p} = 2^q \cdot 3^q$

Divide both sides by 2^q to get $2^{2p-q} = 3^q$.

Since p>q, we'll have 2p>q, so the number on the left side is an even integer. But the number on the right side isn't even, since it only has 3 as a prime factor. This is a contradiction. Hence $\log_4(6)$ can't possibly be rational.

The Boolean Function Problem:

x	y	z	G
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	1
0	1	1	1
0	1	0	0
0	0	1	1
0	0	0	0

a)
sum of products: $G(x, y, z) = xy\bar{z} + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}z$

product of sums: $G(x, y, z) = (\bar{x} + \bar{y} + \bar{z})(x + \bar{y} + z)(\bar{x} + y + \bar{z})(x + y + z)$

x/c problems:

1. regarding this Boolean Function. part b). In the Boolean context a concise expression would be $G(x, y, z) = \bar{x}z + x\bar{z}$, but translating to a logical context with $(x, y, z) \mapsto (P, Q, R)$, you could say $G \equiv P \oplus R$. (I don't remember the Boolean analogue to the logical $\oplus$, but it's probably out there.)

2. (1.19)

a) because $\sqrt[m]{k}$ is a root of the polynomial $x^m - k$. According to the theorem the only rational roots of this are integers. So if k is not an m^{th} power so that $\sqrt[m]{k}$ isn't an integer, it must be irrational.

1.19 b) Suppose r is a root of $x^m + a_{m-1}x^{m-1} + \ldots + a_1x + a_0$.

Case 1: r is irrational. In this case the conclusion of the theorem is satisfied.

Case 2: r is rational. In this case we have some work to do. We need to show that r is an integer. r begin rational means that $r = p/q$ for some integers p and q. We can assume w.l.o.g. that this fraction is reduced, so that p and q have no common factors.

$r = p/q$ being a root means that $\left(\frac{p}{q}\right)^m + a_{m-1}\left(\frac{p}{q}\right)^{m-1} + \ldots + a_1\left(\frac{p}{q}\right) + a_0 = 0.$

Multiply this mess by q^m in order to clear denominators.

$p^m + a_{m-1}p^{m-1}q + \ldots + a_1pq^{m-1} + a_0q^m = 0.$ Subtract p^m from both sides…

$a_{m-1}p^{m-1}q + \ldots + a_1pq^{m-1} + a_0q^m = -p^m.$ Factor out a q on the left side…

$q(a_{m-1}p^{m-1} + \ldots + a_1pq^{m-2} + a_0q^{m-1}) = -p^m.$

This means that q divides p^m which is strange since we assumed that p and q don't have any common factors. The prime factors of p^m are the same as those of p (they're just repeated m times in the number p^m.) So since q and p have no factors in common, q and p^m don't have any common factors either. So with q dividing p^m the only possibility is that $q = 1$. This is great since that means the $r = p/q$ is an integer, as we wanted to show. QED.

3. Prove or disprove that between any two irrational numbers there exists another irrational number. Basically, taking an average of two irrational numbers works - usually - but not always.

The average of the irrational numbers $\sqrt{2}$ and $1 - \sqrt{2}$, for example, is 0.5, which is rational. But nevertheless we can still find an irrational number between two given irrational numbers.

Proof:

Suppose x and y are irrational numbers. We want to find an irrational z between x and y. (The problem should have included the assumption $x \neq y$.)

Case 1: $x + y$ is irrational. In this case $z = (x + y)/2$ will work, just like it did when we were working with two given rational numbers.

Case 2: $x + y$ is rational. In this case $x + 2y$ must be irrational because if $x + 2y$ was also rational along with $x + y$ then we would have a rational number in this difference $(x + 2y) - (x + y) = y$, which contradicts the assumption that y is an irrational number.

So in this case 2, we have $x + 2y$ is irrational. We want a z? Let's try this one: $z = (x + 2y)/3$. You could think of this as the average of the numbers $\{x, y, y\}$.

Notice that this z is irrational, since dividing the irrational $x + 2y$ by 3 won't rationalize it. Also note that z is between the x and y, because

If $x < y$, then $x = \dfrac{x + x + x}{3} < \dfrac{x + y + y}{3} = z < \dfrac{y + y + y}{3} = y$

If $x > y$, then $x = \dfrac{x + x + x}{3} > \dfrac{x + y + y}{3} = z > \dfrac{y + y + y}{3} = y$

So this z is between the given x and y.

4. $\sqrt{3} - \sqrt{2}$ is irrational.

Proof (by contradiction). Suppose it's rational.

Then $\sqrt{3} + \sqrt{2}$ must be irrational since if this number were also rational then the sum of these two rational numbers, $(\sqrt{3} - \sqrt{2}) + (\sqrt{3} + \sqrt{2}) = 2\sqrt{3}$, would also be rational, contrary to the fact that $\sqrt{3}$ is irrational.

The product of the rational number $\sqrt{3} - \sqrt{2}$ and the irrational number $\sqrt{3} + \sqrt{2}$ is a rational number 1.

Said differently, if we divide 1 by $\sqrt{3} - \sqrt{2}$ we get a quotient $\sqrt{3} + \sqrt{2}$. This quotient is irrational as reasoned above, but also rational since dividing the rational number 1 by the rational number $\sqrt{3} - \sqrt{2}$ results in a rational number. This is a contradiction. Thus $\sqrt{3} - \sqrt{2}$ can't be rational.

5. Prove or disprove that $\sqrt{3} - \sqrt[3]{2}$ is irrational.

Note: I'm confident that it's irrational. But I think a proof is not very easy. I think there is a proof amidst some fancy algebra (see Galois Theory), but maybe there's also some justification for this when you think about vector spaces. For example the numbers $\mathbf{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a\,.\,b \in \mathbf{Q}\}$ form a two-dimensional vector space over $\mathbf{Q}$.

If you're curious about more details along these lines, let me know. These details won't be required in our class. Otherwise, there may very well be a good proof out there without any fancy theoretical details. I suspect such a proof might be rather messy, but not impossible.