name: kevin wong
filename: Math115 homework6
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desc: https://courses.csail.mit.edu/6.042/spring18/mcs.pdf (Links to an external site.) please do these problems: 1, 2, 3(7.5)

#1

Here is a recursively defined function, $f : \{ n \in \mathbb{N} : n \geq 1 \} \to \mathbb{N}$.

Base case: $f(1) = 0$.

Recusive case: $f(n+1) = f(n) + 2n - 1$

a) Compute $f(n)$ for $n = 2, 3, 4, 5$.

b) This function is equal to a polynomial function p. Find p and prove that $f(n) = p(n)$ for all $n \geq 1$.

- a) Compute $f(n)$ for $n = 2, 3, 4, 5$.
  - $n = 1$

    $f(1+1) = f(1) + (2 * 1) - 1$

    $f(2) = 0 + 2 - 1$

    $f(2) = 1$
  - $n = 2$

    $f(2+1) = f(2) + (2 * 2) - 1$

    $f(3) = 1 + 4 - 1$

    $f(3) = 4$
  - $n = 3$

    $f(3+1) = f(3) + (2 * 3) - 1$

    $f(4) = 4 + 6 - 1$

    $f(4) = 9$
  - $n = 4$

    $f(4+1) = f(4) + (2 * 4) - 1$

    $f(5) = 9 + 8 - 1$

    $f(5) = 16$
  - $n = 5$

    $f(5+1) = f(5) + (2 * 5) - 1$

    $f(6) = 16 + 10 - 1$

    $f(6) = 25$

- evaluated base cases for $n = 1, 2, 3, 4, 5$ for $f(n)$, so now we want an explicit formula for the recursive function.
- b) finding $p(n)$

  Assume $n \in \mathbb{N}$ and $p(n) = n^2 - 2n + 1$ and $p(n) \in \mathbb{N}$ (found through trial and error).

  Using the recursive case equation:

  $p(n+1) = p(n) + 2n - 1$

  substituting $p(n) = n^2 - 2n + 1$

  $= (n^2 - 2n + 1) + 2n - 1$

  $= n^2$

  so, $p(n+1) = n^2$ ✓

  and $p(n) = (n-1)^2$ ✓

- prove $f(n) = p(n)$

  reminder: $f(n) = f(n+1) - 2n + 1$ and $p(n) = (n-1)^2$

  prove by weak induction

  - base case $n \in \mathbb{N}$ and $n = 1, 2, 3, 4, 5$ for $p(n)$

    $p(1) = (1-1)^2 = 0$ ✓

    $p(2) = (2-1)^2 = 1$ ✓

    $p(3) = (3-1)^2 = 4$ ✓

    $p(4) = (4-1)^2 = 9$ ✓

    $p(5) = (5-1)^2 = 16$ ✓

    so for $n = 1, 2, 3, 4, 5$, $p(n) = f(n)$ in each case.

  - inductive step

    inductive hypothesis: Assume $k \in \mathbb{N}$, $k \geq 5$ and $1 \leq n \leq k$ and $p(k) = (k-1)^2$. Also assume $p(k) \to p(k+1) \in \mathbb{N}$.

    $p(n) = (n-1)^2$

    so $p(k+1) = ((k+1) - 1)^2$

    $= k^2$

    $= (k + 1 - 1)^2$

    $= ((k-1) + 1)^2$

    $= ((k-1) + 1) * ((k-1) + 1)$

    $= (k-1)^2 + 2(k-1) + 1$

    $= (k-1)^2 + 2k - 2 + 1$

    $= (k-1)^2 + 2k - 1$

    $= p(n) + 2k - 1$

so $p(k+1) = p(n) + 2k - 1 = f(k+1)$ ✓

#2

Here is a recursively defined set, S.

Base cases: $5, 9 \in S$

Recursive Case: If $x \in S$ and $y \in S$, then $x + y \in S$

a) Give a non-recursive definition, R, of the set S.

b) Prove that your set R in part (a) equals the set S.

- a) non-recursive definition, R

  $R ::= \{\ x, y \in S : \exists n \in \mathbb{N}_1, \exists m \in \mathbb{N}_0 : nx + my = z : z \in S\ \} \bigcap \{\ x = 5 \text{ or } 9, y = 5 \text{ or } 9\ \}$

  english: "if x, y are both in S, there exist an n in $\mathbb{N}_1$ and a m in $\mathbb{N}_0$ where $nx + my = z$ such that z is in S, and x and y can equal 5 or 9."

- b) prove by structural induction
  - base cases
    - $n = 1, m = 0, x = 5$

      $n * x = 1 * 5 = 5 \in S$ and $\in R$ ✓
    - $n = 1, m = 0, x = 9$

      $n * x = 1 * 9 = 9 \in S$ and $\in R$ ✓
  - induction step
    - case 1: Assume $n \in \mathbb{N}_0, m \in \mathbb{N}_1, m = (m+1)^{th}$ generation, $n = (n+1)^{th}$ generation and x and y can equal 5 or 9

      subcase 1: $nx + my = (n+1)^{th}(5) + (m+1)^{th}(9) \in R$ and any sum of the multiples of 5 and 9 are in S. ✓

      subcase 2: $nx + my = (n+1)^{th}(5) + (m+1)^{th}(5) \in R$ and any sum of multiples of 5 are in S. ✓

      subcase 3: $nx + my = (n+1)^{th}(9) + (m+1)^{th}(9) \in R$ and any sum of multiples of 9 are in S. ✓
    - $S \subseteq R$ and $R \subseteq S$

7.5

Here is a simple recursive definition of the set E of even integers:

Definition. Base case: $0 \in E$.

Constructor cases: If $n \in E$, then so are $n+2$ and $-n$.

Provide similar simple recursive definitions of the following sets:

- a) $S ::= \{2^k 3^m 5^n \in \mathbb{N} | k, m, n \in \mathbb{N}\}$

  Definition. Base case: $f(0,0,0) = 1 \in \mathbb{N}$

  Constructor cases: If $k, m, n \in \mathbb{N}^*$, then $f(k+1, m+1, n+1) = 2^{k+1} 3^{m+1} 5^{n+1} \in \mathbb{N}^*$

- b) $S ::= \{2^k 3^{2k+m} 5^{m+n} \in \mathbb{N} | k, m, n \in \mathbb{N}\}$

  Definition. Base case: $f(0,0,0) = 1 \in \mathbb{N}$

  Constructor cases: If $k, m, n \in \mathbb{N}^*$, then $f(k+1, m+1, n+1) = 2^{k+1} 3^{2k+m+3} 5^{m+n+2} \in \mathbb{N}^*$

- c) $L ::= \{\ (a, b) \in \mathbb{Z}^2 | (a-b)$ is a multiple of 3 $\}$

  Definition. Base case: $(0,0) \in \mathbb{Z}^2$

  Constructor cases: If $a - b$ is a multiple of 3, then $(a+1, b+1) \in \mathbb{Z}^2$

- d) L' is the recursive definition given in (c)

  prove by induction

  - base case

    $a = 0, b = 0$

    $(0,0)$, $a - b$ is a multiple of 3 and in L

  - inductive step

    inductive hypothesis: $m, n \in \mathbb{Z}$, $m - n$ is a multiple of 3, $(m, n) \in \mathbb{Z}^2 \rightarrow (m+1, n+1) \in \mathbb{Z}^2$

    - case 1: $m > 0, n > 0$, $m - n$ is a multiple of 3, $(m+1, n+1) \in L'$ and $\in L$
    - case 2: $m < 0, n < 0$, $m - n$ is a multiple of 3, $(m+1, n+1) \in L'$ and $\in L$
    - case 3: $m > 0, n < 0$, $m - n$ is a multiple of 3, $(m+1, n+1) \in L'$ and $\in L$
    - case for a counter example: let's say $k, l \in \mathbb{Z}$ and $(k, l) \in L'$. Based on the recursive definition set (k, l) would only be in L' and L if $k - l$ is a multiple of 3. So if it's in L' then it's in L. No counter example found.

  - By induction we have all combinations of 2 dimensional sets of positive and negative integers m and n, where $m - n$ is a multiple of 3. So $L' \subseteq L$ and $L \subseteq L'$.
