Toll Road Improvements in a Network

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(Continuous variables, constrained.)

Problem Statement:

Consider a connected graph $G = (E, V)$ that describes a network of toll roads. Customers wishing to travel from point i to point j arrive at point i according to a Poisson process with rate λ_{ij}. Let p_{ij} denote the toll charged for driving on road (i, j). Customers who have completed a trip on a road either choose to take another trip or leave the network, based on a fixed routing matrix R. Travel times are assumed to be triangularly distributed with fixed parameters that are route-specific. Other than the triangularly-distributed travel times, this is an open Jackson network where the roads are interconnected queues.

We assume that the each of the directed edge (representing a road) has capacity for one car to travel on it at a time. Thus if another car is occupying the road, other cars will form a queue and wait for the car to finish using the road.

The operator of the toll network can make improvements to the toll roads to change the distributions of travel times. Suppose that if x_{ij} hundred dollars is spent on improving road (i, j), the resulting mode of the travel-time distribution for road (i, j) will be $a_{ij} + (b_{ij} - a_{ij}) \exp(-x_{ij})$ where a_{ij} and b_{ij} represent the minimum and maximum of the triangular distribution. Therefore if no investment is made to improve road (i, j), the mode will be $c_{ij} = b_{ij}$.

The objective of the network operator is to choose the individual road improvements x_{ij} to maximize the expected profit over a finite horizon of length T:

$$\mathbb{E}[\text{Profit}] = \sum_{(i,j) \in E} p_{ij} \mathbb{E}[N_{ij}(T)] - \sum_{(i,j) \in E} x_{ij},$$

where $N_{ij}(T)$ is the number of trips on road (i, j) completed before time T.

Recommended Parameter Settings: As a small example, let G be a complete graph consisting of three vertices, i.e., the adjacency matrix of G is

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

There are six routes for which to determine investments. For external arrival rates, take $\lambda_{ij} = i + j$ for all $i \neq j = 1, \ldots, 3$. For internal routing, let

$$R = \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/5 & 1/5 & 0 & 3/5 \end{bmatrix},$$

where the last column represents the option of exiting the network. Set $p_{ij} = 1$ for all roads (i, j). For the parameters of the triangular distribution for travel times on route (i, j), take $a_{ij} = 0$ (the minimum) and $b_{ij} = 1$ (the maximum). Set $T = 96$, representing 12 hours of operation.

Starting Solutions: Set $x_{ij} = 1$ for all routes (i, j).

Measurement of Time: One time unit is equal to 10 minute's operation.

Optimal Solutions: Unknown.

Known Structure: None.