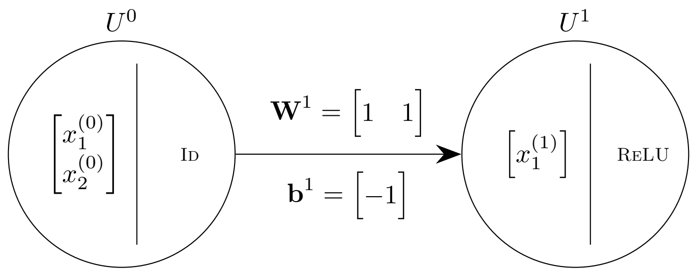
1 Logical AND

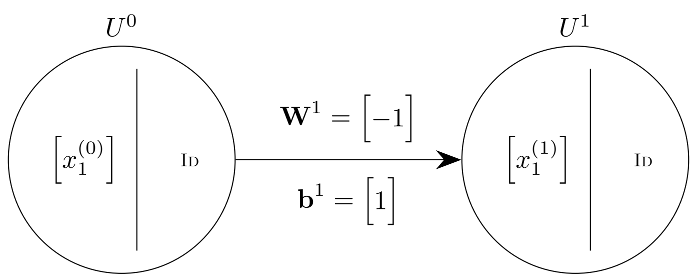
$$\{0,1\}^2 \to \{0,1\}$$

$$U_0 = \mathrm{Id}\left(\begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \end{bmatrix}\right)$$
$$U_1 = \mathrm{ReLU}\left(\left[\mathrm{Id}\left(x_1^{(0)}\right) + \mathrm{Id}\left(x_2^{(0)}\right) - 1\right]\right)$$



2 Logical NOT

$$U_0 = \mathrm{Id}\left(\left[x_1^{(0)}\right]\right)$$
$$U_1 = \mathrm{Id}\left(\left[-\mathrm{Id}\left(x_1^{(0)}\right) + 1\right]\right)$$

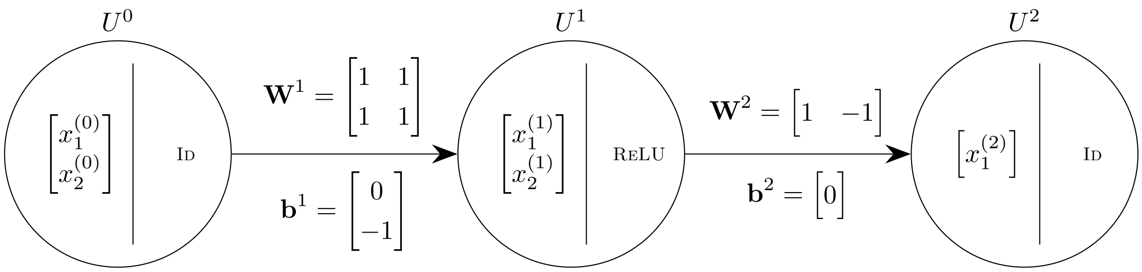


3 Logical OR

$$\{0,1\}^2 \to \{0,1\}$$

Let S be the sum of the inputs, which is $x_1^{(0)} + x_2^{(0)}$. We output whether $S > 0$ by the taking the ReLU of S and $S - 1$. If $S \leq 0$, then $0 - 0 = 0$. If $S > 0$, then $S - (S-1) = 1$.

$$U_0 = \mathrm{Id}\left(\begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \end{bmatrix}\right)$$
$$U_1 = \mathrm{ReLU}\left(\begin{bmatrix} \mathrm{Id}\left(x_1^{(0)} + x_2^{(0)}\right) \\ \mathrm{Id}\left(x_1^{(0)} + x_2^{(0)}\right) - 1 \end{bmatrix}\right)$$
$$U_2 = \mathrm{Id}\left(\left[\mathrm{ReLU}\left(\mathrm{Id}\left(x_1^{(0)} + x_2^{(0)}\right)\right) - \mathrm{ReLU}\left(\mathrm{Id}\left(x_1^{(0)} + x_2^{(0)}\right) - 1\right)\right]\right)$$



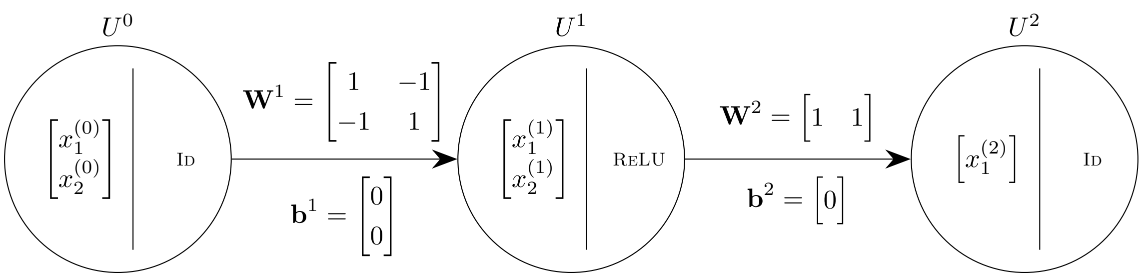
4 Logical XOR

$$\{0,1\}^2 \to \{0,1\}$$

TODO: what if more than 2 inputs?

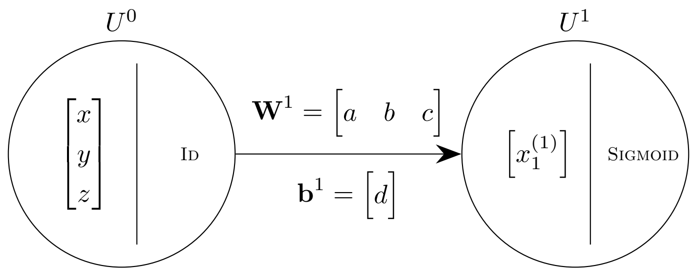
We can think of binary XOR as returning 1 if the inputs are different, and 0 if they are the same. For the second case, we can take advantage of the fact that the difference between two identical numbers is 0. Thus, we take the ReLU of the difference between the two inputs in both directions, that is, $\mathrm{ReLU}\left(x_1^{(0)} - x_2^{(0)}\right)$ and $\mathrm{ReLU}\left(x_2^{(0)} - x_1^{(0)}\right)$. If the inputs are the same, $0 + 0 = 0$. If the inputs are different, the difference in one direction is 1 and the other is -1 rectified to 0, resulting in 1.

$$\begin{aligned}
U_0 &= \mathrm{Id}\left(\begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \end{bmatrix}\right) \\
U_1 &= \mathrm{ReLU}\left(\begin{bmatrix} \mathrm{Id}\left(x_1^{(0)} - x_2^{(0)}\right) \\ \mathrm{Id}\left(x_2^{(0)} - x_1^{(0)}\right) \end{bmatrix}\right) \\
U_2 &= \mathrm{Id}\left(\left[\mathrm{ReLU}\left(\mathrm{Id}\left(x_1^{(0)} - x_2^{(0)}\right)\right) + \mathrm{ReLU}\left(\mathrm{Id}\left(x_2^{(0)} - x_1^{(0)}\right)\right)\right]\right)
\end{aligned}$$



5 Logistic Regression

$$\begin{aligned}
U_0 &= \mathrm{Id}\left(\begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \end{bmatrix}\right) \\
U_1 &= \sigma\left(\left[a \cdot \mathrm{Id}\left(x_1^{(0)}\right) + b \cdot \mathrm{Id}\left(x_2^{(0)}\right) + c \cdot \mathrm{Id}\left(x_3^{(0)}\right) + d\right]\right)
\end{aligned}$$



6 Two-Variable Function