Approximating the Period of a Pendulum

AP Calculus BC: Taylor Series Project

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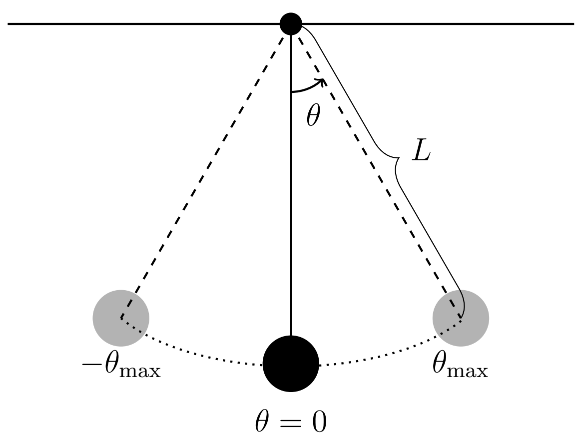

1 Inputs

- L: Length of pendulum (meters)
- $\theta_{\max}$: Maximum angular displacement (radians)

2 Period of a Pendulum

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} \, d\theta$$

where $k = \sin\left(\frac{\theta_{\max}}{2}\right)$ (a constant).

3 Series Expansion

Let $u = k^2 \sin^2\theta$, and the integrand becomes:

$$f(u) = \frac{1}{\sqrt{1-u}} = (1-u)^{-\frac{1}{2}}$$

which is in the form of a binomial series $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$.

Since we have not yet covered binomial series in class, we will instead use the Maclaurin series expansion for $(1-u)^{-1/2}$ around $u = 0$. First, we find the general form of the nth derivative:

$$\begin{aligned}
f^{(0)}(u) &= (1-u)^{-\frac{1}{2}} && \to f^{(0)}(0) = \frac{1}{\cancel{2^0}} \\
f^{(1)}(u) &= -\frac{1}{2}(1-u)^{-\frac{3}{2}}(-1) && \to f^{(1)}(0) = \frac{1}{2^1} \\
f^{(2)}(u) &= -\frac{1\cdot 3}{2^2}(1-u)^{-\frac{5}{2}}(-1) && \to f^{(2)}(0) = \frac{1\cdot 3}{2^2} \\
f^{(3)}(u) &= -\frac{\overbrace{1\cdot 3\cdot 5}^{\text{odd-only factorial}}}{2^3}(1-u)^{-\frac{7}{2}}(-1) && \to f^{(3)}(0) = \frac{1\cdot 3\cdot 5}{2^3} \\
f^{(n)}(u) &= \frac{(2n-1)!!}{2^n}(1-u)^{-\frac{2n+1}{2}} && \to f^{(n)}(0) = \frac{(2n-1)!!}{2^n}
\end{aligned}$$

Plugging in to the Maclaurin series formula $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$, we get:

$$\frac{1}{\sqrt{1-u}} = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^n n!} u^n$$

Since $2^n n!$ is equivalent to multiplying by 2 for every term in the factorial, we can rewrite the series as the following, after substituting $u = k^2 \sin^2\theta = (k\sin\theta)^2$:

$$\sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} (k\sin\theta)^{2n}$$

Keep in mind that for $n = 0$, $(2(0) - 1)!! = (-1)!! = 1$ by convention. [1]

4 Simplifying the Integral

Substitute the above Maclaurin series into the integral:

$$\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}}\, d\theta = \int_0^{\frac{\pi}{2}} \sum_{n=0}^{\infty} \left[\frac{(2n-1)!!}{(2n)!!} (k \sin \theta)^{2n} \right] d\theta$$

Take constant (non-θ) terms out of the integral, since they are independent of θ:

$$= \sum_{n=0}^{\infty} \left[\frac{(2n-1)!!}{(2n)!!} k^{2n} \int_0^{\frac{\pi}{2}} \sin^{2n} \theta \, d\theta \right]$$

We evaluate the first three terms of the series to get a sense of the pattern:

$$1 + \frac{1}{2} k^2 \int_0^{\frac{\pi}{2}} \sin^2 \theta \, d\theta + \frac{3}{8} k^4 \int_0^{\frac{\pi}{2}} \sin^4 \theta \, d\theta + \cdots$$

Find the values of the definite integrals using half-angle identities:

$$\int_0^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{1}{2} \cos(2\theta) \right) d\theta = \left[\frac{\theta}{2} - \frac{1}{4} \sin(2\theta) \right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \sin^4 \theta \, d\theta = \left[\frac{\theta}{4} - \frac{1}{4} \sin(2\theta) + \frac{\theta}{8} + \frac{1}{32} \sin(4\theta) \right]_0^{\frac{\pi}{2}} = \frac{\pi}{8} + \frac{\pi}{16} = \frac{3\pi}{16}$$

The resulting terms are:

$$1 + \frac{1}{2} k^2 \left(\frac{\pi}{4} \right) + \frac{3}{8} k^4 \left(\frac{3\pi}{16} \right) + \cdots$$

The process is tedious, but we see that each integral is in the form of Wallis integrals, defined by the sequence $W_n = \int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta$. Evaluating each

[1] https://mathworld.wolfram.com/DoubleFactorial.html

definite integral gives some multiple of $\frac{\pi}{2}$, and the antiderivative of all the recursive cosines from the half-angle identities all vanish, since $\sin(2n \cdot \frac{\pi}{2}) = \sin(n\pi) = 0$. Through integration by parts and recursive substitution of I, the value of even Wallis integrals are known to be:

$$W_{2n} = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2}$$

After substituting in the value of the Wallis integrals, we can now express the general (approximate) formula for the period of a pendulum using the integrated Maclaurin series:

$$\begin{aligned} T &= 4\sqrt{\frac{L}{g}} \sum_{n=0}^{\infty} \left[\frac{(2n-1)!!}{(2n)!!} k^{2n} \cdot W_{2n} \right] \\ &= 4\sqrt{\frac{L}{g}} \sum_{n=0}^{\infty} \left[\frac{(2n-1)!!}{(2n)!!} k^{2n} \cdot \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2} \right] \\ &= 4 \cdot \frac{\pi}{2} \sqrt{\frac{L}{g}} \sum_{n=0}^{\infty} \left[k^{2n} \left(\frac{(2n-1)!!}{(2n)!!} \right)^2 \right] \\ &= 2\pi \sqrt{\frac{L}{g}} \sum_{n=0}^{\infty} \left[\left(\frac{(2n-1)!!}{(2n)!!} \right)^2 k^{2n} \right] \end{aligned}$$

5 Results

Let $L = 1$ m and $\theta_{\max} = \frac{\pi}{6}$ radians. It follows that $k = \sin\left(\frac{\theta_{\max}}{2}\right) = \sin\left(\frac{\pi}{12}\right)$. The CIPM defines $g = 9.80665\,\mathrm{m/s^2}$.

Using a calculator, the exact period of the pendulum is:

$$T = 4\sqrt{\frac{1}{g}} \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - \sin^2\left(\frac{\pi}{12}\right)\sin^2\theta}}\, d\theta \approx \boxed{2.041339\,\mathrm{s}}$$

$$T_0 = 2\pi\sqrt{\frac{L}{g}}\,(1) \approx \boxed{2.006409\,\mathrm{s}}$$

$$T_1 = 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{1}{4}k^2\right) \approx \boxed{2.040010\,\mathrm{s}}$$

$$T_2 = 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{1}{4}k^2+\frac{9}{64}k^4\right) \approx \boxed{2.041276\,\mathrm{s}}$$

$$T_3 = 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{1}{4}k^2+\frac{9}{64}k^4+\frac{25}{256}k^6\right) \approx \boxed{2.041335\,\mathrm{s}}$$

As the number of terms approaches infinity, the value of T converges to the value using the definite integral.

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{1}{4}k^2+\frac{9}{64}k^4+\frac{25}{256}k^6+\cdots\right)$$

6 Error Bound

Using the ratio test,

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\left(\frac{(2n+1)!!}{(2n+2)!!}\right)^2 k^{2n+2}}{\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 k^{2n}}\right| \\
&= \lim_{n\to\infty}\left|\frac{\left(\frac{(2n+1)\cancel{(2n-1)!!}}{(2n+2)\cancel{(2n)!!}}\right)^2 \cancel{k^{2n}}k^2}{\left(\frac{\cancel{(2n-1)!!}}{\cancel{(2n)!!}}\right)^2 \cancel{k^{2n}}}\right| \\
&= \lim_{n\to\infty}\left|\frac{(2n+1)^2}{(2n+2)^2}k^2\right| = \left|k^2\right| && \text{(ladder of functions)} \\
&\implies \left|k^2\right| < 1 \implies -1 < k < 1 \implies -1 < \sin\left(\frac{\theta_{\max}}{2}\right) < 1 \\
&\implies 0 < \theta_{\max} < \pi && (\theta_{\max} \text{ is nonnegative})
\end{aligned}$$

The maximum error bound is given by the next term in the series.

$$T_\infty = T_j + R_j \quad \Longrightarrow \quad R_j = T_\infty - T_j \leq a_{j+1}$$

$$T_0 \approx 2.006409\,\mathrm{s} \quad \Longrightarrow \quad R_0 \leq 2\pi\sqrt{\frac{L}{g}}\left(\frac{1}{4}k^2\right) \approx \boxed{0.001266\,\mathrm{s}}$$

$$T_1 \approx 2.040010\,\mathrm{s} \quad \Longrightarrow \quad R_1 \leq 2\pi\sqrt{\frac{L}{g}}\left(\frac{9}{64}k^4\right) \approx \boxed{0.000059\,\mathrm{s}}$$

$$T_2 \approx 2.041276\,\mathrm{s} \quad \Longrightarrow \quad R_2 \leq 2\pi\sqrt{\frac{L}{g}}\left(\frac{25}{256}k^6\right) \approx \boxed{0.000003\,\mathrm{s}}$$

$$T_3 \approx 2.041335\,\mathrm{s} \quad \Longrightarrow \quad R_3 \leq 2\pi\sqrt{\frac{L}{g}}\left(\frac{49}{1024}k^8\right) \approx \boxed{0.000000\,\mathrm{s}}$$