Approximating the Period of a Pendulum

AP Calculus BC: Taylor Series Project

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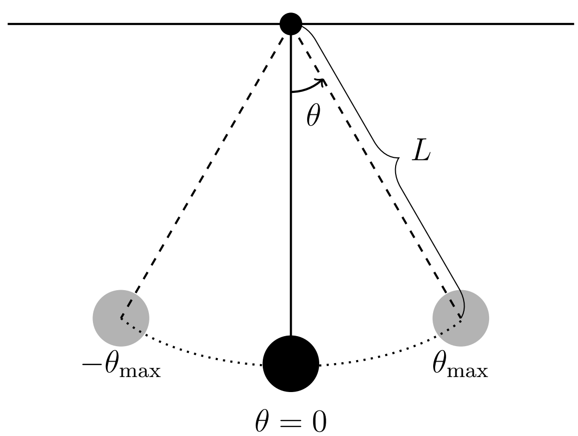

1 Inputs

- L: Length of pendulum (meters)
- $\theta_{\max}$: Maximum angular displacement (radians)

2 Period of a Pendulum

$$T = 2\pi\sqrt{\frac{L}{g}}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2\sin^2\theta}}\,d\theta$$

where $k = \sin\left(\frac{\theta_{\max}}{2}\right)$ (a constant).

3 Series Expansion

Let $u = k^2 \sin^2\theta$, and the integrand becomes:

$$f(u) = \frac{1}{\sqrt{1-u}} = (1-u)^{-\frac{1}{2}}$$

which is in the form of a binomial series $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$.

Since we have not yet covered binomial series in class, we will instead use the Maclaurin series expansion for $(1-u)^{-1/2}$ around $u = 0$. First, we find the general form of the nth derivative:

$$\begin{aligned}
f^{(0)}(u) &= (1-u)^{-\frac{1}{2}} && \to f^{(0)}(0) = \frac{1}{\cancel{2^0}} \\
f^{(1)}(u) &= -\frac{1}{2}(1-u)^{-\frac{3}{2}} && \to f^{(1)}(0) = -\frac{1}{2^1} \\
f^{(2)}(u) &= \frac{1 \cdot 3}{2^2}(1-u)^{-\frac{5}{2}} && \to f^{(2)}(0) = \frac{1 \cdot 3}{2^2} \\
f^{(3)}(u) &= -\frac{1 \cdot 3 \cdot 5}{2^3}(1-u)^{-\frac{7}{2}} && \to f^{(3)}(0) = \underbrace{-}_{\text{alternating}}\frac{\overbrace{1 \cdot 3 \cdot 5}^{\text{odd-only factorial}}}{2^3} \\
f^{(n)}(u) &= \frac{(2n-1)!!}{2^n}(-1)^n
\end{aligned}$$

Plugging in to the Maclaurin series formula $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$, we get:

$$\frac{1}{\sqrt{1-u}} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{2^n n!} u^n$$

Since $2^n n!$ is equivalent to multiplying by 2 for every term in the factorial, we can rewrite the series as the following, after substituting $u = k^2 \sin^2\theta = (k\sin\theta)^2$:

$$\sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{(2n)!!} (k \sin\theta)^{2n}$$

Keep in mind that for $n = 0$, $(2(0) - 1)!! = (-1)!! = 1$ by convention. [1]

4 Simplifying the Integral

Substitute the above Maclaurin series into the integral:

$$\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}}\, d\theta = \int_0^{\frac{\pi}{2}} \sum_{n=0}^{\infty} \left[\frac{(-1)^n (2n-1)!!}{(2n)!!} (k \sin\theta)^{2n} \right] d\theta$$

Take constant (non-θ) terms out of the integral, since they are independent of θ:

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^n (2n-1)!!}{(2n)!!} k^{2n} \int_0^{\frac{\pi}{2}} \sin^{2n}\theta\, d\theta \right]$$

We evaluate the first three terms of the series to get a sense of the pattern:

$$1 - \frac{1}{2}k^2 \int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta + \frac{3}{8}k^4 \int_0^{\frac{\pi}{2}} \sin^4\theta\, d\theta + \cdots$$

Find the values of the definite integrals using half-angle identities:

$$\int_0^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{1}{2}\cos(2\theta) \right) d\theta = \left[\frac{\theta}{2} - \frac{1}{4}\sin(2\theta) \right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \sin^4\theta\, d\theta = \left[\frac{\theta}{4} - \frac{1}{4}\sin(2\theta) + \frac{\theta}{8} + \frac{1}{32}\sin(4\theta) \right]_0^{\frac{\pi}{2}} = \frac{\pi}{8} + \frac{\pi}{16} = \frac{3\pi}{16}$$

The resulting terms are:

$$1 - \frac{1}{2}k^2 \left(\frac{\pi}{4} \right) + \frac{3}{8}k^4 \left(\frac{3\pi}{16} \right) + \cdots$$

The process is tedious, but we see that each integral is in the form of Wallis integrals, defined by the sequence $W_n = \int_0^{\frac{\pi}{2}} \sin^n\theta\, d\theta$. Evaluating each

[1] https://mathworld.wolfram.com/DoubleFactorial.html

definite integral gives some multiple of $\frac{\pi}{2}$, and the antiderivative of all the recursive cosines from the half-angle identities all vanish, since $\sin(2n \cdot \frac{\pi}{2}) = \sin(n\pi) = 0$. Through integration by parts and recursive substitution of I, the value of even Wallis integrals are known to be:

$$W_{2n} = \frac{(2n-1)!!}{2^n} \cdot \frac{\pi}{2}$$