Approximating the Period of a Pendulum

AP Calculus BC: Taylor Series Project

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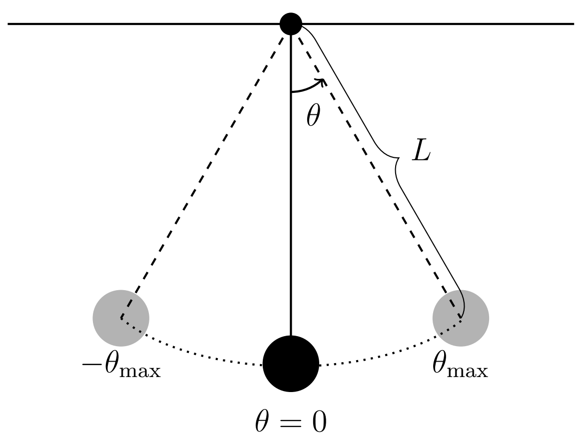

1 Inputs

- L: Length of pendulum (meters)
- $\theta_{\max}$: Maximum angular displacement (radians)

2 Period of a Pendulum

$$T = 2\pi\sqrt{\frac{L}{g}}\int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2\theta}}\, d\theta$$

where $k = \sin^2\left(\frac{\theta_{\max}}{2}\right)$ (a constant).

3 Series Expansion

Let $u = k^2 \sin^2 \theta$, and the integrand becomes:

$$f(u) = \frac{1}{\sqrt{1-u}} = (1-u)^{-\frac{1}{2}}.$$

which is in the form of a binomial series $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$.

Since we have not covered binomial series in class, we will instead use the Maclaurin series expansion for $(1-u)^{-1/2}$ around $u = 0$. First, we find the general form of the nth derivative:

$$\begin{aligned}
f^{(0)}(u) &= (1-u)^{-\frac{1}{2}} && \to f^{(0)}(0) = \frac{1}{\cancel{2^0}} \\
f^{(1)}(u) &= -\frac{1}{2}(1-u)^{-\frac{3}{2}} && \to f^{(1)}(0) = -\frac{1}{2^1} \\
f^{(2)}(u) &= \frac{1 \cdot 3}{2^2}(1-u)^{-\frac{5}{2}} && \to f^{(2)}(0) = \frac{1 \cdot 3}{2^2} \\
f^{(3)}(u) &= \frac{1 \cdot 3 \cdot 5}{2^3}(1-u)^{-\frac{7}{2}} && \to f^{(3)}(0) = \underbrace{-}_{\text{alternating}} \frac{\overbrace{1 \cdot 3 \cdot 5}^{\text{odd-only factorial}}}{2^3} \\
f^{(n)}(u) &= \frac{(2n-1)!!}{2^n}(-1)^n.
\end{aligned}$$

Plugging in to the Maclaurin series formula $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$, we get:

$$\frac{1}{\sqrt{1-u}} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{2^n n!} u^n.$$

Since $2^n n!$ is equivalent to multiplying 2 for every term in the factorial, we can rewrite the series as:

$$\sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{(2n)!!} u^n.$$