Exercises for Introduction to Mathematical Arguments

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Based on notes by M. Hutchings*

2 How to prove things

1. Prove the following statements; what is the negation of each of these statements?

   (a) For every integer x, if x is even, then for every integer y, xy is even.

   Proof. Since x is even, choose an integer w such that $x = 2w$.
   Then, $xy = 2wy$.
   Let $z = wy$; then $xy = 2z$, so xy is even. □

   Negation: x is even, and there is an integer y where xy is odd.

   (b) For every integer x and for every integer y, if x is odd and y is odd then $x + y$ is even.

   Proof. Since x is odd, choose an integer v such that $x = 2v + 1$.
   Since y is odd, choose an integer w such that $y = 2w + 1$.
   Then, $x + y = 2v + 2w + 2$.
   Let $z = v + w + 1$; then $x + y = 2z$, so $x + y$ is even. □

   Negation: There is an integer x and an integer y such that x is odd, y is odd, and $x + y$ is odd.

   (c) For every integer x, if x is odd then x^3 is odd.

   Proof. Since x is odd, choose an integer w such that $x = 2w + 1$.
   Then, $x^3 = 8w^3 + 12w^2 + 6w + 1$.
   Let $z = 4w^3 + 6w^2 + 3w$; then $x^3 = 2z + 1$, so x^3 is odd. □

   Negation: There is an integer x such that x is odd and x^3 is even.

*https://math.berkeley.edu/~hutching/teach/proofs.pdf

2. Prove that for every integer x, $x+4$ is odd if and only if $x+7$ is even.

   Proof. ($\Rightarrow$) Suppose $x+4$ is odd.
   Choose an integer v such that $x+4 = 2v+1$.
   Then, $x+7 = 2v+4$.
   Let $z = v+2$; then $x+7 = 2z$, so $x+7$ is even.
   ($\Leftarrow$) Suppose $x+7$ is even.
   Choose an integer w such that $x+7 = 2w$.
   Then, $x+4 = 2w-3 = 2(w-2)+1$.
   Let $z = w-2$; then $x+4 = 2z+1$, so $x+4$ is odd. $\square$

3. Figure out whether the statement we negated in §1.3 is true or false, and prove it (or its negation).

   The statement from §1.3 is:

$$(\forall x \in \mathbb{Z})\left((\exists y \in \mathbb{Z})x = 3y+1\right) \Rightarrow \left((\exists y \in \mathbb{Z})x^2 = 3y+1\right). \tag{1}$$

   Proof. Let x be an integer such that there exists an integer y where $x = 3y+1$.
   By squaring both sides,

$$\begin{aligned} x^2 &= (3y+1)^2 && (2)\\ &= 9y^2+6y+1 && (3)\\ &= 3(3y^2+2y)+1. && (4) \end{aligned}$$

   Let $z = 3y^2+2y$. Since z must be an integer, there exists an integer z where $x^2 = 3z+1$. $\square$

4. Prove that for every integer x, if x is odd then there exists an integer y such that $x^2 = 8y+1$.

   Proof. Since x is odd, let w be an integer such that $x = 2w+1$.
   By squaring both sides,

$$\begin{aligned} x^2 &= (2w+1)^2 && (5)\\ &= 4w^2+4w+1 && (6)\\ &= 4w(w+1)+1 && (7)\\ &= 8(\frac{1}{2}w(w+1))+1 && (8) \end{aligned}$$

   Since one of w and $w+1$ must be even, their product $w(w+1)$ must also be even.

Therefore, there exists an integer y where $w(w+1) = 2y$, meaning $y = \frac{1}{2}w(w+1)$.

Thus, by substitution, x^2 can be represented in the form $8y+1$, where y is an integer. $\square$

3 More proof techniques

1. Prove that the inverse of a given element $x \in G$ is unique.

   Proof. Let e be the identity element and let v, w be elements of G satisfying $vx = e$ and $xw = e$, by the definition of an inverse.

$$\begin{aligned} v &= ve && \text{(definition of identity element)} \quad (9) \\ &= v(xw) && \text{(substitution)} \quad (10) \\ &= (vx)w && \text{(associative property)} \quad (11) \\ &= ew && \text{(substitution)} \quad (12) \\ &= w && \text{(definition of identity element)} \quad (13) \end{aligned}$$

   Thus, the inverse of a given element $x \in G$ is unique. $\square$