Exercises for Introduction to mathematical arguments (Hutchings)

Kevin Ma
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5 2 How to prove things

- 1. Prove the following statements; what is the negation of each of these statements?
 - (a) For every integer x, if x is even, then for every integer y, xy is even.

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Proof. Since x is even, choose an integer w such that x = 2w.
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Then, xy = 2wy.

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Let z = wy; then xy = 2z, so xy is even.
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- (b) For every integer x and for every integer y, if x is odd and y is odd then x + y is even.
- Proof. Since x is odd, choose an integer v such that x = 2v + 1. Since y is odd, choose an integer w such that y = 2w + 1.

Then, x + y = 2v + 2w + 2.

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Let z = v + w + 1; then x + y = 2z, so x + y is even.
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(c) For every integer x, if x is odd then x^3 is odd.

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Proof. Since x is odd, choose an integer w such that x = 2w + 1.
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Then, $x^3 = 8w^3 + 12w^2 + 6w + 1$.

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Let z = 4w^3 + 6w^2 + 3w; then x^3 = 2z + 1, so x^3 is odd.
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2. Prove that for every integer x, x + 4 is odd if and only if x + 7 is even.

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23 Proof. (\Rightarrow) Suppose x + 4 is odd.
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Choose an integer v such that x + 4 = 2v + 1.

Then, x + 7 = 2v + 4.

Let z = v + 2; then x + 7 = 2z, so x + 7 is even.

 (\Leftarrow) Suppose x+7 is even.

Choose an integer w such that x + 7 = 2w.

Then, x + 4 = 2w - 3 = 2(w - 2) + 1.

Let z = w - 2; then x + 4 = 2z + 1, so x + 4 is odd.

- 3. Figure out whether the statement we negated in §1.3 is true or false, and prove it (or its negation).
- 4. Prove that for every integer x, if x is odd then there exists an integer y such that $x^2=8y+1$.