# Exercises for Introduction to Mathematical Arguments

Kevin Ma

May 23, 2025

Based on notes by M. Hutchings*

## 2 How to prove things

1. Prove the following statements; what is the negation of each of these statements?

   (a) For every integer $x$, if $x$ is even, then for every integer $y$, $xy$ is even.

   *Proof.* Since $x$ is even, choose an integer $w$ such that $x = 2w$.
   Then, $xy = 2wy$.
   Let $z = wy$; then $xy = 2z$, so $xy$ is even. □

   **Negation:** $x$ is even, and there is an integer $y$ where $xy$ is odd.

   (b) For every integer $x$ and for every integer $y$, if $x$ is odd and $y$ is odd then $x + y$ is even.

   *Proof.* Since $x$ is odd, choose an integer $v$ such that $x = 2v + 1$.
   Since $y$ is odd, choose an integer $w$ such that $y = 2w + 1$.
   Then, $x + y = 2v + 2w + 2$.
   Let $z = v + w + 1$; then $x + y = 2z$, so $x + y$ is even. □

   **Negation:** There is an integer $x$ and an integer $y$ such that $x$ is odd, $y$ is odd, and $x + y$ is odd.

   (c) For every integer $x$, if $x$ is odd then $x^3$ is odd.

   *Proof.* Since $x$ is odd, choose an integer $w$ such that $x = 2w + 1$.
   Then, $x^3 = 8w^3 + 12w^2 + 6w + 1$.
   Let $z = 4w^3 + 6w^2 + 3w$; then $x^3 = 2z + 1$, so $x^3$ is odd. □

   **Negation:** There is an integer $x$ such that $x$ is odd and $x^3$ is even.

*https://math.berkeley.edu/~hutching/teach/proofs.pdf

2. Prove that for every integer $x$, $x+4$ is odd if and only if $x+7$ is even.

   *Proof.* ($\Rightarrow$) Suppose $x+4$ is odd.
   Choose an integer $v$ such that $x+4 = 2v+1$.
   Then, $x+7 = 2v+4$.
   Let $z = v+2$; then $x+7 = 2z$, so $x+7$ is even.
   ($\Leftarrow$) Suppose $x+7$ is even.
   Choose an integer $w$ such that $x+7 = 2w$.
   Then, $x+4 = 2w-3 = 2(w-2)+1$.
   Let $z = w-2$; then $x+4 = 2z+1$, so $x+4$ is odd. $\square$

3. Figure out whether the statement we negated in §1.3 is true or false, and prove it (or its negation).

   The statement from §1.3 is:

$$(\forall x \in \mathbb{Z})\,((\exists y \in \mathbb{Z})x = 3y+1) \Rightarrow \big((\exists y \in \mathbb{Z})x^2 = 3y+1\big)\,. \tag{1}$$

   *Proof.* Let $x$ be an integer such that there exists an integer $y$ where $x = 3y+1$.
   By squaring both sides,

$$\begin{aligned} x^2 &= (3y+1)^2 && (2)\\ &= 9y^2+6y+1 && (3)\\ &= 3(3y^2+2y)+1. && (4) \end{aligned}$$

   Let $z = 3y^2+2y$. Since $z$ must be an integer, there exists an integer $z$ where $x^2 = 3z+1$. $\square$

4. Prove that for every integer $x$, if $x$ is odd then there exists an integer $y$ such that $x^2 = 8y+1$.

   *Proof.* Since $x$ is odd, let $w$ be an integer such that $x = 2w+1$.
   By squaring both sides,

$$\begin{aligned} x^2 &= (2w+1)^2 && (5)\\ &= 4w^2+4w+1 && (6)\\ &= 4w(w+1)+1 && (7)\\ &= 8(\frac{1}{2}w(w+1))+1 && (8) \end{aligned}$$

   Since one of $w$ and $w+1$ must be even, their product $w(w+1)$ must also be even.

Therefore, there exists an integer $y$ where $w(w+1) = 2y$, meaning $y = \frac{1}{2}w(w+1)$.

Thus, by substitution, $x^2$ can be represented in the form $8y+1$, where $y$ is an integer. $\square$