Exercises for Introduction to mathematical arguments (Hutchings)

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2 How to prove things

1. Prove the following statements; what is the negation of each of these statements?

   (a) For every integer x, if x is even, then for every integer y, xy is even.

   Proof. Since x is even, choose an integer w such that $x = 2w$.
   Then, $xy = 2wy$.
   Let $z = wy$; then $xy = 2z$, so xy is even. □

   (b) For every integer x and for every integer y, if x is odd and y is odd then $x + y$ is even.

   Proof. Since x is odd, choose an integer v such that $x = 2v+1$. Since y is odd, choose an integer w such that $y = 2w + 1$.
   Then, $x + y = 2v + 2w + 2$.
   Let $z = v + w + 1$; then $x + y = 2z$, so $x + y$ is even. □

   (c) For every integer x, if x is odd then x^3 is odd.

   Proof. Since x is odd, choose an integer w such that $x = 2w + 1$.
   Then, $x^3 = 8w^3 + 12w^2 + 6w + 1$.
   Let $z = 4w^3 + 6w^2 + 3w$; then $x^3 = 2z + 1$, so x^3 is odd. □

2. Prove that for every integer x, $x + 4$ is odd if and only if $x + 7$ is even.

   Proof. ($\Rightarrow$) Suppose $x + 4$ is odd.
   Choose an integer v such that $x + 4 = 2v + 1$.
   Then, $x + 7 = 2v + 4$.
   Let $z = v + 2$; then $x + 7 = 2z$, so $x + 7$ is even.
   ($\Leftarrow$) Suppose $x + 7$ is even.
   Choose an integer w such that $x + 7 = 2w$.
   Then, $x + 4 = 2w - 3 = 2(w - 2) + 1$.
   Let $z = w - 2$; then $x + 4 = 2z + 1$, so $x + 4$ is odd. □

3. Figure out whether the statement we negated in §1.3 is true or false, and prove it (or its negation).

4. Prove that for every integer x, if x is odd then there exists an integer y such that $x^2 = 8y + 1$.