Exercises for Introduction to Mathematical Arguments

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May 27, 2025

Based on notes by M. Hutchings*

2 How to prove things

1. Prove the following statements; what is the negation of each of these statements?

   (a) For every integer x, if x is even, then for every integer y, xy is even.

   Proof. Since x is even, choose an integer w such that $x = 2w$.
   Then, $xy = 2wy$.
   Let $z = wy$; then $xy = 2z$, so xy is even. □

   Negation: x is even, and there is an integer y where xy is odd.

   (b) For every integer x and for every integer y, if x is odd and y is odd then $x + y$ is even.

   Proof. Since x is odd, choose an integer v such that $x = 2v + 1$.
   Since y is odd, choose an integer w such that $y = 2w + 1$.
   Then, $x + y = 2v + 2w + 2$.
   Let $z = v + w + 1$; then $x + y = 2z$, so $x + y$ is even. □

   Negation: There is an integer x and an integer y such that x is odd, y is odd, and $x + y$ is odd.

   (c) For every integer x, if x is odd then x^3 is odd.

   Proof. Since x is odd, choose an integer w such that $x = 2w + 1$.
   Then, $x^3 = 8w^3 + 12w^2 + 6w + 1$.
   Let $z = 4w^3 + 6w^2 + 3w$; then $x^3 = 2z + 1$, so x^3 is odd. □

   Negation: There is an integer x such that x is odd and x^3 is even.

*https://math.berkeley.edu/~hutching/teach/proofs.pdf

2. Prove that for every integer x, $x+4$ is odd if and only if $x+7$ is even.

Proof. ($\Rightarrow$) Suppose $x+4$ is odd.
Choose an integer v such that $x+4 = 2v+1$.
Then, $x+7 = 2v+4$.
Let $z = v+2$; then $x+7 = 2z$, so $x+7$ is even.
($\Leftarrow$) Suppose $x+7$ is even.
Choose an integer w such that $x+7 = 2w$.
Then, $x+4 = 2w-3 = 2(w-2)+1$.
Let $z = w-2$; then $x+4 = 2z+1$, so $x+4$ is odd. $\square$

3. Figure out whether the statement we negated in §1.3 is true or false, and prove it (or its negation).

The statement from §1.3 is:

$$(\forall x \in \mathbb{Z})\,((\exists y \in \mathbb{Z})x = 3y+1) \Rightarrow \big((\exists y \in \mathbb{Z})x^2 = 3y+1\big)\,. \tag{1}$$

Proof. Let x be an integer such that there exists an integer y where $x = 3y+1$.
By squaring both sides,

$$\begin{aligned} x^2 &= (3y+1)^2 && (2)\\ &= 9y^2+6y+1 && (3)\\ &= 3(3y^2+2y)+1. && (4) \end{aligned}$$

Let $z = 3y^2+2y$. Since z must be an integer, there exists an integer z where $x^2 = 3z+1$. $\square$

4. Prove that for every integer x, if x is odd then there exists an integer y such that $x^2 = 8y+1$.

Proof. Since x is odd, let w be an integer such that $x = 2w+1$.
By squaring both sides,

$$\begin{aligned} x^2 &= (2w+1)^2 && (5)\\ &= 4w^2+4w+1 && (6)\\ &= 4w(w+1)+1 && (7)\\ &= 8(\frac{1}{2}w(w+1))+1 && (8) \end{aligned}$$

Since one of w and $w+1$ must be even, their product $w(w+1)$ must also be even.

Therefore, there exists an integer y where $w(w+1) = 2y$, meaning $y = \frac{1}{2}w(w+1)$.

Thus, by substitution, x^2 can be represented in the form $8y+1$, where y is an integer. $\square$

3 More proof techniques

1. Prove that the inverse of a given element $x \in G$ is unique.

   Proof. Let e be the identity element and let v, w be elements of G satisfying $vx = e$ and $xw = e$, by the definition of an inverse.

$$\begin{aligned} v &= ve && \text{(definition of identity element)} && (9) \\ &= v(xw) && \text{(substitution)} && (10) \\ &= (vx)w && \text{(associative property)} && (11) \\ &= ew && \text{(substitution)} && (12) \\ &= w && \text{(definition of identity element)} && (13) \end{aligned}$$

   Thus, the inverse of a given element $x \in G$ is unique. $\square$

4 Proof by induction

1. Fix a real number $x \neq 1$. Show that for every positive integer n,

$$1 + x + x^2 + \ldots + x^n = \frac{x^{n+1} - 1}{x - 1}.$$

   Proof. Base case: For $n = 1$:

$$1 + x = \frac{x^2 - 1}{x - 1} = x + 1.$$

   Inductive step: Add x^{n+1} to both sides.

$$\begin{aligned} 1 + x + x^2 + \ldots + x^n + x^{n+1} &= \frac{x^{n+1} - 1}{x - 1} + x^{n+1} \\ &= \frac{x^{n+1} - 1 + x^{n+1}(x-1)}{x - 1} \\ &= \frac{x^{n+1}(1 + x - 1) - 1}{x - 1} \\ &= \frac{x^{n+2} - 1}{x - 1}. \end{aligned}$$

$\square$

2. Guess a formula for

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n(n+1)}$$

and prove it by induction. *Hint:* Compute the above expression for some small values of n.

For $n = 1$: $S = \frac{1}{2}$.
For $n = 2$: $S = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$.
For $n = 3$: $S = \frac{4}{6} + \frac{1}{12} = \frac{3}{4}$.
For $n = 4$: $S = \frac{3}{4} + \frac{1}{20} = \frac{4}{5}$.
The pattern looks like $\frac{n}{n+1}$.

Proof. Base case:

$$\frac{1}{1 \cdot 2} = \frac{1}{1+1} = \frac{1}{2}.$$

Inductive step: To prove the sum with the $(n+1)$th term is $\frac{n+1}{n+2}$, add the next term $\frac{1}{(n+1)(n+2)}$ to both sides.

$$\begin{aligned}
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} \\
&= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\
&= \frac{n(n+2)+1}{(n+1)(n+2)} \\
&= \frac{n^2+2n+1}{(n+1)(n+2)} \\
&= \frac{n+1}{n+2}.
\end{aligned}$$

□

3. Show that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with L-triominoes. (An **L-triomino** is a shape consisting of three squares joined in an 'L'-shape.)

Proof. Base case: When removing any square of a 2×2 checkerboard, 3 squares remain and must form an L-triomino.

Inductive step: A $2^{n+1} \times 2^{n+1}$ board can be divided into four $2^n \times 2^n$ boards by dividing it in half (by 2) both vertically and horizontally. We will refer to each of these boards as **quadrants**.

A square is removed from one of the four quadrants. By our induction hypothesis, this quadrant (a $2^n \times 2^n$ board) can be tiled with L-triominoes.

Because $n \geq 1$, the board will always have at least 4 squares. Consider the center 4 squares. Since this is a 2×2 "board" with one square from a quadrant that has already been accounted for or "occupied," the base case applies to this region, and an L-triomino can be placed here.

Now, each of the remaining quadrants has one square occupied, which has the same effect as being removed. By our induction hypothesis, the remaining quadrants can also be tiled by L-triominoes. □

4. Show that the smallest element of a nonempty set of positive integers is unique.

   Proof. Base case: Consider an n-element set of positive integers where $n = 1$. The smallest (only) element is unique.

   Inductive step: Our inductive hypothesis states that the smallest element in a set with n positive integers is unique. Let z be the smallest element of the set, excluding element $n + 1$, which we will refer to as m. □

A Sets

1. List separately the elements and the subsets of $\{\{1, \{2\}\}, \{3\}\}$. (There are 2 elements and 4 subsets.)

   Elements: $\{1, \{2\}\}$, $\{3\}$
   Subsets: $\emptyset$, $\{\{1, \{2\}\}\}$, $\{\{3\}\}$, $\{\{1, \{2\}\}, \{3\}\}$

2. Explain why if $A \subset B$ and $B \subset C$, then $A \subset C$.

   Since every element of A is in B, and every element of B is in C, every element in A is in C.

3. If a set has exactly n elements, how many subsets does it have? Why?

   Every element in the set has 2 options, to be included or excluded. Thus, there are 2^n possible subsets.

4. We have repeatedly used the words, '*the* empty set'. Is this justified? If A and B are both sets that contain no elements, then is A necessarily equal to B?

   Yes, because the elements—or lack thereof—of both A and B are identical.

5. Which of the following statements are true, and which are false? Why?

   (a) $\{\emptyset\} \cup \emptyset = \{\emptyset\}$
   (b) $\{\emptyset\} \cup \{\emptyset\} = \{\emptyset\}$
   (c) $\{\emptyset\} \cap \{\{\emptyset\}\} = \emptyset$

(d) $\{\emptyset\} \cap \{\{\emptyset\}\} = \{\emptyset\}$

6. Prove all of the properties of union, intersection, and set difference that we stated without proof in the text.

7. Show that $A \cup (B - C) = (A \cup B) - (A \cap C)$. Is it always true that $A \cup (B - C) = (A \cup B) - (A \cup C)$?

8. Find some more set theoretic identities and prove them.