

Exercises for Introduction to Mathematical Arguments

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Based on notes by M. Hutchings*

2 How to prove things

1. Prove the following statements; what is the negation of each of these statements?

- (a) For every integer x , if x is even, then for every integer y , xy is even.

Proof. Since x is even, choose an integer w such that $x = 2w$.

Then, $xy = 2wy$.

Let $z = wy$; then $xy = 2z$, so xy is even. \square

Negation: x is even, and there is an integer y where xy is odd.

- (b) For every integer x and for every integer y , if x is odd and y is odd then $x + y$ is even.

Proof. Since x is odd, choose an integer v such that $x = 2v + 1$.

Since y is odd, choose an integer w such that $y = 2w + 1$.

Then, $x + y = 2v + 2w + 2$.

Let $z = v + w + 1$; then $x + y = 2z$, so $x + y$ is even. \square

Negation: There is an integer x and an integer y such that x is odd, y is odd, and $x + y$ is odd.

- (c) For every integer x , if x is odd then x^3 is odd.

Proof. Since x is odd, choose an integer w such that $x = 2w + 1$.

Then, $x^3 = 8w^3 + 12w^2 + 6w + 1$.

Let $z = 4w^3 + 6w^2 + 3w$; then $x^3 = 2z + 1$, so x^3 is odd. \square

Negation: There is an integer x such that x is odd and x^3 is even.

*<https://math.berkeley.edu/~hutching/teach/proofs.pdf>

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2. Prove that for every integer x , $x + 4$ is odd if and only if $x + 7$ is even.

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Proof. (\Rightarrow) Suppose $x + 4$ is odd.

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Choose an integer v such that $x + 4 = 2v + 1$.

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Then, $x + 7 = 2v + 4$.

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Let $z = v + 2$; then $x + 7 = 2z$, so $x + 7$ is even.

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(\Leftarrow) Suppose $x + 7$ is even.

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Choose an integer w such that $x + 7 = 2w$.

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Then, $x + 4 = 2w - 3 = 2(w - 2) + 1$.

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Let $z = w - 2$; then $x + 4 = 2z + 1$, so $x + 4$ is odd. \square

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3. Figure out whether the statement we negated in §1.3 is true or false, and prove it (or its negation).

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The statement from §1.3 is:

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$$(\forall x \in \mathbb{Z}) ((\exists y \in \mathbb{Z}) x = 3y + 1) \Rightarrow ((\exists y \in \mathbb{Z}) x^2 = 3y + 1). \quad (1)$$

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Proof. Let x be an integer such that there exists an integer y where $x = 3y + 1$.

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By squaring both sides,

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$$x^2 = (3y + 1)^2 \quad (2)$$

$$= 9y^2 + 6y + 1 \quad (3)$$

$$= 3(3y^2 + 2y) + 1. \quad (4)$$

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Let $z = 3y^2 + 2y$. Since z must be an integer, there exists an integer z where $x^2 = 3z + 1$. \square

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4. Prove that for every integer x , if x is odd then there exists an integer y such that $x^2 = 8y + 1$.

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Proof. Since x is odd, let w be an integer such that $x = 2w + 1$.

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By squaring both sides,

$$x^2 = (2w + 1)^2 \quad (5)$$

$$= 4w^2 + 4w + 1 \quad (6)$$

$$= 4w(w + 1) + 1 \quad (7)$$

$$= 8\left(\frac{1}{2}w(w + 1)\right) + 1 \quad (8)$$

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Since one of w and $w + 1$ must be even, their product $w(w + 1)$ must also be even.

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50 Therefore, there exists an integer y where $w(w+1) = 2y$, meaning $y =$
51 $\frac{1}{2}w(w+1)$.
52 Thus, by substitution, x^2 can be represented in the form $8y+1$, where y
53 is an integer. \square