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Decision Support

A simple method to improve the consistency ratio of the pair-wise comparison matrix in ANP

Daji Ergu [a,b,1], Gang Kou [a,1], Yi Peng [a,*,1], Yong Shi [c,d,1]

[a] School of Management and Economics, University of Electronic Science and Technology of China, Chengdu 610054, China
[b] Southwest University for Nationalities, Chengdu 610200, China
[c] College of Information Science and Technology, University of Nebraska at Omaha, Omaha, NE 68182, USA
[d] Research Center on Fictitious Economy and Data Sciences, Chinese Academy of Sciences, Beijing 100190, China

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ABSTRACT

Tests of consistency for the pair-wise comparison matrices have been studied extensively since AHP was introduced by Saaty in 1970s. However, existing methods are either too complicated to be applied in the revising process of the inconsistent comparison matrix or are difficult to preserve most of the original comparison information due to the use of a new pair-wise comparison matrix. Those methods might work for AHP but not for ANP as the comparison matrix of ANP needs to be strictly consistent. To improve the consistency ratio, this paper proposes a simple method, which combines the theorem of matrix multiplication, vectors dot product, and the definition of consistent pair-wise comparison matrix, to identify the inconsistent elements. The correctness of the proposed method is proved mathematically. The experimental studies have also shown that the proposed method is accurate and efficient in decision maker's revising process to satisfy the consistency requirements of AHP/ANP.

1. Introduction

In 1970s, Saaty (1978, 1979, 1980) proposed an analytical hierarchy process (AHP) to resolve the qualitative and quantitative factor for decision makers in multiple criteria decision making (MCDM). Since then, this method has been applied into many real applications (Wind and Saaty, 1980; Zahedi, 1986; Liberatore, 1987; Ghodsypour and O'brien, 1998; Forman and Gass, 2001; Handfield et al., 2002; Cebeci and Ruan, 2007; Despotis and Derpanis, 2008; Li and Ma, 2008; Aragones-Beltran et al., 2008; Peng et al., 2008; Sueyoshi et al., 2009; Kou et al., 2003; Kou and Lou, 2010; Peng et al., 2010; Lin et al., 2011; Peng et al., 2011a,b,c). In AHP, however, the calculated priorities are plausible only if the pair-wise comparison matrices passed the consistency test, when the transitivity and reciprocity rules are respected within the pair-wise comparison process (Ishizaka and Lusti, 2004). The pair-wise comparison matrix consists of elements expressed on a numerical scale and the values of elements are given by decision makers based on their experiences and expertise. Thereby, the pair-wise comparison matrix could be inconsistent due to the limitations of experiences and expertise as well as the complexity nature of the decision problem.

Because the inconsistency has impacts on the results of priority vector, many studies have been focused on the inconsistency problem for several decades (Saaty, 1986, 1987, 1990; Harker and Vargas, 1987; Liu, 1999; Xu and Wei, 1999; Wei and Zhang, 2000; Li and Ma, 2007; Cao et al., 2008; Dong et al., 2008; Iida, 2009; Linares, 2009; Koczkodaj and Szarek, 2010; Ergu et al., 2011). However, some existing methods are complicated and difficult to use in the revising process of the inconsistent comparison matrix while some are difficult to preserve most of the original comparison information since a new matrix has to be constructed to replace the original comparison matrix. For an example, the inconsistency identification method embedded in the AHP software *Expert Choice* is to construct a matrix based on a ratio of priorities to detect the inconsistencies. This method is described as an example by Saaty (2003). Two similar methods are proposed by Xu and Wei (1999) and Cao et al. (2008) to detect the inconsistencies.

Although these heuristics and approximations of comparison matrix do not affect the priority order and may achieve consistent result in AHP, they are not acceptable in the analytic network process (ANP). In general, there are more pair-wise comparison matrices in ANP than AHP, and all ratio scale priority vectors are columns in supermatrix in ANP (Saaty, 1996, 2006; Caballero-Luque et al., 2010). The priority

* Corresponding author.
E-mail address: pengyicd@gmail.com (Y. Peng).
[1] Authors are alphabetically ordered by their last names.

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result of ANP changes even if the pair-wise comparison matrices are slightly inconsistent (Saaty, 1996, 2005; Lee and Kim, 2000; Mikhailov and Singh, 2003). Therefore, an inconsistency identification method that can retain most of the comparison information in the original pair-wise comparison matrix provided by the experts is a requisite for ANP and AHP.

The objective of this paper is to propose a simple, efficient and accurate method to improve the consistency ratio of the pair-wise comparison matrix in ANP and AHP while preserve the original comparison information as much as possible. In order to identify the inconsistent elements in the original comparison matrix, we introduce an induced matrix based on the original comparison matrix, which preserves the maximum information in the original comparison matrix. The proposed induced matrix is based on the theorems of matrix multiplication and vectors dot product as well as the definitions and notations of the pair-wise comparison matrix. The vector dot product is used to identify the possible inconsistent elements in the comparison matrix and a method of matrix order reduction is introduced to further identify the specific inconsistent elements in the comparison matrix.

The contributions of this paper include: first, a simple and efficient inconsistent elements identification method is proposed to improve the consistency ratio. Second, the inconsistent elements can be identified accurately while more original comparison information from experts can be preserved than existing techniques. Last but not least, this paper provides a new way to analyze the sensitivity of comparison matrix and can also accurately identify the inconsistent elements in high order comparison matrix.

The remaining parts of this paper are organized as follows. The next section briefly describes the analytic network process and presents the definitions and notations for the pair-wise comparison matrix. In Section 3, the inconsistency identification method is introduced. Five numeric examples are illustrated using the proposed method and the results are compared with existing techniques in Section 4. Section 5 concludes the paper and discusses limitations as well as future research directions.

2. Basics of ANP and the pair-wise comparison matrix

2.1. The analytic network process (ANP) overview

AHP has two assumptions (Saaty, 1994): the independence of higher level elements from lower level elements and the independence of the elements within a level. These two assumptions simplify the calculations when analyzing multiple criteria decision making (MCDM) with quantitative and qualitative attributes. However, many decision problems cannot be structured hierarchically due to the complexity and dynamics nature of decision problems. Therefore, the interaction of higher level elements with lower level elements and their dependence should not be neglected. ANP provides a solution for problems which cannot be structured hierarchically and provides a general framework to deal with decisions without making the above assumptions (Saaty, 1996).

The ANP process has two parts. The first part is a control hierarchy or network of criteria and subcriteria that controls the interactions. The second part consists of a network of influences among the elements and clusters (Saaty, 1996). In ANP, there are outer-dependence and/or inner-dependence between the elements and clusters. The priority vectors in ANP are derived from pair-wise comparison matrices and supermatrix is composed of elements which could also be matrices of column priorities. Each of these supermatrices is weighed by the priority of its control criterion and the results are synthesized through addition for all the control criteria (Saaty, 2005).

2.2. Definitions and notations for the pair-wise comparison matrix

Saaty (1978, 2001) suggested a 1–9 fundamental scale to compare two elements with respect to the criteria. For simplicity, let $A = (a_{ij})_{n\times n}$be a pair-wise matrix with n criteria ($n \geqslant 3$, $i, j = 1,2,\ldots,n$), then we have the following definitions and notations.

Definition 1. A comparison matrix A is positive reciprocal matrix if $a_{ii} = 1$, $a_{ij} > 0$ and $a_{ij} = \frac{1}{a_{ji}}$ for all positive integer i and j.

Definition 2. A reciprocal matrix is perfectly consistent if $a_{ik}a_{kj} = a_{ij}$ for all i, j and k.

Definition 3. A reciprocal matrix is approximately consistent if $a_{ik}a_{kj} \approx a_{ij}$ for all i, j and k, where '$\approx$' denotes approximately or close to.

Definition 4. A reciprocal matrix is transitive if $A > C$can be derived from $A > B$ and $B > C$ logically.

Definition 5. The pair-wise comparison matrix can pass the consistency test, if the consistency ratio $C.R. = \frac{C.I.}{R.I.} < 0.1$, where the consistency index $C.I. = \frac{\lambda_{max}-n}{n-1}$, $R.I.$ is the average random index based on matrix size, $\lambda_{\max}$ is the maximum eigenvalue of matrix A, and n is the order of matrix A (Saaty, 1991).

3. The theorems and process of the inconsistency identification method

Although the consistency tests are needed for both ANP and AHP, there are more pair-wise comparison matrices in ANP than in AHP. Since the supermatrix in ANP consists of priority vectors from comparison matrices, the requirements of consistency for comparison matrices in ANP are stricter than they are in AHP. The final priorities might be changed sometimes when the pair-wise comparison matrices pass the consistency test with acceptable C.R., which is close to 0.1. To satisfy the consistency requirement, some existing methods construct a new comparison matrix to replace the original one and may lose the original comparison information provided by the decision makers (Xu and Wei, 1999; Wei and Zhang, 2000). In this paper, we propose an inconsistency identification method, which keeps most of the information in the original pair-wise comparison matrix provided by the experts. The following subsections describe the theorems involved in the inconsistency identification method and the detailed process.

3.1. The theorems of the inconsistency identification method

According to the theorems of matrix multiplication and vectors dot product as well as the definitions for reciprocal matrix, we derive the following statements.

Theorem 1. *The induced matrix $C = AA - nA$ should be a zero matrix if comparison matrix A is perfectly consistent.*

Proof. Let A be a $n \times n$ pair-wise matrix (n rows, n columns), and B also be a $n \times n$ matrix. Multiply A to B, then the product C is also a matrix with n rows and n columns, that is,

$$C_{n\times n} = A_{n\times n}B_{n\times n}. \tag{1}$$

From the theorem of matrix multiplication, we have:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}, \tag{2}$$

where c_{ij} represents the element in the ith row and jth column of matrix C.

Likewise, multiply A to A, the product B becomes:

$$B_{n\times n} = A_{n\times n}A_{n\times n}. \tag{3}$$

Applying formula (1) and (2) to formula (3), obviously we get

$$b_{ij} = \sum_{k=1}^{n} a_{ik} \cdot a_{kj}, \tag{4}$$

where b_{ij} denotes the element with ith row and jth column in matrix B.

According to definition 2 mentioned above, there is $a_{ik}a_{kj} = a_{ij}$ for all i, j and k if pair-wise matrix A is perfectly consistent. That is

$$a_{ik}a_{kj} = \frac{w_i}{w_k} \cdot \frac{w_k}{w_j} = \frac{w_i}{w_j} = a_{ij} \tag{5}$$

where w_i, w_j, and w_k denote the weigh vector of attributes a_i, a_j and a_k, respectively. Applying formula (5) to formula (4), then we have:

$$b_{ij} = \sum_{k=1}^{n} a_{ik} \cdot a_{kj} = na_{ij}. \tag{6}$$

Therefore all elements of the induced matrix C are equal to zero and the induced matrix C is a zero matrix.

$$C = AA - nA = (na_{ij})_{n\times n} - n(a_{ij})_{n\times n} = (0)_{n\times n}. \quad \square \tag{7}$$

In order to demonstrate how the theorem works in our inconsistency identification method, a 3×3 pair-wise matrix (3 rows, 3 columns) is introduced as an example, and the processes of matrix multiplication are exhibited below. Let $A = (a_{ij})$ be a 3×3 pair-wise matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \tag{8}$$

$$\begin{aligned} \text{Then } B = A \cdot A &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \\ &= \begin{pmatrix} a_{11}\cdot a_{11} + a_{12}\cdot a_{21} + a_{13}\cdot a_{31} & a_{11}\cdot a_{12} + a_{12}\cdot a_{22} + a_{13}\cdot a_{32} & a_{11}\cdot a_{13} + a_{12}\cdot a_{23} + a_{13}\cdot a_{33} \\ a_{21}\cdot a_{11} + a_{22}\cdot a_{21} + a_{23}\cdot a_{31} & a_{21}\cdot a_{12} + a_{22}\cdot a_{22} + a_{23}\cdot a_{32} & a_{21}\cdot a_{13} + a_{22}\cdot a_{23} + a_{23}\cdot a_{33} \\ a_{31}\cdot a_{11} + a_{32}\cdot a_{21} + a_{33}\cdot a_{31} & a_{31}\cdot a_{12} + a_{32}\cdot a_{22} + a_{33}\cdot a_{32} & a_{31}\cdot a_{13} + a_{32}\cdot a_{23} + a_{33}\cdot a_{33} \end{pmatrix}. \end{aligned} \tag{9}$$

If A is perfectly consistent, then

$$b_{ij} = \sum_{k=1}^{3} a_{ik} \cdot a_{kj} = 3a_{ij}. \tag{10}$$

$$\text{So} \quad B = A \cdot A = \begin{pmatrix} 3a_{11} & 3a_{12} & 3a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ 3a_{31} & 3a_{32} & 3a_{33} \end{pmatrix} = 3A. \tag{11}$$

$$\text{And} \quad C = A \cdot A - 3A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{12}$$

The scale values in pair-wise matrix are given by experts according to their judgments and expertise. To be consistent, the rank can be transitive but the values of judgment do not necessarily follow the multiplication formula $a_{ik}a_{kj} = a_{ij}$ (Saaty, 2001). Obviously, the closer the values of $a_{ik}a_{kj}$ to the value of a_{ij}, the more consistent the comparison matrix is.

Hence, the values of $a_{ik}a_{kj}$ should be as close to the value of a_{ij}as possible. If the consistency index of this matrix is less than 0.1, as the average of express $\sum_{k=1}^{n} a_{ik} \cdot a_{kj}$ is close to the value of a_{ij}, no correction of judgment matrix is needed. The values of all elements are close to zero in the induced matrix C.

If the absolute value of element in the induced matrix is much larger than zero, it indicates that the deviation between the average of $\sum_{k=1}^{n} a_{ik} \cdot a_{kj}$ and the value of a_{ij} is not negligible. It shows that either some of the values of $a_{ik}a_{kj}$ are too large or the corresponding value of a_{ij} is too small. The deviation could be observed in the induced matrix C, and the possible error elements could also be identified by the vectors dot product method. The decision maker may revise his judgments once the deviation elements are identified. Thereby, two corollaries also are derived as follows.

Corollary 1. *The induced matrix $C = AA - nA$ should be as close as possible to zero matrix if comparison matrix A is approximately consistent.*

Corollary 2. *There must be some inconsistent elements in induced matrix C deviating far away from zero if the pair-wise matrix is inconsistent.*

3.2. The inconsistency identification process

Assuming the pair-wise comparison matrix A with n rows and n columns is inconsistent. Based on the above definitions, theorems and corollaries, the processes to identify inconsistent elements of comparison matrix as well as the methods to analyze and adjust those elements are proposed as the following three major steps which include seven specific identifying steps.

Step I: Identify the location of inconsistent element whose absolute value is the largest in the induced pair-wise comparison matrix.

Step 1: Construct an induced matrix C with the following formula:

$$C = AA - nA. \tag{13}$$

Step 2: Identify the largest absolute value(s) of elements deviating farthest from zero in the induced matrix C, and record the location. For instance, suppose c_{ij} is such an element in matrix C and the location is i^{th} row and j^{th} column.

Step II: Identify the potential inconsistent elements by the bias identifying vector.

Step 3: Let the i^{th} row of the original pair-wise comparison matrix A be represented as a row vector $r_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ and the j^{th} column of the same matrix as a column vector $c_j^T = (a_{1j}, a_{2j}, \ldots, a_{nj})^T$. Where c_j^T is the transpose vector of column vector c_j.

Step 4: Calculate the scalar product of the vectors r_i and c_i^T in n dimension. The dot product b of the two vectors becomes:

$$b = r_i \cdot c_j^T = (a_{i1}, a_{i2}, \ldots, a_{in}) \cdot (a_{1j}, a_{2j}, \ldots, a_{nj}) = \left(a_{i1}a_{1j}, a_{i2}a_{2j}, \ldots, a_{in}a_{nj}\right). \tag{14}$$

Step 5: Compute the deviation elements which are far away from a_{ij} in vector b by the following formula. Let f be the bias identifying vector henceforth.

$$f = b - a_{ij} = \left(a_{i1}a_{1j} - a_{ij}, a_{i2}a_{2j} - a_{ij}, \ldots, a_{ik}a_{kj} - a_{ij}, \ldots, a_{in}a_{nj} - a_{ij}\right). \tag{15}$$

Step III: Identify the inconsistent elements using the identification method and the method of matrix order reduction.

Step 6: Identify the error elements in pair-wise matrix A that might cause the inconsistency by bias identifying vector f using the following three principal identification methods and the method of matrix order reduction.

(a) *Method of Maximum:* If more absolute values in vector f are around zero, and fewer values are deviating from zero, then identify the largest value in vector f. If there are other values close to the largest one, then identify those elements simultaneously.

(b) *Method of Minimum:* If more absolute values of elements in vector f are far away from zero, and fewer values are close to zero, or equal to zero, then identify the smallest value in vector f. If there are other values close or equal to the smallest one, then identify those elements simultaneously.

(c) *Method for identifying* a_{ij}:

1. If the largest value in induced matrix C is negative, then a_{ij} is too large.
2. If there are only two zeroes where the location is ith and jth in bias vector f, and others are positive, then a_{ij} is too small. Otherwise, a_{ij} is too large. In the former case, if a_{ij} is already close to the maximum scale 9, then identify the next largest value in the induced matrix C, and further identify other inconsistent elements using method of matrix order reduction.

Assume the bias value of $a_{ik}a_{kj} - a_{ij}$ in bias vector f is the largest positive one, and others are around zero. Clearly, $a_{ik}a_{kj}$ is larger than a_{ij}, and others are equal or close to a_{ij}. Then, there are following four conditions.

Condition 1: a_{ik} is too large;
Condition 2: a_{kj} is too large;
Condition 3: Both a_{ik} and a_{kj} are too large; or
Condition 4: a_{ij} is too small.

Given the above conditions, the element to be adjusted could be identified as *Step 7:*

Table 1
The identification process for method of matrix order reduction (*Sub-step 1–3*).

Remove	Test	Might problem	Good	Problem
A_k	√	a_{ik}, a_{kj}	a_{ij}	
	×	a_{ik}, a_{kj}		a_{ij}
A_i	√	a_{ik}	a_{kj}	
	×			a_{kj}
A_j	√	a_{kj}	a_{ik}	
	×			a_{ik}

Step 7: Find the values of c_{ik} and c_{kj} in the induced matrix C according to the following procedures:
Based on our assumption, that is, assume the bias value of $a_{ik}a_{kj} - a_{ij}$ in bias vector f is the largest positive one, then one or both a_{ik} and a_{kj} are too large, therefore it is impossible that $c_{ik} > 0$ and $c_{kj} > 0$ simultaneously.
If $c_{ik} < 0$ and $c_{kj} > 0$, then a_{ik} is too large due to $c_{ik} = \frac{1}{n}\sum_{l=1}^{n} a_{il}a_{lk} - a_{ik}$, and a_{kj} is too small. If a_{ik} is too large, then the decision makers should decrease the value of element a_{ik}, so the value of $a_{ik}a_{kj}$ is closer to the value of a_{ij}.
If $c_{ik} > 0$ and $c_{kj} < 0$, similarly, the decision makers should decrease the value of element a_{kj}, so the value of $a_{ik}a_{kj}$ is closer to the value of a_{ij}.
If $c_{ik} < 0$ and $c_{kj} < 0$, and the bias between both absolute values are too large, then the maximum absolute element can be identified using the inconsistency identification method again.
If $c_{ik} < 0$ and $c_{kj} < 0$, and the bias between both absolute values are close to each other, then the following method of matrix order reduction for pair-wise matrix could be used to identify the bias elements. This method could identify the bias elements accurately and keep the comparison information provided by the experts as much as possible, especially for the pair-wise matrix with high order. The method of matrix order reduction could also identify the elements which are close to the largest or smallest simultaneously.

3.2.1. Method of matrix order reduction

As illustrated above, both a_{ik} and a_{kj} are either too large or the value of a_{ij} is too small. It indicates that some attributes or criteria, namely, A_i, A_k or A_j have impacts on other attributes and is an inconsistent element. Therefore, we can test whether it can pass the consistency test or not by removing some attributes one by one from the original pair-wise matrix, which is called method of matrix order reduction. The inconsistent attributes could be identified by this method with the following sub-steps:

Sub-step 1: Test the consistency of the order reduced comparison matrix $A_{(n-1)\times(n1)}$ by removing the attribute A_k, namely, deleting kth row and k^{th} column from the original pair-wise matrix A.

If the consistency test passed, the attribute A_k is inconsistent while a_{ij} is consistent, then go to *Sub-step* 2 to identify a_{ik} and a_{kj}.

If the consistency test failed, there must be other inconsistent attributes in the order reduced pair-wise matrix. Hence, a_{ij} is inconsistent and the value of a_{ij} can be increased so it is closer to the average of $\sum_{k=1}^{n} a_{ik} \cdot a_{kj}$. Meanwhile, both a_{ik} and a_{kj} also might be problematic, and continue to *Sub-step 2*.

Sub-step 2: Test the consistency of the order reduced pair-wise matrix $A_{(n-1)\times(n-1)}$ by removing the attribute A_i from the original pair-wise matrix A.

If the consistency test passed, both attributes A_k and A_j are consistent. There is no need to change a_{kj}. Hence, decrease a_{ik} as a_{ij} was identified in *Sub-step 1*.

If the consistency test failed, at least one of the attributes A_k or A_j is inconsistent, then decrease a_{kj}. Meanwhile, a_{ik} might also be inconsistent, and go to *Sub-step 3*.

Sub-step 3: Test the consistency of order reduced pair-wise matrix $A_{(n-1)\times(n-1)}$ by removing the attribute A_j from the original pair-wise matrix A.

If the consistency test passed, then a_{ik} is consistent; otherwise a_{ik} should be decreased.

If the consistency test failed in both *Sub-steps* 2 and 3, we have to let the decision makers to change both elements a_{ik} and a_{kj} simultaneously.

If the decision makers want to further check whether there exists other inconsistent attributes, we have to test the consistency of the order reduced pair-wise matrix by removing attributes A_i, A_k or A_j simultaneously.

To explain the identification process, Table 1 shows the identification process of a_{ik}, a_{kj} and a_{ij}. In Table 1, "Remove" represents removing the corresponding attributes. "Test" denotes the consistency test for the order reduced pair-wise matrix. "Might Problem" stands for the elements might have inconsistent problem. "Good" denotes the elements are consistent. "Problem" denotes the elements are inconsistent. "×" denotes the consistency test failed while "√" stands for a passed consistency test. A_i, A_k and A_j stand for three different attributes.

4. Illustrative examples

In order to test and compare the inconsistency identification method illustrated above with others methods, this paper applied the proposed method to some public-domain examples. The first example is a pair-wise matrix with unacceptable *C.I*, which was introduced in Liu (1999). The second example that covers a different type of error in pair-wise comparison matrices is provided by an anonymous reviewer of this paper. The third example was used by Iida (2009) as an example of ordinality consistency test, which was first introduced in Kwiesielewicz and Uden (2002) as an example of a pair-wise comparison matrix with *C.R* is 0.1055. In addition, a new pair-wise matrix was generated by adding an attribute with random value in the third example to test the proposed inconsistency identification method. Finally, an example, which was introduced by Cao et al. (2008) to compare their heuristic inconsistency modifying approach with Xu and Wei (1999)'s, is also introduced to demonstrate that the proposed method cannot only preserve more original comparison information than others but also identify the inconsistent elements easier and quicker. Besides, with this example we also want to demonstrate the identification process for some special situations in the original comparison matrix mentioned in method for adjusting a_{ij} section.

Example 1. The 4×4 pair-wise comparison matrix A is inconsistent with $C.R = 0.173 > 0.1$.

$$A = \begin{pmatrix} 1 & 1/9 & 3 & 1/5 \\ 9 & 1 & 5 & 2 \\ 1/3 & 1/5 & 1 & 1/2 \\ 5 & 1/2 & 2 & 1 \end{pmatrix}.$$

The proposed method is applied to test this pair-wise comparison matrix following (*Steps 1–7*) in Section 3.2 using MATLAB (The Matlab codes are listed in Appendix).

Step 1. The induced matrix $C = A^* A - 4^*A$ is

$$\begin{pmatrix} 0 & 0.4778 & -5.0444 & 1.3222 \\ -6.3333 & 0 & 21.0000 & 0.3000 \\ 3.6333 & -0.1130 & 0 & -0.5333 \\ -4.8333 & -0.0444 & 13.5000 & 0 \end{pmatrix}.$$

Step 2. The largest value in matrix C is 21, where location is 2nd row and 3rd column.
Step 3. Draw out all the values in 2nd row and 3rd column of pair-wise matrix A, that is

$$r_2 = (9 \quad 1 \quad 5 \quad 2), \quad \text{and } c_3^T = (3 \quad 5 \quad 1 \quad 2)$$

Step 4. The scalar product b of the vectors r_2 and c_3^T in the dimension 4, that is

$$b = r_2 \cdot c_3^T = (27 \quad 5 \quad 5 \quad 4).$$

Step 5. The bias identifying vector f is

$$f = b - a_{23} = (22 \quad 0 \quad 0 \quad -1).$$

Step 6. The value, 22, is the largest one far from zero, and others are zero or close to zero. It indicates that $a_{23} = 5$ is probably correct while $22 = a_{21}a_{13} - a_{23}$ is the inconsistent element. Therefore, we identified $a_{21}a_{13}$ may have problem.
Step 7. As $c_{21} = -6.3333 < 0$ and $c_{13} = -5.0444 < 0$, whose values are close to each other, the corresponding elements a_{21} and a_{13} are too large. Then, the method of matrix order reduction is applied to identify a_{21} and a_{13}.
Sub-step 1. Remove 2nd row and 2nd column from pair-wise matrix A, and do the consistency test, the $\lambda_{\max} = 3.4683$, and $C.R. = 0.3 > 0.1$, the test failed. Check a_{13}, and decrease the value of a_{13} and let the product value of $a_{21}a_{13}$ as close to $a_{23} = 5$ as possible.
Sub-step 2. Remove 3rd row and 3rd column from the pair-wise matrix A, and do the consistency test, the $\lambda_{\max} = 3.0012$, and $C.R. = 0 < 0.1$, the test passed. So no further correction is needed for a_{21}.
Sub-step 3. Remove 1st row and 1st column from pair-wise matrix A, and do the consistency test, the $\lambda_{\max} = 3.0055$, and $C.R. = 0 < 0.1$, the test passed. No correction is needed for a_{23}. Besides, according to the result in *Step 5*, a_{23} is consistent.

Most of the time, we do not need to finish all Sub-steps for inconsistency test, except some situations when complicated inconsistency identification and adjustment is needed.

As $a_{23} = 5$ and $a_{21} = 9$ are given in the original pair-wise matrix, let $a_{13}=1/2$, and $a_{31} = 2$, then $a_{21}a_{13}$ is equal to 4.5, which is very close to $a_{23} = 5$. Replace the two values from comparison matrix, and the consistency test passed with $C.R. = 0.0028 < 0.1$. This result is the same as the one in Liu (1999). However, in Liu's method (Liu, 1999), two matrices are needed to identify the inconsistent elements including an induced matrix based on priority vector derived from the comparison matrix and another deviation matrix. In the proposed method, there is no need to construct a new induced matrix based on the priority vector derived from the pair-wise matrix and another deviation matrix to identify the inconsistent elements.

Example 2. To demonstrate the example where there are more than one largest value which is equal to each other in the induced bias matrix C, the following inconsistent comparison matrix with an outlying judgment and $C.R. = 1.0242$ is introduced.

$$A = \begin{bmatrix} 1 & 2 & 4 & \frac{1}{8} \\ \frac{1}{2} & 1 & 2 & 4 \\ \frac{1}{4} & \frac{1}{2} & 1 & 2 \\ 8 & \frac{1}{4} & \frac{1}{2} & 1 \end{bmatrix}.$$

Step 1. The induced matrix $C = A^*A - 4^*A$ is

$$\begin{pmatrix} 0 & -1.9688 & -3.9375 & 15.7500 \\ 31.5000 & 0 & 0 & -3.9375 \\ 15.7500 & 0 & 0 & -1.9688 \\ -15.7500 & 15.7500 & 31.5000 & 0 \end{pmatrix}.$$

Step 2. There are two equal largest values in matrix C, 31.5, where locations are 2nd row and 1st column, 4th row and 3rd column. In such case, we can identify one of them first, or identify both of them simultaneously. To compare the identified results, the following steps identify both elements simultaneously.

Step 3. Draw out all the values in 2nd row and 1st column, 4th row and 3rd column of pair-wise matrix A, that is

$$r_2 = (0.5 \quad 1 \quad 2 \quad 4), \quad \text{and } c_1^T = (1 \quad 0.5 \quad 0.25 \quad 8)$$

$$r_4 = (8 \quad 0.25 \quad 0.5 \quad 1), \quad \text{and } c_3^T = (4 \quad 2 \quad 1 \quad 0.5).$$

Step 4. The scalar product b_1 of the vectors r_2 and c_1^T, and the scalar product b_2 of the vectors r_4 and c_3^T in the dimension *4* are

$$b_1 = r_2 \cdot c_1^T = (0.5 \quad 0.5 \quad 0.5 \quad 32),$$

$$b_2 = r_4 \cdot c_3^T = (32 \quad 0.5 \quad 0.5 \quad 0.5).$$

Step 5. The bias identifying vectors $f_i(i = 1,2)$ are

$$f_1 = b_1 - a_{21} = (0 \quad 0 \quad 0 \quad 31.5),$$

$$f_2 = b_2 - a_{43} = (31.5 \quad 0 \quad 0 \quad 0).$$

Step 6. The values, 31.5, in both bias identifying vectors, are the largest one far from zero, and others are zero in both vectors. The results indicate that $a_{21} = 0.5$ and $a_{43} = 0.5$ are correct while $32 = a_{24}a_{41}$ in b_1 and $32 = a_{41}a_{13}$ in b_2 are the inconsistent elements. Therefore, we identified a_{24}, a_{41} and a_{13} may have problems.

Step 7. As $c_{24} = c_{13} = -3.9375 < 0$ and $c_{41} = -15.75 < 0$, the a_{41} is the largest value far from zero, therefore, the corresponding elements a_{41} is too large, and it is suggested to be decreased. The following is the revising process of a_{41}.

Since $a_{24}a_{41} = 32$ should be equal to $a_{21} = \frac{1}{2}$ in b_1, and we have known $a_{24} = 4$. Therefore, $a_{41} = \frac{a_{21}}{a_{24}} = \frac{1}{8}$. According to reciprocal rule, we can get $a_{14} = 8$. Likewise, Since $a_{41}a_{13} = 32$ should be equal to $a_{43} = \frac{1}{2}$ in b_2, and we have known $a_{13} = 4$. Therefore, $a_{41} = \frac{a_{43}}{a_{13}} = \frac{1}{8}$, and we can get $a_{14} = 8$.

Replace the values of a_{14} and a_{41} in the original comparison matrix A with 8 and 1/8, then the induced bias matrix C becomes a zero matrix as follows, and the modified comparison matrix passed the test with $C.R. = 0 < 0.1$.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, in such case where there are two or more than two largest values which are equal to each other in the induced bias matrix, we can identify one of them first, then use another element to validate the identification result, or identify them simultaneously using the proposed method.

Example 3. The 8×8 pair-wise comparison matrix A first introduced in Kwiesielewicz and Uden (2002) as an example of a pair-wise comparison matrix. This pair-wise matrix is slightly inconsistent with $C.R = 0.1055 > 0.1$

$$A = \begin{bmatrix} 1 & 2 & 1/2 & 2 & 1/2 & 2 & 1/2 & 2 \\ 1/2 & 1 & 4 & 1 & 1/4 & 1 & 1/4 & 1 \\ 2 & 1/4 & 1 & 4 & 1 & 4 & 1 & 4 \\ 1/2 & 1 & 1/4 & 1 & 1/4 & 1 & 1/4 & 1 \\ 2 & 4 & 1 & 4 & 1 & 4 & 1 & 4 \\ 1/2 & 1 & 1/4 & 1 & 1/4 & 1 & 1/4 & 1 \\ 2 & 4 & 1 & 4 & 1 & 4 & 1 & 4 \\ 1/2 & 1 & 1/4 & 1 & 1/4 & 1 & 1/4 & 1 \end{bmatrix}.$$

According to the proposed method, we have:

Step 1. The induced matrix $C = A^*A - 8^*A$ is

$$\begin{pmatrix} 0 & -1.8750 & 7.5000 & 0 & 0 & 0 & 0 & 0 \\ 7.5000 & 0 & -22.5000 & 15.0000 & 3.7500 & 15.0000 & 3.7500 & 15.0000 \\ -1.8750 & 22.5000 & 0 & -3.7500 & -0.9375 & -3.7500 & -0.9375 & -3.7500 \\ 0 & -0.9375 & 3.7500 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3.7500 & 15.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.9375 & 3.7500 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3.7500 & 15.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.9375 & 3.7500 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Step 2. The value, 22.5 (3rd row and 2nd column), is the largest one in matrix$C$.
Step 3. The vectors are

$$r_3 = (2 \quad 0.25 \quad 1 \quad 4 \quad 1 \quad 4 \quad 1 \quad 4) \text{ and}$$

$$c_2^T = (2 \quad 1 \quad 0.25 \quad 1 \quad 4 \quad 1 \quad 4 \quad 1)$$

Step 4. The scalar product b is

$$b = r_3 \cdot c_2^T = (4 \quad 0.25 \quad 0.25 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4).$$

Step 5. The bias identifying vector f is

$$f = b - a_{32} = (3.75 \quad 0 \quad 0 \quad 3.75 \quad 3.75 \quad 3.75 \quad 3.75 \quad 3.75).$$

Step 6. According to the method of minimum, most of the values in bias vector f are deviating equally from zero except two values are equal to zero whose location is the same asa_{32}. Thereby, a_{32} is too small. Likewise, the first $0 = a_{32}a_{22} - a_{32}$, the second $0 = a_{33}a_{32} - a_{32}$. We know that $a_{22} = 1$and$a_{33} = 1$, so a_{32} is the inconsistent element as there are six elements equally and slightly more than zero. Then *Step 7* is no longer needed. We can also confirm whether a_{32} has problem using the method of order reduction. For instance, removing 3rd row and 3rd column, or 2nd row and 2nd column, we have the following induced matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Increase a_{32} to 4 from 1/4, and a_{23} to 1/4 in the original pair-wise matrix, then the induced matrix C becomes a zero matrix as follows, and the modified comparison matrix passed the test with $C.R. = 0 < 0.1$.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The identified inconsistent element is the same as one in Iida (2009). However, in Iida's method, decision maker has to calculate the number of circular triads with a tie in pair-wise matrix, and eliminate ties from pair-wise matrix to identify the inconsistent element to find the matrix which has a circular triad with lower order and identify the inconsistent element. This identification process is relatively complicated compared with our method.

Example 4. In order to demonstrate how the proposed method could identify more than two elements in pair-wise matrix with high order, we generated the following pair-wise matrix by adding one row and one column with random value to the comparison matrix in the second example. The new comparison matrix also denoted by A with $\lambda_{\max} = 11.124$ and $C.R. = 0.2328 > 0.1$.

$$A = \begin{bmatrix} 1 & 2 & 1/2 & 2 & 1/2 & 2 & 1/2 & 2 & 1/3 \\ 1/2 & 1 & 4 & 1 & 1/4 & 1 & 1/4 & 1 & 1/4 \\ 2 & 1/4 & 1 & 4 & 1 & 4 & 1 & 4 & 1/7 \\ 1/2 & 1 & 1/4 & 1 & 1/4 & 1 & 1/4 & 1 & 1/6 \\ 2 & 4 & 1 & 4 & 1 & 4 & 1 & 4 & 6 \\ 1/2 & 1 & 1/4 & 1 & 1/4 & 1 & 1/4 & 1 & 1/3 \\ 2 & 4 & 1 & 4 & 1 & 4 & 1 & 4 & 7 \\ 1/2 & 1 & 1/4 & 1 & 1/4 & 1 & 1/4 & 1 & 1/2 \\ 3 & 4 & 7 & 6 & 1/6 & 3 & 1/7 & 2 & 1 \end{bmatrix}.$$

According to the proposed inconsistency identification method, we have:

Table 2
The identification process of a_{23}, a_{72}, a_{79}, and a_{93}.

Remove	Test	Might problem	Good	Problem
A_3	√	a_{23}, a_{93}	a_{72}, a_{79}	
A_2	×	a_{79}, a_{93}	a_{79}	a_{93}
A_7	×	a_{23}, a_{93}		
A_9	×	a_{23}, a_{72}	a_{72}	a_{23}

Step 1. The induced matrix $C = A^*A - 9^*A$ becomes

$$\begin{pmatrix} 0 & -2.5471 & 9.3333 & 0 & -0.4444 & -1.0000 & -0.4524 & -1.3333 & 6.7381 \\ 7.7500 & 0 & -24.7500 & 15.5000 & 3.5417 & 14.7500 & 3.5357 & 14.5000 & 3.2381 \\ -3.4464 & 22.8214 & 0 & -6.8929 & -1.9137 & -7.3214 & -1.9171 & -7.4643 & 16.7292 \\ 0 & -1.2708 & 4.6667 & 0 & -0.2222 & -0.5000 & -0.2262 & -0.6667 & 3.3690 \\ 16.0000 & 16.2500 & 56.0000 & 32.0000 & 0 & 14.0000 & -0.1429 & 8.0000 & -29.1905 \\ 0.5000 & -0.6042 & 5.8333 & 1.0000 & -0.1944 & 0 & -0.2024 & -0.3333 & 2.0357 \\ 19.0000 & 20.2500 & 63.0000 & 38.0000 & 0.1667 & 17.0000 & 0 & 10.0000 & -37.1905 \\ 1.0000 & 0.0625 & 7.0000 & 2.0000 & -0.1667 & 0.5000 & -0.1786 & 0 & 0.7024 \\ 1.1190 & -8.0119 & -28.4405 & 2.2381 & 11.2262 & 26.2381 & 11.4167 & 34.2381 & 0 \end{pmatrix}$$

Step 2. The largest value, 63, is located at 7th row and 3rd column.

Step 3. The vectors are

$$r_7 = (2 \quad 4 \quad 1 \quad 4 \quad 1 \quad 4 \quad 1 \quad 4 \quad 7) \text{ and}$$

$$c_3^T = (0.5 \quad 4 \quad 1 \quad 0.25 \quad 1 \quad 0.25 \quad 1 \quad 0.25 \quad 7).$$

Step 4. The scalar product b is

$$b = r_7 \cdot c_3^T = (1 \quad 16 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 49).$$

Step 5. The bias identifying vector f is

$$f = b - a_{73} = (0 \quad 15 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 48).$$

Step 6. In vectors f and b, we find that all the elements are corresponding to a_{73} except two values are larger than a_{73}. That is, $49 = a_{79}a_{93}$, and $16 = a_{72}a_{23}$. So the decision makers should change the value of a_{23}, a_{72}, a_{79}, and a_{93}.

Furthermore, $c_{23} = -24.75 < 0$ and $c_{72} = 20.25 > 0$. It indicates that a_{23} is too large and a_{72} is too small, so a_{23} should be decreased. Besides, we get $c_{79} = -37.195 < 0$ and $c_{93} = -28.4405 < 0$ from the induced matrix C. Hence, a_{79} and a_{93} may be too large. In Table 2, method of matrix order reduction is applied to identify the inconsistency in a_{23}, a_{72}, a_{79}, and a_{93} for their corresponding attributes A_2, A_3, A_7 and A_9, respectively.

The identification process is as follows:

Sub-step 1. Remove A_3, the order reduced matrix passed test, so a_{72} and a_{79} are consistent while a_{23} and a_{93} might have problem.

Sub-step 2. Remove A_2, the order reduced matrix could not pass test, so a_{79}, a_{93} might be problematic while a_{79} has been identified to be consistent in *Sub-step 1,* so a_{93} is one of the inconsistent elements.

Sub-step 3. Remove A_7, the order reduced matrix could not pass the test, so a_{23}, a_{93} might be problematic while a_{93} has been identified to be inconsistent in *Sub-step 2,* so continue to check a_{23}.

Sub-step 4. Remove A_9, the order reduced matrix could not pass the test, so a_{23}, a_{72} might be problematic while a_{72} has been identified to be consistent in *Sub-step 1.* Hence, the inconsistent element is a_{23}.
Therefore, both a_{23} and a_{93} are the inconsistent elements which have been identified simultaneously.

4.1. Adjusting steps

$a_{79} = 7$ is consistent while $a_{79}a_{93} = 49$, so we should decrease a_{93} and let $a_{79}a_{93}$ as close to $a_{73} = 1$ as possible. Let us assume $a_{93} = \frac{1}{7}$, and $a_{39} = 7$. Likewise, $a_{72} = 4$ is consistent and $a_{72}a_{23} = 16$, so a_{23} should be decreased and let $a_{72}a_{23}$ as close to $a_{73} = 1$as possible. Assume $a_{23} = \frac{1}{4}$, and $a_{32} = 4$. Replace the four values from the original comparison matrix A, then the $\lambda_{\max}$ is 9.8491while the *C.R.* is 0.0732 less than 0.1, so the consistency test passed, and no correction of judgments is needed. However, as the C.R. is close to 0.1, some elements are still large in the induced matrix. Thus, the decision maker can continue to adjust the value using the proposed method until he gets satisfied result.

Example 5. The following 8×8 pair-wise comparison matrix A was first introduced in Xu and Wei (1999) as an example of an inconsistent pair-wise comparison matrix for the selection of a trucking company, which is based on the performance of the following eight attributes including punctuality, delivery time, temperature control, track and trace, error rate, service reputation, damage loss, and GPS features. This pair-wise matrix is inconsistent with $\lambda_{\max} = 9.669$ and $C.R = 0.169 > 0.1$. This example was also used by Saaty (2003) as an example to describe the method embedded in the *Expert Choice* Software detecting the inconsistencies and also by Cao et al. (2008) as an inconsistent pair-wise comparison matrix to test their proposed heuristic approach. We also use this public-domain example to illustrate some special cases in Method for adjusting a_{ij}

$$A=\begin{bmatrix} 1 & 5 & 3 & 7 & 6 & 6 & 1/3 & 1/4 \\ 1/5 & 1 & 1/3 & 5 & 3 & 3 & 1/5 & 1/7 \\ 1/3 & 3 & 1 & 6 & 3 & 4 & 6 & 1/5 \\ 1/7 & 1/5 & 1/6 & 1 & 1/3 & 1/4 & 1/7 & 1/8 \\ 1/6 & 1/3 & 1/3 & 3 & 1 & 1/2 & 1/5 & 1/6 \\ 1/6 & 1/3 & 1/4 & 4 & 2 & 1 & 1/5 & 1/6 \\ 3 & 5 & 1/6 & 7 & 5 & 5 & 1 & 1/2 \\ 4 & 7 & 5 & 8 & 6 & 6 & 2 & 1 \end{bmatrix}.$$

According to the proposed method, we have:

Step 1. The induced matrix $C = A^*A - 8^*A$ is

$$\begin{pmatrix} 0 & -12.1833 & -10.3611 & 47.3333 & 5.5000 & -1.0833 & 20.9000 & 2.8560 \\ 1.7968 & 0 & 1.9310 & -3.0571 & -6.2762 & -10.8595 & 3.0667 & 0.9845 \\ 19.4238 & 18.6000 & 0 & 49.9333 & 34.2000 & 21.20000 & -32.6317 & 4.2286 \\ 0.2642 & 1.7980 & 0.3177 & 0 & 1.9214 & 2.2548 & 0.5971 & -0.4837 \\ 0.9563 & 2.7667 & 0.1028 & -8.4333 & 0 & 3.0833 & 1.7841 & -0.2857 \\ 1.3214 & 3.2167 & 1.3111 & -10.9333 & -5.9167 & 0 & 1.7270 & 0.0726 \\ -12.2778 & -6.2667 & 16.2500 & 44.0000 & 18.8333 & 10.9167 & 0 & 1.0393 \\ -11.7905 & 8.6000 & -10.50000 & 101.0000 & 48.6667 & 44.0000 & 24.2762 & 0 \end{pmatrix}.$$

Step 2. The largest value, 101, is located at 8th row and 4th column.

Step 3. The vectors are

$$r_8 = (4 \quad 7 \quad 5 \quad 8 \quad 6 \quad 6 \quad 2 \quad 1) \text{ and}$$

$$c_4^T = (7 \quad 5 \quad 6 \quad 1 \quad 3 \quad 4 \quad 7 \quad 8).$$

Step 4. The scalar product b is

$$b = r_8 \cdot c_4^T = (28 \quad 35 \quad 30 \quad 8 \quad 18 \quad 24 \quad 14 \quad 8)$$

Step 5. The bias identifying vector f is

$$f = b - a_{84} = (20 \quad 27 \quad 22 \quad 0 \quad 10 \quad 16 \quad 6 \quad 0).$$

Step 6. According to the Method for adjusting a_{ij}, there are only two zeroes where the location is 4th and 8th in bias vector f, and others are positive, so a_{84} is too small. However, the value of a_{84} is 8, which is already close to the maximum scale 9, and the other values are larger than zero, which can not be decreased by increasing the value of a_{84}. In such case, we can remove the 4th attribute or the 8th attribute to test the consistency using the order reduction method. Since the consistency test failed, these two attributes might be consistent. In this case, clearly, the 8th attribute, GPS features with large value are more important than the 4th attribute, track and trace, with small value for the selection of a trucking company.

Therefore, continuing the identification process on the second largest outlier, 49.9333, which is located at 3rd row and 4th column, and repeating *Steps 3–6*:

Step 3′. The vectors are

$$r_3 = (0.3333 \quad 3 \quad 1 \quad 6 \quad 3 \quad 4 \quad 6 \quad 2) \text{ and}$$

$$c_4^T = (7 \quad 5 \quad 6 \quad 1 \quad 3 \quad 4 \quad 7 \quad 8).$$

Step 4′. The scalar product b is

$$b = r_3 \cdot c_4^T = (2.3334 \quad 15 \quad 6 \quad 6 \quad 9 \quad 16 \quad 42 \quad 1.6).$$

Step 5′. The bias identifying vector f is

$$f = b - a_{34} = (-3.6667 \quad 9 \quad 0 \quad 0 \quad 3 \quad 10 \quad 36 \quad -4.4).$$

Step 6′. In vectors f and b, most of the elements are around zero, while the number of 36 in vector f is far away zero, which is corresponding to 42 ($a_{37}a_{74}$) in vector b. Hence, the decision makers should check the values of a_{37} and a_{74}.

Furthermore, $c_{37} = -32.6317 < 0$ and $c_{74} = 44 > 0$. It indicates that a_{37} is too large and a_{74} is too small, so the value of a_{37} should be decreased. The inconsistent element in the pair-wise comparison matrix has been identified.

In order to validate that the inconsistent element is a_{37}, we removed 3rd attribute and 7th attribute respectively to test the consistency. Both consistency tests passed. Therefore, the inconsistent element is a_{37}. Since $a_{37}a_{74}$ is supposed to be equal to 6, hence we can get a_{37} = 6/

a_{74} = 6/7 = 0.86. Either 0.5 or 1 in the 9-point scale could be selected as the optimal value of this inconsistent entry. Assume $a_{37} = \frac{1}{2}$, anda_{73} = 2. Replace these two values in the original pair-wise comparison matrix with the above values and test the consistency, the consistency test passed with $\lambda_{\max}$ = 8.8117and$C.R.$ = 0.0828 < 0.1.

Comparisons have been made among the proposed method, Xu and Wei' method, Saaty's method and Cao's et al. method. In Xu and Wei (1999), a consistent matrix by an auto-adaptive process based on the original inconsistent matrix was proposed instead of revising single elements. For instance, the element a_{ij} in the original inconsistent comparison matrix is replaced by $b_{ij} = a_{ij}^{\lambda}(w_i/w_j)^{1-\lambda}$, and $0 < \lambda < 1$, where w_i and w_j are the priority vector derived from the original inconsistent matrix. Thus, a new consistent matrix $B = (b_{ij})$ was generated by adjusting the parameter λ repeatedly, and the decision makers use this new matrix as a reference for revising the original inconsistent matrix instead of the original comparison matrix. Therefore, Xu and Wei's method lost some original comparison information and made some perturbations when adjusting the parameterλ. Clearly, the more bias values of elements are zeros or close to zero, the more original matrix information will be preserved in the bias matrix between the modified comparison matrix and the original comparison matrix. The bias matrix can be calculated by subtracting the modified comparison matrix from the original comparison matrix. For example, in Xu and Wei (1999) final modified comparison matrix, when the parameter λ = 0.98, the bias matrix becomes:

$$\begin{pmatrix} 0 & 0.4760 & 0.6610 & -0.5230 & 0.1120 & 0.3140 & -0.0917 & -0.0420 \\ -0.0210 & 0 & 0.0073 & 0.4840 & 0.3290 & 0.4200 & -0.0220 & -0.0041 \\ -0.0937 & -0.0670 & 0 & -0.7490 & -0.4600 & -0.1880 & 1.8450 & -0.0490 \\ 0.0099 & -0.0210 & 0.0187 & 0 & -0.0397 & -0.0370 & 0.0089 & 0.0210 \\ -0.0033 & -0.0407 & 0.0443 & 0.3190 & 0 & -0.0610 & 0.0030 & 0.0197 \\ -0.0093 & -0.0547 & -0.0160 & 0.5210 & 0.2160 & 0 & -0.0040 & 0.137 \\ 0.6460 & 0.5030 & -0.0743 & -0.4790 & -0.0730 & 0.1010 & 0 & -0.0010 \\ 0.5810 & 0.2140 & 0.9760 & -1.6240 & -0.7830 & -0.5510 & 0.0040 & 0 \end{pmatrix}.$$

In the bias matrix, there are some relatively larger perturbations such as a_{37}, a_{84}, a_{83}, a_{34}, a_{14} and, etc. Many values in the original matrix have been changed. For example, the value of $a_{84}$is 12.339 and 9.624 for λ = 0.5 and λ = 0.98, respectively in the modified pair-wise comparison matrix, which are higher than the maximum scale 9 (Saaty, 1980).

Likewise, in Cao et al. (2008), the consistent matrix based on the original inconsistent matrix is automatically generated instead of revising single element. The deviation of the generated pair-wise comparison information in inconsistent matrix is expressed as a deviation matrix. The consistency ratio is improved by an iterative process which adjusts the deviation matrix. Although the consistency test passed with $C.R.$ = 0.0997 < 0.1, and they illustrated that their proposed method could retain more original comparison information than Xu and Wei's method did, their method also made some perturbations when adjusting the parameter γ. The following bias matrix is calculated by subtracting the modified comparison matrix when the parameter γ = 0.98 from the original comparison matrix.

$$\begin{pmatrix} 0 & 0.5588 & 0.6318 & -0.6743 & 0.1441 & 0.3921 & -0.0868 & -0.0468 \\ -0.0252 & 0 & 0.0123 & 0.5776 & 0.3825 & 0.4608 & -0.0268 & -0.0057 \\ -0.0890 & -0.1151 & 0 & -0.9149 & -0.5351 & -0.2774 & 1.4895 & -0.0487 \\ 0.0126 & -0.0261 & 0.0504 & 0 & -0.0472 & -0.0427 & 0.0116 & 0.0220 \\ -0.0041 & -0.0487 & 0.0162 & 0.3772 & 0 & -0.0722 & 0.0040 & 0.0223 \\ -0.0116 & -0.0605 & 0.0285 & 0.5834 & 0.2522 & 0 & -0.0055 & 0.0173 \\ 0.6196 & 0.5901 & -0.0550 & -0.6136 & -0.1012 & 0.1339 & 0 & -0.0004 \\ 0.6311 & 0.2707 & 0.9785 & -1.7130 & -0.9235 & -0.6949 & 0.0016 & 0 \end{pmatrix}.$$

In the above bias matrix, there are also some relatively larger perturbations such as a_{84}, a_{37}, a_{83}, a_{85}, a_{34} and, etc. Besides, some values of elements in the original matrix had been changed undesirably. For instance, a_{84} = 11.2647 for γ = 0.5, and a_{84} = 9.7130 for γ = 0.98 in the modified pair-wise comparison matrix, which are higher than the maximum scale 9 (Saaty, 1980).

In addition to the above two methods, a similar matrix $\varepsilon_{ij} = a_{ij}(w_j/w_i)$ is constructed to identify the most inconsistent element in the inconsistency identification method embedded of the AHP software *Expert Choice*. The above comparison matrix A was also introduced by Saaty (2003) as an example (Saaty's method in the following). The inconsistent element a_{37} is identified by Saaty's method, and replaced with the value of w_i/w_j = 1/2. Compared with the other two methods, Saaty's method is the easier to use as it is based on the ratio of priorities and designed for the Perron Eigenvalue Method (EM) (Saaty, 1977) and AHP. However, the 'precise' number recommended by Saaty's method is $a_{ij} = \omega_i/\omega_j$, which is an approximated value since the ω_i and ω_j can be calculated by the different method to derive priority vectors. In the example described by Saaty (2003), the method gives the 'precise' value 1/2 by approximating the $a_{37} = \omega_3/\omega_7 = 1/2.18 \approx 1/2$ to adjust the a_{37} and a_{73}. When we select the following pairs, (1,1), (1/3,3), (1/4,4), (1/5,5), (1/6,6) and (1/7,7) to approximate the pairs (a_{37},a_{73}), we get the following CRs: 0.0886, 0.084, 0.086, 0.0897, 0.093, 0.097, respectively. All consistency ratios are less than 0.1, the priority vector derived by replacing one of the pairs of values are acceptable in the AHP, however, the result will be unacceptable in the ANP. The 'precise' value is no longer precise in the ANP, and it is necessary to show the modification direction and provide some optimal values for the decision makers.

To summarize, the formula used by Xu& Wei's method, Saaty's method and Cao's et al. method are $\varepsilon_{ij} = a_{ij}(w_j/w_i), b_{ij} = a_{ij}^{\lambda}(w_i/w_j)^{1-\lambda}$ and $d_{ij}{}' = \gamma a_{ij}/(w_i/w_j) + (1-\gamma)$, respectively. All three methods are based on the priority vector ratios, which are calculated by the inconsistent comparison matrix. Different methods, other than the EM, have been proposed to derive a priority vector with a given positive reciprocal matrix A, including the Normalization of the Column Sum Method and Arithmetic Mean of Normalized Columns Method (Saaty, 1980), the

Direct Least Squares Method (DLSM) and the Weighted Least Squares Method (WLSM) (Chu et al., 1979), the Logarithmic Least Squares Method (LLSM) (Crawford and Williams, 1985)/Geometric Means Solution (Barzilai, 1997), and the Logarithmic Goal Programming Method (GPM) (Bryson, 1995). Review works on methods to derive priorities can be found in Lin (2007). Different methods may yield different vectors (ω_i, ω_j). The inconsistent entries and the approximated value ω_i/ω_j of the identified inconsistent entry a_{ij} may be different when different methods are selected to calculate ω_i, ω_j.

In the proposed method, the inconsistent element a_{37} is identified by the induced matrix C, which is only based on the original comparison matrix A. The decision maker only needs to adjust a_{37} and a_{73} without changing other elements. After identifying the inconsistent entry, one can use any of the known methods to derive the priority vector. It is more practical and keeps most of the information provided by the original comparison matrix. For instance, as identified above, either 0.5 or 1 in the 9-point scale could be selected as the optimal value of this inconsistent entry a_{37}. Let $a_{37} = \frac{1}{2}$, and $a_{73} = 2$, and the modified comparison matrix could be generated by replacing these two values in the original comparison matrix. Thus, the bias matrix becomes:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5.5000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.8333 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

All values provided by experts in the original comparison matrix have been retained except the inconsistent elements a_{37} and a_{73}. Furthermore, the proposed method does not violate the scale [1, 9], needs fewer computations than Xu and Wei' method and Cao's et al. method, and also preserve more original comparison information than these two methods. Compare with Saaty's method, the proposed method is based on only the original comparison matrix instead of the ratio of priorities. Any of the known methods, such as EM, DLSM, WLSM, LLSM/GMS, and GPM, could be applied to derive the priority vectors for the revised reciprocal comparison matrix by the proposed method and the same inconsistent entries will always be identified. Furthermore, the proposed method can also show the modification direction and provide the optimal values.

5. Conclusion and discussion

In this paper, we have proposed a simple method to identify the inconsistent elements in the pair-wise comparison matrix, which is a critical part in AHP and ANP. The correctness of the proposed method for consistent case has been proved mathematically, which combines the theorems of matrix multiplication and vectors dot product as well as the definitions of the pair-wise comparison matrix. The effectiveness of the method has also been demonstrated through several examples.

We have also provided some specific inconsistency identification processes in different situations. The inconsistency identification processes can be conducted in three major steps. The first step is to identify the location of inconsistent element whose absolute value is the largest in the induced pair-wise comparison matrix. The second step is to identify the potential inconsistent elements by the bias identifying vector. The third step is to identify the inconsistent elements using the identification method and the method of matrix order reduction. The inconsistent elements could be easily and quickly identified in the first step. A number of possible inconsistent elements could be identified directly in the second step. Then the specific inconsistent elements could be identified and revised in the third step. Besides, the proposed method could also identify multiple inconsistent elements simultaneously in both low order and high order matrices.

The results of five illustrative examples show that the proposed method is a simple and effective yet accurate method for inconsistency test and revising the inconsistent elements. Furthermore, those results also show that the proposed method can preserve more original comparison information than some existing methods (Xu and Wei, 1999; citebib5). The proposed method is only based on the original comparison matrix A and any of the known methods, such as EM, DLSM, WLSM, LLSM/GMS, and GPM, could be applied to derive the priority vectors for the revised reciprocal comparison matrix and the same inconsistent entries will always be identified.

We have not tested the pair-wise comparison matrix with order higher than ten using the proposed method, because the order maximum of *R.I.* provided by Saaty is 10. However, researchers have proposed to calculate a *R.I.* with an order larger than 10 (Ishizaka and Labib, 2009). This method may be a new inconsistent test method for higher order matrix, which we leave for further research. In addition, this method could also be used to identify the sensitive elements with acceptable *C.R.* which is slightly less than the maximum 0.1, which is also another topic for further research in future.

6. Endnotes

Recently, Feng (2010) had published a similar research results. However, Feng's research has no detailed descriptions and proofs of the theorems. The process of identification in Feng's paper is also incomplete. The differences between Feng's method and our method include: (1) Our method retains the information in the original comparison matrix provided by experts to the maximum. It also could identify the possible inconsistent elements directly and simultaneously by the vectors dot product in its dimension. The specific inconsistent elements could also further be identified with the method of matrix order reduction for the original pair-wise comparison matrix. It is more accurate and efficient especially for the case when $c_{ik} < 0$ and $c_{kj} < 0$ with close values, while Feng's method did not work correctly in such cases. For instance, a_{21} should be inconsistent in terms of Feng's method, however the inconsistent element is a_{31}. Although the result of Feng's method can pass the consistency test by adjusting one or both of a_{21} and a_{31}, more information in the original pair-wise comparison matrix needs to be revised, this might become a time-consuming yet unwelcome task for the decision maker in ANP. (2) There is only one numeric example of low order pair-wise matrix, which is not enough to demonstrate such method. In our paper, three examples with 4 orders, 8 orders and 8 orders, respectively were introduced to test the proposed method. Those examples were also used in some existing researches (Liu, 1999; Xu and Wei, 1999; Kwiesielewicz and Uden, 2002; Cao et al., 2008; Iida, 2009). The identification results are the same as the results in literatures while the proposed method is the most efficient one comparing with the other three methods described in the literatures, and also can preserve more original comparison information. (3) Moreover, we tested the proposed method in nine order matrix by adding one attribute at random to the pair-wise comparison matrix introduced in example 2. The result shows that this method could identify the inconsistent elements accurately and simultaneously.

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Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at doi:10.1016/j.ejor.2011.03.014.

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