where A and B are constants.

Now it's time to finish off two of the examples from Section 30.4 above, as promised:

$$y' + 2y = 4x + \frac{1}{3}\sin(5x) \qquad \text{and} \qquad y'' - 5y' + 6y = 2x^2e^x.$$

At this point, you should try to solve them both. Once you've done that, read on.

The left-hand example is a first-order equation. The homogeneous version is $y' + 2y = 0$, which has the solution $y = Ae^{-2x}$, where A is a constant. Upon consulting the above table, we see that the form for a particular solution is $y_P = ax + b + C\cos(5x) + D\sin(5x)$. We'll need the derivative, namely $y_P' = a - 5C\sin(5x) + 5D\cos(5x)$. Substituting y_P' and y_P into the original equation, we get

$$(a - 5C\sin(5x) + 5D\cos(5x)) + 2(ax + b + C\cos(5x) + D\sin(5x)) = 4x + \frac{1}{3}\sin(5x),$$

which reduces to

$$2ax + 2b + a + (5D + 2C)\cos(5x) + (2D - 5C)\sin(5x) = 4x + \frac{1}{3}\sin(5x).$$

Now we have to equate coefficients of various components of this expression. The coefficient of x is $2a$ on the left-hand side and 4 on the right-hand side, so $a = 2$. The constant coefficient on the left is $2b + a$, whereas there's no constant on the right, so $2b + a = 0$. This means that $b = -1$. Meanwhile, there's no term in $\cos(5x)$ on the right, so $5D + 2C = 0$. On the other hand, the $\sin(5x)$ terms must match, so we have $2D - 5C = 1/3$. Solving these last two equations simultaneously (try it!) gives $C = -5/87$ and $D = 2/87$. So, we have

$$y_P = 2x - 1 - \frac{5}{87}\cos(5x) + \frac{2}{87}\sin(5x);$$

putting it all together, we get the solution

$$y = y_H + y_P = Ae^{-2x} + 2x - 1 - \frac{5}{87}\cos(5x) + \frac{2}{87}\sin(5x),$$

where A is a constant.

How about the other example above? That's a second-order equation, with homogeneous version given by $y'' - 5y' + 6y = 0$. The characteristic quadratic equation is $t^2 - 5t + 6 = 0$, which has solutions $t = 2$ and $t = 3$. So, $y_H = Ae^{2x} + Be^{3x}$, where A and B are constants. Now it's time to deal with the particular solution. Since the right-hand side of the original differential equation is $2x^2e^x$, the form should be $y_P = (ax^2 + bx + c)e^x$; remember that you don't need a constant outside of the e^x, since that constant could be absorbed into a, b and c. Let's differentiate y_P a couple of times:

$$\begin{aligned}
y_P &= (ax^2 + bx + c)e^x, \\
y_P' &= (ax^2 + bx + c)e^x + (2ax + b)e^x \\
&= (ax^2 + (2a + b)x + (b + c))e^x, \\
y_P'' &= (ax^2 + (2a + b)x + (b + c))e^x + (2ax + (2a + b))e^x \\
&= (ax^2 + (4a + b)x + (2a + 2b + c))e^x.
\end{aligned}$$

Now substitute into the original equation $y' - 5y' + 6y = 2x^2e^x$ to get

$$(ax^2 + (4a + b)x + (2a + 2b + c))e^x - 5(ax^2 + (2a + b)x + (b + c))e^x \\ + 6(ax^2 + bx + c)e^x = 2x^2e^x.$$

This simplifies to

$$(2ax^2 + (-6a + 2b)x + (2a - 3b + 2c))e^x = 2x^2e^x.$$

Now equate coefficients to see that $2a = 2$, $-6a + 2b = 0$ and $2a - 3b + 2c = 0$. This means that $a = 1$, $b = 3$ and $c = \frac{7}{2}$, so $y_P = (x^2 + 3x + \frac{7}{2})e^x$. The solution to the whole equation is therefore

$$y = y_H + y_P = Ae^{2x} + Be^{3x} + \left(x^2 + 3x + \frac{7}{2}\right)e^x,$$

where A and B are constants.

30.4.7 Resolving conflicts between y_P and y_H

The last line of the table in Section 30.4.5 above indicates that there might be conflicts between y_P and y_H. How can this happen? Well, consider the differential equation

$$y'' - 3y' + 2y = 7e^{2x}.$$

The homogeneous version is $y'' - 3y' + 2y = 0$, with characteristic quadratic equation given by $t^2 - 3t + 2 = (t - 1)(t - 2) = 0$, so the homogeneous solution is

$$y_H = Ae^x + Be^{2x}.$$

Here A and B are unknown constants. Now, since the right-hand side of the differential equation is $7e^{2x}$, our table says that the form for the particular solution is $y_P = Ce^{2x}$. The sad fact, alas, is that this choice will crash and burn. Indeed, this y_P is included in y_H by setting $A = 0$ and $B = C$. This means that if you plug $y_P = Ce^{2x}$ into the differential equation, you will get 0 on the left-hand side (try it!), so it doesn't work. Instead, as the final line of the table indicates, you need to introduce an extra power of x to make it work. So, we'll use $y_P = Cxe^{2x}$ instead. Let's see what happens now. First, note that $y_P' = 2Cxe^{2x} + Ce^{2x}$ and $y_P'' = 4Cxe^{2x} + 4Ce^{2x}$, so when you substitute into the differential equation above, you get

$$(4Cxe^{2x} + 4Ce^{2x}) - 3(2Cxe^{2x} + Ce^{2x}) + 2Cxe^{2x} = 7e^{2x}.$$

The terms in xe^{2x} cancel completely, and you're left with $Ce^{2x} = 7e^{2x}$. So $C = 7$, meaning that $y_P = 7xe^{2x}$. Finally, the complete solution is given by $y = y_H + y_P = Ae^x + Be^{2x} + 7xe^{2x}$.

One more example. If you want to solve

$$y'' + 6y' + 9y = e^{-3x},$$

you'll have to go even further than before. Now the homogeneous equation $y'' + 6y' + 9y = 0$ has characteristic quadratic $t^2 + 6t + 9 = (t + 3)^2$, so the homogeneous solution is $y_H = Ae^{-3x} + Bxe^{-3x}$. Since the right-hand side of the differential equation is e^{-3x}, we'd want to take $y_P = Ce^{-3x}$. That won't work, since it's included in y_H (with $A = C$ and $B = 0$). Even $y_P = Cxe^{-3x}$ won't work, since that's also included in y_H (with $A = 0$ and $B = C$). So we have to go all the way up to x^2 and set $y_P = Cx^2e^{-3x}$. Now you can differentiate twice to see that $y_P' = 2Cx^{-3x} - 3Cx^2e^{-3x}$ and $y_P'' = 2Ce^{-3x} - 12Cxe^{-3x} + 9Cx^2e^{-3x}$ (check this!). I leave it to you to plug these quantities into the original equation and show that it all simplifies to $2Ce^{-3x} = e^{-3x}$. This means that $C = \frac{1}{2}$, so the solution to the differential equation is $y = y_H + y_P = Ae^{-3x} + Bxe^{-3x} + \frac{1}{2}x^2e^{-3x}$ for some constants A and B.

30.4.8 Initial value problems (constant-coefficient linear)

Let's see how to deal with initial-value problems (IVPs) involving constant-coefficient linear differential equations. As usual, to solve an IVP, first solve the differential equation, then use the initial conditions to find the remaining unknown constants.

Let's modify the last two examples from Section 30.4.6 above to make them into IVPs, then solve them. For the first example, suppose you are given that $y' + 2y = 4x + \frac{1}{3}\sin(5x)$, and that $y(0) = -1$. Well, ignoring the condition $y(0) = -1$ for the moment, we already saw that the general solution is

$$y = Ae^{-2x} + 2x - 1 - \frac{5}{87}\cos(5x) + \frac{2}{87}\sin(5x).$$

Now we also know that $y(0) = -1$, which means that when $x = 0$, $y = -1$. Substituting this in, we get

$$-1 = Ae^0 + 2(0) - 1 - \frac{5}{87}\cos(0) + \frac{2}{87}\sin(0) = A - 1 - \frac{5}{87}.$$

This reduces to $A = 5/87$, so the solution to the IVP is

$$y = \frac{5}{87}e^{-2x} + 2x - 1 - \frac{5}{87}\cos(5x) + \frac{2}{87}\sin(5x).$$

There are no unknown constants.

To modify the second example, let's suppose that $y'' - 5y' + 6y = 2x^2e^x$ and that $y(0) = y'(0) = 0$. As we saw in Section 30.4.6, the general solution (ignoring the initial conditions $y(0) = 0$ and $y'(0) = 0$) is given by

$$y = Ae^{2x} + Be^{3x} + \left(x^2 + 3x + \frac{7}{2}\right)e^x.$$

We'll need to differentiate this once to take advantage of the fact that we know what $y'(0)$ is; check that

$$y' = 2Ae^{2x} + 3Be^{3x} + \left(x^2 + 5x + \frac{13}{2}\right)e^x.$$

So, when $x = 0$, we know that both y and y' are equal to 0; substituting into the equation for y gives

$$0 = Ae^0 + Be^0 + \left(0^2 + 3(0) + \frac{7}{2}\right)e^0 = A + B + \frac{7}{2},$$