

# The Book of Math (Notes)

Kevin Kuo

December 2, 2020

# Forward and Disclaimer

These are math notes made by a student (with a physics major and math minor) based off text books. It may contain misconceptions and misinterpretations, thus should not be viewed in the same light of a text book. Use at your own risk and mental sanity.

## Symbols

### Logic

Name	Symbol	Comment
Exists	$\exists$	There exists at least one
For all	$\forall$	
Not exists	$\nexists$	There does not exist
Exists one	$\exists!$	There only exists one and only one
And	$\wedge$	
Or	$\vee$	Inclusive or
Not	$\neg$	
Logically implies	$\implies$	If
Logically implied by	$\impliedby$	Only if
Logically equivalent	$\iff$	If and only if
Implies	$\longrightarrow$	
Implied by	$\longleftarrow$	
Double Implication	$\longleftrightarrow$	

### Set Notation

Name	Symbol	Comment
Empty Set	$\emptyset$	The set that is empty
Natural Numbers	$\mathbb{N}$	Set of natural numbers not containing 0, equivalent to the set of positive integers
Integers	$\mathbb{Z}$	Set of integers
Rational Numbers	$\mathbb{Q}$	
Algebraic Numbers	$\mathbb{A}$	
Real Numbers	$\mathbb{R}$	
Complex Numbers	$\mathbb{C}$	
In	$\in$	
Not in	$\notin$	
Owns	$\ni$	Has an element
Proper Subset	$\subset$	Subset that is not itself
Subset	$\subseteq$	
Superset	$\supset$	Superset that is not itself
Proper Superset	$\supsetneq$	

Power set	$\wp$
Union	$\cup$
Intersection	$\cap$
Difference	$\setminus$

## Relationships

Name	Symbol	Comment
Defined	$\doteq$	
Approximate	$\approx$	
Equivalent	$\equiv$	Isomorphic (Group Theory)
Congruent	$\cong$	Homomorphic (Group Theory)
Proportional	$\propto$	

## Operators

Name	Symbol	Comment
	$\oplus$	
	$\otimes$	
	$\odot$	
	$\circ$	Convolution
Dagger	$\dagger$	Complex conjugate transpose of a matrix

## Arrows

Name	Symbol	Comment
Maps to	$\mapsto$	

## Hebrew

Name	Symbol	Comment
Aleph	$\aleph$	Carnality of infinite sets that can be well ordered

## Other

Name	Symbol	Comment
Real part	$\Re$	Real part of a number
Imaginary part	$\Im$	Imaginary part of a number

# Book Constitution

## Intents and Purpose

The goal of this book is to organize mathematical knowledge of topics related to the study of physics or the author's interest. It is meant to be used as a source of for future reference, not as a textbook for students new to the topics. It is a notebook of a student, thus should be treated as one and not as a textbook. At most, it could be used as a study guide alongside a textbook. Definitely not as the main source for acquiring knowledge.

## Layout and Organization

The book is split into parts each containing a field of study mathematics, or a topic large enough to justify giving it its own part. Each part contains chapters that focus on a particular topic required to understand the field, with sections dedicated to describing a particular knowledge required for the topic.

As axioms, definitions, theorems, corollary, and proofs are integral and abundant to the study of mathematics, each will have a unique style. Each environment and its styles are displayed as follows:

### **Axiom 0.1: Axiom name**

*Example Axiom Axioms are the “ground rules” of the set.*

### **Theorem 0.0.1: Theorem name or citation**

*Example Theorem An important logical result from the axioms, with proof.*

### **Conjecture 0.0.1: Name of conjecture or citation**

*Example Conjecture A hypothesis, without proof.*

### **Corollary 0.0.1.1:**

*Example Corollary An implication as a result of a theorem.*

### **Lemma 0.0.1.1:**

*Example Lemma Small theorems that build up to a larger theorem.*

### **Proposition 0.0.1.1:**

*Example Proposition Example proposition.*

*Proof:* Logical deductions that results in a theorem. Proofs I've written will be in grey, which may or may not be correct. □

### **Definition 0.0.1: Word**

*Example Definition The definition of a word.*

**Example 0.0.1** *An example.*

**Remark.** *Remark A comment by the author in the textbooks used.*

**Observation.** *Example Observation A remark by me.*

**Question.** *Example Question A question from me for a mystery to be answered later.*



# Contents

<b>I</b>	<b>Logic</b>	<b>1</b>
1	Proofs	3
<b>II</b>	<b>Numbers</b>	<b>5</b>
2	Natural $\mathbb{N}$	7
3	Integers $\mathbb{Z}$	9
4	Rationals $\mathbb{Q}$	11
5	Constructible	13
6	Algebraic $\mathbb{A}$	15
7	Reals $\mathbb{R}$	17
8	Complex $\mathbb{C}$	19
<b>III</b>	<b>Real Analysis</b>	<b>21</b>
9	Sequences	23
9.1	Limits . . . . .	23
9.1.1	Limit Theorems . . . . .	23
9.2	Monotone and Cauchy Sequences . . . . .	23
9.3	Subsequences . . . . .	23
9.4	$\limsup$ and $\liminf$ . . . . .	23

9.5	Series . . . . .	23
9.6	Alternating Series and Integral Tests . . . . .	23
<b>10</b>	<b>Continuity</b>	<b>25</b>
10.1	Continuous Functions . . . . .	25
10.1.1	Properties . . . . .	25
10.2	Uniform Continuity . . . . .	25
10.3	Limits of Functions . . . . .	25
<b>11</b>	<b>Metric Spaces</b>	<b>27</b>
<b>IV</b>	<b>Complex Analysis</b>	<b>29</b>
<b>12</b>	<b>Basics</b>	<b>31</b>
12.1	Complex Numbers . . . . .	31
12.2	Triangle Inequality . . . . .	32
12.3	Polar and Exponential Form . . . . .	33
12.3.1	Properties of Polar and Exponential Form . . . . .	35
12.3.2	Properties of Arguments . . . . .	35
12.4	Roots of $z$ . . . . .	36
12.5	Complex Conjugate . . . . .	38
12.6	Operations as Transformations . . . . .	39
12.7	Complex Analysis Definitions . . . . .	40
<b>13</b>	<b>Analytic Functions</b>	<b>45</b>
13.1	Functions as mappings . . . . .	45
13.2	Limits . . . . .	47
13.2.1	Limit Theorems . . . . .	49
13.2.2	Limits of Points at Infinity . . . . .	51
13.3	Continuity . . . . .	53
<b>14</b>	<b>Conformal Mapping</b>	<b>55</b>



V	Ordinary Differential Equations	57
VI	Nonlinear Dynamics	59
VII	Partial Differential Equations	61
VIII	Integral Equations	63
IX	Linear Algebra	65
15	Markov Chains	67
X	Tensors	69
XI	Riemann Geometry	71
XII	Abstract Algebra	73
16	Groups	75
17	Rings	77
17.1	Ideals . . . . .	77
18	Integral Domains	79
19	GCD Domains	81
20	Unique Factorization Domains	83
21	Principal Ideal Domains	85
22	Fields	87

<b>XIII</b>	<b>Galois Theory</b>	<b>89</b>
<b>XIV</b>	<b>Lie Theory</b>	<b>91</b>
<b>23</b>	<b>Lie Groups</b>	<b>93</b>
<b>24</b>	<b>Lie Algebra</b>	<b>95</b>
<b>XV</b>	<b>C-Star Algebra</b>	<b>97</b>
<b>XVI</b>	<b>Set Theory</b>	<b>99</b>
<b>XVII</b>	<b>Model Theory</b>	<b>101</b>
<b>XVIII</b>	<b>Statistics</b>	<b>103</b>
<b>XIX</b>	<b>Tips and Tricks</b>	<b>105</b>
<b>25</b>	<b>Integration Techniques</b>	<b>107</b>
25.1	DI Method (Integration Table) . . . . .	107
25.2	Feynman Integration . . . . .	107
<b>XX</b>	<b>Index</b>	<b>109</b>
<b>XXI</b>	<b>Bibliography</b>	<b>111</b>

# Part I

## Logic



# Chapter 1

## Proofs



# **Part II**

## **Numbers**

## Resources used in part II

content...



# Chapter 2

## Natural $\mathbb{N}$



# Chapter 3

## Integers $\mathbb{Z}$



# Chapter 4

## Rationals $\mathbb{Q}$



# Chapter 5

## Constructible





# Chapter 6

## Algebraic $\mathbb{A}$



# Chapter 7

## Reals $\mathbb{R}$



# Chapter 8

## Complex $\mathbb{C}$



# Part III

## Real Analysis

## **Resources used in part III**

1. Kenneth A. Ross - Elementary Analysis (2nd Ed.) [1]



# Chapter 9

## Sequences

### 9.1 Limits

#### 9.1.1 Limit Theorems

### 9.2 Monotone and Cauchy Sequences

### 9.3 Subsequences

### 9.4 $\limsup$ and $\liminf$

### 9.5 Series

### 9.6 Alternating Series and Integral Tests



# Chapter 10

## Continuity

### 10.1 Continuous Functions

#### 10.1.1 Properties

### 10.2 Uniform Continuity

### 10.3 Limits of Functions



# Chapter 11

## Metric Spaces



# Part IV

## Complex Analysis

## **Resources used in part IV**

Primary:

1. Brown and Churchill - Complex Variables and Applications [2]

Supplement:

1. A. David Wunsch - Complex Variables with Applications [3]



# Chapter 12

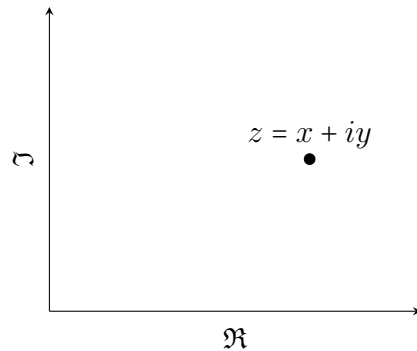
## Basics

### 12.1 Complex Numbers

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}, i = \sqrt{-1}\}$$

Complex numbers are elements of the complex field ( $\mathbb{C}$ ), therefore, they obey all the properties of a field.

We will denote complex numbers by  $z = x + iy$  with  $x, y \in \mathbb{R}$ , and refer the real part as  $\Re(z) = \text{Re}(z) = x$  and imaginary part as  $\Im(z) = \text{Im}(z) = y$ . Complex numbers can also be defined as an ordered pair  $z = (x, y)$  which is interpreted as points in the complex plane.  $(x, 0)$  are points on the real axis while  $(0, y)$  are points in the imaginary axis. This expression is often called a Couple, and was presented in 1833 by mathematician William Rowan Hamilton (1805 - 1865).



Like numbers in  $\mathbb{R}$ , numbers in  $\mathbb{C}$  obey the commutative, distributive, and associative laws. We add and multiply complex numbers in the usual way:

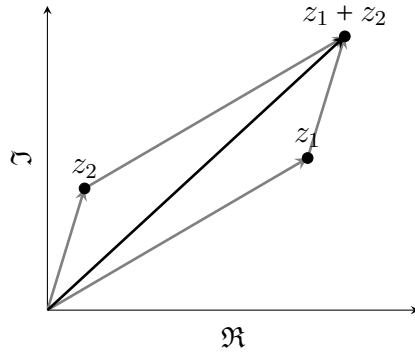
$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) & z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) & &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \end{aligned}$$

$\forall z \in \mathbb{C}$ , there is an unique additive inverse  $(-z)$  and  $\forall z \in \mathbb{C} \setminus \{0\}$ , there is an unique multiplicative inverse  $(z^{-1})$  such that

$$\begin{aligned} z + (-z) &= 0 & zz^{-1} &= 1 \\ \implies -z &= -x - iy & \implies (x_1x_2 - y_1y_2) &= 1 \wedge (x_1y_2 + x_2y_1) = 0 \\ & & \implies z^{-1} &= \frac{x_1}{x_1^2 + y_1^2} - i \frac{y_1}{x_1^2 + y_1^2} \end{aligned}$$

The existence and uniqueness of the inverses can be easily proven.

The addition of complex numbers may also be interpreted as akin to vector addition.



## 12.2 Triangle Inequality

It is not analysis without a section dedicated to the triangle inequality. For any given number  $z_1, z_2 \in \mathbb{C}$  it makes no sense to write an inequality  $z_1 = a_1 + ib_1 < a_2 + ib_2 = z_2$ . Thus, we need have a different notion of size.

### Definition 12.2.1: Modulus

*The modulus of a complex number is a function  $\mathbb{C} \rightarrow \mathbb{R}_{>0}$ :*

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$

It is obvious why the definition is not  $|z| = \sqrt{x^2 + (iy)^2}$  as problems arise when  $x = y$ . The modulus is the distance of  $z$  from  $(0,0)$ .  $\bar{z}$  is the complex conjugate of  $z$ , which is explored in section 12.5

### Theorem 12.2.1: Triangle Inequality

$$\forall z_1, z_2 \in \mathbb{C} [|z_1 + z_2| \leq |z_1| + |z_2|]$$

From the theorem, we can derive a similar inequality:

$$|z_1| = |z_1 + z_2 - z_2| \leq |z_1 + z_2| + |-z_2| \implies |z_1| - |z_2| \leq |z_1 + z_2|$$

An important property of polynomials is observed when theorem 12.2.1 is applied to polynomials.

**Corollary 12.2.1.1:**

Consider the polynomial  $P(z)$  where  $a_n \in \mathbb{C}$ ,  $n \in \mathbb{N}$ ,  $a_0 \neq 0$ , and  $z \in \mathbb{C}$ .

$$P(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$$

Then  $\forall z, \exists R \in \mathbb{R}_{>0}, |z| < R$  such that

$$\left| \frac{1}{P(z)} \right| < \frac{2}{|a_n| R^n}$$

*Proof:* Consider

$$w = \frac{P(z)}{z^n} - a_n = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \dots + \frac{a_{n-1}}{z} \quad z \neq 0$$

$$\implies w z^n = a_0 + a_1 z + \dots + a_{n-1} z^{n-1}$$

$$\implies |w| |z|^n \leq |a_0| + |a_1| |z| + \dots + |a_{n-1}| |z|^{n-1}$$

$$\implies |w| \leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \dots + \frac{|a_{n-1}|}{|z|}$$

$$\implies |w| < n \frac{|a_n|}{2n} = \frac{|a_n|}{2} \quad \exists \text{ sufficiently large } R < |z| \text{ s.t.}$$

$$\forall m, 0 \leq m \leq n-1, \frac{|a_m|}{|z|^{n-m}} < \frac{|a_n|}{2n}$$

$$\implies |a_n + w| \geq |a_n| - |w| > \frac{|a_n|}{2} \quad R < |z|$$

$$\implies |P_n(z)| = |a_n + w| |z|^n > \frac{|a_n|}{2} |z|^n > \frac{|a_n|}{2} R^n \quad R < |z|$$

$$\implies \left| \frac{1}{P(z)} \right| < \frac{2}{|a_n| R^n}$$

□

This tells us that if  $z$  is a solution to a polynomial  $P(z)$ , then the reciprocal of the polynomial  $1/P(z)$  is bounded above by  $R = |z|$ . (i.e. It is bounded by a circle of radius  $|z|$ .)

## 12.3 Polar and Exponential Form

### Definition 12.3.1: Argument of $z$

Consider any  $z \in \mathbb{C}$  where  $z \neq 0$ . Let  $\theta$  be the angle in radians between  $z$  and the real axis. Then  $\forall n \in \mathbb{N}$ ,  $-\pi < \theta \leq \pi$ , the argument of  $z$ :

$$\arg(z) = \theta + 2n\pi$$

We know  $\forall n \in \mathbb{N}, \theta + 2\pi n = \theta$ . This leads us to the definition of the principal argument of  $z$ .

**Definition 12.3.2: Principal Argument of  $z$**

*Consider any  $z \in \mathbb{C}$  where  $z \neq 0$ . Let  $\theta$  be the angle in radians between  $z$  and the real axis. Then for  $-\pi < \theta \leq \pi$ , the principal argument of  $z$ :*

$$\text{Arg}(z) = \theta$$

It is clear that  $\arg(z) = \text{Arg}(z) + 2n\pi$ . It is common for the principal argument to be defined  $-\pi < \theta \leq \pi$ , although other definitions use  $0 \leq \theta < 2\pi$ .

**Definition 12.3.3: Polar Form of  $z$**

*Consider  $z \in \mathbb{C}$ . Let  $r = |z|$ , and  $\theta = \arg(z)$ . Then  $\forall z \in \mathbb{C}, z \neq 0$ :*

$$z = x + iy = r(\cos(\theta) + i \sin(\theta))$$

Notice that all three definitions require that  $z \neq 0$  as  $\theta$  is undefined at  $z = 0$ .

**Theorem 12.3.1: Euler's Formula**

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

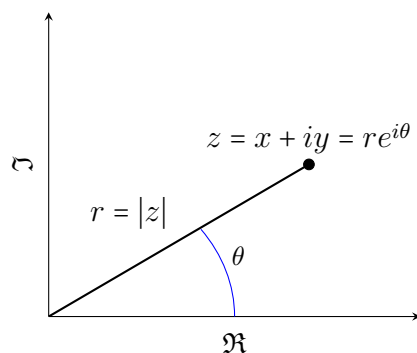
Combining definition 12.3.3 with theorem 12.3.1, we obtain the Exponential Form of  $z$ :

**Definition 12.3.4: Exponential Form of  $z$**

*Consider any  $z \in \mathbb{C}$ , and let  $r = |z|$  and  $\theta = \text{Arg}(z)$ . Then the exponential form of  $z$ :*

$$z = re^{i\theta}$$

Note:  $\theta = \tan^{-1}(y/x)$  and  $r = \sqrt{x^2 + y^2}$ .



### 12.3.1 Properties of Polar and Exponential Form

It would be easier to work with the exponential form of  $z$  then convert it to the polar form later. The exponential form of a complex number is part of the exponential family of functions, thus possess all the properties of the family. Consider any complex number  $z_1 = r_1 e^{i\theta_1}$  and  $z_2 = r_2 e^{i\theta_2}$ .

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} \qquad z^n = r^n e^{in\theta} \qquad \forall n \in \mathbb{Z}$$

A special case arrives for integer exponential of  $z$  on the unit circle.

#### Theorem 12.3.2: de Moivre's Formula

Consider any  $z = e^{i\theta} \in \mathbb{C}$  on the unit circle, and let  $n \in \mathbb{Z}$ .

$$\forall z \in \mathbb{C} \quad \forall n \in \mathbb{Z} \quad [|z| = 1 \implies (\cos(\theta) + i \sin(\theta))^n = \cos(n\theta) + i \sin(n\theta)]$$

*Proof:* Consider  $z = e^{i\theta}$  and let  $n \in \mathbb{Z}$ .

$$z^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i \sin(n\theta)$$

□

The proof hints that theorem 12.3.2 can be generalized to  $\forall n \in \mathbb{R}$ , which we will see shortly in ???. Using theorem 12.3.2, we can obtain the double angle identities.

#### Corollary 12.3.2.1: Double Angle Identities

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \qquad \sin(2\theta) = 2 \sin(\theta) \cos(\theta)$$

*Proof:* Consider any  $z$  on the unit circle, that is  $z = e^{i\theta}$ .

$$\begin{aligned} (\cos(\theta) + i \sin(\theta))^2 &= \cos(2\theta) + i \sin(2\theta) && \text{Theorem 12.3.2} \\ \implies \cos^2(\theta) - \sin^2(\theta) + i 2 \sin(\theta) \cos(\theta) &= \cos(2\theta) + i \sin(2\theta) \end{aligned}$$

Equating the real and imaginary parts yield the desired results.

□

### 12.3.2 Properties of Arguments

Recall from section 12.3.1:

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} \qquad z^n = r^n e^{in\theta} \qquad \forall n \in \mathbb{Z}$$

The arguments for the arguments of products of any  $z_1, z_2 \in \mathbb{C}$  follows immediately from the properties of the exponential.

### Corollary 12.3.2.2: Arguments of Products

$$\begin{aligned}\arg(z_1 z_2) &= \arg(z_1) + \arg(z_2) & \text{Arg}(z_1 z_2) &= \text{Arg}(z_1) + \text{Arg}(z_2) \\ \arg(z^n) &= n \arg(z) & \text{Arg}(z^n) &= n \text{Arg}(z)\end{aligned}$$

*Proof:*

$$\begin{aligned}z_1 z_2 &= r_1 r_2 e^{i(\theta_1 + \theta_2)} \\ \implies \arg(z_1 z_2) &= \arg(z_1) + 2n_1\pi + \arg(z_2) + 2n_2\pi & n_1, n_2 \in \mathbb{Z} \\ \implies \arg(z_1 z_2) &= \arg(z_1) + \arg(z_2) \\ \implies \text{Arg}(z_1 z_2) &= \text{Arg}(z_1) = \text{Arg}(z_2)\end{aligned}$$

$$\begin{aligned}z^n &= r^n e^{in\theta} \\ \implies \arg(z^n) &= n \arg(z) + 2n\pi & n \in \mathbb{Z} \\ \implies \arg(z^n) &= n \arg(z) \\ \implies \text{Arg}(z^n) &= n \text{Arg}(z)\end{aligned}$$

□

It is clear that:

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) \qquad \text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}(z_1) - \text{Arg}(z_2)$$

## 12.4 Roots of $z$

In definition 12.3.4, you might be wondering why  $z^n = r^n e^{in\theta}$  is not for  $n \in \mathbb{R}$ . That is because there is more things to consider, which we will explore in this section. Recall that  $z = re^{i\theta} = re^{i(\theta+2n\pi)}$  for  $n \in \mathbb{Z}$ .

### Definition 12.4.1: Exponential of $z$

Consider any  $z \in \mathbb{C}$  and any  $x \in \mathbb{R}$

$$z^x = \left( r e^{i(\theta+2n\pi)} \right)^x = r^x e^{ix(\theta+2n\pi)}$$

For  $x \notin \mathbb{Z}$ , it is clear that  $z^x = r^x e^{ix(\theta+2n\pi)} \neq r^x e^{ix\theta}$ , since  $2nx\pi = 0 \iff nx \in \mathbb{Z}$ . In order to define the roots of  $z$  we must need a more general and proper definition of  $z$ .

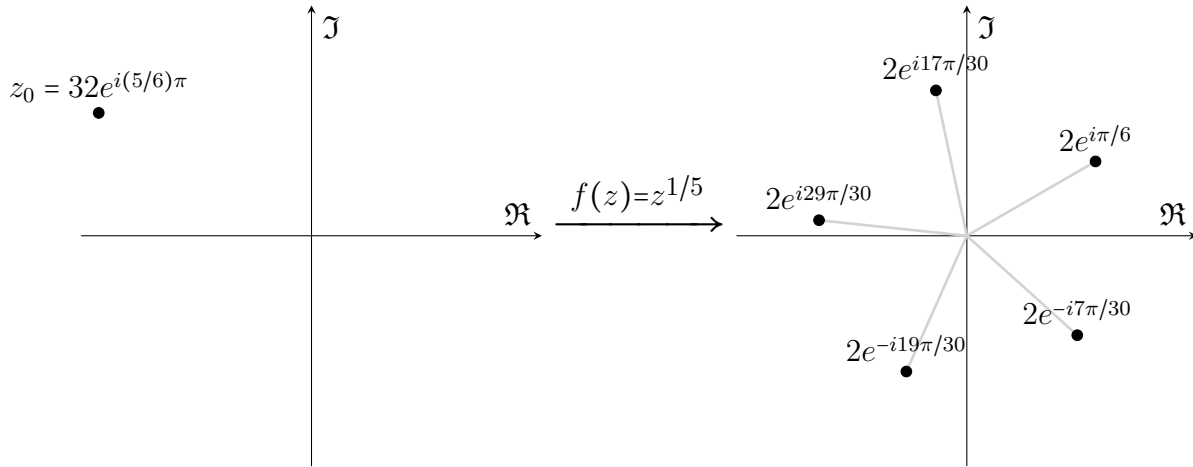
**Definition 12.4.2: Roots of  $z_0$** 

Consider any  $z_0 \in \mathbb{C}$  and any  $m \in \mathbb{N}$ .

$$z_0^{\frac{1}{m}} = r_0^{\frac{1}{m}} e^{i\left(\frac{\theta_0 + 2n\pi}{m}\right)} = r_0^{\frac{1}{m}} e^{i\left(\frac{\theta_0}{m} + \frac{2n\pi}{m}\right)}$$

Taking the  $m$ -th root of  $z_0 \in \mathbb{C}$  scales  $\theta_0$  by  $1/m$ , and provides solutions at equally spaced by  $2\pi/m$  on a circle of radius  $r_0^{1/m}$ . That is, the roots lie on the vertices of a regular  $n$ -sided polygon inscribed in a circle of radius  $|z|^{1/m}$ .

**Example 12.4.1** Consider  $z_0 = 32e^{i(5/6)\pi}$ , then  $z_0^{(1/5)} = 3e^{i(\pi/6) + i(2/5)n\pi}$  for  $n \in \mathbb{Z}$ . The radius went from 32 to  $32^{(1/5)} = 2$ , and five roots appear equally spaced with distance of  $(2/5)\pi$  on a circle with radius 2. Before and after graphs are as follows, note graph on right is zoomed in:



We can see that the roots of  $z_0$  form a set:

**Definition 12.4.3: Set of roots of  $z_0$** 

Consider the  $m$ -th root of any  $z_0 \in \mathbb{C}$ . Let:

$$z_0 = r_0 e^{i\theta_0} \quad c_0 = r_0^{1/m} e^{i\theta_0/m} \quad \omega_n = e^{\frac{i2\pi}{m}} \quad m \in \mathbb{N}$$

Then the set of roots of  $z_0$ :

$$z_0^{1/m} = \{c_k = c_0 \omega_m^k \mid k \in \mathbb{N}, 0 \leq k < m\}$$

$c_0$  is the principal root. The root corresponding to the principal argument of  $z$ .

**Definition 12.4.4: Principal Root**

Consider the  $m$ -th root of any  $z_0 \in \mathbb{C}$ . The principal root of  $z_0$  is defined as:

$$c_0 = r_0^{\frac{1}{m}} e^{i\frac{\theta_0}{m}}$$

**Example 12.4.2** Recall from the previous example:  $z_0 = 32e^{i(5/6)\pi}$ . This gives us

$$c_0 = 32^{1/5}e^{i\pi/6} = 2e^{i\pi/6} \qquad \omega_5 = e^{i2\pi/5}$$

Then

$$\begin{aligned} c_0 &= c_0\omega_5^0 = 2e^{i\pi/6} \\ c_1 &= c_0\omega_5^1 = 2e^{i\pi/6}e^{i2\pi/5} = 2e^{i17\pi/30} \\ c_2 &= c_0\omega_5^2 = 2e^{i\pi/6}e^{i4\pi/5} = 2e^{i29\pi/30} \\ c_3 &= c_0\omega_5^3 = 2e^{i\pi/6}e^{i6\pi/5} = 2e^{i41\pi/30} = 2e^{-i19\pi/30} \\ c_4 &= c_0\omega_5^4 = 2e^{i\pi/6}e^{i8\pi/5} = 2e^{i53\pi/30} = 2e^{-i7\pi/30} \end{aligned}$$



## 12.5 Complex Conjugate

### Definition 12.5.1: Complex Conjugate

The complex conjugate of  $z \in \mathbb{C}$  is denoted  $\bar{z}$ .

$$\bar{z} = x - iy = r(\cos(\theta) - i\sin(\theta)) = re^{-i\theta}$$

Graphically, it is the reflection of  $z$  across the real axis.





It is then easy to see

$$\operatorname{Re}(z) = \frac{z + \bar{z}}{2} \qquad \operatorname{Im}(z) = \frac{z - \bar{z}}{2i} \qquad |z|^2 = z\bar{z}$$

As  $\operatorname{Re}(z) = x = r \cos(\theta)$  and  $\operatorname{Im}(z) = y = r \sin(\theta)$  and using definition 12.3.4, we can obtain the complex forms of sine and cosine:

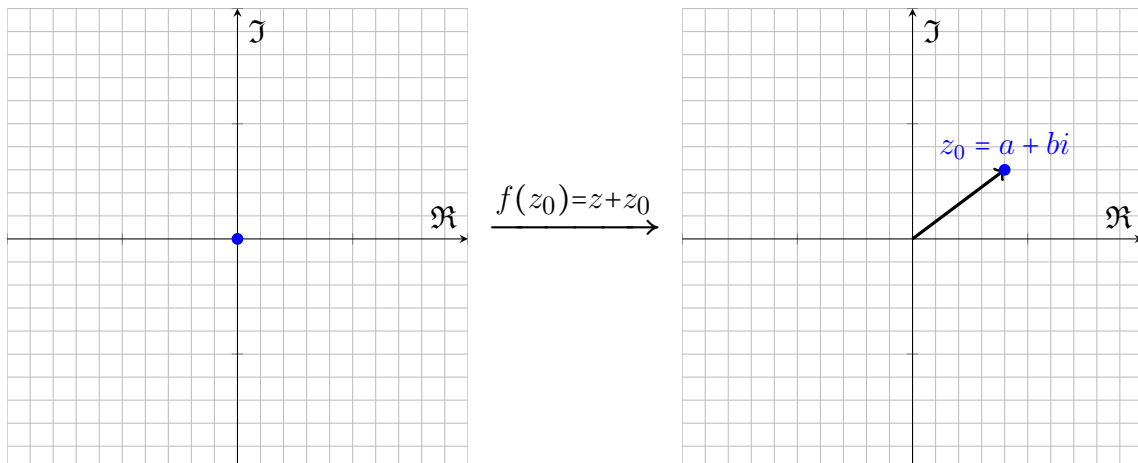
**Definition 12.5.2: Complex Sine and Cosine**

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

## 12.6 Operations as Transformations

Consider any  $z \in \mathbb{C}$ . A function  $f : \mathbb{C} \rightarrow \mathbb{C}$  can be viewed as transformations of the complex plane.

**Example 12.6.1** (Addition as translation) Consider any  $z_0 \in \mathbb{C}$ ,  $z_0 = a + bi$  for  $a, b \in \mathbb{R}$ . Addition by  $z_0$  can be seen as a shift in the complex plane by  $a + bi$ . (i.e. It takes the origin and shifts it by  $z_0$ .)



**Example 12.6.2** (Multiplication as scaling and rotation) Consider any  $z_0 \in \mathbb{C}$ ,  $z_0 = re^{i\theta}$ . Multiplication by  $z_0$  scales the entire complex plane by  $r$  and rotates it by  $\theta$ . (Imagine rotating and stretching out a net.)



## 12.7 Complex Analysis Definitions

### Definition 12.7.1: Neighbourhood

A neighbourhood of a point  $z_0$  is the set of all points  $z$  with distance less than  $\epsilon$ .

$$\{z : |z - z_0| < \epsilon\}$$

i.e. It is the set of all points that lie within a circle centred at  $z_0$  with radius  $\epsilon$ . Points on the circumference not included.



### Definition 12.7.2: Deleted Neighbourhood

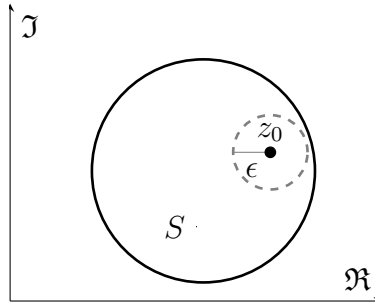
A deleted neighbourhood is the set of all points  $z$  with distance less than  $\epsilon$  from a point  $z_0$ , not including  $z_0$ . That is, it is a neighbourhood of  $z_0$  without  $z_0$ .

$$\{z : |z - z_0| < \epsilon, z \neq z_0\}$$



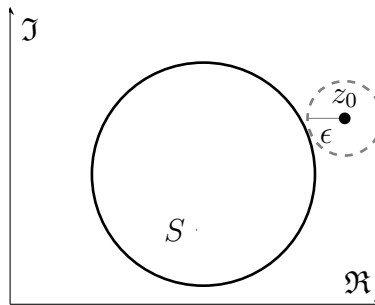
### Definition 12.7.3: Interior Point

Let  $S$  be a set. A point  $z_0$  is an interior point of  $S$  if  $\exists \epsilon$  such that  $\forall z, |z - z_0| < \epsilon \implies z \in S$ . That is,  $z_0$  is an interior point of  $S$  if it has a neighbourhood where all points in the neighbourhood are elements of  $S$ .



### Definition 12.7.4: Exterior Point

Let  $S$  be a set. A point  $z_0$  is an exterior point of  $S$  if  $\exists \epsilon$  such that  $\forall z, |z - z_0| < \epsilon \implies z \notin S$ . That is,  $z_0$  is an exterior point of  $S$  if it has a neighbourhood that does not contain any element of  $S$ .



### Definition 12.7.5: Boundary Point

Let  $S$  be a set. A point  $z_0$  is a boundary point of  $S$  if  $\forall \epsilon, \exists z \in S, z' \notin S$ , such that  $|z - z_0| < \epsilon$  and  $|z' - z_0| < \epsilon$ . That is, for all neighbourhoods of  $z_0$  there exists a point that is in  $S$  and a point not in  $S$ .



Note: A boundary point of  $S$  may or may not be in  $S$ .

**Definition 12.7.6: Boundary of a Set**

*A boundary of a set  $S$  is the set of all boundary points of  $S$ . The set containing all boundary points of  $S$ .*

$$\{z_0 : \forall \epsilon \exists z \in S, z' \notin S (|z - z_0| < \epsilon \wedge |z' - z_0| < \epsilon)\}$$

**Definition 12.7.7: Open Set**

*A set that does not contain any boundary points.*

**Theorem 12.7.1:**

*Set  $S$  is open  $\iff \forall s \in S, s$  is an interior point of  $S$*

*Proof:*  $\implies$ : Suppose  $S$  is open  $\nRightarrow \forall s \in S, s$  is an interior point of  $S$ , for contradiction. That is,  $\exists s \in S$  that is either a boundary point or an exterior point.  $s \in S$  implies  $s$  is not an exterior point of  $S$ , so  $s$  has to be a boundary point of  $S$ . This contradicts that  $S$  is an open set.

$$S \text{ is open} \implies \forall s \in S (s \text{ is an interior point of } S)$$

$\impliedby$ :

$$\begin{aligned} & \forall s \in S (s \text{ is an interior point of } S) \\ & \implies \forall s' \forall \epsilon (|s' - s| < \epsilon \implies s' \in S) \\ & \implies S \text{ does not contain boundary points} \implies S \text{ is open} \end{aligned}$$

□

A set can be neither open or closed. Consider the set  $S = \{z : 0 < |z| \leq 1\}$ .  $S$  is not closed since it does not contain the boundary point 0, and it is not open since it contains boundary points where  $|z| = 1$ . The set  $\mathbb{C}$  is both open and closed since it has no boundary points.

**Definition 12.7.8: Closed Set**

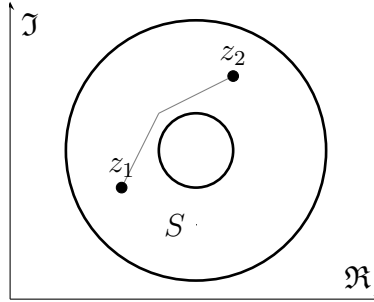
*A set that contains all of its boundary points.*

**Definition 12.7.9: Closure of a Set**

Let  $S$  be a set. The closure of  $S$  is a closed set containing all points of  $S$  and all boundary points of  $S$ .

**Definition 12.7.10: Connected Set**

An opens set  $S$  is connected if  $\forall z_1, z_2 \in S$ ,  $z_1$  and  $z_2$  can be connected by a polygonal line lying within  $S$ .

**Definition 12.7.11: Polygonal Line**

A finite set of line segments joined end to end.

**Definition 12.7.12: Domain**

A nonempty connected set.

Note: All neighbourhoods are domains.

**Definition 12.7.13: Region**

A domain with none, some, or all of its boundary points.

**Definition 12.7.14: Closed Region**

A domain with all of its boundary points.

**Definition 12.7.15: Bounded Set**

A set  $S$  is bounded if  $\exists R$  such that  $\forall s \in S$ ,  $s < R$ .

**Definition 12.7.16: Closed Region**

A bounded and closed region.

**Definition 12.7.17: Accumulation/Limit Point**

A point  $z_0$  is a accumulation point of a set  $S$  if all deleted neighbourhood of  $z_0$  contains an element of  $S$ .

$$\forall \epsilon \exists s \in S (s \neq z_0 \wedge |z - s| < \epsilon)$$

Note: Unlike a boundary point, an accumulation point does not require that all neighbourhood of  $z_0$  contain an element not in  $S$ .

**Theorem 12.7.2:**

*Set  $S$  is closed  $\iff \forall$  accumulation points  $z_0$  of  $S$ ,  $z_0 \in S$*

*Proof:*  $\implies$ : Let  $S$  is closed and  $z_0$  is an accumulation point of a set  $S$  where  $z_0 \notin S$  for contradiction. If  $\exists z_0 \notin S$ , then  $z_0$  is a boundary point of  $S$ . Contradicts closed set contains all boundary points.

$\impliedby$ : Suppose all accumulation points of  $S$  are elements of  $S$  but  $S$  is not closed for contradiction. Then  $S$  does not contain one or more boundary points. Suppose  $z_0$  is a boundary point of  $S$  that is not in  $S$ . Then  $\forall \epsilon \exists s \in S$  where  $|s - z_0| < \epsilon$ , so by considering the deleted neighbourhood of  $z_0$ , this makes  $z_0$  an accumulation point of  $S$ . This contradicts that all accumulation points of  $S$  is in  $S$ .  $\square$

# Chapter 13

## Analytic Functions

### 13.1 Functions as mappings

A function  $f : S \rightarrow S'$  is a function that maps elements from  $S$  to elements on  $S'$ . The value of  $f$  at  $z$  is denoted  $f(z)$  and the set  $S$  is the domain of  $f$  while  $S'$  is the image of  $f$ . Recall section 12.6, a function can likewise be viewed as a transformation or mapping, that maps  $z \in \text{dom}(f) = S$  to values  $z' \in \text{img}(f) = S'$ .

**Definition 13.1.1: Range**

*Let  $f$  be a function with domain  $S$  and image  $S'$ . The range of  $f$  is the entire image of  $S$ .*

Note: Image is a subset of range, and can be a single point or a set of points.

**Definition 13.1.2: Inverse Range**

*The set of all points  $s \in S$  with the value  $f(s) = s'$  for some  $s' \in S'$ .*

$$\{s : f(s) = s', s' \in S'\}$$

Note: The domain of a function is often a domain, but it does not need to be a domain.

We will consider functions  $f : S \rightarrow S'$  where both  $S, S' \subseteq \mathbb{C}$ . For such functions we can break it into a two real valued functions:

$$\begin{aligned} f(z) &= u(x, y) + iv(x, y) & \text{dom}(u) \subseteq \mathbb{R}, \text{dom}(v) \subseteq \mathbb{R} \\ &= u(r, \theta) + iv(r, \theta) \end{aligned}$$

Recall that a real-valued function is a function with a domain that is a subset of  $\mathbb{R}$  (??). If  $\forall z, v(x, y) = 0$ , then  $f$  is called a real-valued function of a complex variable.

**Definition 13.1.3: Polynomial**

*Let  $a_i \in \mathbb{C}$ ,  $0 \leq i \leq n$  where  $i, n \in \mathbb{N} \cup \{0\}$ . If  $a_n \neq 0$ , then a polynomial of degree  $n$  is*

$$P(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n = \sum_{i=0}^n a_iz^i$$

### Definition 13.1.4: Rational Functions

Let  $P(z)$  and  $Q(z)$  are polynomials, then rational functions are quotients:

$$\frac{P(z)}{Q(z)}$$

Defined for all  $z$  where  $Q(z) \neq 0$ .

### Definition 13.1.5: Multiple-Valued Function

Let  $f$  be a function and  $z \in \text{dom}(f)$ .  $f$  is a multiple-valued function if it assigns more than one value to a point  $z$ .

“When multiple-valued functions are studied, usually just one of the possible values assigned at each point is taken, in a systematic manner and a (single-valued) function is constructed from the multiple-valued one” - Brown and Churchill [2]

What this means that for  $z \in \mathbb{C}$  a function  $f$  assigns  $u(z)$  and  $v(z)$  to  $z$ . By taking just  $u$  or  $v$ , we create a single-valued function from a multiple-valued function.

**Example 13.1.1** ( $f(z) = z^2$ )

$$\begin{aligned} f(z) = z^2 &= x^2 - y^2 + i2xy \\ \implies u(x, y) &= x^2 - y^2 \quad v(x, y) = 2xy \end{aligned}$$

By setting  $u = x^2 - y^2 = c_1$  where  $c_1 \in \mathbb{R}_{>0}$  we can see that

$$u = x^2 - y^2 = c_1 \quad v = 2xy = \pm 2y\sqrt{y^2 + c_1}$$

This tells us that in the complex plane of  $u$  and  $v$ , if we fix  $u$  to a constant  $c_1$  and move along  $v = \pm 2y\sqrt{y^2 + c_1}$  by incrementing  $y$  we draw out two hyperbolas in the complex plane of  $x$  and  $y$ . This means that the function  $f(z) = z^2$  takes points on hyperbolas the complex plane of  $x$  and  $y$  and translates them onto a vertical line in the complex plane of  $u$  and  $v$  where  $u$  is a constant.





Likewise if we set  $v = c_2$  where  $c_2 \in \mathbb{R}_{>0}$ , we get:

$$u = x^2 - \frac{c_2^2}{4x^2} \qquad v = 2xy = c_2$$

Taking the limits:

$$\lim_{x \rightarrow 0^+} u = -\infty \qquad \lim_{x \rightarrow \infty, x > 0} u = \infty \qquad (13.1)$$

$$\lim_{x \rightarrow -\infty, x < 0} u = \infty \qquad \lim_{x \rightarrow 0^-} u = -\infty \qquad (13.2)$$

Equation 11.1 tells us as  $x$  goes from 0 to  $\infty$ ,  $u$  moves from  $-\infty$  to  $\infty$ , which corresponds to the hyperbola in the first quadrant of the  $xy$  complex plane. Similarly for equations 11.2.



If we look at  $f$  using the polar representation, we get  $f(z) = r^2 e^{i2\theta}$ . This tells us  $\forall r \geq 0$ ,  $r \mapsto r^2 = \rho \geq 0$ , and  $\forall \theta$ ,  $\theta \mapsto \phi = 2\theta$ . It is worth noting that mapping of points between  $0 \leq \theta < 2\pi$  is not one-to-one, since points in  $0 \leq \theta < \pi$  and points in  $\pi \leq \theta < 2\pi$  both get mapped to  $0 \leq \phi < 2\pi$ .

## 13.2 Limits

### Definition 13.2.1: Limit

Let  $z, z_0, w_0 \in \mathbb{C}$  and  $f$  be a function. We say  $f(z)$  has limit  $w_0$  as  $z$  approaches  $z_0$  if:

$$\forall \epsilon \exists \delta [0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon]$$

We then denote:  $\lim_{z \rightarrow z_0} f(z) = w_0$

This tells us that  $\lim_{z \rightarrow z_0} f(z) = w_0$  if some deleted neighbourhood  $|z - z_0| < \delta$  corresponds to a neighbourhood  $|f(z) - w_0| < \epsilon$ . Note that the mapping of all points  $z$  in  $|z - z_0| < \delta$  to  $|f(z) - w_0| < \epsilon$  need not be surjective. It just needs to be mapped less than distance  $\epsilon$  from  $w_0$ .

Note: Definition 13.2.1 allows us to verify if a limit exists, but it is not a method for determining a limit.



### Theorem 13.2.1: Uniqueness of Limits

*Suppose the limit of  $f$  at  $z_0$  exists, then it is unique.*

*Proof:* Suppose two limits of  $f$  at  $z_0$  exists for contradiction.

$$\begin{aligned} & [\lim_{z \rightarrow z_0} f(z) = w_0] \wedge [\lim_{z \rightarrow z_0} f(z) = w_1] \\ \implies & [0 < |z - z_0| < \delta_0 \implies |f(z) - w_0| < \epsilon] \wedge [0 < |z - z_0| < \delta_0 \implies |f(z) - w_1| < \epsilon] \end{aligned}$$

$$\begin{aligned} w_1 - w_0 &= [f(z) - w_0] + [w_1 - f(z)] \\ \implies & |w_1 - w_0| = |[f(z) - w_0] + [w_1 - f(z)]| \leq |f(z) - w_0| + |f(z) - w_1| \end{aligned}$$

Now choosing  $\delta = \min\{\delta_1, \delta_2\}$ , we get:

$$|w_1 - w_0| < \epsilon + \epsilon = 2\epsilon$$

Choosing  $\epsilon$  to be arbitrary small, we end up with:

$$w_1 - w_0 = 0 \implies w_1 = w_0$$

□

Definition 13.2.1 requires that  $f$  be defined at all points in the deleted neighbourhood of  $z_0$ . That is,  $z_0$  is interior to the region which  $f$  is defined. We can extend the definition by agreeing that  $0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon$  also holds for  $z$  that lie in the region where  $f$  is defined and the deleted neighbourhood of  $z_0$ . That is  $f(z_0)$  need not be defined for a limit at  $z_0$  to exist.

**Example 13.2.1** Show  $(f(z) = iz/2) \wedge (|z| < 1) \implies \lim_{z \rightarrow 1} f(z) = i/2$ .

We can see that we have restricted the domain of  $f$  to the region  $|z| < 1$ , this puts  $z = 1$  right at the boundary of the domain of definition of  $f$ .

$$\begin{aligned} |z| < 1 &\implies \left| f(z) - \frac{i}{2} \right| = \left| \frac{iz}{2} - \frac{i}{2} \right| = \frac{|z - 1|}{2} \\ &\implies \forall \epsilon \forall \epsilon \exists \delta \left[ 0 < |z - 1| < \delta = 2\epsilon \implies \left| f(z) - \frac{i}{2} \right| < \epsilon \right] \\ &\implies \lim_{z \rightarrow 1} f(z) = \frac{i}{2} \end{aligned}$$

This highlights the fact that if the limit exists, then  $z$  is allowed to approach  $z_0$  from any arbitrary direction.

**Example 13.2.2** *Limit of  $f(z) = z/\bar{z}$  does not exist at  $z = 0$*

Consider  $\lim_{z \rightarrow 0} f(z)$ . Let us approach the limit from the  $x$ -axis and the  $y$ -axis.

$$\lim_{z=(x,0) \rightarrow 0} f(z) = \frac{x+i0}{x-i0} = 1 \qquad \lim_{z=(0,y) \rightarrow 0} f(z) = \frac{0+iy}{0-iy} = -1$$

We end up with two different limits. As limits are unique, we conclude that  $\lim_{z \rightarrow 0} f(z)$  does not exist.

### 13.2.1 Limit Theorems

#### Theorem 13.2.2:

Consider  $f(z) = u(x, y) + iv(x, y)$ . Let  $z_0 = x_0 + iy_0$  and  $w_0 = u_0 + iv_0$ .

$$\left[ \lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = u_0 \right] \wedge \left[ \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = v_0 \right] \iff \lim_{z \rightarrow z_0} f(z) = w_0$$

*Proof:*  $\implies$ :

By definition:

$$\begin{aligned} & \left[ \lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = u_0 \right] \wedge \left[ \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = v_0 \right] \\ & \implies \forall \epsilon \exists \delta_1, \delta_2 \left[ \left( 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_1 \implies |u - u_0| < \frac{\epsilon}{2} \right) \right. \\ & \quad \left. \wedge \left( 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_2 \implies |v - v_0| < \frac{\epsilon}{2} \right) \right] \end{aligned} \tag{13.3}$$

Triangle inequality for the distance between points:

$$\begin{aligned} |(u + iv) - (u_0 + iv_0)| &= |(u - u_0) + i(v - v_0)| \leq |u - u_0| + |v - v_0| \\ \sqrt{(x-x_0)^2 + (y-y_0)^2} &= |(x - x_0) + i(y - y_0)| = |(x + iy) - (x_0 + iy_0)| \end{aligned}$$

Let  $\delta = \min\{\delta_1, \delta_2\}$ , it follows from eq. (13.3):

$$0 < |(x + iy) - (x_0 + iy_0)| < \delta \implies |(u + iv) - (u_0 + iv_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus,  $\lim_{z \rightarrow z_0} f(z) = w_0$ .

$\impliedby$ :

Suppose  $\lim_{z \rightarrow z_0} f(z) = w_0$ .

$$\begin{aligned} & \lim_{z \rightarrow z_0} f(z) = w_0 \\ & \implies \forall \epsilon \exists \delta > 0 [ |(x + iy) - (x_0 + iy_0)| < \delta \implies |(u + iv) - (u_0 + iv_0)| < \epsilon ] \end{aligned} \tag{13.4}$$

By the triangle inequality:

$$\begin{aligned}|u - u_0| &\leq |(u - u_0) + i(v - v_0)| = |(u + iv) - (u_0 + iv_0)| \\ |v - v_0| &\leq |(u - u_0) + i(v - v_0)| = |(u + iv) - (u_0 + iv_0)|\end{aligned}$$

$$|(x + iy) - (x_0 + iy_0)| = |(x - x_0) + i(y - y_0)| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

Thus, it follows from the inequalities in eq. (13.4):

$$\begin{aligned}0 &< \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \\ \implies & [|u - u_0| < \epsilon] \wedge [|v - v_0| < \epsilon] \\ \implies & \left[ \lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = u_0 \right] \wedge \left[ \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = v_0 \right]\end{aligned}$$

□

### Theorem 13.2.3:

*Suppose*

$$\left[ \lim_{z \rightarrow z_0} f(z) = w_0 \right] \wedge \left[ \lim_{z \rightarrow z_0} F(z) = W_0 \right]$$

*Then*

$$\begin{aligned}\lim_{z \rightarrow z_0} [f(z) + F(z)] &= w_0 + W_0 \\ \lim_{z \rightarrow z_0} [f(z)F(z)] &= w_0 W_0 \\ \lim_{z \rightarrow z_0} \frac{f(z)}{F(z)} &= \frac{w_0}{W_0} \quad W_0 \neq 0\end{aligned}$$

*Proof:* Let:

$$f(z) = u(x, y) + iv(x, y) \quad F(z) = U(x, y) + iV(x, y)$$

$$z_0 = x_0 + iy_0 \quad w_0 = u_0 + iv_0 \quad W_0 = U_0 + iV_0$$

$$\underline{\lim_{z \rightarrow z_0} [f(z) + F(z)] = w_0 + W_0}$$

From Theorem 13.2.2:

$$\begin{aligned}f(z) + F(z) &= (u + U) + i(v + V) \\ \implies \lim_{(x,y) \rightarrow (x_0,y_0)} f(z)F(z) &= (u_0 + U_0) + i(v_0 + V_0) = w_0 + W_0\end{aligned}$$

$$\underline{\lim_{z \rightarrow z_0} [f(z)F(z)] = w_0 W_0}$$

From Theorem 13.2.2:

$$\begin{aligned} f(z)F(z) &= (uU - vV) + i(vU + uV) \\ \implies \lim_{(x,y) \rightarrow (x_0,y_0)} f(z)F(z) &= (u_0U_0 - v_0V_0) + i(v_0U_0 + u_0V_0) = w_0W_0 \end{aligned}$$

$$\lim_{z \rightarrow z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0} \text{ if } W_0 \neq 0$$

From Theorem 13.2.2:

$$\frac{f(z)}{F(z)} = \frac{u + iv}{U + iV} \implies \lim_{(x,y) \rightarrow (x_0,y_0)} \frac{f(z)}{F(z)} = \frac{u_0 + v_0}{U_0 + iV_0} = \frac{w_0}{W_0}$$

□

### Corollary 13.2.3.1:

Let  $c$  be a constant,  $z, z_0 \in \mathbb{C}$ , and  $P(z)$  be a polynomial. Then

$$\lim_{z \rightarrow z_0} c = c \qquad \lim_{z \rightarrow z_0} z = z_0 \qquad \lim_{z \rightarrow z_0} z^n = z_0^n \qquad n \in \mathbb{N}$$

$$\lim_{z \rightarrow z_0} P(z) = P(z_0)$$

**Observation.** It is surprisingly quick that Brown and Churchill went from  $\epsilon$ - $\delta$  proofs straight to proving with limits. This is different to the approach in Sequences of Limits Theorem for Sequences Section by Kenneth A. Ross. [1]. (Section 9.1.1)

**Question.** It might be possible use a series approach to prove limit theorems for  $z \in \mathbb{C}$  by having separate series for  $x$  and  $y$  (real and imaginary components of  $z$ ), or a series in the form of  $s_n = (x_n, y_n)$ . Which would be the proper approach?

## 13.2.2 Limits of Points at Infinity

### Definition 13.2.2: Extended Complex Plane

The complex plane union with the points at infinity:

$$\mathbb{C} \cup \{\pm\infty, \pm i\infty\}$$

### Definition 13.2.3: Riemann Sphere

A unit sphere centred at the origin of the complex plane, which is consequently bisected by the complex plane.

**Definition 13.2.4: Stereographic Projection**

Consider the Riemann Sphere. Let  $N$  be the northern point of the sphere (the point on the sphere above the origin of the complex plane) and  $z$  be any point in the complex plane. Let  $l$  be a line that goes through  $N$  and  $z$ , then  $l$  will intersect the Riemann Sphere. Let  $P$  be the point where  $l$  intersects the Riemann Sphere. If we let  $N$  correspond to the points at infinity, then there is a one-to-one correspondence between points on the sphere and the points on the extended complex plane. This correspondence called the Stereographic Projection. (Figure 13.1)

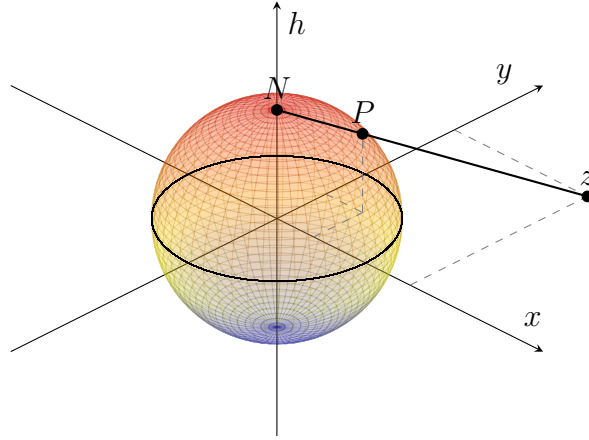


Figure 13.1: Riemann Sphere and Stereographic Projection

The region outside the unit circle enveloped by the Riemann sphere corresponds to the upper hemisphere of the Riemann sphere, with the point  $N$  deleted.  $N$  corresponds to the points at infinity, since  $l$  will be parallel to the complex plane.

**Definition 13.2.5: Neighbourhood of  $\infty$** 

The set:  $\{|z| > 1/\epsilon : \epsilon \in \mathbb{R}_{>0}\}$

Note that since  $\epsilon$  is a small positive number,  $|z| > 1/\epsilon$  corresponds to points far away from the unit circle, hence  $P$  is close to  $N$ .

Note: When referring to any point  $z$ , it is referring to a point in the finite plane. Points at infinity will be specifically mentioned.

**Theorem 13.2.4:**

Let  $z_0, w_0 \in \mathbb{C}$ , then

$$\begin{aligned} \lim_{z \rightarrow z_0} \frac{1}{f(z)} = 0 &\implies \lim_{z \rightarrow z_0} f(z) = \infty \\ \lim_{z \rightarrow 0} f\left(\frac{1}{z}\right) = w_0 &\implies \lim_{z \rightarrow \infty} f(z) = w_0 \\ \lim_{z \rightarrow 0} \frac{1}{f(1/z)} = 0 &\implies \lim_{z \rightarrow \infty} f(z) = \infty \end{aligned}$$

*Proof:*  $\lim_{z \rightarrow z_0} \frac{1}{f(z)} = 0 \implies \lim_{z \rightarrow z_0} f(z) = \infty$

---

$$\begin{aligned} \lim_{z \rightarrow z_0} \frac{1}{f(z)} = 0 &\implies \forall \epsilon \exists \delta > 0 \left[ |z - z_0| < \delta \implies \left| \frac{1}{f(z)} - 0 \right| < \epsilon \right] \\ &\implies \forall \epsilon \exists \delta > 0 \left[ |z - z_0| < \delta \implies |f(z)| > \frac{1}{\epsilon} \right] \\ &\implies \lim_{z \rightarrow z_0} f(z) = \infty \end{aligned}$$

$\lim_{z \rightarrow 0} f\left(\frac{1}{z}\right) = w_0 \implies \lim_{z \rightarrow \infty} f(z) = w_0$

---

$$\begin{aligned} \lim_{z \rightarrow 0} f\left(\frac{1}{z}\right) = w_0 &\implies \forall \epsilon \exists \delta > 0 \left[ |z - 0| < \delta \implies \left| f\left(\frac{1}{z}\right) - w_0 \right| < \epsilon \right] \\ &\implies \forall \epsilon \exists \delta > 0 \left[ |z| > \frac{1}{\delta} \implies |f(z) - w_0| < \epsilon \right] \\ &\implies \lim_{z \rightarrow \infty} f(z) = w_0 \end{aligned}$$

$\lim_{z \rightarrow 0} \frac{1}{f(1/z)} = 0 \implies \lim_{z \rightarrow \infty} f(z) = \infty$

---

$$\begin{aligned} \lim_{z \rightarrow 0} \frac{1}{f(1/z)} = 0 &\implies \forall \epsilon \exists \delta > 0 \left[ |z - 0| < \delta \implies \left| \frac{1}{f(1/z)} - 0 \right| < \epsilon \right] \\ &\implies \forall \epsilon \exists \delta > 0 \left[ |z| > \frac{1}{\delta} \implies |f(z)| > \frac{1}{\epsilon} \right] \\ &\implies \lim_{z \rightarrow \infty} f(z) = \infty \end{aligned}$$

□

Note: As  $\delta$  goes to 0,  $1/\delta$  goes to  $\infty$ , hence  $|z|$  goes to  $\infty$  if  $|z| > 1/\delta$ .

**Observation.** As expected, theorem 13.2.4 is consistent if  $z \in \mathbb{R}$ . (Check: Section 9.1.1).

## 13.3 Continuity





## Chapter 14

# Conformal Mapping



## Part V

# Ordinary Differential Equations



# **Part VI**

## **Nonlinear Dynamics**



## Part VII

# Partial Differential Equations

## Calculus of Variations



# Part VIII

## Integral Equations



# Part IX

## Linear Algebra



# Chapter 15

## Markov Chains



# Part X

## Tensors





# Part XI

## Riemann Geometry



# Part XII

## Abstract Algebra



# Chapter 16

## Groups



# Chapter 17

## Rings

### 17.1 Ideals





## Chapter 18

# Integral Domains



# Chapter 19

## GCD Domains



## Chapter 20

# Unique Factorization Domains



## Chapter 21

# Principal Ideal Domains





# Chapter 22

## Fields



# Part XIII

## Galois Theory



# **Part XIV**

## **Lie Theory**



# Chapter 23

## Lie Groups





# Chapter 24

## Lie Algebra



# Part XV

## C-Star Algebra



# Part XVI

## Set Theory



**Part XVII**

**Model Theory**





# **Part XVIII**

## **Statistics**



# **Part XIX**

## **Tips and Tricks**



# Chapter 25

## Integration Techniques

### 25.1 DI Method (Integration Table)

### 25.2 Feynman Integration



# Part XX

## Index





# Part XXI

## Bibliography



# Bibliography

- [1] Kenneth A. Ross. *Elementary Analysis*. Springer, 2 edition, 2013.
- [2] James Ward Brown and Ruel V. Churchill. *Complex Variables and Applications*. McGraw-Hill Education, 9 edition, 2014.
- [3] A. David Wunsch. *Complex Variables with Applications*. Pearson, 3 edition, 2005.