# The Book of Math (Notes)

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# Forward and Disclaimer

These are math notes made by a student (with a physics major and math minor) based off text books. It may contain misconceptions and misinterpretations, thus should not be viewed in the same light of a text book. Use at your own risk and mental sanity.

# Symbols

## Logic

| Name | Symbol | Comment |
|---|---|---|
| Exists | $\exists$ | There exists at least one |
| For all | $\forall$ | |
| Not exists | $\nexists$ | There does not exist |
| Exists one | $\exists!$ | There only exists one and only one |
| And | $\wedge$ | |
| Or | $\vee$ | Inclusive or |
| Not | $\neg$ | |
| Logically implies | $\Longrightarrow$ | If |
| Logically implied by | $\Longleftarrow$ | Only if |
| Logically equivalent | $\Longleftrightarrow$ | If and only if |
| Implies | $\longrightarrow$ | |
| Implied by | $\longleftarrow$ | |
| Double Implication | $\longleftrightarrow$ | |

## Set Notation

| Name | Symbol | Comment |
|---|---|---|
| Empty Set | $\varnothing$ | The set that is empty |
| Natural Numbers | $\mathbb{N}$ | Set of natural numbers not containing 0, equivalent to the set of positive integers |
| Integers | $\mathbb{Z}$ | Set of integers |
| Rational Numbers | $\mathbb{Q}$ | |
| Algebraic Numbers | $\mathbb{A}$ | |
| Real Numbers | $\mathbb{R}$ | |
| Complex Numbers | $\mathbb{C}$ | |
| In | $\in$ | |
| Not in | $\notin$ | |
| Owns | $\ni$ | Has an element |
| Proper Subset | $\subset$ | Subset that is not itself |
| Subset | $\subseteq$ | |
| Superset | $\supset$ | Superset that is not itself |
| Proper Superset | $\supseteq$ | |

| | | |
|---|---|---|
| Power set | $\wp$ | |
| Union | $\cup$ | |
| Intersection | $\cap$ | |
| Difference | $\setminus$ | |

## Relationships

| Name | Symbol | Comment |
|---|---|---|
| Defined | $\doteq$ | |
| Approximate | $\approx$ | |
| Equivalent | $\equiv$ | Isomorphic (Group Theory) |
| Congruent | $\cong$ | Homomorphic (Group Theory) |
| Proportional | $\propto$ | |

## Operators

| Name | Symbol | Comment |
|---|---|---|
| | $\oplus$ | |
| | $\otimes$ | |
| | $\odot$ | |
| | $\circ$ | Convolution |
| Dagger | $\dagger$ | Complex conjugate transpose of a matrix |

## Arrows

| Name | Symbol | Comment |
|---|---|---|
| Maps to | $\mapsto$ | |

## Hebrew

| Name | Symbol | Comment |
|---|---|---|
| Aleph | $\aleph$ | Carnality of infinite sets that can be well ordered |

## Other

| Name | Symbol | Comment |
|---|---|---|
| Real part | $\Re$ | Real part of a number |
| Imaginary part | $\Im$ | Imaginary part of a number |

# Book Constitution

## Intents and Purpose

The goal of this book is to organize mathematical knowledge of topics related to the study of physics or the author's interest. It is meant to be used as a source of for future reference, not as a textbook for students new to the topics. It is a notebook of a student, thus should be treated as one and not as a textbook. At most, it could be used as a study guide along side a textbook. Definitely not as the main source for acquiring knowledge.

## Layout and Organization

The book is split into parts each containing a field of study mathematics, or a topic large enough to justify giving it its own part. Each part contains chapters that focuses on a particular topic required to understand the field, with sections dedicated to describing a particular knowledge required for the topic.

As axioms, definitions, theorems, corollary, and proofs are integral and abundant to the study of mathematics, each will have a unique style. Each environment and its styles are displayed as follows:

**Axiom 0.1: Axiom name**
*Example Axiom Axioms are the "ground rules" of the set.*

**Theorem 0.0.1: Theorem name or citation**
*Example Theorem An important logical result from the axioms, with proof.*

**Conjecture 0.0.1: Name of conjecture or citation**
*Example Conjecture A hypothesis, without proof.*

**Corollary 0.0.1.1:**
*Example Corollary An implication as a result of a theorem.*

**Lemma 0.0.1.1:**
*Example Lemma Small theorems that build up to a larger theorem.*

**Proposition 0.0.1.1:**
*Example Proposition Example proposition.*

*Proof:* Logical deductions that results in a theorem. Proofs I've written will be in grey, which may or may not be correct. □

**Definition 0.0.1: Word**
*Example Definition The definition of a word.*

**Example 0.0.1** *An example.*

**Remark.** *Remark A comment by the author in the textbooks used.*

**Observation.** *Example Observation A remark by me.*

**Question.** *Example Question A question from me for a mystery to be answered later.*

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# Chapter 7

# Reals $\mathbb{R}$

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# Part III

# Real Analysis

# Resources used in part III

1. Kenneth A. Ross - Elementary Analysis (2nd Ed.) [1]

# Chapter 9

# Metric Spaces

# Part IV

# Complex Analysis

# Resources used in part IV

Primary:

1. Brown and Churchill - Complex Variables and Applications [2]

Supplement:

1. A. David Wunsch - Complex Variables with Applications [3]

# Chapter 10

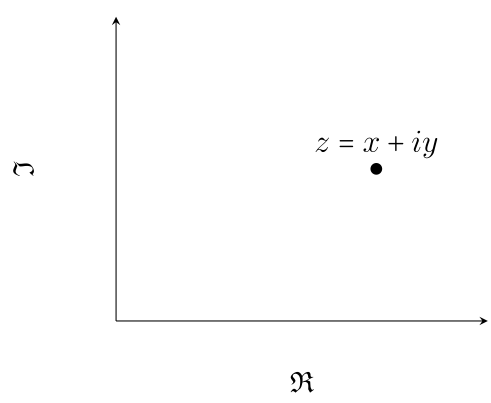
# Basics

## 10.1 Complex Numbers

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}, i = \sqrt{-1}\}$$

Complex numbers are elements of the complex field ($\mathbb{C}$), therefore, they obey all the properties of a field.

We will denote complex numbers by $z = x + iy$ with $x, y \in \mathbb{R}$, and refer the real part as $\mathfrak{R}(z) = \mathrm{Re}(z) = x$ and imaginary part as $\mathfrak{I}(z) = \mathrm{Im}(z) = y$. Complex numbers can also be defined as an ordered pair $z = (x, y)$ which is interpreted as points in the complex plane. $(x, 0)$ are points on the real axis while $(0, y)$ are points in the imaginary axis.




We add and multiply complex numbers in the usual way:

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \end{aligned} \qquad \begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \end{aligned}$$

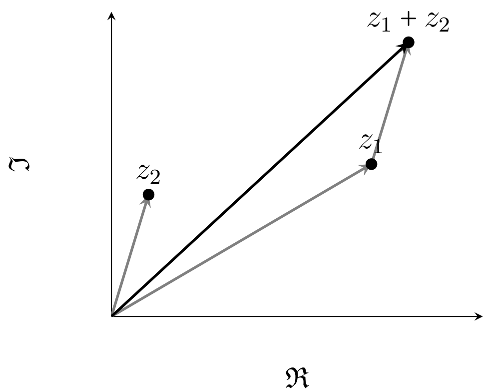
$\forall z \in \mathbb{C}$, there is an unique additive inverse $(-z)$ and $\forall z \in \mathbb{C} \setminus \{0\}$, there is an unique multiplicative inverse $(z^{-1})$ such that

$$z + (-z) = 0 \implies -z = -x - iy$$

$$zz^{-1} = 1 \implies (x_1x_2 - y_1y_2) = 1 \wedge (x_1y_2 + x_2y_1) = 0 \implies z^{-1} = \frac{x_1}{x_1^2 + y_1^2} - i\frac{y_1}{x_1^2 + y_1^2}$$

The existence and uniqueness of the inverses can be easily proven.

The addition of complex numbers may also be interpreted as akin to vector addition.




## 10.2 Triangle Inequality

It is not analysis without a section dedicated to the triangle inequality.

**Definition 10.2.1: Modulus**
$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$

It is obvious why the definition is not $|z| = \sqrt{x^2 + (iy)^2}$ as problems arise when $x = y$. The modulus is the distance of $z$ from $(0, 0)$. $\bar{z}$ is the complex conjugate of $z$, which is explored in section 10.5

**Theorem 10.2.1: Triangle Inequality**
$\forall z_1, z_2 \in \mathbb{C}[|z_1 + z_2| \leq |z_1| + |z_2|]$

From the theorem, we can derive a similar inequality:

$$|z_1| = |z_1 + z_2 - z_2| \leq |z_1 + z_2| + |-z_2| \implies |z_1| - |z_2| \leq |z_1 + z_2|$$

An important property of polynomials is observed when theorem 10.2.1 is applied to polynomials.

**Corollary 10.2.1.1:**
*Consider the polynomial $P(z)$ where $a_n \in \mathbb{C}$, $n \in \mathbb{N}$, $a_0 \neq 0$, and $z \in \mathbb{C}$.*

$$P(z) = a_0 + a_1z + a_2z^2 + \ldots + a_nz^n$$

*Then $\forall z, \exists R \in \mathbb{R}_{>0}, |z| < R$ such that*

$$\left|\frac{1}{P(z)}\right| < \frac{2}{|a_n|R^n}$$

*Proof:* Consider

$$\begin{aligned}
w = \frac{P(z)}{z_n} - a_n = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \ldots + \frac{a_{n-1}}{z} & & z \neq 0 \\
\implies wz^n = a_0 + a_1 z + \ldots + a_{n-1}z^{n-1} & & \\
\implies |w||z|^n \leq |a_0| + |a_1||z| + \ldots + |a_{n-1}||z|^{n-1} & & \\
\implies |w| \leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \ldots + \frac{|a_{n-1}|}{|z|} & & \\
\implies |w| < n\frac{|a_n|}{2n} = \frac{|a_n|}{2} & & \exists \text{ sufficiently large } R < |z| \text{ s.t.} \\
& & \forall m,\ 0 \leq m \leq n-1,\ \frac{|a_m|}{|z|^{n-m}} < \frac{|a_n|}{2n} \\
\implies |a_n + w| \geq ||a_n| - |w|| > \frac{|a_n|}{2} & & R < |z| \\
\implies |P_n(z)| = |a_n + w||z|^n > \frac{|a_n|}{2}|z|^n > \frac{|a_n|}{2}R^n & & R < |z| \\
\implies \left|\frac{1}{P(z)}\right| < \frac{2}{|a_n|R^n} & &
\end{aligned}$$

□

This tells us that if $z$ is a solution to a polynomial $P(z)$, then the reciprocal of the polynomial $1/P(z)$ is bounded above by $R = |z|$. (i.e. It is bounded by a circle of radius $|z|$.)

## 10.3 Polar and Exponential Form

**Definition 10.3.1: Argument of $z$**

*Consider any $z \in \mathbb{C}$ where $z \neq 0$. Let $\theta$ be the angle in radians between $z$ and the real axis . Then $\forall n \in \mathbb{N}$, $0 \leq \theta < 2\pi$, the argument of $z$:*

$$\arg(z) = \theta + 2n\pi$$

We know $\forall n \in \mathbb{N}$, $\theta + 2\pi n = \theta$. This leads us to the definition of the principal argument of $z$.

**Definition 10.3.2: Principal Argument of $z$**

*Consider any $z \in \mathbb{C}$ where $z \neq 0$. Let $\theta$ be the angle in radians between $z$ and the real axis. Then for $0 \leq \theta < 2\pi$, the principal argument of $z$:*

$$\mathrm{Arg}(z) = \theta$$

It is clear that $\arg(z) = \mathrm{Arg}(z) + 2n\pi$.

**Definition 10.3.3: Polar Form of $z$**

*Consider $z \in \mathbb{C}$. Let $r = |z|$, and $\theta = \arg(z)$. Then $\forall z \in \mathbb{C}, z \neq 0$:*

$$z = x + iy = r(\cos(\theta) + i\sin(\theta))$$

Notice that all three definitions require that $z \neq 0$ as $\theta$ is undefined at $z = 0$.

**Theorem 10.3.1: Euler's Formula**

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

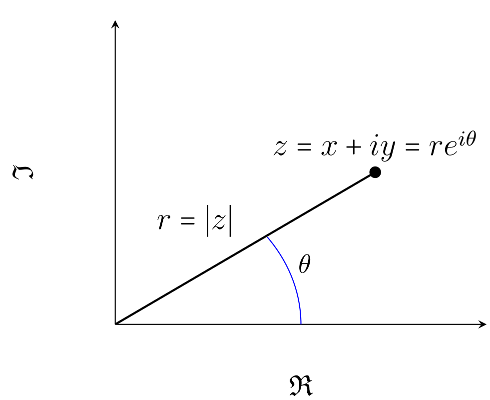
Combining definition 10.3.3 with theorem 10.3.1, we obtain the Exponential Form of $z$:

**Definition 10.3.4: Exponential Form of $z$**

*Consider any $z \in \mathbb{C}$, and let $r = |z|$ and $\theta = \mathrm{Arg}(z)$. Then the exponential form of $z$:*

$$z = re^{i\theta}$$

Note: $\theta = \tan^{-1}(y/x)$ and $r = \sqrt{x^2 + y^2}$.



### 10.3.1 Properties of Polar and Exponential Form

It would be easier to work with the exponential form of $z$ then convert it to the polar form later. The exponential form of a complex number is part of the exponential family of functions, thus possess all the properties of the family. Consider any complex number $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$.

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} \qquad\qquad z^n = r^n e^{in\theta} \qquad \forall n \in \mathbb{Z}$$

A special case arrives for integer exponential of $z$ on the unit circle.

**Theorem 10.3.2: de Moivre's Formula**

*Consider any $z = e^{i\theta} \in \mathbb{C}$ on the unit circle, and let $n \in \mathbb{Z}$.*

$$\forall z \in \mathbb{C}\ \forall n \in \mathbb{Z}\,[|z| = 1 \implies (\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)]$$

*Proof:* Consider $z = e^{i\theta}$ and let $n \in \mathbb{Z}$.

$$z^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

□

The proof hints that theorem 10.3.2 can be generalized to $\forall n \in \mathbb{R}$, which we will see shortly in **??**. Using theorem 10.3.2, we can obtain the double angle identities.

**Corollary 10.3.2.1: Double Angle Identities**

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \qquad\qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

*Proof:* Consider any $z$ on the unit circle, that is $z = e^{i\theta}$.

$$\begin{aligned} &(\cos(\theta) + i\sin(\theta))^2 = \cos(2\theta) + i\sin(2\theta) && \text{Theorem 10.3.2} \\ \implies\ &\cos^2(\theta) - \sin^2(\theta) + i2\sin(\theta)\cos(\theta) = cos(2\theta) + i\sin(2\theta) \end{aligned}$$

Equating the real and imaginary parts yield the desired results. □

### 10.3.2 Properties of Arguments

Recall from section 10.3.1:

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} \qquad\qquad z^n = r^n e^{in\theta} \qquad \forall n \in \mathbb{Z}$$

The arguments for the arguments of products of any $z_1, z_2 \in \mathbb{C}$ follows immediately from the properties of the exponential.

**Corollary 10.3.2.2: Arguments of Products**

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \qquad \operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$$
$$\arg(z^n) = n\arg(z) \qquad \operatorname{Arg}(z^n) = n\operatorname{Arg}(z)$$

*Proof:*

$$\begin{aligned}
z_1 z_2 &= r_1 r_2 e^{i(\theta_1+\theta_2)} \\
&\implies \arg(z_1 z_2) = \arg(z_1) + 2n_1\pi + \arg(z_2) + 2n_2\pi \qquad n_1, n_2 \in \mathbb{Z} \\
&\implies \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \\
&\implies \operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) = \operatorname{Arg}(z_2)
\end{aligned}$$

$$\begin{aligned}
z^n &= r^n e^{in\theta} \\
&\implies \arg(z^n) = n\arg(z) + 2n\pi \qquad n \in \mathbb{Z} \\
&\implies \arg(z^n) = n\arg(z) \\
&\implies \mathrm{z^n} = n\operatorname{Arg}(z)
\end{aligned}$$

□

It is clear that:

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) \qquad \operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \operatorname{Arg}(z_1) - \operatorname{Arg}(z_2)$$

## 10.4 Roots of $z$

In definition 10.3.4, you might be wondering why $z^n = r^n e^{in\theta}$ is not for $n \in \mathbb{R}$. That is because there is more things to consider, which we will explore in this section. Recall that $z = re^(i\theta) = re^{i(\theta+2n\pi)}$ for $n \in \mathbb{Z}$.

**Definition 10.4.1: Exponential of $z$**

*Consider any $z \in \mathbb{C}$ and any $x \in \mathbb{R}$*

$$z^x = \left(re^{i(\theta+2n\pi)}\right)^x = r^x e^{ix(\theta+2n\pi)}$$

For $x \notin \mathbb{Z}$, it is clear that $z^x = r^x e^{ix(\theta+2n\pi)} \neq r^x e^{ix\theta}$, since $2nx\pi = 0 \iff nx \in \mathbb{Z}$. In order to define the roots of $z$ we must need a more general and proper definition of $z$.

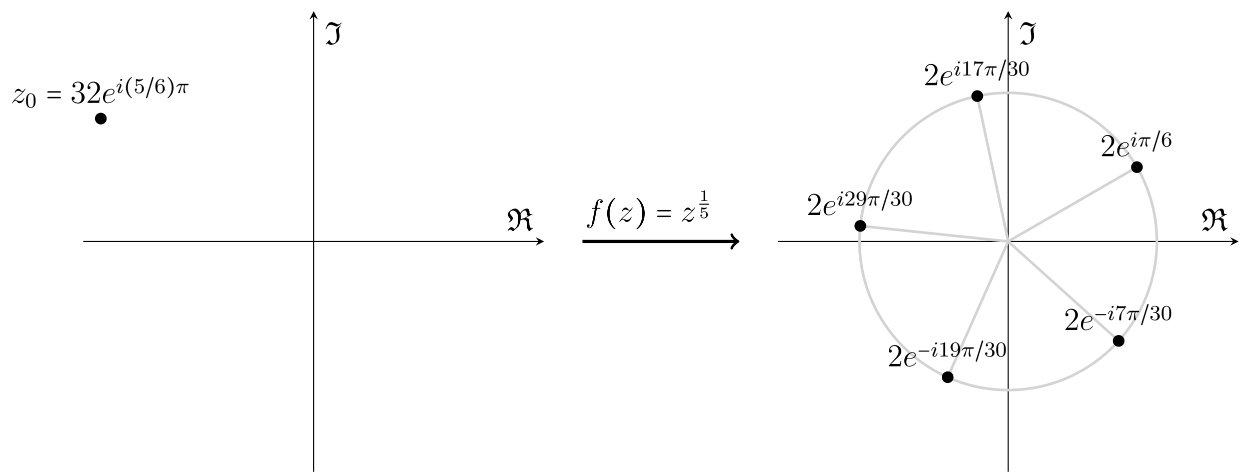
**Definition 10.4.2: Roots of $z_0$**

*Consider any $z_0 \in \mathbb{C}$ and any $m \in \mathbb{N}$.*

$$z_0^{\frac{1}{m}} = r_0^{\frac{1}{m}} e^{i\left(\frac{\theta_0 + 2n\pi}{m}\right)} = r_0^{\frac{1}{m}} e^{i\left(\frac{\theta_0}{m} + \frac{2n\pi}{m}\right)}$$

Taking the $m$-th root of $z_0 \in C$ scales $\theta_0$ by $1/m$, and provides solutions at equally spaced by $2\pi/m$ on a circle of radius $r^{1/m}$. That is, the roots lie on the vertices of a regular n-sided polygon inscribed in a circle of radius $|z|^{1/m}$.

**Example 10.4.1** *Consider $z_0 = 32e^{i(5/6)\pi}$, then $z_0^{(1/5)} = 3e^{i(\pi/6)+i(2/5)n\pi}$ for $n \in \mathbb{Z}$. The radius went from 35 to $35^{(1/5)} = 2$, and five roots appear equally spaced with distance of $(2/5)\pi$ on a circle with radius 2. Before and after graphs are as follows, note graph on right is zoomed in:*




We can see that the roots of $z_0$ form a set:

**Definition 10.4.3: Set of roots of $z_0$**

*Consider the m-th root of any $z_0 \in \mathbb{C}$. Let:*

$$z_0 = r_0 e^{i\theta_0} \qquad c_0 = r_0^{1/m} e^{i\theta_0/m} \qquad \omega_n = e^{\frac{i2\pi}{m}} \qquad m \in \mathbb{N}$$

*Then the set of roots of $z_0$:*

$$z_0^{1/m} = \left\{c_k = c_0 \omega_m^k \mid k \in \mathbb{N},\ 0 \le k < m\right\}$$

$c_0$ is the principal root. The root corresponding to the principal argument of $z$.

**Definition 10.4.4: Principal Root**

*Consider the m-th root of any $z_0 \in \mathbb{C}$. The principal root of $z_0$ is defined as:*

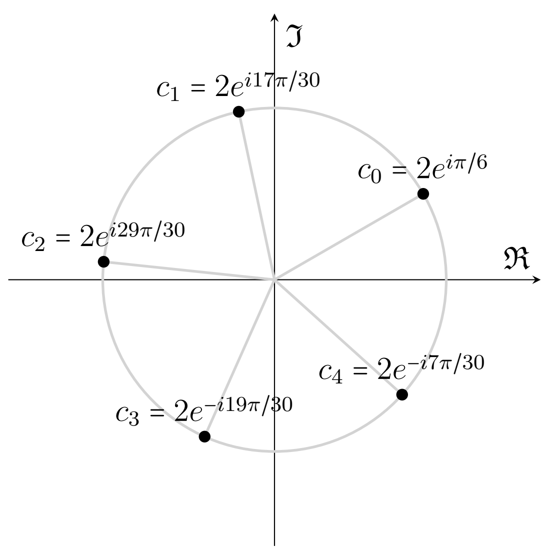
$$c_0 = r_0^{\frac{1}{m}} e^{i\frac{\theta_0}{m}}$$

**Example 10.4.2** *Recall from the previous example:* $z_0 = 32e^{i(5/6)\pi}$. *This gives us*

$$c_0 = 32^{1/5}e^{i\pi/6} = 2e^{i\pi/6} \qquad\qquad \omega_5 = e^{i2\pi/5}$$

*Then*

$$\begin{aligned}
c_0 &= c_0\omega_5^0 = 2e^{i\pi/6} \\
c_1 &= c_0\omega_5^1 = 2e^{i\pi/6}e^{i2\pi/5} = 2e^{i17\pi/30} \\
c_2 &= c_0\omega_5^1 = 2e^{i\pi/6}e^{i4\pi/5} = 2e^{i29\pi/30} \\
c_3 &= c_0\omega_5^1 = 2e^{i\pi/6}e^{i6\pi/5} = 2e^{i41\pi/30} = 2e^{-i19\pi/30} \\
c_4 &= c_0\omega_5^1 = 2e^{i\pi/6}e^{i8\pi/5} = 2e^{i53\pi/30} = 2e^{-i7\pi/30}
\end{aligned}$$



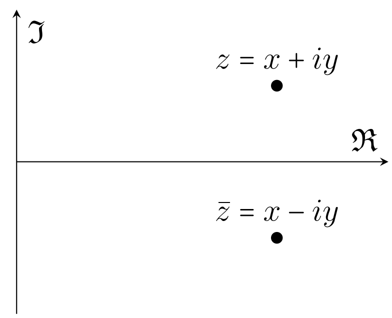

## 10.5 Complex Conjugate

**Definition 10.5.1: Complex Conjugate**

*The complex conjugate of* $z \in \mathbb{C}$ *is denoted* $\bar{z}$.

$$\bar{z} = x - iy = r(\cos(\theta) - i\sin(\theta)) = re^{-i\theta}$$

Graphically, it is the reflection of $z$ across the real axis.




It is then easy to see

$$\mathrm{Re}(z) = \frac{z + \bar{z}}{2} \qquad \mathrm{Im}(z) = \frac{z - \bar{z}}{2i} \qquad |z|^2 = z\bar{z}$$

As $\mathrm{Re}(z) = x = r\cos(\theta)$ and $\mathrm{Im}(z) = y = r\sin(\theta)$ and using definition 10.3.4, we can obtain the complex forms of sine and cosine:

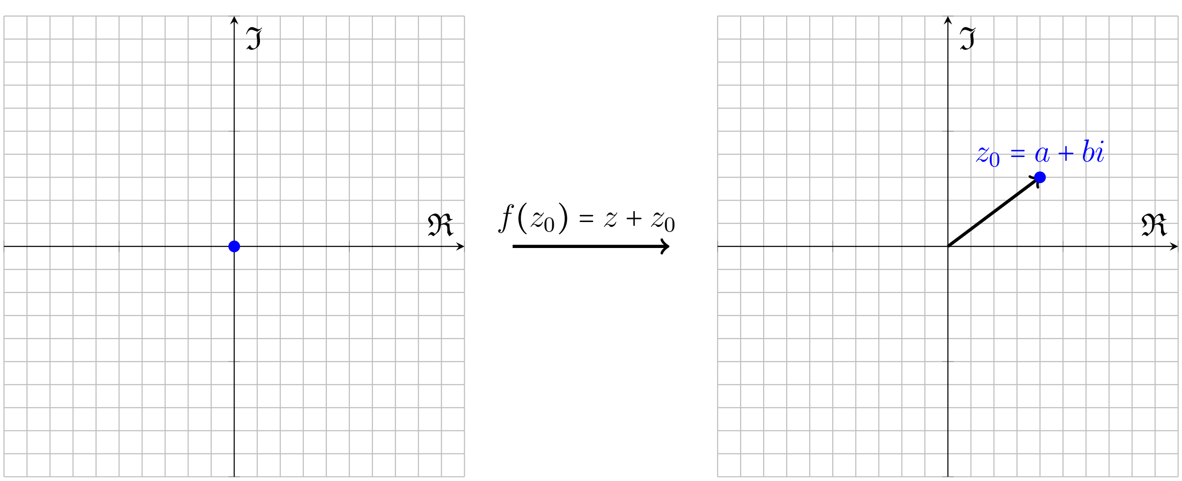
**Definition 10.5.2: Complex Sine and Cosine**

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

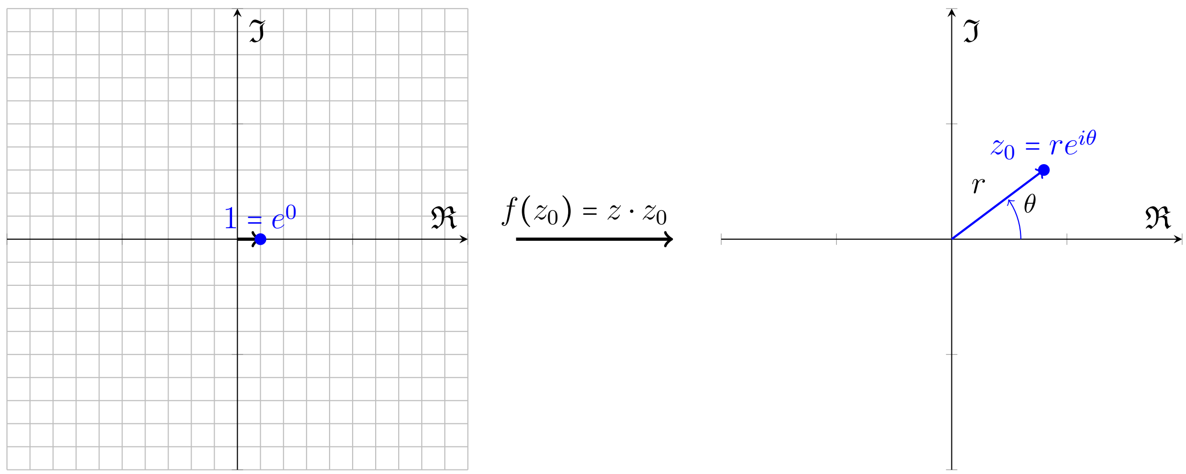
## 10.6 Operations as Transformations

Consider any $z \in \mathbb{C}$. A function $f : \mathbb{C} \to \mathbb{C}$ can be viewed as transformations of the complex plane.

**Example 10.6.1** (Addition as sliding) *Consider any $z_0 \in \mathbb{C}$, $z_0 = a + ib$ for $a, b \in \mathbb{R}$. Addition by $z_0$ can be seen as a shift in the complex plane by $a + bi$. (i.e. It takes the origin and shifts it by $z_0$.)*



**Example 10.6.2** (Multiplication as scaling and rotation) *Consider any $z_0 \in \mathbb{C}$, $z_0 = re^{i\theta}$. Multiplication by $z_0$ scales the entire complex plane by $r$ and rotates it by $\theta$. (Imagine rotating and stretching out a net.)*



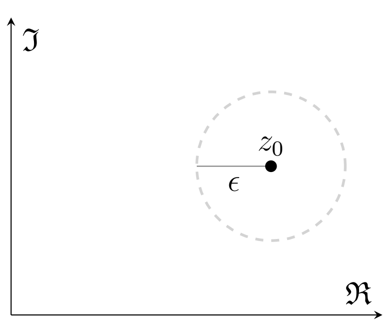

## 10.7 Complex Analysis Definitions

**Definition 10.7.1: Neighbourhood**

*A neighbourhood of a point $z_0$ is the set of all points $z$ with distance less than $\epsilon$.*

$$\{z : |z - z_0| < \epsilon\}$$

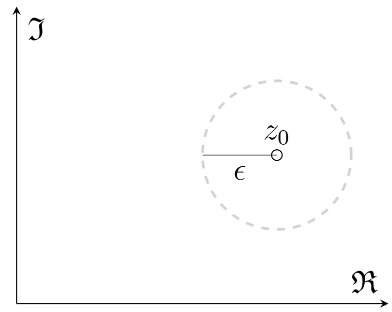
*i.e. It is the set of all points that lie within a circle centred at $z_0$ with radius $\epsilon$. Points on the circumference not included.*




**Definition 10.7.2: Deleted Neighbourhood**

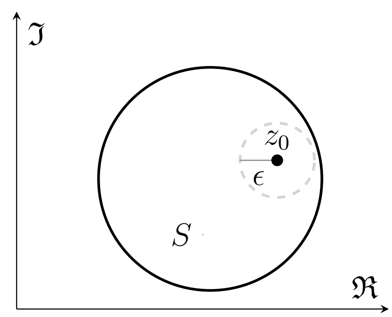
*A deleted neighbourhood is the set of all points $z$ with distance less than $\epsilon$ from a point $z_0$, not including $z_0$. That is, it is a neighbourhood of $z_0$ without $z_0$.*

$$\{z : |z - z_0| < \epsilon,\ z \neq z_0\}$$



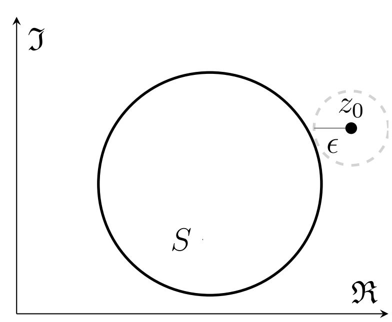

### Definition 10.7.3: Interior Point

*Let $S$ be a set. A point $z_0$ is an interior point of $S$ if $\exists \epsilon$ such that $\forall z$, $|z - z_0| < \epsilon \implies z \in S$. That is, $z_0$ is an interior point of $S$ if it has a neighbourhood where all points in the neighbourhood is an element of $S$.*



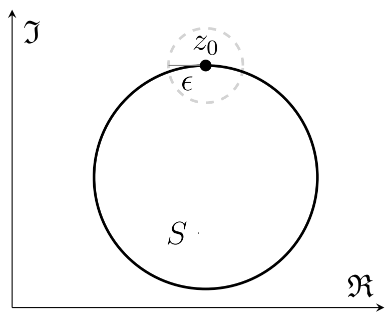

### Definition 10.7.4: Exterior Point

*Let $S$ be a set. A point $z_0$ is an exterior point of $S$ if $\exists \epsilon$ such that $\forall z$, $|z - z_0| < \epsilon \implies z \notin S$. That is, $z_0$is an exterior point of $S$ if it has neighbourhood that does not contain any element of $S$.*




### Definition 10.7.5: Boundary Point

*Let $S$ be a set. A point $z_0$ is a boundary point of $S$ if $\forall \epsilon$, $\exists z \in S, z' \notin S$, such that $|z - z_0| \epsilon$ and $|z' - z_0| < \epsilon$. That is, for all neighbourhoods of $z_0$ there exists a point that is in $S$ and a point not in $S$.*




Note: A boundary point of $S$ may or may not be in $S$.

**Definition 10.7.6: Boundary of a Set**

*A boundary of a set $S$ is the set of all boundary points of $S$. The set containing all boundary points of $S$.*

$$\{z_0 : \forall\epsilon\exists z \in S, z' \notin S(|z - z_0| < \epsilon \wedge |z' - z_0| < \epsilon)\}$$

**Definition 10.7.7: Open Set**

*A set that does not contain any boundary points.*

**Theorem 10.7.1:**

*Set $S$ is open $\iff \forall s \in S$, $s$ is an interior point of $S$*

*Proof:* $\underline{\implies}$: Suppose $S$ is open $\not\Rightarrow \forall s \in S$, $s$ is an interior point of $S$, for contradiction. That is, $\exists s \in S$ that is either a boundary point or an exterior point. $s \in S$ implies $s$ is not an exterior point of $S$, so $s$ has to be a boundary point of $S$. This contradicts that $S$ is an open set.

$$S \text{ is open } \implies \forall s \in S(s \text{ is an interioir point of } S)$$

$\underline{\impliedby:}$

$$\begin{aligned}
&\forall s \in S(s \text{ is an interior point of S})\\
&\implies \forall s'\forall\epsilon(|s' - s| < \epsilon \implies s' \in S)\\
&\implies S \text{ does not contain boundary points} \implies S \text{ is open}
\end{aligned}$$

□

**Definition 10.7.8: Closed Set**

*A set that contains all of its boundary points.*

**Definition 10.7.9: Closure of a Set**

*Let $S$ be a set. The closure of $S$ is a closed set containing all points of $S$ and all boundary points of $S$.*

**Definition 10.7.10: Connected Set**

*content...*

**Definition 10.7.11: Domain**

*content...*

**Definition 10.7.12: Region**

*content...*

**Definition 10.7.13: Bounded Set**

*content...*

**Definition 10.7.14: Accumulation Point**

*content...*

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