

The Book of Math (Notes)

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Forward and Disclaimer

These are math notes made by a student (with a physics major and math minor) based off text books. It may contain misconceptions and misinterpretations, thus should not be viewed in the same light of a text book. Use at your own risk and mental sanity.

Symbols

Logic

Name	Symbol	Comment
Exists	\exists	There exists at least one
For all	\forall	
Not exists	\nexists	There does not exist
Exists one	$\exists!$	There only exists one and only one
And	\wedge	
Or	\vee	Inclusive or
Not	\neg	
Logically implies	\implies	If
Logically implied by	\impliedby	Only if
Logically equivalent	\iff	If and only if
Implies	\longrightarrow	
Implied by	\longleftarrow	
Double Implication	\longleftrightarrow	

Set Notation

Name	Symbol	Comment
Empty Set	\emptyset	The set that is empty
Natural Numbers	\mathbb{N}	Set of natural numbers not containing 0, equivalent to the set of positive integers
Integers	\mathbb{Z}	Set of integers
Rational Numbers	\mathbb{Q}	
Algebraic Numbers	\mathbb{A}	
Real Numbers	\mathbb{R}	
Complex Numbers	\mathbb{C}	
In	\in	
Not in	\notin	
Owns	\ni	Has an element
Proper Subset	\subset	Subset that is not itself
Subset	\subseteq	
Superset	\supset	Superset that is not itself
Proper Superset	\supsetneq	

Power set	\wp
Union	\cup
Intersection	\cap
Difference	\setminus

Relationships

Name	Symbol	Comment
Defined	\doteq	
Approximate	\approx	
Equivalent	\equiv	Isomorphic (Group Theory)
Congruent	\cong	Homomorphic (Group Theory)
Proportional	\propto	

Operators

Name	Symbol	Comment
	\oplus	
	\otimes	
	\odot	
	\circ	Convolution
Dagger	\dagger	Complex conjugate transpose of a matrix

Arrows

Name	Symbol	Comment
Maps to	\mapsto	

Hebrew

Name	Symbol	Comment
Aleph	\aleph	Carnality of infinite sets that can be well ordered

Other

Name	Symbol	Comment
Real part	\Re	Real part of a number
Imaginary part	\Im	Imaginary part of a number

Book Constitution

Intents and Purpose

The goal of this book is to organize mathematical knowledge of topics related to the study of physics or the author's interest. It is meant to be used as a source of for future reference, not as a textbook for students new to the topics. It is a notebook of a student, thus should be treated as one and not as a textbook. At most, it could be used as a study guide along side a textbook. Definitely not as the main source for acquiring knowledge.

Layout and Organization

The book is split into parts each containing a field of study mathematics, or a topic large enough to justify giving it its own part. Each part contains chapters that focuses on a particular topic required to understand the field, with sections dedicated to describing a particular knowledge required for the topic.

As axioms, definitions, theorems, corollary, and proofs are integral and abundant to the study of mathematics, each will have a unique style. Each environment and its styles are displayed as follows:

Axiom 0.1: Axiom name

Example Axiom Axioms are the “ground rules” of the set.

Theorem 0.0.1: Theorem name or citation

Example Theorem An important logical result from the axioms, with proof.

Conjecture 0.0.1: Name of conjecture or citation

Example Conjecture A hypothesis, without proof.

Corollary 0.0.1.1:

Example Corollary An implication as a result of a theorem.

Lemma 0.0.1.1:

Example Lemma Small theorems that build up to a larger theorem.

Proposition 0.0.1.1:

Example Proposition Example proposition.

Proof: Logical deductions that results in a theorem. Proofs I've written will be in grey, which may or may not be correct. □

Definition 0.0.1: Word

Example Definition The definition of a word.

Example 0.0.1 *An example.*

Remark. *Remark A comment by the author in the textbooks used.*

Observation. *Example Observation A remark by me.*

Question. *Example Question A question from me for a mystery to be answered later.*

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Part I

Logic

Chapter 1

Proofs

Part II

Numbers

Resources used in part II

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Chapter 2

Natural \mathbb{N}

Chapter 3

Integers \mathbb{Z}

Chapter 4

Rationals \mathbb{Q}

Chapter 5

Constructible

Chapter 6

Algebraic \mathbb{A}

Chapter 7

Reals \mathbb{R}

Chapter 8

Complex \mathbb{C}

Part III

Real Analysis

Resources used in part III

1. Kenneth A. Ross - Elementary Analysis (2nd Ed.) [1]

Chapter 9

Metric Spaces

Part IV

Complex Analysis

Resources used in part IV

Primary:

1. Brown and Churchill - Complex Variables and Applications [2]

Supplement:

1. A. David Wunsch - Complex Variables with Applications [3]

Chapter 10

Basics

10.1 Complex Numbers

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}, i = \sqrt{-1}\}$$

Complex numbers are elements of the complex field (\mathbb{C}), therefore, they obey all the properties of a field.

We will denote complex numbers by $z = x + iy$ with $x, y \in \mathbb{R}$, and refer the real part as $\Re(z) = \text{Re}(z) = x$ and imaginary part as $\Im(z) = \text{Im}(z) = y$. Complex numbers can also be defined as an ordered pair $z = (x, y)$ which is interpreted as points in the complex plane. $(x, 0)$ are points on the real axis while $(0, y)$ are points in the imaginary axis. This expression is often called a Couple, and was presented in 1833 by mathematician William Rowan Hamilton (1805 - 1865).



Like numbers in \mathbb{R} , numbers in \mathbb{C} obey the commutative, distributive, and associative laws. We add and multiply complex numbers in the usual way:

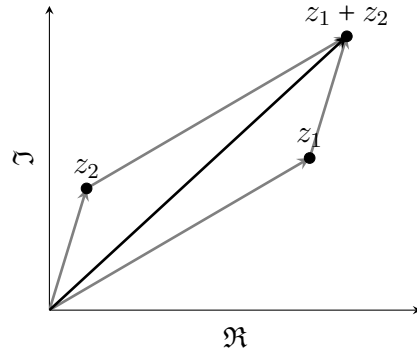
$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) & z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) & &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \end{aligned}$$

$\forall z \in \mathbb{C}$, there is an unique additive inverse $(-z)$ and $\forall z \in \mathbb{C} \setminus \{0\}$, there is an unique multiplicative inverse (z^{-1}) such that

$$\begin{aligned} z + (-z) &= 0 & zz^{-1} &= 1 \\ \implies -z &= -x - iy & \implies (x_1x_2 - y_1y_2) &= 1 \wedge (x_1y_2 + x_2y_1) = 0 \\ & & \implies z^{-1} &= \frac{x_1}{x_1^2 + y_1^2} - i \frac{y_1}{x_1^2 + y_1^2} \end{aligned}$$

The existence and uniqueness of the inverses can be easily proven.

The addition of complex numbers may also be interpreted as akin to vector addition.



10.2 Triangle Inequality

It is not analysis without a section dedicated to the triangle inequality. For any given number $z_1, z_2 \in \mathbb{C}$ it makes no sense to write an inequality $z_1 = a_1 + ib_1 < a_2 + ib_2 = z_2$. Thus, we need have a different notion of size.

Definition 10.2.1: Modulus

The modulus of a complex number is a function $\mathbb{C} \rightarrow \mathbb{R}_{>0}$:

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$

It is obvious why the definition is not $|z| = \sqrt{x^2 + (iy)^2}$ as problems arise when $x = y$. The modulus is the distance of z from $(0,0)$. \bar{z} is the complex conjugate of z , which is explored in section 10.5

Theorem 10.2.1: Triangle Inequality

$$\forall z_1, z_2 \in \mathbb{C} [|z_1 + z_2| \leq |z_1| + |z_2|]$$

From the theorem, we can derive a similar inequality:

$$|z_1| = |z_1 + z_2 - z_2| \leq |z_1 + z_2| + |-z_2| \implies |z_1| - |z_2| \leq |z_1 + z_2|$$

An important property of polynomials is observed when theorem 10.2.1 is applied to polynomials.

Corollary 10.2.1.1:

Consider the polynomial $P(z)$ where $a_n \in \mathbb{C}$, $n \in \mathbb{N}$, $a_0 \neq 0$, and $z \in \mathbb{C}$.

$$P(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$$

Then $\forall z, \exists R \in \mathbb{R}_{>0}, |z| < R$ such that

$$\left| \frac{1}{P(z)} \right| < \frac{2}{|a_n| R^n}$$

Proof: Consider

$$w = \frac{P(z)}{z^n} - a_n = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \dots + \frac{a_{n-1}}{z} \quad z \neq 0$$

$$\implies w z^n = a_0 + a_1 z + \dots + a_{n-1} z^{n-1}$$

$$\implies |w| |z|^n \leq |a_0| + |a_1| |z| + \dots + |a_{n-1}| |z|^{n-1}$$

$$\implies |w| \leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \dots + \frac{|a_{n-1}|}{|z|}$$

$$\implies |w| < n \frac{|a_n|}{2n} = \frac{|a_n|}{2} \quad \exists \text{ sufficiently large } R < |z| \text{ s.t.}$$

$$\forall m, 0 \leq m \leq n-1, \frac{|a_m|}{|z|^{n-m}} < \frac{|a_n|}{2n}$$

$$\implies |a_n + w| \geq |a_n| - |w| > \frac{|a_n|}{2} \quad R < |z|$$

$$\implies |P_n(z)| = |a_n + w| |z|^n > \frac{|a_n|}{2} |z|^n > \frac{|a_n|}{2} R^n \quad R < |z|$$

$$\implies \left| \frac{1}{P(z)} \right| < \frac{2}{|a_n| R^n}$$

□

This tells us that if z is a solution to a polynomial $P(z)$, then the reciprocal of the polynomial $1/P(z)$ is bounded above by $R = |z|$. (i.e. It is bounded by a circle of radius $|z|$.)

10.3 Polar and Exponential Form

Definition 10.3.1: Argument of z

Consider any $z \in \mathbb{C}$ where $z \neq 0$. Let θ be the angle in radians between z and the real axis. Then $\forall n \in \mathbb{N}$, $-\pi < \theta \leq \pi$, the argument of z :

$$\arg(z) = \theta + 2n\pi$$

We know $\forall n \in \mathbb{N}, \theta + 2\pi n = \theta$. This leads us to the definition of the principal argument of z .

Definition 10.3.2: Principal Argument of z

Consider any $z \in \mathbb{C}$ where $z \neq 0$. Let θ be the angle in radians between z and the real axis. Then for $-\pi < \theta \leq \pi$, the principal argument of z :

$$\text{Arg}(z) = \theta$$

It is clear that $\arg(z) = \text{Arg}(z) + 2n\pi$. It is common for the principal argument to be defined $-\pi < \theta \leq \pi$, although other definitions use $0 \leq \theta < 2\pi$.

Definition 10.3.3: Polar Form of z

Consider $z \in \mathbb{C}$. Let $r = |z|$, and $\theta = \arg(z)$. Then $\forall z \in \mathbb{C}, z \neq 0$:

$$z = x + iy = r(\cos(\theta) + i \sin(\theta))$$

Notice that all three definitions require that $z \neq 0$ as θ is undefined at $z = 0$.

Theorem 10.3.1: Euler's Formula

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

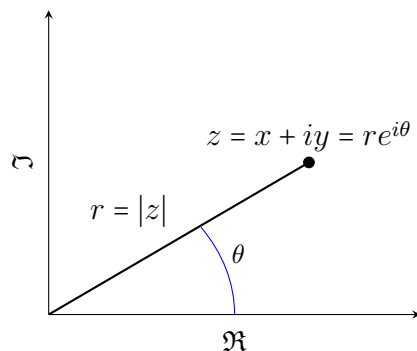
Combining definition 10.3.3 with theorem 10.3.1, we obtain the Exponential Form of z :

Definition 10.3.4: Exponential Form of z

Consider any $z \in \mathbb{C}$, and let $r = |z|$ and $\theta = \text{Arg}(z)$. Then the exponential form of z :

$$z = re^{i\theta}$$

Note: $\theta = \tan^{-1}(y/x)$ and $r = \sqrt{x^2 + y^2}$.



10.3.1 Properties of Polar and Exponential Form

It would be easier to work with the exponential form of z then convert it to the polar form later. The exponential form of a complex number is part of the exponential family of functions, thus possess all the properties of the family. Consider any complex number $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$.

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} \qquad z^n = r^n e^{in\theta} \qquad \forall n \in \mathbb{Z}$$

A special case arrives for integer exponential of z on the unit circle.

Theorem 10.3.2: de Moivre's Formula

Consider any $z = e^{i\theta} \in \mathbb{C}$ on the unit circle, and let $n \in \mathbb{Z}$.

$$\forall z \in \mathbb{C} \quad \forall n \in \mathbb{Z} \quad [|z| = 1 \implies (\cos(\theta) + i \sin(\theta))^n = \cos(n\theta) + i \sin(n\theta)]$$

Proof: Consider $z = e^{i\theta}$ and let $n \in \mathbb{Z}$.

$$z^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i \sin(n\theta)$$

□

The proof hints that theorem 10.3.2 can be generalized to $\forall n \in \mathbb{R}$, which we will see shortly in ???. Using theorem 10.3.2, we can obtain the double angle identities.

Corollary 10.3.2.1: Double Angle Identities

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \qquad \sin(2\theta) = 2 \sin(\theta) \cos(\theta)$$

Proof: Consider any z on the unit circle, that is $z = e^{i\theta}$.

$$\begin{aligned} (\cos(\theta) + i \sin(\theta))^2 &= \cos(2\theta) + i \sin(2\theta) && \text{Theorem 10.3.2} \\ \implies \cos^2(\theta) - \sin^2(\theta) + i 2 \sin(\theta) \cos(\theta) &= \cos(2\theta) + i \sin(2\theta) \end{aligned}$$

Equating the real and imaginary parts yield the desired results.

□

10.3.2 Properties of Arguments

Recall from section 10.3.1:

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} \qquad z^n = r^n e^{in\theta} \qquad \forall n \in \mathbb{Z}$$

The arguments for the arguments of products of any $z_1, z_2 \in \mathbb{C}$ follows immediately from the properties of the exponential.

Corollary 10.3.2.2: Arguments of Products

$$\begin{aligned}\arg(z_1 z_2) &= \arg(z_1) + \arg(z_2) & \text{Arg}(z_1 z_2) &= \text{Arg}(z_1) + \text{Arg}(z_2) \\ \arg(z^n) &= n \arg(z) & \text{Arg}(z^n) &= n \text{Arg}(z)\end{aligned}$$

Proof:

$$\begin{aligned}z_1 z_2 &= r_1 r_2 e^{i(\theta_1 + \theta_2)} \\ \implies \arg(z_1 z_2) &= \arg(z_1) + 2n_1\pi + \arg(z_2) + 2n_2\pi & n_1, n_2 \in \mathbb{Z} \\ \implies \arg(z_1 z_2) &= \arg(z_1) + \arg(z_2) \\ \implies \text{Arg}(z_1 z_2) &= \text{Arg}(z_1) = \text{Arg}(z_2)\end{aligned}$$

$$\begin{aligned}z^n &= r^n e^{in\theta} \\ \implies \arg(z^n) &= n \arg(z) + 2n\pi & n \in \mathbb{Z} \\ \implies \arg(z^n) &= n \arg(z) \\ \implies \text{Arg}(z^n) &= n \text{Arg}(z)\end{aligned}$$

□

It is clear that:

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) \qquad \text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}(z_1) - \text{Arg}(z_2)$$

10.4 Roots of z

In definition 10.3.4, you might be wondering why $z^n = r^n e^{in\theta}$ is not for $n \in \mathbb{R}$. That is because there is more things to consider, which we will explore in this section. Recall that $z = re^{i\theta} = re^{i(\theta+2n\pi)}$ for $n \in \mathbb{Z}$.

Definition 10.4.1: Exponential of z

Consider any $z \in \mathbb{C}$ and any $x \in \mathbb{R}$

$$z^x = \left(r e^{i(\theta+2n\pi)} \right)^x = r^x e^{ix(\theta+2n\pi)}$$

For $x \notin \mathbb{Z}$, it is clear that $z^x = r^x e^{ix(\theta+2n\pi)} \neq r^x e^{ix\theta}$, since $2nx\pi = 0 \iff nx \in \mathbb{Z}$. In order to define the roots of z we must need a more general and proper definition of z .

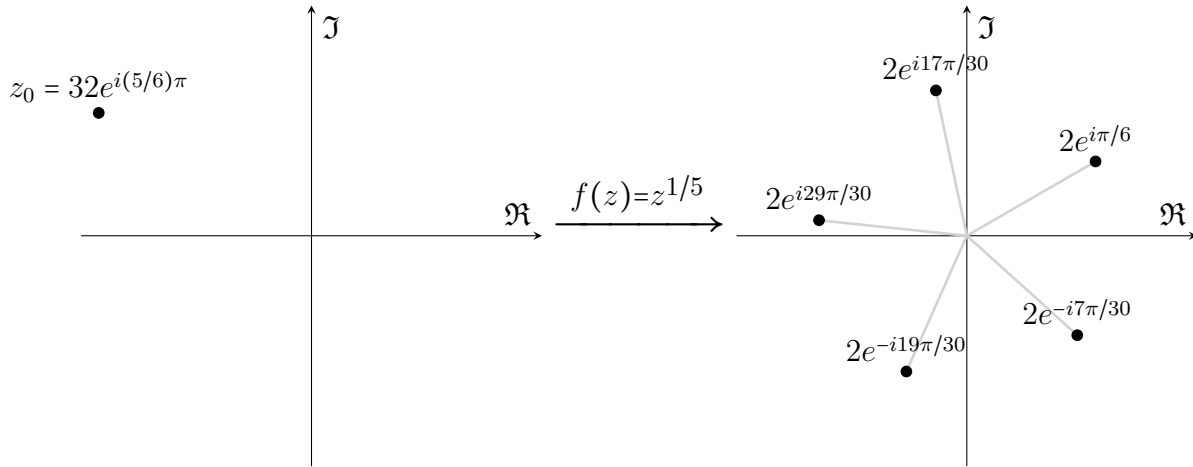
Definition 10.4.2: Roots of z_0

Consider any $z_0 \in \mathbb{C}$ and any $m \in \mathbb{N}$.

$$z_0^{\frac{1}{m}} = r_0^{\frac{1}{m}} e^{i\left(\frac{\theta_0 + 2n\pi}{m}\right)} = r_0^{\frac{1}{m}} e^{i\left(\frac{\theta_0}{m} + \frac{2n\pi}{m}\right)}$$

Taking the m -th root of $z_0 \in \mathbb{C}$ scales θ_0 by $1/m$, and provides solutions at equally spaced by $2\pi/m$ on a circle of radius $r_0^{1/m}$. That is, the roots lie on the vertices of a regular n -sided polygon inscribed in a circle of radius $|z|^{1/m}$.

Example 10.4.1 Consider $z_0 = 32e^{i(5/6)\pi}$, then $z_0^{(1/5)} = 3e^{i(\pi/6) + i(2/5)n\pi}$ for $n \in \mathbb{Z}$. The radius went from 32 to $32^{(1/5)} = 2$, and five roots appear equally spaced with distance of $(2/5)\pi$ on a circle with radius 2. Before and after graphs are as follows, note graph on right is zoomed in:



We can see that the roots of z_0 form a set:

Definition 10.4.3: Set of roots of z_0

Consider the m -th root of any $z_0 \in \mathbb{C}$. Let:

$$z_0 = r_0 e^{i\theta_0} \quad c_0 = r_0^{1/m} e^{i\theta_0/m} \quad \omega_n = e^{\frac{i2\pi}{m}} \quad m \in \mathbb{N}$$

Then the set of roots of z_0 :

$$z_0^{1/m} = \{c_k = c_0 \omega_m^k \mid k \in \mathbb{N}, 0 \leq k < m\}$$

c_0 is the principal root. The root corresponding to the principal argument of z .

Definition 10.4.4: Principal Root

Consider the m -th root of any $z_0 \in \mathbb{C}$. The principal root of z_0 is defined as:

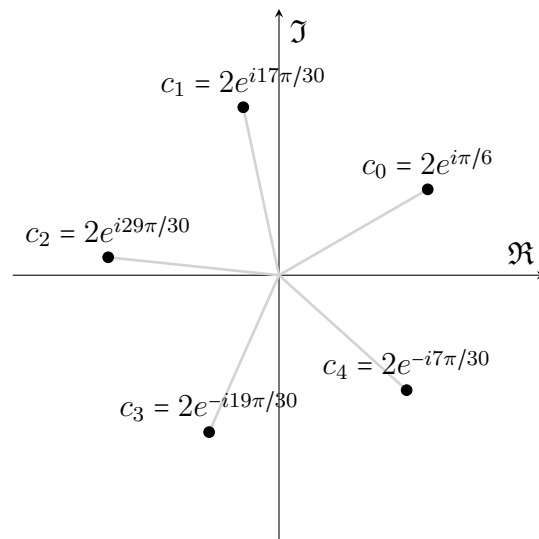
$$c_0 = r_0^{\frac{1}{m}} e^{i\frac{\theta_0}{m}}$$

Example 10.4.2 Recall from the previous example: $z_0 = 32e^{i(5/6)\pi}$. This gives us

$$c_0 = 32^{1/5}e^{i\pi/6} = 2e^{i\pi/6} \qquad \omega_5 = e^{i2\pi/5}$$

Then

$$\begin{aligned} c_0 &= c_0\omega_5^0 = 2e^{i\pi/6} \\ c_1 &= c_0\omega_5^1 = 2e^{i\pi/6}e^{i2\pi/5} = 2e^{i17\pi/30} \\ c_2 &= c_0\omega_5^2 = 2e^{i\pi/6}e^{i4\pi/5} = 2e^{i29\pi/30} \\ c_3 &= c_0\omega_5^3 = 2e^{i\pi/6}e^{i6\pi/5} = 2e^{i41\pi/30} = 2e^{-i19\pi/30} \\ c_4 &= c_0\omega_5^4 = 2e^{i\pi/6}e^{i8\pi/5} = 2e^{i53\pi/30} = 2e^{-i7\pi/30} \end{aligned}$$



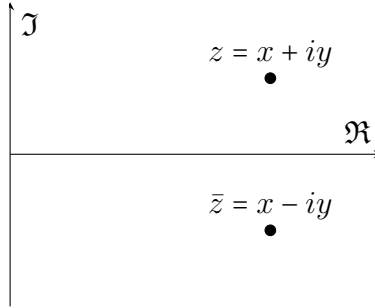
10.5 Complex Conjugate

Definition 10.5.1: Complex Conjugate

The complex conjugate of $z \in \mathbb{C}$ is denoted \bar{z} .

$$\bar{z} = x - iy = r(\cos(\theta) - i\sin(\theta)) = re^{-i\theta}$$

Graphically, it is the reflection of z across the real axis.



It is then easy to see

$$\operatorname{Re}(z) = \frac{z + \bar{z}}{2}$$

$$\operatorname{Im}(z) = \frac{z - \bar{z}}{2i}$$

$$|z|^2 = z\bar{z}$$

As $\operatorname{Re}(z) = x = r \cos(\theta)$ and $\operatorname{Im}(z) = y = r \sin(\theta)$ and using definition 10.3.4, we can obtain the complex forms of sine and cosine:

Definition 10.5.2: Complex Sine and Cosine

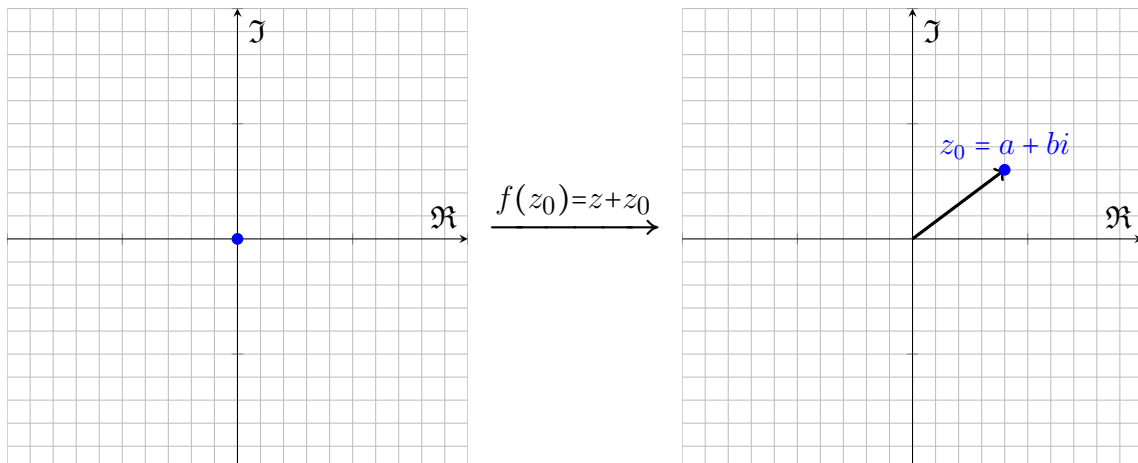
$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

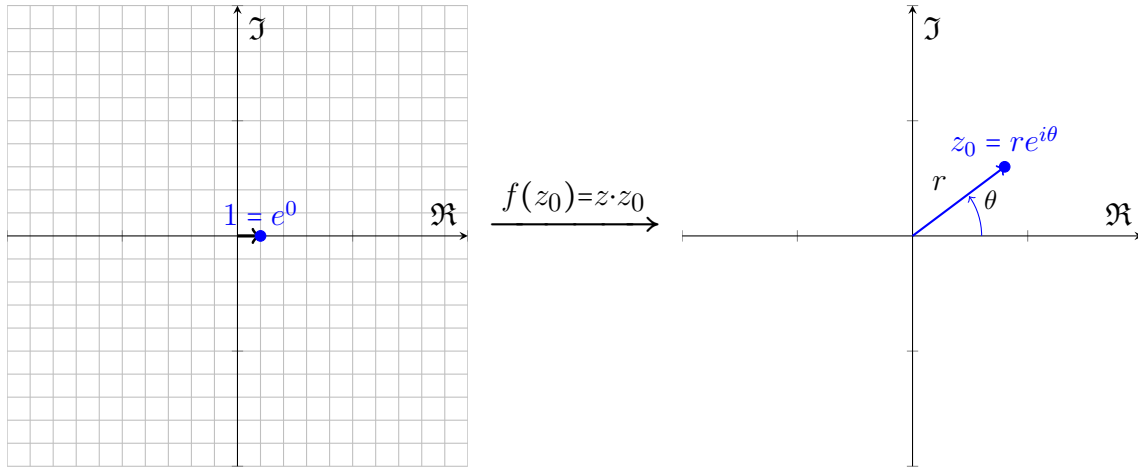
10.6 Operations as Transformations

Consider any $z \in \mathbb{C}$. A function $f : \mathbb{C} \rightarrow \mathbb{C}$ can be viewed as transformations of the complex plane.

Example 10.6.1 (Addition as translation) Consider any $z_0 \in \mathbb{C}$, $z_0 = a + bi$ for $a, b \in \mathbb{R}$. Addition by z_0 can be seen as a shift in the complex plane by $a + bi$. (i.e. It takes the origin and shifts it by z_0 .)



Example 10.6.2 (Multiplication as scaling and rotation) Consider any $z_0 \in \mathbb{C}$, $z_0 = re^{i\theta}$. Multiplication by z_0 scales the entire complex plane by r and rotates it by θ . (Imagine rotating and stretching out a net.)



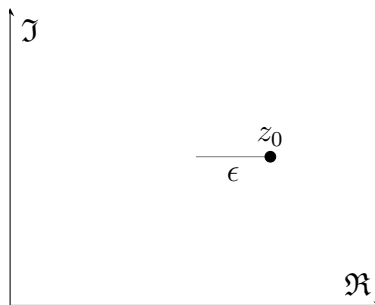
10.7 Complex Analysis Definitions

Definition 10.7.1: Neighbourhood

A neighbourhood of a point z_0 is the set of all points z with distance less than ϵ .

$$\{z : |z - z_0| < \epsilon\}$$

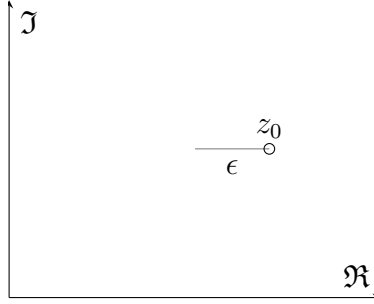
i.e. It is the set of all points that lie within a circle centred at z_0 with radius ϵ . Points on the circumference not included.



Definition 10.7.2: Deleted Neighbourhood

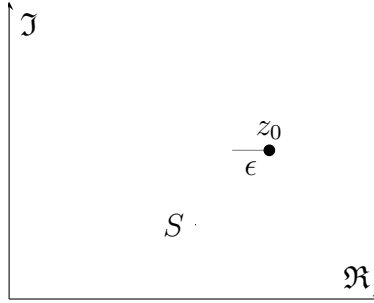
A deleted neighbourhood is the set of all points z with distance less than ϵ from a point z_0 , not including z_0 . That is, it is a neighbourhood of z_0 without z_0 .

$$\{z : |z - z_0| < \epsilon, z \neq z_0\}$$



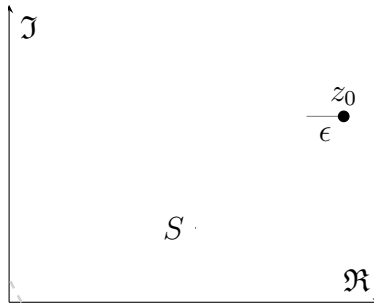
Definition 10.7.3: Interior Point

Let S be a set. A point z_0 is an interior point of S if $\exists \epsilon$ such that $\forall z, |z - z_0| < \epsilon \implies z \in S$. That is, z_0 is an interior point of S if it has a neighbourhood where all points in the neighbourhood are elements of S .



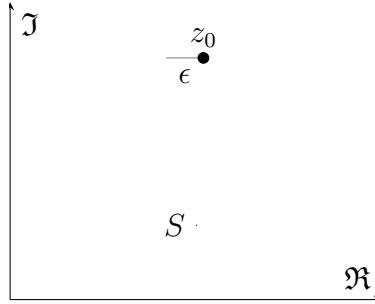
Definition 10.7.4: Exterior Point

Let S be a set. A point z_0 is an exterior point of S if $\exists \epsilon$ such that $\forall z, |z - z_0| < \epsilon \implies z \notin S$. That is, z_0 is an exterior point of S if it has a neighbourhood that does not contain any element of S .



Definition 10.7.5: Boundary Point

Let S be a set. A point z_0 is a boundary point of S if $\forall \epsilon, \exists z \in S, z' \notin S$, such that $|z - z_0| < \epsilon$ and $|z' - z_0| < \epsilon$. That is, for all neighbourhoods of z_0 there exists a point that is in S and a point not in S .



Note: A boundary point of S may or may not be in S .

Definition 10.7.6: Boundary of a Set

A boundary of a set S is the set of all boundary points of S . The set containing all boundary points of S .

$$\{z_0 : \forall \epsilon \exists z \in S, z' \notin S (|z - z_0| < \epsilon \wedge |z' - z_0| < \epsilon)\}$$

Definition 10.7.7: Open Set

A set that does not contain any boundary points.

Theorem 10.7.1:

Set S is open $\iff \forall s \in S, s$ is an interior point of S

Proof: \implies : Suppose S is open $\nRightarrow \forall s \in S, s$ is an interior point of S , for contradiction. That is, $\exists s \in S$ that is either a boundary point or an exterior point. $s \in S$ implies s is not an exterior point of S , so s has to be a boundary point of S . This contradicts that S is an open set.

$$S \text{ is open} \implies \forall s \in S (s \text{ is an interior point of } S)$$

\impliedby :

$$\begin{aligned} & \forall s \in S (s \text{ is an interior point of } S) \\ & \implies \forall s' \forall \epsilon (|s' - s| < \epsilon \implies s' \in S) \\ & \implies S \text{ does not contain boundary points} \implies S \text{ is open} \end{aligned}$$

□

A set can be neither open or closed. Consider the set $S = \{z : 0 < |z| \leq 1\}$. S is not closed since it does not contain the boundary point 0, and it is not open since it contains boundary points where $|z| = 1$. The set \mathbb{C} is both open and closed since it has no boundary points.

Definition 10.7.8: Closed Set

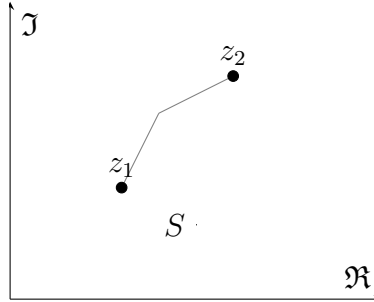
A set that contains all of its boundary points.

Definition 10.7.9: Closure of a Set

Let S be a set. The closure of S is a closed set containing all points of S and all boundary points of S .

Definition 10.7.10: Connected Set

An open set S is connected if $\forall z_1, z_2 \in S$, z_1 and z_2 can be connected by a polygonal line lying within S .

**Definition 10.7.11: Polygonal Line**

A finite set of line segments joined end to end.

Definition 10.7.12: Domain

A nonempty connected set.

Note: All neighbourhoods are domains.

Definition 10.7.13: Region

A domain with none, some, or all of its boundary points.

Definition 10.7.14: Closed Region

A domain with all of its boundary points.

Definition 10.7.15: Bounded Set

A set S is bounded if $\exists R$ such that $\forall s \in S$, $|s| < R$.

Definition 10.7.16: Closed Region

A bounded and closed region.

Definition 10.7.17: Accumulation/Limit Point

A point z_0 is an accumulation point of a set S if all deleted neighbourhood of z_0 contains an element of S .

$$\forall \epsilon \exists s \in S (s \neq z_0 \wedge |z - s| < \epsilon)$$

Note: Unlike a boundary point, an accumulation point does not require that all neighbourhood of z_0 contain an element not in S .

Theorem 10.7.2:

Set S is closed $\iff \forall$ accumulation points z_0 of S , $z_0 \in S$

Proof: \implies : Let S is closed and z_0 is an accumulation point of a set S where $z_0 \notin S$ for contradiction. If $\exists z_0 \notin S$, then z_0 is a boundary point of S . Contradicts closed set contains all boundary points.

\impliedby : Suppose all accumulation points of S are elements of S but S is not closed for contradiction. Then S does not contain one or more boundary points. Suppose z_0 is a boundary point of S that is not in S . Then $\forall \epsilon \exists s \in S$ where $|s - z_0| < \epsilon$, so by considering the deleted neighbourhood of z_0 , this makes z_0 an accumulation point of S . This contradicts that all accumulation points of S is in S . \square

Chapter 11

Analytic Functions

11.1 Functions as mappings

A function $f : S \rightarrow S'$ is a function that maps elements from S to elements on S' . The value of f at z is denoted $f(z)$ and the set S is the domain of f while S' is the image of f . Recall section 10.6, a function can likewise be viewed as a transformation or mapping, that maps $z \in \text{dom}(f) = S$ to values $z' \in \text{img}(f) = S'$.

Definition 11.1.1: Range

Let f be a function with domain S and image S' . The range of f is the entire image of S .

Note: Image is a subset of range, and can be a single point or a set of points.

Definition 11.1.2: Inverse Range

The set of all points $s \in S$ with the value $f(s) = s'$ for some $s' \in S'$.

$$\{s : f(s) = s', s' \in S'\}$$

Note: The domain of a function is often a domain, but it does not need to be a domain.

We will consider functions $f : S \rightarrow S'$ where both $S, S' \subseteq \mathbb{C}$. For such functions we can break it into a two real valued functions:

$$\begin{aligned} f(z) &= u(x, y) + iv(x, y) & \text{dom}(u) \subseteq \mathbb{R}, \text{dom}(v) \subseteq \mathbb{R} \\ &= u(r, \theta) + iv(r, \theta) \end{aligned}$$

Recall that a real-valued function is a function with a domain that is a subset of \mathbb{R} (??). If $\forall z, v(x, y) = 0$, then f is called a real-valued function of a complex variable.

Definition 11.1.3: Polynomial

Let $a_i \in \mathbb{C}$, $0 \leq i \leq n$ where $i, n \in \mathbb{N} \cup \{0\}$. If $a_n \neq 0$, then a polynomial of degree n is

$$P(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n = \sum_{i=0}^n a_iz^i$$

Definition 11.1.4: Rational Functions

Let $P(z)$ and $Q(z)$ are polynomials, then rational functions are quotients:

$$\frac{P(z)}{Q(z)}$$

Defined for all z where $Q(z) \neq 0$.

Definition 11.1.5: Multiple-Valued Function

Let f be a function and $z \in \text{dom}(f)$. f is a multiple-valued function if it assigns more than one value to a point z .

“When multiple-valued functions are studied, usually just one of the possible values assigned at each point is taken, in a systematic manner and a (single-valued) function is constructed from the multiple-valued one” - Brown and Churchill [2]

What this means that for $z \in \mathbb{C}$ a function f assigns $u(z)$ and $v(z)$ to z . By taking just u or v , we create a single-valued function from a multiple-valued function.

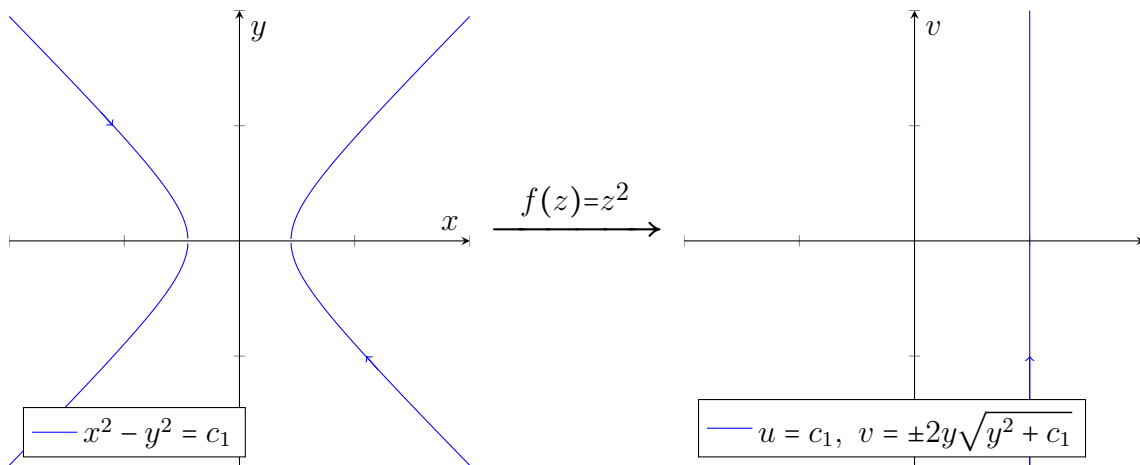
Example 11.1.1 ($f(z) = z^2$)

$$\begin{aligned} f(z) = z^2 &= x^2 - y^2 + i2xy \\ \implies u(x, y) &= x^2 - y^2 \quad v(x, y) = 2xy \end{aligned}$$

By setting $u = x^2 - y^2 = c_1$ where $c_1 \in \mathbb{R}_{>0}$ we can see that

$$u = x^2 - y^2 = c_1 \quad v = 2xy = \pm 2y\sqrt{y^2 + c_1}$$

This tells us that in the complex plane of u and v , if we fix u to a constant c_1 and move along $v = \pm 2y\sqrt{y^2 + c_1}$ by incrementing y we draw out two hyperbolas in the complex plane of x and y . This means that the function $f(z) = z^2$ takes points on hyperbolas the complex plane of x and y and translates them onto a vertical line in the complex plane of u and v where u is a constant.



Likewise if we set $v = c_2$ where $c_2 \in \mathbb{R}_{>0}$, we get:

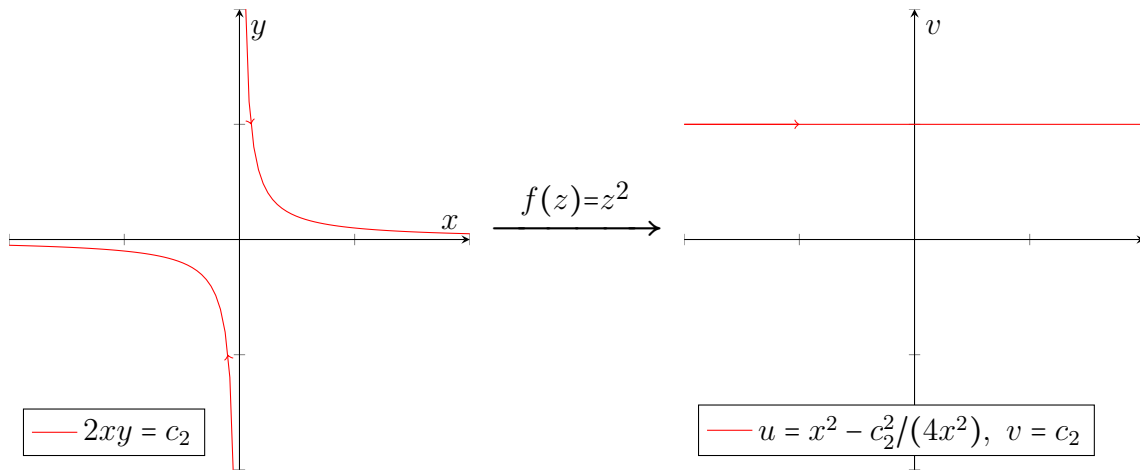
$$u = x^2 - \frac{c_2^2}{4x^2} \qquad v = 2xy = c_2$$

Taking the limits:

$$\lim_{x \rightarrow 0^+} u = -\infty \qquad \lim_{x \rightarrow \infty, x > 0} u = \infty \qquad (11.1)$$

$$\lim_{x \rightarrow -\infty, x < 0} u = \infty \qquad \lim_{x \rightarrow 0^-} u = -\infty \qquad (11.2)$$

Equation 11.1 tells us as x goes from 0 to ∞ , u moves from $-\infty$ to ∞ , which corresponds to the hyperbola in the first quadrant of the xy complex plane. Similarly for equations 11.2.



If we look at f using the polar representation, we get $f(z) = r^2 e^{i2\theta}$. This tells us $\forall r \geq 0$, $r \mapsto r^2 = \rho \geq 0$, and $\forall \theta$, $\theta \mapsto \phi = 2\theta$. It is worth noting that mapping of points between $0 \leq \theta < 2\pi$ is not one-to-one, since points in $0 \leq \theta < \pi$ and points in $\pi \leq \theta < 2\pi$ both get mapped to $0 \leq \phi < 2\pi$.

11.2 Limits

Definition 11.2.1: Limit

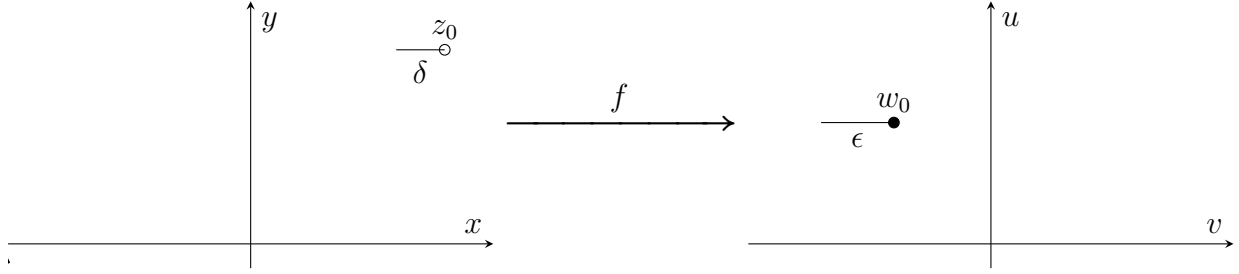
Let $z, z_0, w_0 \in \mathbb{C}$ and f be a function. We say $f(z)$ has limit w_0 as z approaches z_0 if:

$$\forall \epsilon \exists \delta [0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon]$$

We then denote: $\lim_{z \rightarrow z_0} f(z) = w_0$

This tells us that $\lim_{z \rightarrow z_0} f(z) = w_0$ if some deleted neighbourhood $|z - z_0| < \delta$ corresponds to a neighbourhood $|f(z) - w_0| < \epsilon$. Note that the mapping of all points z in $|z - z_0| < \delta$ to $|f(z) - w_0| < \epsilon$ need not be surjective. It just needs to be mapped less than distance ϵ from w_0 .

Note: Definition 11.2.1 allows us to verify if a limit exists, but it is not a method for determining a limit.



Theorem 11.2.1: Uniqueness of Limits

Suppose the limit of f at z_0 exists, then it is unique.

Proof: Suppose two limits of f at z_0 exists for contradiction.

$$\begin{aligned} & [\lim_{z \rightarrow z_0} f(z) = w_0] \wedge [\lim_{z \rightarrow z_0} f(z) = w_1] \\ \implies & [0 < |z - z_0| < \delta_0 \implies |f(z) - w_0| < \epsilon] \wedge [0 < |z - z_0| < \delta_0 \implies |f(z) - w_1| < \epsilon] \end{aligned}$$

$$\begin{aligned} w_1 - w_0 &= [f(z) - w_0] + [w_1 - f(z)] \\ \implies |w_1 - w_0| &= |[f(z) - w_0] + [w_1 - f(z)]| \leq |f(z) - w_0| + |f(z) - w_1| \end{aligned}$$

Now choosing $\delta = \min\{\delta_1, \delta_2\}$, we get:

$$|w_1 - w_0| < \epsilon + \epsilon = 2\epsilon$$

Choosing ϵ to be arbitrary small, we end up with:

$$w_1 - w_0 = 0 \implies w_1 = w_0$$

□

Definition 11.2.1 requires that f be defined at all points in the deleted neighbourhood of z_0 . That is, z_0 is interior to the region which f is defined. We can extend the definition by agreeing that $0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon$ also holds for z that lie in the region where f is defined and the deleted neighbourhood of z_0 . That is $f(z_0)$ need not be defined for a limit at z_0 to exist.

Example 11.2.1 Show $(f(z) = iz/2) \wedge (|z| < 1) \implies \lim_{z \rightarrow 1} f(z) = i/2$.

We can see that we have restricted the domain of f to the region $|z| < 1$, this puts $z = 1$ right at the boundary of the domain of definition of f .

$$\begin{aligned} |z| < 1 &\implies \left| f(z) - \frac{i}{2} \right| = \left| \frac{iz}{2} - \frac{i}{2} \right| = \frac{|z - 1|}{2} \\ &\implies \forall z \forall \epsilon \exists \delta \left[0 < |z - 1| < \delta = 2\epsilon \implies \left| f(z) - \frac{i}{2} \right| < \epsilon \right] \\ &\implies \lim_{z \rightarrow 1} f(z) = \frac{i}{2} \end{aligned}$$

This highlights the fact that if the limit exists, then z is allowed to approach z_0 from any arbitrary direction.

Example 11.2.2 *Limit of $f(z) = z/\bar{z}$ does not exist at $z = 0$*

Consider $\lim_{z \rightarrow 0} f(z)$. Let us approach the limit from the x -axis and the y -axis.

$$\lim_{z=(x,0) \rightarrow 0} f(z) = \frac{x+i0}{x-i0} = 1 \qquad \lim_{z=(0,y) \rightarrow 0} f(z) = \frac{0+iy}{0-iy} = -1$$

We end up with two different limits. As limits are unique, we conclude that $\lim_{z \rightarrow 0} f(z)$ does not exist.

11.2.1 Limit Theorems

Theorem 11.2.2:

Consider $f(z) = u(x, y) + iv(x, y)$. Let $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$.

$$\left[\lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = u_0 \right] \wedge \left[\lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = v_0 \right] \iff \lim_{z \rightarrow z_0} f(z) = w_0$$

Proof: \implies :

By definition:

$$\begin{aligned} & \left[\lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = u_0 \right] \wedge \left[\lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = v_0 \right] \\ & \implies \forall \epsilon \exists \delta_1, \delta_2 \left[\left(0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_1 \implies |u - u_0| < \frac{\epsilon}{2} \right) \right. \\ & \quad \left. \wedge \left(0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_2 \implies |v - v_0| < \frac{\epsilon}{2} \right) \right] \end{aligned} \tag{11.3}$$

Triangle inequality for the distance between points:

$$\begin{aligned} |(u + iv) - (u_0 + iv_0)| &= |(u - u_0) + i(v - v_0)| \leq |u - u_0| + |v - v_0| \\ \sqrt{(x-x_0)^2 + (y-y_0)^2} &= |(x-x_0) + i(y-y_0)| = |(x+iy) - (x_0+iy_0)| \end{aligned}$$

Let $\delta = \min\{\delta_1, \delta_2\}$, it follows from eq. (11.3):

$$0 < |(x+iy) - (x_0+iy_0)| < \delta \implies |(u+iv) - (u_0+iv_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, $\lim_{z \rightarrow z_0} f(z) = w_0$.

\impliedby :

Suppose $\lim_{z \rightarrow z_0} f(z) = w_0$.

$$\begin{aligned} & \lim_{z \rightarrow z_0} f(z) = w_0 \\ & \implies \forall \epsilon \exists \delta > 0 [|(x+iy) - (x_0+iy_0)| < \delta \implies |(u+iv) - (u_0+iv_0)| < \epsilon] \end{aligned} \tag{11.4}$$

By the triangle inequality:

$$\begin{aligned}|u - u_0| &\leq |(u - u_0) + i(v - v_0)| = |(u + iv) - (u_0 + iv_0)| \\ |v - v_0| &\leq |(u - u_0) + i(v - v_0)| = |(u + iv) - (u_0 + iv_0)|\end{aligned}$$

$$|(x + iy) - (x_0 + iy_0)| = |(x - x_0) + i(y - y_0)| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

Thus, it follows from the inequalities in eq. (11.4):

$$\begin{aligned}0 &< \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \\ \implies & [|u - u_0| < \epsilon] \wedge [|v - v_0| < \epsilon] \\ \implies & \left[\lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = u_0 \right] \wedge \left[\lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = v_0 \right]\end{aligned}$$

□

Theorem 11.2.3:

Suppose

$$\left[\lim_{z \rightarrow z_0} f(z) = w_0 \right] \wedge \left[\lim_{z \rightarrow z_0} F(z) = W_0 \right]$$

Then

$$\begin{aligned}\lim_{z \rightarrow z_0} [f(z) + F(z)] &= w_0 + W_0 \\ \lim_{z \rightarrow z_0} [f(z)F(z)] &= w_0 W_0 \\ \lim_{z \rightarrow z_0} \frac{f(z)}{F(z)} &= \frac{w_0}{W_0} \quad W_0 \neq 0\end{aligned}$$

Proof: Let:

$$f(z) = u(x, y) + iv(x, y) \quad F(z) = U(x, y) + iV(x, y)$$

$$z_0 = x_0 + iy_0 \quad w_0 = u_0 + iv_0 \quad W_0 = U_0 + iV_0$$

$$\underline{\lim_{z \rightarrow z_0} [f(z) + F(z)] = w_0 + W_0}$$

From Theorem 11.2.2:

$$\begin{aligned}f(z) + F(z) &= (u + U) + i(v + V) \\ \implies \lim_{(x,y) \rightarrow (x_0,y_0)} f(z)F(z) &= (u_0 + U_0) + i(v_0 + V_0) = w_0 + W_0\end{aligned}$$

$$\underline{\lim_{z \rightarrow z_0} [f(z)F(z)] = w_0 W_0}$$

From Theorem 11.2.2:

$$\begin{aligned} f(z)F(z) &= (uU - vV) + i(vU + uV) \\ \implies \lim_{(x,y) \rightarrow (x_0,y_0)} f(z)F(z) &= (u_0U_0 - v_0V_0) + i(v_0U_0 + u_0V_0) = w_0W_0 \end{aligned}$$

$$\lim_{z \rightarrow z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0} \text{ if } W_0 \neq 0$$

From Theorem 11.2.2:

$$\frac{f(z)}{F(z)} = \frac{u + iv}{U + iV} \implies \lim_{(x,y) \rightarrow (x_0,y_0)} \frac{f(z)}{F(z)} = \frac{u_0 + v_0}{U_0 + iV_0} = \frac{w_0}{W_0}$$

□

Corollary 11.2.3.1:

Let c be a constant, $z, z_0 \in \mathbb{C}$, and $P(z)$ be a polynomial. Then

$$\begin{aligned} \lim_{z \rightarrow z_0} c &= c & \lim_{z \rightarrow z_0} z &= z_0 & \lim_{z \rightarrow z_0} z^n &= z_0^n & n \in \mathbb{N} \\ \lim_{z \rightarrow z_0} P(z) &= P(z_0) \end{aligned}$$

Observation. It is surprisingly quick that Brown and Churchill went from ϵ - δ proofs straight to proving with limits. This is different to the approach in Sequences of Limits Theorem for Sequences Section by Kenneth A. Ross. [1].

Question. It might be possible use a series approach to prove limit theorems for $z \in \mathbb{C}$ by having separate series for x and y (real and imaginary components of z), or a series in the form of $s_n = (x_n, y_n)$. Which would be the proper approach?

11.2.2 Limits of Points at Infinity

Definition 11.2.2: Extended Complex Plane

The complex plane union with the points at infinity:

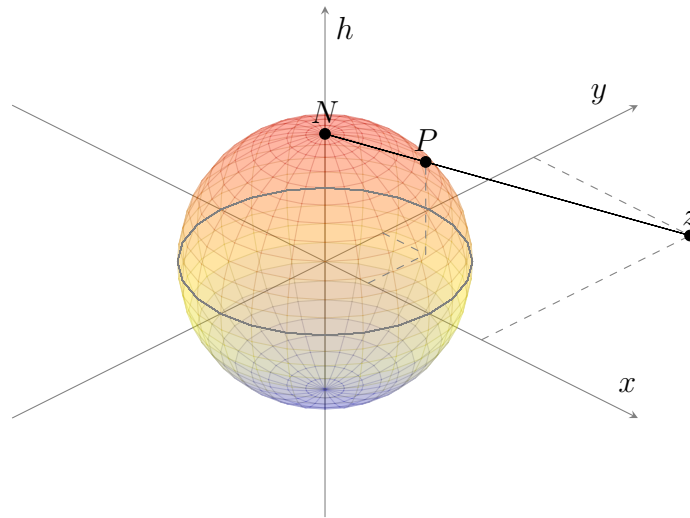
$$\mathbb{C} \cup \{\pm\infty, \pm i\infty\}$$

Definition 11.2.3: Riemann Sphere

A sphere centred at the origin of the complex plane, which is consequently bisected by the complex plane.

Definition 11.2.4: Stereographic Projection

Consider the Riemann Sphere. Let N be the northern point of the sphere (the point on the sphere above the origin of the complex plane) and z be any point in the complex plane. Let l be a line that goes through N and z , then l will intersect the Riemann Sphere. Let P be the point where l intersects the Riemann Sphere. If we let N correspond to the points at infinity, then there is a one-to-one correspondence between points on the sphere and the points on the extended complex plane. This correspondence called the Stereographic Projection.



11.3 Continuity

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Part XXI

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