The Book of Math (Notes)

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Forward and Disclaimer

These are math notes made by a student (with a physics major and math minor) based off text books. It may contain misconceptions and misinterpretations, thus should not be viewed in the same light of a text book. Use at your own risk and mental sanity.

Symbols

Logic

Name	Symbol	Comment
Exists	$\exists$	There exists at least one
For all	$\forall$	
Not exists	$\nexists$	There does not exist
Exists one	$\exists!$	There only exists one and only one
And	$\wedge$	
Or	$\vee$	Inclusive or
Not	$\neg$	
Logically implies	$\implies$	If
Logically implied by	$\impliedby$	Only if
Logically equivalent	$\iff$	If and only if
Implies	$\longrightarrow$	
Implied by	$\longleftarrow$	
Double Implication	$\longleftrightarrow$	

Set Notation

Name	Symbol	Comment
Empty Set	$\varnothing$	The set that is empty
Natural Numbers	$\mathbb{N}$	Set of natural numbers not containing 0, equivalent to the set of positive integers
Integers	$\mathbb{Z}$	Set of integers
Rational Numbers	$\mathbb{Q}$	
Algebraic Numbers	$\mathbb{A}$	
Real Numbers	$\mathbb{R}$	
Complex Numbers	$\mathbb{C}$	
In	$\in$	
Not in	$\notin$	
Owns	$\ni$	Has an element
Proper Subset	$\subset$	Subset that is not itself
Subset	$\subseteq$	
Superset	$\supset$	Superset that is not itself
Proper Superset	$\supseteq$	

Power set	$\wp$	
Union	$\cup$	
Intersection	$\cap$	
Difference	$\setminus$	

Relationships

Name	Symbol	Comment
Defined	$\doteq$	
Approximate	$\approx$	
Equivalent	$\equiv$	Isomorphic (Group Theory)
Congruent	$\cong$	Homomorphic (Group Theory)
Proportional	$\propto$	

Operators

Name	Symbol	Comment
	$\oplus$	
	$\otimes$	
	$\odot$	
	$\circ$	Convolution
Dagger	$\dagger$	Complex conjugate transpose of a matrix

Arrows

Name	Symbol	Comment
Maps to	$\mapsto$	

Hebrew

Name	Symbol	Comment
Aleph	$\aleph$	Carnality of infinite sets that can be well ordered

Other

Name	Symbol	Comment
Real part	Re	Real part of a number
Imaginary part	Im	Imaginary part of a number

Book Constitution

Intents and Purpose

The goal of this book is to organize mathematical knowledge of topics related to the study of physics or the author's interest. It is meant to be used as a source of for future reference, not as a textbook for students new to the topics. It is a notebook of a student, thus should be treated as one and not as a textbook. At most, it could be used as a study guide along side a textbook. Definitely not as the main source for acquiring knowledge.

Layout and Organization

The book is split into parts each containing a field of study mathematics, or a topic large enough to justify giving it its own part. Each part contains chapters that focuses on a particular topic required to understand the field, with sections dedicated to describing a particular knowledge required for the topic.

As axioms, definitions, theorems, corollary, and proofs are integral and abundant to the study of mathematics, each will have a unique style. Each environment and its styles are displayed as follows:

Axiom 0.1: Axiom name

Example Axiom Axioms are the "ground rules" of the set.

Theorem 0.0.1: Theorem name or citation

Example Theorem An important logical result from the axioms, with proof.

Conjecture 0.0.1: Name of conjecture or citation

Example Conjecture A hypothesis, without proof.

Corollary 0.0.1.1:

Example Corollary An implication as a result of a theorem.

Lemma 0.0.1.1:

Example Lemma Small theorems that build up to a larger theorem.

Proposition 0.0.1.1:

Example Proposition Example proposition.

Proof: Logical deductions that results in a theorem. Proofs I've written will be in grey, which may or may not be correct. □

Definition 0.0.1: Word

Example Definition The definition of a word.

Example 0.0.1 *An example.*

Remark. *Remark A comment by the author in the textbooks used.*

Observation. *Example Observation A remark by me.*

Question. *Example Question A question from me for a mystery to be answered later.*

Contents

Part I

Logic [Empty]

Chapter 1

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Part II

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Resources used in part II

content...

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Chapter 4

Rationals $\mathbb{Q}$

Chapter 5

Constructible

Chapter 6

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Reals $\mathbb{R}$

Chapter 8

Complex $\mathbb{C}$

Part III

Real Analysis [Empty]

Resources used in part III

1. Kenneth A. Ross - Elementary Analysis (2nd Ed.) [1]

Chapter 9

Sequences

Corollary 9.0.0.1:
Absolutely convergent series are convergent.

9.1 Limits

9.1.1 Limit Theorems

9.2 Monotone and Cauchy Sequences

9.3 Subsequences

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10.1 Continuous Functions

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10.2 Uniform Continuity

10.3 Limits of Functions

Chapter 11

Metric Spaces

Part IV

Complex Analysis

Resources used in part IV

Primary:

1. Brown and Churchill - Complex Variables and Applications [2]

Supplement:

1. A. David Wunsch - Complex Variables with Applications [3]

Chapter 12

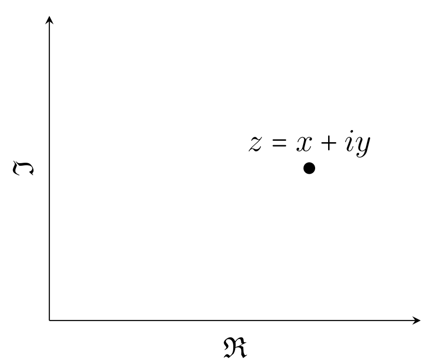
Basics

12.1 Complex Numbers

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}, i = \sqrt{-1}\}$$

Complex numbers are elements of the complex field ($\mathbb{C}$), therefore, they obey all the properties of a field.

We will denote complex numbers by $z = x + iy$ with $x, y \in \mathbb{R}$, and refer the real part as $\mathrm{Re}(z) = \mathrm{Re}(z) = x$ and imaginary part as $\mathrm{Im}(z) = \mathrm{Im}(z) = y$. Complex numbers can also be defined as an ordered pair $z = (x, y)$ which is interpreted as points in the complex plane. $(x, 0)$ are points on the real axis while $(0, y)$ are points in the imaginary axis. This expression is often called a Couple, and was presented in 1833 by mathematician William Rowan Hamilton (1805 - 1865).



Like numbers in $\mathbb{R}$, numbers in $\mathbb{C}$ obey the commutative, distributive, and associative laws. We add and multiply complex numbers in the usual way:

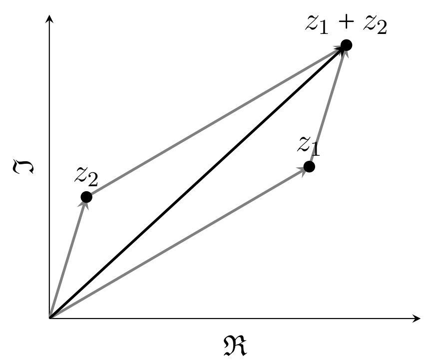
$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \end{aligned} \qquad \begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \end{aligned}$$

$\forall z \in \mathbb{C}$, there is an unique additive inverse $(-z)$ and $\forall z \in \mathbb{C} \setminus \{0\}$, there is an unique multiplicative inverse (z^{-1}) such that

$$z + (-z) = 0 \qquad\qquad zz^{-1} = 1$$
$$\implies -z = -x - iy \qquad \implies (x_1x_2 - y_1y_2) = 1 \wedge (x_1y_2 + x_2y_1) = 0$$
$$\implies z^{-1} = \frac{x_1}{x_1^2 + y_1^2} - i\frac{y_1}{x_1^2 + y_1^2}$$

The existence and uniqueness of the inverses can be easily proven.

The addition of complex numbers may also be interpreted as akin to vector addition.



Note: As a group with addition, $\mathbb{R}^2 \cong \mathbb{C}$, however this is not the case for rings. $\mathbb{C}$ is a field, but $\mathbb{R}^2$ is not. $\mathbb{R}^2$ have non-zero divisors (ie. Take any $a, b \in \mathbb{R}^2$, $(a, 0) \cdot (0, b) = 0$).

12.2 Triangle Inequality

It is not analysis without a section dedicated to the triangle inequality. For any given number $z_1, z_2 \in \mathbb{C}$ it makes no sense to write an inequality $z_1 = a_1 + ib_1 < a_2 + ib_2 = z_2$. Thus, we need have a different notion of size.

Definition 12.2.1: Modulus

The modulus of a complex number is a function $\mathbb{C} \to \mathbb{R}_{>0}$:

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$

It is obvious why the definition is not $|z| = \sqrt{x^2 + (iy)^2}$ as problems arise when $x = y$. The modulus is the distance of z from $(0, 0)$. $\bar{z}$ is the complex conjugate of z, which is explored in section 12.5

Theorem 12.2.1: Triangle Inequality

$\forall z_1, z_2 \in \mathbb{C}[|z_1 + z_2| \leq |z_1| + |z_2|]$

From the theorem, we can derive a similar inequality:

$$|z_1| = |z_1 + z_2 - z_2| \leq |z_1 + z_2| + |-z_2| \implies |z_1| - |z_2| \leq |z_1 + z_2|$$

An important property of polynomials is observed when theorem 12.2.1 is applied to polynomials.

Corollary 12.2.1.1:

Consider the polynomial $P(z)$ where $a_n \in \mathbb{C}$, $n \in \mathbb{N}$, $a_0 \neq 0$, and $z \in \mathbb{C}$.

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

Then $\forall z, \exists R \in \mathbb{R}_{>0}, |z| < R$ such that

$$\left|\frac{1}{P(z)}\right| < \frac{2}{|a_n| R^n}$$

Proof: Consider

$$\begin{aligned}
w = \frac{P(z)}{z_n} - a_n = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \ldots + \frac{a_{n-1}}{z} & \qquad z \neq 0 \\
\implies wz^n = a_0 + a_1 z + \ldots + a_{n-1} z^{n-1} & \\
\implies |w||z|^n \leq |a_0| + |a_1||z| + \ldots + |a_{n-1}||z|^{n-1} & \\
\implies |w| \leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \ldots + \frac{|a_{n-1}|}{|z|} & \\
\implies |w| < n\frac{|a_n|}{2n} = \frac{|a_n|}{2} & \qquad \exists \text{ sufficiently large } R < |z| \text{ s.t.} \\
& \qquad \forall m,\ 0 \leq m \leq n-1,\ \frac{|a_m|}{|z|^{n-m}} < \frac{|a_n|}{2n} \\
\implies |a_n + w| \geq ||a_n| - |w|| > \frac{|a_n|}{2} & \qquad R < |z| \\
\implies |P_n(z)| = |a_n + w||z|^n > \frac{|a_n|}{2}|z|^n > \frac{|a_n|}{2} R^n & \qquad R < |z| \\
\implies \left|\frac{1}{P(z)}\right| < \frac{2}{|a_n| R^n} &
\end{aligned}$$

□

This tells us that if z is a solution to a polynomial $P(z)$, then the reciprocal of the polynomial $1/P(z)$ is bounded above by $R = |z|$. (i.e. It is bounded by a circle of radius $|z|$.)

12.3 Polar and Exponential Form

Definition 12.3.1: Argument of z

Consider any $z \in \mathbb{C}$ where $z \neq 0$. Let θ be the angle in radians between z and the real axis . Then $\forall n \in \mathbb{N}$, $-\pi < \theta \leq \pi$, the argument of z:

$$\arg(z) = \theta + 2n\pi$$

We know $\forall n \in \mathbb{N}$, $\theta + 2\pi n = \theta$. This leads us to the definition of the principal argument of z.

Definition 12.3.2: Principal Argument of z

Consider any $z \in \mathbb{C}$ where $z \neq 0$. Let θ be the angle in radians between z and the real axis. Then for $-pi < \theta \leq \pi$, the principal argument of z:

$$\text{Arg}(z) = \theta$$

It is clear that $\arg(z) = \text{Arg}(z) + 2n\pi$. It is common for the principal argument to be defined $-\pi < \theta \leq \pi$, although other definitions use $0 \leq \theta < 2\pi$.

Definition 12.3.3: Polar Form of z

Consider $z \in \mathbb{C}$. Let $r = |z|$, and $\theta = \arg(z)$. Then $\forall z \in \mathbb{C}, z \neq 0$:

$$z = x + iy = r(\cos(\theta) + i\sin(\theta))$$

Notice that all three definitions require that $z \neq 0$ as θ is undefined at $z = 0$.

Theorem 12.3.1: Euler's Formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

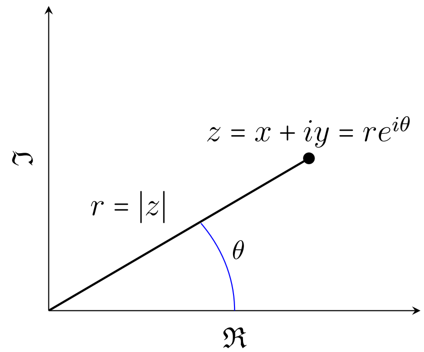
Combining definition 12.3.3 with theorem 12.3.1, we obtain the Exponential Form of z:

Definition 12.3.4: Exponential Form of z

Consider any $z \in \mathbb{C}$, and let $r = |z|$ and $\theta = \text{Arg}(z)$. Then the exponential form of z:

$$z = re^{i\theta}$$

Note: $\theta = \tan^{-1}(y/x)$ and $r = \sqrt{x^2 + y^2}$.



12.3.1 Properties of Polar and Exponential Form

It would be easier to work with the exponential form of z then convert it to the polar form later. The exponential form of a complex number is part of the exponential family of functions, thus possess all the properties of the family. Consider any complex number $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$.

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} \qquad\qquad z^n = r^n e^{in\theta} \quad \forall n \in \mathbb{Z}$$

A special case arrives for integer exponential of z on the unit circle.

Theorem 12.3.2: de Moivre's Formula

Consider any $z = e^{i\theta} \in \mathbb{C}$ on the unit circle, and let $n \in \mathbb{Z}$.

$$\forall z \in \mathbb{C}\ \forall n \in \mathbb{Z}\left[|z| = 1 \implies (\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)\right]$$

Proof: Consider $z = e^{i\theta}$ and let $n \in \mathbb{Z}$.

$$z^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

□

The proof hints that theorem 12.3.2 can be generalized to $\forall n \in \mathbb{R}$, which we will see shortly in section 12.4. Using theorem 12.3.2, we can obtain the double angle identities.

Corollary 12.3.2.1: Double Angle Identities

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \qquad\qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

Proof: Consider any z on the unit circle, that is $z = e^{i\theta}$.

$$\begin{aligned} &(\cos(\theta) + i\sin(\theta))^2 = \cos(2\theta) + i\sin(2\theta) && \text{Theorem 12.3.2} \\ \implies\ &\cos^2(\theta) - \sin^2(\theta) + i2\sin(\theta)\cos(\theta) = cos(2\theta) + i\sin(2\theta) && \end{aligned}$$

Equating the real and imaginary parts yield the desired results. □

12.3.2 Properties of Arguments

Recall from section 12.3.1:

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} \qquad\qquad z^n = r^n e^{in\theta} \quad \forall n \in \mathbb{Z}$$

The arguments for the arguments of products of any $z_1, z_2 \in \mathbb{C}$ follows immediately from the properties of the exponential.

Corollary 12.3.2.2: Arguments of Products

$$\begin{aligned} \arg(z_1 z_2) &= \arg(z_1) + \arg(z_2) \\ \arg(z^n) &= n \arg(z) \end{aligned} \qquad\qquad \begin{aligned} \operatorname{Arg}(z_1 z_2) &= \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) \\ \operatorname{Arg}(z^n) &= n \operatorname{Arg}(z) \end{aligned}$$

Proof:

$$\begin{aligned} z_1 z_2 &= r_1 r_2 e^{i(\theta_1+\theta_2)} \\ &\implies \arg(z_1 z_2) = \arg(z_1) + 2n_1\pi + \arg(z_2) + 2n_2\pi && n_1, n_2 \in \mathbb{Z} \\ &\implies \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \\ &\implies \operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) = \operatorname{Arg}(z_2) \end{aligned}$$

$$\begin{aligned} z^n &= r^n e^{in\theta} \\ &\implies \arg(z^n) = n \arg(z) + 2n\pi && n \in \mathbb{Z} \\ &\implies \arg(z^n) = n \arg(z) \\ &\implies \mathrm{z}^{\mathrm{n}} = n \operatorname{Arg}(z) \end{aligned}$$

□

It is clear that:

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) \qquad\qquad \operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \operatorname{Arg}(z_1) - \operatorname{Arg}(z_2)$$

12.4 Roots of z

In definition 12.3.4, you might be wondering why $z^n = r^n e^{in\theta}$ is not for $n \in \mathbb{R}$. That is because there is more things to consider, which we will explore in this section. Recall that $z = re^(i\theta) = re^{i(\theta+2n\pi)}$ for $n \in \mathbb{Z}$.

Definition 12.4.1: Exponential of z

Consider any $z \in \mathbb{C}$ and any $x \in \mathbb{R}$

$$z^x = \left(re^{i(\theta+2n\pi)}\right)^x = r^x e^{ix(\theta+2n\pi)}$$

For $x \notin \mathbb{Z}$, it is clear that $z^x = r^x e^{ix(\theta+2n\pi)} \neq r^x e^{ix\theta}$, since $2nx\pi = 0 \iff nx \in \mathbb{Z}$. In order to define the roots of z we must need a more general and proper definition of z.

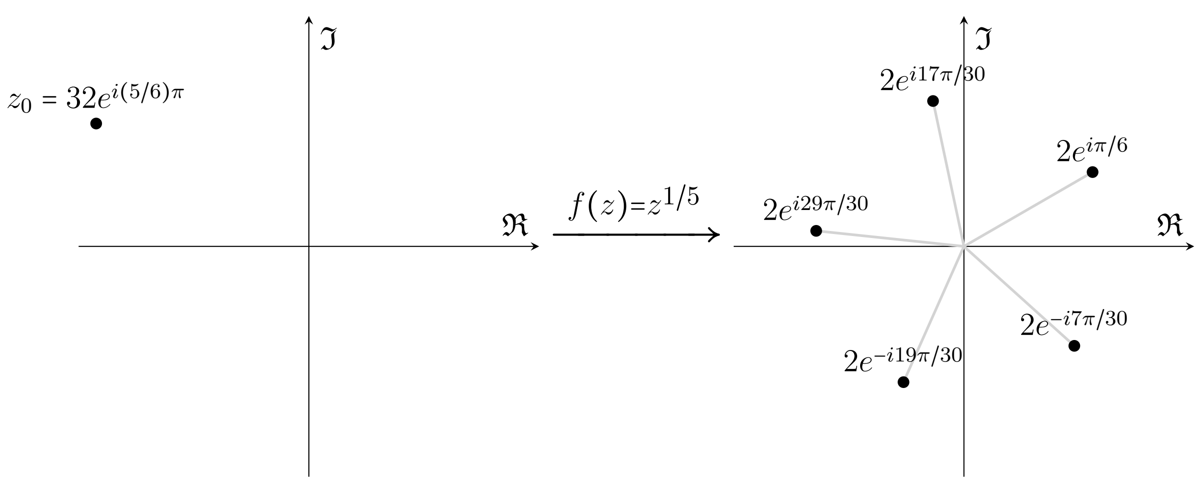
Definition 12.4.2: Roots of z_0

Consider any $z_0 \in \mathbb{C}$ and any $m \in \mathbb{N}$.

$$z_0^{\frac{1}{m}} = r_0^{\frac{1}{m}} e^{i\left(\frac{\theta_0+2n\pi}{m}\right)} = r_0^{\frac{1}{m}} e^{i\left(\frac{\theta_0}{m}+\frac{2n\pi}{m}\right)}$$

Taking the m-th root of $z_0 \in C$ scales θ_0 by $1/m$, and provides solutions at equally spaced by $2\pi/m$ on a circle of radius $r^{1/m}$. That is, the roots lie on the vertices of a regular n-sided polygon inscribed in a circle of radius $|z|^{1/m}$.

Example 12.4.1 *Consider $z_0 = 32e^{i(5/6)\pi}$, then $z_0^{(1/5)} = 3e^{i(\pi/6)+i(2/5)n\pi}$ for $n \in \mathbb{Z}$. The radius went from 35 to $35^{(1/5)} = 2$, and five roots appear equally spaced with distance of $(2/5)\pi$ on a circle with radius 2. Before and after graphs are as follows, note graph on right is zoomed in:*



We can see that the roots of z_0 form a set:

Definition 12.4.3: Set of roots of z_0

Consider the m-th root of any $z_0 \in \mathbb{C}$. Let:

$$z_0 = r_0 e^{i\theta_0} \qquad c_0 = r_0^{1/m} e^{i\theta_0/m} \qquad \omega_n = e^{\frac{i2\pi}{m}} \qquad m \in \mathbb{N}$$

Then the set of roots of z_0:

$$z_0^{1/m} = \left\{c_k = c_0\omega_m^k \mid k \in \mathbb{N},\ 0 \leq k < m\right\}$$

c_0 is the principal root. The root corresponding to the principal argument of z.

Definition 12.4.4: Principal Root

Consider the m-th root of any $z_0 \in \mathbb{C}$. The principal root of z_0 is defined as:

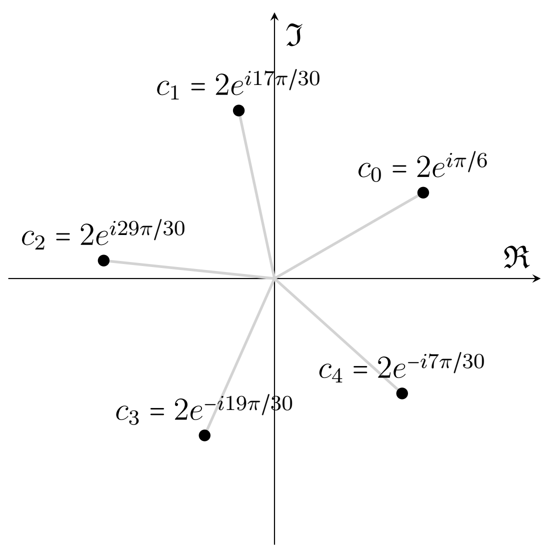
$$c_0 = r_0^{\frac{1}{m}} e^{i\frac{\theta_0}{m}}$$

Example 12.4.2 *Recall from the previous example: $z_0 = 32e^{i(5/6)\pi}$. This gives us*

$$c_0 = 32^{1/5}e^{i\pi/6} = 2e^{i\pi/6} \qquad \omega_5 = e^{i2\pi/5}$$

Then

$$\begin{aligned}
c_0 &= c_0\omega_5^0 = 2e^{i\pi/6} \\
c_1 &= c_0\omega_5^1 = 2e^{i\pi/6}e^{i2\pi/5} = 2e^{i17\pi/30} \\
c_2 &= c_0\omega_5^1 = 2e^{i\pi/6}e^{i4\pi/5} = 2e^{i29\pi/30} \\
c_3 &= c_0\omega_5^1 = 2e^{i\pi/6}e^{i6\pi/5} = 2e^{i41\pi/30} = 2e^{-i19\pi/30} \\
c_4 &= c_0\omega_5^1 = 2e^{i\pi/6}e^{i8\pi/5} = 2e^{i53\pi/30} = 2e^{-i7\pi/30}
\end{aligned}$$



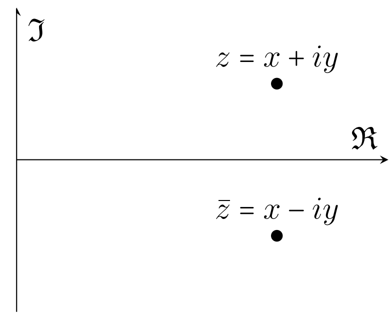
12.5 Complex Conjugate

Definition 12.5.1: Complex Conjugate

The complex conjugate of $z \in \mathbb{C}$ is denoted $\bar{z}$.

$$\bar{z} = x - iy = r(\cos(\theta) - i\sin(\theta)) = re^{-i\theta}$$

Graphically, it is the reflection of z across the real axis.



It is then easy to see

$$\mathrm{Re}(z) = \frac{z + \bar{z}}{2} \qquad \mathrm{Im}(z) = \frac{z - \bar{z}}{2i} \qquad |z|^2 = z\bar{z}$$

As $\mathrm{Re}(z) = x = r\cos(\theta)$ and $\mathrm{Im}(z) = y = r\sin(\theta)$ and using definition 12.3.4, we can obtain the complex forms of sine and cosine:

Definition 12.5.2: Complex Sine and Cosine

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

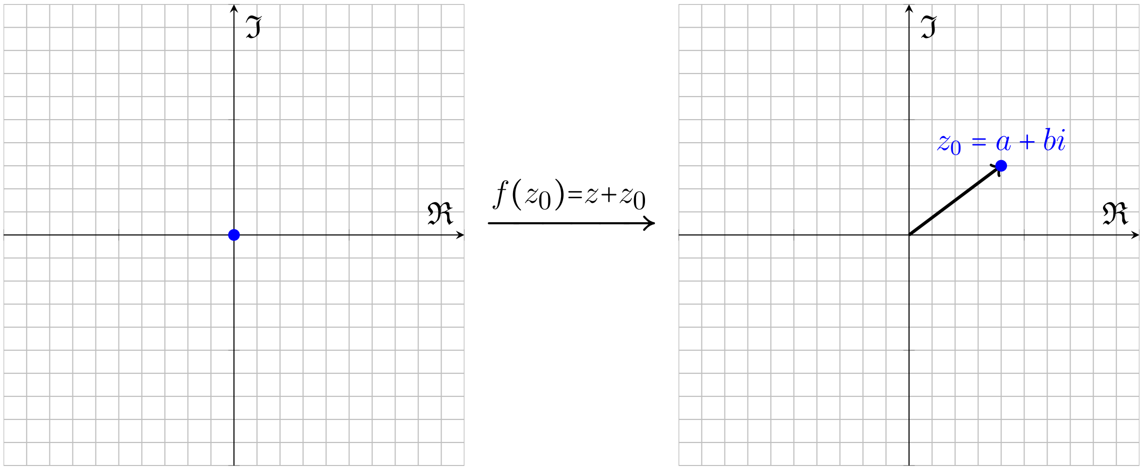
It is easy to prove $\forall z_1, z_2 \in \mathbb{C}$:

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \qquad \overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$$

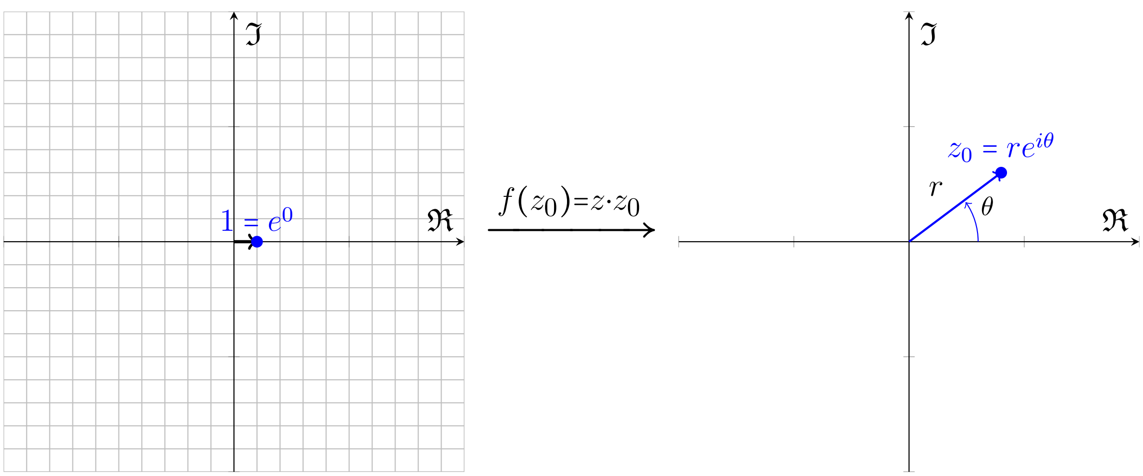
12.6 Operations as Transformations

Consider any $z \in \mathbb{C}$. A function $f : \mathbb{C} \to \mathbb{C}$ can be viewed as transformations of the complex plane.

Example 12.6.1 (Addition as translation) *Consider any $z_0 \in \mathbb{C}$, $z_0 = a + ib$ for $a, b \in \mathbb{R}$. Addition by z_0 can be seen as a shift in the complex plane by $a + bi$. (i.e. It takes the origin and shifts it by z_0.)*



Example 12.6.2 (Multiplication as scaling and rotation) *Consider any $z_0 \in \mathbb{C}$, $z_0 = re^{i\theta}$. Multiplication by z_0 scales the entire complex plane by r and rotates it by θ. (Imagine rotating and stretching out a net.)*



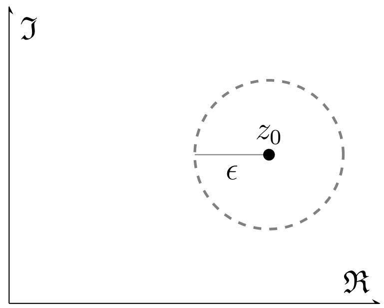
12.7 Complex Analysis Definitions

Definition 12.7.1: Neighbourhood

A neighbourhood of a point z_0 is the set of all points z with distance less than ϵ.

$$\{z : |z - z_0| < \epsilon\}$$

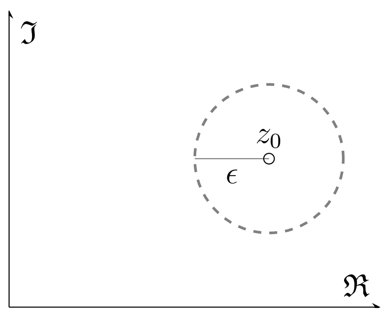
i.e. It is the set of all points that lie within a circle centred at z_0 with radius ϵ. Points on the circumference not included.



Definition 12.7.2: Deleted Neighbourhood

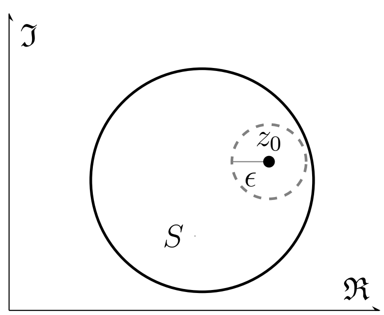
A deleted neighbourhood is the set of all points z with distance less than ϵ from a point z_0, not including z_0. That is, it is a neighbourhood of z_0 without z_0.

$$\{z : |z - z_0| < \epsilon,\ z \neq z_0\}$$



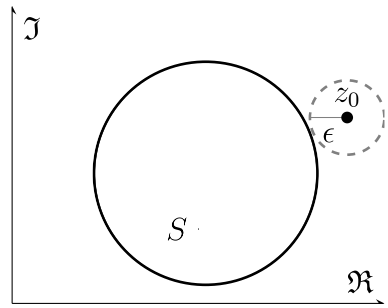
Definition 12.7.3: Interior Point

Let S be a set. A point z_0 is an interior point of S if $\exists \epsilon$ such that $\forall z$, $|z - z_0| < \epsilon \implies z \in S$. That is, z_0 is an interior point of S if it has a neighbourhood where all points in the neighbourhood is an element of S.



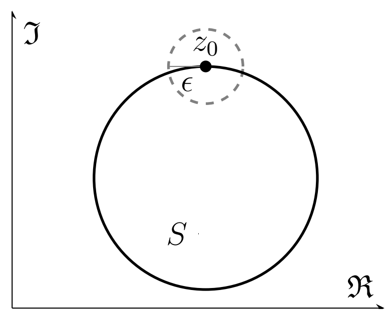
Definition 12.7.4: Exterior Point

Let S be a set. A point z_0 is an exterior point of S if $\exists \epsilon$ such that $\forall z$, $|z - z_0| < \epsilon \implies z \notin S$. That is, $z_0$is an exterior point of S if it has neighbourhood that does not contain any element of S.



Definition 12.7.5: Boundary Point

Let S be a set. A point z_0 is a boundary point of S if $\forall\epsilon$, $\exists z \in S, z' \notin S$, such that $|z - z_0|\epsilon$ and $|z' - z_0| < \epsilon$. That is, for all neighbourhoods of z_0 there exists a point that is in S and a point not in S.



Note: A boundary point of S may or may not be in S.

Definition 12.7.6: Boundary of a Set

A boundary of a set S is the set of all boundary points of S. The set containing all boundary points of S.

$$\{z_0 : \forall\epsilon\exists z \in S, z' \notin S(|z - z_0| < \epsilon \wedge |z' - z_0| < \epsilon)\}$$

Definition 12.7.7: Open Set

A set that does not contain any boundary points.

Theorem 12.7.1:

Set S is open $\iff$ $\forall s \in S$, s is an interior point of S

Proof: $\underline{\implies}$: Suppose S is open $\not\Rightarrow$ $\forall s \in S$, s is an interior point of S, for contradiction. That is, $\exists s \in S$ that is either a boundary point or an exterior point. $s \in S$ implies s is not an exterior point of S, so s has to be a boundary point of S. This contradicts that S is an open set.

$$S \text{ is open } \implies \forall s \in S(s \text{ is an interioir point of } S)$$

$\underline{\Longleftarrow:}$

$$\begin{aligned}&\forall s \in S(s \text{ is an interior point of S})\\ &\implies \forall s' \forall \epsilon(|s'-s| < \epsilon \implies s' \in S)\\ &\implies S \text{ does not contain boundary points} \implies S \text{ is open}\end{aligned}$$

$\square$

A set can be neither open or closed. Consider the set $S = \{z : 0 < |z| \leq 1\}$. S is not closed since it does not contain the boundary point 0, and it is not open since it contains boundary points where $|z| = 1$. The set $\mathbb{C}$ is both open and closed since it has no boundary points.

Definition 12.7.8: Closed Set

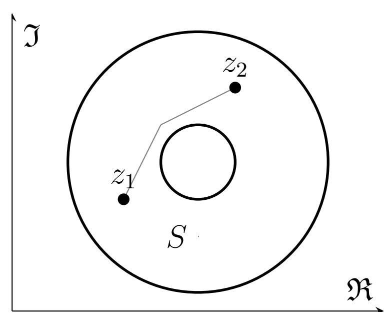
A set that contains all of its boundary points.

Definition 12.7.9: Closure of a Set

Let S be a set. The closure of S is a closed set containing all points of S and all boundary points of S.

Definition 12.7.10: Connected Set

An opens set S is connected if $\forall z_1, z_2 \in S$, z_1 and z_2 can be connected by a polygonal line lying within S.



Definition 12.7.11: Polygonal Line

A finite set of line segments joined end to end.

Definition 12.7.12: Domain

A nonempty connected set.

Note: All neighbourhoods are domains.

Definition 12.7.13: Region

A domain with none, some, or all of its boundary points.

Definition 12.7.14: Bounded Set/Region

A set S is bounded if $\exists R = |z| > 0$ such that $\forall s \in S$, $|s| < R$. That is, S is bounded if $\forall s \in S$, s is contained in some circle of radius R centred at the origin.

Definition 12.7.15: Closed Region

A domain with all its boundary points. A bounded and closed region.

Definition 12.7.16: Accumulation/Limit Point

A point z_0 is a accumulation point of a set S if all deleted neighbourhood of z_0 contains an element of S.

$$\forall \epsilon \exists s \in S(s \neq z_0 \wedge |z - s| < \epsilon)$$

Note: Unlike a boundary point, an accumulation point does not require that all neighbourhood of z_0 contain an element not in S.

Theorem 12.7.2:

Set S is closed $\iff$ $\forall$ accumulation points z_0 of S, $z_0 \in S$

Proof: $\underline{\implies:}$ Let S is closed and z_0 is an accumulation point of a set S where $z_0 \notin S$ for contradiction. If $\exists z_0 \notin S$, then z_0 is a boundary point of S. Contradicts closed set contains all boundary points.

$\underline{\impliedby:}$ Suppose all accumulation points of S are elements of S but S is not closed for contradiction. Then S does not contain one or more boundary points. Suppose z_0 is a boundary point of S that is not in S. Then $\forall \epsilon\ \exists s \in S$ where $|s - z_0| < \epsilon$, so by considering the deleted neighbourhood of z_0, this makes z_0 an accumulation point of S. This contradicts that all accumulation points of S is in S. $\square$

Chapter 13

Analytic Functions

13.1 Functions as mappings

A function $f : S \to S'$ is a function that maps elements from S to elements on S'. The value of f at z is denoted $f(z)$ and the set S is the domain of f while S' is the image of f. Recall section 12.6, a function can likewise be viewed as a transformation or mapping, that maps $z \in \text{dom}(f) = S$ to values $z' \in \text{img}(f) = S'$.

Definition 13.1.1: Range

Let f be a function with domain S and image S'. The range of f is the entire image of S.

Note: Image is a subset of range, and can be a single point or a set of points.

Definition 13.1.2: Inverse Range

The set of all points $s \in S$ with the value $f(s) = s'$ for some $s' \in S'$.

$$\{s : f(s) = s',\ s' \in S'\}$$

Note: The domain of a function is often a domain, but it does not need to be a domain.

We will consider functions $f : S \to S'$ where both $S, S' \subseteq \mathbb{C}$. For such functions we can break it into a two real valued functions:

$$\begin{aligned} f(z) &= u(x,y) + iv(x,y) & \qquad \text{dom}(u) \subseteq \mathbb{R}, \text{dom}(v) \subseteq \mathbb{R} \\ &= u(r,\theta) + iv(r,\theta) \end{aligned}$$

Recall that a real-valued function is a function with a domain that is a subset of $\mathbb{R}$ (**??**). If $\forall z,\ v(x,y) = 0$, then f is called a real-valued function of a complex variable.

Definition 13.1.3: Polynomial

Let $a_i \in \mathbb{C}$, $0 \le i \le n$ where $i, n \in \mathbb{N} \cup \{0\}$. If $a_n \neq 0$, then a polynomial of degree n is

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n = \sum_{i=0}^{n} a_i z^i$$

Definition 13.1.4: Rational Functions

Let $P(z)$ and $Q(z)$ are polynomials, then rational functions are quotients:

$$\frac{P(z)}{Q(z)}$$

Defined for all z where $Q(z) \neq 0$.

Definition 13.1.5: Multiple-Valued Function

Let f be a function and $z \in \text{dom}(f)$. f is a multiple-valued function if it assigns more than one value to a point z.

"When multiple-valued functions are studied, usually just one of the possible values assigned at each point is taken, in a systematic manner and a (single-valued) function is constructed from the multiple-valued one" - Brown and Churchill [2]

What this means that for $z \in \mathbb{C}$ a function f assigns $u(z)$ and $v(z)$ to to z. By taking just u or v, we create a single-valued function from a multiple-valued function.

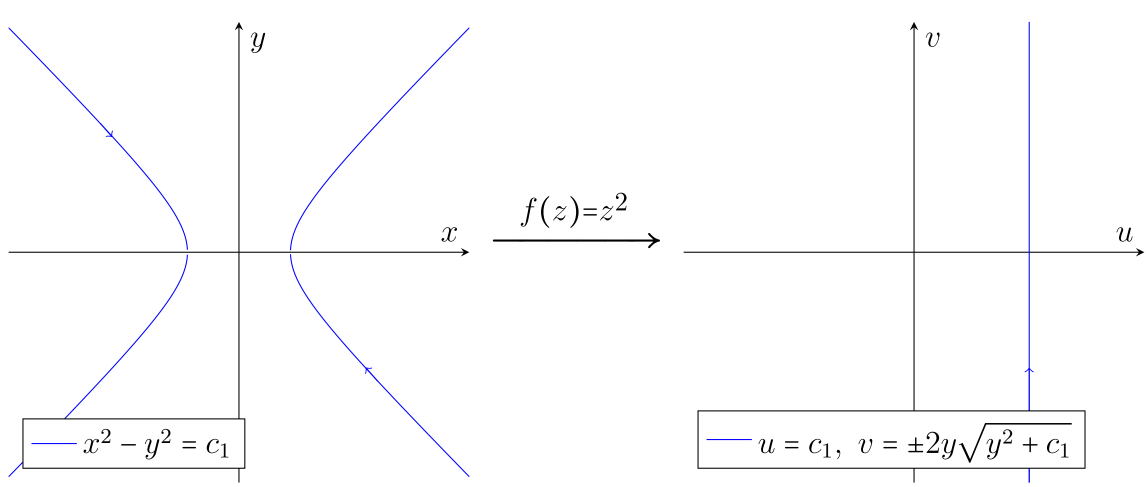
Example 13.1.1 ($f(z) = z^2$)

$$\begin{aligned} f(z) = z^2 = x^2 - y^2 + i2xy \\ \implies u(x, y) = x^2 - y^2 \qquad v(x, y) = 2xy \end{aligned}$$

By setting $u = x^2 - y^2 = c_1$ where $c_1 \in \mathbb{R}_{>0}$ we can see that

$$u = x^2 - y^2 = c_1 \qquad\qquad v = 2xy = \pm 2y\sqrt{y^2 + c_1}$$

This tells us that in the complex plane of u and v, if we fix u to a constant c_1 and move along $v = \pm 2y\sqrt{y^2 + c_1}$ by incrementing y we draw out two hyperbolas in the complex plane of x and y. This means that the function $f(z) = z^2$ takes points on hyperbolas the complex plane of x and y and translates them onto a vertical line in the complex plane of u and v where u is a constant.



Likewise if we set $v = c_2$ where $c_2 \in \mathbb{R}_{>0}$, we get:

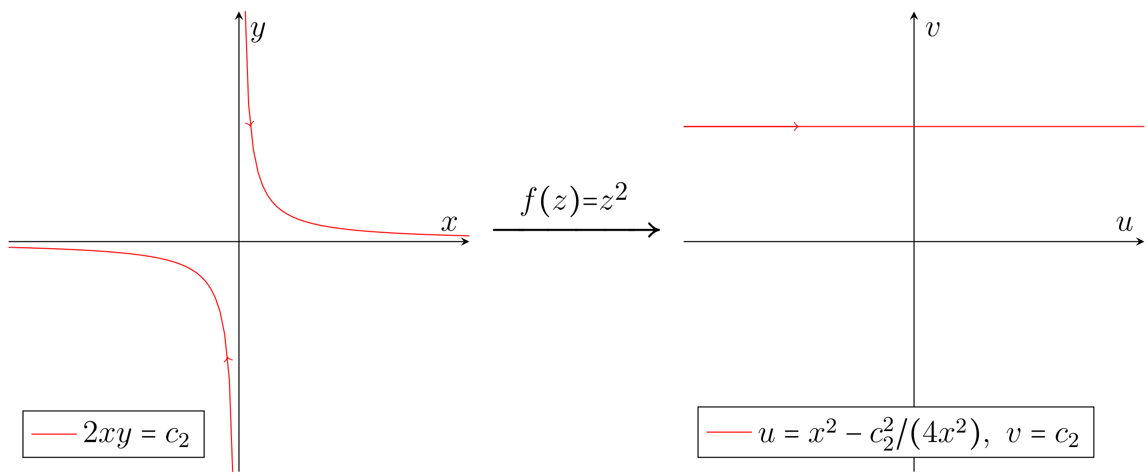
$$u = x^2 - \frac{c_2^2}{4x^2} \qquad\qquad v = 2xy = c_2$$

Taking the limits:

$$\lim_{x\to 0^+} u = -\infty \qquad\qquad \lim_{x\to\infty, x>0} u = \infty \tag{13.1}$$

$$\lim_{x\to -\infty, x<0} u = \infty \qquad\qquad \lim_{x\to 0^-} u = -\infty \tag{13.2}$$

Equation 11.1 tells us as x goes from 0 to ∞, u moves from $-\infty$ to ∞, which corresponds to the hyperbola in the first quadrant of the xy complex plane. Similarly for equations 11.2.



If we look at f using the polar representation, we get $f(z) = r^2 e^{i2\theta}$. This tells us $\forall r \geq 0$, $r \mapsto r^2 = \rho \geq 0$, and $\forall \theta$, $\theta \mapsto \phi = 2\theta$. It is worth noting that mapping of points between $0 \leq 0 < 2\pi$ is not one-to-one, since points in $0 \leq \theta < \pi$ and points in $\pi \leq \theta < 2\pi$ both get mapped to $0 \leq \phi < 2\pi$.

13.2 Limits

Definition 13.2.1: Limit

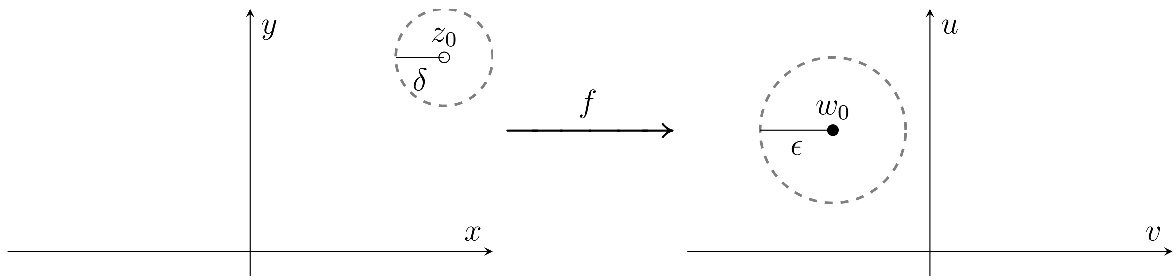
Let $z, z_0, w_0 \in \mathbb{C}$ and f be a function. We say $f(z)$ has limit w_0 as z approaches z_0 if:

$$\forall \epsilon \exists \delta [0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon]$$

We then denote: $\lim_{z\to z_0} f(z) = w_0$

This tells us that $\lim_{z\to z_0} f(z) = w_0$ if some deleted neighbourhood $|z - z_0| < \delta$ corresponds to a neighbourhood $|f(z) - w_0| < \epsilon$. Note that the mapping of all points z in $|z - z_0| < \delta$ to $|f(z) - w_0| < \epsilon$ need not be subjective. It just needs to be mapped less than distance ϵ from w_0.

Note: Definition 13.2.1 allows us to verify if a limit exists, but it is not a method for determining a limit.



Theorem 13.2.1: Uniqueness of Limits

Suppose the limit of f at z_0 exists, then it is unique.

Proof: Suppose two limits of f at z_0 exists for contradiction.

$$[\lim_{z \to z_0} f(z) = w_0] \wedge [\lim_{z \to z_0} f(z) = w_1]$$
$$\implies [0 < |z - z_0| < \delta_0 \implies |f(z) - w_0| < \epsilon] \wedge [0 < |z - z_0| < \delta_0 \implies |f(z) - w_0| < \epsilon]$$

$$w_1 - w_0 = [f(z) - w_0] + [w_1 - f(z)]$$
$$\implies |w_1 - w_0| = |[f(z) - w_0] + [w_1 - f(z)]| \leq |f(z) - w_0| + |f(z) - w_1|$$

Now choosing $\delta = \min\{\delta_1, \delta_2\}$, we get:

$$|w_1 - w_0| < \epsilon + \epsilon = 2\epsilon$$

Choosing ϵ to be arbitrary small, we end up with:

$$w_1 - w_0 = 0 \implies w_1 = w_0$$

□

Definition 13.2.1 requires that f be defined at all points in the deleted neighbourhood of z_0. That is, z_0 is interior to the region which f is defined. We can extend the definition by agreeing that $0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon$ also holds for z that lie in the region where f is defined and the deleted neighbourhood of z_0. That is $f(z_0)$ need not be defined for a limit at z_0 to exist.

Example 13.2.1 *Show* $(f(z) = iz/2) \wedge (|z| < 1) \implies \lim_{z \to 1} f(z) = i/2$.

We can see that we have restricted the domain of f to the region $|z| < 1$, this puts $z = 1$ right at the boundary of the domain of definition of f.

$$\begin{aligned} |z| < 1 &\implies \left| f(z) - \frac{i}{2} \right| = \left| \frac{iz}{2} - \frac{i}{2} \right| = \frac{|z - 1|}{2} \\ &\implies \forall z \forall \epsilon \exists \delta \left[0 < |z - 1| < \delta = 2\epsilon \implies \left| f(z) - \frac{i}{2} \right| < \epsilon \right] \\ &\implies \lim_{z \to 1} f(z) = \frac{i}{2} \end{aligned}$$

This highlights the fact that if the limit exists, then z is allowed to approach z_0 from any arbitrary direction.

Example 13.2.2 *Limit of $f(z) = z/\bar{z}$ does not exist at $z = 0$*

Consider $\lim_{z\to 0} f(z)$. Let us approach the limit from the x-axis and the y-axis.

$$\lim_{z=(x,0)\to 0} f(z) = \frac{x+i0}{x-i0} = 1 \qquad\qquad \lim_{z=(0,y)\to 0} f(z) = \frac{0+iy}{0-iy} = -1$$

We end up with two different limits. As limits are unique, we conclude that $\lim_{z\to 0} f(z)$ does not exist.

13.2.1 Limit Theorems

Theorem 13.2.2:

Consider $f(z) = u(x,y) + iv(x,y)$. Let $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$.

$$\left[\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0\right] \wedge \left[\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0\right] \iff \lim_{z\to z_0} f(z) = w_0$$

Proof: $\underline{\implies:}$

By definition:

$$\begin{aligned} &\left[\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0\right] \wedge \left[\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0\right] \\ &\implies \forall\epsilon\exists\delta_1,\delta_2\Bigg[\left(0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta_1 \implies |u-u_0| < \frac{\epsilon}{2}\right) \\ &\qquad \wedge \left(0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta_2 \implies |v-v_0| < \frac{\epsilon}{2}\right)\Bigg] \end{aligned} \tag{13.3}$$

Triangle inequality for the distance between points:

$$\begin{aligned} |(u+iv)-(u_0-iv_0)| &= |(u-u_0)+i(v-v_0)| \le |u-u_0| + |v-v_0| \\ \sqrt{(x-x_0)^2+(y-y_0)^2} &= |(x-x_0)+i(v-v_0)| = |(x+iy)-(x_0-iy_0)| \end{aligned}$$

Let $\delta = \min\{\delta_1,\delta_2\}$, it follows from eq. (13.3):

$$0 < |(x+iy)-(x_0+iy_0)| < \delta \implies |(u+iv)-(u_0-iv_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, $\lim_{z\to z_0} f(z) = w_0$.

$\underline{\impliedby:}$

Suppose $\lim_{z\to z_0} f(z) = w_0$.

$$\begin{aligned} &\lim_{z\to z_0} f(z) = w_0 \\ &\implies \forall\epsilon\exists\delta > 0[|(x+iy)-(x_0-iy_0)| < \delta \implies |(u+iv)-(u_0+iv_0)| < \epsilon] \end{aligned} \tag{13.4}$$

By the triangle inequality:

$$|u - u_0| \leq |(u - u_0) + i(v - v_0)| = |(u + iv) - (u_0 + iv_0)|$$
$$|v - v_0| \leq |(u - u_0) + i(v - v_0)| = |(u + iv) - (u_0 + iv_0)|$$

$$|(x + iy) - (x_0 + iy_0)| = |(x - x_0) + i(y - y_0)| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

Thus, it follows from the inequalities in eq. (13.4):

$$\begin{aligned}
&0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \\
&\implies [|u - u_0| < \epsilon] \wedge [|v - v_0| < \epsilon] \\
&\implies \left[\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0\right] \wedge \left[\lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0\right]
\end{aligned}$$

$\square$

Theorem 13.2.3:

Suppose

$$\left[\lim_{z\to z_0} f(z) = w_0\right] \wedge \left[\lim_{z\to z_0} F(z) = W_0\right]$$

Then

$$\lim_{z\to z_0}[f(z) + F(z)] = w_0 + W_0$$
$$\lim_{z\to z_0}[f(z)F(z)] = w_0 W_0$$
$$\lim_{z\to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0} \qquad W_0 \neq 0$$

Proof: Let:

$$f(z) = u(x, y) + iv(x, y) \qquad F(z) = U(x, y) + iV(x, y)$$

$$z_0 = x_0 + iy_0 \qquad w_0 = u_0 + iv_0 \qquad W_0 = U_0 + iV_0$$

$\lim_{z\to z_0}[f(z) + F(z)] = w_0 + W_0$

From Theorem 13.2.2:

$$\begin{aligned}
&f(z) + F(z) = (u + U) + i(v + V) \\
&\implies \lim_{(x,y)\to(x_0,y_0)} f(z)F(Z) = (u_0 + U_0) + i(v_0 + V_0) = w_0 + W_0
\end{aligned}$$

$\lim_{z\to z_0}[f(z)F(z)] = w_0 W_0$

From Theorem 13.2.2:

$$f(z)F(z) = (uU - vV) + i(vU + uV)$$
$$\implies \lim_{(x,y)\to(x_0,y_0)} f(z)F(Z) = (u_0U_0 - v_0V_0) + i(v_0U_0 + u_0V_0) = w_0W_0$$

$\lim_{z\to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}$ if $W_0 \neq 0$

From Theorem 13.2.2:

$$\frac{f(z)}{F(z)} = \frac{u+iv}{U+iV} \implies \lim_{(x,y)\to(x_0,y_0)} \frac{f(z)}{F(z_0)} = \frac{u_0+v_0}{U_0+iV_0} = \frac{w_0}{W_0}$$

□

Corollary 13.2.3.1:

Let c be a constant, $z, z_0 \in \mathbb{C}$, and $P(z)$ be a polynomial. Then

$$\lim_{z\to z_0} c = c \qquad \lim_{z\to z_0} z = z_0 \qquad \lim_{z\to z_0} z^n = z_0^n \qquad n \in \mathbb{N}$$

$$\lim_{z\to z_0} P(z) = P(z_0)$$

Observation. *It is surprisingly quick that Brown and Churchill went from ϵ−δ proofs straight to proving with limits. This is different to the approach in Sequences of Limits Theorem for Sequences Section by Kennith A. Ross. [1]. (Section 9.1.1)*

Question. *It might be possible use a series approach to prove limit theorems for $z \in \mathbb{C}$ by having separate series for x and y (real and imaginary components of z), or a series in the form of $s_n = (x_n, y_n)$. Which would be the proper approach?*

13.2.2 Limits of Points at Infinity

Definition 13.2.2: Extended Complex Plane

The complex plane union with the points at infinity:

$$\mathbb{C} \cup \{\pm\infty, \pm i\infty\}$$

Definition 13.2.3: Riemann Sphere

A unit sphere centred at the origin of the complex plane, which is consequently bisected by the complex plane.

Definition 13.2.4: Stereographic Projection

Consider the Riemann Sphere. Let N be the northern point of the sphere (the point on the sphere above the origin of the complex plane) and z be any point in the complex plane. Let l be a line that goes through N and z, then l will intersect the Riemann Sphere. Let P be the point where l intersects the Riemann Sphere. If we let N correspond to the points at infinity, then there is a one-to-one correspondence between points on the sphere and the points on the extended complex plane. This correspondence is called the Stereographic Projection. (Figure 13.1)

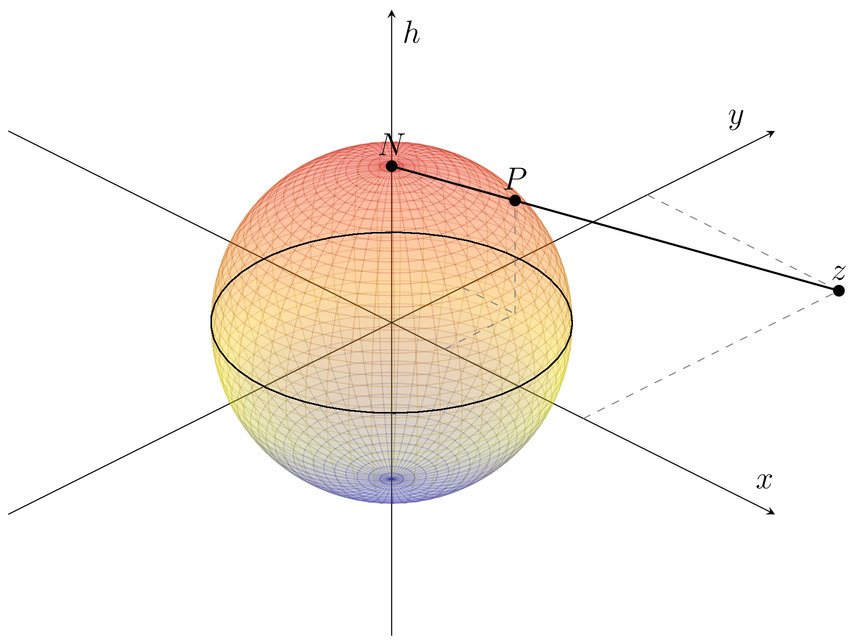


Figure 13.1: Riemann Sphere and Stereographic Projection

The region outside the unit circle enveloped by the Riemann sphere corresponds to the upper hemisphere of the Riemann sphere, with the point N deleted. N corresponds to the points at infinity, since l will be parallel to the complex plane.

Note: In some texts, the Riemann Sphere is a sphere of unit diameter (not a unit sphere, which is of unit radius) sitting on top of the Complex Plane. That is, with the south pole sitting at $(0, 0)$. The definitions for line L, and points N, P, and z remains the same. In either case, the Stereographic Projection maps to a unique point P on the sphere, and the definition of the point at infinity remains unchanged.

Definition 13.2.5: Neighbourhood of ∞

The set: $\{|z| > 1/\epsilon : \epsilon \in \mathbb{R}_{>0}\}$

Note that since ϵ is a small positive number, $|z| > 1/\epsilon$ corresponds to points far away from the unit circle, hence P is close to N.

Note: When referring to any point z, it is referring to a point in the finite plane. Points at infinity will be specifically mentioned.

Definition 13.2.6: Limit at Infinity

Let $f(z)$ be a function, and $z, z_0 \in \mathbb{C}$.

$$\forall \epsilon \in \mathbb{R}_{>0}, \exists r \in \mathbb{R}_{>0}[|z| > r \implies |f(z) - z_0| < \epsilon] \iff \lim_{z\to\infty} f(z) = z_0$$

That is, if $\forall z$ in the neighbourhood of infinity implies $|f(z) - z_0| < \epsilon$, then $\lim_{z\to\infty} f(z) = z_0$.

Theorem 13.2.4:

Let $z_0, w_0 \in \mathbb{C}$, then

$$\lim_{z\to z_0} \frac{1}{f(z)} = 0 \implies \lim_{z\to z_0} f(z) = \infty$$
$$\lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0 \implies \lim_{z\to\infty} f(z) = w_0$$
$$\lim_{z\to 0} \frac{1}{f(1/z)} = 0 \implies \lim_{z\to\infty} f(z) = \infty$$

Proof: $\underline{\lim_{z\to z_0} \frac{1}{f(z)} = 0 \implies \lim_{z\to z_0} f(z) = \infty}$

$$\begin{aligned} \lim_{z\to z_0} \frac{1}{f(z)} = 0 &\implies \forall\epsilon\exists\delta > 0\left[|z - z_0| < \delta \implies \left|\frac{1}{fz} - 0\right| < \epsilon\right] \\ &\implies \forall\epsilon\exists\delta > 0\left[|z - z_0| < \delta \implies |f(z)| > \frac{1}{\epsilon}\right] \\ &\implies \lim_{z\to z_0} f(z) = \infty \end{aligned}$$

$\underline{\lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0 \implies \lim_{z\to\infty} f(z) = w_0}$

$$\begin{aligned} \lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0 &\implies \forall\epsilon\exists\delta > 0\left[|z - 0| < \delta \implies \left|f\left(\frac{1}{z}\right) - w_0\right| < \epsilon\right] \\ &\implies \forall\epsilon\exists\delta > 0\left[|z| > \frac{1}{\delta} \implies |f(z) - w_0| < \epsilon\right] \\ &\implies \lim_{z\to\infty} f(z) = w_0 \end{aligned}$$

$\underline{\lim_{z\to 0} \frac{1}{f(1/z)} = 0 \implies \lim_{z\to\infty} f(z) = \infty}$

$$\begin{aligned} \lim_{z\to 0} \frac{1}{f(1/z)} = 0 &\implies \forall\epsilon\exists\delta > 0\left[|z - 0| < \delta \implies \left|\frac{1}{f(1/z)} - 0\right| < \epsilon\right] \\ &\implies \forall\epsilon\exists\delta > 0\left[|z| > \frac{1}{\delta} \implies |f(z)| > \frac{1}{\epsilon}\right] \\ &\implies \lim_{z\to\infty} f(z) = \infty \end{aligned}$$

□

Note: As δ goes to 0, $1/\delta$ goes to ∞, hence $|z|$ goes to ∞ if $|z| > 1/\delta$.

Observation. *As expected, theorem 13.2.4 is consistent if $z \in \mathbb{R}$. (Check: Section 9.1.1).*

13.3 Continuity

Definition 13.3.1: Continuous

Let f be a function. We say f is continuous at all point $z_0 \in \mathbb{C}$ if it satisfies the following:

$$\lim_{z \to z_0} f(z) \text{ exists } \wedge f(z_0) \text{ exists } \wedge \lim_{z \to z_0} f(z) = f(z_0)$$

Note:

$$\lim_{z \to z_0} f(z) = f(z_0) \implies \lim_{z \to z_0} f(z) \text{ exists } \wedge f(z_0) \text{ exists}$$
$$\forall \epsilon \exists \delta > 0 \left[|z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon \right] \iff \lim_{z \to z_0} f(z) = f(z_0)$$

Definition 13.3.2: Continuous at a Region

Let f be a function, $R \subset \mathbb{C}$ be a region, and $z \in R$:

$$f \text{ is continuous in } R \iff \forall z \in R (f \text{ is continuous})$$

Theorem 13.3.1:

Let $f(z)$ and $g(z)$ be continuous functions at $z_0 \in \mathbb{C}$. Then the following are also continuous at z_0:

$$f(z_0) + g(z_0) \qquad f(z_0)g(z_0) \qquad \frac{f(z_0)}{g(z_0)} \qquad g(z_0) \neq 0$$

Proof: Consequence of theorem 13.2.3. □

Corollary 13.3.1.1:

Let $P(z)$ be a polynomial, then $P(z)$ is continuous $\forall z \in \mathbb{C}$. That is $P(z)$ is continuous in the entire plane of $\mathbb{C}$.

Proof: Consequence of corollary 13.2.3.1. □

Observation. *Both theorem 13.3.1 and corollary 13.3.1.1 rely on definition 13.3.1, which state for a function f and point $z_0 \in \mathbb{C}$:*

$$\lim_{z \to z_0} f(z) \text{ exists } \implies f(z) \text{ is continuous at } z_0$$

This is why the proofs cite the results of theorem 13.2.3 and corollary 13.2.3.1.

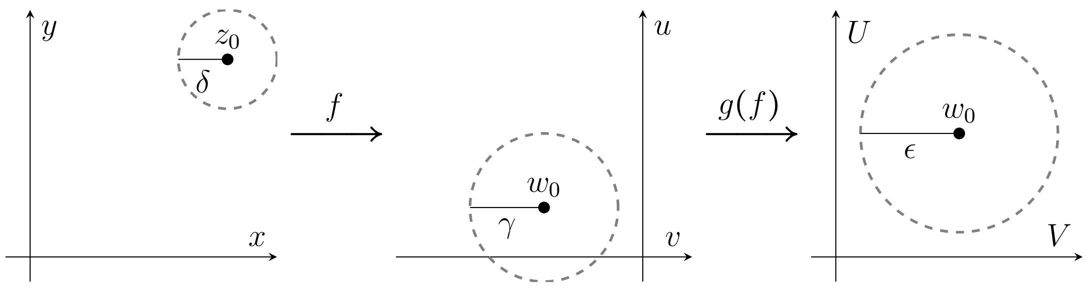
Theorem 13.3.2:
Let $f(z)$ and $g(z)$ be functions.

$$f(z) \text{ and } g(z) \text{ continuous} \implies g(f(z)) \text{ continuous}$$

Proof: Let $f(z) = w$ be defined in the neighbourhood $\forall z[|z - z - 0| < \delta]$, and $g(w) = W$ where $\text{dom}(g) = \text{img}(f)$. Suppose that f is continuous at z_0 and g is continuous at $f(z_0)$.

$$\begin{aligned} f \text{ continuous at } z_0 &\iff \forall\gamma\exists\delta > 0\,[|z - z_0| < \delta \implies |f(z) - f(z_0)| < \gamma] \\ &\implies \forall\epsilon\exists\gamma > 0\,[|f(z) - f(z_0)| < \gamma \implies |g(f(z)) - g(f(z_0))| < \epsilon] \end{aligned}$$

We can always find a small enough δ for γ to satisfy $|g(f(z)) - g(f(z_0))| < \epsilon$. □



Theorem 13.3.3:
Let $f(z)$ be a function and $f(z_0) \neq 0$.

$$f(z_0) \neq 0 \implies \exists\epsilon\forall z[|f(z) - f(z_0)| < \epsilon \implies f(z) \neq 0]$$

That is, if $f(z_0) \neq 0$ then it has a neighbourhood where $f(z) \neq 0$.

Proof: Suppose $f(z)$ is continuous and non-zero at z_0, and let $\epsilon = |f(z_0)|/2$:

$$\exists z[f(z) = 0] \wedge \forall\epsilon\exists\delta > 0\left[|z - z_0| < \delta \implies |f(z) - f(z_0)| < \frac{|f(z_0)|}{2}\right]$$

$$\implies |f(z_0)| < \frac{|f(z_0)|}{2} \qquad \text{Contradiction!}$$

□

Theorem 13.3.4:
Let $f(z) = u(x,y) + iv(x,y)$ be a function, and $z = x + iy$, $z \in \mathbb{C}$.

$$f \text{ continuous at } z_0 \iff [u \text{ continuous at } z_0] \wedge [v \text{ continuous at } z_0]$$

Proof: Direct consequence of theorem 13.2.2 □

Theorem 13.3.5:
Let f be continuous in a closed and bounded region R, then

$$\forall z \in R, \exists M \in \mathbb{R}_{>0}[|f(z) \leq M|] \wedge |\{z : f(z) = M\}| \geq 1$$

That is, for $\forall z \in R$, $|f(z)| \leq M$ and there is at least one point z where $|f(z)| = M$. $f(z)$ is bounded in R.

Proof: Let $f(z) = u(x, y) + iv(x, y)$ be continuous, then

$$|f(z)| = \sqrt{[u(x,y)]^2 + [v(x,y)]^2} \text{ is continuous in } R \implies \exists M \in \mathbb{R}_{>0}[|f(z)| \leq M]$$

□

13.3.1 Exercises

Example 13.3.1 *Prove:*

$$\lim_{z \to z_0} f(z) = w_0 \implies \lim_{z \to z_0} |f(z)| = |w_0|$$

Note: $||f(z_0)| - |w_0|| \leq |f(z) - w_0|$

Proof: Use definition of limit, then plug and chug. □

Example 13.3.2 *Prove: Limits involving points at infinity are unique.*

Proof: Suppose that limit of the point at infinity is not unique, that is there is two neighbourhoods of infinity. Using he definition of the limit, we will arrive at a contradiction where the two neighbourhoods are the same. □

Example 13.3.3 *Prove:*

$$S \text{ is unbounded} \iff \forall \epsilon \exists z \left[z \in S : |z| > \frac{1}{\epsilon} \right]$$

That is, S is unbounded $\iff$ every neighbourhood of the point at infinity contains at least one point in S

Proof: Proof Sketch: Recall the Riemann Sphere. (Definition 13.2.3). The set $|z| > 1/\epsilon$ corresponds to the points close to N, which is the neighbourhood of the point at infinity. If we let $\gamma = 2\epsilon$, $\exists z$ where $|z| > 1/\gamma$ holds. This along with $z \in \mathbb{C}$ (which is S in our case), implies the direction $\impliedby$ is true. That is, we can still find elements in S as we shrink the circle around N.

S is unbounded implies that for all circle with radius R centred at the origin there is at least one element of $s \in S$ where $|s| > R$. Suppose for contradiction that there is a neighbourhood of the point at infinity that does not contain any points in S. We will arrive at a contradiction, where there is $M \in \mathbb{R}_{>0}$ such that $\forall s \in S[|s| < M]$. Thus S is bounded, a contradiction. This implies that the direction $\implies$ is true. □

13.4 Differentiation

Definition 13.4.1: Derivative

Let f be a function where $|z - z_0| < \epsilon$ and $z \in \operatorname{dom}(f)$. Then the derivative of f at point z_0:

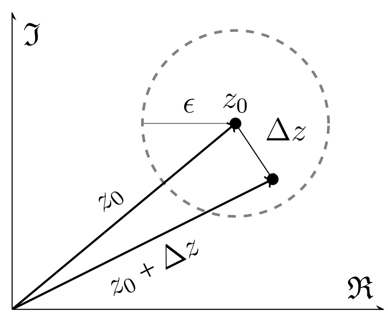
$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Definition 13.4.2: Differentiable

A function f is differentiable at $z_0 \in \mathbb{C}$ if $f'(z_0)$ exists.

If we let $\Delta z = z - z_0$ where $z \neq z_0$:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$



There's another notation by letting $\Delta w = f(z + \Delta z) - f(z)$:

$$f'(z) = \frac{\mathrm{d}w}{\mathrm{d}z} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

Observation. *The definition of a derivative in definition 13.4.1 looks similar to that of a derivative for the real numbers:*

$$F'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

However, the existence of $f'(z)$ possesses a much stronger requirement than the existence of $F'(z)$. That is, let $f(z) = u(x,y) + iv(x,y)$. The existence of $f'(z)$ at point z_0 requires the existence of both $u'(x,y)$ and $v'(x,y)$.

$$f'(z_0) = \lim_{(x,y) \to (x_0,y_0)} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{(x,y) \to (x_0,y_0)} \frac{u(z) - u(z_0)}{z - z_0} + i\frac{v(z) - v(z_0)}{z - z_0}$$

and that

$$\lim_{(x,y_0)\to(x_0,y_0)} \frac{u(x,y_0)-u(x_0,y_0)}{x-x_0} + i\frac{v(x_0,y_0)-v(x_0,y_0)}{x-x_0}$$
$$= \lim_{(x_0,y)\to(x_0,y_0)} \frac{u(x_0,y)-u(x_0,y_0)}{x-x_0} + i\frac{v(x_0,y)-v(x_0,y_0)}{x-x_0}$$

That is

$$\lim_{(\Delta x,0)\to(0,0)} \frac{u(x_0+\Delta x,y_0)-u(x_0,y_0)}{\Delta x} = \lim_{(0,\Delta y)\to(0,0)} \frac{u(x_0,y_0+\Delta y)-u(x_0,y_0)}{\Delta y}$$
$$\lim_{(\Delta x,0)\to(0,0)} \frac{v(x+\Delta x,y_0)-v(x_0,y_0)}{\Delta x} = \lim_{(0,\Delta y)\to(0,0)} \frac{v(x_0,y+\Delta y)-v(x_0,y_0)}{\Delta y}$$

This tells us that the existence of a derivative for a real valued function $F(x)$ does not imply the existence of a derivative for a similar function $f(z)$ in the complex plane, which we will see later. (i.e. Take $f(z) = |z|^2$ and $F(x) = |x|^2$.) We are dealing with a two-dimensional limit instead of a one dimensional limit.

Question. *Under what conditions will differentiability in $\mathbb{C}$ imply differentiability in $\mathbb{R}$, and vice versa?*

Example 13.4.1 *Let $f(z) = \bar{z}$:*

$$\frac{\Delta w}{\Delta z} = \frac{\overline{z+\Delta z}-\bar{z}}{\Delta z} = \frac{\bar{z}+\overline{\Delta z}-\bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}$$

Consider $\Delta z = (\Delta x, \Delta y) \to (0,0)$. If we move on the real axis, that is $(\Delta x, 0)$:

$$\overline{\Delta z} = \overline{\Delta x + i0} = \Delta x - i0 = \Delta x + i0 = \Delta z \implies \frac{\Delta w}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta z}{\Delta z} = 1$$

If we move on the imaginary axis, that is $(0, \Delta y)$:

$$\overline{\Delta z} = \overline{0 + i\Delta y} = 0 - i\Delta y = -\Delta z \implies \frac{\Delta w}{\Delta z} = \frac{\overline{\Delta z}}{\Delta y} = \frac{-\Delta z}{\Delta z} = -1$$

Limits are unique, so the limit of $\mathrm{d}w/\mathrm{d}z$ does not exist anywhere.

Example 13.4.2 *Consider $f(z) = |z|^2$:*

$$\frac{\Delta w}{\Delta z} = \frac{|z+\Delta z|^2 - |z|^2}{\Delta z} = \frac{(z+\Delta z)(\overline{z+\Delta z}) - z\bar{z}}{\Delta z}$$
$$= \frac{(z+\Delta z)(\bar{z}+\overline{\Delta z}) - z\bar{z}}{\Delta z} = \frac{z\bar{z} + \Delta z\bar{z} + \overline{\Delta z}z + \overline{\Delta z}\Delta z - z\bar{z}}{\Delta z} = \bar{z} + \overline{\Delta z} + z\frac{\overline{\Delta z}}{\Delta z}$$

As in the previous example, as $(\Delta x, \Delta y) \to (0,0)$:

$\overline{\Delta z} = \Delta z$	*From the real axis*
$\overline{\Delta z} = -\Delta z$	*From the imaginary axis*

Thus

$$\frac{\Delta w}{\Delta z} = \bar{z} + \Delta z + z \qquad\qquad \Delta z = (\Delta x, 0)$$
$$\frac{\Delta w}{\Delta z} = \bar{z} - \Delta z - z \qquad\qquad \Delta z = (0, \Delta y)$$

Therefore, by uniqueness of limits as $\Delta z \to 0$*:*

$$\lim_{\Delta z \to 0}(\bar{z} + \Delta z + z) = \lim_{\Delta z \to 0}(\bar{z} - \Delta z - z) \implies z = -z \implies z = 0$$

Hence, $\mathrm{d}w/\mathrm{d}z$ *does not exist for* $z \neq 0$*. We can also see that:*

$$\frac{\Delta w}{\Delta z} = \bar{z} + \overline{\Delta z} + z\frac{\overline{\Delta z}}{\Delta z} = \overline{\Delta z} \qquad\qquad z = 0$$

Thus, $\mathrm{d}w/\mathrm{d}z$ *only exists at* $z = 0$*:*

$$\left.\frac{\mathrm{d}w}{\mathrm{d}z}\right|_{z=0} = 0$$

Remark. *The following are facts:*

(1) A function $f(z)$ can be differentiable at a point z_0, but nowhere else in the neighbourhood of z_0.

(2) $f(z) = |z|^2 \implies u(x,y) = x^2 + y^2 \wedge v(x,y) = 0$, hence $u(x,y)$ and $v(x,y)$ can have continuous partial derivatives of all orders at a point z_0, even though f may not be differentiable at z_0.

(3) $f(z)$ differentiable at $z_0 \implies f(z)$ continuous at z_0

Proof: Assume $f'(z_0)$ exists:

$$\lim_{z \to z_0}[f(z) - f(z_0)] = \lim_{z \to z_0}\frac{f(z) - f(z_0)}{z - z_0}\lim_{z \to z_0}(z - z_0) = f'(z_0) \cdot 0 = 0$$
$$\implies \lim_{z \to z_0} f(z) = f(z_0)$$

So, f is differentiable at $z_0 \implies f$ is continuous at z_0. □

Note: Continuity of a function at $z_0 \in \mathbb{C} \not\Rightarrow$ existence of derivative at point z_0.

Ex: $f(z) = |z|^2$ is continuous everywhere in $\mathbb{C}$ for $z_0 \neq 0$, but $f(z_0)$ does not exist at z_0.

13.4.1 Differentiation Rules

Definition of derivative in $\mathbb{C}$ (definition 13.4.1) is the same of that in $\mathbb{R}$, so rules remain the same.

Let $c \in \mathbb{C}$ be a constant and functions f and g be differentiable at point z. Then

$$\frac{\mathrm{d}}{\mathrm{d}z}c = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}z}z = 1 \qquad \frac{\mathrm{d}}{\mathrm{d}z}[cf(z)] = cf'(z) \qquad \frac{\mathrm{d}}{\mathrm{d}z}z^n = nz^{n-1} \qquad n \in \mathbb{Z} \setminus \{0\}$$

Let functions f and g be differentiable at point z. Then

$$\frac{\mathrm{d}}{\mathrm{d}z}[f(z) + g(z)] = f'(z) + g'(z) \qquad \frac{\mathrm{d}}{\mathrm{d}z}[f(z)g(z)] = f(z)g'(z) + f'(z)g(z)$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2}$$

Proof: Deriving: $\frac{\mathrm{d}}{\mathrm{d}z}[f(z)g(z)] = f(z)g'(z) + f'(z)g(z)$

Let $w = f(z)g(z)$:

$$\begin{aligned}\Delta w &= f(z+\Delta z)g(z+\Delta z) - f(z)g(z)\\ &= f(z)[g(z+\Delta z) - g(z)] + [f(z+\Delta z) - f(z)]g(z+\Delta z)\end{aligned}$$

Thus

$$\frac{\Delta w}{\Delta z} = f(z)\frac{g(z+\Delta z) - g(z)}{\Delta z} + \frac{f(z+\Delta z) - f(z)}{\Delta z}g(z+\Delta z)$$

Hence

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \lim_{\Delta z \to 0}\frac{\Delta w}{\Delta z} = f(z)g'(z) + f'(z)g(z)$$

□

Theorem 13.4.1: Chain Rule for Composite Functions

Let function f be differentiable at z_0 and function g be differentiable at $f(z_0)$. Then $F(z) = g[f(z)]$ is differentiable at z_0.

$$F'(z_0) = g'[f(z_0)]f'(z_0)$$

Proof: Suppose f is differentiable at z_0. Let $w_0 = f(z_0)$ and assume that $g'(w_0)$ exists. Then

$$\forall w \exists \epsilon [|w - w_0| < \epsilon \implies \Phi(w_0) = 0]$$

Where

$$\Phi(w) = \frac{g(w) - g(w_0)}{w - w_0} - g'(w_0) \qquad w \neq w_0$$

Note: $\lim_{w \to w_0} \Phi(w) = 0$, so Φ is continuous at w_0. Then

$$g(w) - g(w_0) = [g'(w_0) + \Phi(w)](w - w_0) \qquad |w - w_0| < \epsilon$$

Note: This is valid for $w = w_0$.

$$\begin{aligned} f'(z_0) \text{ exists} &\implies f \text{ continuous at } z_0 \\ &\implies \forall\epsilon\exists\delta > 0[|z - z_0| < \delta \implies |w - w_0| < \epsilon] \end{aligned}$$

Hence, we can replace w by $f(z)$ when $|z - z_0| < \delta$. Subbing $w = f(z)$ and $w_0 = f(z_0)$:

$$\frac{g[f(z)] - g[f(z_0)]}{z - z_0} = \{g'[f(z_0)] + \Phi[f(z)]\}\frac{f(z) - f(z_0)}{z - z_0} \qquad 0 < |z - z_0| < \delta, \ z \neq z_0$$

Then

$$(f \text{ continous at } z_0) \wedge (\Phi \text{ continuoust at } w_0 = f(z_0)) \implies \Phi[f(z)] \text{ continuous at } z_0$$

$$\Phi(w_0) = 0 \implies \lim_{z\to z_0} \Phi[f(z) = 0]$$

Thus

$$\begin{aligned} \lim_{z\to z_0} \frac{g[f(z)] - g[f(z_0)]}{z - z_0} &= \lim_{z\to z_0} \{g'[f(z_0)] + \Phi[f(z)]\}\frac{f(z) - f(z_0)}{z - z_0} \\ &= g'[f(z_0)]f'(z_0) \end{aligned}$$

We then get

$$F'(z_0) = g'[f(z_0)]f'(z_0)$$

□

Alternatively, if we let $w = f(z)$ and $W = F(z)$, then the Chain Rule becomes:

$$\frac{\mathrm{d}W}{\mathrm{d}z} = \frac{\mathrm{d}W}{\mathrm{d}w}\frac{\mathrm{d}w}{\mathrm{d}z}$$

Note: Although this looks like a fraction, it is not a fraction and should not be treated as such! (Logical inconsistency when infinitesimals when viewed as ratios.)

Theorem 13.4.2: L'Hopital's Rule

Suppose $f(z_0) = 0$ and $g(z_0) = 0$, $f'(z_0)$ and $g(z_0)$ exists, with $g'(z_0) \neq 0$. Then

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

Proof: Let $f(z_0) = 0$, $g(z_0) = 0$, and $z \neq z_0$.

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{\frac{f(z)-f(z_0)}{z-z_0}}{\frac{g(z)-g(z_0)}{z-z_0}} = \lim_{\Delta z\to 0} \frac{\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}}{\frac{g(z_0+\Delta z)-g(z_0)}{\Delta z}} = \frac{f'(z_0)}{g'(z_0)}$$

□

13.4.2 Exercises

Example 13.4.3 *Show that $f'(z)$ does not exist for all points $z \in \mathbb{C}$ when:*

(a) $f(z) = \mathrm{Re}\{z\}$

(b) $f(z) = \mathrm{Im}\{z\}$

Proof: Let $f(z) = u(x,y) + iv(x,y)$, $\Delta w = f(x+\Delta x, y+\Delta y) - f(x,y)$.

$\underline{f(z) = \mathrm{Re}\{z\}}$

Recall $\mathrm{Re}\{z\} = x + i0$.

$$\frac{\Delta w}{\Delta z} = \frac{\mathrm{Re}\{z+\Delta z\} - \mathrm{Re}\{z\}}{\Delta z} = \frac{x+\Delta x - x}{\Delta z} = \frac{\Delta x}{\Delta x + \Delta y}$$

Now as $(\Delta x, 0) \to (0,0)$:

$$\lim_{(\Delta x,0)\to(0,0)} \frac{\Delta w}{\Delta z} = \lim_{(\Delta x,0)\to(0,0)} \frac{\Delta x}{\Delta x} = 1$$

Now as $(0, \Delta y) \to (0,0)$:

$$\lim_{(0,\Delta y)\to(0,0)} \frac{\Delta w}{\Delta z} = \lim_{(0,\Delta y)\to(0,0)} \frac{0}{\Delta y} = 0$$

Limits are unique, but this isn't the case, so we conclude that $f'(z)$ when $f(z) = \mathrm{Re}\{z\}$ does not exist.

$\underline{f(z) = \mathrm{Im}\{z\}}$

Recall $\mathrm{Im}\{z\} = 0 + iy$.

$$\frac{\Delta w}{\Delta z} = \frac{\mathrm{Im}\{z+\Delta z\} - \mathrm{Im}\{z\}}{\Delta z} = \frac{y+\Delta y - y}{\Delta z} = \frac{\Delta y}{\Delta x + \Delta y}$$

Now as $(\Delta x, 0) \to (0,0)$:

$$\lim_{(\Delta x,0)\to(0,0)} \frac{\Delta w}{\Delta z} = \lim_{(\Delta x,0)\to(0,0)} \frac{0}{\Delta x} = 0$$

Now as $(0, \Delta y) \to (0,0)$:

$$\lim_{(0,\Delta y)\to(0,0)} \frac{\Delta w}{\Delta z} = \lim_{(0,\Delta y)\to(0,0)} \frac{\Delta y}{\Delta y} = 1$$

Limits are unique, but this isn't the case, so we conclude that $f'(z)$ when $f(z) = \mathrm{Im}\{z\}$ does not exist. $\square$

13.5 Cauchy-Riemann Equations

Theorem 13.5.1: Cauchy-Riemann Equations (Cartesian)
Let $f(z) = u(x,y) + iv(x,y)$. If $f'(z)$ exists at a point $z_0 = x_0 + iy_0$, then $u'(x_0,y_0)$ and $v'(x_0,y_0)$ exists and satisfy Cauchy-Riemann equations:

$$u_x = v_y \qquad\qquad u_y = -v_x$$

Also, as a result of evaluating $f'(z)$ from the horizontal and vertical direction:

$$f'(z_0) = [u_x + iv_x]\Big|_{(x_0,y_0)} = [v_y - iu_y]\Big|_{(x_0,y_0)}$$

Proof: Let $f(z) = u(x,y) + iv(x,y)$, and suppose $f'(z)$ exists at z_0. Then

$$z_0 = x_0 + iy_0 \qquad \Delta z = \Delta x + i\Delta y \qquad \Delta w = f(z_0 + \Delta z) - f(z_0)$$

So that

$$\Delta w = [u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y)] - [u(x_0,y_0) + iv(x_0,y_0)]$$

Therefore

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0,y_0)}{\Delta x + i\Delta y} + i\frac{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0,y_0)}{\Delta x + i\Delta y}$$

Note: This equation remains valid as $(\Delta x, \Delta y) \to (0,0)$.

Horizontal Approach:

Let $(\Delta x, 0) \to (0,0)$ in the horizontal direction, then

$$\begin{aligned} f'(z_0) &= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0,y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0,y_0)}{\Delta x} \\ &\implies f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0) \end{aligned}$$

Vertical Approach:

Let $(0, \Delta y) \to (0,0)$ in the vertical direction, then

$$\begin{aligned} f'(z_0) &= \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0,y_0)}{i\Delta y} + i \lim_{\Delta x \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0,y_0)}{i\Delta y} \\ &= -i \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0,y_0)}{\Delta y} + \lim_{\Delta x \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0,y_0)}{\Delta y} \\ &\implies f'(z_0) = v_y(x_0,y_0) - iu_y(x_0,y_0) \end{aligned}$$

Putting it together:

For $f'(z)$ to exists at z_0, $f(z_0)$ from the horizontal approach must equal that of the vertical approach. By equating the real and imaginary parts:

$$\begin{aligned}
&u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0) \\
&\implies (u_x = v_y) \wedge (u_y = -v_x)
\end{aligned}$$

$\square$

Theorem 13.5.2: Cauchy-Riemann Equations (Polar)

Let $f(z) = u(r, \theta) + iv(r, \theta)$ be defined in some neighbourhood ϵ of $z_0 = r_0 e^{i\theta_0}$, $z_0 \neq 0$. If the first order partials derivatives of u and v with respect to r and θ exists and are continuous at z_0, and satisfies the polar form of the Cauchy-Riemann equations:

$$ru_r = v_\theta \qquad\qquad u_\theta = -rv_r$$

Then $f'(z_0)$ exists:

$$f'(z_0) = e^{-i\theta}(u_r + iv_r)\Big|_{(r_0,\theta_0)} = \frac{-i}{z_0}(u_\theta + iv_\theta)\Big|_{(r_0,\theta_0)}$$

Proof: Let $f(z) = u(r, \theta) + iv(r, \theta)$. Suppose that the first order partial derivatives of u and v exists in some neighbourhood ϵ of z_0 and is continuous at z_0. By differentiating u with respect to x and y:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \qquad\qquad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

Likewise for v. As $x = r\cos\theta$ and $y = r\sin\theta$:

$$\begin{aligned}
u_r &= u_x \cos\theta + u_y \sin\theta & u_\theta &= -u_x r \sin\theta + u_y r\cos\theta \\
v_r &= v_x \cos\theta + v_y \sin\theta & v_\theta &= -v_x r \sin\theta + v_y r\cos\theta
\end{aligned}$$

From theorem 13.5.1 we have:

$$u_x = v_y \qquad\qquad u_y = -v_x$$

Subbing the Cauchy-Riemann equations into v_r and v_θ:

$$\begin{aligned}
u_r &= u_x \cos\theta + u_y \sin\theta & u_\theta &= -u_x r \sin\theta + u_y r\cos\theta \\
v_r &= -u_y \cos\theta + u_x \sin\theta & v_\theta &= u_y r \sin\theta + u_x r\cos\theta
\end{aligned}$$

We can see that:

$$ru_r = v_\theta \qquad\qquad u_\theta = -rv_r$$

Which are the Cauchy Riemann equations in polar form. Let's verify it without relying on the Cauchy-Riemann equations in Cartesian form:

Recall:

$$u_r = u_x \cos\theta + u_y \sin\theta \qquad u_\theta = -u_x r \sin\theta + u_y r \cos\theta$$
$$v_r = v_x \cos\theta + v_y \sin\theta \qquad v_\theta = -v_x r \sin\theta + v_y r \cos\theta$$

Writing u_r and v_r in matrix notation:

$$\begin{bmatrix} u_r \\ u_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

Solving for u_x and u_y:

$$\begin{aligned} \begin{bmatrix} u_x \\ u_y \end{bmatrix} &= \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}^{-1} \begin{bmatrix} u_r \\ u_\theta \end{bmatrix} \\ &= \frac{1}{r\cos^2\theta + r\sin^2\theta} \begin{bmatrix} r\cos\theta & -\sin\theta \\ -r\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_r \\ u_\theta \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{r} \begin{bmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_r \\ u_\theta \end{bmatrix} \end{aligned}$$

It is clear that for u_x and u_y, and likewise for v_x and v_y:

$$u_x = u_r \cos\theta - \frac{1}{r} u_\theta \sin\theta \qquad u_y = u_r \sin\theta + \frac{1}{r} u_\theta \cos\theta \tag{13.5}$$
$$v_x = v_r \cos\theta - \frac{1}{r} v_\theta \sin\theta \qquad v_y = v_r \sin\theta + \frac{1}{r} v_\theta \cos\theta \tag{13.6}$$

Using the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$, we see:

$$u_r \cos\theta - \frac{1}{r} u_\theta \sin\theta = v_r \sin\theta + \frac{1}{r} v_\theta \cos\theta$$
$$u_r \sin\theta + \frac{1}{r} u_\theta \cos\theta = -v_r \cos\theta + \frac{1}{r} v_\theta \sin\theta$$

Clearly, the equations are equal only if

$$r u_r = v_\theta \qquad u_\theta = -r v_r$$

Which are the polar forms of the Cauchy-Riemann equations.

Show $f'(z_0) = e^{-i\theta}(u_r + i v_r)$:

Recall from theorem 13.5.1:

$$f'(z_0) = u_x + i v_y$$

Using eq. (13.5) and eq. (13.6) from before and substituting them into $f'(z_0)$:

$$
\begin{aligned}
f'(z_0) &= \left(u_r \cos\theta - \frac{1}{r} u_\theta \sin\theta + i v_r \cos\theta - \frac{i}{r} v_\theta \sin\theta\right)\Bigg|_{(r_0,\theta_0)} \\
&= \left(u_r \cos\theta + v_r \sin\theta + i v_r \cos\theta - i u_r \sin\theta\right)\Big|_{(r_0,\theta_0)} \\
&= \left[u_r(\cos\theta - i\sin\theta) + v_r(\sin\theta + i\cos\theta)\right]\Big|_{(r_0,\theta_0)} \\
&= \left[u_r(\cos\theta - i\sin\theta) + i v_r(\cos\theta - i\sin\theta)\right]\Big|_{(r_0,\theta_0)} \\
&= \left[\left(\frac{e^{i\theta}+e^{-i\theta}}{2} - \frac{e^{i\theta}-e^{-i\theta}}{2}\right)(u_r + i v_r)\right]\Bigg|_{(r_0,\theta_0)} \\
&= e^{-i\theta}(u_r + i v_r)\Big|_{(r_0,\theta_0)} \\
&= \frac{-i}{re^{i\theta}}(u_\theta + i v_\theta)\Big|_{(r_0,\theta_0)} = \frac{-i}{z_0}(u_\theta + i v_\theta)\Big|_{(r_0,\theta_0)} \qquad (r u_r = v_\theta) \wedge (u_\theta = -r v_r)
\end{aligned}
$$

Thus

$$
f'(z_0) = e^{-i\theta}(u_r + i v_r)\Big|_{(r_0,\theta_0)} = \frac{-i}{z_0}(u_\theta + i v_\theta)\Big|_{(r_0,\theta_0)}
$$

$\square$

Question. *When comparing the Cartesian form to the polar form of the Cauchy-Riemann equations:*

$$
\begin{gathered}
f'(z_0) \text{ exists} \implies \forall z_0[(u_x = v_y) \wedge (u_y = -v_x)] \\
(z_0 \neq 0) \wedge \forall z_0[(r u_r = v_\theta) \wedge (u_\theta = -r v_r)] \implies f'(z_0) \text{ exists}
\end{gathered}
$$

Should both be $\iff$ instead of $\implies$? No, satisfying Cauchy-Riemann equations does not guarantee differentiability at a point as we will see in example 13.5.3. However, satisfying certain conditions allows allows differentiability to exist (theorem 13.5.4).

Example 13.5.1 (Solving the $f'(z)$ using the partial derivative with respect to one variable) *Recall in theorem 13.5.1:*

$$
f'(z_0) = [u_x + i v_x]\Big|_{(x_0,y_0)} = [v_y - i u_y]\Big|_{(x_0,y_0)}
$$

This implies we can solve $\mathrm{d}f(z)/\mathrm{d}z$ by taking the partial of $f(z)$ with respect to x or y. Consider $f(z) = z^2$:

$$
f(z) = z^2 = x^2 - y^2 + i2xy
$$

We then have:

$$
u(x,y) = x^2 - y^2 \qquad\qquad v(x,y) = 2xy
$$

Hence

$$u_x = 2x = v_y \qquad\qquad u_y = -2y = -v_x$$

Thus

$$f'(z) = 2x + i2y = 2(x + iy) = 2z$$

Example 13.5.2 (Using Cauchy-Riemann equations to find where $f(z)$ is not differentiable) *Using the contrapositive of* $f'(z_0)$ *exists* $\implies \exists u' \exists v' [(u_x = v_y) \wedge (u_y = -v_x)]$*:*

$$\exists z_0 [(u_x \neq v_y) \vee (u_y \neq -v_x)] \implies f(z) \text{ not differentiable at } z_0$$

Consider $f(z) = |z|^2$*:*

$$u(x,y) = x^2 + y^2 \qquad\qquad v(x,y) = 0$$

By Cauchy-Riemann:

$$2x = 0 \qquad\qquad 2y = 0$$

Therefore, $f'(z)$ *only exists at* $(0,0)$ *and does not exist elsewhere.*

Note: Theorem 13.5.1 does not guarantee the existence of $f'(z)$ at z_0.

Example 13.5.3 ($f(z)$ satisfy Cauchy-Riemann equations at (0,0), but $f'(0)$ does not exist) *Consider*

$$f(z) = \begin{cases} \bar{z}^2/z & z \neq 0 \\ 0 & z = 0 \end{cases}$$

Then

$$u(x,y) = \frac{x^3 - 3xy^2}{x^2 + y^2} \qquad v(x,y) = \frac{y^3 - 3x^2y}{x^2 + y^2} \qquad (x,y) \neq (0,0)$$

Checking differentiability at $(0,0)$*, note* $u(0,0) = 0$ *and* $v(0,0) = 0$*:*

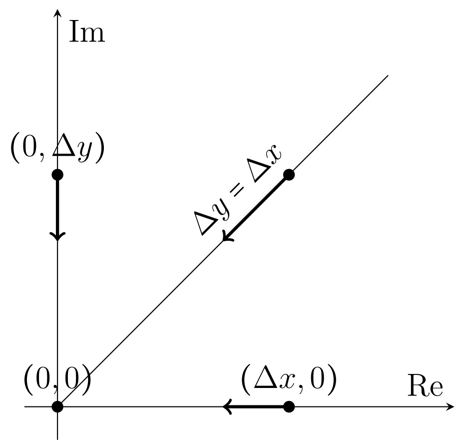
$$\begin{aligned}
u_x(0,0) &= \lim_{\Delta x \to 0} \frac{u(0 + \Delta x, 0) - u(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1 \\
v_y(0,0) &= \lim_{\Delta y \to 0} \frac{v(0, 0 + \Delta y) - v(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y}{\Delta y} = 1 \\
u_y(0,0) &= \lim_{\Delta y \to 0} \frac{u(0, 0 + \Delta y) - u(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0/(\Delta y)^2}{\Delta y} = 0 \\
v_x(0,0) &= \lim_{\Delta x \to 0} \frac{v(0 + \Delta x, 0) - v(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0/(\Delta x)^2}{\Delta x} = 0
\end{aligned}$$

We can see that the Cauchy-Riemann equations are satisfied:

$$u_x = v_y = 1 \qquad\qquad u_y = -v_x = 0$$

However, $f'(0)$ does not exist: (Brown and Churchill - Complex Variables and Applications, Section 20, Exercise 9 [2])

Let $\Delta w = f(z+\Delta z) - f(z)$. We need to show for all nonzero points on the real and imaginary axis, $\Delta w/\Delta z = -1$, but for all nonzero points on the line $\Delta x = \Delta y$, $\Delta w/\Delta z = -1$. Hence, a contradiction, so $f('0)$ does not exist.



$$\frac{\Delta w}{\Delta z} = \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{u(x+\Delta x, y+\Delta y) + v(x+\Delta x, y+\Delta y)}{\Delta x + \Delta y} - \frac{u(x,y) + v(x,y)}{\Delta x + \Delta y}$$

Along the real axis:

Evaluating along $(\Delta x, 0) \to (0, 0)$.

$$\begin{aligned}\lim_{(\Delta x,0)\to(0,0)} \frac{\Delta w}{\Delta z} &= \frac{u(\Delta x,0) + v(\Delta x,0)}{\Delta x} - \frac{u(0,0)+v(0,0)}{\Delta x} \\ &= \frac{1}{\Delta x}\left[\frac{(\Delta x)^3}{(\Delta x)^2} + \frac{0}{(\Delta x)^2}\right] - 0 = \frac{\Delta x}{\Delta x} = 1\end{aligned}$$

Along the imaginary axis:

Evaluating along $(0, \Delta y) \to (0, 0)$.

$$\begin{aligned}\lim_{(0,\Delta y)\to(0,0)} \frac{\Delta w}{\Delta z} &= \frac{u(0,\Delta y) + v(0,\Delta y)}{\Delta y} - \frac{u(0,0)+v(0,0)}{\Delta y} \\ &= \frac{1}{\Delta y}\left[\frac{0}{(\Delta y)^2} + \frac{(\Delta y)^3}{(\Delta y)^2}\right] - 0 = \frac{\Delta y}{\Delta y} = 1\end{aligned}$$

Along the axis $\Delta x = \Delta y$:

Evaluating along $(\Delta x, \Delta x) \to (0,0)$.

$$\begin{aligned}
\lim_{(\Delta x, \Delta x)\to(0,0)} \frac{\Delta w}{\Delta z} &= \frac{u(\Delta x, \Delta x)}{\Delta x + \Delta x} - \frac{u(0,0)+v(0,0)}{\Delta x + \Delta x} \\
&= \frac{1}{2\Delta x}\left[\frac{(\Delta x)^3 - 3(\Delta x)^3}{2(\Delta x)^2} + \frac{(\Delta x)^3 - 3(\Delta x)^3}{2(\Delta x)^2}\right] \\
&= \frac{1}{2\Delta x}\left[-\frac{2(\Delta x)^3}{2(\Delta x)^2} - \frac{2(\Delta x)^3}{2(\Delta x)^2}\right] = \frac{1}{2\Delta x}\left[-\Delta x - \Delta x\right] = -\frac{2\Delta x}{2\Delta x} = -1
\end{aligned}$$

As we can see, the limits are not unique regardless of the path we take to approach $(0,0)$, hence $f'(0)$ does not exist. Therefore, an equation can satisfy the Cauchy-Riemann equations at $0,0$, yet have a derivative that does not exist. The Cauchy-Riemann equations does not guarantee differentiability at z_0.

Example 13.5.4 *(Any branch of $f(z) = z^{1/2}$ is differentiable everywhere in domain of definition) Let*

$$f(z) = z^{1/2} = \sqrt{r}e^{i\theta} \qquad\qquad r > 0,\ \alpha < \theta < \alpha + 2\pi$$

Hence

$$u(r,\theta) = \sqrt{r}\cos\left(\frac{\theta}{2}\right) \qquad\qquad v(r,\theta) = \sqrt{r}\sin\left(\frac{\theta}{2}\right)$$

By Cauchy-Riemann:

$$ru_r = \frac{\sqrt{r}}{2}\cos\left(\frac{\theta}{2}\right) = v_\theta \qquad\qquad u_\theta = -\frac{\sqrt{r}}{2}\sin\left(\frac{\theta}{2}\right) = -rv_r$$

Thus, the derivative exists wherever $f(z)$ is defined. Also, by theorem 13.5.2:

$$\begin{aligned}
f'(z) &= e^{i\theta}(u_r + iv_r)\Big|_{(r_0,\theta_0)} \\
&= e^{-i\theta}\left[\frac{1}{2\sqrt{r}}\cos\left(\frac{\theta}{2}\right) + i\frac{1}{2\sqrt{r}}\sin\left(\frac{\theta}{2}\right)\right] = \frac{1}{2\sqrt{r}}e^{-i\theta}\left[\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right] \\
&= \frac{1}{2\sqrt{r}e^{i\theta/2}} = \frac{1}{2f(z)} = \frac{1}{2}z^{-1/2}
\end{aligned}$$

13.5.1 Complex Form of the Cauchy-Riemann Equations

Theorem 13.5.3: Cauchy-Riemann Equation (Complex Form)

Let $f(z) = u(x,y) + iv(x,y)$. If the first order partial derivatives of u and v with respect to x and y exists and satisfy the Cauchy-Riemann equations. Then

$$\frac{\partial}{\partial \bar{z}} f(z) = 0$$

Proof: Recall:

$$x = \frac{z + \bar{z}}{2} \qquad\qquad y = \frac{z - \bar{z}}{2i}$$

Let F be a real valued function, that is $x, y \in \mathbb{R}$. Then

$$\frac{\partial F}{\partial \bar{z}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \bar{z}}$$

Substituting $\frac{\partial x}{\partial \bar{z}} = 1/2$ and $\frac{\partial y}{\partial \bar{z}} = i/2$:

$$\frac{\partial F}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right)$$

Define the operator:

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Then

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) \\ &= \frac{1}{2}\left[(u_x - v_y) + i(u_y + v_x)\right]\end{aligned}$$

We can see that if $\frac{\partial f}{\partial \bar{z}}$ satisfies the Cauchy-Riemann equations (theorem 13.5.1):

$$\frac{\partial}{\partial \bar{z}} f(z) = 0 \qquad\qquad \frac{\partial}{\partial x} f = -i\frac{\partial f}{\partial y} \implies i\frac{\partial}{\partial x} f = \frac{\partial f}{\partial y}$$

□

13.5.2 Conditions for Differentiability

Theorem 13.5.4:
Let $f(z) = u(x, y) + iv(x, y)$ be defined in some neighbourhood ϵ of point $z_0 = x_0 + iy_0$. Consider the first order partial derivatives of u and v with respect to x and y. If they

(1) Exist for all z, $|z - z_0| < \epsilon$.

(2) Are continuous at z_0.

(3) Satisfies the Cauchy-Riemann equations at z_0.

Then $f'(z_0)$ exists:

$$f'(z_0) = (u_x + iv_x)\Big|_{(x_0, y_0)}$$

Proof: Assume the first order partial derivatives of u and v with respect to x and y exists $\forall z[|z - z_0| < \epsilon]$, are continuous at z_0, and satisfies the Cauchy-Riemann equations. Let $\Delta z = \Delta x + i\Delta y$, $0 < |\Delta z| < \epsilon$, and $\Delta w = f(z_0 + \Delta z) - f(z_0)$. We then have

$$\Delta w = \Delta u + i\Delta v$$

Where

$$\begin{aligned}\Delta u &= u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)\\ \Delta v &= v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)\end{aligned}$$

Since first order partials of u and v are continuous at z_0:

$$\begin{aligned}\Delta u &= u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y\\ \Delta v &= v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \epsilon_3\Delta x + \epsilon_4\Delta y\end{aligned}$$

$$(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \to (0,0,0,0) \text{ as } (\Delta x, \Delta y) \to (0,0)$$

Substituting Δu and Δv into Δw:

$$\begin{aligned}\Delta w =& u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y\\ &+ i[v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \epsilon_3\Delta x + \epsilon_4\Delta y]\end{aligned}$$

Using the Cauchy-Riemann equations and dividing by Δz:

$$\frac{\Delta w}{\Delta z} = u_x(x_0, y_0) + iv_x(x_0, y_0) + (\epsilon_1 + i\epsilon_3)\frac{\Delta x}{\Delta z} + (\epsilon_2 + i\epsilon_4)\frac{\Delta y}{\Delta z}$$

From the inequalities $|\Delta x| \le |\Delta z|$ and $|\Delta y| \le |\Delta z|$:

$$\left|\frac{\Delta x}{\Delta z}\right| \le 1 \qquad\qquad \left|\frac{\Delta y}{\Delta z}\right| \le 1$$

So

$$\begin{aligned}\left|(\epsilon_1 + i\epsilon_3)\frac{\Delta x}{\Delta z}\right| &\le |\epsilon_1 + i\epsilon_3| \le |\epsilon_1| + |\epsilon_3|\\ \left|(\epsilon_2 + i\epsilon_4)\frac{\Delta y}{\Delta z}\right| &\le |\epsilon_2 + i\epsilon_4| \le |\epsilon_2| + |\epsilon_4|\end{aligned}$$

Then $|\epsilon_2| + |\epsilon_4| \to 0$ and $|\epsilon_1| + |\epsilon_3| \to 0$ as $\Delta z = \Delta x + i\Delta y \to 0$.

$$\implies \frac{\Delta w}{\Delta z} = u_x(x_0, y_0) + iv_x(x_0, y_0) \implies f'(z_0) \text{ exists}$$

$\square$

Example 13.5.5 (All 3 conditions must be satisfied for $f'(z_0)$ to exist) *Do not use expression of $f'(z)$ before existence of $f'(z_0)$ is established. Consider $f(z) = x^3 + i(1-y)^3$.*

$$u(x, y) = x^3 \qquad\qquad v(x, y) = (1 - y)^3$$

Taking the partial derivatives:

$$u_x = 3x^2 \qquad v_x = 0$$
$$u_y = 0 \qquad v_y = -3(1-y)^2$$

It would be foolish to ignore Cauchy-Riemann and directly use:

$$f'(z) = u_x + iv_x = 3x^2$$

We can see that the Cauchy-Riemann equations are satisfied only if:

$$3x^2 = -3(1-y)^2 \implies x^2 + (1-y)^2 = 0 \implies (x=0) \wedge (y=1)$$

Therefore, $f'(z)$ exists only if $z = i$, and that $f'(i) = 0$

13.6 Analytic Functions

Definition 13.6.1: Analytic/Regular/Holomorphic
Let S be an open set, $S \subset \mathbb{C}$. Let f be a function.

$$f \text{ is analytic in } S \iff \forall z \in S[f'(z) \text{ exists }]$$

We say $f(z)$ is analytic at a point z_0 if it is analytic in some neighbourhood of z_0. If we say that $f(z)$ is analytic in a closed set S' then we mean that it is analytic in an open set S where $S' \subset S$.

Definition 13.6.2: Entire
A function $f(z)$ is entire if it is analytic at all points in the plane.

Example 13.6.1

$$\text{Derivative of polynomial exists everywhere} \implies \text{All polynomials are entire functions}$$

See section 13.5.2 for conditions for a function te be differentiable, hence analytic in a set S.

Corollary 13.6.0.1:
Let $f(z)$ and $g(z)$ be analytic in a domain D. Then the following are analytic in D:

$$f(z) + g(z)$$
$$f(z)g(z)$$
$$\frac{f(z)}{g(z)} \qquad g(z) \neq 0 \forall z \in D$$

Likewise, if $P(z)$ and $Q(z)$ are polynomials, then $P(z)/Q(z)$ is analytic if $\forall z \in D[Q(z) \neq 0]$.

Corollary 13.6.0.2:
Let w be the image of D under $f(z)$ and w be the domain of g. Then $g(f(z))$ is analytic in D and

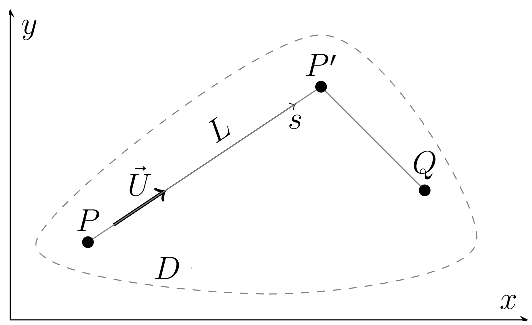
$$\frac{\mathrm{d}}{\mathrm{d}z}g[f(z)] = g'[f(z)]f'(z)$$

Theorem 13.6.1:
Let D be the domain of a function $f(z)$.

$$\forall z \in D[f'(z) = 0] \implies f(z) \text{ is constant in } D$$

Proof: Let $f(z) = u(x,y) + iv(x,y)$ with domain D, and P, P', and Q be points in D. Let $\vec{U}$ be the unit vector on the line segment L connecting P and P', and s be the distance along L.

$$f'(z) = 0 \implies \forall z \in D[u_x = u_y = v_x = v_y = 0]$$



We know that the directional derivative:

$$\frac{\mathrm{d}u}{\mathrm{d}s} = \nabla u \cdot \vec{U} \qquad\qquad \nabla u = u_x\hat{\imath} + u_y\hat{\jmath}$$

Previously, $u_x = u_y = 0$, so for all points on L:

$$u_x = u_y = 0 \implies \nabla u = 0 \implies \frac{\mathrm{d}u}{\mathrm{d}s} = 0 \implies u \text{ constant on } L$$

Now, that we have established that u is constant on any given line L in D, we can see that since D is simply connected and there are finitely many lines connecting P and Q, the values of u at P and Q must be equal and constant. Hence, $\exists a \in \mathbb{R}$ such that $u(x,y) = a$ in D. Likewise, $v(x,y) = b$ in D. Thus

$$f(z) = a + bi = c \qquad\qquad c \text{ is constant}$$

□

Definition 13.6.3: Singular Point
Let ϵ be a neighbourhood of point z_0, and $f(z)$ be a function. z_0 is a singular point if $f'(z_0)$ does not exist, but $f(z)$ is differentiable in all neighbourhoods of z_0.

13.6.1 Examples

Example 13.6.2 (Determining analyticity using Cauchy-Riemann equations) *Consider* $f(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$.

$$u(x,y) = \sin(x)\cosh(y) \qquad v(x,y) = \cos(x)\sinh(y)$$

Cauchy-Riemann:

$$u_x = \cos(x)\cosh(y) = v_y \qquad u_y = \sin(x)\sinh(y) = -v_x$$

Therefore, it is clear that $f(z)$ *is entire.*

$$f'(z) = u_x + iv_x = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

Another application of Cauchy-Riemann see that $f'(z)$ *is also entire.*

Example 13.6.3 ($f(z)$ and $\overline{f(z)}$ is analytic in $D \implies f(z)$ is constant in D) *Let*

$$f(z) = u(x,y) + iv(x,y) \qquad \overline{f(z)} = u(x,y) - iv(x,y) = U(x,y) + iV(x,y)$$

Because of $f(z)$ *and* $\overline{f(z)}$ *is analytic in* D*, the Cauchy-Riemann equations hold:*

$$\begin{aligned} u_x &= v_y & u_y &= -v_x \\ U_x &= V_y & U_y &= -V_x \end{aligned}$$

We can see that:

$$u_x = -v_y = v_y \qquad u_y = v_x = -v_x$$

Hence, $u_x = 0$ *and* $v_x = 0$*, then we can conclude*

$$f'(z) = 0 \implies f(z) \text{ is constant in } D$$

Example 13.6.4 ($f(z)$ is analytic in D and $|f(z)|$ is constant in $D \implies f(z)$ is constant in D) *Let* $\forall z \in D[|f(z)| = c]$*, where* c *is a constant. It is easy to see that* $c = 0 \implies \forall z \in D[f(z) = 0]$*, so consider* $c \neq 0$*. Then*

$$f(z)\overline{f(z)} = c^2 \neq 0 \implies \forall z \in D[f(z) \neq 0]$$

Thus

$$\overline{f(z)} = \frac{c^2}{f(z)} \qquad \forall z \in D$$

Hence $\overline{f(z)}$ *is analytic everywhere in* D*, so* $f(z)$ *is constant in* D*.*

13.7 Harmonic Functions

Harmonic functions are functions where the curvature in each component direction cancels each other out.

Definition 13.7.1: Laplace's Equation

Let $F(x,y)$ be a real-valued function. That is $x,y \in \mathbb{R}$. Laplace's equation:

$$\frac{\partial^2}{\partial x^2}F + \frac{\partial^2}{\partial y^2}F = 0$$

In polar form:

$$r^2 u_{rr}(r,\theta) + r u_r(r,\theta) + u_{\theta\theta}(r,\theta) = 0$$
$$r^2 v_{rr}(r,\theta) + r v_r(r,\theta) + v_{\theta\theta}(r,\theta) = 0$$

See example 13.7.1

Definition 13.7.2: Harmonic

A real-valued function $F(x,y)$ is harmonic in the xy-plane if it satisfies Laplace's equation.

Theorem 13.7.1:

Let D be the domain of a function $f(z) = u(x,y) + iv(x,y)$.

$$f(z) \text{ is analytic in } D \implies u(x,y) \wedge v(x,y) \text{ are harmonic in } D$$

Proof: f is analytic in D, so its component functions must satisfy the Cauchy-Riemann equations:

$$(u_x = v_y) \wedge (u_y = -v_x) \implies (u_{xy} = v_{yy}) \wedge (u_{yx} = -v_{xx})$$
$$(u_x = v_y) \wedge (u_y = -v_x) \implies (u_{xx} = v_{yx}) \wedge (u_{yy} = -v_{xy})$$

Now, we know from calculus that $u_{xy} = u_{yx}$ and $v_{yx} = v_{xy}$, so we conclude

$$u_{xx} + u_{yy} = 0 \qquad\qquad v_{xx} + v_{yy} = 0$$

$\square$

Note: The converse ($\Longleftarrow$) is true for simply connected domains, hence, theorem 13.7.1 becomes $\iff$ in simply connected domains. (R, Boas - Invitation to Complex Analysis. (1987) Section 19.)

Corollary 13.7.1.1:

Let $F(x,y)$ is a real-valued function in a simply connected domain D. Then there exists a function $f(z)$ and $g(z)$ in D such that $f(z) = F(x,y) + iv(x,y)$ and $g(z) = u(x,y) + iF(x,y)$. That is, there exists a function where the real part equals F and a function where the imaginary part equals F.

Definition 13.7.3: Harmonic Conjugate

If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then $v(x,y)$ is the harmonic conjugate of $u(x,y)$. This is not to be confused with the complex conjugate.

Example 13.7.1 *Let $f(z) = u(r,\theta) + iv(r,\theta)$ be analytic in domain $D' = D \setminus \{0\}$. Show $u(r,\theta)$ and $v(r,\theta)$ satisfies the polar form of Laplace's equation.*

Proof: We know from the Polar form of the Cauchy-Riemann equation:

$$ru_r = v_\theta \qquad\qquad u_\theta = -rv_r$$

Operating by $r\,\partial/\partial r$ and $\partial/\partial\theta$, we obtain:

$$\begin{aligned}
r\frac{\partial}{\partial r}ru_r &= ru_r + r^2u_{rr} = rv_{\theta r}\\
r\frac{\partial}{\partial r}u_\theta &= ru_{\theta r} = r\frac{\partial}{\partial r}(-rv_r) = -rv_r - r^2v_{rr}\\
\frac{\partial}{\partial\theta}ru_r &= ru_{\theta r} = v_{\theta\theta}\\
\frac{\partial}{\partial\theta}u_\theta &= u_{\theta\theta} = -rv_{r\theta}
\end{aligned}$$

We can see that

$$\begin{cases} ru_r + r^2u_{rr} = -u_{\theta\theta} \\ rv_r + r^2v_{rr} = -v_{\theta\theta} \end{cases} \implies \begin{cases} r^2u_{rr} + ru_r + u_{\theta\theta} = 0 \\ r^2v_{rr} + rv_r + v_{\theta\theta} = 0 \end{cases}$$

□

Example 13.7.2 *Let $f(z) = u(x,y)+iv(x,y)$ be analytic in domain D. Consider the families of level curves $u(x,y) = c_1$ and $v(x,y) = c_2$, with $c_1, c_2 \in \mathbb{R}$ being constants. Show for $z_0 = (x_0,y_0) \in \mathbb{C}$ common to $u(x,y) = c_1$ and $v(x,y) = c_2$ and $f'(z_0) \neq 0$, then the lines tangent to $u(x,y) = c_1$ and $v(x,y) = c_2$ at z_0 are orthogonal.*

Note:

$$[u(x,y) = c_1] \wedge [v(x,y) = c_2] \implies \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0\right) \wedge \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0\right)$$

Proof: The tangent lines of $u(x,y)$ and $v(x,y)$ are

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = (u_x, u_y) \qquad\qquad \nabla v = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) = (v_x, v_y)$$

Taking the dot product, and applying the Cauchy-Riemann equations:

$$u_xv_x + u_yv_y = u_x(-u_y) + u_y(u_x) = 0$$

Hence, $u(x,y)$ and $v(x,y)$ are orthogonal.

Note:

$$\begin{aligned} f'(z_0) = 0 &\implies u_x + iv_x = 0 \implies v_y - iu_y = 0 \\ &\implies u_x = u_y = v_x = v_y = 0 \end{aligned}$$

Hence, we can see that $f'(z_0) = 0$ is required for $u(x, y)$ and $v(x, y)$ to exist and be orthogonal. □

13.8 Uniquely Determined Analytic Functions

Lemma 13.8.0.1:
Suppose a function f is analytic throughout domain D, and $f(z) = 0$ $\forall z \in D' \subset D$ or line segment contained in D. Then $f(z) \equiv 0$ throughout D.

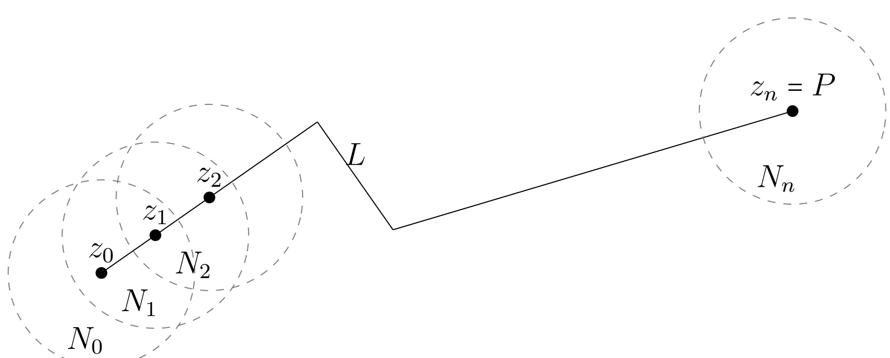
Proof: Let f be a function analytic in domain D and $f(z) = 0$ for all point or line segment in D. Let z_0 in the subdomain of D or on a line segment in D.

D is connected open set, so there is a polygonal line L jointing any point P in D to z_0 lying entirely in D. (Recall: A polygonal line consists of a finite number of lines connected end-to-end.) Let d be the shortest distance from points on L to the boundary on D, so $d > 0$, unless D is the entire plane. Then there is a sequence of points along L:

$$\{z_0, z_1, z_2, \ldots, z_{n-1}, z_n = P\} \qquad |z_k - z_{k+1}| < d \qquad k \in \mathbb{N}$$

That is, each point is sufficiently close to each other. We construct neighbourhoods of each point with radius d, all of which are in D, so points z_{k-1} and z_{k+1} lie in the neighbourhood of z_k, $k \in \mathbb{N}$:

$$\{N_0, N_1, N_2, \ldots, N_{n-1}, N_n\}$$



Now as f is analytic in N_0 and $f(z) = 0$ in a domain or line segment containing z_0, then $f(z) \equiv 0$ in N_0. z_1 is in N_0, so $f(z_0) \equiv 0$ in N_1. Continuing this we can see that $f(z_n) \equiv 0$ in N_n, hence, $f(z) \equiv 0$ in D. □

Theorem 13.8.1:

Let f be analytic in domain D. Then it's uniquely determined over D by its values in D or along a line segment in D.

Proof: Let functions f and g be analytic in some domain D, and $f(z) = g(z)\ \forall z \in D$. Then $h(z) = f(z) - g(z))$ is also analytic in D, and $h(z) = 0$ in the subdomain or along the line segment, so $h(z) \equiv 0$ throughout D. □

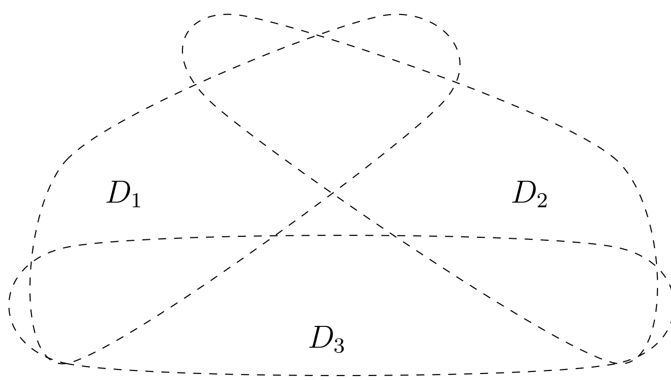
Theorem 13.8.2: Coincidence Principle

If functions f and g are analytic in D and $f(z) = g(z)$ in $D' \subset D$ with limit point $z_0 \in D$, then $f(z) = g(z)$ everywhere in D.

This is a more generalized version of theorem 13.8.1

Definition 13.8.1: Analytic continuation

Consider the domains D_1 and D_2 with intersection $D_1 \cap D_2$, and functions f_1 and f_2. If f_1 is analytic in D_1, and there exists f_2 that is analytic in D_2 such that $f_1(z) = f_2(2)$ for all $z \in D_1 \cap D_2$. Then f_2 is the analytic continuation of f_1.



Theorem 13.8.1 tells us that if such analytic continuation exists, then it is unique. Now if there exists f_3 in D_3 that is an analytic continuation of f_2, then it is not necessarily true that $f_3(z) = f_1(z)$ for all $z \in D_1 \cap D_3$. (See example 13.8.1.)

Definition 13.8.2: Elements of a function

Let f_2 be the analytic continuation of a function f_1 in D_1 into domain D_2, and let $F(z)$ be analytic in $D_1 \cup D_2$.

$$F(z) = \begin{cases} f_1(z) & z \in D_1 \\ f_2(z) & z \in D_2 \end{cases}$$

Then F is the analytic continuation of f_1 and f_2 into $D_1 \cup D_2$, and f_1 and f_2 are elements of F.

13.8.1 Reflection Principle

Generally, $\overline{f(z)} \neq f(\bar{z})$ for all z, but....

Theorem 13.8.3: Reflection Principle

Let f be a function with domain D containing a segment of the real axis $R \subset D$. Then

$$\forall z \in D[\overline{f(z)} = f(\bar{z})] \iff \forall x \in R[f(x) \in \mathbb{R}]$$

See example 13.8.2 for the case when $f(x)$ is purely imaginary.

Proof: Let $f(z)$ and $F(z)$ be analytic functions:

$$f(z) = u(x,y) + iv(x,y) \qquad\qquad F(z) = U(x,y) + iV(x,y)$$

$\underline{(\Longleftarrow):}$

Suppose $\forall x \in R[f(x) \in \mathbb{R}]$, and that $F(z) = \overline{f(\bar{z})}$.

$$f(z) = u(x,y) + iv(x,y) \qquad\qquad F(z) = U(x,y) + iV(x,y)$$

Then

$$\overline{f(\bar{z})} = u(x,-y) - iv(x,-y)$$

Therefore

$$U(x,y) = u(x,t) \qquad\qquad V(x,y) = -v(x,t) \qquad\qquad t = -y$$

$f(x,t)$ is analytic, so it satisfies the Cauchy-Riemann equations:

$$u_x = v_t \qquad\qquad u_t = -v_x$$

Hence

$$U_x = u_x \qquad\qquad V_y = -v_t\frac{\mathrm{d}t}{\mathrm{d}y} = v_t$$

$$U_y = u_t\frac{\mathrm{d}t}{\mathrm{d}y} = -u_t \qquad\qquad V_x = -v_x$$

Thus, we can see that $F(z)$ also satisfies the Cauchy Riemann equations

$$U_x = V_y \qquad\qquad U_y = -V_x$$

Since, the partial derivatives of U and V are continuous in D, we can say hat $F(z)$ is analytic in D. On the segment of the real axis $R \subset D$, $f(z)$ is real, so $v(x,0) = 0$.

$$\begin{aligned} F(x) &= U(x,0) + iV(x,0) = u(x,0) - iv(x,0) = u(x,0) \\ &\Longrightarrow \forall z \in R[F(z) = f(z)] \\ &\Longrightarrow \forall z \in D[\overline{f(\bar{z})} = f(z)] \qquad\qquad \textit{Theorem } 13.8.1 \end{aligned}$$

$\underline{(\implies):}$

Suppose $\overline{f(z)} = f(\bar{z})$. Then

$$u(x,-y) - iv(x,-y) = u(x,y) + iv(x,y)$$

Consider any point $(x,0) \in R \subset D$:

$$u(x,0) - iv(x,0) = u(x,0) + iv(x,0) \implies v(x,0) = 0$$

Hence, $f(x)$ is real $\forall x \in R \subset D$. $\square$

Theorem 13.8.3 tells us that if a complex function is real for all points on the real axis, then it will obey the Reflection Principle, and vice versa.

13.8.2 Examples

Example 13.8.1 *Consider*

$$\begin{aligned} f_1(z) &= \sqrt{r}e^{i\theta/2} && r>0,\ 0<\theta<\pi \\ f_2(z) &= \sqrt{r}e^{i\theta/2} && r>0,\ \frac{\pi}{2}<\theta<2\pi \\ f_3(z) &= \sqrt{r}e^{i\theta/2} && r>0,\ \pi<\theta<\frac{5\pi}{2} \end{aligned}$$

It is clear that $f_1,\ f_2,\ f_3$ are continuous and satisfies the Cauchy-Riemann equations throughout their domain of definition, since they have a derivative everywhere in their domain of definition. Hence, they are analytic continuations of each other. Let $D_1,\ D_2$, and D_3 be the domains of $f_1,\ f_2$, and f_3, respectively. Consider f_1 and f_3 in the domain $D_1 \cap D_3$, and any z in the first quadrant of the complex plane. Then $z = re^{i\theta} = re^{i(\theta+2\pi)}$ and we have

$$\begin{aligned} f_1(z) &= \sqrt{r}e^{i(\theta/2)} && 0<\theta<\pi \\ f_3(z) &= \sqrt{r}e^{i(\theta/2+\pi)} && 0<\theta<\pi \end{aligned}$$

Hence,

$$\begin{aligned} f_1(z) &= \sqrt{r}[\cos(\theta/2) + i\sin(\theta/2)] && 0<\theta<\pi \\ f_3(z) &= \sqrt{r}[\cos(\theta/2+\pi) + i\sin(\theta/2+\pi)] && 0<\theta<\pi \\ &= -\sqrt{r}[\cos(\theta/2) + i\sin(\theta/2)] \end{aligned}$$

Thus we can see that $f_1 = -f_3$ in $D_1 \cap D_3$.

Example 13.8.2 *Consider theorem 13.8.3, but $f(x)$ is purely imaginary $\forall x \in \mathbb{R}$. We know that $\Longleftarrow$ holds, and that $F(z) = \overline{f(\bar{z})}$ satisfies the Cauchy-Riemann equations. We have*

$$F(x) = U(x,0) + iV(x,0) = u(x,0) - iv(x,0) = -iv(x,0) = -f(x)$$

Hence

$$\overline{f(\bar{z})} = -f(z) \implies \overline{f(z)} = -f(\bar{z})$$

Chapter 14

Elementary Functions

14.1 Exponential Function

Definition 14.1.1: Exponential Function

Consider $z \in \mathbb{C}$, the exponential function is defined:

$$f(z) = e^z = e^{x+iy} = e^x[\cos(y) + i\sin(y)]$$

Where y is taken in radians.

Note: This is not the same as the polar form of a complex number (definition 12.3.3).

It is clear that the set of n-th roots of e:

$$\{e^{1/n} : n \in \mathbb{N}\}$$

and

$$|e^z| = e^x \qquad \arg(e^z) = y + 2n\pi \qquad n \in \mathbb{N} \cup \{0\}$$

The exponential function follows from the usual properties of exponentials. We also know that

$$\frac{\mathrm{d}}{\mathrm{d}z}e^z = e^z \qquad \forall z \in \mathbb{C}$$

so, e^z is entire. We should also note that e^z is periodic due to e^{iy}.

14.2 Logarithmic Function

Definition 14.2.1: Logarithmic Function

Consider any $z \in \mathbb{C}$ in exponential form:

$$\log(z) = \ln(r) + i(\theta + 2n\pi) = \ln(|z|) + i\arg(z) \qquad n \in \mathbb{Z}$$

Note: This is a multi-valued function.

Definition 14.2.2: Principal Value of the Logarithmic Function

Let $z \in \mathbb{C}$, the principal value of the logarithmic function is denoted by $\mathrm{Log}(z)$.

$$\mathrm{Log}(z) = \ln(r) + i\theta$$

It is clear that

$$\log(z) = \mathrm{Log}(z) + 2n\pi \qquad n \in \mathbb{Z}$$

and for any z on the real axis, the logarithmic function reduces to

$$\mathrm{Log}(z) = \ln(x) \qquad x \in \mathbb{R}$$

14.2.1 Branches and Derivatives of Logarithms

$\log(z)$ is a multi-valued function. Let $\alpha \in \mathbb{R}$:

$$\log(z) = \ln(r) + i\theta = u(r,\theta) + iv(r,\theta) \qquad\qquad r > 0, \alpha < \theta < \alpha + 2\pi$$

Note: If $\log(z)$ is defined on $\theta = \alpha$, then it is not continuous there, as there is a discontinuity between points near α and $\alpha + 2\pi$.

The first order partials of u and v are continuous in the domain, and satisfies the Cauchy-Riemann equations:

$$ru_r = v_\theta \qquad\qquad u_\theta = -rv_\theta$$

So its derivative exists everywhere in the domain.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}\log(z) = e^{-i\theta}(u_r + iv_r) = e^{i\theta}\left(\frac{1}{r} + i0\right) = \frac{1}{re^{i\theta}}& \\
\implies \frac{\mathrm{d}}{\mathrm{d}z}\log(z) = \frac{1}{z} \qquad\qquad & |z| > 0,\ \alpha < \arg(z) < \alpha + 2\pi \\
\implies \frac{\mathrm{d}}{\mathrm{d}z}\mathrm{Log}(z) = \frac{1}{z} \qquad\qquad & |z| > 0,\ -\pi < \mathrm{Arg}(z) < \pi
\end{aligned}$$

Definition 14.2.3: Branch

A branch is a single-valued function F of a multi-valued function f. F is analytic throughout some domain of f and assumes the one of the values of f.

Definition 14.2.4: Principal Branch

$$\mathrm{Log}(z) = \ln(r) + i\theta \qquad\qquad r > 0,\ -\pi < \theta < \pi$$

Definition 14.2.5: Branch Cut

A portion of a line or curved introduced to define a branch F of a multi-valued function f. Points on the branch cut of F are singular points of F.

Definition 14.2.6: Branch Point
A singular point common to all branch cuts of a multi-valued function f.

Example 14.2.1 *The branch cut for* $\text{Log}(z) = \ln(r) + i\theta$, $r > 0$, $-\pi < \theta < \pi$, *is the origin and* $\theta = \pi$.

Branch points for all branches of $\log(z)$ *is the origin.*

Different branches may result in different values.

Example 14.2.2 *Consider* $\log(i^2)$ *in the branch:*

$$\log(z) = \ln(r) + i\theta \qquad r > 0,\ \frac{\pi}{4} < \theta < \frac{9\pi}{4}$$

Then

$$\log\left(i^2\right) = \log(-1) = \ln(1) + i\pi = i\pi$$
$$2\log(i) = 2\left(\ln(1) + i\frac{\pi}{2}\right) = \pi i$$

Therefore

$$\log\left(i^2\right) = 2\log(i) \qquad r > 0,\ \frac{\pi}{4} < \theta < \frac{9\pi}{4}$$

Now consider the branch:

$$\log(z) = \ln(r) + i\theta \qquad r > 0,\ \frac{3\pi}{4} < \theta < \frac{11\pi}{4}$$

Then

$$\log\left(i^2\right) = \log(-1) = \ln(1) + i\pi = i\pi$$
$$2\log(i) = 2\left(\ln(1) + i\frac{5\pi}{2}\right) = 5\pi i$$

Therefore,

$$\log\left(i^2\right) \neq 2\log(i) \qquad r > 0,\ \frac{3\pi}{4} < \theta < \frac{11\pi}{4}$$

14.2.2 Identities of Logarithms

Let $z_1, z_2 \in \mathbb{C}$, then

$$\log(z_1 z_2) = \log(z_1) + \log(z_2)$$

can be interpreted as

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

therefore

$$\ln|z_1 z_2| + i\arg(z_1 z_2) = (\ln|z_1| + i\arg(z_1)) + (\ln|z_2| + i\arg(z_2))$$

Rest of the identities are the same as for elements in $\mathbb{R}$, but beware of branches and arguments.

Example 14.2.3 *Show* $\forall z_1, z_2 \in \mathbb{C}$

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log}(z_1) + \operatorname{Log}(z_2) + 2N\pi i \qquad N \in \{0, \pm 1\}$$

Consider:

$$\begin{aligned}
\log(z_1 z_2) &= \ln|z_1 z_2| + i\arg(z_1 z_2) \\
&= \ln(r_1) + \ln(r_2) + i\arg(z_1) + i\arg(z_2) \\
&= \ln(r_1) + \ln(r_2) + i\theta_1 + i\theta_2 + 2n\pi i && n \in \mathbb{Z} \\
&= \ln(r_1) + \ln(r_2) + i\operatorname{Arg}(z_1) + i\operatorname{Arg}(z_2) + 2n\pi i && n \in \mathbb{Z}
\end{aligned}$$

Then, since $-\pi < \operatorname{Arg}(z_1) < \pi$ *and* $-\pi < \operatorname{Arg}(z_2) < \pi$*:*

$$\begin{aligned}
\operatorname{Log}(z_1 z_2) &= \ln(r_1) + \ln(r_2) + i\operatorname{Arg}(z_1) + i\operatorname{Arg}(z_2) + 2N\pi i && N \in \{0, \pm 1\} \\
&= \operatorname{Log}(z_1) + \operatorname{Log}(z_2) + 2N\pi i && N \in \{0, \pm 1\}
\end{aligned}$$

14.2.3 Power Function

Definition 14.2.7: Power Function

Let $z, c \in \mathbb{C}$*. The Power Function:*

$$z^c = e^{c\log(z)} \qquad z \neq 0$$

Likewise

$$c^z = e^{z\log(c)} \qquad c \neq 0$$

The logarithm is multi-valued $\implies$ the power function is multi-valued.

The principle branch of the Power Function is log being replaced by Log:

$$\begin{aligned}
z^c &= e^{c\operatorname{Log}(z)} && z \neq 0 \\
c^z &= e^{z\operatorname{Log}(c)} && c \neq 0
\end{aligned}$$

When a branch is specified, $\log(z)$ becomes single-valued and analytic. Hence the derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}z}z^c = \frac{\mathrm{d}}{\mathrm{d}z}e^{c\log(z)} = \frac{c}{z}e^{c\log(z)} = cz^{z-1} \qquad |z| > 0,\ \alpha < \arg(z) < \alpha + 2\pi,\ \alpha \in \mathbb{R}$$

When value of $\log(c)$ is specified, c^z is entire function of z and

$$\frac{\mathrm{d}}{\mathrm{d}z}c^z = \frac{\mathrm{d}}{\mathrm{d}z}e^{z\log(c)} = e^{z\log(c)}\log(c) = c^z\log(c)$$

14.3 Trigonometric Functions

Recall: Definition 12.5.2. Likewise for any $z \in \mathbb{C}$:

Definition 14.3.1: Complex Sine and Cosine Functions

For any $z \in \mathbb{C}$:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

Sine and cosine are entire functions as e^{iz} and e^{-iz} are entire.

Taking the derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}z} e^{iz} = ie^{iz} \implies \left(\frac{\mathrm{d}}{\mathrm{d}z}\sin(z) = \cos z\right) \wedge \left(\frac{\mathrm{d}}{\mathrm{d}z}\cos(z) = -\sin(z)\right)$$

It's also easy to see that:

$$\sin(-z) = \sin(z) \qquad \cos(-z) = -\cos(z) \qquad e^{iz} = \cos(z) + i\sin(z)$$

The usual trigonometric identities apply, such as:

$$\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2)$$
$$\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin(z_2)\sin(z_2)$$

Now suppose $y \in \mathbb{R}$, and take the hyperbolic functions:

$$\sinh(y) = \frac{e^y - e^{-y}}{2} \qquad \cosh(y) = \frac{e^y + e^{-y}}{2}$$

Then we get:

$$\sin(iy) = i\sinh(y) \qquad \cos(iy) = \cosh(y)$$

If we let $z = x + iy$, we can define:

$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$
$$\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

and that

$$|\sin(z)|^2 = \sin^2(x) + \sinh^2(y)$$
$$|\cos(z)|^2 = \cos^2(x) + \sinh^2(y)$$

Note: Unlike in $\mathbb{R}$ where sine and cosine are bounded by 1 and -1, it is clear that sine and cosine are not bounded in the complex plane, since sinh is unbounded for all values of y.

14.3.1 Zeros and Singularities

Definition 14.3.2: Zero (function)
Let $f(z)$ be a function. A zero of f is a point z_0 such that

$$f(z_0) = 0$$

Theorem 14.3.1:
The zeros of $\cos(z)$ and $\sin(z)$ for $z \in \mathbb{C}$ is the same as the zeros of $\cos(x)$ and $\sin(x)$ for $x \in \mathbb{R}$, that is

$$\forall x \in \mathbb{R} \forall z \in \mathbb{C} \forall n \in \mathbb{Z} \left[(\cos(x) = 0) \wedge (\cos(z) = 0) \iff z = x = \frac{\pi}{2} + n\pi \right]$$
$$\forall x \in \mathbb{R} \forall z \in \mathbb{C} \forall n \in \mathbb{Z} [(\sin(x) = 0) \wedge (\sin(z) = 0) \iff z = x = n\pi]$$

Proof: Let $z = x + iy$ and consider $\sin(z) = 0$:

$$\begin{aligned} \sin(z) = 0 &\implies \sin^2(x) + \sinh^2(y) = 0 && |\sin(z)|^2 = \sin^2(x) + \sinh^2(y) \\ &\implies [\sin(x) = 0] \wedge [\sinh(y) = 0] \\ &\implies [x = n\pi] \wedge [y = 0] && n \in \mathbb{Z} \\ &\implies z = x = n\pi && n \in \mathbb{Z} \end{aligned}$$

As for cosine, we know that:

$$\cos(z) = \sin\left(z + \frac{\pi}{2}\right)$$

Thus

$$\cos(z) = 0 \implies z = x = n\pi + \frac{\pi}{2} \qquad n \in \mathbb{Z}$$

□

Example 14.3.1 *Show $\forall z \in \mathbb{C}$:*
The Reflection Principle:

$$\forall z \in D \subset \mathbb{C}[\overline{f(z)} = f(\bar{z})] \iff \forall x \in \mathbb{R}[f(x) \in \mathbb{R}]$$

(a) $\overline{\cos(z)} = \cos(\bar{z})$

Proof: It is clear that $\forall x \in \mathbb{R}$, $\sin(x) \in \mathbb{R}$. The result follows from the Reflection Principle. Also

$$\overline{\sin(z)} = \overline{\frac{z - \bar{z}}{2i}} = \frac{\bar{z} - z}{2i} = \sin(\bar{z})$$

□

(b) $\overline{\sin(z)} = \sin(\bar{z})$

Proof: It is clear that $\forall x \in \mathbb{R}$, $\cos(x) \in \mathbb{R}$. The result follows from the Reflection Principle. Also

$$\overline{\cos(z)} = \overline{\frac{z+\bar{z}}{2}} = \frac{\bar{z}+z}{2} = \cos(\bar{z})$$

$\square$

Example 14.3.2 *Show:*

(a) $\forall z \in \mathbb{C}[\overline{\cos(iz)} = \cos(i\bar{z})]$

Proof:

$$\overline{\cos(iz)} = \overline{\frac{iz+\overline{iz}}{2}} = \frac{\overline{iz}+iz}{2} = \frac{-i\bar{z}+iz}{2} = i\frac{z-\bar{z}}{2} = i\,\mathrm{Im}\{z\}$$
$$\cos(i\bar{z}) = \frac{i\bar{z}+\overline{i\bar{z}}}{2} = \frac{i\bar{z}-iz}{2} = i\frac{\bar{z}-z}{2} = i\,\mathrm{Im}\{z\}$$

Hence $\forall z \in \mathbb{C}$

$$\overline{\cos(iz)} = \cos(i\bar{z})$$

$\square$

(b) $\forall z \in \mathbb{C}\forall n \in \mathbb{Z}[\overline{\sin(iz)} = \sin(i\bar{z}) \iff z = n\pi i]$

Proof:

$$\overline{\sin(iz)} = \overline{\frac{iz-\overline{iz}}{2i}} = \frac{-i\bar{z}-iz}{2i} = -\frac{z+\bar{z}}{2} = -\mathrm{Re}\{z\}$$
$$\sin(i\bar{z}) = \frac{i\bar{z}-\overline{i\bar{z}}}{2i} = \frac{i\bar{z}+iz}{2i} = \frac{z+\bar{z}}{2} = \mathrm{Re}\{z\}$$

We know that

$$\mathrm{Re}\{z\} = -\mathrm{Re}\{z\} \implies \mathrm{Re}\{z\} = 0 \implies \overline{\sin(iz)} = \sin(i\bar{z}) = 0 \iff z = n\pi i$$

$\square$

14.4 Hyperbolic Functions

Definition 14.4.1: Hyperbolic Sine and Cosine Functions

Let $z \in \mathbb{C}$:

$$\sinh(z) = \frac{e^z - e^{-z}}{2} \qquad\qquad \cosh(z) = \frac{e^z + e^{-z}}{2}$$

It is clear that the derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}z}\sinh(z) = \cosh(z) \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}z}\cosh(z) = \sinh(z)$$

The relationships with sine and cosine:

$$\begin{aligned} -i\sinh(iz) &= \sin(z) & \cosh(iz) &= \cos(z) \\ -i\sin(iz) &= \sinh(z) & \cos(iz) &= \cosh(z) \end{aligned}$$

Hence in the complex plane, sinh and cosh are periodic with period $2\pi i$.

Identities:

$$\sinh(-z) = -\sinh(z) \qquad \cosh(-z) = \cosh(z) \qquad \cosh^2(z) - \sinh^2(z) = 1$$

$$\begin{aligned} \sinh(z_1 + z_2) &= \sinh(z_1)\cosh(z_2) + \cosh(z_1)\sinh(z_2) \\ \cosh(z_1 + z_2) &= \cosh(z_2)\cosh(z_2) + \sinh(z_1)\sinh(z_2) \end{aligned}$$

$$\begin{aligned} \sinh(z) &= \sinh(x)\cos(y) + i\cosh(x)\sin(y) \\ \cosh(z) &= \cosh(x)\cos(y) + i\sinh(x)\sin(y) \end{aligned}$$

$$\begin{aligned} |\sinh(z)|^2 &= \sinh^2(x) + \sin^2(y) \\ |\cosh(z)|^2 &= \sinh^2(x) + \cos^2(y) \end{aligned}$$

Theorem 14.4.1:

The zeros of hyperbolic zine and cosine:

$$\begin{aligned} \sinh(z) = 0 &\iff z = n\pi i & n \in \mathbb{Z} \\ \cosh(z) = 0 &\iff z = \left(\frac{\pi}{2} + n\pi\right)i & n \in \mathbb{Z} \end{aligned}$$

Example 14.4.1 *Show:*

(a) $\sinh(z + \pi i) = -\sinh(z)$

Proof:

$$\sinh(z + \pi i) = \frac{e^{z+\pi i} - e^{-z-\pi i}}{2} = \frac{-e^z + e^{-z}}{2} = -\frac{e^z - e^{-z}}{2} = -\sinh(z)$$

□

(b) $\cosh(z + \pi i) = -\cosh(z)$

Proof:

$$\cosh(z+\pi i) = \frac{e^{z+\pi i} + e^{-z-\pi i}}{2} = -\frac{e^{z} + e^{-z}}{2} = -\cosh(z)$$

□

(c) $\tanh(z+\pi i) = \tanh(z)$

Proof:

$$\tanh(z+\pi i) = \frac{\sinh(z+\pi i)}{\cosh(z+\pi i)} = \frac{-\sinh(z)}{-\cosh(z)} = \tanh(z)$$

□

Example 14.4.2 *Show* $\forall z \in \mathbb{C}$*:*

$$\overline{\sinh(z)} = \sinh(\bar{z}) \qquad \overline{\cosh(z)} = \cosh(\bar{z}) \qquad \forall z \neq 0\left[\overline{\tanh(z)} = \tanh(\bar{z})\right]$$

Proof: We can see that $\forall x \in \mathbb{R}$, $\sinh(x) \in \mathbb{R}$ and $\cosh(x) \in \mathbb{R}$, so we can conclude from the Reflection Principle (theorem 13.8.3) that $\forall z \in \mathbb{C}$:

$$\overline{\sinh(z)} = \sinh(\bar{z}) \qquad \overline{\cosh(z)} = \cosh(\bar{z})$$

Thus it follows that

$$\forall z \neq 0\left[\overline{\tanh(z)} = \tanh(\bar{z})\right]$$

□

14.5 Inverse Trigonometric and Hyperbolic Functions

Definition 14.5.1: Inverse Trigonometric Functions

Let $z \in \mathbb{C}$*:*

$$\sin^{-1}(z) = -i\log[iz + (1-z^2)^{1/2}]$$
$$\cos^{-1}(z) = -i\log[z + i(i-z^2)^{1/2}]$$
$$\tan^{-1}(z) = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right)$$

$\cos^{-1}(z)$ *and* $\tan^{-1}(z)$ *are multi-valued. All inverse trigonometric functions become single-valued and analytic when in specific branches of the square root and logarithmic functions.*

Proof: $\underline{\sin^{-1}(z) = -i\log[iz + (1-z^2)^{1/2}]}$
Let $w = \sin^{-1}(z)$ whenever $z = \sin(w)$

$$z = \sin(w) \implies z = \frac{e^{iw} - e^{-iw}}{2i} \implies (e^{iw})^2 - 2ize^{iw} - 1 = 0$$

Using the quadratic formula to solve for e^{iw}:

$$\begin{aligned} e^{iw} = iz + (1-z^2)^{1/2} &\implies iw = \log\left(iz + (1-z^2)^{1/2}\right) \\ &\implies \sin^{-1}(z) = -i\log\left[iz + (1-z^2)^{1/2}\right] \end{aligned}$$

$\underline{\cos^{-1}(z) = -i\log[z + i(i-z^2)^{1/2}]}$
Likewise, let $w = \cos^{-1}(z)$ whenever $z = \cos(w)$

$$z = \cos(w) \implies z = \frac{e^{iw} + e^{-iw}}{2} \implies (e^{iw})^2 - 2ze^{iw} + 1 = 0$$

Using the quadratic formula to solve for e^{iw}:

$$\begin{aligned} e^{iw} &= \frac{2z \pm \sqrt{4z^2 - 4}}{2} = z \pm \sqrt{z^2 - 1} = z \pm i(1-z^2)^{1/2} \\ &\implies iw = \log\left[z \pm i(1-z^2)^{i/2}\right] \\ &\implies w = -i\log\left[z \pm i(1-z^2)^{i/2}\right] \end{aligned}$$

$\underline{\tan^{-1}(z) = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right)}$
Again, let $w = \tan^{-1}(z)$ whenever $z = \tan(w)$

$$\begin{aligned} z = \tan(w) &= \frac{e^{iw} - e^{-iw}}{i(e^{iw} + e^{iw})} \\ &\implies iz = \frac{e^{iw} - e^{-iw}}{e^{iw} + e^{iw}} \implies ize^{iw} + ize^{-iw} = e^{iw} - e^{-iw} \\ &\implies (iz-1)e^{iw} + (iz+1)e^{-iw} = 0 \implies (iz-1)e^{2iw} + (iz+1) = 0 \\ &\implies e^{iw} = \left(\frac{-iz-1}{iz-1}\right)^{\frac{1}{2}} \implies iw = \frac{1}{2}\log\left(\frac{-iz-1}{iz-1}\right) \implies w = -\frac{i}{2}\log\left(\frac{-(iz+1)}{iz-1}\right) \\ &\implies w = \tan^{-1}(z) = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right) \end{aligned}$$

$\square$

Derivatives:

$\frac{\mathrm{d}}{\mathrm{d}z}\sin^{-1}(z) = \frac{1}{(1-z^2)^{1/2}}$	Depends on value coosen for square root
$\frac{\mathrm{d}}{\mathrm{d}z}\cos^{-1}(z) = -\frac{1}{(1-z^2)^{1/2}}$	Depends on value chosen for square root
$\frac{\mathrm{d}}{\mathrm{d}z}\tan^{-1}(z) = \frac{1}{1+z^2}$	Independent on value chosen for square root

Using the same procedures on the hyperbolic functions, we obtain the inverse hyperbolic functions:

Definition 14.5.2: Inverse Hyperbolic Functions
Let $z \in \mathbb{C}$:

$$\sinh^{-1}(z) = \log\left[z + (z^2+1)^{1/2}\right]$$
$$\cosh^{-1}(z) = \log\left[z + (z^2-1)^{1/2}\right]$$
$$\tanh^{-1}(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

14.6 Phasors

Definition 14.6.1: Phasor
Consider the function

$$f(t) = \mathrm{Re}\{Fe^{st}\} \qquad F = F_0 e^{i\theta} \qquad s = \sigma + i\omega,\ \sigma, \omega \in \mathbb{R}$$

F is the phasor associated with $f(t)$.

We can see tha $f(t)$ grows exponentially according to the value of σ, has a phase of θ, and a phase frequency of ω. (It is clear why engineers love this.)

Properties:

1. $$f(t) = \mathrm{Re}\{Fe^{st}\} \implies \begin{cases} F \text{ is unique} & \omega \neq 0 \\ \text{Only } \mathrm{Re}\{F\} \text{ is unique} & \omega = 0 \end{cases}$$

2. $$\forall t \in \mathbb{R}[f(t) = g(t)] \implies \begin{cases} F = G & \omega \neq 0 \\ \mathrm{Re}\{F\} = \mathrm{Re}\{G\} & \omega = 0 \end{cases}$$

3. For any given $s = \sigma + i\omega$, there is only one function of t corresponding to a phasor.

4. Let $f(t)$ and $g(t)$ have the same complex frequency, that is, $\omega_1 = \omega_2$, then the phasor for $f(t) + g(t)$ is $F + G$.

5. $\forall M \in \mathbb{R}$. The phasor for $Mf(t)$ is MF.

6. "For a given complex frequency, the function of t corresponding to the sum of two or more phasors is the sum of the time functions for each." -Wunsch

7. Let $n \in \mathbb{N}$.

$$(f(t) \text{ has phasor } F) \wedge (\mathrm{d}f/\mathrm{d}t \text{ has phasor } sF) \implies \mathrm{d}^n f/\mathrm{d}t^n \text{ has phasor } s^n F$$

8.

$$f(t) \text{ has phasor } F \implies \int^t f(t')dt' \text{ has phasor } \frac{F}{s} \qquad\qquad s \neq 0$$

These properties follow from the properties of e.

Example 14.6.1 *Consider:*

$$R\mathtt{i}(t) + L\frac{\mathrm{d}\mathtt{i}}{\mathrm{d}t} = V_0 \cos(\omega t)$$

Suppose the complex frequency is $s = i\omega$. If I is the phasor for $\mathtt{i}$, then substituting it into the differential equation:

$$RI + iwLI = (R + i\omega L)I = V_0 \qquad\qquad \textit{Phasors on both side must equal}$$

Then solving for the phasor:

$$I = \frac{V_0}{R + i\omega L} = \frac{V_0 e^{i\theta}}{\sqrt{R^2 + \omega^2 L^2}} \qquad\qquad \theta = -\tan^{-1}\left(\frac{\omega L}{R}\right)$$

Using $s = i\omega$ to obtain $\mathtt{i}(t)$, we have

$$\mathtt{i}(t) = \mathrm{Re}\left\{\frac{V_0 e^{i\theta}}{\sqrt{R^2 + (\omega L)^2}} e^{i\omega t}\right\}$$

Chapter 15

Integrals

15.1 Derivatives of Functions

Definition 15.1.1: Derivative of Complex-Valued Function

Consider a complex-valued function $w(t) = u(t) + iv(t)$, with u and v being real-valued functions. If $w(t)$ is differentiable at t, then it's derivative with respect to t:

$$w'(t) = \frac{\mathrm{d}}{\mathrm{d}t} w(t) = u'(t) + iv'(t)$$

The rules for calculus in $\mathbb{R}$ still applies.

Note: The mean value theorem for derivatives no longer apply for complex-valued functions.

Example 15.1.1 *Let $w(t) = e^{it}$ be continuous on $[0, 2\pi]$, so $u(t)$ and $v(t)$ are also continuous on $[0, 2\pi]$. For the mean value theorem to hold, there must exist $a, b, c \in \mathbb{C}$, where $a < c < b$, such that*

$$w'(c) = \frac{w(b) - w(a)}{b - a}$$

We can see that

$$|w'(t)| = \left|ie^{it}\right| = 1 \qquad \frac{w(b) - w(a)}{b - a} = \frac{w(2\pi) - w(0)}{2\pi - 0} = \frac{e^{i2\pi} - e^{i0}}{2\pi} = \frac{1 - 1}{2\pi} = 0$$

So we can see that there does not exist a $c \in \mathbb{C}$ such that the mean value theorem holds.

15.2 Definite Integrals of Functions

Definition 15.2.1: Definite Integral of Complex-Valued Function

Consider a complex-valued function $w(t) = u(t) + iv(t)$, with u and v being real-valued *functions. If u and v are piecewise continuous on interval $[a,b]$, then the definite integral of $w(t)$ over interval $[a,b]$:*

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

The rules for integrals in $\mathbb{R}$ and the Fundamental Theorem of Calculus still applies.

Likewise with derivatives of complex-valued functions, the mean value theorem does not hold for complex-valued integrals.

Example 15.2.1 *Let $w(t) = e^{it}$ be a complex-valued function of t. For $w(t)$ to hold on $[a,b]$, this must hold for some $a < c < b$:*

$$\int_a^b w(t)dt = w(c)(b-a)$$

Consider $w(t)$ on $[0, 2\pi]$. Then

$$\int_a^b w(t)dt = \int_0^{2\pi} e^{it}dt = \frac{e^it}{i}\bigg|_0^{2\pi} = 0 \qquad\qquad |w(c)(b-a)| = \left|e^{ic}\right|2\pi = 2\pi$$

We can see there does not exist c, $0 < c < 2\pi$, such that both sides of the equations are equal.

15.3 Contours

Definition 15.3.1: Arc

An arc is a set of points dependent on a parameter $t \in \mathbb{R}$.

$$\{z = (x(t), y(t)) : t \in [a,b]\}$$

Where $x(t)$ and $y(t)$ are continuous functions.

It is convenient in $\mathbb{C}$ to use:

$$z = z(t) = x(t) + iy(t) \qquad\qquad a \le t \le b$$

Definition 15.3.2: Simple/Jordan Arc

An arc is simple if it does not cross itself. That is: $t_1 \neq t_2 \implies z(t_1) \neq z(t_2)$

Definition 15.3.3: Simple Closed Curve / Jordan Curve

A simple curve, but endpoints gets mapped to equal values. That is, $z(a) = z(b)$ for $a \le t \le b$. It is positively oriented if it is counterclockwise.

The interval for which the arc is parameterized is not unique. Consider

$$t = \phi(\tau) \qquad\qquad \alpha \le \tau \le \beta$$

Then $\phi(\alpha) = a$ and $\phi(\beta) = b$. We have

$$z(t) = Z(\tau) = z[\phi(\tau)] \qquad\qquad \alpha \le \tau \le \beta$$

Definition 15.3.4: Differentiable Arc

Suppose $\mathrm{d}/\mathrm{dt}\ z(t) = z'(t) = x'(t) + iy'(t)$ *exists and is continuous. Then* $z'(t)$ *is a differentiable arc.*

We can integrate over the differential arc in the interval $[a, b]$:

$$L = \int_a^b |z'(t)|dt \qquad\qquad |z'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

Note: $|z'(t)|$ is a real-valued function.

Again, due to the curve being invariant under the representation for the arc:

$$L = \int_a^b |z'(t)|dt = \int_\alpha^\beta |z'[\phi(t)]|\phi'(t)dt = \int_\alpha^\beta |Z'(\tau)|d\tau \qquad Z'(\tau) = z'[\phi(\tau)]\phi'(\tau)$$

If the differentiable arc $z'(t) \neq 0$ in the interval $[a, b]$, then the unit tangent vector is defined in said interval:

$$\hat{\mathbf{T}} = \frac{z'(t)}{|z'(t)|}$$

Recall: The gradient of a function is perpendicular to the function, so $\hat{\mathbf{T}}$ is normal to $z(t)$, over the interval $[a, b]$.

Definition 15.3.5: Smooth

An arc $z(t)$ *is smooth in the interval* $[a, b]$ *if* $z'(t)$ *is continuous* $\forall t \in [a, b]$ *and non-zero* $\forall t \in (a, b)$.

Definition 15.3.6: Contour

A piecewise smooth arc.

Definition 15.3.7: Simple Closed Contour

A contour where only $z(a) = z(b)$ *in the interval* $[a, b]$.

Theorem 15.3.1: Jordan Curve Theorem

All points on a simple close curve or simple closed contour $z(t)$ *are boundary points of two distinct domains. One is bounded and interior to* $z(t)$ *and the other is unbounded and exterior to* $z(t)$*. That is,* $z(t)$ *as a boundary line of two domains.*

Proof: Good luck! □

i.e. If $z(t)$ is a circle, then one domain is within the circle and contains the points on the parameter, and one is exterior to the circle.

15.4 Contour Integrals

Evaluating an integral over a contour. It is common to assume that a line integral represents an area under a curve. Generally, this is a far too simplistic approach, even for real-valued functions. Consider the limit definition of the line integral.

Definition 15.4.1: Line Integral (Real)

Let f be a real-valued function.

$$\int_a^b f(z)ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\Delta s_k \qquad \Delta s_k = (x_k - x_{k-1}) + (y_k - y_{k-1})$$

Note: The integral exists only if the limit exists.

We can see that Δs_k acts like a vector, and the definition of the line integral assigns a value according to the weighting function f to each Δs_k. The contour integral is then the weighted sum of these vectors from a to b as $n \to \infty$.

The complex line integral is defined similarly.

Definition 15.4.2: Line Integral (Complex)

Let f be a complex-valued function.

$$\int_a^b f(z)dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(z_k)\Delta z_k \qquad \Delta z = (x_k - x_{k-1}) + i(y_k - y_{k-1})$$

Upon expanding, we can see that:

$$\begin{aligned}
\int_a^b f(z)dz &= \lim_{n\to\infty} \sum_{k=1}^{n} f(z_k)\Delta z_k \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} [u(x_k, y_k) + iv(x_k, y_k)](\Delta x_k + i\Delta y_k) \\
&= \int_a^b (u + iv)(dx + idy) = \int_a^b u\ dx - v\ dy + i \int_a^b v\ dx + u\ dy
\end{aligned}$$

These are the integrals taken in each direction when evaluating from a to b for $a, b \in \mathbb{C}$.

Note: This reduces to a regular integral $\int_a^b u\ dx$ in the reals when $v = 0$ and $dy = 0$.

A contour integral is a line integral over a contour. Here we define it parameterized by t.

Definition 15.4.3: Contour Integral

Let $z(t)$ in $C = [a, b]$ be a contour, and $f[z(t)]$ be a piecewise continuous function on C. Then the contour integral:

$$\int_C f(z)dz = \int_a^b f[z(t)]z'(t)dt$$

Note: C is contour $\implies$ $z'(t)$ is piecewise continuous on C $\implies$ Existence of integral on C

Notation: Let $f(z)$ be a function evaluated over the contour C.

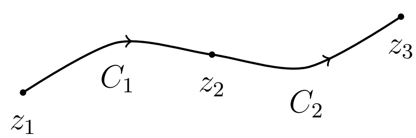
$$\int_C f(z)dz \qquad\qquad \int_{z_1}^{z_2} f(z)dz$$

$\int_{z_1}^{z_2}$ is often used when the integral is independent of the path between end points. $\int_{-C}$ represents the same contour, but in reverse.

Following from the contours, the integral is invariant under change in representation of the contour.

Definition 15.4.4: Sum (Contour)

Let C_1 be contour from z_1 to z_2, and C_2 be from z_2 to z_3, then the sum is contour C from z_1 to z_3.



Some properties (which follows from integrals):

$$\int_C = z_0 f(z)dz = z_0 \int_C f(z)dz \qquad \int_C f(z) + g(z)dz = \int_C f(z)dz + \int_C g(z)dz$$

$$\begin{aligned}\int_{-C} f(z)dz &= \int_{-b}^{-a} f([z(-t)])\frac{\mathrm{d}}{\mathrm{d}t}z(-t)dt = -\int_{-b}^{-a} f[z(-t)]z'(-t)dt \\ &= -\int_{-b}^{-a} f[z(\tau)]z'(\tau)d\tau \\ &= -\int_C f(z)dz\end{aligned}$$

$$\begin{aligned}\int_a^b f[z(t)]z'(t)dt &= \int_a^c f[z(t)]z'(t)dt + \int_c^b f[z(t)]z'(t)dt \\ \int_C f(z)dz &= \int_{C_1} f(z)dz + \int_{C_2} f(z)dz\end{aligned}$$

15.4.1 Upper Bounds for the Moduli

It is not analysis without inequality involving modulus.

Lemma 15.4.0.1:

Let $w(t)$ be a piecewise smooth function defined on $[a, b]$. Then

$$\left|\int_a^b w(t)dt\right| \leq \int_a^b |w(t)|dt$$

Proof: Assume $\int_a^b w(t)dt$ is non-zero, otherwise the inequality is trivial.

$$\begin{aligned}\int_a^b w(t)dt = r_0e^{i\theta_0} &\implies r_0 = e^{-i\theta_0}\int_a^b w(t)dt \\ &\implies r_0 = \operatorname{Re}\Big\{e^{-i\theta_0}\int_a^b w(t)dt\Big\} \qquad\qquad r_0 \in \mathbb{R} \\ &\implies r_0 = \int_a^b \operatorname{Re}\big\{e^{i\theta_0}w(t)\big\}dt\end{aligned}$$

Now

$$\begin{aligned}\operatorname{Re}\big\{e^{-i\theta_0}w(t)\big\} \le \big|e^{-i\theta_0}w(t)\big| &= \big|e^{-i\theta_0}\big||w(t)| = |w(t)| \\ \implies r_0 &\le \int_a^b |w(t)|dt \\ \implies \left|\int_a^b w(t)dt\right| &\le \int_a^b |w(t)|dt \qquad\qquad r_0 = \left|\int_a^b w(t)dt\right|\end{aligned}$$

□

Theorem 15.4.1: ML Inequality

Let $f(z)$ be piecewise continuous function on contour C with length L.

$$\forall z \in C\ \exists M \in \mathbb{R}\,[|f(z)| \le M] \implies \left|\int_C f(z)dz\right| \le ML$$

That is, if $f(z)$ is bounded on the contour, then the value of it's integral is bounded.

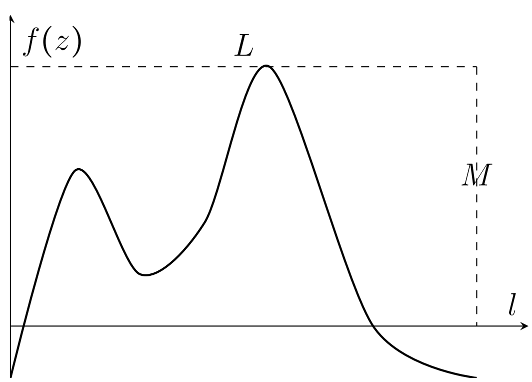
Proof: Let $z = z(t)$ in $[a,b]$ be a parametric representation of C. Then

$$\begin{aligned}\left|\int_C f(z)dz\right| = \left|\int_a^b f[z(t)]z'(t)dt\right| \le \int_a^b |f[z(t)]z'(t)|dt &= \int_a^b |f[z(t)]||z'(t)|dt \\ \le M\int_a^b |z'(t)|dt = M|z(t)| &= ML\end{aligned}$$

□

Note: According to the Extreme Value Theorem, any continuous real-valued function on a closed interval is bounded, so such $M \in \mathbb{R}$ will always exist.

Observation. *Let l be the length along the contour C. Graphically, what this is telling us:*



This is useful in evaluating the size of the integral, and if we are lucky:

Example 15.4.1 *Let C_R be the semicircle:*

$$z = Re^{i\theta} \qquad\qquad 0 \le \theta \le \pi,\ \ R > 3$$

Consider

$$\lim_{r\to\infty} \int_{C_R} \frac{z+1}{(z^2+4)(z^2+9)} dz$$

We know that:

$$\begin{aligned}
|z+1| &\le |z| + 1 = R + 1 \\
\left|z^2+4\right| &\ge \left||z|^2 - 4\right| = R^2 - 4 \\
\left|z^2+9\right| &\ge \left|\left|z^2\right| - 9\right| = R^2 - 9
\end{aligned}$$

Then

$$|f(z)| = \left|\frac{z+1}{(z^2+4)(z^2+9)}\right| = \frac{|z+1|}{|z^2+4||z^2+9|} \le \frac{R+1}{(R^2-4)(R^2-9)} = M_R$$

Now we have

$$M_R = \frac{R+1}{(R^2-4)(R^2-9)} \qquad\qquad L = \pi R$$

Since

$$\left|\int_{C_R} \frac{z+1}{(z^2+4)(z^2+9)} dz\right| \le M_R L$$

Hence

$$\lim_{R\to\infty} M_R L = \lim_{R\to\infty} \frac{R^2+R}{(R^2-4)(R^2-9)}\pi = \lim_{R\to\infty} \frac{\frac{1}{R^2}+\frac{1}{R^3}}{\left(1-\frac{4}{R^2}\right)\left(1-\frac{9}{R^2}\right)}\pi = 0$$

Thus

$$\lim_{r\to\infty} \int_{C_R} \frac{z+1}{(z^2+4)(z^2+9)} dz = 0$$

15.5 Antiderivatives

Definition 15.5.1: Antiderivative

Let $f(z)$ be a continuous function on domain D, the antiderivative is a function $F(z)$ such that

$$F'(z) = f(z) \qquad\qquad \forall z \in D$$

Note: By definition, $F(z)$ is an analytic function, and an antiderivative is unique up to an additive constant. An indefinite integral is the family of functions that are the antiderivative of a particular function.

Theorem 15.5.1:

Let $f(z)$ be a continuous function on domain D. TFAE:

(a) *There is a function $F(z)$ such that*

$$\forall z \in D[F'(z) = f(z)]$$

($f(z)$ has an antiderivative throughout D.)

(b) *All contours of $f(z)$ in D from any point z_1 to z_2 all have the same value. That is*

$$\int_{z_1}^{z_2} f(z)dz = F(z)\Big|_{z_1}^{z_2} = F(z_2) - F(z_1)$$

(c) *For all closed contours C lying in D:*

$$\oint_C f(z)dz = 0$$

Simply:

$$\forall z \in D[F'(z) = f(z)] \iff \int_{z_1}^{z_2} f(z)dz = F(z)\Big|_{z_1}^{z_2} \iff \oint_C f(z)dz = 0$$

Proof: (a) $\implies$ (b):

Suppose $F'(z)$ exists for $f(z)$ for all $z \in D$. We know:

$$\frac{\mathrm{d}}{\mathrm{d}t}F[z(t)] = F'(z(t))z'(t) = f[z(t)]z'(t) \qquad t \in [a,b]$$

Then

$$\int_C f(z)dz = \int_a^b f[z(t)]z'(t)dt = F[z(t)]\Big|_a^b = F[z(b)] - F[z(a)] = F(z_2) - F(z_1)$$

If C consists of a finite number of smooth arc C_k, $k \in \{1,2,3,\ldots,n\}$:

$$\int_C f(z)dz = \sum_{k=1}^{n} \int_C f(z)dz = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} f(z)dz = \sum_{k=1}^{n} [F(z_{k+1}) - F(z_k)]$$

Then

$$\int_C f(z)dz = F(z_{n+1}) - F(z_1)$$

Thus

$$\forall z \in D[F'(z) = f(z)] \implies \int_{z_1}^{z_2} f(z)dz = F(z)\Big|_{z_1}^{z_2}$$

<u>(b) $\implies$ (c):</u>
Let C_1 and C_2 be contours with endpoints z_1 and z_2. Suppose that integration is path independent, then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz \implies \int_{C_1} f(z)dz + \int_{-C_2} f(z)dz = 0 \implies \int_{C=C_1-C_2} f(z)dz = 0$$

Thus

$$\int_{z_1}^{z_2} f(z)dz = F(z)\Big|_{z_1}^{z_2} \implies \oint_C f(z)dz = 0$$

<u>(c) $\implies$ (a):</u>
Suppose integration around a closed contour C in D is zero. Since integration is path independent in D, we can define the function:

$$F(z) = \int_{z_0}^{z} f(z)ds$$

Let $z + \Delta z$ be any point in the neighbourhood of z contained in D, then

$$F(z+\Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(z)ds - \int_{z_0}^{z} f(z)ds = \int_{z}^{z+\Delta z} f(s)ds$$

Since

$$\int_{z}^{z+\Delta z} ds = \Delta z \implies f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)ds$$

Then

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(s) - f(z)ds$$

As f is continuous at the point z:

$$\forall \epsilon \exists \delta [|s - z| < \delta \implies |f(s) - f(z)| < \epsilon]$$

If $|\Delta z| < \delta$:

$$\left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right| < \frac{1}{|\Delta z|}\epsilon|\Delta z| = \epsilon$$

Then

$$\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z) \implies F'(z) = f(z)$$

Thus

$$\oint_C f(z)dz = 0 \implies \forall z \in D[F'(z) = f(z)]$$

□

Example 15.5.1 *Let $f(z) = z^{-2}$. We can see that f is continuous everywhere except at the origin, and has antiderivative $F(z) = -z^{-1}$ in $|z| > 0$. Thus around the unit circle:*

$$\int_C z^{-2} dz = 0 \qquad\qquad z = e^{i\theta},\ \theta \in [-\pi, \pi]$$

Example 15.5.2 *Consider $f(z) = z^{-1}$. It has an antiderivative $F(z) = \log(z)$, which is not differentiable or defined along its branch cut. To evaluate the integral along the unit circle, we can break it up into two domains to avoid this issue. First consider C_1:*

$$z = e^{i\theta} \qquad\qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

Then

$$\begin{aligned}\int_{C_1} z^{-1} dz = \int_{-i}^{i} z^{-1} dz = \mathrm{Log}(z)\Big|_{-i}^{i} &= \mathrm{Log}(i) - \mathrm{Log}(-i) \\ &= \left(\ln(1) + i\frac{\pi}{2}\right) - \left(\ln(1) - i\frac{\pi}{2}\right) = \pi i\end{aligned}$$

Now consider C_2:

$$z = e^{i\theta} \qquad\qquad \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$$

Then

$$\begin{aligned}\int_{C_2} = z^{-1} dz = \int_{i}^{-i} z^{-1} dz = \log(z)\Big|_{i}^{-i} &= \log(-i) - \log(i) \\ &= \left(\ln(1) + i\frac{3\pi}{2}\right) - \left(\ln(1) + i\frac{\pi}{2}\right) = \pi i\end{aligned}$$

Thus around the circle $C = C_1 + C_2$:

$$\int_C z^{-1} dz = \int_{C_1} z^{-1} dz + \int_{C_2} z^{-1} dz = \pi i + \pi i = 2\pi i$$

15.6 Cauchy-Goursat Theorem

Previously: A function f that has an antiderivative in any domain D, then the integral of f around any closed contour in D is zero. (Theorem 15.5.1) Now, it's for simple closed contours.

Recall:

$$\begin{aligned}\int f(z) dz = \int (u + iv)(dx + idy) &= \int u\ dx - v\ dy + i \int v\ dx + u\ dy \\ &= \iint -v_x - u_x\ dx\ dy + i \iint u_x - v_y dx\ dy && \textit{Theorem } 19.4.1 \\ &= 0 && \textit{Theorem } 13.5.1\end{aligned}$$

This result requires $f'(z)$ be continuous, due to the requirement of Green's Theorem (theorem 19.4.1). The Cauchy-Goursat theorem eliminates this requirement.

Theorem 15.6.1: Cauchy-Goursat Theorem

Let C be a simple closed contour. If a function f is analytic for all set of points z on and in C, then

$$\int_C f(z)dz = 0$$

Proof: First, a lemma:

Lemma 15.6.1.1:

Let C be a closed contour, R denote the region enclosed and on the contour, and f be a function analytic in R. The region R can be covered by a finite number of squares or partial squares, indexed $j = 1, 2, \dots, n$, such that for some $\epsilon > 0$:

$$\left|\frac{f(z) - f(z_j)}{z - z_j} - f'(z_j)\right| < \epsilon$$

Holds for all points z other than a fixed point z_j in that square or partial square. We let a square denote a region with boundary points included with points interior to it. If a square has points not in R, then we remove those points and it becomes a partial square.

Proof: Suppose there does not exist a z_j where

$$\left|\frac{f(z) - f(z_j)}{z - z_j} - f'(z_j)\right| < \epsilon$$

holds after subdividing a square a finite number of times for contradiction. Let σ_0 denote the original square or the entire square of the partial square, σ_1 denote the squares after subdividing σ_0 into four equal smaller squares, and so on. After subdividing σ_0, one of the σ_1 must contain points of R but still no such z_j exists, so we continue to subdivide such σ_1 since the inequality does not hold. We will then obtain an infinite sequence

$$\sigma_0, \sigma_1, \sigma_2, \dots, \sigma_{k-1}, \sigma_k, \dots$$

There is a point z_0 that is common to each of these squares and each of these squares contain points of R other than z_0. As the size of the squares are decreasing, there exists a neighbourhood $\delta > |z - z_0|$ containing the squares with diagonals less than δ, so each neighbourhood δ contains points of R distinct from z_0. Thus z_0 is an accumulation point of R (definition 12.7.16), and since R is a closed set, $z_0 \in R$.

Now, since f is analytic in R and z_0, $f'(z_0)$ exists and according to definition 13.4.1:

$$\forall \epsilon \exists \delta > 0 \left[|z - z_0| < \delta \implies \left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \epsilon \right]$$

However, such neighbourhood $|z - z_0| < \delta$ contains σ_K for some sufficiently large K, so z_0 serves as z_j for a the subregion of σ_K or part of σ_K, thus there is no need to subdivide σ_K. We have reached a contradiction. $\square$

<u>Upper bound for modulus of an integral:</u>

Given some ϵ we cover region R with squares such that

$$\left|\frac{f(z) - f(z_j)}{z - z_j} - f'(z_j)\right| < \epsilon$$

We define a neighbourhood $\delta_j(z)$ enclosing the j-th square or partial square by

$$\delta_j(z) = \begin{cases} \frac{f(z)-f(z_j)}{z-z_j} - f'(z_j) & z \neq z_j \\ 0 & z = z_j \end{cases}$$

Then

$$\forall z \in \sigma \subset R[|\delta_j(z)| < \epsilon]$$

As $f(z)$ is continuous throughout subregion σ, $\delta_j(z)$ is continuous in σ and

$$\lim_{z \to z_j} \delta_j(z) = f'(z_j) - f'(z_j) = 0$$

Now, let C_j denote the positively oriented contours on the boundaries of the squares and partial squares covering R. Then on any C_j be definition of $\delta_j(z)$:

$$\begin{aligned} &f(z) = f(z_j) - z_j f'(z_j) + f'(z_j)z + (z - z_j)\delta_j(z) \\ &\implies \int_{C_j} f(z)dz = [f(z_j) - z_j f'(z_j)] \int_{C_j} dz + f'(z_j) \int_{C_j} zdz + \int_{C_j} (z - z_j)\delta_j(z)dz \end{aligned}$$

However, according to theorem 15.5.1:

$$\int_{C_j} dz = 0 \qquad\qquad \int_{C_j} zdz = 0$$

So

$$\begin{aligned} &\int_{C_j} f(z)dz = \int_{C_j} (z - z_j)\delta_j(z)dz && j = 1, 2, 3, \ldots, n \\ &\implies \sum_{j=1}^{n} \int_{C_j} f(z)dz = \sum_{j=1}^{n} \int_{C_j} (z - z_j)\delta_j(z)dz \end{aligned}$$

Now as boundaries of adjacent subregions cancel each other out, since they are taken along opposite senses to each other, only those on C remain, so

$$\int_C f(z)dz = \sum_{j=1}^{n} \int_{C_j} (z - z_j)\delta_j(z)dz \implies \left|\int_C f(z)dz\right| \leq \sum_{j=1}^{n} \left|\int_{C_j} (z - z_j)\delta_j(z)dz\right|$$

Endgame:

Let s_j denote the length of the sides of the square or partial square σ_j, since C_j is on the boundary or part of the boundary of the square.

$$|z - z_j| \leq \sqrt{2}s_j \qquad\qquad \sqrt{2}s_j \text{ is diagonal of square}$$

Then

$$|\delta_j(z)| < \epsilon \implies |(z - z_j)\delta_j(z)| = |z - z_j||\delta_j(z)| < \sqrt{2}s_j\epsilon$$

Let A_j be the area of the square. If C_j is the boundary of a square, then the length of C_j is $4s_j$ and we have

$$\left|\int_{C_j} (z - z_j)\delta_j(z)dz\right| < \sqrt{2}s_j\epsilon 4s_j = 4\sqrt{2}A_j\epsilon$$

Now, if C_j is the boundary of a partial square, then the length of C_j is less than $4s_j + L_j$, where L_j is the length of C_j that is a part of C. Let S be the length of the sides of some square that entirely encloses C, so sum of A_j is less then S^2. Then we have:

$$\left|\int_{C_j} (z - z_j)\delta_j(z)dz\right| < \sqrt{2}s_j\epsilon(4s_j + L_j) < 4\sqrt{2}A_j\epsilon + \sqrt{2}SL_j\epsilon$$

If L is the length of C, then

$$\begin{aligned}\left|\int_C f(z)dz\right| &\leq \sum_{j=1}^{n}\left|\int_{C_j} (z - z_j)\delta_j(z)dz\right| < \sum_{j=1}^{n}\left(4\sqrt{2}A_j\epsilon + \sqrt{2}SL_j\epsilon\right)\\ &< \left(4\sqrt{2}S^2 + \sqrt{2SL}\right)\epsilon\end{aligned}$$

Since ϵ is arbitrary, we can choose it to be as small as we like, so

$$\forall\epsilon > 0\left[\left|\int_C f(z)dz\right| < \left(4\sqrt{2}S^2 + \sqrt{2SL}\right)\epsilon\right] \implies \left|\int_C f(z)dz\right| = 0$$

Hence, if function f is analytic on all $z \in C$ where C is a simple closed contour, then

$$\int_C f(z)dz = 0$$

TLDR:
We found that the upper bound for f around the contour integral C is less than or equal to the sum of all the contours around the squares covering the region bounded by C. Since f is analytic in R, the sum of the contours of the squares in R is a function of the neighbourhood $\delta_j(z)$ surrounding z_j in each square, which is chosen to be less than some ϵ, the error between the derivative of f and the finite difference of f. As ϵ can be made arbitrary small and the inequality must hold for all values of ϵ, we find $\int_C f(z)dz = 0$. □

15.6.1 Morera's Theorem

A converse to the Cauchy-Goursat theorem.

Lemma 15.6.1.2:

Suppose $P(x,y)$, $Q(x,y)$, P_y, and Q_x are continuous in a simply connected domain D. Then for all simple closed contour C in D:

$$\int_C P\ dx + Q\ dy = 0 \implies \frac{\partial}{\partial x}Q = \frac{\partial}{\partial y}P$$

Proof: Suppose $\exists x_0, y_0 \in D$ such that $\partial Q/\partial x - \partial P/\partial y > 0$. Then there exists a circle C in D centred at (x_0, y_0) such that $\partial Q/\partial x - \partial P/\partial y > 0$ in and on C. That is, there exists a neighbourhood that $\partial Q/\partial x - \partial P/\partial y > 0$ holds. Then by Green's Theorem (theorem 19.4.1):

$$\int_C P\ dx + Q\ dy = \iint \left(\frac{\partial}{\partial x}Q - \frac{\partial}{\partial y}P\right)\ dx\ dy$$

Then

$$\iint \left(\frac{\partial}{\partial x}Q - \frac{\partial}{\partial y}P\right)\ dx\ dy > 0 \implies \int_C P\ dx + Q\ dy > 0$$

But by hypothesis

$$\int_C P\ dx + Q\ dy = 0$$

We have a contradiction, so

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \leq 0$$

Using a similar argument, we have $\partial Q/\partial x - \partial P/\partial y \geq 0$ for $\partial Q/\partial x - \partial P/\partial y < 0$, so we have

$$\frac{\partial}{\partial x}Q = \frac{\partial}{\partial y}P$$

□

Theorem 15.6.2: Morera's Theorem

Let $f(x,y) = u(x,y) + iv(x,y)$ where u and v are continuous in a domain D. Then for every simple closed contour C in D:

$$\int_C f(z)dz = 0 \implies f(z) \text{ is analytic in } D$$

Proof:

$$\begin{aligned}\int_C f(z)dz = 0 &\implies \int_C u\ dx - v\ dy + iv\ dx + iu\ dy = 0\\ &\implies \left[\int_C u\ dx - v\ dy = 0\right] \wedge \left[\int_C v\ dx + u\ dy = 0\right]\\ &\implies \left[\frac{\partial}{\partial y}u = -\frac{\partial}{\partial x}v\right] \wedge \left[\frac{\partial}{\partial y}v = \frac{\partial}{\partial x}u\right]\end{aligned}$$

These are the Cauchy-Riemann equations (theorem 13.5.1), thus $f(z)$ is analytic in D. □

Note: This proof requires that the partial derivatives to be continuous in D. However, there is a proof that eliminates this requirement.

15.6.2 Simply Connected Domains

Definition 15.6.1: Simply Connected Domain

A domain D which every simple closed contour that lies within it only encloses points in D.

Theorem 15.6.3:

Let f be a function that is analytic throughout a simply connected domain D. Then for every closed contour C lying in D:

$$\int_C f(z)dz = 0$$

We will later learn in theorem 15.7.4 that this is $\iff$, due to theorem 15.5.1.

Proof: Suppose C is simple and lies entirely in D. The result follows from the Cauchy-Goursat theorem (theorem 15.6.1).

Suppose that C is closed, but intersects itself a finite number of times, then it consists of a finite number of simple closed contours. Result again follows from the Cauchy-Goursat Theorem.

Note: There are subtleties for infinite number of self-intersection points. □

Corollary 15.6.3.1:

If f is a function analytic throughout a simply connected domain D, then it has antiderivatives everywhere in D.

Proof: If f is analytic in a simply connected domain D then it is continuous in D. Then

$$\int_C f(z)dz = 0 \iff \forall z \in D\ \exists F(z)[F'(z) = f(z)] \qquad \textit{Theorem } 15.5.1$$

□

Corollary 15.6.3.2:
Entire functions have antideriviatives everywhere in their domain of definition.

Proof: Consequence of previous corollary and that finite plane is simply connected. □

15.6.3 Multiply Connected Domains

Definition 15.6.2: Multiply Connected Domains
A domain that is not simply connected.

Theorem 15.6.4:
Let C be a simple closed contour in the positive direction, and C_k ($k \in \{1, 2, 3, \ldots, n\}$) be simple closed contours in C taken in the negative direction that are disjoint with no common interior points. If a function f is analytic on C and C_k and throughout the multiply connected domain consisting of points inside C but exterior to all C_k, then

$$\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0$$

Proof: Let a polygonal path L_1 connect C to the inner contour C_1, L_2 connecting C_1 to C_2, and continue in this manner. Finally, let L_{n+1} connect C_n to C. Then we have two contours Γ_1 and Γ_2. Γ_1 consisting of parts of the contours C, C_k, and L_k. Γ_2 consisting of the remaining parts of contours C, C_k, and $-L_k$. If we apply the Cauchy-Goursat theorem to Γ_1 and Γ_2, then

$$\int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz = 0$$

Now, since the integrals along L_k cancel (due to being taken in the opposite direction), only integrals along C and C_k remain. Hence

$$\begin{aligned}
&\int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz = 0 \\
\implies &\int_C f(z)dz + \sum_{k=1}^{n+1} \int_{L_k} f(z)dz - \sum_{k=1}^{n+1} \int_{L_k} f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0 \\
\implies &\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0
\end{aligned}$$

□

Observation. *Basically, imagine a slice of Swiss cheese. We took a knife and cut a single path though all of the holes in order. Another way of saying it is that we cut through each hole only once. This way we end up with two slices each consisting a part of the outer edge of the original slice, a part of the edge of the holes, and the edges introduced by our cut. Since we have cut through all of the holes, our two slices will not have holes so we can integrate along the outer edge of each of those slices. The Cauchy-Goursat theorem tells us that the value for the sum will be zero, since they are now simply connected domains.*

Question. *We pretty much end up with two simply connected domains, right? If so then shouldn't it be:*

$$\left(\int_{\Gamma_1} f(z)dz = 0\right) \wedge \left(\int_{\Gamma_2} f(z)dz = 0\right) \implies \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz = 0$$

As apposed to just

$$\int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz = 0$$

which does not imply

$$\left(\int_{\Gamma_1} f(z)dz = 0\right) \wedge \left(\int_{\Gamma_2} f(z)dz = 0\right)$$

Corollary 15.6.4.1: Principle of Deformation of Paths

Let C_1 and C_2 be positively oriented simple closed contours, with C_1 interior to C_2. If a function f is analytic on C_1, C_2, and the regions between them, then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

Proof: It follows from theorem 15.6.4 that

$$\int_{C_1} f(z)dz - \int_{-C_2} f(z)dz = 0 \implies \int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

□

Observation. *It is easy to see that for any contour C_3 lying between contours C_1 and C_2 and on points on C_1 and C_2:*

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz = \int_{C_3} f(z)dz$$

In fact, this theorem more powerful then it seems. It is implying that the contour integrals of a function f over any contour is equal, given that one contour can continuously deform into the other without crossing any singular points of $f(z)$. It is the key to choosing a simpler path for integration.

15.6.4 Examples

Example 15.6.1 *(Beware of domain of definition and analyticity of function when using Cauchy-Goursat Theorem) Let C be a positively oriented curve of a semicircle from of radius $|z| \le 1$ from $\theta \in [0, \pi]$, and consider a function:*

$$f(z) = z^{1/2} \begin{cases} 0 & z = 0 \\ \sqrt{r}e^{i\theta/2} & r > 0,\ \theta \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right) \end{cases}$$

Show:

$$\int_C f(z)dz = 0$$

Proof: From the domain of definition of the function, we can see that is is not defined on the negative real axis. Now, let's check the analyticity. We have the polar form of the Cauchy-Riemann Equations (theorem 13.5.2):

$$ru_r = v_\theta \qquad\qquad u_\theta = -rv_r$$

Knowing $f(z) = \sqrt{r}e^{i\theta/2} = \sqrt{r}[\cos(\theta/2) + i\sin(\theta/2)]$:

$$\begin{aligned} ru_r &= r\left[\frac{1}{2}r^{-1/2}\cos\left(\frac{\theta}{2}\right)\right] = \frac{1}{2}\sqrt{r}\cos\left(\frac{\theta}{2}\right) & v_\theta &= \sqrt{r}\cos\left(\frac{\theta}{2}\right)\frac{1}{2} = \frac{1}{2}\sqrt{r}\cos\left(\frac{\theta}{2}\right) \\ u_\theta &= -\sqrt{r}\sin\left(\frac{\theta}{2}\right)\frac{1}{2} = -\frac{1}{2}\sqrt{r}\sin\left(\frac{\theta}{2}\right) & -rv_r &= -r\left[\frac{1}{2}r^{-1/2}\sin\left(\frac{\theta}{2}\right)\right] = -\frac{1}{2}\sqrt{r}\sin\left(\frac{\theta}{2}\right) \end{aligned}$$

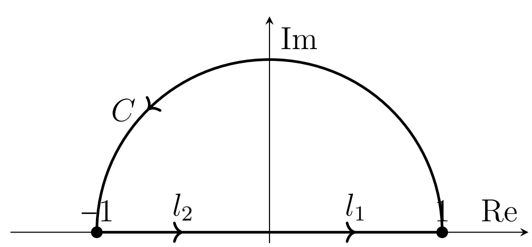
While the polar form of the Cauchy-Riemann equations are satisfied, keep in mind that it does not apply at $z = 0$. $f(z)$ fails to be analytic at $z = 0$. Due to these, we can not use the Cauchy-Goursat Theorem. We must integrate $f(z)$ over the contour directly. Let l_1 be the line from 0 to 1, l_2 be a line from -1 to 0, and C_R be a semicircular contour from 0 to π with radius $|z| = 1$. The parameterizations are then:

$$l_1: \ z = r \qquad\qquad l_2: \ z = r \qquad\qquad C_R: \ z = e^{i\theta}$$

The contour integral:

$$\begin{aligned} \int_C f(z)dz = \int_0^1 \sqrt{r}dr + \int_{-1}^0 \sqrt{r}dr + \int_{C_R} f(z)dz &= \int_{-1}^1 \sqrt{r}dr + \int_0^\pi e^{i\theta/2}ie^{i\theta}d\theta \\ &= \frac{1}{2}\,r^{-1/2}\bigg|_{-1}^1 + i\int_0^\pi e^{i3\theta/2}d\theta = \frac{1}{2}(1+i) + i\frac{2}{3}\,e^{i3\theta/2}\bigg|_0^\pi \\ &= \frac{1}{2}(1+i) + i\frac{2}{3}(-i-1) = 0 \end{aligned}$$

□



Example 15.6.2 *Show*

$$\int_0^\infty \exp(-x^2)dx = \frac{\sqrt{\pi}}{2}$$

Proof: Consider

$$\begin{aligned}
\int_0^\infty \exp(-x^2)dx \int_0^\infty \exp(-y^2)dy &= \int_0^\infty \int_0^\infty \exp[-(x^2+y^2)]dxdy \\
&= \int_0^{\pi/2} \int_0^\infty \exp(-r^2)r\ drd\theta \qquad r^2 = x^2+y^2 \\
&= \frac{\pi}{2}\int_0^\infty \exp(-r^2)r\ dr \\
&= \frac{\pi}{4}\int_0^\infty \exp(-u)\ du \qquad u = r^2 \\
&= \frac{\pi}{4}\ e^{-u}\Big|_0^\infty = \frac{\pi}{4}
\end{aligned}$$

Now, if we let $x = y$, we can see that

$$\int_0^\infty \exp(-x^2)dx \int_0^\infty \exp(-y^2)dy = \left[\int_0^\infty \exp(-x^2)dx\right]^2$$

Thus

$$\int_0^\infty \exp(-x^2)dx = \frac{\sqrt{\pi}}{2}$$

$\square$

Example 15.6.3 *Show*

$$\int_0^\infty \exp(-x^2)\cos(2bx)dx = \frac{\sqrt{\pi}}{2}\exp(-b^2) \qquad b > 0$$

Proof: We will consider the simple closed contour formed by the lines: l_1 from $-a$ to a, l_2 from a to $a+bi$, l_3 from $a+bi$ to $-a+bi$, and l_4 from $-a+bi$ to $-a$. The parameterization are given by:

$$l_1 : z = x \qquad l_2 : z = a+yi \qquad l_3 : z = x+bi \qquad l_4 : z = -a+yi$$

Evaluating the contour, and using the Cauchy-Goursat Theorem (theorem 15.6.1):

$$\begin{aligned}
&\int_C \exp(-z^2)dz \\
&= \int_{-a}^a e^{-x^2}dx + \int_0^b e^{-(a+yi)^2}idy + \int_a^{-a} e^{-(x+bi)^2}dx + \int_b^0 e^{-(-a+yi)^2}idy \\
&= \int_{-a}^a e^{-x^2} - e^{-x^2+b^2-2bxi}dx + i\int_0^b e^{-a^2+y^2-2ayi} - e^{-a^2+y^2+2ayi}dy \\
&= 2\int_{-a}^0 e^{-x^2} - e^{b^2}e^{-x^2}[\cos(-2bx) + i\sin(-2bx)]dx + ie^{-a^2}\int_0^b e^{y^2}[e^{-2ayi} - e^{2ayi}]dy \\
&= 0
\end{aligned}$$

Since we are evaluating over a symmetric integral, only even functions remain, so the sine term is zero. We also know that

$$\cos(-2bx) = \cos(2bx) \qquad\qquad e^{-2ayi} - e^{2ayi} = -2i\sin(2ay)$$

Making these substitutions and rearranging:

$$2\int_0^a e^{b^2}e^{-x^2}\cos(2bx)dx = 2\int_0^a e^{-x^2}dx + 2e^{-a^2}\int_0^b e^{y^2}\sin(2ay)dy$$

Dividing everything by e^{b^2} since $b > 0$ is a constant:

$$\int_0^a e^{-x^2}\cos(2bx)dx = e^{-b^2}\int_0^a e^{-x^2}dx + e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin(2ay)dy$$

Taking the limit as $a \to \infty$:

$$\lim_{a\to\infty}\int_0^a e^{-x^2}\cos(2bx)dx = \lim_{a\to\infty}\left[e^{-b^2}\int_0^a e^{-x^2}dx + e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin(2ay)dy\right]$$

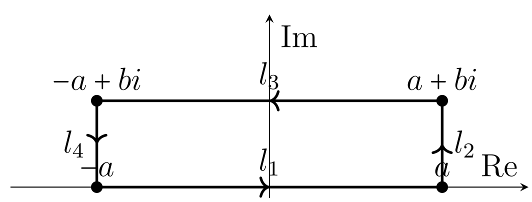
Using the knowledge from example 15.6.2, and also:

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2} \qquad\qquad \left|\int_0^b e^{y^2}\sin(2ay)dy\right| \le \int_0^b e^{y^2}dy$$

We get

$$\int_0^\infty \exp\left(-x^2\right)\cos(2bx)dx = \frac{\sqrt{\pi}}{2}\exp\left(-b^2\right) \qquad\qquad b > 0$$

□



Example 15.6.4 *Show for a positively oriented simple closed contour C, the area of the region enclosed by C is*

$$\frac{1}{2i}\int_C \bar{z}dz$$

Proof: The parameterization of $f(z)$:

$$\begin{aligned}\int_C f(z)dz &= \int_a^b f[z(t)]z'(t)dt \\ &= \int_a^b [u(x,y) + iv(x,y)][x'(t) + y'(t)]dt \\ &= \int_a^b (ux' - vy')dt + i\int_a^b (vx' + uy')dt \qquad = \int_C u\ dx - v\ dy + i\int_C v\ dx + u\ dy\end{aligned}$$

Using Green's Theorem (theorem 19.4.1):

$$\int_C f(z)dz = \iint_R(-v_x - u_y)dA + i\iint_R(u_x - v_y)dA$$

From $\bar{z} = x - iy$:

$$u_x = 1 \qquad v_y = -1 \qquad v_x = 0 \qquad u_x = 0$$

Substituting these in, we get:

$$\int_C \bar{z}dz = i\iint_R 2\ dA = 2i\iint_R dA = 2iA$$

A is the area of the region enclosed by C, thus:

$$A = \frac{1}{2i}\int_C \bar{z}dz$$

□

15.7 Cauchy Integral Formula

The Cauchy-Goursat theorem (theorem 15.6.1) gives us the value of a contour integral without singularities in the interior. What if there is a singularity in the interior? See below.

Theorem 15.7.1: Cauchy Integral Formula

Let a function f be analytic on and inside a simple closed contour C oriented positively. Then for all z_0 interior to C:

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}dz$$

That is, if f is analytic within and on a simple closed contour C, then values of f interior to C is determined by values of f on C.

Proof: Let C be a positively oriented contour and z_0 be any point interior to C. Let C_ρ be a positively oriented circular contour lying inside C centred at z_0. That is, C_ρ lies on points $|z - z_0| = \rho$. Then from corollary 15.6.4.1, we can write:

$$\int_C \frac{f(z)}{z - z_0}dz = \int_{C_\rho} \frac{f(z)}{z - z_0}dz$$
$$\implies \int_C \frac{f(z)}{z - z_0}dz - f(z_0)\int_{C_\rho} \frac{1}{z - z_0}dz = \int_C \frac{f(z) - f(z_0)}{z - z_0}dz$$

Now

$$\int_{C_\rho} \frac{1}{z - z_0}dz = \int_{C_\rho} \frac{1}{re^{i\theta}} ire^{i\theta}d\theta \qquad z = z_0 + re^{i\theta}$$
$$= \int_{C_\rho} id\theta = 2\pi i$$

So

$$\int_C \frac{f(z)}{z-z_0}dz - 2\pi i f(z_0) = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}dz$$

Since f is analytic, thus continuous:

$$\forall \epsilon > 0, \exists \delta > 0[|z-z_0| < \delta \implies |f(z)-f(z_0)| < \epsilon]$$

Now, $|z-z_0| = \rho < \delta$ for all z on C_ρ, so according to theorem 15.4.1:

$$\left|\int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0}dz\right| < \frac{\epsilon}{\rho}(2\pi\rho) = 2\pi\epsilon \qquad\qquad \left|\frac{f(z)-f(z_0)}{z-z_0}\right| < \frac{\epsilon}{\delta}$$

Then

$$\left|\int_C \frac{f(z)}{z-z_0}dz - 2\pi i f(z_0)\right| < 2\pi\epsilon$$

This inequality must hold for all values of $\epsilon > 0$, so

$$\int_C \frac{f(z)}{z-z_0}dz - 2\pi i f(z_0) = 0 \implies \int_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$$

□

The Cauchy Integral Theorem links $f(z_0)$ to a contour integral. The extension of the Cauchy Integral Formula links the n-th derivative of f, $f^{(n)}(z_0)$, to the contour integral of f at z_0.

Theorem 15.7.2: Cauchy Integral Formula (Extension)

Let a function f be analytic on and inside a simple closed contour C oriented positively. Then for all z_0 interior to C:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz \qquad\qquad n \in \mathbb{N} \cup \{0\}$$

Proof: Proof that is not a proof, but a verification in Brown and Churchill [2].
Taking the original Cauchy Integral formula:

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s-z}ds \implies f'(z) = \frac{1}{2\pi i}\int_C f(z)\frac{\partial}{\partial z}(s-z)^{-1}ds = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}ds$$

Continued differentiation under the integral sign yields the desired result...or does it? Verification is needed.

<u>Verification</u>

Let z be any point interior to a simple closed contour C, and d denote the smallest distance from z to points s on C. Assume $0 < |\Delta z| < d$, then

$$\begin{aligned}\frac{f(z+\Delta z)-f(s)}{\Delta z} &= \frac{1}{2\pi i}\int_C \left(\frac{1}{s-z-\Delta z} - \frac{1}{s-z}\right)\frac{f(s)}{\Delta z}ds \\ &= \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z-\Delta z)(s-z)}ds\end{aligned}$$

Now

$$\frac{1}{(s-z-\Delta z)(s-z)} = \frac{1}{(s-z)^2} + \frac{\Delta z}{(s-z-\Delta z)(s-z)^2}$$

Hence

$$\frac{f(z+\Delta z)-f(s)}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}ds = \frac{1}{2\pi i}\int_C \frac{f(s)\Delta z}{(s-z-\Delta z)(s-z)^2}ds$$

Let $M = \max|f(s)|$ on C, since $|s-z| \geq d$ and $|\Delta z| < d$:

$$|s-z-\Delta z| = |(s-z)-\Delta z| \geq ||s-z|-|\Delta z|| \geq d-|\Delta z| > 0$$

Then letting L be the length of C and using theorem 15.4.1:

$$\left|\int_C \frac{f(s)\Delta z}{(s-z-\Delta z)(s-z)^2}\right| \leq \frac{|\Delta z|M}{(d-|\Delta z|)d^2}L$$

Taking the limit $\Delta z \to 0$:

$$\lim_{\Delta z\to 0}\left|\int_C \frac{f(s)\Delta z}{(s-z-\Delta z)(s-z)^2}\right| \leq \lim_{\Delta z\to 0}\frac{|\Delta z|M}{(d-|\Delta z|)d^2}L = 0$$

Hence

$$\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(s)}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}ds = \lim_{\Delta z\to 0}\frac{1}{2\pi i}\int_C \frac{f(s)\Delta z}{(s-z-\Delta z)(s-z)^2}ds = 0$$

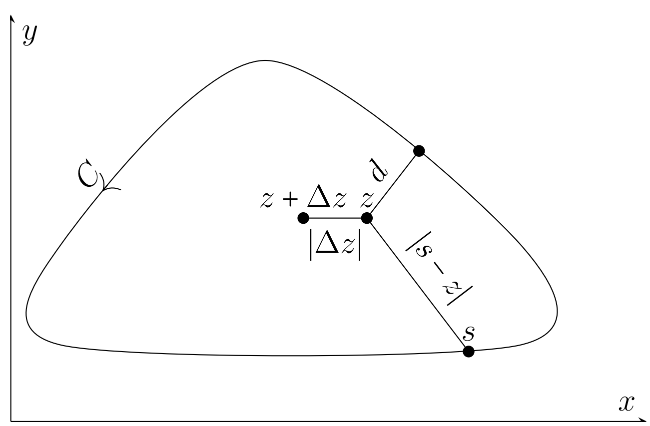
Thus

$$\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(s)}{\Delta z} = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}ds$$

By induction, we get:

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(s)}{(s-z)^{n+1}}ds \qquad\qquad n \in \mathbb{N}\cup\{0\}$$

$\square$



Example 15.7.1 *We can then rewrite the Legendre Polynomials:*

$$P_n(z) = \frac{1}{n!2^n}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(z^2-1\right)^2 = \frac{1}{2^{n+1}\pi i}\int_C \frac{(s^2-1)^n}{(s-z)^{n+1}}ds \qquad\qquad n \in \mathbb{N}\cup\{0\}$$

15.7.1 Consequences

Theorem 15.7.3:

Let a function f be analytic at a point z_0, then $f^{(n)}$ exists at z_0 for all $n \in \mathbb{N}$. That is, the derivative of f of all orders are analytic at z_0.

Proof: Suppose a function f is analytic at point z_0, then there exists a neighbourhood $\epsilon > |z - z_0|$ where f is analytic. By extension, there is a positively oriented circular contour C_0 centred at z_0 with radius $\epsilon/2$ where f is analytic on and inside C_0. Then by theorem 15.7.2:

$$f''(z) = \frac{1}{\pi i} \int_{C_0} \frac{f(s)}{(s-z)^3} ds \qquad \forall z \text{ interior to } C_0$$

The existence of of $f''(z)$ in $|z - z_0| < \epsilon \implies f'$ is analytic at z_0. The same argument on f' implies f'' is also analytic. □

Note: Suppose $f(z) = u(x,y) + iv(x,y)$ is analytic at $z = (x,y)$. Then it is also continuous:

$$\begin{aligned} f(z) = u(x,y) + iv(x,y) &\implies [f'(z) = u_x + iv_x = v_y - iu_y] \wedge [f' \text{ is continuous }] \\ &\implies [f''(z) = u_{xx} + iv_{xx} = v_{xy} - iu_{yx}] \wedge [f'' \text{ is continuous}] \end{aligned}$$

Corollary 15.7.3.1:

Let a function $f(z) = u(x,y) + iv(x,y)$ be analytic at a point z_0. Then u and v have continuous partial derivatives of all orders at z_0.

Theorem 15.7.4:

Let a function f be continuous on domain D, and C be any closed contour lying in D.

$$\forall C \left[\int_C f(z) dz = 0 \right] \implies f \text{ is analytic throughout } D$$

If D is simply connected, this is the converse of theorem 15.6.3.

Proof:

$$\begin{aligned} f \text{ is continous in } D &\implies \forall z \in D, \exists F(z)[F'(z) = f(z)] && \textit{Theorem } 15.5.1 \\ &\implies f \text{ is analytic in } D && \textit{Theorem } 15.7.3 \end{aligned}$$

□

Theorem 15.7.5: Cauchy's Inequality

Let a function f be analytic on and inside a positively oriented circular contour C_R centered at z_0 with radius R.

$$M_R = \max |f(z)| \Big|_{C_R} \implies \left| f^{(n)}(z_0) \right| \leq \frac{n! M_R}{R^n} \qquad n \in \mathbb{N}$$

Proof: From theorem 15.4.1:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_R}\frac{f(z)}{(z-z_0)^{n+1}} \qquad n \in \mathbb{N}$$
$$\implies \left|f^{(n)}(z_0)\right| \le \frac{n!}{2\pi}\cdot\frac{M_R}{R^{n+1}}\cdot 2\pi R = \frac{n!M_R}{R^n} \qquad |z-z_0| \le R,\ n \in \mathbb{N}$$

□

15.8 Liouville's Theorem and the Fundamental Theorem of Algebra

Theorem 15.8.1: Liouville's Theorem
Let f be a function in the complex plane

$$f \text{ is entire and bounded in } \mathbb{C} \implies f(z) \text{ is constant in } \mathbb{C}$$

Proof: f is entire so $\forall z \in \mathbb{C}\ f'(z)$ exists. Then From theorem 15.7.5, and f being bounded:

$$|f'(z_0)| \le \frac{M_R}{R} = \frac{M}{R} \qquad n = 1, \forall z \in C, \exists M[|f(z)| \le M]$$

This inequality must hold for all values of R (R can be arbitrarily large), so we find:

$$|f'(z_0)| = 0 \implies f \text{ is constant}$$

□

Observation. *Liouville's Theorem implies that any non-constant function in $\mathbb{C}$ is either not entire or unbounded. See the Maximum Modulus Principle (theorem 15.9.2).*

Question. *Shouldn't Liouville's theorem be $\iff$ since a constant function is also entire and bounded?*

Intuitively, this tells us that if f is bounded by $f(z_0)$, then $f(z_0)$ must either be a maximum or a minimum. This can not be the case since it violates f being a harmonic function, where the sum of curvatures in each component direction is zero.

Theorem 15.8.2: Fundamental Theorem of Algebra
Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be any polynomial, then

$$\forall n \in \mathbb{N}, \exists z_0 \in \mathbb{C}[P(z_0) = 0]$$

Proof: Suppose for contradiction $\nexists z_0 \in \mathbb{C}$ such that $P(z_0) = 0$ Recall from corollary 12.2.1.1 that $\exists R \in \mathbb{R}$ such that

$$\left|\frac{1}{P(z)}\right| < \frac{2}{|a_n|R^n} \qquad \forall z \in \mathbb{C}[|z| > R]$$

$1/P(z)$ is bounded on for $|z| > R$, but $P(z)$ is continuous on $|z| \leq R$, which implies that $1/P(z)$ is bounded for $|z| \leq R$. Thus $P(z)$ is bounded in the entire complex plane. (Theorem 13.3.5) It follows from theorem 15.8.1 that $1/P(z)$ is constant $\implies P(z)$ is constant, but $P(z)$ is not constant. Contradiction! □

Theorem 15.8.2 tells us that any polynomial of degree $n \geq 1$ can be expressed as a product of linear factors:

$$P(z) = c\prod_{i=1}^{i=n}(z - z_i)$$

Since the existence of a zero z_1 implies

$$P(z) = (z - z_1)Q_1(z) \qquad \deg(Q_1) = n - 1$$

Result follows from induction.

15.9 Maximum Modulus Principle

Theorem 15.9.1: Gauss's Mean Value Theorem

Let f be a function analytic in and on a circular contour C_ρ centred at z_0, then $f(z_0)$ is the arithmetic mean of the values on the circle:

$$f(z_0) = \frac{1}{2\pi i}\int_0^{2\pi} f(z_0 + \rho e^{i\theta})d\theta \qquad z = z_0 + \rho e^{i\theta},\ 0 \leq \theta \leq 2\pi$$

That is, the value of $f(z_0)$ is the average of the values of $f(z)$ in some neighbourhood with radius ρ around z_0.

Proof: Let C_ρ be a circular contour centred at z_0, and $|z - z_0| = \rho$. Then by the Cauchy Integral formula (theorem 15.7.1):

$$f(z_0) = \frac{1}{2\pi i}\int_{C_\rho}\frac{f(z)}{z - z_0}dz = \frac{1}{2\pi i}\int_0^{2\pi} f(z_0 + \rho e^{i\theta})d\theta \qquad z = z_0 + \rho e^{i\theta},\ 0 \leq \theta \leq 2\pi$$

□

Lemma 15.9.1.1:

Let f be a function analytic in some neighbourhood $|z-z_0|<\epsilon$:

$$\forall z \in \{z : |z-z_0| < \epsilon\}[|f(z)| \leq |f(z_0)|] \implies \forall z \in \{|z-z_0|<\epsilon\}[f(z) = f(z_0)]$$

That is, if f is bounded by its value at z_0 in the neighbourhood of z_0, then it is constant throughout the neighbourhood with value $f(z_0)$.

Proof: Following theorem 15.9.1, we have

$$f(z_0) = \frac{1}{2\pi i}\int_0^{2\pi} f(z_0+\rho e^{i\theta})d\theta \implies |f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \left|f(z_0+\rho e^{i\theta})\right|d\theta$$

However, since we have condition $|f(z)| = |f(z_0+\rho e^{i\theta})| \leq |f(z_0)|$:

$$\int_0^{2\pi} \left|f(z_0+\rho e^{i\theta})\right|d\theta \leq \int_0^{2\pi} |f(z_0)|d\theta = 2\pi|f(z_0)| \qquad 0 \leq \theta \leq 2\pi$$
$$\implies |f(z_0)| \geq \frac{1}{2\pi}\int_0^{2\pi} \left|f(z_0+\rho e^{i\theta})\right|d\theta$$

The inequalities tells us:

$$|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} \left|f(z_0+\rho e^{i\theta})\right|d\theta \implies \int_0^{2\pi} |f(z_0)| - \left|f(z_0+\rho e^{i\theta})\right|d\theta = 0$$

Our condition $|f(z)| = |f(z_0+\rho e^{i\theta})| \leq |f(z_0)|$ tells us that

$$\int_0^{2\pi} \left|f(z_0+\rho e^{i\theta})\right|d\theta \leq \int_0^{2\pi} |f(z_0)|d\theta \implies \int_0^{2\pi} |f(z_0)| - \left|f(z_0+\rho e^{i\theta})\right|d\theta \geq 0$$

So for $\int_0^{2\pi} |f(z_0)| - |f(z_0+\rho e^{i\theta})|d\theta = 0$, the integrand must be zero:

$$\left|f(z_0+\rho e^{i\theta})\right| - |f(z_0)| = 0 \qquad 0 \leq \theta \leq 2\pi$$
$$\implies \forall z \in \{z : |z-z_0| = \rho\}[|f(z)| = |f(z_0)|]$$

Since $0 < |z-z_0| < \epsilon$ and $|f(z)| = |f(z_0)|$ for all $0 < \rho < \epsilon$, we have $|f(z)| = |f(z_0)|$ for $|z-z_0| < \epsilon$. We know that if a function is analytic in a domain and its modulus is constant in the domain, then the function is constant (example 13.6.4), thus

$$\forall z \in \{z : |z-z_0| < \epsilon\}[f(z) = f(z_0)]$$

That is $f(z)$ is constant in the neighbourhood $|z-z_0| < \epsilon$ with value $f(z_0)$. □

Theorem 15.9.2: Maximum Modulus Principle

Let f be an analytic function in a domain D.

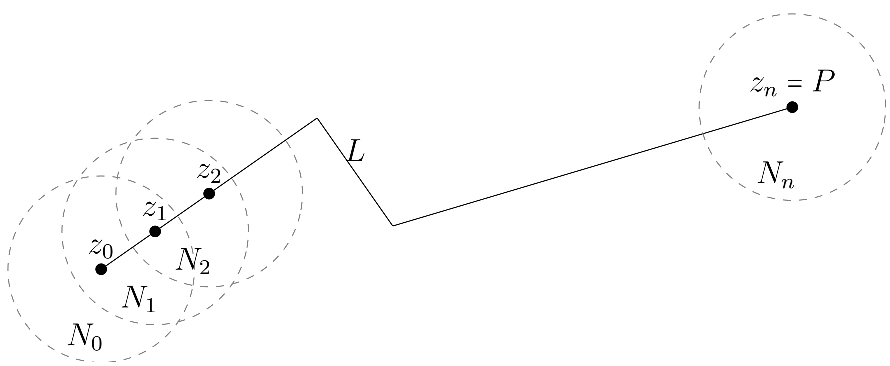
$$f \text{ not constant in } D \implies \forall z \in D, \nexists z_0 \in D[|f(z)| \leq |f(z_0)| = M]$$

That is, if an analytic function is not constant in a domain D, then it is not bound.

Proof: Suppose for contradiction f is bounded in domain D. That is $\forall z \in D[|f(z)| \leq M]$. Let L be a polygonal line lying in D extending from z_0 to any arbitrary $z_n = P$ in D, and d be the shortest distance from points on L to the boundary of D. Then for each point z_k ($k \in [0, n]$), we have $|z_k - z_{k-1}| < d$ and neighbourhoods N_k.

Each neighbourhood N_k has radius d and the center of each neighbourhood N_k lies in the neighbourhood of N_{k-1}.

Since $\max|f(z)| = |f(z_0)|$, by lemma 15.9.1.1, all points in N_0 has value $f(z_0)$. The neighbourhoods overlap, so by extension $\forall k \in [0, n]$, $f(z_k) = f(z_0)$, and we have $f(z)$ is constant in D with value $f(z_0)$. f is then bounded in D and we have a contradiction! Thus, if an analytic function f is not constant in domain D, then it is not bounded. $\square$



For a closed bounded region R, the Maximum Modulus Principle may seem to contradict theorem 13.3.5. It is important to realize that we are working with a domain and the differences between a domain (definition 12.7.12) and region (definition 12.7.13).

Corollary 15.9.2.1:

Let f be an analytic function on a closed bounded region R that is not constant in the interior of R. Then $\max|f(z)|$ in R is always reached and only reached at some boundary of R, never in the interior of R.

Proof: Consider

$$f(x, y) = u(x, y) + iv(x, y)$$

Then as f as analytic in R, u is harmonic in R and can not assume maximum value in the interior of R. The same logic applies to v. (See Maximum Principle.)

More precisely, consider $g(z) = e^{f(z)}$, then g is analytic, continuous, and non-constant in the interior of R. Hence $|g(z)| = |e^{u(x,y)}|$ must assume its maximum value at the boundary of R, so $f(z)$ must also obtain its maximum at the boundary of R. $\square$

Note: Same is true for $\min|f(z)|$ (Example 15.9.2).

Note: Corollary 15.9.2.1 follows from the properties of $f(z) = u(x,y) + iv(x,y)$ being able to be expressed in terms of real valued functions and that harmonic real valued functions only have maximum and minimum values occurring at the boundaries of a closed and bounded region. We will soon see this in example 15.9.2 and example 15.9.3.

Note: There is a difference between complex-valued functions and real-valued functions. For complex valued functions $f(z)$, the maximums and minimums of the modulus $|f(z)|$ occur at some boundary of R, while for some real-valued function $u(x,y)$ (no modulus) the maximum and minimum occurs only at some boundary of R.

Note: This can be seen a result of Gauss's Mean Value Theorem (theorem 15.9.1).

15.9.1 Examples

Example 15.9.1 *Suppose that $f(z)$ is entire and that the function $u(x,y) = \mathrm{Re}\{f(z)\}$ is harmonic and has upper bound $u_0 \geq u(x,y)$ for all $(x,y) \in \mathbb{R}^2$. Then $u(x,y)$ is constant in $\mathbb{R}^2$.*

Proof: Consider $g(z) = e^{f(z)} = e^{u(x,y)}e^{iv(x,y)}$. $e^{iv(x,y)}$ is a phase, so we ignore it and focus on $e^{u(x,y)}$:

$$\begin{aligned}
&e^{u(x,y)} \text{ is entire} \\
&\implies \exists (x_0,y_0) \in \mathbb{R}^2 \left[\left|e^{u(x,y)}\right| \leq u_0 = u(x_0,y_0)\right] && \text{Cauchy's Inequalitey } (\mathit{Theorem}\ 15.7.5) \\
&\implies e^{u(x,y)} \text{ is constant} && \text{Liouville's Theorem } (\mathit{Theorem}\ 15.8.1)
\end{aligned}$$

□

Example 15.9.2 *Let a function f be continuous on a closed bounded region R, and be analytic and not constant throughout the interior of R. Also, let $f(z) \neq 0$ for all $z \in \mathbb{R}$. Prove $\min|f(z)|$ only occurs at the boundaries and never in the interiors of R.*

Proof: Consider $g(z) = 1/f(z)$.

$$\begin{aligned}
&f \text{ is continuous, analytic, non-constant, and } \forall z/inR[f(z) \neq 0] \\
&\implies g \text{ is continuous, analytic, non-constant} \\
&\implies \max|g(z)| \text{ only occurs at boundary of } R && \mathit{Corollary}\ 15.9.2.1 \\
&\implies \min|f(z)| \text{ occurs only at some boundary of } R
\end{aligned}$$

As for why $f(z) \neq 0$ for all $z \in R$ is required:
Suppose $f(z_0) = 0$ for some z_0 in the interior of R.

$$\begin{aligned}
&\implies g(z) = 1/f(z) \text{ is not countinuous in } R && \mathit{Corollary}\ 15.9.2.1 \\
&\implies \max|g(z)| \text{ does not occur only some boundary of } R \\
&\implies \min|f(z)| \text{ exists in the interior of } R
\end{aligned}$$

□

Question. *If* $f(z_0) = 0$ *for some* z_0 *in the interior of* R*, then does that mean* $\min|f(z_0)| = 0$ *for all* $z_0 \in \{z : f(z) = 0\}$*?*

Example 15.9.3 *Let* $f(z) = u(x,y) + iv(x,y)$ *be a function continuous on a closed bounded region* R *be analytic and not constant in the interior of* R*. Prove* $\min u(x,y)$ *occurs only at some boundary of* R*.*

Proof: Let $f(z) = u_1(x,y) + iv_1(x,y)$ and $g(z) = e^{1/f(z)} = e^{u_2(x,y)+iv_2(x,y)}$, which is continuous, analytic, and not constant in the interior of R. Then $|g(z)| = e^{u_2(x,y)}$ is continuous in R must have a maximum at the some boundary of R. Hence, $f(z)$ has a minimum at some boundary of R. □

15.10 Poisson Integral Formula

We are looking to solve the Dirichlet Problem (definition 19.4.4). Finding a function in a harmonic domain that assumes preassigned values at the boundary.

Definition 15.10.1: Poisson Integral Formula (Circle Interior)

Let f *be a complex-valued function on a circular simple closed domain with radius* R*, and* C *be a circular contour an the boundary of the domain. Then the Dirichlet Problem is solved in the domain by*

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{u(R,\phi)(R^2 - r^2)}{R^2 + r^2 - 2Rr\cos(\phi - \theta)} d\phi$$

Proof: Let z be any point inside a circular domain with radius R, f be an analytic function throughout the interior of the domain, and C be a circular contour on the boundary of the domain. Consider the Cauchy Integral Formula (theorem 15.7.1):

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w - z} dw$$

Define a point that lies outside the circle: $z_1 = R^2/\bar{z}$

Where

$$|z_1| = \frac{R^2}{|\bar{z}|} = \frac{R^2}{|z|} > R \qquad\qquad |z| < R$$
$$\arg(z_1) = \arg(z)$$

Then by Cauchy-Goursat (theorem 15.6.1)

$$\frac{1}{2\pi i}\int_C \frac{f(w)}{w - z_1} dw = \frac{1}{2\pi i}\int_C \frac{f(w)}{w - (R^2/\bar{z})} dw = 0$$

Subtracting the two equations:

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_C f(w)\left[\frac{1}{w-z} - \frac{1}{w-(R^2/\bar{z})}\right]dw \\
&= \frac{1}{2\pi i}\int_C f(w)\left[\frac{z-(R^2/\bar{z})}{(w-z)[w-(R^2/\bar{z})]}\right]dw \\
&= \frac{1}{2\pi i}\int_C f(R,\phi)\left[\frac{re^{i\theta}-[R^2/(re^{-i\theta})]}{(Re^{i\phi}-re^{i\theta})\{Re^{i\phi}-[R^2/(re^{-i\theta})]\}}\right]Re^{i\phi}i\,d\phi \qquad w = Re^{i\phi} \\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{f(R,\phi)(R^2-r^2)}{R^2+r^2-2Rr\cos(\phi-\theta)}d\phi
\end{aligned}
$$

Separating into real and imaginary parts:

$$
u(r,\theta)+iv(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\frac{[u(R,\phi)+iv(R,\phi)][R^2-r^2]}{R^2+r^2-2Rr\cos(\phi-\theta)}d\phi
$$

Taking the real part yields the desired result. □

Definition 15.10.2: Poisson Integral Formula (Upper Half-Plane)

Consider the Dirichlet problem for the upper half of the complex plane ($y>0$) which satisfies the boundary condition $U(X,0)$ on $Y=0$. Then the Dirichlet Problem is solved by

$$
v(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{u(X,0)}{(X-x)^2+y^2}dX
$$

Proof: Let $f(w) = U(X,Y)+iV(X,Y)$ be analytic for $Y>0$, and C be the contour comprised of the semi-circular arc from R to $-R$ on the upper half of the complex, and the line from $-R$ to R on the real axis. Then from the Cauchy Integral Formula (theorem 15.7.1):

$$
f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw
$$

Then for all z with $\text{Im}\{z\}>0$ and by the Cauchy-Goursat theorem (theorem 15.6.1):

$$
\frac{1}{2\pi i}\int_C \frac{f(w)}{(w-\bar{z})}dw = 0
$$

Subtracting the two equations:

$$
f(z) = \frac{1}{2\pi i}\int_C f(w)\left[\frac{1}{w-z}-\frac{1}{w-\bar{z}}\right]dw = \frac{1}{2\pi i}\int_C\frac{z-\bar{z})}{(w-z)(w-\bar{z})}dw
$$

By breaking up C into the line L and arc A components:

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_L\frac{(z-\bar{z})f(w)}{(w-z)(w-\bar{z})}dw + \frac{1}{2\pi i}\int_C\frac{(z-\bar{z})f(w)}{(w-z)(w-\bar{z})}dw \\
&= \frac{y}{\pi}\int_{-R}^{R}\frac{f(X)}{(X-x)^2+y^2}dX + \frac{y}{\pi}\int_A\frac{f(w)}{(w-z)(w-\bar{z})}dw \qquad w = X+iY
\end{aligned}
$$

Taking the limit as $R \to \infty$, letting $z = x + iy$ and $M = \max|f(w)|$ on A, and knowing $y \le R$ and $r \le R$ (since $z = re^{i\theta}$ is within the semi-circle):

$$\begin{aligned}
&\lim_{R\to\infty}\left|\int_A \frac{f(w)}{(w-z)(w-\bar{z})}dw\right| \\
&\le \lim_{R\to\infty}\left|\frac{M}{(Re^{i\phi}-re^{i\theta})(Re^{i\phi}-re^{-i\theta})}\right||\pi R| \qquad\qquad Theorem\ 15.4.1 \\
&= \lim_{R\to\infty}|\pi R|\left|\frac{M}{R^2e^{2i\phi}-Rre^{i(\phi-\theta)}-Rre^{i(\phi+\theta)}+r^2}\right| \\
&\le \lim_{R\to\infty}|\pi R|\frac{|M|}{|R^2+r^2|} = 0
\end{aligned}$$

Hence, the contour integral on the arc disappears and we are left with:

$$f(z) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(X)}{(X-x)^2+y^2}dX$$

Thus

$$U(x,y) + iV(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{U(X,0)+iV(X,0)}{(X-x)^2+y^2}dX$$

Taking the real part yields the desired results. $\square$

Chapter 16

Series

16.1 Convergence

Definition 16.1.1: Limit (Sequence)

Let $z_1, z_2, z_3, \dots$ be an infinite sequence of complex numbers. We say a limit exists for the sequence if

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}[n > n_0 \implies |z_n - z| < \epsilon]$$

Definition 16.1.2: Converge (Sequence)

If such limit exists, we say the sequence converges to z and that

$$\lim_{n \to \infty} z_n = z$$

Definition 16.1.3: Diverge (Sequence)

If a limit does not exists for a series.

Theorem 16.1.1:

Let $z_n = x_n + iy_n$ for $n \in \mathbb{N}$ and $z = x + iy$. Then

$$\lim_{n \to \infty} z_n = z \iff \left[\lim_{n \to \infty} x_n = x\right] \wedge \left[\lim_{n \to \infty} y_n = y\right]$$

That is, a complex series converges $\iff$ real and imaginary parts of the sequence converge.

Proof: $\underline{\Longleftarrow:}$

$$\begin{aligned}&\left[\lim_{n \to \infty} x_n = x\right] \wedge \left[\lim_{n \to \infty} y_n = y\right] \\ &\implies \forall \epsilon > 0, \exists n_1, n_2 \in \mathbb{N}\left[\left(n > n_1 \implies |x_n - x| < \frac{\epsilon}{2}\right) \wedge \left(n > n_2 \implies |y_n - y| < \frac{\epsilon}{2}\right)\right]\end{aligned}$$

Let $n_0 = \max(n_1, n_2)$, then:

$$n > n_0 \implies \left[|x_n - x| < \frac{\epsilon}{2}\right] \wedge \left[|y_n - y| < \frac{\epsilon}{2}\right]$$

Since

$$|(x_n - iy_n) - (x + iy)| = |(x_n - x) + i(y_n - y)| \leq |x_n - x| + |y_n - y|$$

Thus

$$n > n_0 \implies |z_n - z| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\underline{\implies:}$

$$\lim_{n\to\infty} z_n = z \implies [n > n_0 \implies |(x_n + iy_n) - (x + iy)| < \epsilon]$$

However,

$$\begin{aligned}|x_n - x| \leq |(x_n - x) + i(y_n - y)| = |(x_n + iy_n) - (x + iy)| \\ |y_n - y| \leq |(x_n - x) + i(y_n - y)| = |(x_n + iy_n) - (x + iy)|\end{aligned}$$

Hence

$$[n > n_0 \implies (|x_n - x| < \epsilon) \wedge (|y_n - y| < \epsilon)] \implies \left(\lim_{n\to\infty} x_n = x\right) \wedge \left(\lim_{n\to\infty} y_n = y\right)$$

Thus

$$\lim_{n\to\infty} z_n = z \iff \left[\lim_{n\to\infty} x_n = x\right] \wedge \left[\lim_{n\to\infty} y_n = y\right]$$

$\square$

Be extra careful when converting to polar coordinates:

Example 16.1.1 *It is easy to see that*

$$\lim_{n\to\infty} z_n = -1 + i\frac{(-1)^n}{n^2} = -1 \qquad\qquad n \in \mathbb{N}$$

Converting to polar:

$$r_n = |z_n| \qquad\qquad \Theta_n = \operatorname{Arg}(z_n) \qquad\qquad n \in \mathbb{N},\ -\pi < \Theta_n \leq \pi$$

Then

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} \sqrt{1 + \frac{1}{n^4}} = 1$$

But

$$\lim_{n\to\infty} \Theta_{2n} = \pi \qquad\qquad \lim_{n\to\infty} \Theta_{2n-1} = -\pi \qquad\qquad n \in \mathbb{N}$$

The limit of Θ_n does not exist as $n \to \infty$.

Definition 16.1.4: Partial Sum (Series)

Consider a infinite series

$$\sum_{n=1}^{\infty} z_n$$

A partial sum is a finite sum of of the first N terms of the infinite series, that is:

$$S_N = \sum_{n=1}^{N} z_n \qquad\qquad N \in \mathbb{N}$$

Definition 16.1.5: Converge (Series)

We say a series converge to sum S if the sequence of partial sums converges to S.

Definition 16.1.6: Divergence (Series)

A series diverge if its sequence of partial sums does not converge to sum S.

Theorem 16.1.2:

Let $z_n = x_n + iy_n$ for $n \in \mathbb{N}$ and $S = X + iY$, then

$$\sum_{n=1}^{\infty} z_n = S \iff \left[\sum_{n=1}^{\infty} x_n = X\right] \wedge \left[\sum_{n=1}^{\infty} y_n = Y\right]$$

Proof:

$$S_N = X_N + iY_N = \sum_{n=1}^{N} x_n + i\sum_{n=1}^{\infty} y_n$$

Then

$$\lim_{N\leftarrow\infty} S_N = S \iff \left[\lim_{N\to\infty} X_N = X\right] \wedge \left[\lim_{N\to\infty} Y_N = Y\right]$$

□

Corollary 16.1.2.1:

Let $z_1, z_2, z_3, \ldots$ be an infinite sequence of complex numbers. Then $z_n \to 0$ as $n \to \infty$.

Proof: Follows from the n-th term of a convergent series of real number tends to zero as $n \to \infty$. □

Corollary 16.1.2.2:

Convergent series are bounded, that is:

$$\forall n \in \mathbb{N}, \exists M \in \mathbb{R}_{>0}[|z_n| \leq M]$$

Definition 16.1.7: Absolutely Convergent (Series)

A series is absolutely convergent if

$$\sum_{n=1}^{\infty} |z_n| = \sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2} \qquad\qquad z_n = x_n + iy_n$$

converges to some real number.

Corollary 16.1.2.3:

Absolute Convergence $\implies$ *Convergence*

Proof: Assume a complex series converges absolutely.

$$|x_n| \le \sqrt{x_n^2 + y_n^2} \qquad\qquad |y_n| \le \sqrt{x_n^2 + y_n^2}$$

Then by comparison test, the following must converge

$$\sum_{n=1}^{\infty} |x_n| \qquad\qquad \sum_{n=1}^{\infty} |y_n|$$

Result follows from the absolute convergence of real series implies convergence of real series (corollary 9.0.0.1). □

Definition 16.1.8: Conditional Convergence

A series is conditionally convergent if it converges, but does not converge absolutely.

Definition 16.1.9: Remainder (Series Definition - Complex)

Let S be an infinite series, the remainder after N terms:

$$\rho_N = S - S_N$$

Corollary 16.1.2.4:

Series tend to S $\iff$ $\rho_N \to 0$ *as* $N \to \infty$

Proof:

$$S = S_N + \rho_N \implies |S_N - S| = |\rho_N - 0|$$

Therefore

$$\forall \epsilon > 0, \exists N_0 \in \mathbb{N}[N > N_0 \implies |S_N - S| = |\rho_N - 0| < \epsilon]$$

□

Example 16.1.2 *Verify following using remainders:*

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \qquad\qquad |z| < 1$$

Consider the series:

$$Z = \sum_{n=0}^{N} z^n \qquad\qquad z \neq 1$$

Then

$$Z - Zz = \sum_{n=0}^{N} z^n - \sum_{n=1}^{N+1} z^n = 1 - z^{N+1} \implies Z = \sum_{n=0}^{N} z^n = \frac{1 - z^{N+1}}{1-z}$$

Now, we can use this to write the partial sum:

$$S_N(z) = \sum_{n=0}^{N-1} z^n = \frac{1-z^n}{1-z}$$

If we let:

$$S(z) = \frac{1}{1-z} \implies \rho_N(z) = S(z) - S_N(z) = \frac{z^N}{1-z}$$
$$\implies |\rho_N(z)| = \frac{|z|^N}{|1-z|}$$

It is clear that $\rho_N(z) \to 0$ *for* $z < 1$*, so* $\sum_{n=0}^{\infty} z^n = 1/(1-z)$ *is established.*

Theorem 16.1.3:

The sum and product of two absolutely convergent series is absolutely convergent, and is independent of the order of terms taken. The value of the product/sum is equal to the value of the product/sum of the values of the original series.

Theorem 16.1.4: Ratio Test

Let $\sum_{n=0}^{\infty} S_N$ *be an infinite series, and*

$$\Gamma = \lim_{n\to\infty} \left| \frac{S_{N+1}}{S_N} \right|$$

Then

1. $\Gamma < 1 \implies$ *Series converges absolutely*
2. $\Gamma > 1 \implies$ *Series diverges*
3. $\Gamma = 1 \implies$ *No info on convergence of series*

16.2 Taylor Series

Definition 16.2.1: Power Series

Let R be a region containing a point z_0. A power series is a series of the form

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad\qquad z_0, a_n \in \mathbb{C},\ z \in R$$

Theorem 16.2.1: Taylor's Theorem

Let f be a function analytic throughout a disk $|z-z_0| < R_0$ with radius R_0. Then $f(z)$ has the power series representation:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad a_n = \frac{f^{(n)}(z_0)}{n!} \qquad |z-z_0| < R_0,\ n \in \mathbb{N} \cup \{0\}$$

Note: By extension, f must also be analytic at z_0.

Proof: Let C_0 be a circular contour with radius $|s| = r_0$ and z be any point inside the circle, so $|z| = r$. Now suppose there is a bigger circle enveloping C_0 with radius R_0 such that $r < r_0 < R_0$. Let f be analytic in and on C_0.

$\underline{z_0 = 0}$:
f is analytic in and on C_0, we can use the Cauchy-Integral formula (theorem 15.7.1):

$$f(z) = \frac{1}{2\pi i}\int_{C_0} \frac{f(s)}{s-z} ds$$

Recall from example 16.1.2:

$$\frac{1}{1-z} = \sum_{n=0}^{N-1} z^n + \frac{z^n}{1-z} \implies \frac{1}{s-z} = \sum_{n=0}^{N-1} \frac{1}{s^{n+1}} z^n + \frac{z^N}{(s-z)s^N}$$

We can then write the Cauchy Integral formula:

$$f(z) = \frac{1}{2\pi i}\left[\sum_{n=0}^{N-1}\left(\int_{C_0} \frac{f(s)}{s^{n+1}} ds\right) z^n + z^N \int_{C_0} \frac{f(s)}{(s-z)s^N} ds\right]$$

Using the theorem 15.7.2:

$$\frac{1}{2\pi i}\int_{C_0} \frac{f(z)}{s^{n+1}} ds = \frac{f^{(n)}(0)}{n!} \qquad\qquad n \in \mathbb{N} \cup \{0\}$$

We get

$$f(z) = \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} z^n + \rho_N(z) \qquad\qquad \rho_N(z) = \frac{z^N}{2\pi i}\int_{C_0} \frac{f(s)}{(s-z)s^N} ds$$

This is the Maclaurin series (definition 16.2.2) if we let $N \to \infty$. To prove it:

$$|s - z| \geq ||s| - |z|| = r_0 - r$$

Letting $M = \max|f(s)|$ on C_0:

$$|\rho_N(z)| \leq \frac{r^N}{2\pi} \cdot \frac{M}{(r_0 - r)r_0^N} 2\pi r_0 = \frac{M r_0}{r_0 - r}\left(\frac{r}{r_0}\right)^N$$

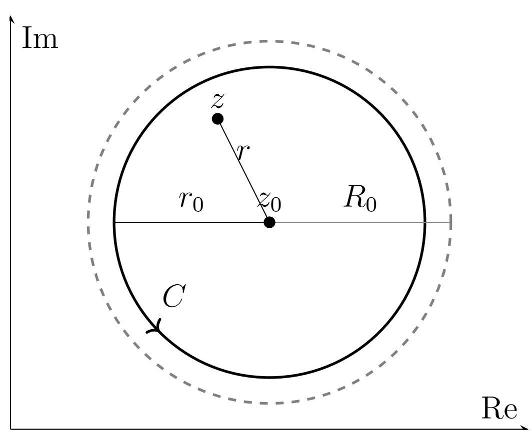
Thus

$$\frac{r}{r_0} < 1 \implies \lim_{N\to\infty}\left(\frac{r}{r_0}\right)^N = 0 \implies \lim_{N\to\infty}\rho_N(z) = 0$$

$\underline{z_0 \neq 0}$:
Suppose f is analytic in $|z - z_0| < R_0$, then $f(z+z_0)$ must be analytic in $|(z + z_0) - z_0| < R_0$. Then

$$\begin{aligned}
f(z + z_0) = g(z) &= \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n && |z| < R_0 \\
\implies f(z + z_0) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} z^n && |z| < R_0 \\
\implies f(z) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n && |z| < R_0
\end{aligned}$$

$\square$



Definition 16.2.2: Maclaurin Series

A Taylor series with $z_0 = 0$:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \qquad\qquad |z| < R_0$$

Useful identities:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad |z| < 1$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad |z| < \infty$$

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \qquad \cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \qquad |z| < \infty$$

$$\sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \qquad \cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad |z| < \infty$$

16.3 Laurent Series

In Taylor's theorem (theorem 16.2.1), f is required to be analytic at z_0. What about the case where f does not need to be analytic at z_0? Well...

Theorem 16.3.1: Laurent's Theorem

Let f be analytic throughout an annular domain $R_1 < |z - z_0| < R_2$ centred at z_0, and C be any positively oriented simple closed contour in said domain. Then at any z in the domain:

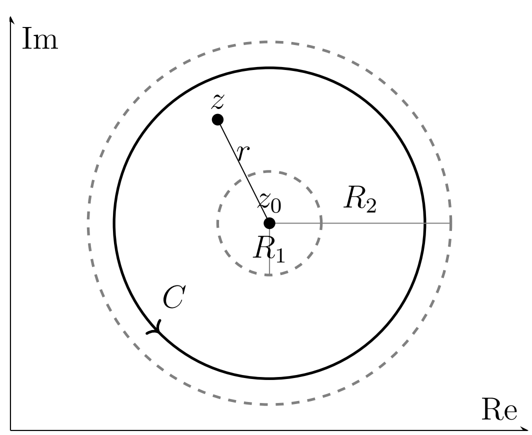
$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \qquad R_1 < |z-z_0| < R_2$$

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{f^{(n)}(z_0)}{n!} \qquad n \in \mathbb{N} \cup \{0\}$$

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{-n+1}} dz \qquad n \in \mathbb{N}$$

Or (by replacing n by $-n$ in b_n):

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n \qquad c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}} \qquad R_1 < |z-z_0| < R_2,\ n \in \mathbb{Z}$$



Proof: Consider the annular regions $R_1 < r_1 \le |z| \le r_2 < R_2$ containing the point z and simple closed contour C. Let C_1 and C_2 denote the circular contours with radius r_1 and r_2, respectively. All the contours are positively oriented. Let f be an analytic function between the region enclosed by C_1 and C_2 and on C_1 and C_2.

$\underline{z_0 = 0}$:
Let γ be a circular contour centred at z and small enough to fit in the annular region $r_1 \le |z| \le r_2$. Using the Cauchy-Goursat theorem (theorem 15.6.1) on multiply connected domains (section 15.6.3):

$$\int_{C_2} \frac{f(s)}{s-z} ds - \int_{C_1} \frac{f(s)}{s-z} ds - \int_{\gamma} \frac{f(s)}{s-z} ds = 0$$

By Cauchy Integral formula (theorem 15.7.1):

$$\int_{\gamma} \frac{f(z)}{s-z} ds = 2\pi i f(z)$$

Hence

$$2\pi i f(z) = \int_{C_2} \frac{f(s)}{s-z} ds - \int_{C_1} \frac{f(s)}{s-z} ds = \int_{C_2} \frac{f(s)}{s-z} ds + \int_{C_1} \frac{f(s)}{z-s} ds$$

Using example 16.1.2:

$$\frac{1}{s-z} = \sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}} + \frac{z^N}{(s-z)s^N}$$
$$\frac{1}{z-s} = \sum_{n=0}^{N-1} \frac{1}{s^{-n}} \cdot \frac{1}{z^{n+1}} + \frac{1}{z^N} \cdot \frac{s^N}{z-s} = \sum_{n=1}^{N} \frac{1}{s^{-n+1}} \cdot \frac{1}{z^n} + \frac{1}{z^N} \cdot \frac{s^N}{z-s}$$

This implies

$$2\pi i f(z) = \int_{C_2} f(s) \left[\sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}} + \frac{z^N}{(s-z)s^N} \right] ds + \int_{C_1} f(s) \left[\sum_{n=1}^{N} \frac{1}{s^{-n+1}} \cdot \frac{1}{z^n} + \frac{1}{z^N} \cdot \frac{s^N}{z-s} \right] ds$$

Interchanging the integral and summation, and dividing by $2\pi i$:

$$f(z) = \sum_{n=0}^{N-1} a_n z^n + \rho_N(z) + \sum_{n=1}^{N} \frac{b_n}{z^n} + \sigma_N(z)$$

Where

$$a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s^{n+1}} ds \qquad \rho_N(z) = \frac{z^N}{2\pi i} \int_{C_2} \frac{f(s)}{(s-z)s^N} ds$$
$$b_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{s^{-n+1}} ds \qquad \sigma_N(z) = \frac{1}{2\pi i z^N} \int_{C_1} \frac{s^N f(s)}{z-s} ds$$

This gives us the Laurent series in $R_1 \le |z| \le R_2$ given that

$$\lim_{N\to\infty} \rho_N(z) = 0 \qquad \lim_{N\to\infty} \sigma_N(z) = 0$$

Which we can prove by letting $|z| = r$, $r_1 < r < r_2$, and $M = \max|f(S)|$ on C_1 and C_2. Then

$$|\rho_N(z)| \leq \frac{Mr_2}{r_2 - r}\left(\frac{r}{r_2}\right)^N \qquad |s - z| \geq r_2 - r \text{ for } s \in C_2$$
$$|\sigma_N(z)| \leq \frac{Mr_1}{r - r_1}\left(\frac{r_1}{r}\right)^N \qquad |z - s| \geq r - r_1 \text{ for } s \in C_1$$

Since $r_1 < r < r_2$, we can see that

$$\lim_{N\to\infty}\left(\frac{r}{r_2}\right)^N = 0 \qquad \lim_{N\to\infty}\left(\frac{r_1}{r}\right)^N = 0$$

By corollary 15.6.4.1, we can replace C_1 and C_2 by a positively oriented closed contour C, giving us the desired expression for the Laurent series.

<u>$z_0 \neq 0$:</u>
Let f be analytic in $R_1 < |z - z_0| < R_2$, then $g(z) = f(z + z_0)$ is analytic in $R_1 < |(z + z_0) - z_0| = |z| < R_2$. Now let t parameterize the path Γ on C so that:

$$z = z(t) - z_0 \qquad R_1 < |z(t) - z_0| = |z| < R_2 \qquad a \leq t \leq b$$

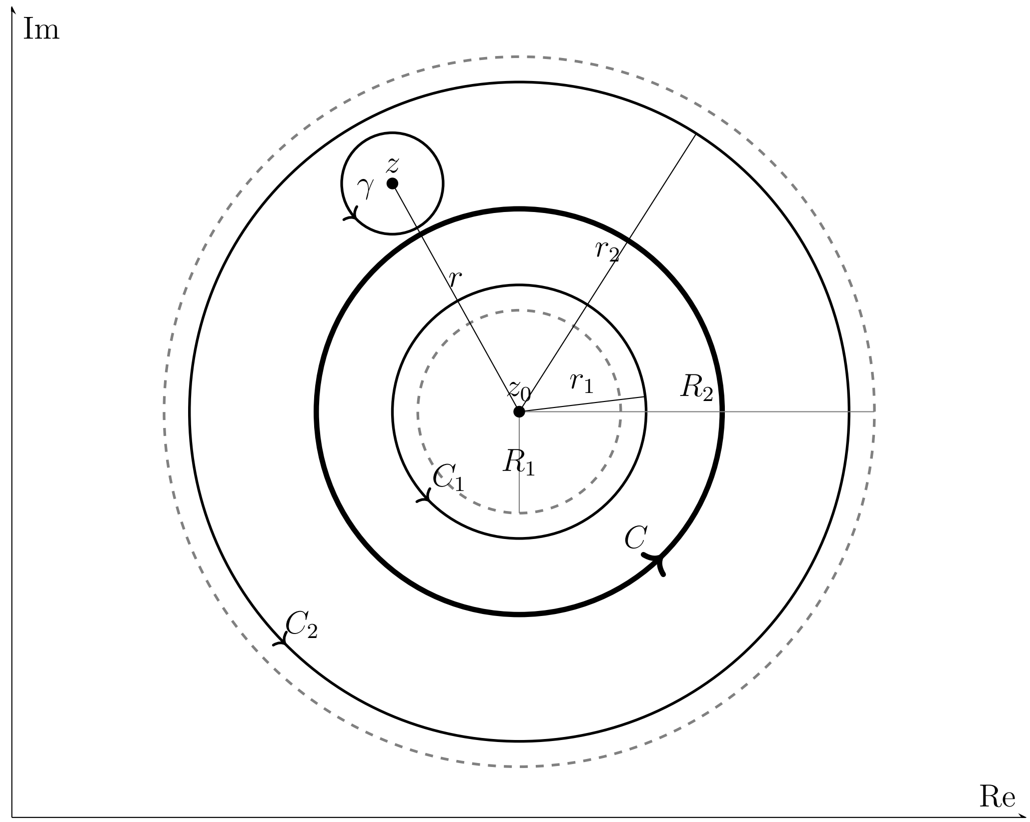
Then g has the Laurent series representation:

$$g(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b^n}{z^n} \qquad R_2 < |z| < R_2$$
$$a_n = \frac{1}{2\pi i}\int_\Gamma \frac{g(z)}{z^{n+1}}dz \qquad n \in \mathbb{N} \cup \{0\}$$
$$b_n = \frac{1}{2\pi i}\int_\Gamma \frac{g(z)}{z^{-n+1}}dz \qquad n \in \mathbb{N}$$

Replacing z by $z - z_0$ and using corollary 15.6.4.1 to replace Γ by C yields the desired result. Note that:

$$2\pi i a_n = \int_\Gamma \frac{g(z)}{z^{n+1}}dz = \int_C \frac{f[z(t)]z'(t)}{[z(t) - z_0]^{n+1}}dt = \int_C \frac{f(z)}{(z - z_0)^{n+1}}dz \qquad z(t) = z$$

Similarly for b_n. □



Note: If f is analytic throughout the disk $|z - z_0| < R_2$, $b_n = (2\pi i)^{-1} \int_C f(z)(z - z_0)^{n-1} dz$ becomes analytic, so $b_n = 0$ due to Cauchy-Goursat theorem (theorem 15.6.1). The Laurent series then becomes a Taylor series about z_0.

Note: In the case where f is not analytic at z_0, then R_1 can become arbitrarily small, so the Laurent series is valid for the punctured disk $0 < |z - z_0| < R_2$. Likewise, if f is only analytic for points outside R_1, then the Laurent series is valid for the region $R_1 < |z - z_0| < \infty$.

Question. *c_n in Laurent's Theorem strongly represents the extended Cauchy Integral formula (theorem 15.7.2). Where*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \qquad\qquad n \in \mathbb{N} \cup \{0\}$$

The only difference is $n \in \mathbb{Z}$ for c_n, while $n \in \mathbb{N} \cup \{0\}$ for the extended Cauchy Integral formula. Is there a connection?

Ans: Generally, no. The Cauchy Integral theorem requires the function to be analytic throughout the domain enclosed by the contour. This is not necessarily true for a Laurent series since we have a deleted neighbourhood that excludes z_0 so, in general, $c_n \neq f^{(n)}(z_0)/n!$.

16.3.1 Examples

Example 16.3.1 (Finding Laurent Series via Known Series) *Find series representation of*

$$f(z) = \frac{1}{z(1+z^2)}$$

We have singularities at $z = 0, \pi 1$, *and since* $|-z^2| < 1 \implies |z| < 1$, *we may use*

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad |z| < 1$$

Substituting $-z^2$ *for* z*:*

$$\begin{aligned} f(z) &= \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n z^{2n} = \sum_{n=0}^{\infty}(-1)^n z^{2n-1} && |z| < 1 \\ &= \frac{1}{z} + \sum_{n=1}^{\infty}(-1)^n z^{2n-1} \\ &= \sum_{n=0}^{\infty}(-1)^{n+1} z^{2n+1} + \frac{1}{z} && \textit{(Standard form)} \end{aligned}$$

Note: This is valid in the region $|z| < 1$, *there is another representation for* $|z - i| < 1$ *and* $|z+1| < 1$.

Example 16.3.2 (z-Transform) *Suppose*

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad R_1 < |z| < R_2, \ n \in \mathbb{Z}$$

Show that if the Laurent series contains the unit circle $|z| = 1$ *then*

$$X^{-1}(z) = x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{i\theta}) e^{in\theta} d\theta \qquad n \in \mathbb{Z}$$

Proof: We can write:

$$\sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n](z - z_0)^{-n} \qquad z_0 = 0$$

It is clear that

$$\begin{aligned} x[n] = b_n &= \frac{1}{2\pi i}\int_C \frac{X(z)}{(z-z_0)^{-n+1}} dz && |z| < 1 \\ &= \frac{1}{2\pi i}\int_C \frac{X(z)}{z^{-n+1}} dz && z_0 = 0 \\ &= \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{X(e^{i\theta})}{e^{i\theta(-n+1)}} \frac{\mathrm{d}}{\mathrm{d}\theta} e^{i\theta} d\theta && z = e^{i\theta} \\ &= \frac{1}{2\pi i}\int_{-\pi}^{\pi} X(e^{i\theta}) e^{in\theta - i\theta} i e^{i\theta} d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{i\theta}) e^{in\theta} d\theta \end{aligned}$$

$\square$

Example 16.3.3 (Bessel Functions of the First Kind) *Let $z \in \mathbb{C}$ and C be the unit circle $w = e^{i\phi}$, $-\pi < \phi < \pi$, in the w-plane. Show for the Laurent series about the origin in the w-plane:*

$$\exp\left[\frac{z}{2}\left(w - \frac{1}{w}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z) w^n \qquad 0 < |w| < \infty$$
$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp[-i(n\phi - z\sin(\phi))]d\phi \qquad n \in \mathbb{Z}$$

And that

$$\operatorname{Re}\{J_n(z)\} = \frac{1}{\pi}\int_{0}^{\infty} \cos[n\phi - z\sin(\phi)]d\phi \qquad n \in \mathbb{Z} \tag{16.1}$$

Proof: We know for a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \qquad R_1 < |z - z_0| < R_2$$
$$c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \qquad n \in \mathbb{Z}$$

For $z_0 = 0$ (since we are taking the series about the origin), we can write

$$\begin{aligned}
J_n(z) = c_n &= \frac{1}{2\pi i}\int_C \frac{\exp[z2^{-1}(w - w^{-1})]}{w^{n+1}} dw && n \in \mathbb{Z} \\
&= \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{\exp[z2^{-1}(e^{i\phi} - e^{-i\phi})]}{e^{i\phi(n+1)}}(ie^{i\phi})d\phi && w = e^{i\phi} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp[zi\sin(\phi)](e^{-in\phi})d\phi \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp[-i(n\phi - z\sin(\phi))]d\phi
\end{aligned}$$

Which is what we are looking for. As for $\operatorname{Re}\{J_n(z)\}$:

$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos[-i(n\phi - z\sin(\phi))] + i\sin[-i(n\phi - z\sin(\phi))]d\phi$$

This implies

$$\begin{aligned}
\operatorname{Re}\{J_n(z)\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos[-i(n\phi - z\sin(\phi))]d\phi \\
&= \frac{1}{\pi}\int_{0}^{\pi} \cos[-i(n\phi - z\sin(\phi))]d\phi && \text{Cosine is an even function}
\end{aligned}$$

$\square$

Example 16.3.4 (Fourier Series) *Let $f(z)$ be a function in some annular domain about the origin that includes the unit circle $z = e^{i\phi}$, $-\pi \le \phi \le \pi$. Show that in the Laurent series representation for any $z \in \mathbb{C}$:*

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})d\phi + \frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(e^{i\phi})\left[\left(\frac{z}{e^{i\phi}}\right)^n + \left(\frac{e^{i\phi}}{z}\right)^n\right] \qquad -\pi \le \phi \le \pi$$

and that for $u(\theta) = \mathrm{Re}\{f(e^{i\phi})\}$*:*

$$u(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\phi)d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} u(\phi)\cos[n(\theta-\phi)]d\phi \qquad -\pi \le \theta \le \pi$$

which is the Fourier Series of $u(\theta)$ *about* $-\pi \le \phi \le \pi$*. Restrictions on* $u(\theta)$ *is more severe than necessary in order for it to be represented by a Fourier series, because it needs to be piecewise continuous on* $[-\pi,\pi]$*, and periodic with period of* 2π *and be everywhere differentiable in* $\mathbb{R}\cup\{-\infty,\infty\}$ *(theorem 19.4.2).*

Proof: We know that for the Laurent series representation of a function:

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n} && R_1 < |z| < R_2 \\ a_n &= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz && n \in \mathbb{N}\cup\{0\} \\ b_n &= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{-n+1}}dz && n \in \mathbb{N} \end{aligned}$$

This tells us that

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\left(\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz\right)(z-z_0)^n + \frac{1}{2\pi i}\sum_{n=1}^{\infty}\left(\int_C \frac{f(z)}{(z-z_0)^{-n+1}}dz\right)(z-z_0)^{-n}$$

Let $z_0 = 0$ since we are in an annular domain about the origin:

$$\begin{aligned} 2\pi i f(z) &= \left[\int_C \frac{f(z)}{z^{n+1}}dz\right] + \sum_{n=1}^{\infty}\left(\int_C \frac{f(z)}{z^{n+1}}dz\right)z^n + \sum_{n=1}^{\infty}\left(\int_C \frac{f(z)}{z^{-n+1}}dz\right)z^{-n} \\ &= \left[\int_C \frac{f(z)}{z^{n+1}}dz\right] + \sum_{n=1}^{\infty}\left[\left(\int_C \frac{f(z)}{z^{n+1}}dz\right)z^n + \left(\int_C \frac{f(z)}{z^{-n+1}}dz\right)z^{-n}\right] \end{aligned}$$

We know that

$$\begin{aligned} \int_C \frac{f(z)}{z^{n+1}}dz &= \int_{-\pi}^{\pi}\frac{f(e^{i\phi})}{e^{i\phi(n+1)}}(ie^{i\phi})d\phi = i\int_{-\pi}^{\pi} f(e^{i\phi})d\phi \\ \int_C \frac{f(z)}{z^{n+1}}dz &= \int_{-\pi}^{\pi}\frac{f(e^{i\phi})}{e^{i\phi(n+1)}}(ie^{i\phi})d\phi = i\int_{-\pi}^{\pi}\frac{f(e^{i\phi})}{e^{in\phi}}d\phi \\ \int_C \frac{f(z)}{z^{-n+1}}dz &= \int_{-\pi}^{\pi}\frac{f(e^{i\phi})}{e^{i\phi(-n+1)}}(ie^{i\phi})d\phi = i\int_{-\pi}^{\pi} f(e^{i\phi})e^{in\phi}d\phi \end{aligned}$$

This gives us

$$2\pi i f(z) = i\int_{-\pi}^{\pi} f(e^{i\phi})d\phi + \sum_{n=1}^{\infty}\left[\left(i\int_{-\pi}^{\pi}\frac{f(e^{i\phi})}{e^{in\phi}}d\phi\right)z^n + \left(i\int_{-\pi}^{\pi} f(e^{i\phi})e^{in\phi}d\phi\right)z^{-n}\right]$$

Bringing z into the integral (we can do this since z is any point in the domain, while the z in the integral is any on C. They represent two different sets of points. Bad notation.)

$$2\pi f(z) = \int_{-\pi}^{\pi} f(e^{i\phi})d\phi + \sum_{n=1}^{\infty}\left[\left(\int_{-\pi}^{\pi} f(e^{i\phi})\frac{e^{in\phi}}{z^n}d\phi\right) + \left(\int_{-\pi}^{\pi} f(e^{i\phi})\frac{e^{in\phi}}{z^{-n}}d\phi\right)\right]$$
$$= \int_{-\pi}^{\pi} f(e^{i\phi})d\phi + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(e^{i\phi})\left[\left(\frac{z}{e^{i\phi}}\right)^n + \left(\frac{e^{i\phi}}{z}\right)^n\right]d\phi$$

Giving us our desired equation. Now let $u(\theta) = \text{Re}\{f(e^{i\theta})\}$:

$$2\pi f(e^{i\theta}) = \int_{-\pi}^{\pi} f(e^{i\phi})d\phi + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(e^{i\phi})\left[\left(e^{in(\theta-\phi)}\right) + \left(e^{-in(\theta-\phi)}\right)\right]d\phi$$
$$= \int_{-\pi}^{\pi} f(e^{i\phi})d\phi + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(e^{i\phi})\left[2\cos[n(\theta-\phi)]\right]d\phi$$

$\theta, \phi \in \mathbb{R} \implies$ Cosine is real-valued function:

$$\text{Re}\left\{2\pi f(e^{i\phi})\right\} = \int_{-\pi}^{\pi} u(e^{i\phi})d\phi + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} u(e^{i\phi})\left[2\cos[n(\theta-\phi)]\right]d\phi$$
$$\text{Re}\left\{f(e^{i\phi})\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\phi)d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} u(\phi)\cos[n(\theta-\phi)]d\phi$$

□

16.4 Absolute and Uniform Convergence of Power Series

Definition 16.4.1: Circle of Convergence

The greatest circle centred at z_0 for which the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for all z interior to the circle. That is, if the circle has radius R and is centred at z_0, then the power series converges $\forall z$ where $|z-z_0| < R$.

Note: As a result, the series can not converge at any point outside of the circle of convergence.

Definition 16.4.2: Center of Expansion

The center of the circle of convergence z_0.

Theorem 16.4.1: Absolute Convergence of Power Series

Consider a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$

$$\sum_{n=0}^{\infty} a_n(z_1-z_0)^n \text{ converges for } z_1 \neq z_0$$
$$\implies \sum_{n=0}^{\infty} a_n(z-z_0)^n \text{ converges absolutely } \forall z \in \{z : |z-z_0| < R_1 = |z_1-z_0|\}$$

That is if $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for some z_1, then it converges absolutely for all points interior to the neighbourhood $|z_1 - z_0|$.

Proof: Assume the series converges, and is therefore bounded.

$$\left(\sum_{n=0}^{\infty} a_n(z_1-z_0)^n\right) \wedge (z_1 \neq z_0) \implies \exists M \in \mathbb{R}_{>0}[|a_n(z_1-z_0)^n| \leq M] \qquad n \in \mathbb{N} \cup \{0\}$$

If we let

$$|z-z_0| < R_1 \qquad\qquad \rho = \frac{|z-z_0|}{z_1-z_0}$$

Then we get

$$|a_n(z-z_0)^n| = |a_n(z_1-z_0)^n|\left(\frac{|z-z_0|}{z_1-z_0}\right)^n \leq M\rho^n \qquad\qquad n \in \mathbb{N} \cup \{0\}$$

We get a geometric series which converges since $\rho < 1$

$$\sum_{n=0}^{\infty} M\rho^n$$

By comparison test, this implies that

$$\sum_{n=0}^{\infty} |a_n(z-z_0)^n| \text{ converges on open disk } |z-z_0| < R_1$$

□

Definition 16.4.3: Uniform Convergence of Power Series

Let

$$S(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \qquad S_N(z) = \sum_{n=0}^{N-1} a_n(z-z_0)^n \qquad |z-z_0| < R$$

and the remainder

$$\rho_N(z) = S(z) - S_N(z) \qquad\qquad |z-z_0| < R$$

Convergence is uniform if

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}[N > N_\epsilon \implies |\rho_N(z)| < \epsilon]$$

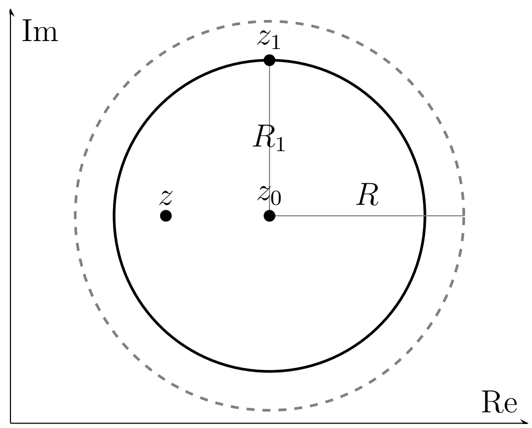
Where N_ϵ is only dependent on ϵ and independent of z in the circle of convergence.

Uniform Convergence is telling us that given any ϵ we can find a $N_\epsilon \in \mathbb{N}$ such that the error for $\rho_N(z)$ for all $N > N_\epsilon$ is always within ϵ. That is, $S_N(z)$ is always within ϵ of the value S_N converges to for all $N_\epsilon > N$. All values of $S_N(z)$ for $N > N_\epsilon$ fall within a static error box ϵ.

Theorem 16.4.2:
Let z_1 be any point inside the circle of convergence $|z - z_0| < R$ of

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Then the series must uniformly converge inside the closed disk $|z - z_0| \le R_1 = |z_1 - z_0|$. It is clear that $R_1 < R$.



Proof: Follows from definition 16.4.3 that the series converges

$$\sum_{n=0}^{\infty} |a_n (z_1 - z_0)^n|$$

Let $m, N \in \mathbb{N}$ and $m > N$, then the remainders:

$$\rho_N = \lim_{m\to\infty} \sum_{n=N}^{m} a_n (z - z_0)^n \qquad \sigma_N = \lim_{m\to\infty} \sum_{n=N}^{m} |a_n (z_1 - z_0)^n|$$

It is clear that

$$|\rho_N(z)| = \lim_{m\to\infty} \left| \sum_{n=N}^{m} a_n (z - z_0)^n \right| \qquad |z - z_0| \le |z_1 - z_0|$$

Then

$$\left| \sum_{n=N}^{m} a_n (z - z_0)^n \right| \le \sum_{n=N}^{m} |a_n||z - z_0|^n \le \sum_{n=N}^{m} |a_n||z_1 - z_0|^n = \sum_{n=N}^{m} |a_n (z_1 - z_0)^n|$$

This implies

$$|z - z_0| \le R_1 \implies |\rho_N(z)| \le \sigma_N$$

Then because σ_N are remainders of a convergent series, $\sigma_N \to 0$ as $N \to \infty$, so

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}[N > N_\epsilon \implies \sigma_N < \epsilon]$$

$$\implies \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ converges uniformly in } |z - z_0| \le R_1$$

$\square$

16.5 Continuity of Sums of Power Series

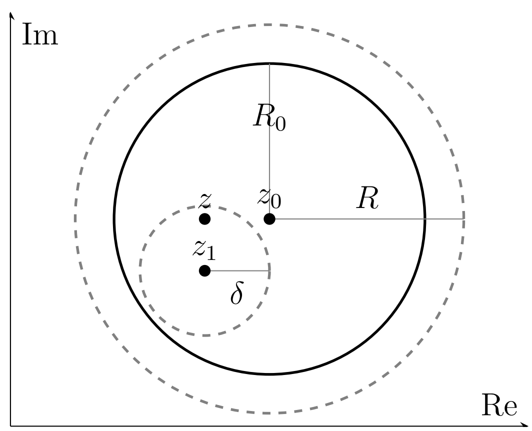
Theorem 16.5.1:
Let

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

If z_1 is a point within the circle of convergence $|z - z_0| = R$, then

$$\forall \epsilon > 0, \exists \delta > 0[|z - z_1| < \delta \implies |S(z) - S(z_1) < \epsilon|]$$

That is, the power series represents a continuous function $S(z)$ for all points inside its circle of convergence.



Proof: Let

$$\begin{aligned} S_N(z) &= \sum_{n=0}^{N} a_n (z - z_0)^n \\ \rho_N(z) &= S(z) - S_N(z) && |z - z_0| < R \\ S(z) &= S_N(z) + \rho_N(z) && |z - z_0| < R \end{aligned}$$

Then

$$\begin{aligned} |S(z) - S(z_1)| &= |S_N(z) - S_N(z_1) + \rho_N(z) - \rho_N(z_1)| \\ |S(z) - S(z_1)| &\leq |S_N(z) - S_N(z_1)| + |\rho_N(z)| + |\rho_N(z_1)| \end{aligned}$$

Suppose that z is any point is a closed disk $|z - z_0| \leq R_0$, where $|z_1 - z_0| < R_0 < R$ and R is the radius of the circle of convergence. By theorem 16.4.2:

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}\left[N > N_\epsilon \implies |\rho_N(z)| < \frac{\epsilon}{3}\right]$$

This also holds for any point z in the neighbourhood $|z - z_1| < \delta$ that is contained in the disk $|z - z_0| \leq R_0$.

$S_N(z)$ is a polynomial, so it is continuous at z_1 for all N. Then we can choose a δ such that for $N = N_\epsilon + 1$:

$$|z - z_1| < \delta \implies |S_N(z) - S_N(z_1)| < \frac{\epsilon}{3}$$

Then

$$\begin{gathered} |S(z) - S(z_1)| \le |S_N(z) - S_N(z_1)| + |\rho_N(z)| + |\rho_N(z_1)| \\ \implies \left[|z - z_1| < \delta \implies |S(z) - S(z_1)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \right] \end{gathered}$$

□

Theorem 16.4.1, theorem 16.4.2, and theorem 16.5.1 can all apply to series of the type

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

by letting $w = 1/(z - z_0)$. Then

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} = \sum_{n=1}^{\infty} b_n w^n$$

Which must converge absolutely to a continuous function when

$$|w| < \frac{1}{|z_1 - z_0|}$$

Now, since $|z - z_0| > |z_1 - z_0|$, the series must converge absolutely to a continuous function exterior to the circle $|z - z_0| = R_2$ where $|z_1 - z_0| = R_1$. We also know that a Laurent Series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

is valid in an annulus $R_1 < |z - z_0| < R_2$, so both series much converge uniformly in the closed annulus.

16.6 Integration and Differentiation of Power Series

Theorem 16.6.1: Integration of Power Series

Let C be any contour lying inside the circle of convergence for the series $S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ and $g(z)$ be any function continuous on C. Then

$$\int_C g(z) S(z) dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z - z_0)^n dz$$

Proof: Let $\rho_N(z)$ be the remainder of $S(z)$, then

$$g(z)S(z) = \sum_{n=0}^{N-1} a_n g(z)(z - z_0)^n + g(z)\rho_N(z)$$

Because the finite sum is continuous over C, their integral over C exists. Likewise with $g(z)\rho_N(z)$. Then

$$\int_C g(z)S(z)dz = \sum_{n=0}^{N-1} a_n \int_C g(z)(z - z_0)^n dz + \int_C g(z)\rho_N(z)dz$$

Since the power series converge uniformly

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}[N > N_\epsilon \implies |\rho_N(z)| < \epsilon]$$

Letting $M = \max|g(z)|$ over C and L be the length of C. Using theorem 15.4.1:

$$N > N_\epsilon \implies \left|\int_C g(z)\rho_N(z)dz\right| < M\epsilon L$$

Then

$$\lim_{N\to\infty} \int_C g(z)\rho_N(z)dz = 0$$

Which gives us

$$\lim_{N\to\infty}\left[\sum_{n=0}^{N-1} a_n \int_C g(z)(z - z_0)^n dz + \int_C g(z)\rho_N(z)dz\right]$$
$$= \lim_{N\to\infty}\left[\sum_{n=0}^{N-1} a_n \int_C g(z)(z - z_0)^n dz\right] = \int_C g(z)S(z)dz$$

□

Corollary 16.6.1.1:
$S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ is analytic for all points z interior to its circle of convergence.

Proof: Let $|g(z)| = 1$ for all z interior to the circle of convergence of $S(z)$. Then for every closed contour C lying in the circle of convergence

$$\int_C g(z)(z - z_0)^n dz = \int_C (z - z_0)^n dz = 0 \qquad n \in \mathbb{N} \cup \{0\}$$

Thus by theorem 16.6.1:

$$\int_C S(z)dz = 0$$

By Morera's Theorem (theorem 15.6.2), $S(z)$ is analytic in the circle of convergence.

□

Theorem 16.6.2: Differentiation of Power Series

Let $S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, then for all points z interior to its circle of convergence:

$$S'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$$

Proof: Let z be any point interior to the circle of convergence of $S(z)$ and C be any positively oriented simple closed contour lying in the circle of convergence surrounding z. Define $\forall s \in C$:

$$g(s) = \frac{1}{2\pi i} \cdot \frac{1}{(s - z)^2}$$

As $g(s)$ is continuous on C

$$\int_C g(s) S(s) ds = \sum_{n=0}^{\infty} a_n \int_C g(s)(s - z_0)^n ds \qquad s \in C$$

Now as $S(z)$ is analytic inside and on C, looking at the left-hand side of the equation, we can use the Extended Cauchy Integral formula (theorem 15.7.2):

$$\int_C g(s) S(s) ds = \frac{1}{2\pi i} \int_C \frac{S(s)}{(s - z)^2} ds = S'(z)$$

As for the right-hand side:

$$\int_C g(s)(s - z_0)^n ds = \frac{1}{2\pi i} \int_C \frac{(s - z_0)^n}{(s - z)^2} ds = \frac{\mathrm{d}}{\mathrm{d}z}(z - z_0)^n$$

Putting this together, we have

$$S'(z) = \sum_{n=0}^{\infty} a_n \frac{\mathrm{d}}{\mathrm{d}z}(z - z_0)^n = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}$$

□

Note: In order to integrate or differentiate a power series, we must first make sure we are doing it at within its circle of convergence. This makes sense since the series no longer converges to the function outside the circle of convergence, so integrating or differentiating it outside does not make much sense.

16.7 Uniqueness of Series Representations

Theorem 16.7.1: Uniqueness of Taylor Series

If $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges to some function $f(z)$ $\forall z$ interior to some circle $|z - z_0| = R$. Then it is the Taylor Series expansion of f for powers of $z - z_0$.

Proof: Let

$$f(z) = \sum_{m=0}^{\infty} a_m (z - z_0)^m \qquad\qquad |z - z_0| < R$$

By theorem 16.6.1:

$$\begin{aligned}
\int_C g(z) f(z) dz &= \sum_{m=0}^{\infty} a_m \int_C g(z)(z - z_0)^m dz \\
g(z) &= \frac{1}{2\pi i} \cdot \frac{1}{(z - z_0)^{n+1}} \qquad\qquad n \in \mathbb{N} \cup \{0\}
\end{aligned}$$

Where C is some circle centred at z_0 with radius less than R. By the Extended Cauchy Integral formula (theorem 15.7.2):

$$\int_C g(z) f(z) dz = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} = \frac{f^{(n)}(z_0)}{n!}$$

This means that

$$\begin{aligned}
\int_C g(z)(z - z_0)^m dz &= \frac{1}{2\pi i} \int_C \frac{1}{(z - z_0)^{n-m+1}} dz \\
&= \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{iRe^{i\theta}}{(Re^{i\theta})^{(n-m+1)}} d\theta \qquad\qquad z = z_0 + Re^{i\theta} \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{(Re^{i\theta})^{(n-m)}} d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta(m-n)}}{R^{(n-m)}} d\theta
\end{aligned}$$

We can see that if $m = n$ the integral is evaluated over 1 so it becomes 1. If $m \neq n$, $e^{i\theta(m-n)}$ is analytic, so the integral becomes zero by theorem 15.6.1. Hence we have:

$$\int_C g(z)(z - z_0)^m dz = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

Then

$$\sum_{m=0}^{\infty} a_m \int_C g(z)(z - z_0)^m dz = a_n$$

So

$$\begin{aligned}
\int_C g(z) f(z) dz &= \sum_{m=0}^{\infty} a_m \int_C g(z)(z - z_0)^m dz \\
\implies \frac{f^{(n)}(z_0)}{n!} &= a_n
\end{aligned}$$

This implies that $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is the Taylor series for f at z_0. $\square$

Theorem 16.7.2: Uniqueness of Laurent Series

Suppose a series

$$\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

converges to $f(z)$ for all points in an annular domain around z_0. Then it is the Laurent series expansion for f in powers of $z-z_0$ in that domain.

Proof: Let

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$$

Then using the techniques in theorem 16.7.1, but for $n, m \in \mathbb{Z}$, we arrive at:

$$\int_C g(z)f(z)dz = \sum_{m=-\infty}^{\infty} \int_C g(z)(z-z_0)^m dz$$

Therefore

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \sum_{m=-\infty}^{\infty} c_m \int_C g(z)(z-z_0)^m dz$$

Which reduces to

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz = c_n \qquad n \in \mathbb{Z}$$

Which implies the series is the Laurent series expansion for f in the annular domain.

□

16.8 Multiplication and Division of Power Series

Definition 16.8.1: Leibniz's Rule

The n-th derivative of the product of two differentiable functions $f(z)$ and $g(z)$:

$$[f(z)g(z)]^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z)g^{(n-k)}(z)$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad n \in \mathbb{N},\ k \in \{0,1,2,\ldots,n\}$$

For proof see example 16.8.1

Definition 16.8.2: Cauchy Product

Suppose within some circle $|z - z_0| = R$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad a_n = \frac{f^{(n)}(z_0)}{n!}$$

$$g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n \qquad b_n = \frac{g^{(n)}(z_0)}{n!}$$

Then the Cauchy Product of the series can be obtained by the Leibniz's Rule:

$$f(z)g(z) = \left[\sum_{n=0}^{\infty} a_n (z - z_0)^n\right] \cdot \left[\sum_{n=0}^{\infty} b_n (z - z_0)^n\right] = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

$$c_n = \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!} \cdot \frac{g^{(n-k)}(z_0)}{(n-k)!} = \sum_{k=0}^{n} a_k b_{n-k}$$

For the division of power series, again suppose two series converges to functions $f(z)$ and $g(z)$ within some circle $|z - z_0| = R$. It may be best to find the series representation of the reciprocal of one function then take the product using the Cauchy Product. That is, write

$$\frac{f(z)}{g(z)} = f(z) \cdot \frac{1}{g(z)}$$

Then find the series representation of $1/g(z)$ using long division. After that is done, use the Cauchy Product to multiply $f(z)$ with $1/g(z)$.

Now, if $f(z)$ and $g(z)$ are both polynomial with $\deg[f(z)] < \deg[g(z)]$, then we can use the method of partial fractions to turn $f(z)/g(z)$ into a sum of partial fractions. We can then substitute a geometric (or another known series) into each of the partial fractions to obtain a series representation for $f(z)/g(z)$. That is

$$\frac{f(z)}{g(z)} = \sum_{n=0}^{N} \frac{A_n}{(z - z_n)^{m_n}} \qquad A_n, z_n \in \mathbb{C},\ m_n \in \mathbb{N}$$

Example 16.8.1 (Deriving Leibniz's Rule) *Show*

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)} \qquad n \in \mathbb{N}$$

Proof: In the case of $n = 1$, we have the usual chain rule:

$$(fg)' = f'g + fg'$$

Now, assume this is true up until some $m \geq 1$, we will show the case for $n = m+1$

$$\begin{aligned}
(fg)^{(m+1)} &= [(fg)']^{(m)} = [f'g + fg']^{(m)} = (f'g)^{(m)} + (fg')^{(m)} \\
&= \sum_{k=0}^{m} \binom{m}{k} f^{(k+1)} g^{(m-k)} + \sum_{k=0}^{m} \binom{m}{k} f^{(k)} g^{(m-k+1)} \\
&= f^{(k+1)} g + \sum_{k=0}^{m-1} \binom{m}{k} f^{(k+1)} g^{(m-k)} + \sum_{k=1}^{m} \binom{m}{k} f^{(k)} g^{(m-k+1)} + f g^{(m+1)} \\
&= f^{(k+1)} g + \sum_{k=1}^{m} \binom{m}{k-1} f^{(k)} g^{(m-k+1)} + \sum_{k=1}^{m} \binom{m}{k} f^{(k)} g^{(m-k+1)} + f g^{(m+1)} \\
&= f^{(m+1)} g + \sum_{k=1}^{m} \left[\binom{m}{k} + \binom{m}{k-1} \right] f^{(k)} g^{(m-k+1)} + f g^{(m+1)}
\end{aligned}$$

Now

$$\begin{aligned}
\binom{m}{k} + \binom{m}{k-1} &= \frac{m!}{k!(m-k)!} + \frac{m!}{(k-1)!(m-k+1)!} \\
&= \frac{m!(m-k+1)}{k!(m-k+1)!} + \frac{m!(k)}{k!(m-k+1)!} \\
&= \frac{m!(m+1)}{k!(m-k+1)!} = \frac{(m+1)!}{k!(m+1-k)!} = \binom{m+1}{k}
\end{aligned}$$

Hence

$$\begin{aligned}
(fg)^{(m+1)} &= f^{(m+1)} g + \sum_{k=1}^{m+1} \binom{m+1}{k} f^{(k)} g^{m-k+1} + f g^{(m+1)} \\
&= \sum_{k=0}^{m+1} \binom{m+1}{k} f^{(k)} g^{m+1-k} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{n-k}
\end{aligned}$$

□

16.9 z-Transform

Take a sequence of numbers and make an analytic function.

Definition 16.9.1: z-Transform

$$\mathbb{Z}[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n} \qquad T \in \mathbb{R}_{>\not{0}},\ z \in \mathbb{C}$$

Notice we are taking n samples of function f at select intervals T. This is the Laurent series with no positive exponential terms with $b_n = f(nT)$. We can also have $w = 1/z$, which we then get the Taylor series with all its usual properties:

$$\mathbb{Z}[f(t)] = \sum_{n=0}^{\infty} f(nT) w^n$$

Definition 16.9.2: Inverse z-Transform

Let C be any circular contour with radius greater than R, where $F(z)$ is analytic $\forall |z| > R$. Then the Inverse z-Transform of $F(z)$:

$$\mathbb{Z}^{-1}[F(z)] = f(nT) = \frac{1}{2\pi i}\int_C F(z)z^{n-1}dz \qquad\qquad n \in \mathbb{N}\cup\{0\}$$

For derivation see example 16.3.2.

Since the z-Transform is linear, we have:

$$\begin{aligned}\mathbb{Z}[cf(t)] &= c\mathbb{Z}[f(t)]\\ \mathbb{Z}[f(t)+g(t)] &= \mathbb{Z}[f(t)] + \mathbb{Z}[g(t)]\\ \mathbb{Z}^{-1}[F(t)+G(t)] &= \mathbb{Z}^{-1}[F(t)] + \mathbb{Z}^{-1}[G(t)]\end{aligned}$$

16.9.1 Product of z-Transforms

Definition 16.9.3: Product of z-Transform

Let C be a circular contour with radius $\rho = |w| = |1/z|$, and

$$\mathbb{Z}[f(t)] = \sum_{n=0}^{\infty} c_n z^{-n} = F(z) \qquad\qquad \mathbb{Z}[g(t)] = \sum_{n=0}^{\infty} d_n z^{-n} = G(z)$$

Then

$$\mathbb{Z}[f(t)g(t)] = \sum_{n=0}^{\infty} f(nT)g(nT)z^{-n} = \sum_{n=0}^{\infty} c_n d_n z^{-n} = \frac{1}{2\pi i}\int_C \frac{F(w)G(z/w)}{w}dw$$

Proof: By definition

$$\mathbb{Z}[f(t)g(t)] = \sum_{n=0}^{\infty} f(nT)g(nT)z^{-n} = \sum_{n=0}^{\infty} c_n d_n z^{-n}$$

Letting $F(z)$ and $G(z)$ be analytic in domain $|z| > R$ we have

$$\begin{aligned}F(w) &= \sum_{m=0}^{\infty} c_m w^{-m} && |w| > R\\ G\left(\frac{z}{w}\right) &= \sum_{n=0}^{\infty} d_n \left(\frac{z}{w}\right)^{-n} = \sum_{n=0}^{\infty} d_n w^n z^{-n} && \left|\frac{z}{w}\right| > R\end{aligned}$$

Hence

$$F(w)G(z/w) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_m d_n w^{n-m} z^{-n} \qquad\qquad |w| > R, \left|\frac{z}{w}\right| > R$$

Now consider a circle of radius $\rho = |w|$ centred at the origin. Taking $\rho > R$, so that $|z| > R\rho$, the Laurent series expansion is then uniformly convergent in the domain containing the circle. So multiplying by $1/(2\pi i w)$:

$$\frac{1}{2\pi i w}F(w)G(z/w) = \frac{1}{2\pi i}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_m d_n \frac{z^{-n}w^{n-m}}{w} \qquad |w| > R, \left|\frac{z}{w}\right| > R$$

Taking the contour integral C around $|w| = \rho$:

$$\frac{1}{2\pi i}\int_C \frac{F(w)G(z/w)}{w}dw = \frac{1}{2\pi i}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_C c_m d_n z^{-n}\frac{w^{n-m}}{w}dw$$

Now by Cauchy-Goursat Theorem *theorem* 15.6.1:

$$\int_C w^k dw = \begin{cases} 0 & k \neq -1 \\ 2\pi i & k = -1 \end{cases} \implies \int_C \frac{w^{n-m}}{w}dw = \begin{cases} 0 & n \neq m \\ 2\pi i & n = m \end{cases}$$

Hence taking the non-trivial solution

$$\frac{1}{2\pi i}\int_C \frac{F(w)G(z/w)}{w}dw = \sum_{n=0}^{\infty} c_n d_n z^{-n} = \mathbb{Z}[f(t)g(t)] \qquad |z| > R\rho,\ \rho > R$$

□

Definition 16.9.4: Convolution

Let $f(t)$ and $g(t)$ be functions, then the convolution of the two functions:

$$f(t) * g(t) = \sum_{k=0}^{\infty} f(kT)g[(n-k)T] \qquad n \in \mathbb{N} \cup \{0\}$$

Definition 16.9.5: Convolution of Products of z-Transform and Inverse z-Transform

$$\mathbb{Z}[h(t)] = \mathbb{Z}[f(t) \circ g(t)] = \mathbb{Z}[f(t)]\mathbb{Z}[G(t)] = F(z)G(z)$$
$$\mathbb{Z}^{-1}[H(z)] = \mathbb{Z}^{-1}[F(z) \circ G(z)] = \mathbb{Z}^{-1}[F(z)]\mathbb{Z}^{-1}[G(z)] = f(t)g(t)$$

Proof: Let $h(t) = f(t) \circ g(t)$, then

$$\mathbb{Z}[h(t)] = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{\infty} f(kT)g[(n-k)T]\right]z^{-n} = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{\infty} a_k b_{n-k}\right]z^{-n}$$
$$=$$

We know that

$$\mathbb{Z}[f(t)] = \sum_{k=0}^{\infty} a_k z^{-k} = F(z) \qquad \mathbb{Z}[g(t)] = \sum_{j=0}^{\infty} b_j z^{-j} = G(z)$$

So

$$F(z)G(z) = \sum_{k=0}^{\infty} a_k z^{-k} \sum_{j=0}^{\infty} b_j z^{-j} = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_k b_j z^{-(k+j)} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k b_{n-k} z^{-n} \qquad n = j + k$$

This tells us that

$$\mathbb{Z}[h(t)] = \mathbb{Z}[f(t) \circ g(t)] = F(z)G(z)$$

$\square$

Chapter 17

Residues and Poles

Cauchy-Goursat theorem (theorem 15.6.1) states the value of a integral over a simple closed contour is zero if the function being evaluated is analytic at all points within and on the contour. What if the function is not analytic for some finite points interior to C? Well...it leaves some "residues" at those points which will contribute to the value of the integral.

17.1 Residues

Definition 17.1.1: Isolated Singular Points

A singular point z_0 is isolated if there exists a deleted neighbourhood ϵ, $0 < |z - z_0| < \epsilon$, where the function f is analytic for all points in the neighbourhood.

Note: It is convenient to consider the point at infinity as an Isolated Singular Point. That is, if a function f is analytic in $0 < R_1 \le |z| < \infty$. Then it has a isolated singular point at $z_0 = \infty$.

Definition 17.1.2: Residue

Recall the Laurent series (theorem 16.3.1), and let C be any positively oriented simple closed contour around z_0 on punctured disk $0 < |z - z_0| < R_2$.

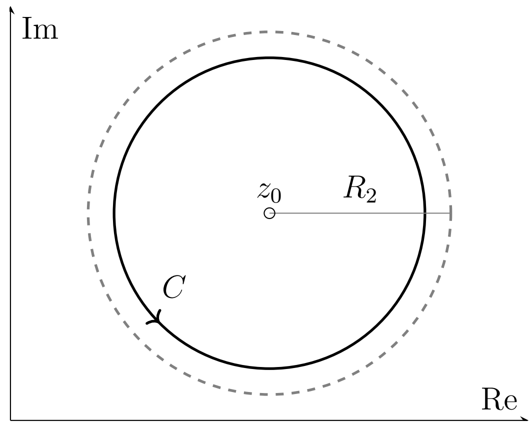
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \qquad 0 < |z - z_0| < R_2$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz$$

The residue is defined as b_n for $n = 1$:

$$\operatorname*{Res}_{z=z_0} [f(z)] = \text{Res}[f(z), z_0] = b_1 = \frac{1}{2\pi i} \int_C f(z) dz$$

Hence, we can use the series expansion of $f(z)$ to find the value of $\int_C f(z)dz$ at $z = z_0$ by finding the coefficient of $1/(z - z_0)$ in the series. That is, series expand, find coefficient of $1/(z - z_0)$, multiply by $2\pi i$ to find contour integral of $f(z)$.



Theorem 17.1.1: Cauchy's Residue Theorem

Let C be a positively oriented simple closed contour, and f be a function analytic inside and on C, except at a finite number of singular points z_k ($k \in \mathbb{N}$) inside C, then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}[f(z), z_k]$$

Proof: Let C be a positively oriented simple closed contour surrounding the singular points z_k, and C_k be positively oriented simple closed contours lying completely in C surrounding each z_k that is small enough they don't share common points. This creates a multiply connected domain consisting of points inside C but exterior to each C_k (section 15.6.3), and using the Cauchy-Goursat Theorem (theorem 15.6.1):

$$\begin{aligned}\int_C f(z)dz - \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0 &\implies \int_C f(z)dz - \sum_{k=1}^{n} 2\pi i \operatorname{Res}[f(z), z_k] = 0 \\ &\implies \int_C f(z)dz = \sum_{k=1}^{n} 2\pi i \operatorname{Res}[f(z), z_k]\end{aligned}$$

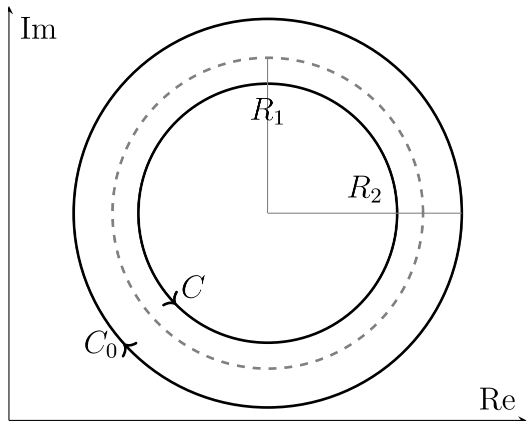
□

17.1.1 Residue at Infinity

Definition 17.1.3: Residue of f at Infinity

Let f be analytic throughout the plane except by a finite number of singular points enclosed inside a positively oriented simple closed contour C, and R_1 be the radius of a circle enclosing C. Then f is analytic in $R_1 < |z| < \infty$. Let C_0 be a circular contour oriented ***negatively*** *with radius R_0 enclosing the previous circle with radius R_1. That is $R_1 < R_0$. Then the residue of f at infinity:*

$$\int_{C_0} f(z)dz = 2\pi i \operatorname{Res}[f(z), \infty]$$



Theorem 17.1.2: Residue at Infinity

Let a function f be analytic everywhere except for a finite number of singular points enclosed by a positively oriented simple closed contour C, then

$$\int_C f(z)dz = 2\pi i \operatorname{Res}\left[\frac{f(1/z)}{z^2}, 0\right]$$

Proof: Following our definition (definition 17.1.3), the point of infinity lies outside of C_0. Since f is anaytic throughout the region bounded by C and C_0, using the Principle of Deformation of Paths (corollary 15.6.4.1):

$$\int_C f(z)dz = \int_{-C_0} f(z)dz = -\int_{C_0} f(z)dz = -2\pi i \operatorname{Res}[f(z), \infty]$$

To find the residue, consider the Laurent series:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \qquad R_1 < |z| < \infty$$

$$c_n = \frac{1}{2\pi i}\int_{-C_0} \frac{f(z)}{z^{n+1}}dz \qquad n \in \mathbb{Z}$$

Replacing z by $1/z$ and multiplying by $1/z^2$:

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} \frac{c_n}{z^{n+2}} = \sum_{n=-\infty}^{\infty} \frac{c_{n-2}}{z^n} \qquad 0 < |z| < \frac{1}{R_1}$$

$$c_{-1} = \operatorname{Res}\left[\frac{f(1/z)}{z^2}, 0\right] = \frac{1}{2\pi i}\int_{-C_0} f(z)dz$$

Thus

$$\int_{C_0} f(z)dz = 2\pi i \operatorname{Res}_{z=\infty}[f(z)] \implies \operatorname{Res}_{z=\infty}[f(z)] = -\operatorname{Res}\left[\frac{f(1/z)}{z^2}, 0\right]$$

$$\implies \int_C f(z)dz = 2\pi i \operatorname{Res}\left[\frac{f(1/z)}{z^2}, 0\right]$$

□

17.2 Three Types of Isolated Singular Points

Definition 17.2.1: Principal Part of f

Let f be a function with an isolated singular point at z_0 in the punctured disk $0 < |z - z_0| < R_2$. Then the Principal Part of of f is this part of the Laurent Series (theorem 16.3.1):

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

Definition 17.2.2: Analytic Part of f

Let f be a function with an isolated singular point at z_0 in the punctured disk $0 < |z - z_0| < R_2$. Then the Analytic Part of of f is this part of the Laurent Series (theorem 16.3.1):

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

It is clear why this is the analytic part, since this part is analytic throughout $|z - z_0| < R_2$, and converges to an analytic function. (It's the Taylor series (theorem 16.2.1) part!)

Definition 17.2.3: Removable Singular Points

When the Principal Part of a Function is zero ($\forall n \in \mathbb{N}[b_n = 0]$). That is,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad\qquad 0 < |z - z_0| < R_2$$

The residue of a removable singular point is always zero. If we define f at z_0 such that $f(z_0) = a_0$ in the expansion, then it becomes a Taylor series and the expansion becomes valid throughout the disk $|z - z_0| < R_2$. The singularity in a sense is removed, since we assigned $f(z_0) = a_0$.

Definition 17.2.4: Essential Singular Points

If an infinite number of b_n in the Principle Part is non-zero, z_0 is an essential singular point of f. That is, they can not be removed.

Definition 17.2.5: Poles of Order m

If Principal Part of f contains more than one, but only finitely many non-zero terms, then

$$\exists m \in \mathbb{N}[(b_m \neq 0) \wedge (b_{m+1} = b_{m+2} = b_{m+3} = \cdots = 0)]$$

That is

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{m} \frac{b_n}{(z - z_0)^n}$$

Definition 17.2.6: Simple Pole

Pole of order 1.

Example 17.2.1 *Consider:*

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{z^n}$$

It's easy to see that $e^{1/z}$ *has an essential singularity at* $z_0 = 0$ *where* $b_1 = 1$*. It follows that*

$$e^{1/z} = -1 \qquad z = \frac{1}{(2n+1)\pi i} \cdot \frac{i}{i} = -\frac{i}{(2n+1)n} \qquad n \in \mathbb{Z}$$

Since

$$e^z = -1 \qquad z = (2n+1)\pi i \qquad n \in \mathbb{Z}$$

Hence, $e^{1/z}$ *assumes the value* -1 *an infinite number of times in each neighbourhood of the origin. So for large enough* n*, an infinite number of such points lie in any given neighbourhood* ϵ *of the origin, except for zero. This illustrates Picard's Theorem.*

Theorem 17.2.1: Picard's Theorem
In each neighbourhood of an essential singular point, a function assumes every finite value an infinite number of times, except with one possible exception.

17.3 Residue at Poles

Theorem 17.3.1:
Let z_0 *be an isolated singular point of function* f*. The following are equivalent:*

1. z_0 *is a pole of order* m $(m \in \mathbb{N})$
2. f *can be written as*

$$f(z) = \frac{\phi(z)}{(z-z_0)^m} \qquad m \in \mathbb{N}$$

   $\phi(z_0)$ *is analytic and* $\phi(z_0) \neq 0$*.*

3.

$$\operatorname{Res}[f(z), z_0] = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} (z-z_0)^m f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \qquad m \in \mathbb{N}$$

Proof: Assume that z_0 is a pole of order m, then f has the Laurent series expansion in the punctured disk $0 < |z - z_0| < R_2$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{m} \frac{b_n}{(z-z_0)^n} \qquad b_m \neq 0$$

Now, define a function ϕ such that

$$\phi(z) = \begin{cases} (z-z_0)^m f(z) & z \neq z_0 \\ b_m & z = z_0 \end{cases}$$

Same as $f(z)$, $\phi(z)$ is analytic throughout $|z - z_0| < R_2$, and has the Laurent series expansion.

$$\phi(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m} + \sum_{n=1}^{m} \frac{b_n}{(z-z_0)^{m-n}}$$

Dividing $\phi(z)$ by $(z-z_0)^m$ gives us

$$f(z) = \frac{\phi(z)}{(z-z_0)^m}$$

Now, since $\phi(z)$ is analytic at z_0, it has the Taylor series expansion in some neighbourhood $|z - z_0| < \epsilon$ of z_0:

$$\phi(z) = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^n$$

Therefore, for $f(z)$ in the same neighbourhood:

$$f(z) = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^{n-m}$$

The Laurent series with $\phi(z_0) \neq 0$ tells us that z_0 is a pole of order m of $f(z)$, and the coefficient of $1/(z-z_0)$, $(n = m-1)$, tells us the coefficient of $\text{Res}[f(z), z_0]$.

$$\text{Res}[f(z), z_0] = \frac{\phi^{(m-1)}}{(m-1)!}$$

$\square$

Warning! Resort to Laurent series for to obtain residues in cases where $\phi(z_0) = 0$ or $\phi(z_0)$ is undefined!

$\phi(z_0) \neq 0$ is essential since $\phi(z_0) = 0$ implies z_0 is a removable singular point which theorem 17.3.1 does not apply. **Always check** $\phi(z_0) \neq 0$ for calculations!

If $\phi(z_0)$ is undefined, then again theorem 17.3.1 does not apply.

From theorem 17.3.1, it is easy to see that if f has a pole of order m at z_0:

1. $[\lim_{z\to z_0}(z-z_0)^n f(z) \neq 0] \vee [\lim_{z\to z_0}(z-z_0)^n f(z) \neq \infty] \implies f(z)$ has pole of order m
2. If f has a pole of order m at z_0:

$$\lim_{z\to z_0}(z-z_0)^n f(z) = \frac{\phi(z)}{(z-z_0)^{m-n}} = \begin{cases} 0 & n > m \\ \infty & n < m \end{cases}$$

On the term "pole":
Consider a function $f(z)$ with a pole of order m at z_0. Then by theorem 17.3.1:

$$f(z) = \frac{\phi(z)}{(z-z_0)^m} \qquad \phi(z) = \sum_{n=0}^{N} c_{-(m-n)}(z-z_0)^n \qquad c_{-m} \neq 0$$

Therefore

$$\lim_{z\to z_0} f(z) \approx \frac{\phi(z_0)}{(z-z_0)^m} = \frac{c_{-m}}{(z-z_0)^m} \implies \lim_{z\to z_0} |f(z)| \approx \frac{|c_{-m}|}{|z-z_0|^m}$$

For higher orders of m, the points around z goes to infinity faster. Hence, a pole of higher order means a "thicker pole" in a graphical sense.

Example 17.3.1 *Consider*

$$f(z) = \frac{1}{z^2\sinh(z)}$$

$z^2\sinh(z)$ *has zeros* $z = n\pi i$, *where* $n \in \mathbb{Z}$, *so* $z = 0$ *is an isolated singularity. It would be a mistake to write*

$$f(z) = \frac{\phi(z)}{z^2} \qquad \phi(z) = \frac{1}{\sinh(z)}$$

since $\phi(z=0)$ *is undefined. We must use the Laurent Series expansion:*

$$\frac{1}{z^2\sinh(z)} = \frac{1}{z^3} - \frac{1}{6z} + \frac{7z}{360} + \cdots \qquad 0 < |z| < \pi$$

So $\mathrm{Res}_{z=0}[f(z)] = -1/6$, *and the singularity is a pole of order* 3, *not* 2.

Example 17.3.2 (Residue to Find Derivative) *Due to the extended Cauchy Integral Formula (theorem 15.7.2), we can use it to find the residue of a function, or use a derivative to find the residue of a function. Let* z_0 *be an isolated singular point of function* f *and suppose*

$$f(z) = \frac{\phi(z)}{(z-z_0)^m} \qquad m \in \mathbb{N}$$

where $\phi(z_0)$ *is analytic and non-zero. Since there is a neighbourhood* $|z-z_0| < \epsilon$ *where* $\phi(z)$ *is analytic throughout, the contour in the extended Cauchy Integral formula can be the positively oriented circle* $|z-z_0| < \epsilon/2$. *By the extended Cauchy Integral formula:*

$$\phi^{(m-1)}(z_0) = \frac{(m-1)!}{2\pi i}\int_C \frac{\phi(z)}{(z-z_0)^m}dz \qquad m = n+1$$
$$\implies \frac{\phi^{(m-1)}(z_0)}{(m-1)!} = \frac{1}{2\pi i}\int_C \frac{\phi(z)}{(z-z_0)^m}dz = \frac{1}{2\pi i}\int_C f(z)dz = \mathrm{Res}[f(z), z_0]$$

17.4 Zeros of Analytic Functions

Definition 17.4.1: Zero of Order m

Let a function f be analytic at point z_0, then all derivative $f^{(n)}(z)$, $n \in \mathbb{N}$, exist at z_0 (theorem 15.7.3). We say that f has a zero of order m if there exists $m \in \mathbb{N}$ such that

$$\left[f(z_0) = f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0\right] \wedge \left[f^{(m)}(z_0) \neq 0\right]$$

That is, it is the lowest order derivative of f at z_0 where $f^m(z_0) \neq 0$.

Theorem 17.4.1:

Let f be a function that is analytic at point z_0, then the following are equivalent:

1. *f has a zero of order m*
2. *There exists a function g, that is analytic and non-zero at z_0, such that*

$$f(z) = (z - z_0)^m g(z)$$

Proof: $\underline{\implies:}$

Suppose f has a zero of order m at z_0, then the analyticity of f at z_0 tells us there is a Taylor series representation in some neighbourhood $|z - z_0| < \epsilon$:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(m+n)}(z_0)}{(m+n)!}(z - z_0)^{m+n} = (z - z_0)^m \sum_{n=0}^{\infty} \frac{f^{m+n}(z_0)}{(m+n)!}(z - z_0)^n$$

Then

$$g(z) = \sum_{n=0}^{\infty} \frac{f^{(m+n)}(z_0)}{(m+n)!}(z - z_0)^n$$

Since $f(z)$ converges in $|z - z_0| < \epsilon$, g is analytic throughout $|z - z_0| < \epsilon$. Also

$$g(z_0) = \frac{f^{(m)}(z_0)}{m!} \neq 0$$

$\underline{\impliedby:}$

Assume there exists a function g that is analytic and non-zero at z_0, such that

$$f(z) = (z - z_0)^m g(z)$$

Then it has the Taylor series representation in some neighbourhood $|z - z_0| < \epsilon$ of z_0:

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z - z_0)^n \implies f(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z - z_0)^{m+n}$$

Hence

$$f^{(m)}(z_0) = m! g(z_0) \neq 0$$

Therefore, z_0 is a zero of order m of f. $\square$

Note: It is clear to see this result by thinking of $f(z)$ and $g(z)$ as polynomials.

Definition 17.4.2: Isolated Zeros

Let f be an analytic function. If there exists z_0 such that $f(z_0) = 0$, then there exists a deleted neighbourhood $0 < |z - z_0| < \epsilon$ where $f(z) \neq 0$. We then call z_0 and isolated zero of $f(z)$.

Theorem 17.4.2:

Let f be a function, and suppose

1. *f is analytic at z_0*
2. *$f(z_0) = 0$, but $f(z) \not\equiv 0$ in any neighbourhood of z_0*

Then $f(z) \neq 0$ in some deleted neighbourhood $0 < |z - z_0| < \epsilon$ of z_0. That is, $f(z) \not\equiv 0 \implies f(z)$ only has isolated zeros.

Proof: Not all derivatives of f is zero at z_0, otherwise all coefficients in the Taylor series of f would be zero, then f would be identically equal to zero in some neighbourhood of z_0. It is clear from definition 17.4.1, that f has a zero of order m at z_0, then by theorem 17.4.1:

$$f(z) = (z - z_0)^m g(z)$$

$g(z)$ analytic and nonzero at z_0. Then there exists some neighbourhood $|z - z_0| < \epsilon$ where $g(z) \neq 0$, thus $f(z) \neq 0$ in deleted neighbourhood $0 < |z - z_0| < \epsilon$. □

Theorem 17.4.3:

Let f be a function, and D and L be a domain and line segment containing point z_0. Suppose

1. *f analytic throughout neighbourhood N_0 of z_0*
2. *$\forall z \in D \cup L[f(z) = 0]$*

Then $\forall z \in N_0[f(z) \equiv 0]$. That is, $f(z)$ is identically zero in N_0 if it does not have isolated zeros.

Proof: Let $f(z) \equiv 0$ in some neighbourhood N of z_0. Otherwise theorem 17.4.2 would imply a contradiction. Then

$$\begin{aligned} \forall z \in N[f(z) \equiv 0] &\implies a_n = \frac{f^{(n)}(z_0)}{n!} = 0 \qquad\qquad n \in \mathbb{N} \cup \{0\} \\ &\implies \forall z \in N_0[f(z) \equiv 0] \end{aligned}$$

Since the Taylor series represents $f(z)$ in N_0. □

Note: We say a function is identically zero $f(z) \equiv 0$ if it becomes the zero function and not merely zero at some point in the domain.

Note: From theorem 17.4.2 and theorem 17.4.3, either a zero is isolated, or it is zero throughout a domain.

17.5 Zeros and Poles

Theorem 17.5.1:
Suppose

1. *Functions p and q are analytic at z_0*
2. *$p(z_0) \neq 0$, and q has zero of order m at z_0*

Then $p(z)/q(z)$ has pole of order m at z_0. That is, the order of the pole of the function takes on the order of the zero of the quotient if the conditions are satisfied.

For converse, see example 17.5.1.

Proof: Suppose p and q are analytic at z_0, $p(z_0) \neq 0$, and $q(z_0)$ is a pole of order m.

$$\begin{aligned} & q(z_0) \text{ is a pole of order } m \\ & \implies q(z) \neq 0 \text{ in some deleted neighbourhood of } z_0 && \text{Theorem 17.4.2} \\ & \implies z_0 \text{ isolated singular point of } \frac{p(z)}{q(z)} \end{aligned}$$

Now, theorem 17.4.1 tells us there exists a function $g(z)$ that is analytic and nonzero at z_0 such that

$$q(z) = (z - z_0)^m g(z)$$

Then

$$\frac{p(z)}{q(z)} = \frac{\phi(z)}{(z - z_0)^m} \qquad\qquad \phi(z) = \frac{p(z)}{g(z)}$$

Therefore

$$\phi(z_0) \text{ analytic and nonzero} \implies z_0 \text{ pole of order } m \text{ of } \frac{p(z)}{q(z)} \qquad \text{Theorem 17.3.1}$$

□

Theorem 17.5.2:
Let function p and q be analytic at z_0.

$$\begin{aligned} & [p(z_0) \neq 0] \wedge [q(z_0) = 0] \wedge [q'(z_0) \neq 0] \\ & \implies \left(z_0 \textit{ simple pole of } \frac{p(z)}{q(z)} \right) \wedge \left(\operatorname{Res}\left[\frac{p(z)}{q(z)}, z_0 \right] = \frac{p(z_0)}{q'(z_0)} \right) \end{aligned}$$

Proof: Suppose $[p(z_0) \neq 0] \wedge [q(z_0) = 0] \wedge [q'(z_0) \neq 0]$, then

$$\begin{aligned} [q(z_0) = 0] \wedge [q'(z_0) \neq 0] &\implies z_0 \text{ zero of order 1 of } q(z) && \text{Definition 17.4.1} \\ &\implies q(z) = (z - z_0)g(z) && \text{Theorem 17.4.1} \end{aligned}$$

Where $g(z)$ analytic and nonzero at z_0. Then

$$\frac{p(z)}{q(z)} = \frac{\phi(z)}{z - z_0} \qquad\qquad \phi(z) = \frac{p(z)}{g(z)}$$

$\phi(z)$ is analytic and nonzero at z_0, so

$$\text{Res}\left[\frac{p(z)}{q(z)}, z_0\right] = \frac{p(z_0)}{q(z_0)} \qquad\qquad \text{Theorem 17.3.1}$$

We know that

$$\begin{aligned} q(z) = (z - z_0)g(z) &\implies q'(z) = g(z) + zg'(z) - z_0 g'(z) \\ &\implies q'(z_0) = g'(z_0) \end{aligned}$$

Thus

$$\text{Res}\left[\frac{p(z)}{q(z)}, z_0\right] = \frac{p(z_0)}{q'(z_0)}$$

□

Example 17.5.1 *Let p and q be analytic functions at z_0. Show*

$$[p(z_0) \neq 0] \wedge [q(z_0) = 0] \wedge \left[\frac{p(z)}{q(z)} \text{ has pole of order } m \text{ at } z_0\right]$$
$$\implies z_0 \text{ pole of order } m \text{ of } q$$

Proof:

$$\frac{p(z)}{q(z)} = \frac{\phi(z)}{(z - z_0)^m} \implies \frac{p(z)}{\phi(z)}(z - z_0)^m = q(z) \qquad\qquad \text{Theorem 17.3.1}$$

Now, both $p(z)$ and $\phi(z)$ are analytic and nonzero at z_0, so let $f(z) = p(z)/\phi(z)$ which is also analytic and nonzero at z_0. Then

$$f(z)(z - z_0)^m = q(z)$$

By theorem 17.4.1, we have $q(z)$ has a zero of order m at z_0. □

Theorem 17.5.3: Bolzano-Weierstrass Theorem

A finite set of points lying in a closed bounded region R has at least one accumulation points in R.

Example 17.5.2 *Show if function f is analytic in region R consisting of all points inside and on simple closed contour C, except for poles inside C, and all zeros of f in R are interior to C are finite in order, then the zeros must be finite in number.*

Proof: Suppose for contradiction that zeros are infinite in number, that is, the set of zeros for f is an infinite set. Then $\exists z_0 \in \{z : f(z) = 0\}$ where z_0 is an accumulation point by Bolzano-Weierstrauss Theorem (theorem 17.5.3).

If all points in the deleted neighbourhood of z_0 which is also in R is a zero of f, then that implies that $f(z) = 0$ for all $z \in R$. This contradicts that we have poles of f in R, so this can not be the case.

Then there must exists a deleted neighbourhood $0 < |z - z_0| < \epsilon$ where a point $x \in R$ is not a zero of f ($f(x) \neq 0$). This implies $f(z) \not\equiv 0$ in all neighbourhoods of z_0, so $f(z) \neq 0$ throughout some deleted neighbourhood of z_0 (theorem 17.4.2). We also know that f only has isolated zeros.

Now, take $\{z : f(z) = 0\} \setminus \{z_0\}$. This is still an infinite set, so we can apply Bolzano-Weierstrauss theorem to obtain another accumulation point $z_1 \neq z_0$ that is exterior to the deleted neighbourhood of z_0 where $f(z) \neq 0$. Applying the same logic as before, we get $f(z) \neq 0$ throughout some deleted neighbourhood of z_1.

We repeat this process to get isolated zeros $z_0, z_1, z_2, \cdots \in R$. Since the zeros are all isolated, the radius each deleted neighbourhood of each point is the distance of each isolated zero to their closest isolated zero. Eventually, R will be contained by the union of finitely many deleted neighbourhoods. This implies the zeros of f are finite in number, hence we have a contradiction to our original assumptions that zeros are infinite in number. □

Example 17.5.3 *Let R be a region consisting of all points interior and on a simple closed contour C. Show if f is analytic in region R except to poles inside C, then the poles must be finite in number.*

Proof: Poles are isolated singular points, so f is analytic in the deleted neighbourhood of poles. Using the same logic as before, R will be contained by finitely many unions of the deleted neighbourhoods of the poles (which are inside R), so we must have a finite number of poles in R. □

17.6 Behaviour of Functions Near Isolated Singular Points

17.6.1 Removable Singular Points

Theorem 17.6.1:

Let f be a function.

z_0 *isolated singular point of* f

$\implies$ f *bounded and analytic in some deleted neighbourhood* $0 < |z - z_0| < \epsilon$ *of* z_0

Proof:

$$\begin{aligned}
&f \text{ analytic throughout } |z - z_0| < R_2 \\
&\implies f \text{ continuous in } |z - z_0| \le \epsilon < R_2 \\
&\implies f \text{ bounded in } |z - z_0| \le \epsilon < R_2 && \text{Theorem 13.3.5} \\
&\implies f \text{ bounded in deleted neighbourhood } 0 < |z - z_0| \le \epsilon
\end{aligned}$$

□

Theorem 17.6.2: Riemann's Theorem

Let function f be bounded and analytic in some deleted neighbourhood $0 < |z - z_0| < \epsilon$ of z_0

$$f \text{ not analytic at } z_0 \implies z_0 \text{ removable singularity of } f$$

Proof: Assume f not analytic at z_0, but bounded and analytic in some deleted neighbourhood $0 < |z - z_0| < \epsilon$ of z_0. Then z_0 is an isolated singularity of f. Consider the Laurent series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \qquad \text{In } 0 < |z - z_0| < \epsilon$$

Let C be a positively oriented circle $|z - z_0| < \rho < \epsilon$, then

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} \qquad n \in \mathbb{N}$$

Since f is bounded, $\exists M \in \mathbb{R}_{>0}[|f(z)| \le M]$. Hence

$$|b_n| \le \frac{1}{2\pi} \cdot \frac{M}{\rho^{-n+1}} 2\pi\rho = M\rho^n \qquad n \in \mathbb{N}$$

ρ can be chosen to be arbitrarily small, therefore $b_n = 0$, and z_0 is a removable singularity of f □

17.6.2 Essential Singular Points

Theorem 17.6.3: Casorati-Weierstrass Theorem

Let z_0 be an essential singularity of function f, and $w_0 \in \mathbb{C}$. Then

$$\forall \epsilon > 0[0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon]$$

Proof: Assume for contradiction that $0 < |z - z_0| < \delta$ does not imply $|f(z) - w_0| < \epsilon$. Then $0 < |z - z_0| < \delta \implies |f(z) - w_0| \geq \epsilon$. Thus

$$g(z) = \frac{1}{f(z) - w_0} \qquad\qquad 0 < |z - z_0| < \delta$$

is bounded and analytic. By Riemann's Theorem (theorem 17.6.2), z_0 is removable singularity of g, so g is defined and analytic at z_0. If $g(z_0) \neq 0$

$$f(z) = \frac{1}{g(z)} + w_0 \qquad\qquad 0 < |z - z_0| < \delta$$

Then $f(z)$ is analytic at z_0, so z_0 is removable singularity of f. Contradicts assumption z_0 is essential singularity of f. □

If $g(z_0) = 0$ then g has zero of order m at z_0 ($g(z_0) \not\equiv 0$ in $|z - z_0| < \delta$), then f has pole of order m at z_0 (theorem 17.5.1). Hence, contradiction!

17.6.3 Poles of Order m

Theorem 17.6.4:

Let f be a function:

$$z_0 \text{ pole of } f \implies \lim_{z \to z_0} f(z) = \infty$$

That is, $|f(z)| \to \infty$ as $z \to \infty$ creating a "pole" in a non-mathematical sense.

Proof: Suppose f has pole of order m at z_0. Then

$$f(z) = \frac{\phi(z)}{(z - z_0)^m} \qquad\qquad \text{Theorem 17.3.1}$$

$\phi(z)$ is analytic and nonzero at z_0. Then

$$\lim_{z \to z_0} \frac{1}{f(z)} = \lim_{z \to z_0} \frac{(z - z_0)^m}{\phi(z)} = \frac{\lim_{z \to z_0}(z - z_0)^m}{\lim_{z \to z_0} \phi(z)} = \frac{0}{\phi(z_0)} = 0$$

By section 13.2.2, $\lim_{z \to z_0} f(z) = \infty$. □

17.7 Application of Residues

17.7.1 Evaluation of Improper Integrals

Definition 17.7.1: Converge (Infinite Integral)

A semi-infinite integral exists if the improper integral converges to a limit:

$$\int_0^\infty f(x)dx = \lim_{R\to\infty}\int_0^R f(x)dx$$

Likewise, for an infinite integral

$$\int_{-\infty}^\infty f(x)dx = \lim_{R_1\to\infty}\int_{R_1}^0 f(x)dx + \lim_{R_2\to\infty}\int_0^{R_2} f(x)dx$$

Definition 17.7.2: Cauchy Principal Value

The Cauchy Principal Value exists if the given limit exists:

$$\text{P.V.}\int_{-\infty}^\infty f(x)dx = \lim_{R\to\infty}\int_{-R}^R f(x)dx$$

Note: Integral Converges $\implies$ Cauchy Principal Value Exists, since

$$\lim_{R\to\infty}\int_{-R}^R f(x)dx = \lim_{R\to\infty}\int_{-R}^0 f(x)dx + \lim_{R\to\infty}\int_0^R f(x)dx$$

Example 17.7.1 *Cauchy Principal Value exists does not imply integral converges*

$$\text{P.V.}\int_{-\infty}^\infty x\ dx = \lim_{R\to\infty}\int_{-R}^R x\ dx = \lim_{R\to\infty}\frac{x^2}{2}\bigg|_{-R}^R = \lim_{R\to\infty} 0 = 0$$

However,

$$\int_{-\infty}^\infty = \lim_{R_1\to\infty}\int_{-R_1}^0 x\ dx + \lim_{R_2\to\infty} x\ dx = \lim_{R_1\to\infty}\frac{x^2}{2}\bigg|_{-R_1}^0 + \lim_{R_2\to\infty}\frac{x^2}{2}\bigg|_0^{R_2} = -\lim_{R_1\to\infty}\frac{R_1^2}{2} + \lim_{R_2\to\infty}\frac{R_2^2}{2}$$

Which the limits do not exist.

Definition 17.7.3: Rational Function

$$f(z) = \frac{p(z)}{q(z)}$$

where $p(z)$ and $q(z)$ are polynomials with coefficients in $\mathbb{R}$, and no common factors. $q(z)$ have no real zeros, but at least one above the real axis.

Theorem 17.7.1:

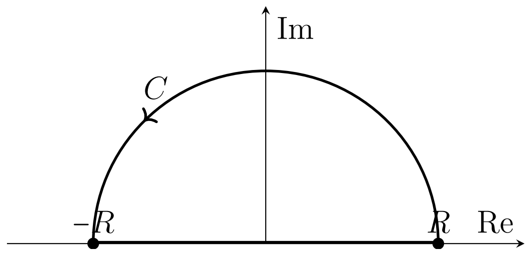
Consider the rational function

$$f(z) = \frac{p(z)}{q(z)}$$

Then

$$\text{P.V.}\int_{-\infty}^\infty f(x)dx = 2\pi i\sum_{k=1}^n \text{Res}[f(z), z_k]$$

Proof: We consider the positively oriented contour C consisting of the line segment from $-R$ to R on the real axis and the top half of the circle with radius $|z| = R$ (C_R), which is large enough where the zeros of the function $z_1, z_2, z_3, \dots, z_n$ all lie within the region enclosed by the contour.



By Cauchy's Residue Theorem (theorem 17.1.1):

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}[f(z), z_k]$$

If

$$\lim_{R\to\infty} \int_{C_R} f(z)dz = 0$$

Then

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{k=1}^{n} \text{Res}[f(z), z_k]$$

□

Question. *Is the choice of the region being above or below the real axis is arbitrary? It seems to be the case. The only thing that will change is the direction of the line integral on the real axis.*

Theorem 17.7.2:

Consider the function $f(z)$ in the upper half-plane. Let C be a semicircular contour $Re^{i\theta}$ in said plane, where $\theta \in [0, \pi]$ with radius R.

$$\exists k > 1, R_0 < R, \mu \in \mathbb{R}, \forall |z| \geq R_0 \left[|f(z)| \leq \frac{\mu}{|z|^k} \implies \lim_{R\to\infty} \int_C f(z)dz = 0 \right]$$

That is, if $|f(z)|$ falls off faster than the reciprocal of the radius R of C, then the integral of $f(z)$ around C vanish as $R \to \infty$.

Proof: Assume $R > R_0$, then by the ML inequality (theorem 15.4.1):

$$\left| \int_C f(z)dz \right| \leq ML = M\pi R$$

Since $|f(z)| \le \mu/|z|^k = \mu/R^k$ on C:

$$\left|\int_C f(z)dz\right| \le \frac{\pi R\mu}{R^k} = \frac{\mu\pi}{R^{k-1}} \qquad\qquad k > 1$$

Thus

$$\lim_{R\to\infty}\left|\int_C f(z)dz\right| = 0$$

□

Example 17.7.2 *Consider the integral*

$$\int_0^\infty f(x)dx = \int_0^\infty \frac{1}{x^6+1}dx$$

The function

$$f(z) = \frac{1}{z^6+1} \qquad\qquad \text{Roots: } c_k = \exp\left[i\left(\frac{\pi}{6}\right) + \left(\frac{\pi}{3}\right)\right]$$

Of these roots, $c_0 = e^{i\pi/6},\ c_1 = i,\ c_2 = e^{i5\pi/6}$ *lie above the real axis. Letting* $B_0,\ B_1$*, and* B_2 *be their residues respectively and using Cauchy's Residue Theorem (theorem 17.1.1):*

$$\int_{-R}^R f(x)dx + \int_{C_R} f(z)dz = 2\pi i(B_0 + B_1 + B_2)$$

Theorem 17.5.2 tells us that these are simple poles, so using said theorem:

$$\text{Res}\left[\frac{p(z)}{q(z)}, z = z_0\right] = \frac{p(z_0)}{q(z_0)} \implies B_k = \text{Res}\left[\frac{1}{z^6+1}, z_k\right] = \frac{1}{6c_k^5} = -\frac{c_k}{6}$$

Hence

$$\int_{-R}^R f(x)dx + \int_{C_R} f(z)dz = 2\pi i(B_0 + B_1 + B_2) = 2\pi i\cdot\frac{1}{6}(e^{i\pi/6} + i + e^{i5\pi/6}) = \frac{2\pi}{3}$$

Now, using the ML Inequality Theorem (theorem 15.4.1), we know

$$\left|z^6+1\right| \ge \left||z|^6 - 1\right| = R^6 - 1$$

So

$$|f(z)| = \frac{1}{|z^6+1|} \le M_R = \frac{1}{R^6-1} \implies \left|\int_{C_R} f(z)dz\right| \le M_R\pi R = \frac{\pi R}{R^6-1}$$

It's clear that

$$\lim_{R\to\infty}\int_{C_R} f(z)dz = 0$$

Hence

$$\lim_{R\to\infty}\int_{-R}^R \frac{1}{x^6+1}dx = \frac{2\pi}{3}$$

Since $f(x)$ *is an even function:*

$$\int_0^\infty \frac{1}{x^6+1}dx = \frac{\pi}{3}$$

17.7.2 Improper Integrals from Fourier Analysis

Evaluating integrals of the form

$$\int_{-\infty}^{\infty} f(x)\sin(ax)dx \qquad \int_{-\infty}^{\infty} f(x)\cos(ax)dx \qquad a \in \mathbb{R}_{>0}$$

We can use

$$\int_{-R}^{R} f(x)\cos(ax)dx + i\int_{-R}^{R} f(x)\sin(ax)dx = \int_{-R}^{R} f(x)e^{iax}dx$$

and that

$$\left|e^{iaz}\right| = \left|e^{iax}e^{-ay}\right| = e^{-ay}$$

is bounded to the upper half of the complex plane to find the integrals using the method in the previous subsection (section 17.7.1).

17.7.3 Jordan's Lemma

Lemma 17.7.2.1: Jordan's Inequality

$$\int_{0}^{\pi} e^{-R\sin(\theta)}d\theta < \frac{\pi}{R} \qquad R > 0$$

Proof: Consider the functions

$$f(\theta) = \sin(\theta) \qquad g(\theta) = \frac{2\theta}{\pi}$$

Now

$$\sin(\theta) \geq \frac{2\theta}{\pi} \implies e^{-R\sin(\theta)} \leq e^{-2R\theta/\pi} \qquad \theta \in \left[0, \frac{\pi}{2}\right],\ R > 0$$

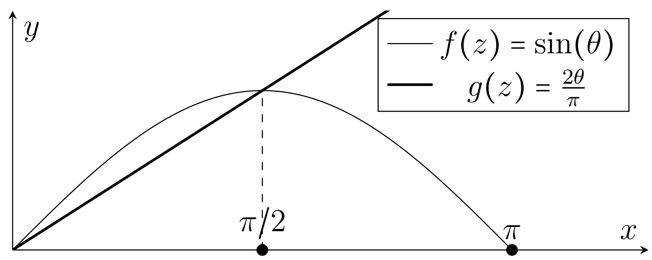
Thus

$$\int_{0}^{\pi/2} e^{-R\sin(\theta)}d\theta \leq \int_{0}^{\pi/2} e^{-2R\theta/\pi}d\theta = \frac{\pi}{2R}(1 - e^{-R}) \leq \frac{\pi}{2R}$$

Since $\sin(\theta)$ is symmetric about $\pi/2$ on $\theta \in [0, \pi]$:

$$\int_{0}^{\pi} e^{-R\sin(\theta)}d\theta < \frac{\pi}{R} \qquad R > 0$$

□



Theorem 17.7.3: Jordan's Lemma

Suppose

1. *Function $f(z)$ is analytic at all points in the upper half plane exterior to circle $|z| = R_0$*
2. *C_R is a semicircle $z = Re^{i\theta}$, $\theta \in [0, \pi]$ and $R > R_0$*
3. *$\forall z \in C_R, \exists M_R > 0$ such that*

$$|f(z)| \leq M_R \qquad\qquad \lim_{R\to\infty} M_R = 0$$

Then $\forall a > 0$:

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{iaz}dz = 0$$

Proof: Assuming the statements in the theorem are true:

$$\int_{C_R} f(z)e^{iaz}dz = \int_0^\pi f(Re^{i\theta})\exp\left(iaRe^{i\theta}\right)Rie^{i\theta}d\theta$$

Since

$$\left|f(Re^{i\theta})\right| \leq M_R \qquad\qquad \left|\exp\left(iaRe^{i\theta}\right)\right| \leq e^{-aR\sin(\theta)}$$

Using Jordan's Inequality (lemma 17.7.2.1):

$$\left|\int_{C_R} f(z)e^{iaz}dz\right| \leq M_R R\int_0^\pi e^{-aR\sin(\theta)}d\theta < \frac{M_R\pi}{a}$$

Since $\lim_{R\to\infty} M_R = 0$

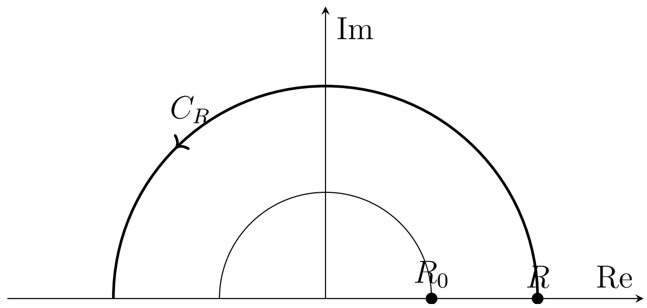
$$\lim_{R\to\infty} \int_{C_R} f(z)e^{iaz}dz = 0$$

□

Note: Jordan's lemma holds true for a quarter circle. This is easily seen by looking at the proof of Jordan's inequality where we have

$$\int_0^{\pi/2} e^{-R\sin(\theta)} d\theta \le \frac{\pi}{2R}$$

then substituting it into a modified proof of Jordan's lemma.



Example 17.7.3 (Fresnel Integrals) *Show*

$$\int_0^{\infty} \cos\left(x^2\right) dx = \int_0^{\infty} \sin\left(x^2\right) dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

Consider the function:

$$f(z) = \exp\left(iz^2\right)$$

We will consider $f(z)$ over the contour C composed of the curves: l_1 line from (0) to R, C_R semicircular arc from R to $Re^{i\pi/4}$, and l_2 line from $Re^{i\pi/4}$ back to 0. We will parameterize the curves:

$$l_1 : z = x \qquad\qquad l_2 : z = re^{i\pi/4} \qquad\qquad C_R : z = re^{i\theta}$$

The integral over the contour is the sum of the curves, which is also zero by Cauchy-Goursat Theorem (theorem 15.6.1):

$$\begin{aligned}\int_C f(z)dz &= \int_0^R \exp\left(ix^2\right)dx + \int_R^0 \exp\left(ir^2 e^{i\pi/2}\right)dz + \int_{C_R} f(z)dz \\ &= \int_0^R \exp\left(ix^2\right)dx - \int_0^R \exp\left(-r^2\right)e^{i\pi/4}dr + \int_{C_R} f(z)dz = 0\end{aligned}$$

Rearranging, we have:

$$\begin{aligned}&\int_0^R \exp\left(ix^2\right)dx = e^{i\pi/4}\int_0^R \exp\left(-r^2\right)dr + \int_{C_R} f(z)dz \\ \implies &\int_0^R \cos\left(x^2\right) + i\sin\left(x^2\right)dx = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\int_0^R \exp\left(-r^2\right)dr + \int_{C_R} f(z)dz\end{aligned}$$

Equating the real and imaginary parts:

$$\int_0^R \cos\left(x^2\right) dx = \frac{1}{\sqrt{2}} \int_0^R \exp\left(-r^2\right) dr + \mathrm{Re}\left\{ \int_{C_R} f(z) dz \right\}$$
$$\int_0^R \sin\left(x^2\right) dx = \frac{1}{\sqrt{2}} \int_0^R \exp\left(-r^2\right) dr + \mathrm{Im}\left\{ \int_{C_R} f(z) dz \right\}$$

Now, using Jordan's Inequality:

$$\begin{aligned}
\int_{C_R} \exp\left(iz^2\right) dz &= \int_{C_R} \exp\left(ir^2 e^{i2\theta}\right) \frac{\partial}{\partial\theta} re^{i\theta} d\theta = \int_{C_R} rie^{i\theta} \exp\left(ir^2 e^{i2\theta}\right) d\theta \\
&= \int_{C_R} rie^{i\theta} \exp\left(ir^2\cos(2\theta) - r^2\sin(2\theta)\right) d\theta \\
&\le R \int_0^{\pi/4} \exp\left(-r^2 \sin(2\theta)\right) d\theta = \frac{R}{2} \int_0^{\pi/2} \exp\left(-r^2\sin(\phi)\right) d\phi && \phi = 2\theta \\
&\le \frac{R}{2} \cdot \frac{\pi}{2r^2} = \frac{\pi}{2r} && \textit{Lemma 17.7.2.1}
\end{aligned}$$

Taking the limit as $R \to \infty$:

$$\lim_{R\to\infty} \int_{C_R} \exp\left(iz^2\right) dz = \lim_{R\to\infty} \frac{\pi}{2R} = 0$$

Using this and the knowledge:

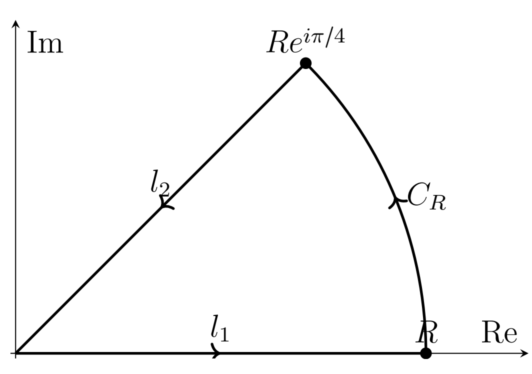
$$\int_0^\infty \exp\left(-r^2\right) dr = \frac{\sqrt{\pi}}{2} \qquad \textit{Example 15.6.2}$$

We get:

$$\lim_{R\to\infty} \int_0^R \cos\left(x^2\right) dx = \lim_{R\to\infty} \frac{1}{\sqrt{2}} \int_0^R \exp\left(-r^2\right) dr = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$
$$\lim_{R\to\infty} \int_0^R \sin\left(x^2\right) dx = \lim_{R\to\infty} \frac{1}{\sqrt{2}} \int_0^R \exp\left(-r^2\right) dr = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

Thus

$$\int_0^\infty \cos\left(x^2\right) dx = \int_0^\infty \sin\left(x^2\right) dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$



17.7.4 Indented Paths

Definition 17.7.4: Cauchy Principal Value of Integral with Singularity

Consider the real-valued function $f(x)$ in $x \in [a,b] \setminus \{x_0\}$ where $x_0 \in (a,b)$ is a singularity of $f(x)$. Let ϵ be some neighbourhood of x_0, then

$$\text{P.V.} \int_a^b f(x)dx = \lim_{\epsilon \to 0} \left[\int_a^{x_0 - \epsilon} f(x)dx + \int_{x_0+\epsilon}^b f(x)dx \right]$$

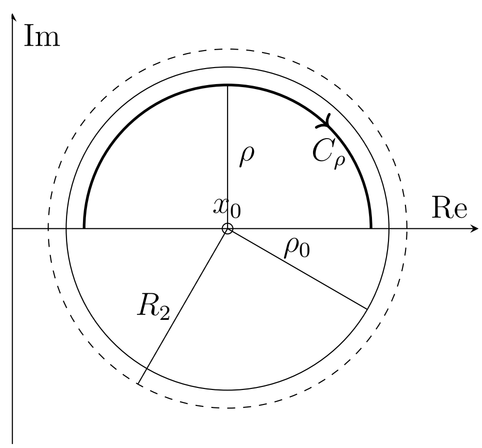
Theorem 17.7.4:

Let

1. *Function $f(z)$ be a function with:*
   (a) *Simple pole at $z = x_0$ on the real axis*
   (b) *Laurent series representation in $0 < |z - x_0| < R_2$*
   (c) *Residue B_0*
2. *C_ρ be a semicircular **negatively** oriented contour with radius $|z - x_0| < \rho$, $\rho < R_2$, in the upper half plane.*

Then

$$\lim_{\rho \to 0} \int_{C_\rho} f(z)dz = -B_0 \pi i$$



Proof: Assume the conditions of the theorem are true, since $f(z)$ has a simple pole at $z = x_0$:

$$f(z) = g(z) + \frac{B_0}{z - x_0} \qquad g(z) = \sum_{n=0}^{\infty} a_n (z - x_0)^n \qquad |z - x_0| \in (0, R_2)$$

Integrating the Laurent series over C_ρ:

$$\int_{C_\rho} f(z)dz = \int_{C_\rho} g(z)dz + B_0 \int_{C_\rho} \frac{1}{z - x_0} dz$$

Since $g(z)$ is continuous in $|z - z_0| < R_2$, it must be bounded on the closed disk $|z - x_0| \leq \rho_0$, $\rho < \rho_0 < R_2$, by theorem 13.3.5. Then

$$\exists M \in \mathbb{R}_{>0}[|z - x_0| \leq \rho_0 \implies |g(z)| \leq M]$$

Using the ML Inequality (theorem 15.4.1):

$$\int_{C_\rho} g(z)dz \leq ML = M\pi\rho \implies \lim_{\rho\to 0}\int_{C_\rho} g(z)dz = 0$$

For the second integral, C_ρ has the parameterization $z = x_0 + \rho e^{i\theta}$, $\theta \in [0, \pi]$. Then

$$\int_{C_\rho} \frac{1}{z - x_0}dz = -\int_{-C_\rho} \frac{1}{z - x_0}dz = -\int_0^\pi \frac{1}{\rho e^{i\theta}}\rho e^{i\theta}d\theta = -i\int_0^\pi d\theta = -\pi i$$

Thus

$$\lim_{\rho\to 0}\int_{C_\rho} f(z)dz = \lim_{\rho\to 0}\left[\int_{C_\rho} g(z)dz + B_0\int_{C_\rho}\frac{1}{z - x_0}dz\right] = -B_0\pi i$$

□

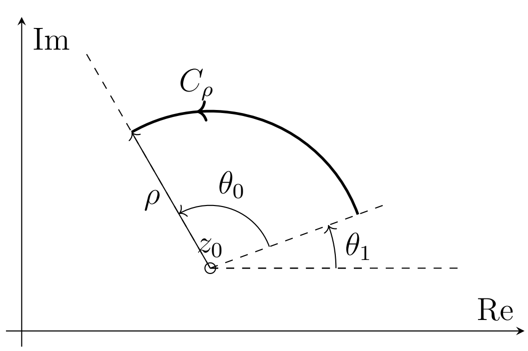
Note: We can use theorem 17.7.4 to integrate around a isolated singular point on the real axis, or a branch cut involving the origin and $\theta \in (-\pi + 2n\pi, 2\pi + 2n\pi)$ (that is, a branch cut below the real axis).

A more general result involving simple poles is as follows:

Theorem 17.7.5:

Let function $f(z)$ has a simple pole at z_0, with z_0 as the center of arc C_ρ ($\theta \in [\theta_0, \theta_0 + \theta_1]$) with radius ρ. If the arc subtends to angle θ_0 at z_0, then

$$\lim_{\rho\to\infty}\int_{C_\rho} f(z)dz = 2\pi i\left(\frac{\theta_0}{2\pi}\operatorname{Res}[f(z), z_0]\right) = i\theta_0\operatorname{Res}[f(z), z_0]$$



Proof: Consider the Laurent series of $f(z)$ around z_0. Since $f(z)$ has a simple pole at z_0:

$$f(z) = g(z) + \frac{B_0}{z - z_0} \qquad\qquad g(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

Note that $g(z)$ is the Taylor series of f about z_0. Integrating the expansion over C_ρ:

$$\int_{C_\rho} f(z)dz = \int_{C_\rho} g(z)dz + \int_{C_\rho} \frac{B_0}{z - z_0}dz$$

$g(z)$ is continuous at z_0, so $|g(z)|$ is bounded in some neighbourhood of z_0, that is, $\exists M \in \mathbb{R}[|g(z)| \leq M]$ in the neighbourhood. By the ML inequality (theorem 15.4.1):

$$\left|\int_{C_\rho} g(z)dz\right| \leq M\rho\theta_0 \implies \lim_{\rho\to 0} \int_{C_\rho} g(z)dz = 0$$

Thus taking the limit as $\rho \to 0$:

$$\begin{aligned}\lim_{\rho\to 0} \int_{C_\rho} f(z)dz &= \int_{C_\rho} \frac{B_0}{z - z_0}dz = \int_{\theta_1}^{\theta_1+\theta_0} \frac{B_0}{(z_0 + re^{i\theta}) - z_0} ire^{i\theta} d\theta \\ &= B_0\theta_0 i = i\theta_0 \operatorname{Res}[f(z), z_0]\end{aligned}$$

□

Observation. *When $\theta = 2\pi$, we get the Cauchy's Residue Theorem (theorem 17.1.1).*

Example 17.7.4 (Dirichlet's Integral) *Show*

$$\int_0^\infty \frac{\sin(x)}{x}dx = \frac{\pi}{2}$$

Proof: Consider integrating $f(z) = e^{iz}/z$ over a the simple closed contour C composed of a line l_1 from ρ to R, a semicircular contour C_R with radius R from 0 to π, a line l_2 from $-R$ to $-\rho$, and a semicircular contour C_ρ with radius $\rho < R$ from $-\pi$ to 0. Notice C_ρ is defined to avoid the singularity of $f(z)$.

By Cauchy-Goursat Theorem (theorem 15.6.1):

$$\begin{aligned}&\int_C f(z)dz = \int_{l_1} \frac{e^{iz}}{z}dz + \int_{C_R} \frac{e^{iz}}{z}dz + \int_{C_\rho} \frac{e^{iz}}{z}dz + \int_{l_2} \frac{e^{iz}}{z}dz = 0 \\ \implies &\int_{l_1} \frac{e^{iz}}{z}dz + \int_{l_2} \frac{e^{iz}}{z}dz = -\int_{C_R} \frac{e^{iz}}{z}dz - \int_{C_\rho} \frac{e^{iz}}{z}dz\end{aligned}$$

Parameterizing the lines:

$$l_1: \ z = re^{i0} = 0 \qquad l_2: \ z = re^{-i\pi} = -r \qquad \rho \leq r \leq R$$

The integral over the lines:

$$\int_{l_1} \frac{e^{iz}}{z}dz + \int_{l_2} \frac{e^{iz}}{z}dz = \int_\rho^R \frac{e^{ir}}{r}dr - \int_\rho^R \frac{e^{-ir}}{r}dr = \int_\rho^R \frac{e^{ir} - e^{-ir}}{r}dr = 2i\int_\rho^R \frac{\sin(r)}{r}dr$$

The integral over the semicircles can be found by using the Laurent Series:

$$\frac{e^{iz}}{z} = \frac{1}{z}\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{i^n z^{n-1}}{n!}$$

$f(z) = e^{iz}/z$ has a simple pole at $z = 0$ with $\text{Res}[f(z), 0] = 1$, then:

$$\lim_{\rho \to 0} \int_{C_\rho} \frac{e^{iz}}{z} dz = -\pi i \qquad \text{Theorem 17.7.4}$$

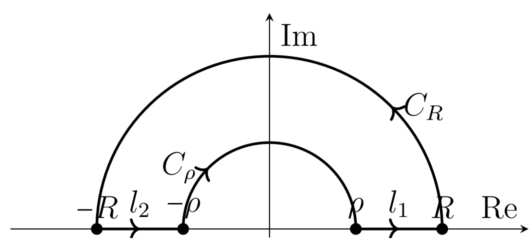
and

$$\left| \frac{1}{z} \right| = \frac{1}{|z|} = \frac{1}{R} \implies \lim_{R \to \infty} \frac{e^{iz}}{z} dz = 0 \qquad \text{Theorem 17.7.3}$$

Hence

$$\int_{l_1} \frac{e^{iz}}{z} dz + \int_{l_2} \frac{e^{iz}}{z} dz = -\int_{C_R} \frac{e^{iz}}{z} dz - \int_{C_\rho} \frac{e^{iz}}{z} dz$$
$$\implies 2i \int_0^\infty \frac{\sin(r)}{r} dr = \pi i \implies \int_0^\infty \frac{\sin(r)}{r} dr = \frac{\pi}{2}$$

□



17.7.5 Integration Along a Branch Cut

TLDR: Integration along both sides of a branch cut does not cancel out, due to different arguments (θ and $\theta + 2\pi$). We must do integration along branch cut with these different values of arguments taken into consideration.

Example 17.7.5 *Show for $a, x \in \mathbb{R}$ with $x > 0$ and $a \in (0, 1)$:*

$$\int_0^\infty \frac{x^{-a}}{x+1} dx = \frac{\pi}{\sin(a\pi)}$$

Proof: Let C_ρ and C_R denote the circular contours with radius ρ and R, respectively, with $\rho < 1 < R$. We will integrate

$$f(z) = \frac{z^{-a}}{z+1} \qquad |z| > 0, \ \arg(z) \in (0, 2\pi)$$

along the simple closed contour composed of a line from ρ to R, positively along C_R, line from R to ρ, and negatively along C_ρ back to ρ. The lines are taken along a branch cut, so we have

$$f(z) = \frac{\exp[-a\log(z)]}{z+1} = \frac{\exp[-a(\ln(r) + i\theta]}{re^{i\theta}+1} = \begin{cases} \dfrac{\exp[-a(\ln(r) + i0]}{r+1} = \dfrac{r^{-a}}{r+1} & \theta = 0 \\[2ex] \dfrac{\exp[-a(\ln(r) + i2\pi]}{r+1} = \dfrac{r^{-a}e^{-i2a\pi}}{r+1} & \theta = 2\pi \end{cases}$$

for paths above and below the branch cut. Then by Cauchy's Residue Theorem (theorem 17.1.1):

$$\int_\rho^R \frac{r^{-a}}{r+1}dr + \int_{C_R} f(z)dz - \int_\rho^R \frac{r^{-a}e^{-2a\pi}}{r+1}dr + \int_{C_\rho} f(z)dz = 2\pi i \,\mathrm{Res}[f(z), -1]$$

Note: This equation is only formal since $f(z)$ is not analytic or defined on the branch cut, but it is valid as we will soon see later in the next example. Now, since there is an isolated singularity at $z = -1$, we define a function $\phi(z)$:

$$\begin{aligned} &\phi(z) = z^{-a} = \exp[-a\log(z)] = \exp[-a(\ln(r) + i\theta)] \qquad r > 0,\ \theta \in (0, 2\pi) \\ &\implies \phi(-1) = \exp[-a(\ln(1) + i\pi)] = e^{-ia\pi} \neq 0 \end{aligned}$$

By theorem 17.3.1:

$$\mathrm{Res}[f(z), -1] = e^{-ia\pi}$$

We then get:

$$(1 - e^{-i2a\pi}) \int_\rho^R \frac{r^{-a}}{r+1}dr + 2\pi i e^{-ia\pi} - \int_{C_\rho} f(z)dz - \int_{C_R} f(z)dz$$

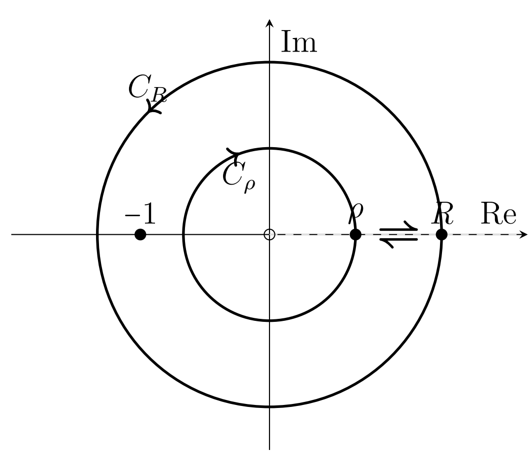
Using the ML Inequality (theorem 15.4.1):

$$\begin{aligned} \left|\int_{C_\rho} f(z)dz\right| &\le \frac{\rho^{-a}}{1-\rho} \cdot 2\pi rho = \frac{\rho}{\rho^a(1-\rho)} &&\implies \lim_{\rho\to\infty} \left|\int_{C_\rho} f(z)dz\right| = 0 \\ \left|\int_{C_R} f(z)dz\right| &\le \frac{R^{-a}}{R-1} \cdot 2\pi R = \frac{2\pi R}{R-1} \cdot \frac{1}{R^a} &&\implies \lim_{\rho\to\infty} \left|\int_{C_R} f(z)dz\right| = 0 \end{aligned}$$

Hence

$$\int_0^\infty \frac{r^{-a}}{r+1}dr = 2\pi i e^{-ia\pi} \implies \int_0^\infty \frac{r^{-a}}{r+1}dr = 2\pi i \frac{e^{-ia\pi}}{1 - e^{-i2a\pi}} = 2\pi i \frac{1}{e^{ia\pi} - e^{-ia\pi}} = \frac{\pi}{\sin(a\pi)}$$

$\square$



Example 17.7.6 *Why the integrals in the previous example is valid, despite $f(z)$ being strictly formal and not analytic or defined on the branch cut.*

Proof: Let $\pi < \theta_0 < 3\pi/2$. Consider two simple closed contours C_1 and C_2. C_1 is composed of line from ρ to R, semicircular contour Γ_R with radius $R > 1$ from $\theta = 0$ to $\theta = \theta_0$, line L on $\theta = \theta_0$, and semicircular contour Γ_ρ with radius $\rho < 1$ from θ_0 to 0. C_2 a contour with line from R to ρ, semicircular contour γ_ρ with radius $\rho < 1$ from 2π to θ_0, line $-L$ on θ_0, and semicircular contour with radius γ_R from θ_0 to 0.

Applying $f_1(z)$ to C_1:

$$f_1(z) = \frac{z^{-a}}{z+1} \qquad |z| > 0,\ -\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}$$

Then knowing $z = re^{i0} = r$ on the line from ρ to R, and using Cauchy's Residue Theorem (theorem 17.1.1):

$$\begin{aligned}\int_{C_1} f_1(z)dz &= \int_\rho^R \frac{z^{-a}}{z+1}dz + \int_{\Gamma_1} f_1(z)dz + \int_L f_1(z)dz + \int_{\Gamma_\rho} f_1(z)dz \\ &= \int_\rho^R \frac{r^{-a}}{r+1}dz + \int_{\Gamma_1} f_1(z)dz + \int_L f_1(z)dz + \int_{\Gamma_\rho} f_1(z)dz = 2\pi i \operatorname{Res}[f_1(z), -1]\end{aligned}$$

Applying $f_2(z)$ to C_2:

$$f_2(z) = \frac{z^{-a}}{z+1} \qquad |z| > 0,\ \frac{\pi}{2} < \arg(z) < \frac{5\pi}{2}$$

Then knowing $z = re^{i2\pi}$ on the line from R to ρ, and using Cauchy-Goursat Theorem (theorem 15.6.1):

$$\begin{aligned}\int_{C_2} f_2(z)dz &= \int_\rho^R \frac{z^{-a}}{z+1}dz + \int_{\gamma_1} f_2(z)dz + \int_{-L} f_2(z)dz + \int_{\gamma_\rho} f_2(z)dz \\ &= \int_\rho^R \frac{r^{-a}e^{-i2a\pi}}{r+1}dz + \int_{\gamma_1} f_2(z)dz + \int_{-L} f_2(z)dz + \int_{\gamma_\rho} f_2(z)dz = 0\end{aligned}$$

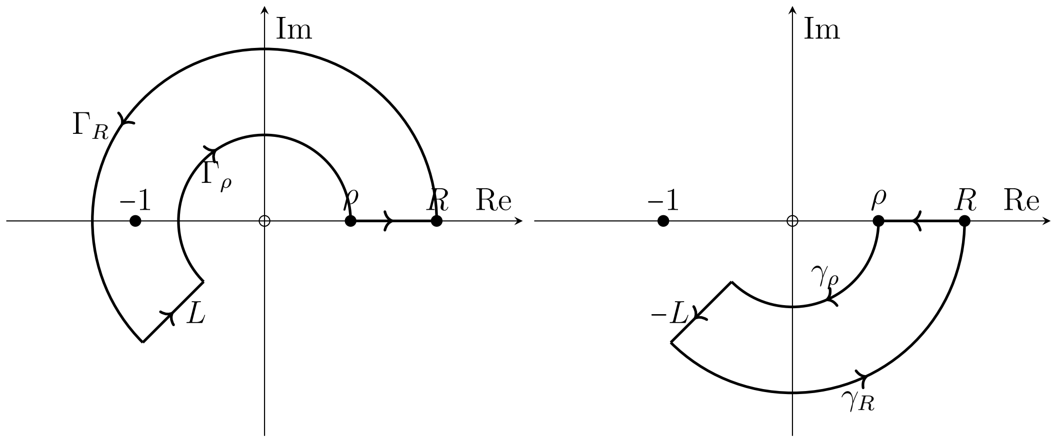
Since the integrals from ρ to R is defined and the other integrals are defined for $(0, 2\pi)$, we can replace $f_1(z)$ and $f_2(z)$ by

$$f(z) = \frac{z^{-a}}{z+1} \qquad |z| > 0,\ \arg(z) \in (0, 2\pi)$$

Adding the integrals over the C_1 and C_2, the integrals on line L cancels out due to being opposite orientations, and we get back our original integral.

$$\int_{C_1} f_1(z)dz + \int_{C_2} f_2(z)dz = \int_C f(z)dz$$

□



Example 17.7.7 (Beta Function) *Show that the Beta Function*

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt = \frac{\pi}{\sin(p\pi)} \qquad\qquad p,q \in \mathbb{R}_{>0}$$

Proof: Let $t = (x+1)^{-1}$, then

$$\begin{aligned} B(p,q) &= \int_0^1 t^{p-1}(1-t)^{q-1}dt = \int_\infty^0 (x+1)^{1-p}[1-(x+1)^{-1}]^{q-1}[-(x+1)^{-2}]dx \\ &= -\int_\infty^0 (x+1)^{-1-p}\left[\frac{x}{x+1}\right]^{q-1} dx = \int_0^\infty \frac{x^{q-1}}{(x+1)^{q+p}}dx \end{aligned}$$

For $q = 1-p$, and using our knowledge from before:

$$B(p,1-p) = \int_0^\infty \frac{x^p}{x+1}dx = \frac{\pi}{\sin(p\pi)}$$

□

17.7.6 Indefinite Integrals Involving Sines and Cosines

Evaluating integrals of the type:

$$\int_0^{2\pi} F[\sin(\theta),\cos(\theta)]d\theta$$

Consider a positively oriented circular simple closed contour C on the unit circle. This suggests a parametric representation:

$$z = e^{i\theta} \qquad\qquad \theta \in [0,2\pi]$$

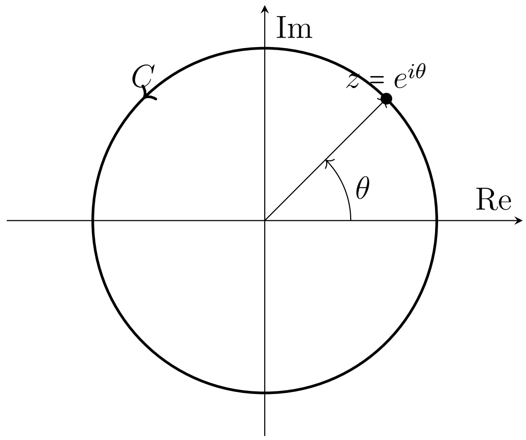
Using the complex representations of sine and cosine (definition 12.5.2):

$$\sin(\theta) = \frac{z-z^{-1}}{2i} \qquad\qquad \cos(\theta) = \frac{z+z^{-1}}{2}$$

We obtain the contour integral:

$$\int_C \frac{1}{iz} F\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right) dz$$

If the integrand reduces to a rational function of z, we can use Cauchy's Residue theorem (theorem 17.1.1) to evaluate the integral, provided we know where the singularities are and conditions are satisfied. That is, substitute the trigonometric identities into the integrand and use Cauchy's Residue Theorem.



17.7.7 Argument Principle

Definition 17.7.5: Meromorphic

Let f be a function with domain D. f is meromorphic if it is analytic throughout D, except at the poles.

Let C be a positively oriented simple closed contour, $w = f(z)$ be a function that is analytic and nonzero on C and meromorphic in domain interior to C, with $\Gamma = \text{img}[f(C)]$. That is, Γ is the image of C under $f(z)$. Γ is also a closed contour, but not necessarily simple.

Since f is nonzero on C, Γ does not pass through the origin.

Suppose we now travel along C starting on z_0, likewise, we are travelling on Γ starting at w_0 under the transformation. Eventually, we will come back to z_0, which is w_0 under the transformation. Let $\arg(w_0) = \phi_0$ and the new argument after travelling Γ back to w_0 be ϕ_1. It is clear that the change in argument is:

$$\Delta_C \arg[f(z)] = \phi_1 - \phi_0$$

It is clear that Δ_C is an integer multiple of 2π, which leads us to the winding number.

Definition 17.7.6: Winding Number

Let C, Γ, z_0, and w_0 be as stated above. The winding number of Γ:

$$\frac{1}{2\pi} \Delta_C \arg[f(z)] \in \mathbb{Z}$$

Note: Γ does not enclose origin $\implies \frac{1}{2\pi}\Delta_C \arg[f(z)] = 0$

It is clear that the zeros of f inside the domain enclosed by C are points that are mapped to the origin (and poles are those that are mapped to points at ∞). Hence, we can use the number of zeros and poles to determine the winding number of Γ.

Question. *The number of zeros of f in C corresponds the the number of times a point in the domain interior to Γ gets mapped to the origin. It is clear this represents the amount of "loops" around the origin. What does the poles represent? They represents the number of times a point in the domain interior to Γ get mapped to points at infinity, which corresponds to loops around the point at infinity. Therefore, the contour is getting "flipped inside-out"?*

Theorem 17.7.6: Argument Principle

Let C be a simple closed contour, $f(z)$ be a function, P be number of poles and Z be number of zeros of $f(z)$ inside C. If

1. *$f(z)$ meromorphic in domain interior to C*
2. *$f(z)$ analytic and nonzero on C*

Then

$$\frac{1}{2\pi}\Delta_C \arg[f(z)] = Z - P$$

Proof: Let C be a simple closed contour and $z = z(t)$ ($t \in [a, b]$) be a parametric representation for C. Consider:

$$\int_C \frac{f'(z)}{f(z)} dz = \int_a^b \frac{f'[z(t)]z'(t)}{f[z(t)]} dt$$

$f(z)$ is nonzero on C, so it does not pass through the origin, thus we can use the exponential form:

$$f[z(t)] = \rho(t)e^{i\phi(t)}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t)e^{i\phi(t)} = \rho'(t)e^{i\phi(t)} + i\rho(t)e^{i\phi(t)}\phi'(t)$$

Substituting:

$$\int_C \frac{f'(z)}{f(z)} dz = \int_a^b \frac{\rho'(t)}{\rho(t)} dt + i\int_a^b \phi'(t) dt = \ln[\rho(t)]\Big|_a^b + i\phi(t)\Big|_a^b$$

Since we have a closed contour:

$$a = b \qquad \rho(a) = \rho(b) \qquad \phi(b) - \phi(a) = \Delta_C \arg[f(z)]$$

Hence

$$\int_C \frac{f'(z)}{f(z)} dz = i\Delta_C \arg[f(z)]$$

Using Cauchy's Residue Theorem (theorem 17.1.1), if f has a zero of order m_0 at z_0:

$$\begin{aligned} f(z) &= (z - z_0)^{m_0} g(z) && \text{Theorem 17.4.1} \\ f'(z) &= m_0(z - z_0)^{m_0 - 1} g(z) + (z - z_0)^{m_0} g'(z) \end{aligned}$$

Thus

$$\frac{f'(z)}{f(z)} = \frac{m}{z - z_0} + \frac{g'(z)}{g(z)}$$

$g'(z)/g(z)$ is analytic at z_0, so it has a Taylor series at z_0. This implies that $f'(z)/f(z)$ has a simple pole at z_0 with residue m_0.

Likewise, if f has a pole of order p_0 at z_0. We get

$$f(z) = (z - z_0)^{-p_0} \phi(z) \qquad \text{Theorem 17.3.1}$$

Using the same procedure as before, $f'(z)/f(z)$ has a simple pole at z_0 with residue $-p_0$. We know the poles and zeros have to be finite (example 17.5.3 and example 17.7.11), therefore, Cauchy's Residue Theorem (theorem 17.1.1):

$$\int_C \frac{f'(z)}{f(z)} dz = 2\pi i \left[\sum_{k=0}^{N} m_k + \sum_{j=0}^{J} (-p_j) \right]$$

Since m_0 corresponds to the number of zeros at z_0, and p_0 is the number of poles at z_0:

$$\left[Z = \sum_{k=0}^{N} m_k \right] \wedge \left[P = \sum_{j=0}^{J} (-p_j) \right] \implies \int_C \frac{f'(z)}{f(z)} dz = 2\pi i (Z - P)$$

Equating the two expressions for the contour integral together and rearranging:

$$\frac{1}{2\pi} \Delta_C \arg[f(z)] = Z - P$$

□

Observation. *The reason we choose to integrate over $f'(z)/f(z)$ is because*

$$\frac{\mathrm{d}}{\mathrm{d}z} \log[f(z)] = \frac{f'(z)}{f(z)}$$

and from the definition of the logarithm (definition 14.2.1):

$$\log(z) = \ln(|z|) + i \arg(z)$$

which is how we are able to "extract" the phase component.

Example 17.7.8 *Consider $f(z)$ over C.*

$$f(z) = \frac{z^3+2}{z = z^2 + \frac{2}{z}}$$

The zeros of the function are $z = \sqrt[3]{2} > 1$ and the poles are $z = 0$. If C is a contour over the unit circle in the positive direction, then by theorem 17.7.6:

$$\Delta_C \arg[f(z)] = 2\pi(0-1) = -2\pi$$

so Γ winds around the origin once in the negative direction.

17.7.8 Rouché's Theorem

Theorem 17.7.7: Rouché's Theorem

Let C be a simple closed contour, and $f(z)$ and $g(z)$ be analytic functions inside and on C, then

$\forall z \in C[|f(z)| > |g(z)|]$
$\implies [f(z)] \wedge [f(z)+g(z)]$ *have same number of zeros, including multiplicities, inside C*

Proof: We know $\forall z \in C$:

$$|f(z)| > |g(z)| \geq 0 \qquad\qquad |f(z)+g(z)| \geq ||f(z)| - |g(z)|| > 0$$

Letting Z_f and Z_{f+g} be the number of zeros, with multiplicities, of $f(z)$ and $f(z)+g(z)$ inside C. From (theorem 17.7.6):

$$Z_f = \frac{1}{2\pi}\Delta_C \arg[f(z)] \qquad Z_{f+g} = \frac{1}{2\pi}\Delta_C \arg[f(z)+g(z)] \qquad z \in C$$

Then

$$\begin{aligned} Z_{f+g} &= \frac{1}{2\pi}\Delta_C \arg[f(z)+g(z)] = \frac{1}{2\pi}\Delta_C \arg\left[f(z) + \left(1 + \frac{g(z)}{f(z)}\right)\right] \\ &= \frac{1}{2\pi}\Delta_C \arg[f(z)] + \frac{1}{2\pi}\Delta_C \arg\left[1 + \frac{g(z)}{f(z)}\right] \end{aligned}$$

Substituting Z_f:

$$Z_{f+g} = Z_f + \frac{1}{2\pi}\Delta_C \arg[F(z)] \qquad\qquad F(z) = 1 + \frac{g(z)}{f(z)}$$

This implies

$$|F(z) - 1| = \frac{|g(z)|}{|f(z)|} < 1$$

This implies the image of C under $F(z)$ lies on open disk $|F(z) - 1| < 1$, hence, does not enclose the origin $F(z) = 0$. Therefore by example 17.7.10:

$$\Delta_C \arg[F(z)] = 0 \implies Z_{f+g} = Z_f$$

$\square$

Theorem 17.7.7 allows us to locate the number of zeros of a function $F(z)$ inside a simple closed contour by smartly breaking it up into two functions $f(z)$ and $g(z)$. Then evaluating the number of zeros of either $f(z)$ or $g(z)$ inside said contour.

Example 17.7.9 (Proof of Fundamental Theorem of Algebra via Rouché's Theorem) *Fundamental Theorem of Algebra: (Theorem 15.8.2).*

Consider a polynomial:

$$P(z) = \sum_{k=0}^{n} a_k z^k \qquad\qquad a_n \neq 0$$

Let

$$f(z) = a_n z^n \qquad\qquad g(z) = = \sum_{k=0}^{n-1} a_k z^k \qquad\qquad |z| = R > 1$$

Then

$$|f(z)| = |a_n| R^n \qquad\qquad |g(z)| \leq \sum_{k=0}^{n-1} |a_k| R^k \leq \sum_{k=0}^{n-1} |a_k| R^{n-1} \qquad\qquad R > 1$$

Then

$$\frac{|g(z)|}{|f(z)|} \leq \sum_{k=0}^{n-1} \frac{|a_k|}{|a_n| R} < 1 \implies \sum_{k=0}^{n-1} \frac{|a_k|}{|a_n|} < R$$

Therefore

$$R > 1 \implies |f(z)| > |g(z)|$$

Now that $|f(z)| > |g(z)|$ is established, we can use Rouché's Theorem. Which tells us that $f(z)$ and $f(z) + g(z) = P(z)$ has the same number of zeros. It is clear that $f(z) = a_n z^n$ has n zeros, which implies $P(z)$ has n zeros.

Note: Liouville's Theorem (theorem 15.8.1) ensures the existence of at least one zero for a polynomial, while Rouché's Theorem ensures the existence of n zeros, including multiplicities.

Example 17.7.10 *Let Γ be a closed contour that is the image of a simple closed contour C under the transformation $f(z)$, and Γ does not enclose the origin $w = 0$, show*

$$\Delta_C arg[f(z)] = 0$$

Proof:

$$\begin{aligned}
&\Gamma \text{ does not enclose } w = 0 \\
&\implies \forall z \in \Gamma[0 \le \arg[f(z)] - \arg[f(z_9)] < 2\pi] \\
&\implies 0 \le \Delta_C \arg[f(z)] < 1 \\
&\implies \Delta_C \arg[f(z)] = 0 && \Delta_C \arg[f(z)] \in \mathbb{Z}
\end{aligned}$$

□

Example 17.7.11 ($\forall z \in D, f(z) \not\equiv 0 \implies$ zeros of f in D finite in order and number) *Let f be a meromorphic function in domain D interior to a simple closed contour C which f is analytic and nonzero. Suppose D_0 consists of all points in D except for the poles. Show that if $\forall z \in D[f(z) \neq 0]$, then zeros of f in D are all finite in order and number.*

Note: $z_0 \in D$ is a zero of f that is not finite in order $\implies \exists \epsilon > |z - z_0|[f(z) \equiv 0]$

Proof: Suppose $\exists z_0 \in D$ where z_0 is a zero of f that is not finite in order

$$\begin{aligned}
&\implies \exists \epsilon > |z - z_0|[f(z) \equiv 0] && \text{Note} \\
&\implies \forall z \in D[f(z) \equiv 0] && \text{Lemma 13.8.0.1} \\
&\implies \text{Contradiction with having poles in } D
\end{aligned}$$

By example 17.5.2: If f is analytic in region R and $\forall z \in D_0[f(z) \not\equiv 0]$, then zeros in R are finite in order and number. □

Example 17.7.12 *Let function f be analytic inside and on a positively oriented simple closed contour C, with no zeros in C. Show if f has n zeros z_k ($k = 1, 2, \ldots, n$) inside C with multiplicity m_k, then*

$$\int_C \frac{zf'(z)}{f(z)} dz = 2\pi i \sum_{k=1}^{n} m_k z_k$$

Proof: $f(z)$ has zero of order m_k at z_k

$$\begin{aligned}
&\implies f(z) = (z - z_k)^{m_k} g(z) && \text{Theorem 17.4.1} \\
&\implies f'(z) = m_k(z - z_k)^{m_k - 1} g(z) + (z - z_k)^{m_0} g'(z)
\end{aligned}$$

Then

$$z\left[\frac{f'(z)}{f(z)}\right] = z\left[\frac{m_k}{z - z_k} + \frac{g'(z)}{g(z)}\right] = \frac{m_k(z - z_k) + m_k z_k}{z - z_k} + \frac{zg'(z)}{g(z)} = m_k + \frac{m_k z_k}{z - z_k} + \frac{zg'(z)}{g(z)}$$

Since $g(z)$ is analytic at z_k, $g'(z)/g(z)$ has a Taylor series at z_0, so it is clear that:

$$\text{Res}\left[\frac{zf'(z)}{f(z)}, z_k\right] = m_k z_k$$

By Cauchy's Residue theorem (theorem 17.1.1):

$$\int_C \frac{zf'(z)}{f(z)}dz = 2\pi i \sum_{k=1}^{n} m_k z_k$$

$\square$

Example 17.7.13 *Show if $c \in \mathbb{C}$, $|c| > e$, then $cz^n = e^z$ has n roots, including multiplicities, inside $|z| = 1$.*

Proof: Rearranging, we have:

$$cz^n - e^z = 0$$

Then

$$|cz^n||c| > e \qquad\qquad |e^z| = \left|e^x e^{iy}\right| = |e^x| = \left|e^{\cos(\theta)}\right| < e$$

Letting $f(z) = cz^n$ and $g(z) = -e^z$, by Rouché's theorem (theorem 17.7.7):

$$f(z) \text{ has } n \text{ zeros} \implies f(z) + g(z) \text{ has } n \text{ zeros} \implies e^z \text{ has } n \text{ zeros}$$

$\square$

Example 17.7.14 (Rouché Theorem Alternate Proof) *Let $f(z)$ and $g(z)$ be functions with $|f(z)| > |g(z)|$, and C be a positively oriented simple closed contour. Consider*

$$\Phi(t) = \frac{1}{2\pi i}\int_C \frac{f'(z) + tg'(z)}{f(z) + tg(z)}dz \qquad\qquad t \in [0,1]$$

Prove Rouché Theorem.

Proof: Prove $\Phi(t)$ exists:
We have $|f(z)| > |g(z)|$, C is as positively oriented simple closed contour, and

$$\Phi(t) = \frac{1}{2\pi i}\int_C \frac{f'(z) + tg'(z)}{f(z) + tg(z)}dz \qquad\qquad t \in [0,1]$$

Since $\forall z \in C[|f(z)| > |g(z)|]$:

$$\begin{aligned}
&\implies |f(z)| > |g(z)| \le |tg(z)| = |t||g(z)| \qquad\qquad t \in [0,1]\\
&\implies \text{Integrand denominator is never zero}\\
&\implies \Phi(t) \text{ exists}
\end{aligned}$$

Prove $\Phi(t)$ continuous:
Let $t, t_0 \in [0,1]$:

$$\begin{aligned}
&\Phi(t) - \Phi(t_0)\\
&= (2\pi i)^{-1}\int_C \frac{f' + tg'}{f + tg} - \frac{f' + t_0 g'}{f + t_0 g}dz\\
&= (2\pi i)^{-1}\int_C \frac{[f + t_0 g][f' + tg'] + [f + tg][f' + t_0 g']}{[f + tg][f + t_0 g]}dz
\end{aligned}$$

Now

$$[f+t_0g][f'+tg'] = ff' + tfg' + t_0f'g + tt_0gg'$$
$$[f+tg][f'+t_0g'] = ff' + tf'g + t_0fg' + tt_0gg'$$

So

$$[f+t_0g][f'+tg'] + [f+tg][f'+t_0g'] = (t-t_0)fg' + (t_0-t)f'g$$

Hence

$$\Phi(t) - \Phi(t_0) = (2\pi i)^{-i}(t-t_0)\int_C \frac{fg'-f'g}{(f+tg)(f+t_0g)}dz$$
$$\implies |\Phi(t) - \Phi(t_0)| = (2\pi i)^{-1}|t-t_0|\left|\int_C \frac{fg'-f'g}{(f+tg)(f+t_0g)}dz\right|$$

Now

$$\frac{fg'-f'g}{(f+tg)(f-t_0g)} = \frac{fg'-f'g}{f^2+tt_0g^2+tfg+t_0fg}$$

Since $t, t_0 \in [0,1]$:

$$\left|tt_0g^2\right| < \left|g^2\right| \qquad |tfg| < |fg| \qquad |t_0fg| < |fg|$$

Therefore

$$\left|f^2+tt_0g^2+tfg+t_0fg\right| \leq \left|f^2\right| + \left|g^2\right| + |2fg| = (|f|+|g|)^2$$
$$\implies \left|\frac{fg'+f'g}{(f+tg)(f+t_0g)}\right| \leq \frac{|fg'-f'g|}{(|f|+|g|)^2}$$
$$\implies |\Phi(t)-\Phi(t_0)| \leq (2\pi i)^{-1}|t-t_0|\int_C \frac{|fg'-f'g|}{(|f|+|g|)^2}dz < |t-t_0|\int_C \frac{|fg'-f'g|}{(|f|+|g|)^2}dz$$

Since C is closed and bounded, and $(fg'+f'g)(|f|+|g|)^{-2}$ is continuous on C, by theorem 13.3.5, $(fg'+f'g)(|f|+|g|)^{-2}$ is bounded on C. Therefore

$$\exists A \in \mathbb{R}_{>0}[|\Phi(t)-\Phi(t_0)| \leq A|t-t_0|] < \epsilon$$

We can choose this neighbourhood ϵ to be arbitrarily small, so $\Phi(t)$ is continuous by definition (chapter 10).

Prove Rouché Theorem (theorem 17.7.7):
Let $F(z) = f(z) + tg(z)$, in the proof of (theorem 17.7.6), we know that:

$$\int_C \frac{F'(z)}{F(z)}dz = 2\pi i(Z-P)$$

Hence

$$\Phi(t) = (2\pi i)^{-1} \int_C \frac{f'(z) + tg(z)}{f(z) + tg(z)} dz = (2\pi i)^{-1} \int_C \frac{F'(z)}{F(z)} dz = Z - P$$
$$= \frac{1}{2\pi} \Delta_C \arg[f(z) + tg(z)]$$

We can then see that $\Phi(t)$ represents the difference of the number of zeros and numbers of poles of $f(z) + tg(z)$. This number is independent of t, hence

$$\Phi(1) = \frac{1}{2\pi} \Delta_C \arg[f(z) + g(z)] = Z - P$$
$$\Phi(0) = \frac{1}{2\pi} \Delta_C \arg[f(z)] = Z - P$$

Thus, the number of zeros for $f(z)$ and $f(z) + g(z)$ are equal given $|f(z)| > |g(z)|$, proving Rouché Theorem. □

17.7.9 Inverse Laplace Transforms

Definition 17.7.7: Bromwich Integral

Let $F(s)$ be a complex function, and L_R be a vertical line segment from $s = \gamma - iR$ to $s = \gamma + iR$ large enough that singularities of F lie to the left of L_R. Consider

$$f(t) = \frac{1}{2\pi i} \lim_{R \to \infty} \int_{L_R} e^{st} F(s) ds \qquad t > 0$$

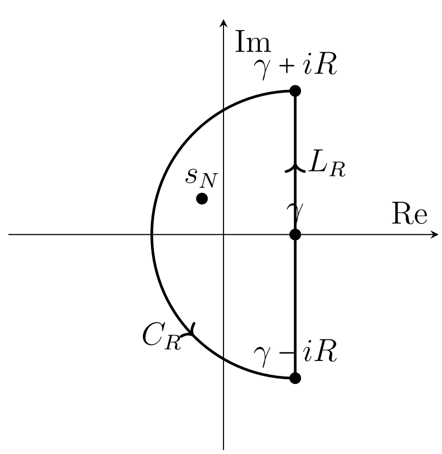
If the limit exists, then the Bromwich Integral:

$$f(t) = \frac{1}{2\pi i} \text{P.V.} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s) ds \qquad t > 0$$

The Bromwich integral is the inverse Laplace transform.

Definition 17.7.8: Laplace Transform

$$F(s) = \int_0^\infty e^{st} f(s) dt$$



Deriving the Inverse Laplace Transform.

Let $F(s)$ and L_R be defined as above, and C_R be a semicircular contour from $\gamma + iR$ to $\gamma - iR$ enveloping the zeros of $F(s)$ $(s_k,\ k = 1, 2, \ldots, n)$. By Cauchy's Residue Theorem (theorem 17.1.1):

$$\int_{L_R} e^{st}F(s)ds = 2\pi i \sum_{n=1}^{N} \mathrm{Res}[e^{st}F(s), s_n] - \int_{C_R} e^{st}F(s)ds$$

Assuming

$$\lim_{R\to\infty} \int_{C_R} e^{st}F(s)ds = 0$$

Then

$$f(t) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{L_R} e^{st}F(s)ds = \sum_{n=1}^{N} \mathrm{Res}[e^{st}F(s), s_n] \qquad t > 0$$

Example 17.7.15 *Let*

$$F(s) = \frac{s}{s^2+4} = \frac{s}{(s+2i)(s-2i)}$$

Then

$$f(t) = \sum_{n=1}^{N} \mathrm{Res}[e^{st}F(s), s_n] = \mathrm{Res}\left[\frac{e^{st}s}{(s+2i)(s-2i)}, 2i\right] + \mathrm{Res}\left[\frac{e^{st}s}{(s+2i)(s-2i)}, -2i\right]$$

The singularities are simple poles, so by theorem 17.3.1:

$$\begin{aligned} f(t) &= \mathrm{Res}\left[\frac{\phi_1(s)}{s-2i}, 2i\right] + \mathrm{Res}\left[\frac{\phi_2(s)}{s+2i}, -2i\right] \qquad \phi_1(s) = \frac{e^{st}s}{s+2i} \qquad \phi_1(s) = \frac{e^{st}s}{s-2i} \\ &= \phi_1(2i) + \phi(-2i) = \frac{e^{2it}(2i)}{4i} + \frac{e^{-2it}(-2i)}{-4i} \\ &= \frac{e^{i2t} + e^{-i2t}}{2} = \cos(2t) \end{aligned}$$

Example 17.7.16 *Find the inverse Laplace transform of*

$$F(s) = \frac{1}{s^2} - \frac{1}{s\sinh(s)}$$

Proof: The isolated singularities of $F(s)$:

$$s_0 = 0 \qquad s_n = n\pi i \qquad \overline{s_n} = -n\pi i \qquad n \in \mathbb{N}$$

Obtaining the series expansion:

$$\sinh(z) = \frac{e^z - e^{-z}}{2} = \frac{1}{2}\left[\sum_{n=0}^{N} \frac{z^n}{n!} - \frac{(-z^n)}{n!}\right] = \frac{1}{2}\left[\sum_{n=0}^{N} \frac{z^n}{n!}[1-(-1)^n]\right] = \sum_{n=0}^{N} \frac{z^{2n+1}}{(2n+1)!}$$

Then

$$\frac{1}{z\sinh(z)} = \frac{1}{z}\left[\sum_{n=0}^{N}\frac{z^{2n+1}}{(2n+1)!}\right]^{-1} = \frac{1}{z}\left(\frac{1}{z} - \frac{z}{3!} - \frac{z^4}{5! - 3!3!} - \cdots\right) = \frac{1}{z^2} - \frac{1}{3!} - \cdots$$

Hence

$$F(s) = \frac{1}{s^2} - \frac{1}{s\sinh(s)} = \frac{1}{3!} - \frac{z^3}{5! - 3!3!} - \cdots$$

Thus, we can see that $F(s)$ has a removable singularity at $s_0 = 0$ with $\operatorname{Res}[F(s), 0] = 0$. Now, let $p(s) = e^{st}$ and $q(s) = [F(s)]^{-1}$, then

$$\begin{aligned}
q'(s) &= \frac{\mathrm{d}}{\mathrm{d}s}[F(s)]^{-1} = \frac{\mathrm{d}}{\mathrm{d}s}\frac{s^2\sinh(s)}{\sinh(s) - s} \\
&= \frac{2s\sinh(s) + s^2\cosh(s)}{\sinh(s) - s} - \frac{[s^2\sinh(s)][\cosh(s) - 1]}{[\sinh(s) - s]^2} \\
&= \frac{2s\sinh^2(s) - s^2\sinh(s) - s^3\cosh(s)}{[\sinh(s) - s]^2}
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{1}{q'(n\pi i)} &= \frac{[\sinh(s) - s]^2}{2s\sinh^2(s) - s^2\sinh(s) - s^3\cosh(s)} \\
&= \frac{[\sinh(n\pi i) - n\pi i]^2}{2(n\pi i)\sinh^2(n\pi i) - (n\pi i)^2\sinh(n\pi i) - (n\pi i)^3\cosh(n\pi i)} \\
&= \frac{(n\pi i)^2}{-(n\pi i)^3\cosh(n\pi i)} = \frac{-1}{(n\pi i)(-1)^n} = \frac{(-1)^n i}{n\pi} \\
\frac{1}{q'(-n\pi i)} &= -\frac{(-1)^n i}{n\pi}
\end{aligned}$$

Since

$$\sinh(n\pi i) = \sin(n\pi) = 0 \qquad\qquad \cosh(n\pi i) = \cos(n\pi) = (-1)^n$$

Also

$$p(n\pi i) = e^{n\pi i} \neq 0 \qquad\qquad q(n\pi i) = \frac{(n\pi i)^2\sinh(n\pi i)}{\sinh(n\pi i) - n\pi i} = 0$$

$$p(-n\pi i) = e^{-n\pi i} \neq 0 \qquad\qquad q(-n\pi i) = \frac{(-n\pi i)^2\sinh(-n\pi i)}{\sinh(-n\pi i) + n\pi i} = 0$$

Thus, we can see that $e^{st}F(s)$ has a simple pole at $s_n = n\pi i$ and $\overline{s_n} = -n\pi i$ ($n \in \mathbb{N}$), so by theorem 17.5.2:

$$\operatorname{Res}[e^{st}F(s), n\pi i] = \operatorname{Res}\left[\frac{p(s)}{q(s)}, n\pi i\right] = \frac{p(n\pi i)}{q'(n\pi i)} = e^{n\pi i t}\frac{(-1)^n i}{n\pi}$$

$$\operatorname{Res}[e^{st}F(s), n\pi i] = \operatorname{Res}\left[\frac{p(s)}{q(s)}, -n\pi i\right] = \frac{p(-n\pi i)}{q'(-n\pi i)} = -e^{-n\pi i t}\frac{(-1)^n i}{n\pi}$$

The inverse Laplace transform of $F(s)$ is then

$$\begin{aligned} f(t) &= \sum_{n=1}^{\infty} \operatorname{Res}[e^{st}F(s), s_n] & t>0 \\ &= \sum_{n=1}^{\infty} \left(\operatorname{Res}[e^{st}F(s), n\pi i] + \operatorname{Res}[e^{st}F(s), -n\pi i]\right) \\ &= \sum_{n=1}^{\infty} \left(e^{n\pi it}\frac{(-1)^n i}{n\pi} - e^{-n\pi it}\frac{(-1)^n i}{n\pi}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n i}{n\pi}(2i)\sin(n\pi t) \\ &= \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(n\pi t) \end{aligned}$$

□

17.7.10 Hilbert Transform

Definition 17.7.9: Hilbert Transform

Let $g(t)$ be a real-valued function. The Hilbert Transform:

$$\mathcal{H}[g(t)] = \hat{g}(t) = \mathrm{P.V.}\,\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(x)}{t-x}dx \qquad x, t \in \mathbb{R}$$

Definition 17.7.10: Hilbert Transform Pairs

The functions $g(t) = \mathcal{H}^{-1}[g(t)]$ and $\hat{g}(t) = \mathcal{H}[g(t)]$ in a Hilbert Transform.

Theorem 17.7.8: Inverse Hilbert Transform

The inverse Hilbert transform of function $\hat{g}(t)$:

$$\mathcal{H}^{-1}[\hat{g}(t)] = g(t) = -\frac{1}{\pi}\,\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{\hat{g}(x)}{t-x}dx$$

Proof: Let $f(z) = u(x,y) + iv(x,y)$ be a function analytic in the upper half-plane ($y = \mathrm{Im}(z) \geq 0$), and $x = t \in \mathbb{R}$. Consider the simple closed contour C composed of a line L_1 from point $-R$ to $\tau - \epsilon$ on the real axis, a semicircular contour C_ϵ in the upper half-plane with radius ϵ centred at t, a line L_2 from $\tau + \epsilon$ to R, and a semicircular contour centred at the origin with radius R.

From the Cauchy-Goursat Theorem (theorem 15.6.1):

$$\frac{1}{\pi}\int_{-R}^{\tau-\epsilon}\frac{f(x)}{t-x}dx + \frac{1}{\pi}\int_{C_\epsilon}\frac{f(z)}{t-z}dz + \frac{1}{\pi}\int_{\tau+\epsilon}^{R}\frac{f(x)}{t-x}dx + \frac{1}{\pi}\int_{C_R}\frac{f(z)}{t-z}dz = 0$$

Looking at C_ϵ, and using theorem 17.7.5 and theorem 17.5.2:

$$\lim_{\epsilon\to 0}\frac{1}{\pi}\int_{C_\epsilon}\frac{f(z)}{t-z}dz = -\frac{\pi i}{\pi}\operatorname{Res}\left[\frac{f(z)}{t-z}, t\right] = if(t)$$

Assuming the integral over C_R disappears, that is

$$\lim_{R\to\infty} \frac{1}{\pi} \int_{C_R} \frac{f(z)}{t-z} dz = 0$$

Taking the limit as $R \to \infty$ and $\epsilon \to 0$ of C:

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x,0)}{t-x} dx + i f(t) = 0 \quad \Longrightarrow \quad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(x,0)+iv(x,0)}{t-x} dx = -iu(t,0) + v(t,0)$$

Equating the real and imaginary parts:

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(x,0)}{t-x} dx = v(t,0) \qquad\qquad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{v(x,0)}{t-x} dx = -u(t,0)$$

Taking the limit as $R \to \infty$ Setting $g(x) = g(x,0)$ and $\hat{g}(x) = h(t,0)$, we obtain the Hilbert and Inverse Hilbert transforms:

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(x)}{t-x} dx = \hat{g}(t) \qquad\qquad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{g}(x)}{t-x} dx = -g(t)$$

□

Note: The inverse Hilbert Transform is valid if there exists an analytic function $f(z) = u(x,y) + iv(x,y)$ in the upper half-plane ($\text{Im}\{z\} \geq 0$) with $u(x,0) = g(x)$ and $v(x,0) = \hat{g}(x)$, or in the lower half-plane ($\text{Im}\{z\} \leq 0$) with $u(x,0) = g(x)$ and $v(x,0) = -\hat{g}(x)$ such that

$$\lim_{R\to\infty} \frac{1}{\pi} \int_{C_R} \frac{f(z)}{t-z} dz = 0$$

Observation. *The Hilbert transform is the convolution of the function $g(x)$ with $h(\tau) = 1/(\pi\tau)$. That is*

$$\mathcal{H}[g(t)] = g(t) * \frac{1}{\pi t} = \int_{-\infty}^{\infty} g(\tau) h(t-\tau) d\tau \qquad\qquad h(\tau) = \frac{1}{\pi t}$$

Definition 17.7.11: Analytic Signal

A complex-valued function which the imaginary part is the Hilbert transform of its real part. That is

$$f(t) = g(t) + i\hat{g}(t)$$

Example 17.7.17 (Hilbert Transform of a Constant is Zero)

$$\text{P.V.} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{c}{t-x} dx = 0 \qquad\qquad c, x, t \in \mathbb{R}$$

Proof:

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{c}{t-x} dx = \frac{c}{\pi} \int_{-\infty}^{\infty} \frac{1}{t-x} dx$$

Now, set up the integral so it does not become indefinite:

$$\lim_{R\to\infty} \left[\int_{1/R}^{R} \frac{1}{t-x} dx + \int_{-R}^{-1/R} \frac{1}{t-x} dx \right] = 0 \implies \frac{c}{\pi} \int_{-\infty}^{\infty} \frac{1}{t-x} dx = 0$$

□

Lemma 17.7.8.1: Fourier Transform of Hilbert Transform
Let $\mathcal{F}$ denote a Fourier transform, and sgn *denote the signum function (function that extracts a sign).*

$$\mathcal{F}[\mathcal{H}[g(t)]] = -i\,\mathrm{sgn}(\omega)\mathcal{F}[g(t)]$$

Proof:

$$\begin{aligned}\mathcal{F}[\hat{g}(t)] &= \mathcal{F}\left[g(t) * \frac{1}{\pi t}\right] = \mathcal{F}[g(t)] \cdot \mathcal{F}\left[\frac{1}{\pi t}\right] \\ &= 2\pi\mathcal{F}[g(t)]\left[-\frac{i}{2\pi}\,\mathrm{sgn}(\omega)\right] = -i\,\mathrm{sgn}(\omega)\mathcal{F}[g(t)]\end{aligned}$$

□

Another way of obtaining the inverse Hilbert transform is by multiplying both sides of the Fourier transform by $i\,\mathrm{sgn}(\omega)$:

$$\begin{aligned}&\mathrm{sgn}^2(\omega)\mathcal{F}[g(t)] = i\mathcal{F}[\hat{g}(t)]\,\mathrm{sgn}(\omega) \\ &\implies \mathcal{F}[g(t)] = i\mathcal{F}[\hat{g}(t)]\,\mathrm{sgn}(\omega) && \mathrm{sgn}^2(\omega) \in \{0, 1\} \\ &\implies g(t) = \mathcal{F}^{-1}[i\mathcal{F}[\hat{g}(t)]\,\mathrm{sgn}(\omega)] \\ &\implies g(t) = \frac{1}{2\pi}\mathcal{F}^{-1}[\mathcal{F}[g(t)]] * \mathcal{F}^{-1}[i\,\mathrm{sgn}(\omega)] && \mathcal{F}\left[\frac{1}{t}\right] = -\frac{i}{2}\,\mathrm{sgn}(\omega) \\ &\implies g(t) = \frac{1}{2\pi}g(t) * \frac{-2}{t} = \frac{-1}{\pi}\int_{-\infty}^{\infty}\frac{\mathcal{H}[g(t)]}{t-x}dx\end{aligned}$$

Lemma 17.7.8.2: Modulus of Fourier Transform of Hilbert Transform
The magnitude of the Hilbert transform is preserved in a Fourier transform.

$$|\mathcal{F}[\mathcal{H}[g(t)]]| = |\mathcal{F}[g(t)]|$$

Proof: From the Fourier transform, $|-i\,\mathrm{sgn}(\omega)| = 1$, therefore, the answer is obvious. □

Theorem 17.7.9: Fourier Transform of an Analytic Signal
Consider the Fourier transform of an analytic signal $g_a(t) = g(t) + i\hat{g}(t)$. Then

$$\mathcal{F}[g_a(t)] = \mathcal{F}[g(t)] + \mathrm{sgn}(\omega)\mathcal{F}[g(t)]$$

Proof:

$$\begin{aligned}\mathcal{F}[g_a(t)] &= \mathcal{F}[g(t)] + \mathcal{F}[i\hat{g}(t)] \\ &= \mathcal{F}[g(t)] + \mathrm{sgn}(\omega)\mathcal{F}[g(\omega)] && \mathcal{F}[\hat{g}(\omega)] = -i\,\mathrm{sgn}(\omega)\mathcal{F}[g(t)]\end{aligned}$$

□

It is clear that:

$$\mathcal{F}[g_a(t)] = \begin{cases} 2\mathcal{F}[g(t)] & \omega > 0 \\ 0 & \omega < 0 \end{cases}$$

Corollary 17.7.9.1:

$$g_a(t) = g(t) + i\hat{g}(t) = \int_0^\infty 2\mathcal{F}[g(t)]e^{i\omega t}d\omega$$

Causality

Definition 17.7.12: Transfer/System Function
Consider the function:

$$Y(\omega) = G(\omega)X(\omega)$$

$G(\omega)$ is the transfer function (aka. excitation). $X(\omega)$ is the excitation. $Y(\omega)$ is the output.

Definition 17.7.13: Green's Function / Impulse Response
Let $G(\omega)$ be a transfer function. The impulse response of the system is the inverse Fourier transform of the transfer function:

$$g(t) = \int_{-\infty}^\infty G(\omega)e^{i\omega t}d\omega$$

Definition 17.7.14: Causal System
A system with impulse response $\forall t < 0[g(t) = 0]$

Theorem 17.7.10: Hilbert Transform and Causal Systems
Let $G(\omega)$ be a transfer function and $G_e(\omega)$ be an even transfer function. A transfer function of a causal system must be of the form:

$$G(\omega) = G_e(\omega) - iG_e(\omega) * \frac{1}{\pi\omega} = G_e(\omega) - \frac{i}{\pi}\int_{-\infty}^\infty \frac{G_e(\omega')}{\omega - \omega'}d\omega$$

Proof: Any real-valued function g is composed of a even g_e and odd g_o part.

$$g(t) = g_e(t) + g_o(t)$$

Then

$$g(t) = g_e(t) + g_o(t) \qquad\qquad g_o(t) = \text{sgn}(t)g_e(t)$$

Applying the Fourier transform:

$$\begin{aligned} \mathcal{F}[g(t)] &= \mathcal{F}[g_e(t)] - i\mathcal{F}[g_e(t)] * \frac{1}{\pi i} \qquad\qquad \mathcal{F}[\text{sgn}(t)] = -\frac{1}{\pi\omega} \\ &= G_e(\omega) - \frac{i}{\pi}\int_{-\infty}^\infty \frac{G_e(\omega')}{\omega - \omega'}d\omega \end{aligned}$$

$\square$

17.7.11 Gamma Function

Definition 17.7.15: Uniform Convergence of Integrals

We say the integral

$$\int_a^\infty f(s,z)ds$$

converges uniformly to $F(z)$ in a closed region R, if

$$\forall z \in R, \forall \epsilon > 0, \exists \tau \in \mathbb{R}\left[b \geq \tau \implies \left|F(z) - \int_a^b f(t,z)dz\right| < \epsilon\right]$$

*(τ is independent of z, as in the requirement for uniform convergence (**??**))*

Theorem 17.7.11: Analyticity of Function Defined by Integration to Infinity

Let $f(z,t)$ be continuous for $t \in [a,b]$, for z interior a simple closed contour C.

$$\int_a^\infty f(t,z)dz \text{ converges uniformly to } F(z) \text{ for } z \text{ interior to } C$$
$$\implies F(z) \text{ analytic in domain with boundary } C$$

also in this domain

$$F^{(n)}(z) = \int_a^\infty \frac{\partial^n}{\partial z^n} f(t,z)dt \qquad n \in \mathbb{N}$$

Theorem 17.7.12: M-Test for Integration to Infinity

Let $f(z,t)$ be continuous for $t \geq a$ for all $z \in R$, where R is a closed region, and $M(t)$ be a positive function of t be independent of z. Suppose $\forall z \in R, \forall t \geq a[|f(z,t)| \leq M(t)]$:

$$\forall z \in R, \forall t \geq a[|f(z,t)| \leq M(t)] \wedge \int_a^\infty M(t)dt \text{ Converges}$$
$$\implies \forall z \in R\left[\int_a^\infty f(t,z)dt \text{ Converges uniformly}\right]$$

That is, if $|f(z,t)|$ is bounded by a function $M(t)$ and the integral of $M(t)$ converges, then the integral of $f(t,z)$ in the same interval converges uniformly.

Definition 17.7.16: Gamma Function

The Gamma function is the analytic function:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \qquad t \in \mathbb{R}$$

The gamma function is continuous and converges for all $x \in \mathbb{R}_{>0}$. If $x \in (\nvdash, \nVdash)$, then it is discontinuous at $t = 0$.

Theorem 17.7.13: Gamma Function and Factorials

$$\Gamma(n+1) = n! \qquad n \in \mathbb{N}$$

Proof: Consider $x+1$, then applying integration by parts:

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt \implies \Gamma(x+1) = -e^{-t}t^x\Big|_{t=0}^{\infty} + x\int_0^\infty t^{x-1}e^{-t}dt$$

We know that

$$e^{-t}t^x\Big|_{t=0} = 0$$

$$e^{-t}t^x\Big|_{\infty} = \lim_{R\to\infty} e^{-R}R^x = \lim_{R\to\infty} \frac{R^x}{\sum_{n=0}^{\infty} R^n/n!} = \lim_{R\to\infty} \left[\sum_{n=0}^{\infty} \frac{R^{n-x}}{n!}\right]^{-1} = 0$$

Hence, we are left with

$$\Gamma(x+1) = \int_0^\infty t^{x-1}e^{-t}dt = x\Gamma(x) \qquad x > 0$$

Now

$$\Gamma(1) = \int_0^\infty e^{-t}dt = -e^{-t}\Big|_0^\infty = 1$$

Using $\Gamma(1) = 1$ as the base case, and induction on $\Gamma(x+1) = x\Gamma(x)$:

$$\Gamma(n+1) = n! \qquad n \in \mathbb{N}$$

□

Note: Since $\Gamma(1) = 1$, we can see why $\Gamma(1) = 0! := 1$.

Definition 17.7.17: Gamma Function (Complex)

Consider $z \in \mathbb{C}$:

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt = \int_0^\infty e^{(z-1)\,\mathrm{Log}(z)}e^{-t}dt \qquad t \in \mathbb{C}$$

Note the principal part of the logarithm.

Consistent with the real definition, the complex Gamma function is analytic for $t > 0$ and $\mathrm{Re}\{z\} > 0$. By differentiating under the integral sign:

$$\Gamma'(z) = \frac{\mathrm{d}}{\mathrm{d}z}\int_0^\infty t^{z-1}e^{-t}dt = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}z}e^{(z-1)\,\mathrm{Log}(z)}e^{-t}dt = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{Log}(t)dt$$

Hence, $\Gamma(z)$ and its derivatives are continuous for $\mathrm{Re}\{z\} > 0$.

Following the same procedure in the proof of theorem 17.7.13, we get:

$$\Gamma(z+1) = z\Gamma(z)$$

Reordering allows us analytically extend the function into negative values of $\mathrm{Re}\{z\}$:

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} \implies \Gamma(z) = \frac{\Gamma(z+2)}{(z+1)z}$$

Continuing with induction:

$$\Gamma(z) = \frac{\Gamma(z+n)}{\prod_{m=1}^{n}(z+n-m)} \qquad n \in \mathbb{N}$$

Hence, we can see that $\Gamma(z)$ has simple poles at $z \in \{0, -1, -2, \ldots\}$.

Theorem 17.7.14:

Let f be a function that is real on the real axis, then it displays opposite signs on the real axis on either side of the simple pole.

Proof: Let f be a function with a simple pole on $x_0 \in \mathbb{R}$. Then from *theorem* 17.3.1:

$$f(x) = \frac{\phi(x)}{x - x_0}$$

Assume that $f(x)$ is continuous on both sides of x_0 and $\phi(x)$ is continuous. Consider $f(x_0 + c)$ and $f(x_0 - c)$:

$$f(x_0 + c) = \frac{\phi(x_0 + c)}{c} \implies \phi(x_0 + c) = cf(x_0 + c)$$
$$f(x_0 - c) = \frac{\phi(x_0 - c)}{-c} \implies \phi(x_0 - c) = -cf(x_0 - c)$$

Since $\phi(x)$ is nonzero at x_0 and is continuous, we can choose c such that $\phi(x_0 + c)$ and $\phi(x_0 - c)$ have the same sign. Therefore

$$f(x_0 + c) = -f(x_0 - c)$$

$\square$

From theorem 17.7.14, we can see that $\Gamma(z)$ alternates signs across its simple poles.

Theorem 17.7.15: Reflection Formula

$$\Gamma(z)\Gamma(z-1) = \frac{\pi}{\sin(\pi z)}$$

Proof: Consider the Gamma function with the change of variable $t = y^2$, $dt = 2ydy$:

$$\Gamma(z) = 2\int_0^\infty y^{2z-1} e^{-y^2} dy$$

Then letting $x = y$:

$$\Gamma(1-z) = 2\int_0^\infty x^{1-2z} e^{-x^2} dx$$

Multiplying the two equations together:

$$\Gamma(z)\Gamma(1-z) = 4\int_0^\infty \int_0^\infty y^{2z-1}x^{1-2z}e^{-x^2}e^{-y^2}\,dxdy$$

Switching to polar coordinates:

$$\begin{aligned}\Gamma(z)\Gamma(1-z) &= 4\int_0^{\pi/2}\int_0^\infty [r\sin(\theta)]^{2z-1}[r\cos(\theta)]^{1-2z}e^{-r^2\cos^2(\theta)}e^{-r^2\sin^2(\theta)}r\,drd\theta \\ &= 4\int_0^\infty e^{-r^2}r\,dr\int_0^{\pi/2}[\tan(\theta)]^{2z-1}d\theta\end{aligned}$$

Letting $u = -r^2$:

$$\int_0^\infty e^{-r^2}dr = -\frac{1}{2}\int_0^{-\infty} e^u du = \frac{1}{2}\,e^u\Big|_{-\infty}^{0} = \frac{1}{2}$$

Therefore

$$\Gamma(z)\Gamma(1-z) = 2\int_0^{\pi/2}[\tan(\theta)]^{2z-1}d\theta$$

Letting $x = \tan^2(\theta)$, so $dx = 2\tan(\theta)[1+\tan^2(\theta)]d\theta$:

$$\begin{aligned}\Gamma(z)\Gamma(1-z) &= 2\int_0^{\pi/2}[\tan(\theta)]^{2z-1}d\theta \\ &= 2\int_0^\infty x^z\frac{1}{\tan(\theta)}\cdot\frac{1}{2\tan(\theta)[1+\tan^2(\theta)]}dx \\ &= \int_0^\infty \frac{x^z}{x(1+x)}dx = \int_0^\infty \frac{x^{z-1}}{1+x}dx\end{aligned}$$

From example 17.7.5, we know that

$$\int_0^\infty \frac{x^{-a}}{x+1}dx = \frac{\pi}{\sin(a\pi)} \qquad\qquad z \in (0,1)$$

Thus

$$\Gamma(z)\Gamma(1-z) = \int_0^\infty \frac{x^{z-1}}{1+x}dx = \int_0^\infty \frac{x^{-(1-z)}}{x+1}dx = \frac{\pi}{\sin((1-z)\pi)} = \frac{\pi}{\sin(z\pi)}$$

□

This is useful for speeding up numerical calculations.

Lemma 17.7.15.1:

The Reflection Formula is not restricted to $z \in (0,1)$.

Proof: We know that the Reflection Formula has zeros at $z \in \mathbb{Z}$, which are the simple poles of either $\Gamma(z)$ or $\Gamma(1-z)$. If we arrange the reflection formula such that

$$F(z) = \sin(\pi z)\Gamma(z)\Gamma(1-z) - \pi = 0$$

It has to be satisfied for all $z \in (0,1)$, so it has removable singularities at $z \in \mathbb{Z}$, and can be made analytic everywhere using the integral definition of the Gamma Function.

Zeros on the other hand, $\forall z \in (0,1)[F(z) = 0]$, hence the zeros are not isolated, and $\forall z \in \mathbb{C}[F(z) = 0]$.

Thus, the Reflection Formula fails to be analytic only when $z \in \mathbb{Z}$. □

Lemma 17.7.15.2:

$\Gamma(z)$ has non zeros in the complex plane, and $1/\Gamma(z)$ is entire. .

Proof: Poles:

$$\Gamma(z) \text{ at } z \in \mathbb{Z} \setminus \mathbb{N}$$
$$\Gamma(1-z) \text{ at } z \in \mathbb{N}$$

$F(z) == \sin(\pi z)\Gamma(z)\Gamma(1-z) - \pi = 0$ tells us that if

$$\begin{aligned} \exists z_0 \in \mathbb{C}[\Gamma(z_0) = 0] &\implies \Gamma(1-z_0) \text{ must be pole to satisfy equation} \\ &\implies z_0 \in \mathbb{N} \end{aligned}$$

But

$$\Gamma(z_0) = (z_0 - 1)! \qquad\qquad z_0 \in \mathbb{N}$$

Contradiction! Thus $\Gamma(z)$ has no zeros in the complex plane and $1/\Gamma(z)$ is entire. □

Computing $\Gamma(z)$ for odd half integers of z:
Consider $z = 1/2$, then

$$\Gamma(1/2)\Gamma(1-1/2) = \Gamma^2(1/2) = \frac{\pi}{\sin(\pi/2)} = \pi \implies \Gamma(1/2) = \pm\sqrt{\pi}$$

Integral definition of Gamma Function is positive for $x > 0$:

$$\Gamma(1/2) = \sqrt{\pi}$$

Then using

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$

We can quickly compute $\Gamma(z)$ for $z \in \{\pm 1/2, \pm 3/2, \pm 5/2, \ldots\}$

Chapter 18

Mapping by Elementary Functions

18.1 Linear Transformations

Definition 18.1.1: Linear Transformation

Let $A, B \in \mathbb{C}$:

$$w = Az + B$$

We can see that $A = |A|e^{i\arg(A)}$ which scales by A and rotates by $\arg(A)$, while B is a translation. The linear transformation scales, rotates, and translates the elements in the complex plane.

18.2 Transformation $w = 1/z$

$f(z) = 1/z$ is a 1-1 mapping from $\mathbb{C} \setminus \{0\} \mapsto \mathbb{C} \setminus \{0\}$. If we consider:

$$|z|^2 = z\bar{z} \implies \frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

We can see that $1/z$ maps the points exterior to the unit circle onto nonzero point within the unit circle (from $1/|z|$), followed by an reflection on the real axis (from $\bar{z}$).

To make the transformation continuous on the extended complex plane, we can define:

$$T(z) = \frac{1}{z} \qquad\qquad z \neq 0$$

Then

$$\lim_{z\to 0} \frac{1}{T(z)} = \lim_{z\to 0} z = 0 \implies \lim_{z\to 0} T(z) = \infty$$
$$\lim_{z\to 0} \frac{1}{T(1/z)} = \lim_{z\to 0} z = 0 \implies \lim_{z\to 0} T(z) = 0$$

To make it continuous, we can define:

$$T(0) = \infty \qquad T(\infty) = 0 \qquad T(z) = \frac{1}{z}$$

18.2.1 Mapping by $1/z$

Letting $w = u + iv$ and $z = x + iy = 1/w$:

$$w = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2} \implies \left[u = \frac{x}{x^2+y^2}\right] \wedge \left[v = \frac{-y}{x^2+y^2}\right]$$
$$z = \frac{1}{w} = \frac{\bar{w}}{w\bar{w}} = \frac{\bar{w}}{|w|^2} \implies \left[x = \frac{u}{u^2+v^2}\right] \wedge \left[y = \frac{-v}{u^2+v^2}\right]$$

We can see that $w = 1/z$ transforms circles and lines into circles and lines by letting the equations represents an arbitrary circle or line:

$$A(x^2+y^2) + Bx + Cy + D = 0 \qquad B^2 + C^2 > 4AD \qquad A, B, C, D \in \mathbb{R}$$

We need $B^2 + C^2 > 4AD$ since completing the squares tells us:

$$\left(x + \frac{B}{2A}\right)^2 + \left(y + \frac{C}{2A}\right)^2 = \left(\frac{\sqrt{B^2 + C^2 - 4AD}}{2A}\right)^2$$

After substituting u and v, and rearranging:

$$A(x^2+y^2) + Bx + Cy + D = 0 \qquad B^2 + C^2 > 4AD \qquad A, B, C, D \in \mathbb{R}$$
$$D(u^2+v^2) + Bu - Cv + A = 0$$

It is clear that transforming from z to w-plane:

1. Circle not passing through origin $(A \neq 0,\ D \neq 0) \mapsto$ Circle not passing through origin
2. Circle through origin $(A \neq 0,\ D = 0) \mapsto$ Line not passing through origin
3. Line not passing through origin $(A = 0,\ D \neq 0) \mapsto$ Circle through origin
4. Line through origin $(A = 0,\ D = 0) \mapsto$ Line through origin

See Desmos graph: www.desmos.com/calculator/j6xj51risy

18.3 Linear Fractional Transformations

Definition 18.3.1: Linear Fractional / Bilinear / Möbius Transform

The transform

$$w = \frac{az+b}{cz+d} \qquad\qquad ad-bc \neq 0$$

Alternate form

$$Azw + Bz + Cw + D = 0 \qquad\qquad AD - BC \neq 0$$

We can see that:

$$w = \frac{az+b}{cz+d} \iff w = \frac{a}{c} + \frac{bc-ad}{c} \cdot \frac{1}{cz+d} \qquad\qquad ad-bc \neq 0$$

Thus the Linear Fractional Transformation is composed of the transformations:

$$Z = cz+d \qquad\qquad W = \frac{1}{Z} \qquad\qquad w = \frac{a}{c} + \frac{bc-ad}{c} \cdot W$$

So any linear fractional transformation transforms circles and lines into circles and lines.

If we solve for z:

$$z = \frac{-dw+b}{cw-a} \qquad\qquad ad-bc \neq 0$$

We can extend the transformation to the extended complex plane by defining:

$$\begin{aligned}
T(z) &= \frac{az+d}{cz+d} & & & ad-bc \neq 0 \\
T(\infty) &= \infty & & & c = 0 \\
T(\infty) &= \frac{a}{c} & T\left(-\frac{d}{c}\right) &= \infty & c \neq 0
\end{aligned}$$

This then becomes a 1-1 mapping from the extended z-plane to the extended w-plane. Hence there is an inverse transformation:

$$T^{-1}(w) = z \iff T(z) = w$$

Hence

$$\begin{aligned}
T^{-1}(w) &= \frac{dw+b}{cw-a} & & & ad-bc \neq 0 \\
T^{-1}(\infty) &= \infty & & & c = 0 \\
T^{-1}\left(\frac{a}{c}\right) &= \infty & T^{-1}(\infty) &= -\frac{d}{c} & c \neq 0
\end{aligned}$$

Example 18.3.1 *Suppose we need to find the transformation corresponding to the mapping:*

$$z_1 = 1 \mapsto w_1 = i \qquad\qquad z_2 = 0 \mapsto w_2 = \infty \qquad\qquad z_3 = -1 \mapsto w_3 = 1$$

Therefore

$$z_2 = 0 \mapsto w_2 = \infty \implies (c \neq 0) \wedge (d = 0) \implies w = \frac{az+b}{cz} \qquad\qquad bc \neq 0$$

Since $z_1 = 1 \mapsto w_1 = i$ *and* $z_3 = -1 \mapsto w_3 = 1$*:*

$$[ic = a + b] \wedge [-c = -a + b] \implies [2a = (1+i)c] \wedge [2b = (i-1)c]$$

Subbing into $w = (az+b)/(cz)$*:*

$$w = \frac{(i+1)z + (i-1)}{2z}$$

18.3.1 Implicit Form

Definition 18.3.2: Implicit Form of Linear Fractional Transformation

A linear fractional transformation that maps $z_1 \mapsto w_1$*,* $z_2 \mapsto w_2$*, and* $z_3 \mapsto w_3$*:*

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

We can verify the mapping by writing

$$(z-w_3)(w-w_1)(w_2-w_3) = (z-z_1)(w-w_3)(z_2-z_3)(w_2-w_1)$$

It follows that $z = z_1 \implies w = w_1$ and $z = z_3 \implies w = w_3$. If $z = z_2$, then

$$(w-w_1)(w_2-w_3) = (w-w_3)(w_2-w_1) \implies w = w_2$$

Since it is a linear fractional transformation, we can write it in the form:

$$Azw + Bz + Cw + D = 0 \qquad\qquad AD - BC \neq 0$$

Example 18.3.2 *Consider again finding the transformation corresponding to the mapping:*

$$z_1 = 1 \mapsto w_1 = i \qquad z_2 = 0 \mapsto w_2 = \infty \qquad z_3 = -1 \mapsto w_3 = 1$$

Substituting into the implicit form of the linear fractional transformation:

$$\frac{w-w_1}{w-w_3} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \implies \frac{w-i}{w-1} = \frac{(z-1)(0+1)}{(z+1)(0-1)} \implies w = \frac{(i-1)z+(i-1)}{2z}$$

Example 18.3.3 *Show every linear fractional transformation has at most 2 fixed points, unless it is the identity transform.*

Proof: A point of transformation $T(z)$ is fixed if $T(z_0) = z_0$, then

$$z = \frac{az+b}{cz+d} \implies cz^2 + dz = az + b \implies cz^2 + (d-a)z - b = 0$$

$$\implies z = \frac{a - d \pm \sqrt{(d-a)^2 + 4cb}}{2c}$$

Thus $T(z)$ has at most 2 fixed points. $\square$

Example 18.3.4 *Linear fractional transformation that maps 3 distinct points in the extended z-plane to the extended w-plane are unique.*

Proof: Let T and S be two linear fractional transformations.

$$T(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \qquad a_1 d_1 - b_1 c_1 \neq 0$$
$$S^{-1}(w) = \frac{a_2 w + b_2}{c_2 w + d_2} \qquad a_2 d_2 - b_2 c_2 \neq 0$$

Then

$$\begin{aligned} S^{-1}[T(z)] &= \left[a_2 \frac{a_1 z + b_1}{c_1 z + d_1} + b_2\right]\left[c_2 \frac{a_1 z + b_1}{c_1 z + d_1} + d_2\right]^{-1} \\ &= \left[\frac{a_1 a_2 z + a_2 b_1 + b_2 c_1 z + b_2 d_1}{c_1 z + d_1}\right]\left[\frac{c_1 z + d_1}{a_1 c_2 z + b_1 c_2 + c_1 d_2 z + d_1 d_2}\right] \\ &= \frac{(a_1 a_2 + b_2 c_1)z + (a_2 b_1 + b_2 d_1)}{(a_1 c_2 + c_1 d_2)z + (b_1 c_2 + d_1 d_2)} = \frac{a_3 z + b_3}{c_3 z + d_3} \end{aligned}$$

Where

$$\begin{aligned} &a_3 d_3 - b_3 c_3 \\ &= (a_1 a_2 + b_2 c_1)(b_1 c_2 + d_1 d_2) - (a_2 b_1 + b_2 d_1)(a_1 c_2 + c_1 d_2) \\ &= a_1 a_2 b_1 c_2 + a_1 a_2 d_1 d_2 + b_1 b_2 c_1 c_2 + b_2 c_1 d_1 d_2 - a_1 a_2 b_1 c_2 - a_2 b_1 c_1 d_2 - a_1 b_2 c_2 d_1 - b_2 c_1 d_1 d_2 \\ &= a_1 a_2 d_1 d_2 + b_1 b_2 c_1 c_2 - a_2 b_1 c_1 d_2 - a_1 b_2 c_2 d_1 \\ &= (a_1 d_1 - b_1 c_1)(a_2 d_2 - b_2 c_2) \neq 0 \end{aligned}$$

Suppose that for the three unique points z_k, $k = 1, 2, 3$:

$$S^{-1}[T(z_k)] = \frac{a_3 z_k + b_3}{c_3 z_k + d_3} = z_k$$

We know from previous example that any linear fractional transformation has at most two fixed points, unless it is the identity transformation. Since we have three fixed points, this implies $\forall z$, $S^{-1}[T(z)] = z$, which implies $T(z) = S(z)$, so transformations are unique. □

Example 18.3.5 *Prove a linear fractional transformation is an automorphism of the real axis $\iff$ coefficients in the transformation are all real, except possibly for a common complex factor.*

Proof:

$$T(x) = \frac{ax + b}{cx + d} = u \in \mathbb{R}$$

This implies

$$ax + b = cxu + du \in \mathbb{R}$$

It is clear that $a, b, c, d \in \mathbb{R}$, unless they share a common complex factor. □

Example 18.3.6 *Show for*

$$T(z) = \frac{az+b}{cz+d} \qquad ad - bc \neq 0$$

which is any linear fractional transformation other than the identity.

$$T^{-1} = T \iff d = -a$$

Proof:

$$T(z) = \frac{az+b}{cz+d} = w \implies az + b = czw + dw \implies az - czw = dw - b$$
$$\implies z(a - cw) = dw - b \implies z = \frac{dw - b}{-cw + a}$$

Therefore

$$\begin{aligned}
T = T^{-1} &\implies \frac{az+b}{cz+d} = \frac{dz-b}{-cz+a} \\
&\implies -acz^2 + a^2 z - bcz + ab = cdz^2 + d^2 z - bcz - bd \\
&\implies c(a+d)z^2 + (d^2 - a^2)z - b(a+d) = 0 \\
&\implies (a+d)[cz^2 + (d-a)z - b] = 0 \\
&\implies d = -a
\end{aligned}$$

□

18.4 Mappings of the Upper Half Plane

Theorem 18.4.1:
A linear fractional transformation that maps $\mathrm{Im}\{z\} > 0 \mapsto |w| < 1$ *and* $\mathrm{Im}\{z\} = 0 \mapsto |z| = 1$ *is of the form:*

$$w = e^{i\alpha} \cdot \frac{z - z_0}{z - \overline{z_0}} \qquad \mathrm{Im}\{z_0\} > 0, \alpha \in \mathbb{R}$$

Proof: **Mapping $\implies$ equation:**

Consider mapping $z = 0, 1, \infty \mapsto |w| = 1$

$$w = \frac{az+b}{cz+d} \qquad ad - bc \neq 0$$

So

$$\begin{aligned}
[z = 0 \mapsto |w| = 1 \implies |b/d| = 0] &\implies |b| = |d| = 0 \\
[z = \infty \mapsto w \in \mathbb{C} \impliedby c \neq 0] &\implies w = \frac{a}{c} \\
&\implies \left[|w| = 1 \implies \left|\frac{a}{c}\right| = 1\right] \\
&\implies |a| = |c| \neq 0
\end{aligned}$$

Then, we can write:

$$w = \frac{a}{c} \cdot \frac{z + (b/a)}{z + (d/c)} = e^{i\alpha} \cdot \frac{z - z_0}{z - z_1} \qquad \left|\frac{a}{c}\right| = 1 \implies \left|\frac{b}{a}\right| = \left|\frac{d}{c}\right| \neq 0 \implies |z_1| = |z_0| \neq 0$$

Imposing $z = 1 \mapsto |w| = 1$:

$$|1 - z_1| = |1 - z_0| \implies (1 - z_1)(1 - \overline{z_1}) = (1 - z_0)(1 - \overline{z_0})$$

$|z_1| = |z_0| \implies z_1\overline{z_1} = z_0\overline{z_0}$:

$$\begin{aligned} z_1 + \overline{z_1} = z_0 + \overline{z_0} &\implies \mathrm{Re}\{z_1\} = \mathrm{Re}\{z_0\} \\ &\implies [z_1 = z_0] \vee [z_1 = \overline{z_0}] \end{aligned}$$

Since $z_1 = z_0 \implies w = e^{i\alpha}$, we choose

$$z_1 = \overline{z_0} \implies w = e^{i\alpha} \cdot \frac{z - z_0}{z - \overline{z_0}}$$

Equation $\implies$ mapping:

$$w = e^{i\alpha}\frac{z - z_0}{z - \overline{z_0}} \implies |w| = \frac{|z - z_0|}{|z - \overline{z_0}|}$$

If $\mathrm{Im}\{z\} > 0$, then z and z_0 lie on the same side of the axis. Then $|z - z_0| < |z - \overline{z_0}|$, so $|w| < 1$. If $\mathrm{Im}\{z\} < 0$, then z and z_0 lie on the opposite side of the axis. Then $|z - z_0| > |z - \overline{z_0}|$, so $|w| > 1$. If $z \in \mathbb{R}$. Then $|z - z_0| = |z - \overline{z_0}|$, so $|w| = 1$. The statement then follows. □

Example 18.4.1 *Consider*

$$w = \frac{z - 1}{z + 1}$$

Then letting $z = x + iy$ and $w = u + iv$:

$$w = \frac{z - 1}{z + 1} \cdot \frac{\bar{z} + 1}{\bar{z} - 1} = \frac{z\bar{z} + z - \bar{z} - 1}{z\bar{z} - z + \bar{z} - 1} = u + iv$$

Not that $z \in \mathbb{R} \implies w \in \diagdown$, so $y = 0 \mapsto v = 0$. For any point w in the finite w-plane:

$$v = \mathrm{Im}\{w\} = \mathrm{Im}\left\{\frac{(z - 1)(\bar{z} + 1)}{(z + 1)(\overline{z + 1})}\right\} = \frac{2y}{|z + 1|^2} \qquad\qquad z \neq -1$$

We can see y and v have the same sign, so $\mathrm{Im}\{z\} > 0 \mapsto \mathrm{Im}\{w\} > 0$, and $\mathrm{Im}\{z\} < 0 \mapsto \mathrm{Im}\{w\} < 0$. Since a linear fractional transformation is 1-1, the mapping is 1-1.

Example 18.4.2 *Consider*

$$w = \text{Log}\left(\frac{z-1}{z+1}\right)$$

This is composed of the two transformations:

$$Z = \frac{z-1}{z+1} \qquad\qquad w = \text{Log}(Z)$$

The first mapping follows from the previous example. The second mapping:

$$\text{Log}(Z) = \ln(R) + i\Theta \qquad\qquad R > 0,\ -\pi < \Theta < \pi$$

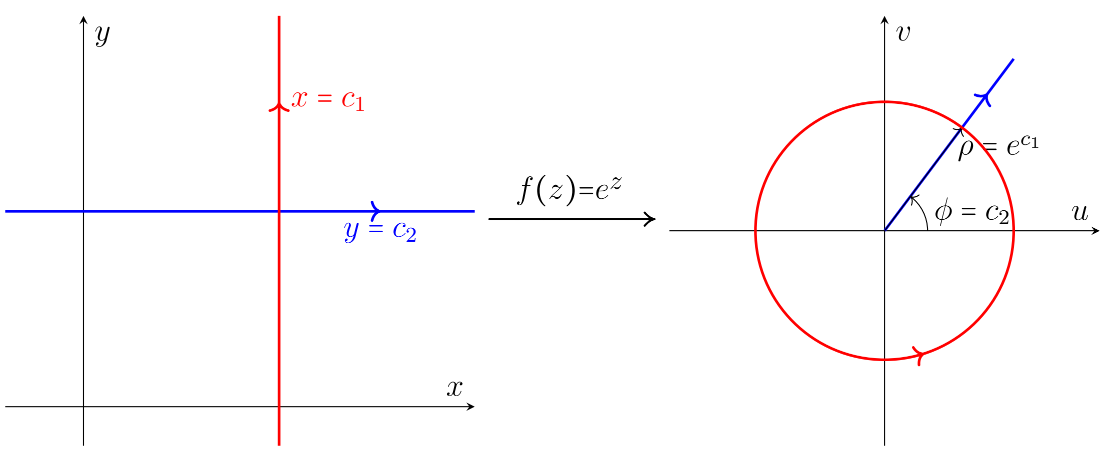
So letting $Z = X + iY$. The second mapping maps $Y > 0 \mapsto v \in [0, \pi]$ and $X, Y \mapsto \ln(R)$, and $Y < 0 \mapsto v \in [-\pi, 0]$ and $X, Y \mapsto \ln(R)$. (The upper and lower half plane is mapped to values between $-\pi i$ and πi, and real values to $\ln|z|$).

18.5 Mappings by the Exponential Function

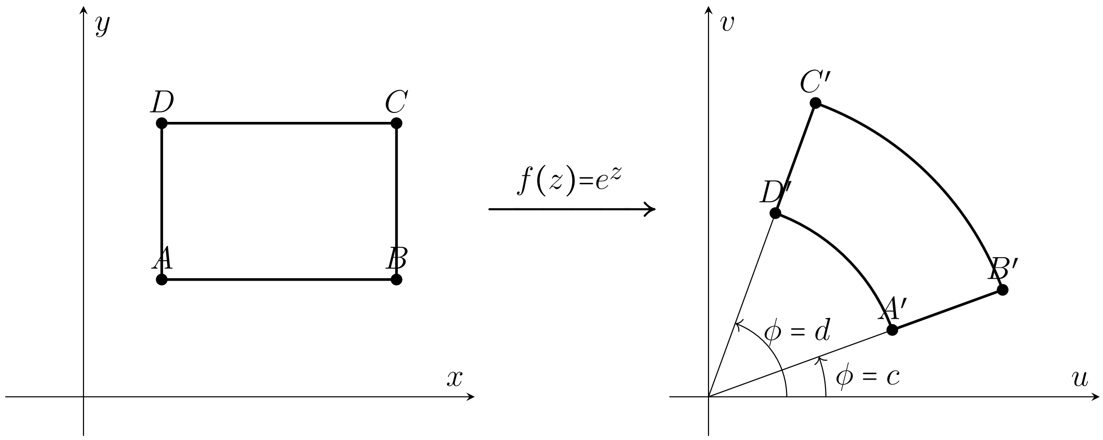
We can see

$$w = e^z = e^x e^{iy} = \rho e^{i\phi}$$

Thus $\text{Re}\{z\}$ gets mapped to rays emanating from the origin, while $\text{Im}\{z\}$ determines the angle of said ray.



Example 18.5.1 *Consider the mapping of $x \in [a, b]$ and $y \in [c, d]$ onto the region $\rho \in [e^a, e^b]$ and $\phi \in [c, d]$. Then the mapping is as follows:*



18.6 Mapping by $w = \sin(z)$

Mappings of Vertical Line Segments

Recall: $\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$, so the transformation $w = \sin(z)$ for $z = x + iy$:

$$w = u + iv = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

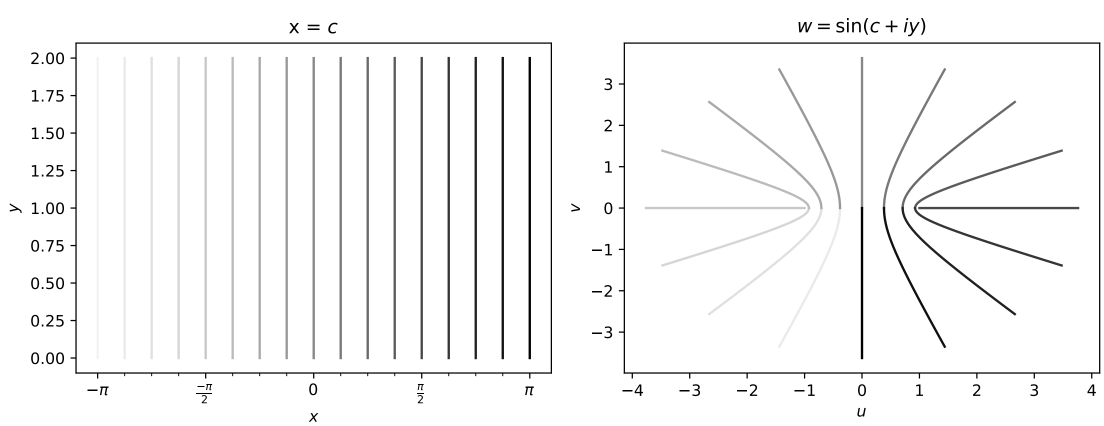
Looking at vertical line segments, we set $x = c_1$ where $c_1 \in \mathbb{R}$.

$$[u = \sin(c_1)\cosh(y)] \wedge [v = \cos(c_1)\sinh(y)] \implies \frac{u^2}{\sin^2(c_1)} - \frac{v^2}{\cos^2(c_1)} = 1$$

This gives a hyperbola with foci at points

$$w = \pm\sqrt{\sin^2(c_1) + \cos^2(c_1)} = \pm 1$$

If we look at the y and v-axis, $(0, y) \mapsto (0, \sinh(y))$, so the mapping is 1-1. As y increases, so does $\sinh(y)$. If we include negative values of y, then the mapping is no longer 1-1.



Mappings of the Horizontal Line Segments

As before, we have:

$$u = \sin(x)\cosh(c_2) \qquad\qquad v = \cos(x)\sinh(c_2)$$

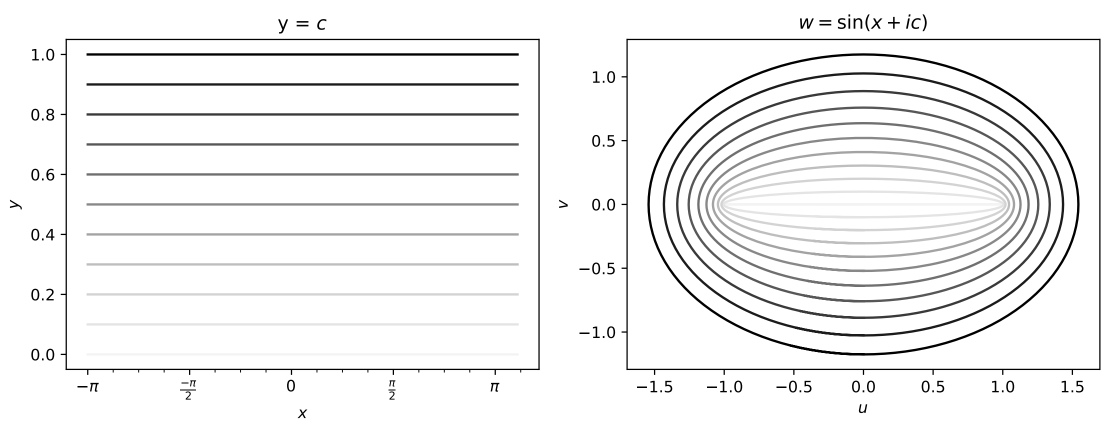
Which gives us the ellipse

$$\frac{u^2}{\cosh^2(c_2)} + \frac{v^2}{\sinh^2(c_2)} = 1$$

with foci

$$w = \pm\sqrt{\cosh^2(c_2) - \sinh^2(c_2)} = \pm 1$$

Since $u = \sin(x)\cosh(c_2)$ and $v = \cos(x)\sinh(c_2)$ we can see that as x increases, the ellipse goes around in a clockwise direction. Since Sine is a cyclic function, we only have 1-1 mappings for constant values of y in $x = 2\pi$ intervals. Likewise, these 1-1 mappings only hold true for $y > 0$ or $y < 0$.

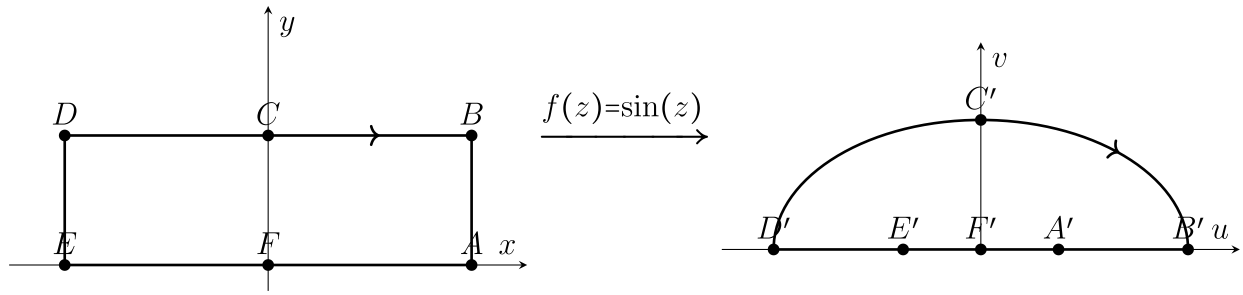


Summary

Basically, the mapping $w = \sin(z)$ maps vertical line segments to hyperbolas and horizontal line segments to ellipses.

Example 18.6.1 *The following rectangular region is mapped onto the semi-elliptical region in a 1-1 manner. We have the mapping of the points:*

$$A = (\pi/2, 0) \mapsto A' = (1, 0) \qquad B = (\pi/2, bi) \mapsto B' = (\cosh(b), 0)$$
$$C = (0, bi) \mapsto C' = (0, \sinh(b)) \qquad D = (-\pi/2, bi) \mapsto D' = (-\cosh(b), 0)$$
$$E = (-\pi/2, 0) \mapsto E' = (-1, 0) \qquad F = (0, 0) \mapsto F' = (0, 0)$$



18.6.1 Related Mappings

Example 18.6.2 *Consider*

$$w = \cos(z) = \sin\left(z + \frac{\pi}{2}\right)$$

It is clear the cosine is just the sine function translated to the right by $\pi/2$. It is composed of the transformations:

$$Z = z + \frac{\pi}{2} \qquad\qquad w = \sin(Z)$$

Example 18.6.3 *Consider*

$$w = \sinh(z) = -i\sin(iz)$$

Which is the composite of transformations

$$Z = iz \qquad\qquad W = \sin(Z) \qquad\qquad w = -iW$$

Hence, it is the sine transformation with rotations by $\pi/2$ and $-\pi/2$.

Example 18.6.4 *Consider*

$$w = \cosh(z) = \cos(iz)$$

Which is the a rotation by $\pi/2$ followed by a cosine transform. Using

$$\sin\left(z + \frac{\pi}{2}\right) = \cos(z) \qquad\qquad \cos(iz) = \cosh(z)$$

We can write $w = \cosh(z)$:

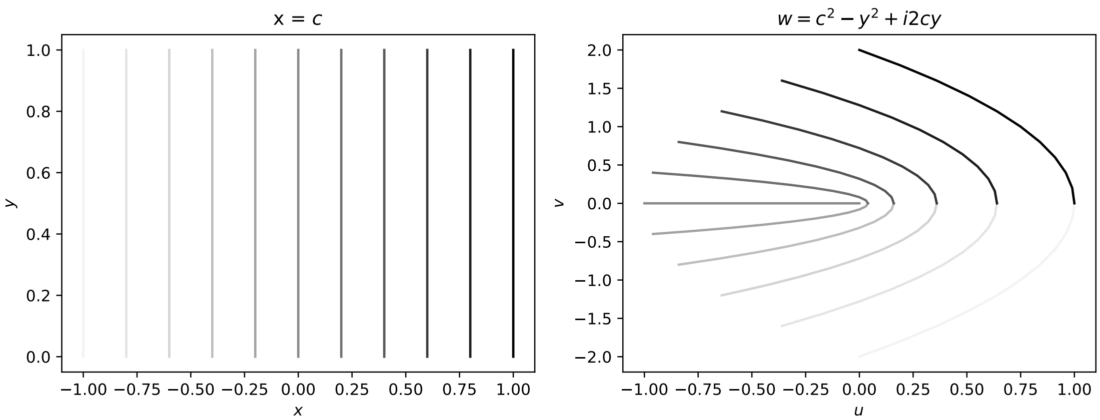
$$w = \sin(Z) \qquad\qquad Z = iz + \frac{\pi}{2}$$

18.7 Mappings by z^2

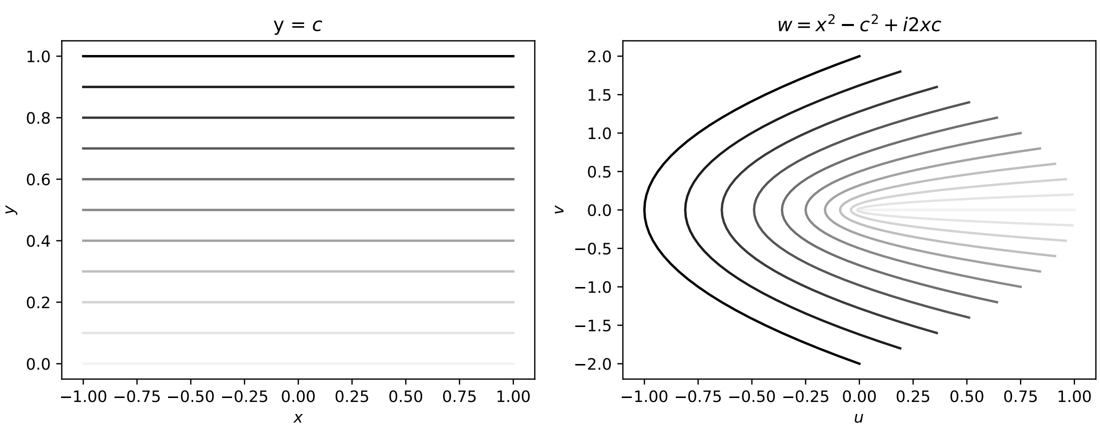
We can write the transformation:

$$w = z^2 = x^2 - y^2 + 2xy = u + iv \qquad\qquad u = x^2 - y^2 \qquad\qquad v = 2xy$$

We can see that if x is constant, as y increases, the transformation curves to the left due to the decreasing value of u. It is clear that if we include negative values of y, then the mapping is not 1-1.



If y is held constant, the increasing value of x curves the transformation to the right. More accurately, while x is negative, it moves left until $x = 0$, then it curves to the left due to the parabolic nature of $u = x^2 - y^2$. The mapping is not 1-1, if we include negative values of y.



18.8 Mappings by Branches of $z^{1/n}$

Recall section 12.4.

Square root transformation:
For $z \neq 0$ and in polar coordinates,

$$z^{1/2} = \sqrt{r}e^{i(\Theta+2\pi x)/2} \qquad k \in \mathbb{N}$$

Looking at the Principal Branch, the transformation becomes:

$$F_0(z) = \sqrt{r}e^{i\Theta/2} \qquad r > 0,\ -\pi < \Theta < \pi$$

It is clear in the principal branch, the mapping is a 1-1 mapping of points in $-\pi < \Theta < \pi$ to $-\pi/2 < \Phi < \pi/2$.

When $\theta = \alpha$ is used to define a branch cut:

$$f_\alpha(z) = \sqrt{r}\exp\left(\frac{i\theta}{2}\right) \qquad r > 0,\ \alpha < \theta < \alpha + 2\pi$$

We can extend this to nonzero points on the branch cut by defining $f_\alpha(0) = 0$, however, such extensions are not continuous on the entire complex plane.

n-th roots of z:

$$z^{1/n} = \exp\left(\frac{1}{n}\log(z)\right) = \sqrt[n]{r}e^{i(\Theta+2k\pi)/n} \qquad n \in \mathbb{N},\ k \in \{0, 1, 2, \ldots, n-1\}$$

Then the transformation of each branch $k$of $z^{1/n}$:

$$F_k(z) = \sqrt[n]{r}\exp\left(\frac{i(\Theta + 2k\pi)}{n}\right) \qquad k = \{0, 1, 2, \ldots, n-1\}$$

It is a 1-1 mapping from the domain:

$$r \mapsto \rho = \sqrt[n]{r} \qquad [-\pi < \Theta < \pi] \mapsto \left[\frac{(2k-1)\pi}{n} < \phi < \frac{(2k+1)\pi}{n}\right]$$

Likewise, for the principal branch ($k = 0$), we can construct transformations $f_\alpha(z)$ branch cuts at $\theta = \alpha$ as before.

18.9 Square Roots of Polynomials

Example 18.9.1 *If we consider* $Z^{1/2} = (z - z_0)^{1/2}$*:*

$$Z^{1/2} = \sqrt{R}\exp\left(\frac{i\theta}{2}\right) \qquad R > 0,\ \alpha < \theta < \alpha + 2\pi$$

By writing

$$R = |z - z_0| \qquad \Theta = \mathrm{Arg}(z - z_0) \qquad \theta = \arg(z - z_0)$$

We have the two branches of $(z - z_0)^{1/2}$*:*

$$G_0(z) = \sqrt{R}\exp\left(\frac{i\Theta}{2}\right) \qquad R > 0,\ -\pi < \Theta < \pi$$

$$g_0(z) = \sqrt{R}\exp\left(\frac{i\theta}{2}\right) \qquad R > 0,\ 0 < \Theta < 2\pi$$

We can see that for $w = G_0$, there is a 1-1 mapping:

$$|z - z_0| \mapsto \sqrt{|z - z_0|} \qquad [-\pi < \operatorname{Arg}(z - z_0) < \pi] \mapsto \left[-\frac{\pi}{2} < \frac{\operatorname{Arg}(z - z_0)}{2} < \frac{\pi}{2}\right]$$

For $w = g_0(z)$, the 1-1 mapping:

$$|z - z_0| \mapsto \sqrt{|z - z_0|} \qquad [0 < \arg(z - z_0) < 2\pi] \mapsto \left[0 < \frac{\arg(z - z_0)}{2} < \pi\right]$$

Example 18.9.2 *Consider*

$$w = (z^2 - 1)^{1/2}$$

Then we can write

$$\begin{aligned}(z^2 - 1)^{1/2} &= \exp\left(\frac{1}{2}\log(z^2 - 1)\right) = \exp\left(\frac{1}{2}\log(z - 1) + \frac{1}{2}\log(z + 1)\right) \\ &= (z - 1)^{1/2}(z + 1)^{1/2} \qquad\qquad z \neq \pm 1\end{aligned}$$

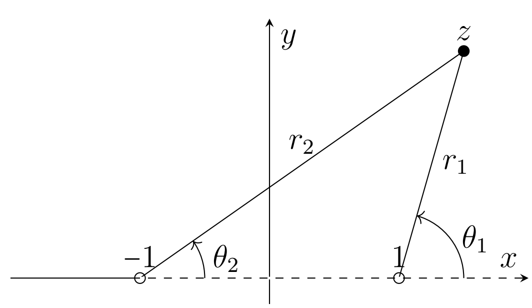
Note: If $f_1(z)$ is a branch of $(z-1)^{1/2}$ defined on domain D_1 and $f_2(z)$ is a branch of $(z+1)^{1/2}$ defined on domain D_2, then $f(z) = f_1(z)f_2(z)$ is a branch of $(z^2 - 1)^{1/2}$ defined at all points in $D_1 \cup D_2$.

For branches of $(z - 1)^{1/2}$ and $(z - 1)^{1/2}$:

$$\begin{aligned} f_1(z) &= \sqrt{r_1}\exp\left(\frac{i\theta_1}{2}\right) \qquad r_1 = |z - 1| \qquad \theta_1 = \arg(z - 1) \qquad r_1 > 0,\ \theta_1 \in (0, 2\pi) \\ f_2(z) &= \sqrt{r_2}\exp\left(\frac{i\theta_2}{2}\right) \qquad r_2 = |z + 1| \qquad \theta_2 = \arg(z + 1) \qquad r_2 > 0,\ \theta_2 \in (0, 2\pi)\end{aligned}$$

Then the branch f of $(z^2 - 1)^{1/2}$:

$$f(z) = \sqrt{r_1 r_2}\exp\left(\frac{i(\theta_1 + \theta_2)}{2}\right) \qquad r_k > 0,\ \theta_k \in (0, 2\pi),\ k \in \{1, 2\}$$



We can extend this to a function that is analytic everywhere in $\mathbb{C}$ except $x \in [-1, 1]$:

$$F(z) = \sqrt{r_1 r_2}\exp\left(\frac{i(\theta_1 + \theta_2)}{2}\right) \qquad r_k > 0,\ r_1 + r_2 > 2,\ \theta_k \in (0, 2\pi),\ k \in \{1, 2\}$$

We need to show F is analytic on $r_1 > 0$, $\theta_1 = 0$, since $f(z) = F(z)$ everywhere except there. Consider:

$$G(z) = \sqrt{r_1 r_2}\exp\left(\frac{i(\Theta_1\Theta_2)}{2}\right) \qquad \begin{array}{l} r_1 = |z-1|,\ r_2 = |z+1|,\ \Theta_1 = \mathrm{Arg}(z-1), \Theta_2 = \mathrm{Arg}(z+1) \\ r_k > 0,\ \Theta_k \in (-\pi, \pi),\ k \in \{1, 2\} \end{array}$$

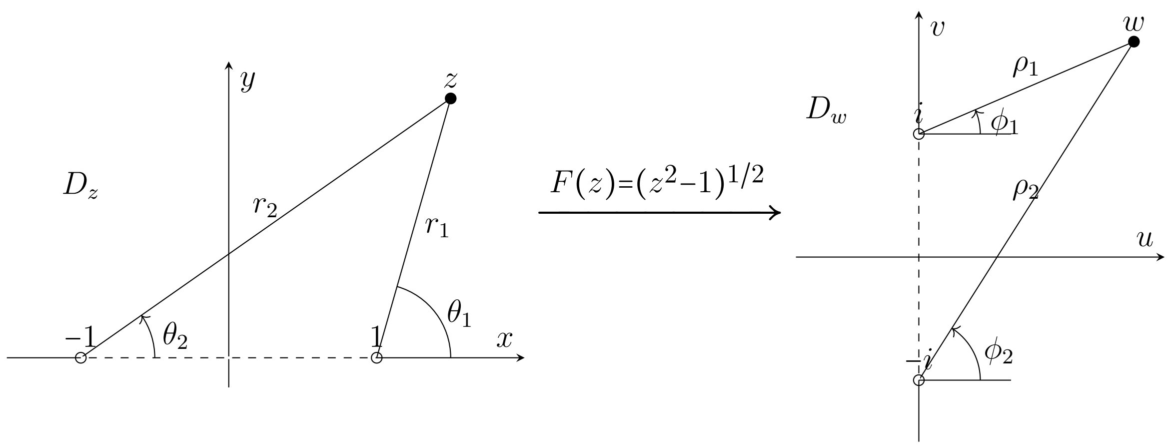
G analytic in entire z-plane except for $r_1 \geq 0$, $\Theta_1 = \pi$. $F(z) = G(z)$ when $r_1 > 0$ and $\Theta_1 \in [0, \pi)$, so $\theta_k = \Theta_k$. When $\Theta_1 \in (0, -\pi)$, $\theta_k = \Theta_k + 2\pi$. Then

$$\exp(i\theta_k/2) = -\exp(i\Theta_k/2) \implies \exp\left(\frac{i(\theta_1+\theta_2)}{2}\right) = \exp\left(\frac{i(\Theta_2+\Theta_2)}{2}\right)$$

So $F(z) = G(z)$, in the domain containing $r_1 > 0$, $\Theta_1 = 0$, so G being analytic in the domain implies F being analytic in the domain. Thus, F is analytic everywhere except $x \in [-1, 1]$.

$F(z)$ cannot be extended to a function analytic at points $x \in [-1, 1]$, because $F(z)$ jumps from $i\sqrt{r_1 r_2}$ to $-i\sqrt{r_1 r_2}$ as z moves across the line segment, thus is not continuous.

$F(z)$ is a 1-1 mapping of $D_z \mapsto D_w$ except for $v \in [-1, 1]$.



Note:

$$[z = iy] \wedge [y > 0] \implies [r_1 = r_2 > 1] \wedge [\theta_1 + \theta_2 = \pi]$$

Hence the mappings:

$$\begin{array}{cc} [y > 0] \wedge [x = 0] \mapsto v > 1 & [y < 0] \wedge [x = 0] \mapsto v < -1 \\ y > 0 \mapsto v > 0 & y < 0 \mapsto v > 0 \\ [r_1 > 0] \wedge [\theta_1 = 0] \mapsto [u > 0] \wedge [v = 0] & [r_2 > 0] \wedge [\theta_2 = \pi] \mapsto [u < 0] \wedge [v = 0] \end{array}$$

To show $w = F(z)$ is 1-1:

$$F(z_1) = F(z_2) \implies z_1^2 - 1 = z_2^2 - 1 \implies [z_1 = z_2] \vee [z_1 = -z_2]$$

Since F maps upper half plane to upper half plane and lower half planes to lower half planes, and the way portions of the real axis in D_z is mapped, $z_1 = -z_2$ is impossible, so $z_1 = z_2$. Thus, $F(z_1) = F(z_2) \implies F$ is 1-1.

We can show $F : D_z \mapsto D_w$ by finding $H : D_w \mapsto D_z$ which $z = H(w) \implies w = F(z)$. That is, $H = F^{-1}$. We have

$$w = (z^2 - 1)^{1/2} \implies z = (w^2 + 1)^{1/2} = (w - i)^{1/2}(w - i)^{1/2} \qquad w \neq \pm i$$

Writing

$$w - i = \rho_1 e^{i\phi_1} \qquad w + i = \rho_2 e^{i\phi_2} \qquad \rho_k > 0,\ \rho_1 + \rho_2 > 2,\ \phi_k \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right],\ k \in \{1, 2\}$$

Hence

$$H(w) = \sqrt{\rho_1 \rho_2} \exp\left(\frac{i(\phi_1 + \phi_2)}{2}\right) \qquad \text{Domain: } D_w$$

with mappings:

$$\begin{array}{cc} v > 0 \mapsto y > 0 & v < 0 \mapsto y < 0 \\ [u > 0] \wedge [y = 0] \mapsto [x > 1] \wedge [y = 0] & [u < 0] \wedge [y = 0] \mapsto [x < -1] \wedge [y = 0] \end{array}$$

Then

$$z = H(w) \implies z^2 = w^2 + 1 \implies w^2 = z^2 - 1$$

Since $z \in D_z$ and $w = \pm(z^2 - 1)^{1/2}$, we have $w = F(z)$ and $w = -F(z)$. By $H(w)$ mapping upper half plane to upper half plane and lower half planes to lower half planes, and the mapping of segments of the real axis in the domains, we have $w = F(z)$.

For mappings by branches of double-valued functions:

$$w = (z^2 + Az + B)^{1/2} = [(z - z_0)^2 - z_1^2]^{1/2} \qquad A = -2z_0 \qquad B = z_0^2 = z_1^2 z_1 \neq 0$$

We can use methods in the above example on the successive transformations:

$$Z = \frac{z - z_0}{z_1} \qquad W = (Z^2 - 1)^{1/2} \qquad w = z_1 W$$

18.10 Riemann Surface

Definition 18.10.1: Riemann Surface (Informal Definition)
A generalized complex plane with more than one "sheet". It's a one dimensional complex manifold.

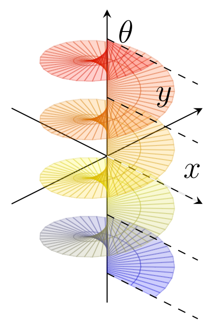
On a Riemann surface, a multi-valued function is assigned a single-value. As such, the theory of single-valued function applies.

Example 18.10.1 *Consider the multi-valued function:*

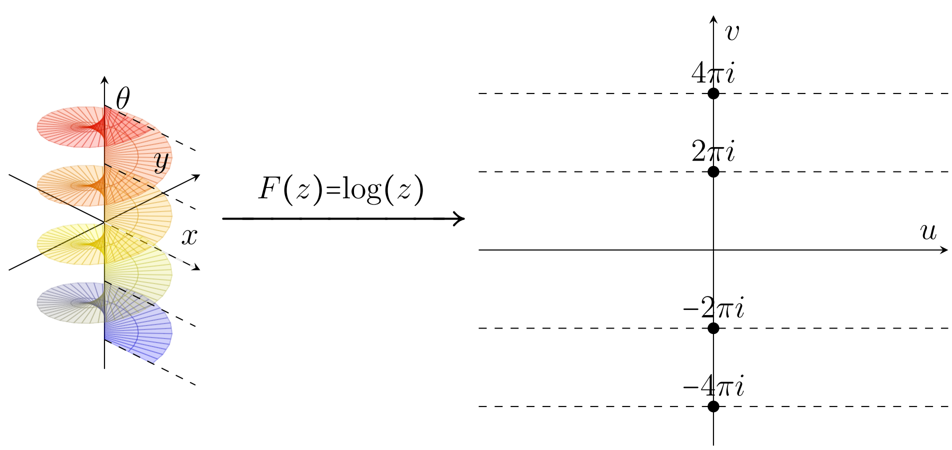
$$\log(z) = \ln(r) + i\theta$$

The Riemann surface of $\log(z)$ *is then the complex plane with a deleted origin, and a cut made along the positive real axis. The first sheet* R_0 *is defined from* $\theta \in [0, 2\pi]$*, the second sheet* R_1 *consists of* $\theta \in [2\pi, 4\pi]$*, the third sheet* R_3 *consists of* $\theta \in [4\pi, 6\pi]$*, and so on. Likewise, the sheet* R_{-1} *consists of* $\theta \in [-2\pi, 0]$.

Essentially, we get an infinite "spiral staircase" with sheets in multiples of 2π*. The radial component, of course, is also infinite.*



The transformation $w = \log(z)$ *maps the Riemann surface onto the* w*-plane in an 1-1 manner.*



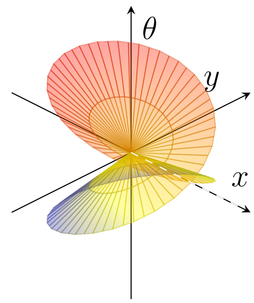
$\log(z)$ *on* R_1 *is an analytic continuation of* $f(z) = \ln(r) + i\theta$, $\theta \in (0, 2\pi)$. $\log(z)$ *is then not only single-valued on all points* z *on the Riemann surface, but also analytic at all points.*

The "spiral staircase" can be cut along any ray from the origin to for other Riemann surfaces.

Example 18.10.2 *Consider the square root function on the* z*-plane with a deleted origin:*

$$z^2 = \sqrt{r}e^{i\theta/2}$$

The function maps $\theta \in [0, 2\pi]$ *to* $\phi \in [0, \pi]$*, and* $\theta \in [2\pi, 4\pi]$ *to* $\phi \in [\pi, 2\pi]$*. Thus the Riemann surface is composed of two sheets,* R_1 *and* R_2.



The function is now single-valued. The choice of values of θ from 0 to 2π or 4π to 6π does not affect the value of $z^{1/2}$. Note, the value of $z^{1/2}$ a point where the sheet passes from R_0 to R_1 is different than that passing from R_1 to R_0. R_0 for $\theta \in (0, 2\pi)$, and R_1 for $\theta \in (2\pi, 4\pi)$.

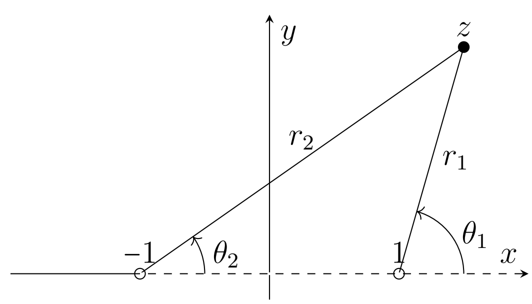
The origin is common to both R_0 and R_1, thus is a branch point of the Riemann Surface.

The function is an analytic continuation of the function defined on the other sheet, thus points of $z^{1/2}$ on the Riemann surface are all analytic except at the origin.

18.10.1 Surfaces for Related Functions

Example 18.10.3 *Riemann surface for*

$$f(z) = (z^2 - 1)^{1/2} = \sqrt{r_1 r_2}\exp\left(\frac{i(\theta_1 + \theta_2)}{2}\right) \qquad z - 1 = r_1 e^{i\theta_1} \qquad z + 1 = r_2 e^{i\theta_2}$$



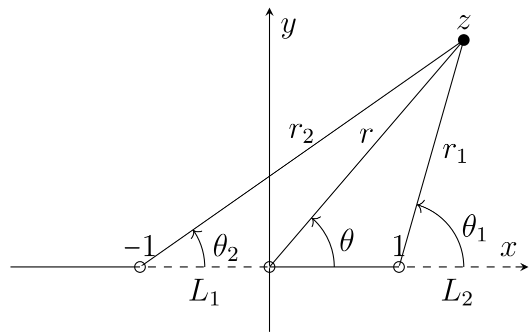
We have a branch cut between the points $z = \pm 1$, thus the Riemann surface has two sheets cut between the points. The lower edge of R_0 is connected to the upper edge of R_1, and lower edge of R_1 is connected to the upper edge of R_0. We have two layers where the line between $z = \pm 1$ serves as a connection between the layers.

Any simple closed curve enclosing $z = \pm 1$ on a sheet will return to its original position as θ_1 and θ_2 go from 0 to 2π, and will not cross into the other sheet.

If a contour encircles $z = 1$ twice, but does not enclose $z = -1$, then the contour passes from R_0 to R_1 and back to R_0. θ_1 then changes by 4π while θ_2 changes by 0. Likewise for a similar case enclosing $z = -1$.

Example 18.10.4 *Consider the double-valued function:*

$$f(z) = [z(z^2 - 1)]^{1/2} = \sqrt{rr_1r_2}\exp\left(\frac{i(\theta + \theta_1 + \theta_2)}{2}\right)$$



The branch points are $z \in \{0, \pm 1\}$. Since the function is double-valued, we will have two sheets, R_0 and R_1. We can define a cut L_1 from -1 to 0 and a cut L_2 from 1 to a point at infinity. The sheets R_0 and R_1 is then joined along L_1 and L_2, with the lower edge of R_0 joined to the upper edge of R_1 and the lower edge of R_1 joined to the lower edge of R_0.

Question. *Is the choice of the cuts L_1 and L_2 arbitrary? Can we define L_1 as the point at infinity to -1 and L_2 from 0 to 1?*

Chapter 19

Conformal Mapping

A map that locally conforms to the original shape of a region.

19.1 Preserving Angles and Scale Factors

Definition 19.1.1: Conformal

A transformation $w = f(z)$ is conformal at point z_0 if f is analytic at z_0 and $f'(z_0) \neq 0$. That is, the orientation and magnitude of the angles between curves are preserved.

$$[f \text{ analytic at } z_0] \wedge [f'(z_0) \neq 0] \implies f \text{ Conformal}$$

Consider an arc C_1 parameterized by $z_1(t)$ and a function f defined by all points in C:

$$z_1 = z_1(t) \qquad w_1 = f[z_1(t)] \qquad t \in [a, b]$$

Thus w is the parametric representation of image Γ_1 of C_1. Suppose there exists $z_0 = z_1(t_0) \in (a, b)$ such that f is analytic and $f'(z_0) \neq 0$. Then

$$\begin{aligned} w_1'(t_0) = f'[z_1(t_0)]z_1'(t_0) &\implies \arg[w_1'(t_0)] = \arg[f'[z_1(t_0)]] + \arg[z_1'(t_0)] \\ &\implies \arg[f'[z_1(t_0)]] = \arg[w_1'(t_0)] - \arg[z_1'(t_0)] \end{aligned}$$

Hence, we can see that $w'(t_0)$ and $z'(t_0)$ differs by an angle of rotation $\varphi_0 = \arg[f'(z_0)]$.

Definition 19.1.2: Angle of Rotation

Let $z_0 = z(t_0)$ be a point in an arc C, and $w = f(z)$ be a conformal transformation. Then the angle of rotation is the difference between the angle of C at z_0 and its image Γ under w at $f(z_0)$.

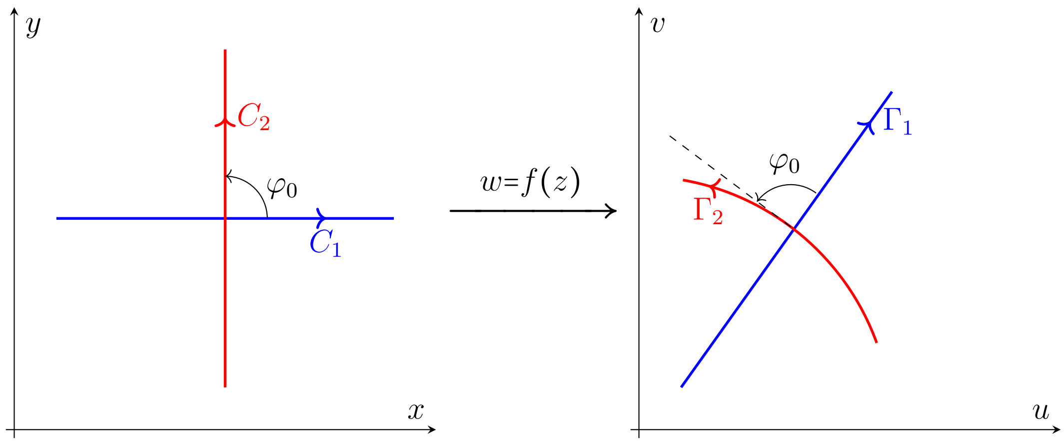
If we consider another arc C_2 passing through $z_0 = z_2(t_0)$ with image Γ_2 under the same transformation w. We obtain:

$$\begin{aligned} \arg[w_2'(t_0)] &= \arg[f'[z_2(t_0)]] + \arg[z_2'(t_0)] && \text{For } C_2 \\ \arg[w_1'(t_0)] &= \arg[f'[z_1(t_0)]] + \arg[z_1'(t_0)] && \text{For } C_1 \end{aligned}$$

Since we have $z_1(t_0) = z_2(t_0) = z_0$, if we subtract the two, we have

$$\arg[w_1'(t_0)] - \arg[w_2'(t_0)] = \arg[z_1'(t_0)] - \arg[z_2'(t_0)]$$

Hence, the angle between C_1 and C_2 at z_0 is the same as the angle between Γ_1 and Γ_2 at $f(z_0)$. The angles between the curves at z_0 and orientation of the angles are preserved in the transformation.



From theorem 17.4.2 we can see that w is also conformal in some neighbourhood of z_0. If this applies to an entire domain:

Definition 19.1.3: Conformal Mapping/Transformation

A transformation $w = f(z)$ is conformal if $\forall z \in D$, $f(z)$ is analytic and $f'(z) \neq 0$.

$$\forall z \in D[f \text{ analytic } \wedge f(z) \neq 0] \implies f \text{ Conformal Mapping}$$

Example 19.1.1 *Consider two smooth arcs that are level curves of*

$$f(z) = u(x, y) + iv(x, y)$$

Suppose $u(x, y) = c_1$ and $v(x, y) = c_2$ intersect at point z_0, where f is analytic and nonzero at z_0. Then f must be conformal at z_0. If the two curves are orthogonal at z_0 then they are orthogonal at $w_0 = f(z_0)$.

Definition 19.1.4: Isogonal Mapping

A mapping that preserves the angle between two curves, but not the orientation of the angle.

Example 19.1.2 *Consider the transformation*

$$w = \bar{z}$$

It is isogonal, not conformal, due to $w = \bar{z}$ not being an analytic function. If followed by an conformal map f, the result $w = f(\bar{z})$ is isognal. That is, ***conformal transformations preserves isogonal transformations****.*

Definition 19.1.5: Critical Point

A function f that is non-constant and analytic at z_0, and $f(z_0) = 0$. Then z_0 is a critical point of transformation $w = f(z)$.

Example 19.1.3 *Consider*

$$w = 1 + z^2$$

Which is a composition of mappings

$$Z = z^2 \qquad\qquad w = 1 + Z$$

$z_0 = 0$ is a critical point of w. We can see that $z_0 = 0 \mapsto w_0 = 1$, and that w doubles any angle at z_0.

Corollary 19.1.0.1:

Suppose z_0 is a critical point of $w = f(z)$. Let Γ_1 and Γ_2 be the image of curves C_1 and C_2 under the transformation respectively. If the angle between C_1 and C_2 at z_0 is α, then the angle between Γ_1 and Γ_2 at $w_0 = f(z_0)$ becomes $m\alpha$ under w for $m \geq 2$, $m \in \mathbb{N}$. Also, m is the smallest natural number such that $f^{(m)}(z_0) \neq 0$.

That is, if z_0 is a critical point of $w = f(z)$, then angles between curves α at z_0 becomes $m\alpha$ under $f(z)$ for $m \geq 2$, $m \in \mathbb{N}$, and $f(z)$ has a zero of order m at z_0.

Proof: Example 19.2.1 □

Definition 19.1.6: Scale Factor

The Scale Factor, $|f'(z_0)|$, is the amount of scaling inflicted on the distances under the transformation $w = f(z)$. Consider the modulus of the derivative of the transformation $w = f(z)$:

$$|f'(z_0)| = \left|\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}\right| = \lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}$$

For any z close to z_0:

$$|f'(z_0)| \approx \frac{|f(z) - f(z_0)|}{|z - z_0|}$$

which represents the ratio between the distances $f(z) - f(z_0)$ and $z - z_0$ under the transformation.

It is clear that $|f'(z_0)| > 1$ represents an expansion while $|f'(z_0) < 1|$ represents a contraction.

From the continuity of $f'(z)$ we see that for z close to z_0:

$$\arg[f'(z)] \approx \arg[f'(z_0)] \qquad\qquad |f'(z)| \approx |f'(z_0)|$$

Hence, the image of a conformal transformation approximates that of the original neighbourhood locally. That is, it is *conforms* to the shape of the original region. Notice, it is *locally* not *globally*.

Example 19.1.4 *Consider*

$$f(z) = z^2 = x^2 - y^2 + i2xy \implies f'(z) = 2z$$

The function is entire and $f'(z)$ is zero only at the origin. Consider the half lines

$$y = x \qquad\qquad x = 1 \qquad\qquad x, y \geq 0$$

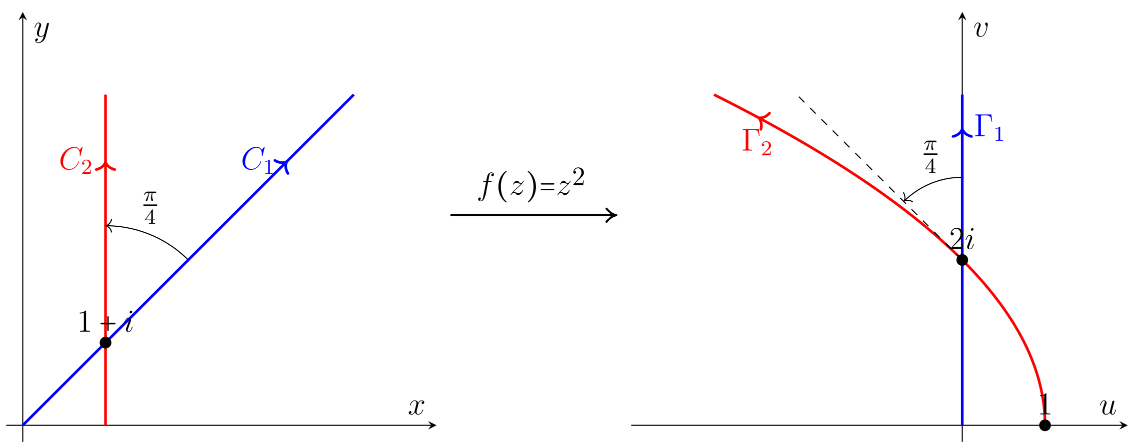
which is denoted as the curves C_1 and C_2 that intersect at $z_0 = 1+i$. It is clear that the angle between the curves are $\pi/4$.

Under the transformation, $C_1 \mapsto \Gamma_1$, with Γ_1 parameterization:

$$u = 0 \qquad\qquad v = 2x \qquad\qquad 0 \leq x < \infty$$

$C_2 \mapsto \Gamma_2$, with Γ_2 parameterization:

$$u = 1 - y^2 \qquad\qquad v = 2y \qquad\qquad 0 \leq y < \infty$$



We then have

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{\mathrm{d}v/\mathrm{d}y}{\mathrm{d}u/\mathrm{d}y} = \frac{2}{-2y} = -\frac{2}{v}$$

$v = 2 \implies \mathrm{d}v/\mathrm{d}u = -1$, *so the angle between Γ_1 and Γ_2 at $w = f(1+i) = 2i$ is $\pi/4$. Hence, we have conformality of the mapping.*

The angle of rotation:

$$\arg[f'(1+i)] = \arg[2(1+i)] = \frac{\pi}{4} + 2n\pi \qquad\qquad n \in \mathbb{Z}$$

Scale factor:

$$|f'(1+i)| = |2(1+i)| = 2\sqrt{2}$$

19.2 Local Inverses

Definition 19.2.1: Local Inverse

Suppose a transformation $w = f(z)$ be conformal and $w_0 = f(z_0)$. Then a local inverse of the transformation is a unique transformation $z = g(w)$ defined and analytic in a neighbourhood N of w_0 such that $\forall w \in N$, $g(w_0) = z_0$ and $f[g(w)] = w$. The derivative:

$$g'(w) = \frac{1}{f'(w)}$$

Proof: Prove:

$$g'(w) = \frac{1}{f'(w)}$$

Let $w = f[g(w)]$, then

$$\frac{\mathrm{d}}{\mathrm{d}w} w = \frac{\mathrm{d}}{\mathrm{d}w} f[g(w)] \implies 1 = f'[g(w)]g'(w) \implies g'(w) = \frac{1}{f'(w)}$$

□

Note: From the definition, $z = g(w)$ is conformal at w_0.

Existence of the Inverse:

Conformality of the transformation $w = f(z)$ at z_0 implies there exist a neighbourhood of z_0 that f is analytic. Hence

$$f(z) = u(x,y) + iv(x,y) \qquad z = x + iy \qquad z_0 = x_0 + iy_0$$

Then there exists some neighbourhood of x_0, y_0 where u and v and their partial derivatives of all orders are continuous (theorem 15.7.3). The Jacobian is then nonzero at z_0:

$$\begin{aligned} \mathrm{J} &= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - v_x u_y = (u_x)^2 + (v_x)^2 && u_x = v_y,\ u_y = -v_x \\ &= |f'(z)|^2 \neq 0 && \text{Conformal at } z_0,\ \text{Theorem 13.5.4} \end{aligned}$$

The condition on the Jacobian and derivatives of u and v are sufficient conditions to ensure invertibility of transform. So, if

$$u_0 = u(x_0, y_0) \qquad v_0 = v(x_0, y_0)$$

then there exists a unique continuous transformation

$$x = x(u,v) \qquad y = y(u,v)$$

that $(u_0, v_0) \mapsto (x_0, y_0)$ in the neighbourhood N. In addition, the first order partial derivatives throughout N satisfy

$$x_u = \frac{v_y}{\mathrm{J}} \qquad x_v = -\frac{u_y}{\mathrm{J}} \qquad y_u = -\frac{v_x}{\mathrm{J}} \qquad y_v = \frac{u_x}{\mathrm{J}}$$

which shows that $g(w)$ is analytic in N.

Proof: Proving $g(w)$ is analytic in N:

We know that $f(z)$ is analytic, so it satisfies the Cauchy-Riemann equations (theorem 13.5.1). Hence, we can write the above four equations as:

$$x_u = \frac{u_x}{\mathrm{J}} \qquad x_v = -\frac{u_y}{\mathrm{J}} \qquad y_u = \frac{u_y}{\mathrm{J}} \qquad y_v = \frac{u_x}{\mathrm{J}}$$

It is then clear that

$$x_u = y_v \qquad x_v = -v_u$$

Which are the Cauchy-Riemann equations. Thus $g(w)$ is analytic in N. □

Letting $z = x + iy$ and $w = u + iv$:

$$\begin{aligned} z &= g(w) = x(u,v) + iy(u,v) \\ w &= f(z) = u(x,y) + iv(x,y) \end{aligned}$$

Thus the inverse exists.

Example 19.2.1 *Suppose that f has a zero of order $m \in \mathbb{N}$ at z_0, that is*

$$f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) \qquad f^{(m)} \neq 0$$

Let $w_0 = f(z_0)$. Using the Taylor expansion of $f(z)$ about z_0 (theorem 16.2.1):

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = f(z_0) + \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \\ &= f(z_0) + \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m + \sum_{n=m+1}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \end{aligned}$$

This implies that

$$\begin{aligned} f(z) - f(z_0) = f(z) - w_0 &= \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m \left[1 + \sum_{n=m+1}^{\infty} \frac{f^{(n)}(z_0)}{f^{(m)}(z_0)} \cdot \frac{m!}{n!}(z-z_0)^{n-m}\right] \\ &= \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m[1+g(z)] \end{aligned}$$

We can see that in $g(z)$, $n > m$, so it is a Taylor series which is analytic at z_0. Also, $g(z_0) = 0$. Let C_1 be a curve with image Γ_1 under the transformation f, and θ_0 be the angle of inclination of C_1 at z_0. Likewise with ϕ_0 for Γ_1. Then we have

$$\theta_0 = \lim_{z\to z_0} \arg(z-z_0) \qquad \phi_0 = \lim_{z\to z_0} \arg[f(z) - w_0]$$

Using the results obtained previously:

$$\begin{aligned} \phi_0 &= \lim_{z\to z_0} \arg\left[\frac{f^{(m)}(z_0)}{m!}(z-z_0)^m[1+g(z)]\right] \\ &= \lim_{z\to z_0}\left[m\arg(z-z_0) + \arg[f^{(m)}(z_0)] + \arg\left(\frac{1+g(z)}{m!}\right)\right] \end{aligned}$$

Since

$$\lim_{z\to z_0}\frac{1+g(z)}{m!}=\frac{1+g(z_0)}{m!}=\frac{1}{m!}\in\mathbb{R}\implies \arg\left[\frac{1}{m}\right]=0 \qquad\qquad g(z_0)=0$$

we have

$$\phi_0=\lim_{z\to z_0}[m\arg(z-z_0)]+\arg[f^{(m)}(z_0)]=m\theta_0+\arg[f^{(m)}(z_0)]$$

Now, let another curve C_2 intersect C_1 at z_0, and Γ_2 intersect Γ_1 at w_0. Then the angle of inclination of Γ_2 at w_0:

$$\phi_1=m\theta_1+\arg[f^{(m)}(z_0)]$$

Let α be the angle between C_1 and C_2 at z_0. Since we are still under the transformation f, the angle between Γ_1 and Γ_2 at w_0:

$$\phi_1-\phi_0=m(\theta_1-\theta_0)+\arg[f^{(m)}(z_0)]-\arg[f^{(m)}(z_0)]=m\alpha \qquad\qquad \alpha=\theta_1-\theta_0$$

If the mapping is conformal, $\phi_1-\phi_0=\theta_1-\theta_0=\alpha$, so $m=1$.

Note: Since f has a zero of order m, $f'(z_0)=0$ for all $m\geq 2$, so z_0 is a critical point of f for $m\geq 2$ by definition 19.1.5. Thus the transformed angle is an natural number multiple of the original angle at a critical point.

19.3 Harmonic Conjugates

Review section 13.7.

Recall from the definition of a Harmonic Conjugate (definition 13.7.3). If function $f(z)=u(x,y)+iv(x,y)$ is analytic in a domain D, then $v(x,y)$ is a harmonic conjugate of $u(x,y)$.

Theorem 19.3.1:
Let $f(z)=u(x,y)+iv(x,y)$ be a function in domain D.

$$f \text{ analytic in } D \iff v \text{ is harmonic conjugate of } u$$

Proof: $\underline{\Longleftarrow}$:

$$\begin{aligned}&v \text{ is harmonic conjugate of } u\\ &\implies \text{Cauchy-Riemann Equations Satisfied}\\ &\implies f \text{ analytic in } D && \text{Theorem 13.5.4}\end{aligned}$$

$\underline{\Longrightarrow}$:

$$\begin{aligned}
& f \text{ analytic in } D && \\
& \Longrightarrow u \text{ and } v \text{ harmonic} && \text{Theorem 13.7.1} \\
& \Longrightarrow \text{Cauchy-Riemann Equations satisfied} && \text{Theorem 13.5.1}
\end{aligned}$$

$\square$

Example 19.3.1 (v harmonic conjugate of u does **not imply** u harmonic conjugate of v) *Consider*

$$u(x,y) = x^2 - y^2 \qquad\qquad v(x,y) = 2xy$$

We know that v is a harmonic conjugate of u since $f(z) = x^2 - y^2 + i2xy = z^2$ is entire. However, u is not a harmonic conjugate of v, since $g(z) = 2xy + i(x^2 - y^2)$ does not satisfy the Cauchy-Riemann equations and are not analytic anywhere:

$$\begin{aligned}
u_x &= 2y & v_y &= -2y \\
u_y &= 2x & -v_x &= -2x
\end{aligned}$$

Example 19.3.2 *(Finding Harmonic Conjugates) Consider*

$$u(x,y) = 2x(1-y)$$

By Cauchy-Riemann equation, we have $u_x = v_y = 2 - 2y$. By integrating:

$$v(x,y) = 2y - y^2 + g(x)$$

By $u_y = -v_x = -2x \Longrightarrow g_x = 2x$, then

$$v(x,y) = 2y - y^2 + x^2 + C$$

Therefore

$$f(z) = 2x(1-y) + i(2y - y^2 + x^2 + C) = 2z + i(z^2 + C)$$

It is customary to write $C = 0$, since C is arbitrary, so

$$f(z) = 2z + iz^2$$

Theorem 19.3.2:

Let $f(z) = u(x,y) + iv(x,y)$ be a harmonic function in a simply connected domain D, then u has a harmonic conjugate v in D.

Proof: Supposes $P(x, y)$ and $Q(x, y)$ have continuous first-order partial derivatives in simply connected domain D. If $P_y = Q_x$ everywhere in D, then the integral over contour C from (x_0, y_0) to (x, y):

$$\int_C P(s,t)ds + Q(s,t)dt$$

is path independent in D. If (x_0, y_0) is fixed, then

$$F(x,y) = \int_{(x_0,y_0)}^{(x,y)} P(s,t)ds + Q(s,t)dt$$

is single-valued, with first-order partial derivatives:

$$F_x(x,y) = P(x,y) \qquad F_y(x,y) = Q(x,y)$$

It follows from Laplace's Equation, that everywhere in D:

$$u_{xx} + u_{yy} = 0 \implies (-u_y)_y = (u_x)_x$$

Since the second-order partial derivatives of v are continuous in D, the first-order partial derivatives are continuous in D as well. Then for a fixed $(x_0, y_0) \in D$:

$$v(x,y) = \int_{(x_0,y_0)}^{(x,y)} -u_t(s,t)ds + u_s(s,t)dt$$

is well defined $\forall(x, y) \in D$. Therefore

$$\begin{aligned}&[F_x(x,y) = P(x,y)] \wedge [F_y(x,y) = Q(x,y)] \\ \implies &[v_x(x,y) = -u_y(x,y)] \wedge [v_y(x,y) = u_x(x,y)]\end{aligned}$$

Hence, the first-order partial derivatives of u are also continuous in D. Thus $u(x, y) + iv(x, y)$ is analytic in D and v is a harmonic conjugate of u.

Note: v is not the only harmonic conjugate of u, since the conditions for a harmonic conjugate is satisfied by a family of functions $v(x, y) + C$ for all $C \in \mathbb{R}$. □

Example 19.3.3 *Consider $u(r, \theta) = \ln(r)$, we can see that it satisfies the polar form of Laplace's equation:*

$$r^2 u_{rr}(r,\theta) + r u_r(r,\theta) + u_{\theta\theta}(r,\theta) = r^2(-r^{-2}) + r(r^{-1}) = 0$$

Hence it is harmonic in domain $r > 0$, $\theta \in (0, 2\pi)$. By the polar form of the Cauchy-Riemann equations (theorem 13.5.2):

$$r u_r = v_\theta = 1 \qquad u_\theta = -r v_r = 0$$

We can integrate and obtain:

$$v(r,\theta) = \theta + C \qquad C \in \mathbb{R}$$

Hence the analytic function is

$$f(r,\theta) = \ln(r) + i(\theta + C)$$

19.3.1 Transformation of Harmonic Functions

Theorem 19.3.3:
Let $f(z) : D_z \mapsto D_w$ be an analytic function $w = f(z) = u(x,y) + iv(x,y)$.

$$h(u,v) \text{ harmonic in } D_w \implies H(x,y) = h[u(x,y), v(x,y)] \text{ harmonic in } D_z$$

That is, a transformation of an harmonic function by an analytic function is harmonic.

(For a more general case of $h(u,v)$ and alternate proof of the theorem, see example 19.3.8.)

(For the case where $h(u,v)$ satisfies Poisson's Equation, see example 19.3.9.)

Proof: D_w **is simply connected:**
$h(u,v)$ is harmonic in $D_w \implies h(u,v)$ has harmonic conjugate $g(u,v)$ (theorem 19.3.2). Hence, we have function

$$\Phi(w) = h(u,v) + ig(u,v) \qquad \text{Analytic in } D_w$$

$f(z)$ is analytic in $D_w \implies \phi[f(z)]$ is analytic in D_z. Hence, $u(x,y)$ is harmonic in $H[u(x,y), v(x,y)]$.

D_w **is not simply connected:**
$\forall w_0 \in D_w$, w_0 has a neighbourhood $|w - w_0| < \epsilon$ lying entirely in D_w. The neighbourhood is simply connected, we have a function:

$$\Phi(w) = h(u,v) + ig(u,v) \qquad \text{Analytic in } D_w$$

f is continuous at $z_0 \in D_z \implies \forall \epsilon, \exists \delta > 0[|w - w_0| < \epsilon \implies |z - z_0| < \delta]$. Hence $\Phi[f(z)]$ is analytic in neighbourhood $|z - z_0| < \delta$, so $h[u(x,y), v(x,y)]$ is analytic in the neighbourhood. Since $\forall z_0 \in D_w$, $z_0 \mapsto w_0 \in D_w$ by function $w = f(z)$, $h[u(x,y), v(x,y)]$ is harmonic in D_z. □

Example 19.3.4 *Consider*

$$w = e^z = e^x \cos(y) + ie^x \sin(y)$$

Which maps $y \in (0, \pi)$ to $v > 0$. w^2 is analytic in the upper-half plane, so the function

$$h(u,v) = \mathrm{Re}\{w^2\} = u^2 - v^2$$

is harmonic in the upper-half plane. By the theorem:

$$H(x,y) = (e^x \cos(y))^2 - (e^x \sin(y))^2 = e^{2x}[\cos^2(y) - sin[2](y)] = e^{2x}\cos(2y)$$

is harmonic throughout $y \in (0, \pi)$.

19.3.2 Transformation of Boundary Conditions

This theorem allows us to transform complex problems in the z-plane into simpler problems in the w-plane, solve it there, and transform back.

Theorem 19.3.4:

Suppose

1. $w = f(z) = u(x,y) + iv(x,y)$ *is conformal* $\forall z \in C$ *and* $\Gamma = \mathrm{Img}_f(C)$.
2. $h(u,v)$ *satisfies*

$$\forall w \in \Gamma \left[(h = h_0) \vee \left(\frac{\mathrm{d}h}{\mathrm{d}n} = 0 \right) \right]$$

where $h_0 \in \mathbb{R}$ *and* $\mathrm{d}h/\mathrm{d}n$ *is the directional derivative of* h *normal to* Γ.

Then

$$H(x,y) = h[u(x,y), v(x,y)]$$

satisfies

$$\forall z \in C \left[(H = h_0) \vee \left(\frac{\mathrm{d}H}{\mathrm{d}N} = 0 \right) \right]$$

where $\mathrm{d}H/\mathrm{d}N$ *is the directional derivative of* H *normal to* C.

Proof: $h = h_0$ **on** $\Gamma \implies H = h_0$ **on** C**:**

$$H(x,y) = h[u(x,y), v(x,y)] = h_0$$

$\mathrm{d}h/\mathrm{d}n = 0$ **on** $\Gamma \implies \mathrm{d}H/\mathrm{d}N = 0$ **on** C**:**

$$\frac{\mathrm{d}h}{\mathrm{d}n} = \nabla h \cdot \vec{n}$$

$\vec{n}$ is unit vector normal to Γ at (u,v).

$$\begin{aligned} \left.\frac{\mathrm{d}h}{\mathrm{d}n}\right|_{(u,v)} = 0 &\implies \nabla h \text{ orthogonal to } \vec{n} \text{ at } (u,v) \\ &\implies \nabla h \text{ tangent to } \Gamma \text{ at } (u,v) \\ &\implies \Gamma \text{ orthogonal to } h(u,v) = c \end{aligned}$$

Now

$$H(x,y) = c \implies h[u(x,y), v(x,y)] = c$$

$C \mapsto \Gamma$ and Γ orthogonal to $h(x,y) = c$ implies C orthogonal to $H(x,y) = c$ by conformality of transform. Thus ∇H is tangent to C at (x,y).

If $\vec{N}$ is unit vector normal to C at (x,y), then ∇H is orthogonal to N:

$$\nabla H \cdot \vec{N} = 0 \implies \frac{\mathrm{d}H}{\mathrm{d}N} = \nabla H \cdot \vec{N} = 0$$

$\square$

The proof assumed $\nabla h = 0$. If $\nabla h = 0$, then

$$|\nabla H(x,y)| = |\nabla h(u,v)||f'(z)| = 0 \implies \nabla h(u,v) = 0$$

We also assumed ∇h and ∇H exists and $H(x,y) = c$ is smooth when $\nabla h \neq 0$ at (u,v), which ensures angles are preserved by $w = f(z)$.

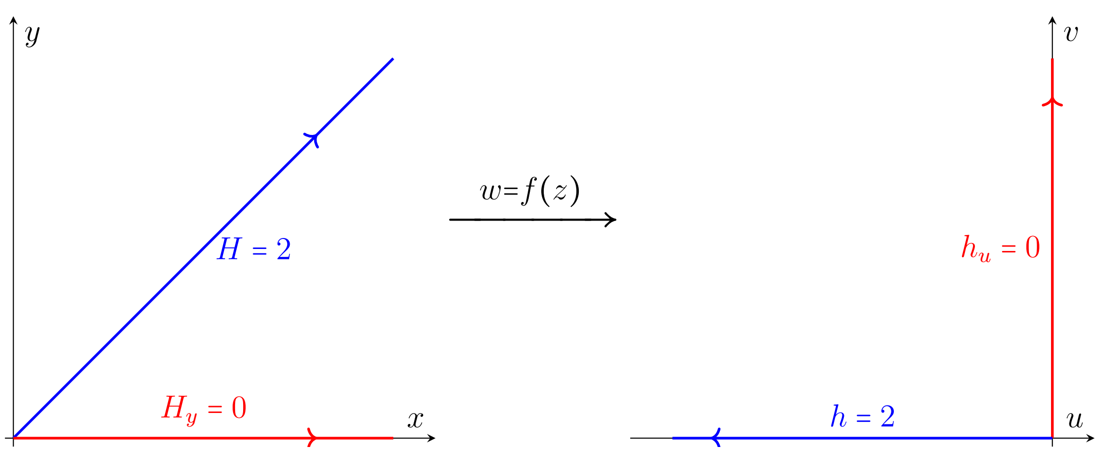
Example 19.3.5 *Consider $h(u,v) = v + 2$, with transformation*

$$w = iz^2 = i(x+iy)^2 = -2xy + i(x^2 - y^2)$$

which is conformal for $z \neq 0$. We have $h = 2$ for $x = y$ $(x > 0)$, and $h_u = 0$. Then

$$H(x,y) = x^2 - y^2 + 2$$

satisfies $H = 2$ for $x = y$ $(x > 0)$ and $H_y = 0$ for $x > 0$ by the theorem.



Under a conformal transformation, the ratio of the directional derivative of H along a smooth arc C in the z-plane to the directional derivative of h along the image Γ in the w-plane is $|f'(z)|$.

Example 19.3.6 *The transformation $w = f(z) = z^2$ which maps the positive x and y axis and the origin in the z plane onto the u axis in the w plane. Consider the harmonic function:*

$$h(u,v) = \mathrm{Re}\{e^{-w}\} = e^{-u}\cos(v)$$

The normal derivative $h_v = 0$ along the u axis. Show the normal derivative of $H(x,y)$ is zero along both positive axis is the z plane. Note: $w = z^2$ is not conformal at the origin.

Proof:

$$w = z^2 = x^2 - y^2 + i2xy \implies H(x,y) = e^{-x^2+y^2}\cos(2xy)$$

Taking the gradient:

$$\begin{aligned}\nabla H &= e^{-x^2+y^2}\{[(-2x)\cos(2xy) - (2y)\sin(2xy)]\hat{x} + [(2y)\cos(2xy) - (2x)\sin(2xy)]\hat{y}\} \\ &= 2e^{-x^2+y^2}\{[(-x)\cos(2xy) - (y)\sin(2xy)]\hat{x} + [(y)\cos(2xy) - (x)\sin(2xy)]\hat{y}\}\end{aligned}$$

Taking the normal derivative along $(x,0)$:

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \nabla H \cdot \hat{y} = (y)\cos(2xy) - (x)\sin(2xy) \implies \nabla H(x,0)\cdot\hat{y} = 0$$

Taking the normal derivative along $(0,y)$:

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \nabla H \cdot \hat{x} = (-x)\cos(2xy) - (y)\sin(2xy) \implies \nabla H(0,y)\cdot\hat{y} = 0$$

Hence, the normal derivative of $H(x,y)$ along both positives axes in the z plane is zero. $\square$

Example 19.3.7 *Let $h(u,v) = \mathrm{Re}\{-2iw + e^{-w}\} = 2v + e^{-u}\cos(v)$ with the transformation $w = f(z) = z^2$. Show $h_v = 2$ along the u axis, but $H_y = 4x$ along the positive x axis and $H_x = 4y$ along the positive y axis. Then this illustrates:*

$$\frac{\mathrm{d}h}{\mathrm{d}n} = h_0 \neq 0 \textit{ not neccessarily transformed to } \frac{\mathrm{d}H}{\mathrm{d}N} = h_0$$

Proof: Applying the transformation to $h(u,v)$:

$$H(x,y) = 4xy + e^{-x^2+y^2}\cos(2xy)$$

For h_v along the u axis:

$$h_v = 2 - e^{-u}\sin(v) \implies h_v(u,0) = 2 - e^{-u}\sin(0) = 2$$

For the partial derivatives of H:

$$\begin{aligned}H_y &= 4x + 2ye^{-x^2+y^2}\cos(2xy) - e^{-x^2+y^2}(2x)\sin(2xy) \\ H_x &= 4y + (-2x)e^{-x^2+y^2}\cos(2xy) - e^{-x^2+y^2}(2y)\sin(2xy)\end{aligned}$$

Hence, H_y along the positive x axis and H_x along the positive y axis:

$$H_y(x>0,0) = 4x \qquad\qquad H_x(0,y>0) = 4y$$

The statement is proven because despite $f(z) = z^2$ mapping the positive x and y axis in the z plane to the positive u axis in the w plane, the value of the normal derivatives is not preserved in the transformation. That is:

$$\frac{\mathrm{d}H}{\mathrm{d}y} = H_y = 4x \neq h_v = 2 \qquad\qquad \frac{\mathrm{d}H}{\mathrm{d}x} = H_x = 4y \neq h_v = 2$$

$\square$

Example 19.3.8 *Suppose an analytic function $w = f(z) = u(x,y) + iv(x,y)$ maps $D_z \mapsto D_w$. Let $h(u,v)$ have continuous first and second order partial derivatives defined on D_w. Show*

$$H(x,y) = h[u(x,y), v(x,y)] \implies H_{xx}(x,y) + H_{yy}(x,y) = [h_{uu}(u,v) + h_{vv}(u,v)]|f'(z)|^2$$

Proof: Calculating H_x:

$$H_x = h_u u_x + h_v v_x$$

Calculating the second order partials:

$$\begin{aligned} H_{xx} &= h_{uu}u_x^2 + h_u u_{xx} + h_{uv}u_x v_x + h_{vv}v_x^2 + h_v v_{xx} + h_{vu}u_x v_x \\ H_{yy} &= h_{uu}u_y^2 + h_u u_{yy} + h_{uv}u_y v_y + h_{vv}v_y^2 + h_v v_{yy} + h_{vu}u_y v_y \end{aligned}$$

$h(u,v)$ have continuous first and second order partial derivatives, so $h_{uv} = hvu$. Since $f(z)$ is analytic, u and v satisfy the Cauchy-Riemann equations (theorem 13.5.1) and Laplace's equation (definition 13.7.1). Therefore, $u_x = v_y$, $u_y = -v_x$, $u_{xx} + u_{yy} = 0$, and $v_{xx} + v_{yy} = 0$, and we have

$$\begin{aligned} &H_{xx} + H_{yy} \\ &= h_{uu}(u_x^2 + u_y^2) + h_u(u_{xx} + u_{yy}) + 2h_{uv}(u_x v_x + u_y v_y) + h_{vv}(v_x^2 + v_y^2) + h_v(v_{xx} + v_{yy}) \\ &= h_{uu}(u_x^2 + v_x^2) + h_{vv}(u_x^2 + v_y^2) + 2h_{uv}(u_x v_x - v_x u_y) \\ &= (h_{uu} + h_{vv})(u_x^2 + v_x^2) \end{aligned}$$

Since

$$|f'(z)|^2 = |u_x + iv_x|^2 = u_x^2 + v_x^2$$

Hence

$$H_{xx} + H_{yy} = [h_{uu} + h_{vv}]|f'(z)|^2$$

By extension, this tells us that in the nontrivial case where $|f'(z)| \neq 0$:

$$H_{xx} + H_{yy} = 0 \implies h_{uu} + h_{vv} = 0$$

which is another proof for theorem 19.3.3. □

Example 19.3.9 *Let $p(u,v)$ be function with continuous first and second order partial derivatives and satisfies Poisson's Equation:*

$$p_{uu}(u,v) + p_{vv}(u,v) = \Phi(u,v)$$

Show if $w = f(z) = u(x,y) + iv(x,y)$ where $f(z) : D_z \mapsto D_w$, then

$$P(x,y) = p[u(x,y), v(x,y)] \implies P_{xx}(x,y) + P_{yy}(x,y) = \Phi[u(x,y), v(x,y)]|f'(z)|^2$$

Proof: By the example above (example 19.3.8), it is clear that

$$P_{xx} + P_{yy} = [p_{uu} + p_{vv}]|f'(z)|^2 = \Phi[u(x,y), v(x,y)]|f'(z)|^2$$

□

Example 19.3.10 *Let $w = f(z) = u(x,y) + iv(x,y)$, $f(z) : D_z \mapsto D_w$, conformal map smooth arc C to smooth arc Γ. Let $h(u,v)$ be defined on Γ and*

$$H(x,y) = h[u(x,y), v(x,y)]$$

Let s and σ be distances along C and Γ, respectively, and $\hat{t}$ and $\hat{\tau}$ be unit tangent vectors at (x,y) on C and (u,v) on Γ in direction of increasing distance. Show using fact that

$$\frac{\mathrm{d}H}{\mathrm{d}s} = \nabla H \cdot \hat{t} \qquad \frac{\mathrm{d}H}{\mathrm{d}\sigma} = \nabla H \cdot \hat{\tau}$$

That the transformed directional derivative along Γ is

$$\frac{\mathrm{d}H}{\mathrm{d}s} = \frac{\mathrm{d}h}{\mathrm{d}\sigma}|f'(z)|$$

Proof: Taking the gradient of H and using the Cauchy-Riemann equations (theorem 13.5.1):

$$\begin{aligned}
\nabla H &= H_x\hat{x} + H_y\hat{y} \\
&= (h_u u_x + h_v v_x)\hat{x} + (h_u u_y + h_v v_y)\hat{y} \\
&= (h_u u_x + h_v v_x)\hat{x} + (-h_u v_x + h_v u_x)\hat{y}
\end{aligned}$$

Taking the modulus of ∇H:

$$\begin{aligned}
|\nabla H|^2 &= (h_u u_x + h_v v_x)^2 + (-h_u v_x + h_v u_x)^2 \\
&= (h_u u_x)^2 + (h_v v_x)^2 + 2h_u h_v u_x v_x + (h_u v_x)^2 + (h_v u_x)^2 + 2h_u h_v u_y v_y \\
&= (h_u^2 + h_v^2)(u_x^2 + v_x^2) + 2h_u h_v (u_x v_x - u_x v_x) \\
&= (h_u^2 + h_v^2)(u_x^2 + v_x^2) = |\nabla h|^2 |f'(z)|^2
\end{aligned}$$

It follows that

$$|\nabla H| = |\nabla h||f'(z)|$$

Since the mapping is conformal, we know that the angle between ∇H and t is equal to the angle between ∇h and τ at the image (u,v) of point (x,y). Letting θ denote the angle:

$$\begin{aligned}
\frac{\mathrm{d}H}{\mathrm{d}s} &= \nabla H \cdot t = |\nabla H||t|\cos(\theta) \\
&= |\nabla h||f'(z)||t|\cos(\theta) && |\nabla H| = |\nabla h||f'(z)| \\
&= |f'(z)|[|\nabla h||\tau|\cos(\theta)] && \text{Conformality of transform and } |t| = |\tau| = 1 \\
&= |f'(z)|[\nabla h \cdot \tau] = \frac{\mathrm{d}h}{\mathrm{d}\sigma}|f'(z)|
\end{aligned}$$

□

19.4 Applications of Conformal Mapping

19.4.1 Time Independent Temperatures

Definition 19.4.1: Fourier's Law

Flux across a surface satisfies

$$\Phi = -K\frac{\mathrm{d}T}{\mathrm{d}N}$$

In terms of thermal flux, temperature is T, thermal conductivity is K, and surface normal vector is N.

Consider a rectangular prism of unit height with base $\Delta x \Delta y$ perpendicular to the xy plane within a solid. Here we assume:

1. The conservation of energy
2. Flow is in a steady state (temperature is time independent)
3. No heat sources or sinks in the solid

The heat flow across the surface $\Delta x \Delta y$ from x to $x + \Delta x$ is then

$$\begin{aligned} \text{Right-hand side: } & -KT_x(x,y)\Delta y \\ \text{Left-hand side: } & -KT_x(x+\Delta x, y)\Delta y \end{aligned}$$

The net heat loss though the faces can be written as

$$-K\left[\frac{T_x(x+\Delta x,y) - T_x(x,y)}{\Delta x}\right]\Delta x\Delta y = -KT_{xx}(x,y)\Delta x \Delta y$$

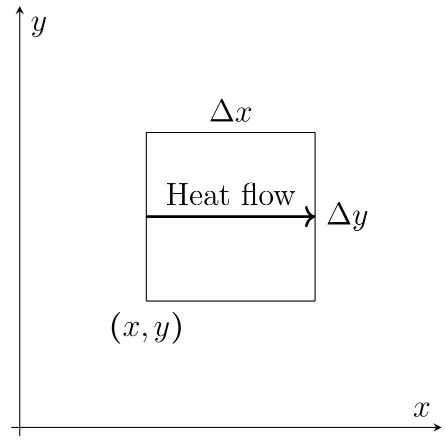
Likewise, for the heat flow across $\Delta x \Delta y$ from y to $y + \Delta y$

$$-KT_{yy}(x,y)\Delta x\Delta y$$

The flux through the surface is zero since temperatures are steady and no heat sources and sinks exist within $\Delta x \Delta y$, so the sum is

$$-KT_{xx}(x,y)\Delta x\Delta y - KT_{yy}\Delta x \Delta y = 0 \implies T_{xx}(x,y) + T_{yy}(x,y) = 0$$

Hence, T is harmonic function of x and y.



Definition 19.4.2: Isotherm

Level curves of the function T. Surfaces where

$$T(x,y) = c_1 \qquad\qquad c_1 \in \mathbb{R}$$

Definition 19.4.3: Lines of Flow

The curves where $S(x,y) = c_2$, $c_2 \in \mathbb{R}$, has $\nabla T(x,y)$ as the tangent vector where $T(x,y) + iS(x,y)$ is conformal. $S(x,y)$ is the harmonic conjugate of $T(x,y)$.

Note: The normal $\mathrm{d}T/\mathrm{d}N = 0$ on where the heat flux is zero. The part is thermally insulated and is a line of flow.

19.4.2 Steady Temperatures in a Half Plane

The temperature $T(x,y)$ in a thin semi-infinite plate $y \geq 0$ where $T = 0$ for $y = 0$ except for $y \in (-1,1)$ where $T = 1$. We can consider the half plane as the limiting case of the plate $0 \leq y \leq y_0$. We can assume $T(x,y) \to 0$ as $y \to \infty$.

We can write the boundary value problem as

$$T_{xx}(x,y) + T_{yy}(x,y) = 0 \qquad\qquad x \in (-\infty, \infty),\ y > 0$$

$$T(x,0) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

Note: $|T(x,y)| < M$, $M \in \mathbb{R}$. That is, T is bounded.

This is a Dirichlet problem for upper half plane $(y > 0)$. We will transform this problem into a Dirichlet problem in the uv plane, while preserving all the properties in that of the xy plane. The region will be the image of the half plane under the transformation $w = f(z)$ that is analytic in $y > 0$, conformal along $y = 0$ except at $(0, \pm 1)$, where $f(z)$ is undefined.

A harmonic function of u and v will be transformed into a harmonic function of x and y, with boundary conditions in the uv plane preserved on corresponding portions of the boundary in the xy plane.

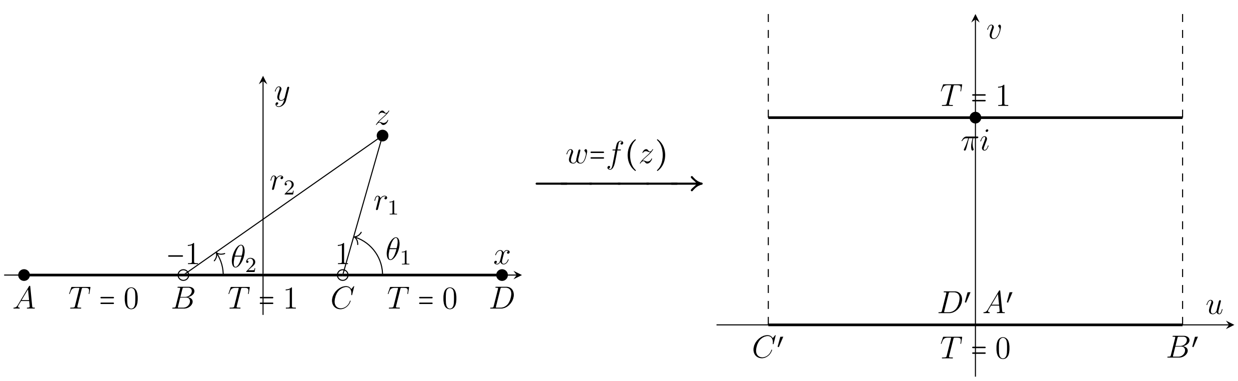
Let

$$z - 1 = r_1 e^{i\theta_1} \qquad z + 1 = r_2 e^{i\theta_2} \qquad 0 \le \theta_k \le \pi,\ \ k \in \{1, 2\}$$

Transformation defined for $y > 0$ except for $z = \pm 1$

$$w = \log\left(\frac{z-1}{z+1}\right) = \ln\left(\frac{r_1}{r_2}\right) + i(\theta_1 + \theta_2) \qquad \frac{r_1}{r_2} > 0,\ -\frac{\pi}{2} < \theta_1 - \theta_2 < \frac{3\pi}{2}$$

is analytic and conformal (see definition 19.1.1). The value of the logarithm is principle, so the upper half plane $y > 0$ is mapped to $v \in (0, \pi)$. The line between -1 and 1, ($x \in (-1, 1),\ y = 0$), is mapped to $\theta_1 - \theta_2 = \pi$, the upper edge of the strip. The rest of the x axis, $y = 0$ is mapped to $\theta_1 - \theta_2 = 0$, the lower edge of the strip.



A bounded harmonic function, $T(u, v)$, with conditions $T(u, 0) = 0$ and $T(u, \pi) = 1$.

$$T = \frac{1}{\pi} v$$

Which is harmonic due to it being the imaginary component of the entire function w/π. Changing to x and y coordinates:

$$w = \ln\left|\frac{z-1}{z+1}\right| + i \arg\left(\frac{z-1}{z+1}\right)$$

Then

$$v = \arg\left[\frac{(z-1)(\bar{z}+1)}{(z+1)(\overline{z+1})}\right] = \arg\left[\frac{x^2 + y^3 - 1 + i2y}{(x+1)^2 + y^2}\right] = \arctan\left(\frac{2y}{x^2 + y^2 - 1}\right) \qquad v \in [0, \pi]$$

We have $v \in [0, \pi]$ since

$$\arg\left(\frac{z-1}{z+1}\right) = \theta_1 - \theta_2 \qquad \theta_1 - \theta_2 \in [0, \pi]$$

The transformation

$$T = \frac{1}{\pi} \arctan\left(\frac{2y}{x^2 + y^2 - 1}\right) \qquad \arctan(t) \in [0, \pi]$$

We can then apply theorem 19.3.3 to show that T is harmonic in the half plane. Boundary conditions are the same on the boundaries due to they are the type $h = h_0$ (theorem 19.3.4).

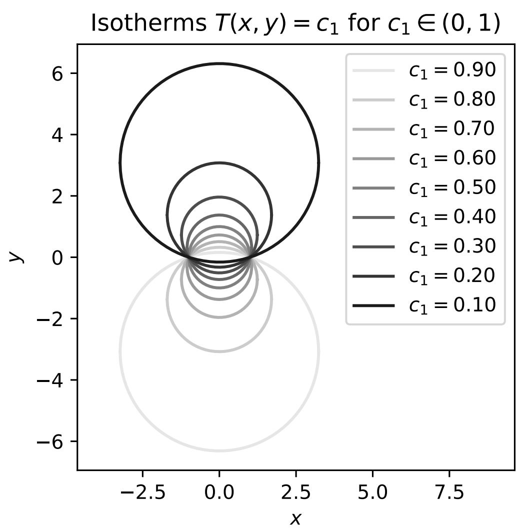
Isotherms $T(x, y) = c_1$, $c_1 \in (0, 1)$ are arcs of circles:

$$x^2 + [y - \cot(\pi c_1)]^2 = \csc^2(\pi c_1)$$

passing through $(\pm 1, 0)$ with centers on the y axis.

Product of harmonic function with a constant is harmonic, so if we replace $T = 1$ with $T = T_0$ along the line segment $x \in (-1, 0)$:

$$T = \frac{T_0}{\pi} \arctan\left(\frac{2y}{x^2 + y^2 - 1}\right) \qquad \arctan(t) \in [0, \pi]$$



Related Problem

Part V

Ordinary Differential Equations [Empty]

Theorem 19.4.1: Green's Theorem

Let $F = P(x,y)\hat{i} + Q(x,y)\hat{j}$ be a vector field on a simple closed contour C, R be the region enclosed and on C, and s be the path along C.

$$\int_C F \cdot ds = \iint_R \nabla \times F \; dx \; dy$$

Part VI

Nonlinear Dynamics [Empty]

Part VII

Partial Differential Equations [Empty]

Definition 19.4.4: Dirichlet Problem

Finding a function in an harmonic domain that assumes preassigned values at the boundary of the domain.

Theorem 19.4.2: Fourier Theorem

Let a function f:

1. *Piecewise continuous on $[-\pi, \pi]$*
2. *Periodic with period 2π $\forall x \in \mathbb{R} \cup \{-\infty, \infty\}$*
3. *$\forall x \in \mathbb{R} \cup \{-\infty, \infty\}$, $f'_+(x)$ and $f'_-(x)$ both exist*

Then the Fourier series converges to the mean value

$$\frac{f(x+) + f(x-)}{2}$$

of one-sided limit of f at x

Calculus of Variations [Empty]

Part VIII

Integral Equations [Empty]

Part IX

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Chapter 20

Markov Chains

Part X

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Part XI

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Part XII

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22.1 Ideals

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Tips and Tricks [Empty]

Chapter 30

Integration Techniques

30.1 DI Method (Integration Table)

30.2 Feynman Integration

Part XX

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