# The Book of Math (Notes)

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# Forward and Disclaimer

These are math notes made by a student (with a physics major and math minor) based off text books. It may contain misconceptions and misinterpretations, thus should not be viewed in the same light of a text book. Use at your own risk and mental sanity.

# Symbols

## Logic

| Name | Symbol | Comment |
|---|---|---|
| Exists | $\exists$ | There exists at least one |
| For all | $\forall$ | |
| Not exists | $\nexists$ | There does not exist |
| Exists one | $\exists!$ | There only exists one and only one |
| And | $\wedge$ | |
| Or | $\vee$ | Inclusive or |
| Not | $\neg$ | |
| Logically implies | $\implies$ | If |
| Logically implied by | $\impliedby$ | Only if |
| Logically equivalent | $\iff$ | If and only if |
| Implies | $\longrightarrow$ | |
| Implied by | $\longleftarrow$ | |
| Double Implication | $\longleftrightarrow$ | |

## Set Notation

| Name | Symbol | Comment |
|---|---|---|
| Empty Set | $\varnothing$ | The set that is empty |
| Natural Numbers | $\mathbb{N}$ | Set of natural numbers not containing 0, equivalent to the set of positive integers |
| Integers | $\mathbb{Z}$ | Set of integers |
| Rational Numbers | $\mathbb{Q}$ | |
| Algebraic Numbers | $\mathbb{A}$ | |
| Real Numbers | $\mathbb{R}$ | |
| Complex Numbers | $\mathbb{C}$ | |
| In | $\in$ | |
| Not in | $\notin$ | |
| Owns | $\ni$ | Has an element |
| Proper Subset | $\subset$ | Subset that is not itself |
| Subset | $\subseteq$ | |
| Superset | $\supset$ | Superset that is not itself |
| Proper Superset | $\supseteq$ | |

| | | |
|---|---|---|
| Power set | $\wp$ | |
| Union | $\cup$ | |
| Intersection | $\cap$ | |
| Difference | $\setminus$ | |

## Relationships

| Name | Symbol | Comment |
|---|---|---|
| Defined | $\doteq$ | |
| Approximate | $\approx$ | |
| Equivalent | $\equiv$ | Isomorphic (Group Theory) |
| Congruent | $\cong$ | Homomorphic (Group Theory) |
| Proportional | $\propto$ | |

## Operators

| Name | Symbol | Comment |
|---|---|---|
| | $\oplus$ | |
| | $\otimes$ | |
| | $\odot$ | |
| | $\circ$ | Convolution |
| Dagger | $\dagger$ | Complex conjugate transpose of a matrix |

## Arrows

| Name | Symbol | Comment |
|---|---|---|
| Maps to | $\mapsto$ | |

## Hebrew

| Name | Symbol | Comment |
|---|---|---|
| Aleph | $\aleph$ | Carnality of infinite sets that can be well ordered |

## Other

| Name | Symbol | Comment |
|---|---|---|
| Real part | $\Re$ | Real part of a number |
| Imaginary part | $\Im$ | Imaginary part of a number |

# Book Constitution

## Intents and Purpose

The goal of this book is to organize mathematical knowledge of topics related to the study of physics or the author's interest. It is meant to be used as a source of for future reference, not as a textbook for students new to the topics. It is a notebook of a student, thus should be treated as one and not as a textbook. At most, it could be used as a study guide along side a textbook. Definitely not as the main source for acquiring knowledge.

## Layout and Organization

The book is split into parts each containing a field of study mathematics, or a topic large enough to justify giving it its own part. Each part contains chapters that focuses on a particular topic required to understand the field, with sections dedicated to describing a particular knowledge required for the topic.

As axioms, definitions, theorems, corollary, and proofs are integral and abundant to the study of mathematics, each will have a unique style. Each environment and its styles are displayed as follows:

**Axiom 0.1: Axiom name**
*Example Axiom Axioms are the "ground rules" of the set.*

**Theorem 0.0.1: Theorem name or citation**
*Example Theorem An important logical result from the axioms, with proof.*

**Conjecture 0.0.1: Name of conjecture or citation**
*Example Conjecture A hypothesis, without proof.*

**Corollary 0.0.1.1:**
*Example Corollary An implication as a result of a theorem.*

**Lemma 0.0.1.1:**
*Example Lemma Small theorems that build up to a larger theorem.*

**Proposition 0.0.1.1:**
*Example Proposition Example proposition.*

*Proof:* Logical deductions that results in a theorem. Proofs I've written will be in grey, which may or may not be correct. $\square$

**Definition 0.0.1: Word**
*Example Definition The definition of a word.*

**Example 0.0.1** *An example.*

**Remark.** *Remark A comment by the author in the textbooks used.*

**Observation.** *Example Observation A remark by me.*

**Question.** *Example Question A question from me for a mystery to be answered later.*

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# Proofs

# Part II

# Numbers

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# Chapter 2

# Natural $\mathbb{N}$

# Chapter 3

# Integers $\mathbb{Z}$

# Chapter 4

# Rationals $\mathbb{Q}$

# Chapter 5

# Constructible

# Chapter 6

# Algebraic $\mathbb{A}$

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# Reals $\mathbb{R}$

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# Complex $\mathbb{C}$

# Part III

# Real Analysis

# Resources used in part III

1. Kenneth A. Ross - Elementary Analysis (2nd Ed.) [1]

# Chapter 9

# Sequences

### 9.0.1 Limits

#### Limit Theorems

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# Metric Spaces

# Part IV

# Complex Analysis

# Resources used in part IV

Primary:

1. Brown and Churchill - Complex Variables and Applications [2]

Supplement:

1. A. David Wunsch - Complex Variables with Applications [3]

# Chapter 11

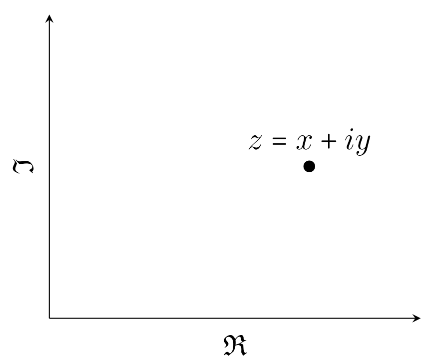
# Basics

## 11.1 Complex Numbers

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}, i = \sqrt{-1}\}$$

Complex numbers are elements of the complex field ($\mathbb{C}$), therefore, they obey all the properties of a field.

We will denote complex numbers by $z = x + iy$ with $x, y \in \mathbb{R}$, and refer the real part as $\mathfrak{R}(z) = \mathrm{Re}(z) = x$ and imaginary part as $\mathfrak{I}(z) = \mathrm{Im}(z) = y$. Complex numbers can also be defined as an ordered pair $z = (x, y)$ which is interpreted as points in the complex plane. $(x, 0)$ are points on the real axis while $(0, y)$ are points in the imaginary axis. This expression is often called a Couple, and was presented in 1833 by mathematician William Rowan Hamilton (1805 - 1865).




Like numbers in $\mathbb{R}$, numbers in $\mathbb{C}$ obey the commutative, distributive, and associative laws. We add and multiply complex numbers in the usual way:

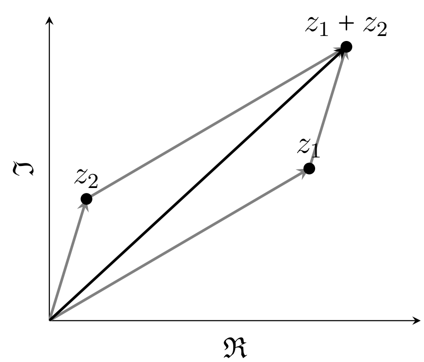
$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \end{aligned} \qquad \begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \end{aligned}$$

$\forall z \in \mathbb{C}$, there is an unique additive inverse $(-z)$ and $\forall z \in \mathbb{C} \setminus \{0\}$, there is an unique multiplicative inverse $(z^{-1})$ such that

$$z + (-z) = 0 \qquad\qquad zz^{-1} = 1$$
$$\implies -z = -x - iy \qquad \implies (x_1x_2 - y_1y_2) = 1 \wedge (x_1y_2 + x_2y_1) = 0$$
$$\implies z^{-1} = \frac{x_1}{x_1^2 + y_1^2} - i\frac{y_1}{x_1^2 + y_1^2}$$

The existence and uniqueness of the inverses can be easily proven.

The addition of complex numbers may also be interpreted as akin to vector addition.




## 11.2 Triangle Inequality

It is not analysis without a section dedicated to the triangle inequality. For any given number $z_1, z_2 \in \mathbb{C}$ it makes no sense to write an inequality $z_1 = a_1 + ib_1 < a_2 + ib_2 = z_2$. Thus, we need have a different notion of size.

**Definition 11.2.1: Modulus**

*The modulus of a complex number is a function $\mathbb{C} \to \mathbb{R}_{>0}$:*

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$

It is obvious why the definition is not $|z| = \sqrt{x^2 + (iy)^2}$ as problems arise when $x = y$. The modulus is the distance of $z$ from $(0,0)$. $\bar{z}$ is the complex conjugate of $z$, which is explored in section 11.5

**Theorem 11.2.1: Triangle Inequality**

$\forall z_1, z_2 \in \mathbb{C}[|z_1 + z_2| \leq |z_1| + |z_2|]$

From the theorem, we can derive a similar inequality:

$$|z_1| = |z_1 + z_2 - z_2| \leq |z_1 + z_2| + |-z_2| \implies |z_1| - |z_2| \leq |z_1 + z_2|$$

An important property of polynomials is observed when theorem 11.2.1 is applied to polynomials.

**Corollary 11.2.1.1:**
*Consider the polynomial $P(z)$ where $a_n \in \mathbb{C}$, $n \in \mathbb{N}$, $a_0 \neq 0$, and $z \in \mathbb{C}$.*

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

*Then $\forall z, \exists R \in \mathbb{R}_{>0}, |z| < R$ such that*

$$\left| \frac{1}{P(z)} \right| < \frac{2}{|a_n| R^n}$$

*Proof:* Consider

$$\begin{aligned}
w = \frac{P(z)}{z_n} - a_n = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \ldots + \frac{a_{n-1}}{z} && z \neq 0 \\
\implies wz^n = a_0 + a_1 z + \ldots + a_{n-1} z^{n-1} \\
\implies |w||z|^n \leq |a_0| + |a_1||z| + \ldots + |a_{n-1}||z|^{n-1} \\
\implies |w| \leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \ldots + \frac{|a_{n-1}|}{|z|} \\
\implies |w| < n \frac{|a_n|}{2n} = \frac{|a_n|}{2} && \exists \text{ sufficiently large } R < |z| \text{ s.t.} \\
&& \forall m,\ 0 \leq m \leq n-1,\ \frac{|a_m|}{|z|^{n-m}} < \frac{|a_n|}{2n} \\
\implies |a_n + w| \geq ||a_n| - |w|| > \frac{|a_n|}{2} && R < |z| \\
\implies |P_n(z)| = |a_n + w||z|^n > \frac{|a_n|}{2}|z|^n > \frac{|a_n|}{2} R^n && R < |z| \\
\implies \left| \frac{1}{P(z)} \right| < \frac{2}{|a_n| R^n}
\end{aligned}$$

□

This tells us that if $z$ is a solution to a polynomial $P(z)$, then the reciprocal of the polynomial $1/P(z)$ is bounded above by $R = |z|$. (i.e. It is bounded by a circle of radius $|z|$.)

## 11.3 Polar and Exponential Form

**Definition 11.3.1: Argument of $z$**
*Consider any $z \in \mathbb{C}$ where $z \neq 0$. Let $\theta$ be the angle in radians between $z$ and the real axis . Then $\forall n \in \mathbb{N}$, $-\pi < \theta \leq \pi$, the argument of $z$:*

$$\arg(z) = \theta + 2n\pi$$

We know $\forall n \in \mathbb{N}$, $\theta + 2\pi n = \theta$. This leads us to the definition of the principal argument of $z$.

**Definition 11.3.2: Principal Argument of $z$**

*Consider any $z \in \mathbb{C}$ where $z \neq 0$. Let $\theta$ be the angle in radians between $z$ and the real axis. Then for $-pi < \theta \leq \pi$, the principal argument of $z$:*

$$\text{Arg}(z) = \theta$$

It is clear that $\arg(z) = \text{Arg}(z) + 2n\pi$. It is common for the principal argument to be defined $-\pi < \theta \leq \pi$, although other definitions use $0 \leq \theta < 2\pi$.

**Definition 11.3.3: Polar Form of $z$**

*Consider $z \in \mathbb{C}$. Let $r = |z|$, and $\theta = \arg(z)$. Then $\forall z \in \mathbb{C}, z \neq 0$:*

$$z = x + iy = r(\cos(\theta) + i\sin(\theta))$$

Notice that all three definitions require that $z \neq 0$ as $\theta$ is undefined at $z = 0$.

**Theorem 11.3.1: Euler's Formula**

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

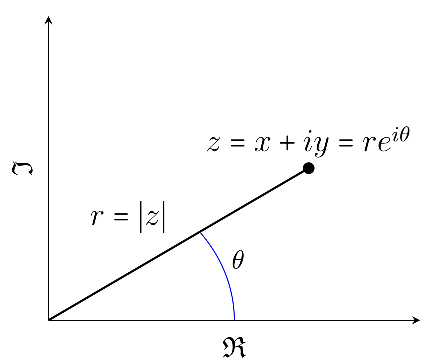
Combining definition 11.3.3 with theorem 11.3.1, we obtain the Exponential Form of $z$:

**Definition 11.3.4: Exponential Form of $z$**

*Consider any $z \in \mathbb{C}$, and let $r = |z|$ and $\theta = \text{Arg}(z)$. Then the exponential form of $z$:*

$$z = re^{i\theta}$$

Note: $\theta = \tan^{-1}(y/x)$ and $r = \sqrt{x^2 + y^2}$.



### 11.3.1 Properties of Polar and Exponential Form

It would be easier to work with the exponential form of $z$ then convert it to the polar form later. The exponential form of a complex number is part of the exponential family of functions, thus possess all the properties of the family. Consider any complex number $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$.

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} \qquad\qquad z^n = r^n e^{in\theta} \qquad \forall n \in \mathbb{Z}$$

A special case arrives for integer exponential of $z$ on the unit circle.

**Theorem 11.3.2: de Moivre's Formula**

*Consider any $z = e^{i\theta} \in \mathbb{C}$ on the unit circle, and let $n \in \mathbb{Z}$.*

$$\forall z \in \mathbb{C}\ \forall n \in \mathbb{Z}\left[|z| = 1 \implies (\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)\right]$$

*Proof:* Consider $z = e^{i\theta}$ and let $n \in \mathbb{Z}$.

$$z^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

□

The proof hints that theorem 11.3.2 can be generalized to $\forall n \in \mathbb{R}$, which we will see shortly in **??**. Using theorem 11.3.2, we can obtain the double angle identities.

**Corollary 11.3.2.1: Double Angle Identities**

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \qquad\qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

*Proof:* Consider any $z$ on the unit circle, that is $z = e^{i\theta}$.

$$\begin{aligned} &(\cos(\theta) + i\sin(\theta))^2 = \cos(2\theta) + i\sin(2\theta) && \text{Theorem 11.3.2} \\ \implies &\cos^2(\theta) - \sin^2(\theta) + i2\sin(\theta)\cos(\theta) = cos(2\theta) + i\sin(2\theta) \end{aligned}$$

Equating the real and imaginary parts yield the desired results. □

### 11.3.2 Properties of Arguments

Recall from section 11.3.1:

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} \qquad\qquad z^n = r^n e^{in\theta} \qquad \forall n \in \mathbb{Z}$$

The arguments for the arguments of products of any $z_1, z_2 \in \mathbb{C}$ follows immediately from the properties of the exponential.

**Corollary 11.3.2.2: Arguments of Products**

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \qquad \operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$$
$$\arg(z^n) = n\arg(z) \qquad \operatorname{Arg}(z^n) = n\operatorname{Arg}(z)$$

*Proof:*

$$\begin{aligned}
z_1 z_2 &= r_1 r_2 e^{i(\theta_1+\theta_2)} \\
&\implies \arg(z_1 z_2) = \arg(z_1) + 2n_1\pi + \arg(z_2) + 2n_2\pi \qquad n_1, n_2 \in \mathbb{Z} \\
&\implies \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \\
&\implies \operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) = \operatorname{Arg}(z_2)
\end{aligned}$$

$$\begin{aligned}
z^n &= r^n e^{in\theta} \\
&\implies \arg(z^n) = n\arg(z) + 2n\pi \qquad n \in \mathbb{Z} \\
&\implies \arg(z^n) = n\arg(z) \\
&\implies \mathrm{z^n} = n\operatorname{Arg}(z)
\end{aligned}$$

□

It is clear that:

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) \qquad \operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \operatorname{Arg}(z_1) - \operatorname{Arg}(z_2)$$

## 11.4 Roots of $z$

In definition 11.3.4, you might be wondering why $z^n = r^n e^{in\theta}$ is not for $n \in \mathbb{R}$. That is because there is more things to consider, which we will explore in this section. Recall that $z = re^(i\theta) = re^{i(\theta+2n\pi)}$ for $n \in \mathbb{Z}$.

**Definition 11.4.1: Exponential of $z$**

*Consider any $z \in \mathbb{C}$ and any $x \in \mathbb{R}$*

$$z^x = \left(re^{i(\theta+2n\pi)}\right)^x = r^x e^{ix(\theta+2n\pi)}$$

For $x \notin \mathbb{Z}$, it is clear that $z^x = r^x e^{ix(\theta+2n\pi)} \neq r^x e^{ix\theta}$, since $2nx\pi = 0 \iff nx \in \mathbb{Z}$. In order to define the roots of $z$ we must need a more general and proper definition of $z$.

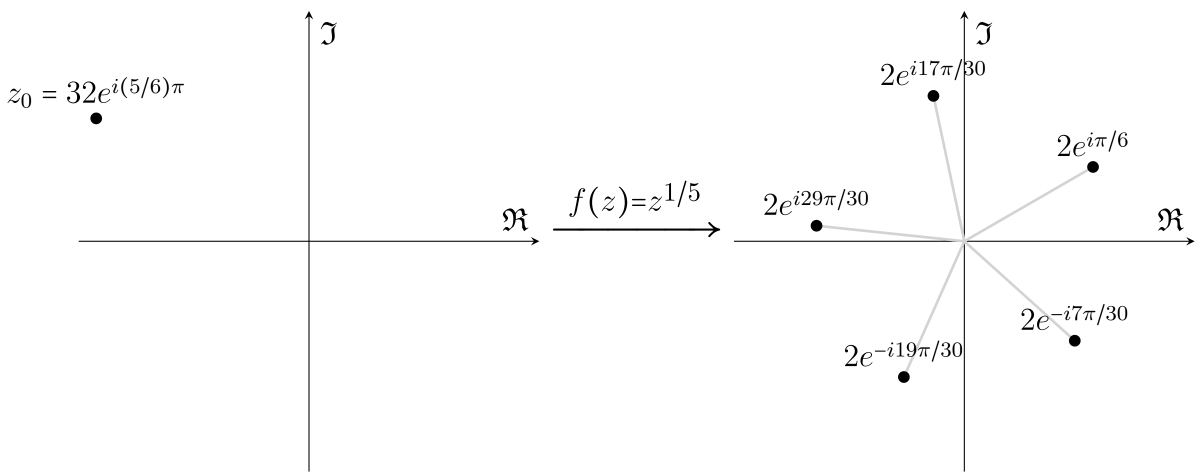
**Definition 11.4.2: Roots of $z_0$**

*Consider any $z_0 \in \mathbb{C}$ and any $m \in \mathbb{N}$.*

$$z_0^{\frac{1}{m}} = r_0^{\frac{1}{m}} e^{i\left(\frac{\theta_0 + 2n\pi}{m}\right)} = r_0^{\frac{1}{m}} e^{i\left(\frac{\theta_0}{m} + \frac{2n\pi}{m}\right)}$$

Taking the $m$-th root of $z_0 \in C$ scales $\theta_0$ by $1/m$, and provides solutions at equally spaced by $2\pi/m$ on a circle of radius $r^{1/m}$. That is, the roots lie on the vertices of a regular n-sided polygon inscribed in a circle of radius $|z|^{1/m}$.

**Example 11.4.1** *Consider $z_0 = 32e^{i(5/6)\pi}$, then $z_0^{(1/5)} = 3e^{i(\pi/6)+i(2/5)n\pi}$ for $n \in \mathbb{Z}$. The radius went from 35 to $35^{(1/5)} = 2$, and five roots appear equally spaced with distance of $(2/5)\pi$ on a circle with radius 2. Before and after graphs are as follows, note graph on right is zoomed in:*




We can see that the roots of $z_0$ form a set:

**Definition 11.4.3: Set of roots of $z_0$**

*Consider the $m$-th root of any $z_0 \in \mathbb{C}$. Let:*

$$z_0 = r_0 e^{i\theta_0} \qquad c_0 = r_0^{1/m} e^{i\theta_0/m} \qquad \omega_n = e^{\frac{i2\pi}{m}} \qquad m \in \mathbb{N}$$

*Then the set of roots of $z_0$:*

$$z_0^{1/m} = \left\{ c_k = c_0 \omega_m^k \mid k \in \mathbb{N},\ 0 \le k < m \right\}$$

$c_0$ is the principal root. The root corresponding to the principal argument of $z$.

**Definition 11.4.4: Principal Root**

*Consider the $m$-th root of any $z_0 \in \mathbb{C}$. The principal root of $z_0$ is defined as:*

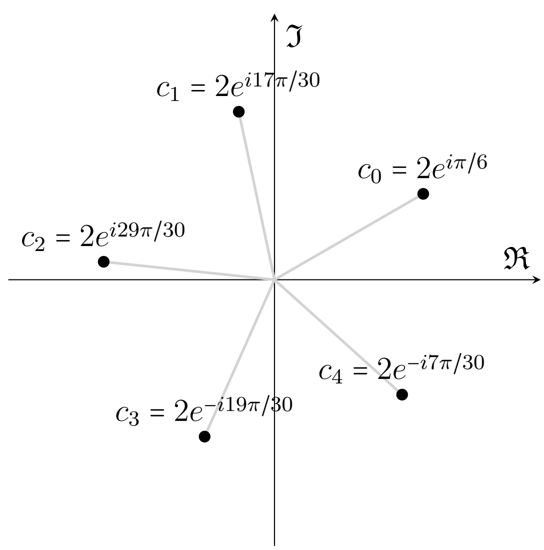
$$c_0 = r_0^{\frac{1}{m}} e^{i\frac{\theta_0}{m}}$$

**Example 11.4.2** *Recall from the previous example:* $z_0 = 32e^{i(5/6)\pi}$. *This gives us*

$$c_0 = 32^{1/5}e^{i\pi/6} = 2e^{i\pi/6} \qquad\qquad \omega_5 = e^{i2\pi/5}$$

*Then*

$$\begin{aligned}
c_0 &= c_0\omega_5^0 = 2e^{i\pi/6}\\
c_1 &= c_0\omega_5^1 = 2e^{i\pi/6}e^{i2\pi/5} = 2e^{i17\pi/30}\\
c_2 &= c_0\omega_5^1 = 2e^{i\pi/6}e^{i4\pi/5} = 2e^{i29\pi/30}\\
c_3 &= c_0\omega_5^1 = 2e^{i\pi/6}e^{i6\pi/5} = 2e^{i41\pi/30} = 2e^{-i19\pi/30}\\
c_4 &= c_0\omega_5^1 = 2e^{i\pi/6}e^{i8\pi/5} = 2e^{i53\pi/30} = 2e^{-i7\pi/30}
\end{aligned}$$



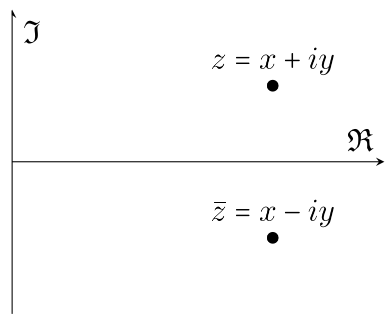

## 11.5 Complex Conjugate

**Definition 11.5.1: Complex Conjugate**

*The complex conjugate of* $z \in \mathbb{C}$ *is denoted* $\bar{z}$.

$$\bar{z} = x - iy = r(\cos(\theta) - i\sin(\theta)) = re^{-i\theta}$$

Graphically, it is the reflection of $z$ across the real axis.




It is then easy to see

$$\mathrm{Re}(z) = \frac{z + \bar{z}}{2} \qquad \mathrm{Im}(z) = \frac{z - \bar{z}}{2i} \qquad |z|^2 = z\bar{z}$$

As $\mathrm{Re}(z) = x = r\cos(\theta)$ and $\mathrm{Im}(z) = y = r\sin(\theta)$ and using definition 11.3.4, we can obtain the complex forms of sine and cosine:

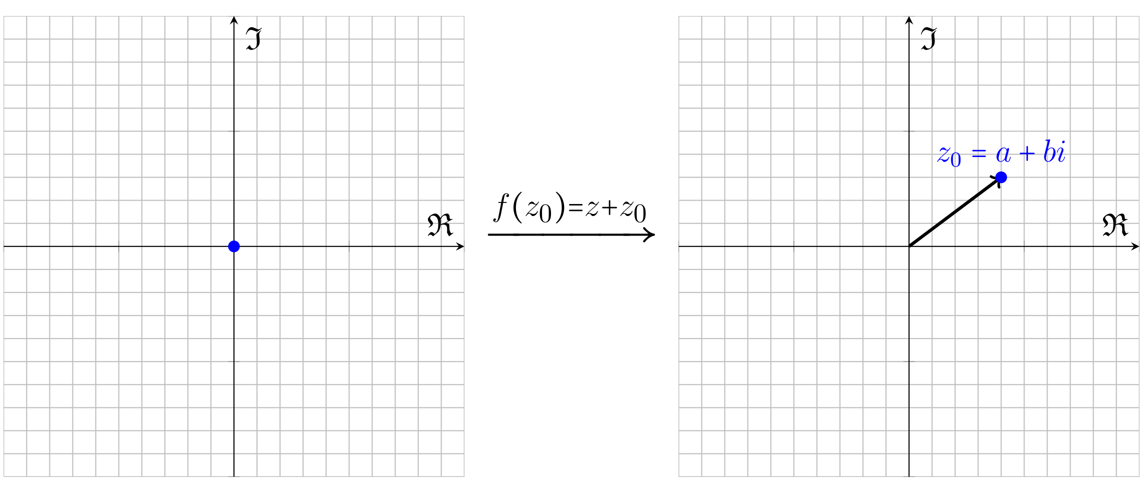
**Definition 11.5.2: Complex Sine and Cosine**

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

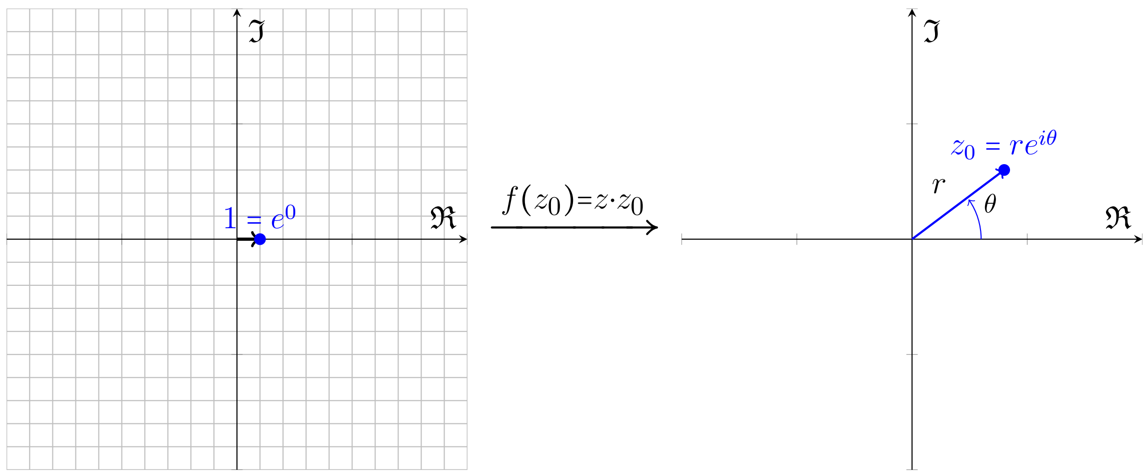
## 11.6 Operations as Transformations

Consider any $z \in \mathbb{C}$. A function $f : \mathbb{C} \to \mathbb{C}$ can be viewed as transformations of the complex plane.

**Example 11.6.1** (Addition as translation) *Consider any $z_0 \in \mathbb{C}$, $z_0 = a + ib$ for $a, b \in \mathbb{R}$. Addition by $z_0$ can be seen as a shift in the complex plane by $a + bi$. (i.e. It takes the origin and shifts it by $z_0$.)*



**Example 11.6.2** (Multiplication as scaling and rotation) *Consider any $z_0 \in \mathbb{C}$, $z_0 = re^{i\theta}$. Multiplication by $z_0$ scales the entire complex plane by $r$ and rotates it by $\theta$. (Imagine rotating and stretching out a net.)*



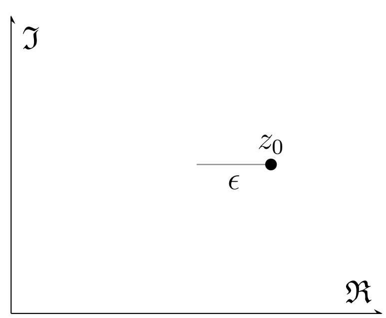

## 11.7 Complex Analysis Definitions

### Definition 11.7.1: Neighbourhood

*A neighbourhood of a point $z_0$ is the set of all points $z$ with distance less than $\epsilon$.*

$$\{z : |z - z_0| < \epsilon\}$$

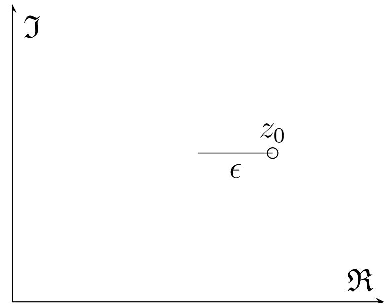
*i.e. It is the set of all points that lie within a circle centred at $z_0$ with radius $\epsilon$. Points on the circumference not included.*




### Definition 11.7.2: Deleted Neighbourhood

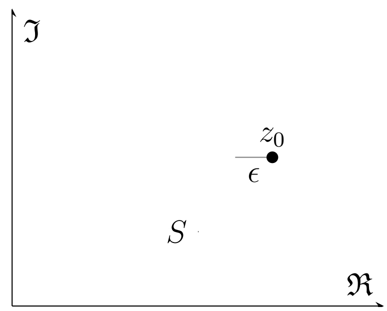
*A deleted neighbourhood is the set of all points $z$ with distance less than $\epsilon$ from a point $z_0$, not including $z_0$. That is, it is a neighbourhood of $z_0$ without $z_0$.*

$$\{z : |z - z_0| < \epsilon,\ z \neq z_0\}$$



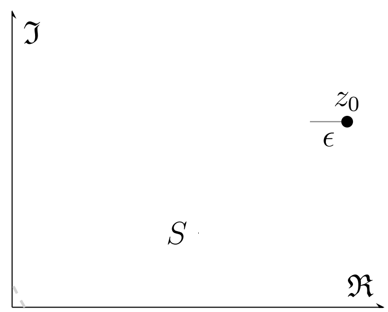

**Definition 11.7.3: Interior Point**

*Let $S$ be a set. A point $z_0$ is an interior point of $S$ if $\exists\epsilon$ such that $\forall z$, $|z - z_0| < \epsilon \implies z \in S$. That is, $z_0$ is an interior point of $S$ if it has a neighbourhood where all points in the neighbourhood is an element of $S$.*



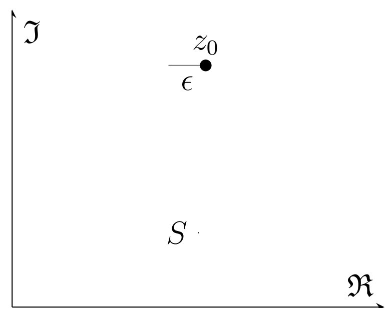

**Definition 11.7.4: Exterior Point**

*Let $S$ be a set. A point $z_0$ is an exterior point of $S$ if $\exists\epsilon$ such that $\forall z$, $|z - z_0| < \epsilon \implies z \notin S$. That is, $z_0$is an exterior point of $S$ if it has neighbourhood that does not contain any element of $S$.*




**Definition 11.7.5: Boundary Point**

*Let $S$ be a set. A point $z_0$ is a boundary point of $S$ if $\forall\epsilon$, $\exists z \in S, z' \notin S$, such that $|z - z_0|\epsilon$ and $|z' - z_0| < \epsilon$. That is, for all neighbourhoods of $z_0$ there exists a point that is in $S$ and a point not in $S$.*




Note: A boundary point of $S$ may or may not be in $S$.

**Definition 11.7.6: Boundary of a Set**

*A boundary of a set $S$ is the set of all boundary points of $S$. The set containing all boundary points of $S$.*

$$\{z_0 : \forall\epsilon\exists z \in S, z' \notin S(|z - z_0| < \epsilon \wedge |z' - z_0| < \epsilon)\}$$

**Definition 11.7.7: Open Set**

*A set that does not contain any boundary points.*

**Theorem 11.7.1:**

*Set $S$ is open $\iff$ $\forall s \in S$, $s$ is an interior point of $S$*

*Proof:* $\underline{\implies}$: Suppose $S$ is open $\not\Rightarrow \forall s \in S$, $s$ is an interior point of $S$, for contradiction. That is, $\exists s \in S$ that is either a boundary point or an exterior point. $s \in S$ implies $s$ is not an exterior point of $S$, so $s$ has to be a boundary point of $S$. This contradicts that $S$ is an open set.

$$S \text{ is open} \implies \forall s \in S(s \text{ is an interioir point of } S)$$

$\underline{\impliedby:}$

$$\begin{aligned}
&\forall s \in S(s \text{ is an interior point of S})\\
&\implies \forall s'\forall\epsilon(|s' - s| < \epsilon \implies s' \in S)\\
&\implies S \text{ does not contain boundary points} \implies S \text{ is open}
\end{aligned}$$

□

A set can be neither open or closed. Consider the set $S = \{z : 0 < |z| \le 1\}$. $S$ is not closed since it does not contain the boundary point 0, and it is not open since it contains boundary points where $|z| = 1$. The set $\mathbb{C}$ is both open and closed since it has no boundary points.

**Definition 11.7.8: Closed Set**

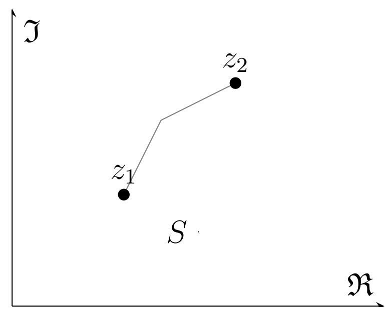
*A set that contains all of its boundary points.*

**Definition 11.7.9: Closure of a Set**

*Let $S$ be a set. The closure of $S$ is a closed set containing all points of $S$ and all boundary points of $S$.*

**Definition 11.7.10: Connected Set**

*An opens set $S$ is connected if $\forall z_1, z_2 \in S$, $z_1$ and $z_2$ can be connected by a polygonal line lying within $S$.*




**Definition 11.7.11: Polygonal Line**

*A finite set of line segments joined end to end.*

**Definition 11.7.12: Domain**

*A nonempty connected set.*

Note: All neighbourhoods are domains.

**Definition 11.7.13: Region**

*A domain with none, some, or all of its boundary points.*

**Definition 11.7.14: Closed Region**

*A domain with all of its boundary points.*

**Definition 11.7.15: Bounded Set**

*A set $S$ is bounded if $\exists R$ such that $\forall s \in S$, $s < R$.*

**Definition 11.7.16: Closed Region**

*A bounded and closed region.*

**Definition 11.7.17: Accumulation/Limit Point**

*A point $z_0$ is a accumulation point of a set $S$ if all deleted neighbourhood of $z_0$ contains an element of $S$.*

$$\forall \epsilon \exists s \in S (s \neq z_0 \wedge |z - s| < \epsilon)$$

Note: Unlike a boundary point, an accumulation point does not require that all neighbourhood of $z_0$ contain an element not in $S$.

**Theorem 11.7.2:**

*Set $S$ is closed $\iff$ $\forall$ accumulation points $z_0$ of $S$, $z_0 \in S$*

*Proof:* $\underline{\implies:}$ Let $S$ is closed and $z_0$ is an accumulation point of a set $S$ where $z_0 \notin S$ for contradiction. If $\exists z_0 \notin S$, then $z_0$ is a boundary point of $S$. Contradicts closed set contains all boundary points.

$\underline{\impliedby:}$ Suppose all accumulation points of $S$ are elements of $S$ but $S$ is not closed for contradiction. Then $S$ does not contain one or more boundary points. Suppose $z_0$ is a boundary point of $S$ that is not in $S$. Then $\forall \epsilon\ \exists s \in S$ where $|s - z_0| < \epsilon$, so by considering the deleted neighbourhood of $z_0$, this makes $z_0$ an accumulation point of $S$. This contradicts that all accumulation points of $S$ is in $S$. $\square$

# Chapter 12

# Analytic Functions

## 12.1 Functions as mappings

A function $f : S \to S'$ is a function that maps elements from $S$ to elements on $S'$. The value of $f$ at $z$ is denoted $f(z)$ and the set $S$ is the domain of $f$ while $S'$ is the image of $f$. Recall section 11.6, a function can likewise be viewed as a transformation or mapping, that maps $z \in \mathrm{dom}(f) = S$ to values $z' \in \mathrm{img}(f) = S'$.

**Definition 12.1.1: Range**

*Let $f$ be a function with domain $S$ and image $S'$. The range of $f$ is the entire image of $S$.*

Note: Image is a subset of range, and can be a single point or a set of points.

**Definition 12.1.2: Inverse Range**

*The set of all points $s \in S$ with the value $f(s) = s'$ for some $s' \in S'$.*

$$\{s : f(s) = s', \;\; s' \in S'\}$$

Note: The domain of a function is often a domain, but it does not need to be a domain.

We will consider functions $f : S \to S'$ where both $S, S' \subseteq \mathbb{C}$. For such functions we can break it into a two real valued functions:

$$\begin{aligned} f(z) &= u(x,y) + iv(x,y) & \qquad \mathrm{dom}(u) \subseteq \mathbb{R}, \mathrm{dom}(v) \subseteq \mathbb{R} \\ &= u(r,\theta) + iv(r,\theta) \end{aligned}$$

Recall that a real-valued function is a function with a domain that is a subset of $\mathbb{R}$ (**??**). If $\forall z,\ v(x,y) = 0$, then $f$ is called a real-valued function of a complex variable.

**Definition 12.1.3: Polynomial**

*Let $a_i \in \mathbb{C}$, $0 \le i \le n$ where $i, n \in \mathbb{N} \cup \{0\}$. If $a_n \ne 0$, then a polynomial of degree $n$ is*

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n = \sum_{i=0}^{n} a_i z^i$$

**Definition 12.1.4: Rational Functions**

*Let $P(z)$ and $Q(z)$ are polynomials, then rational functions are quotients:*

$$\frac{P(z)}{Q(z)}$$

*Defined for all $z$ where $Q(z) \neq 0$.*

**Definition 12.1.5: Multiple-Valued Function**

*Let $f$ be a function and $z \in \mathrm{dom}(f)$. $f$ is a multiple-valued function if it assigns more than one value to a point $z$.*

"When multiple-valued functions are studied, usually just one of the possible values assigned at each point is taken, in a systematic manner and a (single-valued) function is constructed from the multiple-valued one" - Brown and Churchill [2]

What this means that for $z \in \mathbb{C}$ a function $f$ assigns $u(z)$ and $v(z)$ to to $z$. By taking just $u$ or $v$, we create a single-valued function from a multiple-valued function.

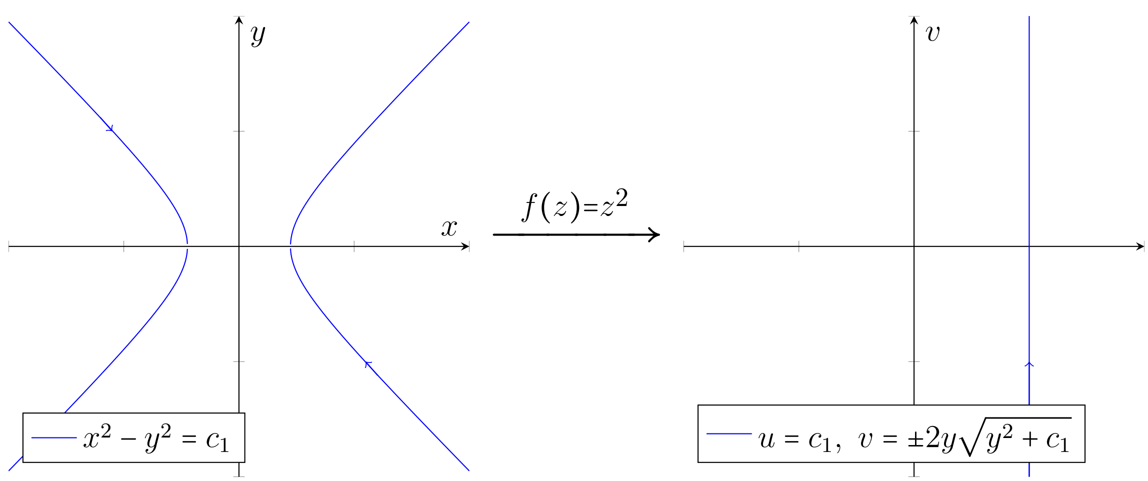
**Example 12.1.1** ($f(z) = z^2$)

$$\begin{aligned} f(z) &= z^2 = x^2 - y^2 + i2xy \\ &\implies u(x,y) = x^2 - y^2 \qquad v(x,y) = 2xy \end{aligned}$$

*By setting $u = x^2 - y^2 = c_1$ where $c_1 \in \mathbb{R}_{>0}$ we can see that*

$$u = x^2 - y^2 = c_1 \qquad\qquad v = 2xy = \pm 2y\sqrt{y^2 + c_1}$$

*This tells us that in the complex plane of $u$ and $v$, if we fix $u$ to a constant $c_1$ and move along $v = \pm 2y\sqrt{y^2 + c_1}$ by incrementing $y$ we draw out two hyperbolas in the complex plane of $x$ and $y$. This means that the function $f(z) = z^2$ takes points on hyperbolas the complex plane of $x$ and $y$ and translates them onto a vertical line in the complex plane of $u$ and $v$ where $u$ is a constant.*



*Likewise if we set $v = c_2$ where $c_2 \in \mathbb{R}_{>0}$, we get:*

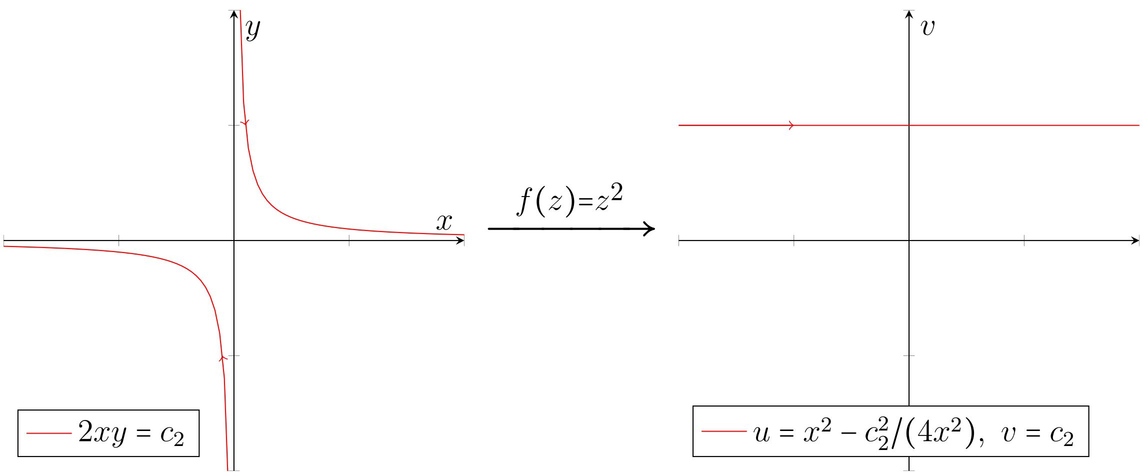
$$u = x^2 - \frac{c_2^2}{4x^2} \qquad\qquad v = 2xy = c_2$$

*Taking the limits:*

$$\lim_{x\to 0^+} u = -\infty \qquad\qquad \lim_{x\to\infty, x>0} u = \infty \tag{12.1}$$

$$\lim_{x\to -\infty, x<0} u = \infty \qquad\qquad \lim_{x\to 0^-} u = -\infty \tag{12.2}$$

*Equation 11.1 tells us as $x$ goes from 0 to $\infty$, $u$ moves from $-\infty$ to $\infty$, which corresponds to the hyperbola in the first quadrant of the $xy$ complex plane. Similarly for equations 11.2.*




*If we look at $f$ using the polar representation, we get $f(z) = r^2 e^{i2\theta}$. This tells us $\forall r \geq 0$, $r \mapsto r^2 = \rho \geq 0$, and $\forall \theta$, $\theta \mapsto \phi = 2\theta$. It is worth noting that mapping of points between $0 \leq 0 < 2\pi$ is not one-to-one, since points in $0 \leq \theta < \pi$ and points in $\pi \leq \theta < 2\pi$ both get mapped to $0 \leq \phi < 2\pi$.*

## 12.2 Limits

**Definition 12.2.1: Limit**

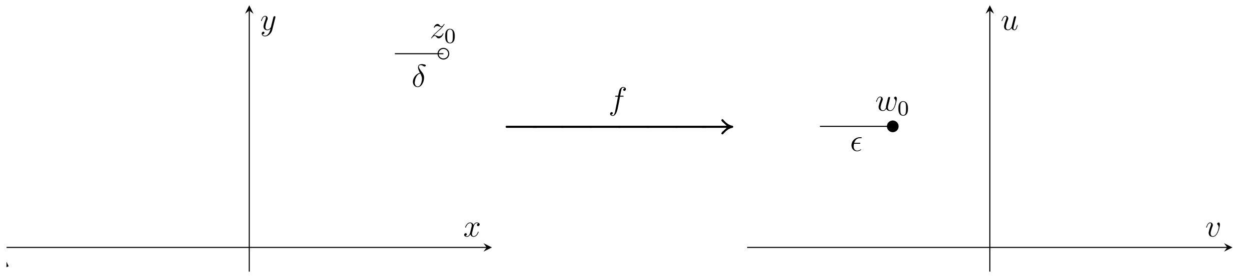
*Let $z, z_0, w_0 \in \mathbb{C}$ and $f$ be a function. We say $f(z)$ has limit $w_0$ as $z$ approaches $z_0$ if:*

$$\forall \epsilon \exists \delta [0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon]$$

*We then denote: $\lim_{z\to z_0} f(z) = w_0$*

This tells us that $\lim_{z\to z_0} f(z) = w_0$ if some deleted neighbourhood $|z - z_0| < \delta$ corresponds to a neighbourhood $|f(z) - w_0| < \epsilon$. Note that the mapping of all points $z$ in $|z - z_0| < \delta$ to $|f(z) - w_0| < \epsilon$ need not be subjective. It just needs to be mapped less than distance $\epsilon$ from $w_0$.

Note: Definition 12.2.1 allows us to verify if a limit exists, but it is not a method for determining a limit.




**Theorem 12.2.1: Uniqueness of Limits**
*Suppose the limit of $f$ at $z_0$ exists, then it is unique.*

*Proof:* Suppose two limits of $f$ at $z_0$ exists for contradiction.

$$\left[\lim_{z\to z_0} f(z) = w_0\right] \wedge \left[\lim_{z\to z_0} f(z) = w_1\right]$$
$$\implies [0 < |z - z_0| < \delta_0 \implies |f(z) - w_0| < \epsilon] \wedge [0 < |z - z_0| < \delta_0 \implies |f(z) - w_0| < \epsilon]$$

$$w_1 - w_0 = [f(z) - w_0] + [w_1 - f(z)]$$
$$\implies |w_1 - w_0| = |[f(z) - w_0] + [w_1 - f(z)]| \leq |f(z) - w_0| + |f(z) - w_1|$$

Now choosing $\delta = \min\{\delta_1, \delta_2\}$, we get:

$$|w_1 - w_0| < \epsilon + \epsilon = 2\epsilon$$

Choosing $\epsilon$ to be arbitrary small, we end up with:

$$w_1 - w_0 = 0 \implies w_1 = w_0$$

□

Definition 12.2.1 requires that $f$ be defined at all points in the deleted neighbourhood of $z_0$. That is, $z_0$ is interior to the region which $f$ is defined. We can extend the definition by agreeing that $0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon$ also holds for $z$ that lie in the region where $f$ is defined and the deleted neighbourhood of $z_0$. That is $f(z_0)$ need not be defined for a limit at $z_0$ to exist.

**Example 12.2.1** *Show* $(f(z) = iz/2) \wedge (|z| < 1) \implies \lim_{z\to 1} f(z) = i/2$.

*We can see that we have restricted the domain of $f$ to the region $|z| < 1$, this puts $z = 1$ right at the boundary of the domain of definition of $f$.*

$$|z| < 1 \implies \left|f(z) - \frac{i}{2}\right| = \left|\frac{iz}{2} - \frac{i}{2}\right| = \frac{|z-1|}{2}$$
$$\implies \forall z \forall \epsilon \exists \delta \left[0 < |z - 1| < \delta = 2\epsilon \implies \left|f(z) - \frac{i}{2}\right| < \epsilon\right]$$
$$\implies \lim_{z\to 1} f(z) = \frac{i}{2}$$

This highlights the fact that if the limit exists, then $z$ is allowed to approach $z_0$ from any arbitrary direction.

**Example 12.2.2** *Limit of $f(z) = z/\bar{z}$ does not exist at $z = 0$*

*Consider* $\lim_{z\to 0} f(z)$. *Let us approach the limit from the $x$-axis and the $y$-axis.*

$$\lim_{z=(x,0)\to 0} f(z) = \frac{x+i0}{x-i0} = 1 \qquad\qquad \lim_{z=(0,y)\to 0} f(z) = \frac{0+iy}{0-iy} = -1$$

*We end up with two different limits. As limits are unique, we conclude that* $\lim_{z\to 0} f(z)$ *does not exist.*

### 12.2.1 Limit Theorems

**Theorem 12.2.2:**

*Consider $f(z) = u(x,y) + iv(x,y)$. Let $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$.*

$$\left[\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0\right] \wedge \left[\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0\right] \iff \lim_{z\to z_0} f(z) = w_0$$

*Proof:* $\underline{\implies:}$

By definition:

$$\begin{aligned}&\left[\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0\right] \wedge \left[\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0\right] \\ &\implies \forall\epsilon\exists\delta_1,\delta_2\left[\left(0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta_1 \implies |u-u_0| < \frac{\epsilon}{2}\right)\right. \\ &\qquad\qquad \left.\wedge\left(0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta_2 \implies |v-v_0| < \frac{\epsilon}{2}\right)\right]\end{aligned} \tag{12.3}$$

Triangle inequality for the distance between points:

$$|(u+iv)-(u_0-iv_0)| = |(u-u_0)+i(v-v_0)| \le |u-u_0| + |v-v_0|$$
$$\sqrt{(x-x_0)^2+(y-y_0)^2} = |(x-x_0)+i(v-v_0)| = |(x+iy)-(x_0-iy_0)|$$

Let $\delta = \min\{\delta_1,\delta_2\}$, it follows from eq. (12.3):

$$0 < |(x+iy)-(x_0+iy_0)| < \delta \implies |(u+iv)-(u_0-iv_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, $\lim_{z\to z_0} f(z) = w_0$.

$\underline{\impliedby:}$

Suppose $\lim_{z\to z_0} f(z) = w_0$.

$$\begin{aligned}&\lim_{z\to z_0} f(z) = w_0 \\ &\implies \forall\epsilon\exists\delta>0[|(x+iy)-(x_0-iy_0)| < \delta \implies |(u+iv)-(u_0+iv_0)| < \epsilon]\end{aligned} \tag{12.4}$$

By the triangle inequality:

$$|u - u_0| \leq |(u - u_0) + i(v - v_0)| = |(u + iv) - (u_0 + iv_0)|$$
$$|v - v_0| \leq |(u - u_0) + i(v - v_0)| = |(u + iv) - (u_0 + iv_0)|$$

$$|(x + iy) - (x_0 + iy_0)| = |(x - x_0) + i(y - y_0)| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

Thus, it follows from the inequalities in eq. (12.4):

$$\begin{aligned} &0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \\ &\implies [|u - u_0| < \epsilon] \wedge [|v - v_0| < \epsilon] \\ &\implies \left[\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0\right] \wedge \left[\lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0\right] \end{aligned}$$

□

**Theorem 12.2.3:**

*Suppose*

$$\left[\lim_{z\to z_0} f(z) = w_0\right] \wedge \left[\lim_{z\to z_0} F(z) = W_0\right]$$

*Then*

$$\lim_{z\to z_0}[f(z) + F(z)] = w_0 + W_0$$
$$\lim_{z\to z_0}[f(z)F(z)] = w_0 W_0$$
$$\lim_{z\to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0} \qquad W_0 \neq 0$$

*Proof:* Let:

$$f(z) = u(x, y) + iv(x, y) \qquad F(z) = U(x, y) + iV(x, y)$$

$$z_0 = x_0 + iy_0 \qquad w_0 = u_0 + iv_0 \qquad W_0 = U_0 + iV_0$$

$\underline{\lim_{z\to z_0}[f(z) + F(z)] = w_0 + W_0}$

From Theorem 12.2.2:

$$\begin{aligned} &f(z) + F(z) = (u + U) + i(v + V) \\ &\implies \lim_{(x,y)\to(x_0,y_0)} f(z)F(Z) = (u_0 + U_0) + i(v_0 + V_0) = w_0 + W_0 \end{aligned}$$

$\underline{\lim_{z\to z_0}[f(z)F(z)] = w_0 W_0}$

From Theorem 12.2.2:

$$f(z)F(z) = (uU - vV) + i(vU + uV)$$
$$\implies \lim_{(x,y)\to(x_0,y_0)} f(z)F(Z) = (u_0U_0 - v_0V_0) + i(v_0U_0 + u_0V_0) = w_0W_0$$

$\lim_{z\to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}$ if $W_0 \neq 0$

From Theorem 12.2.2:

$$\frac{f(z)}{F(z)} = \frac{u+iv}{U+iV} \implies \lim_{(x,y)\to(x_0,y_0)} \frac{f(z)}{F(z_0)} = \frac{u_0+v_0}{U_0+iV_0} = \frac{w_0}{W_0}$$

□

**Corollary 12.2.3.1:**

*Let $c$ be a constant, $z, z_0 \in \mathbb{C}$, and $P(z)$ be a polynomial. Then*

$$\lim_{z\to z_0} c = c \qquad \lim_{z\to z_0} z = z_0 \qquad \lim_{z\to z_0} z^n = z_0^n \qquad n \in \mathbb{N}$$

$$\lim_{z\to z_0} P(z) = P(z_0)$$

**Observation.** *It is surprisingly quick that Brown and Churchill went from $\epsilon$−$\delta$ proofs straight to proving with limits. This is different to the approach in Sequences of Limits Theorem for Sequences Section by Kennith A. Ross. [1]. (Section 9.0.1)*

**Question.** *It might be possible use a series approach to prove limit theorems for $z \in \mathbb{C}$ by having separate series for $x$ and $y$ (real and imaginary components of $z$), or a series in the form of $s_n = (x_n, y_n)$. Which would be the proper approach?*

### 12.2.2 Limits of Points at Infinity

**Definition 12.2.2: Extended Complex Plane**

*The complex plane union with the points at infinity:*

$$\mathbb{C} \cup \{\pm\infty, \pm i\infty\}$$

**Definition 12.2.3: Riemann Sphere**

*A unit sphere centred at the origin of the complex plane, which is consequently bisected by the complex plane.*

**Definition 12.2.4: Stereographic Projection**

*Consider the Riemann Sphere. Let $N$ be the northern point of the sphere (the point on the sphere above the origin of the complex plane) and $z$ be any point in the complex plane. Let $l$ be a line that goes through $N$ and $z$, then $l$ will intersect the Riemann Sphere. Let $P$ be the point where $l$ intersects the Riemann Sphere. If we let $N$ correspond to the points at infinity, then there is a one-to-one correspondence between points on the sphere and the points on the extended complex plane. This correspondence called the Stereographic Projection. (Figure 12.1)*

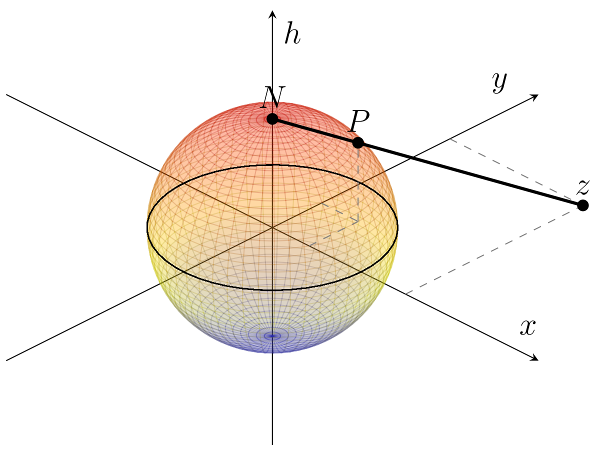



Figure 12.1: Riemann Sphere and Stereographic Projection

The region outside the unit circle enveloped by the Riemann sphere corresponds to the upper hemisphere of the Riemann sphere, with the point $N$ deleted. $N$ corresponds to the points at infinity, since $l$ will be parallel to the complex plane.

**Definition 12.2.5: Neighbourhood of $\infty$**

*The set:* $\{|z| > 1/\epsilon : \epsilon \in \mathbb{R}_{>0}\}$

Note that since $\epsilon$ is a small positive number, $|z| > 1/\epsilon$ corresponds to points far away from the unit circle, hence $P$ is close to $N$.

Note: When referring to any point $z$, it is referring to a point in the finite plane. Points at infinity will be specifically mentioned.

**Theorem 12.2.4:**

*Let $z_0, w_0 \in \mathbb{C}$, then*

$$\lim_{z\to z_0} \frac{1}{f(z)} = 0 \implies \lim_{z\to z_0} f(z) = \infty$$
$$\lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0 \implies \lim_{z\to\infty} f(z) = w_0$$
$$\lim_{z\to 0} \frac{1}{f(1/z)} = 0 \implies \lim_{z\to\infty} f(z) = \infty$$

*Proof:* $\underline{\lim_{z\to z_0} \frac{1}{f(z)} = 0 \implies \lim_{z\to z_0} f(z) = \infty}$

$$\begin{aligned}
\lim_{z\to z_0} \frac{1}{f(z)} = 0 &\implies \forall\epsilon\exists\delta > 0\left[|z - z_0| < \delta \implies \left|\frac{1}{fz} - 0\right| < \epsilon\right] \\
&\implies \forall\epsilon\exists\delta > 0\left[|z - z_0| < \delta \implies |f(z)| > \frac{1}{\epsilon}\right] \\
&\implies \lim_{z\to z_0} f(z) = \infty
\end{aligned}$$

$\underline{\lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0 \implies \lim_{z\to\infty} f(z) = w_0}$

$$\begin{aligned}
\lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0 &\implies \forall\epsilon\exists\delta > 0\left[|z - 0| < \delta \implies \left|f\left(\frac{1}{z}\right) - w_0\right| < \epsilon\right] \\
&\implies \forall\epsilon\exists\delta > 0\left[|z| > \frac{1}{\delta} \implies |f(z) - w_0| < \epsilon\right] \\
&\implies \lim_{z\to\infty} f(z) = w_0
\end{aligned}$$

$\underline{\lim_{z\to 0} \frac{1}{f(1/z)} = 0 \implies \lim_{z\to\infty} f(z) = \infty}$

$$\begin{aligned}
\lim_{z\to 0} \frac{1}{f(1/z)} = 0 &\implies \forall\epsilon\exists\delta > 0\left[|z - 0| < \delta \implies \left|\frac{1}{f(1/z)} - 0\right| < \epsilon\right] \\
&\implies \forall\epsilon\exists\delta > 0\left[|z| > \frac{1}{\delta} \implies |f(z)| > \frac{1}{\epsilon}\right] \\
&\implies \lim_{z\to\infty} f(z) = \infty
\end{aligned}$$

$\square$

Note: As $\delta$ goes to 0, $1/\delta$ goes to $\infty$, hence $|z|$ goes to $\infty$ if $|z| > 1/\delta$.

**Observation.** *As expected, theorem 12.2.4 is consistent if $z \in \mathbb{R}$. (Check: Section 9.0.1).*

## 12.3 Continuity

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# Bibliography

# Bibliography


[1] Kenneth A. Ross. *Elementary Analysis.* Springer, 2 edition, 2013.

[2] James Ward Brown and Ruel V. Churchill. *Complex Variables and Applications.* McGraw-Hill Education, 9 edition, 2014.

[3] A. David Wunsh. *Complex Variables with Applications.* Pearson, 3 edition, 2005.