The Book of Math (Notes)

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Forward and Disclaimer

These are math notes made by a student (with a physics major and math minor) based off text books. It may contain misconceptions and misinterpretations, thus should not be viewed in the same light of a text book. Use at your own risk and mental sanity.

Symbols

Logic

Name	Symbol	Comment
Exists	$\exists$	There exists at least one
For all	$\forall$	
Not exists	$\nexists$	There does not exist
Exists one	$\exists!$	There only exists one and only one
And	$\wedge$	
Or	$\vee$	Inclusive or
Not	$\neg$	
Logically implies	$\implies$	If
Logically implied by	$\impliedby$	Only if
Logically equivalent	$\iff$	If and only if
Implies	$\longrightarrow$	
Implied by	$\longleftarrow$	
Double Implication	$\longleftrightarrow$	

Set Notation

Name	Symbol	Comment
Empty Set	$\varnothing$	The set that is empty
Natural Numbers	$\mathbb{N}$	Set of natural numbers not containing 0, equivalent to the set of positive integers
Integers	$\mathbb{Z}$	Set of integers
Rational Numbers	$\mathbb{Q}$	
Algebraic Numbers	$\mathbb{A}$	
Real Numbers	$\mathbb{R}$	
Complex Numbers	$\mathbb{C}$	
In	$\in$	
Not in	$\notin$	
Owns	$\ni$	Has an element
Proper Subset	$\subset$	Subset that is not itself
Subset	$\subseteq$	
Superset	$\supset$	Superset that is not itself
Proper Superset	$\supseteq$	

Power set	$\wp$	
Union	$\cup$	
Intersection	$\cap$	
Difference	$\setminus$	

Relationships

Name	Symbol	Comment
Defined	$\doteq$	
Approximate	$\approx$	
Equivalent	$\equiv$	Isomorphic (Group Theory)
Congruent	$\cong$	Homomorphic (Group Theory)
Proportional	$\propto$	

Operators

Name	Symbol	Comment
	$\oplus$	
	$\otimes$	
	$\odot$	
	$\circ$	Convolution
Dagger	$\dagger$	Complex conjugate transpose of a matrix

Arrows

Name	Symbol	Comment
Maps to	$\mapsto$	

Hebrew

Name	Symbol	Comment
Aleph	$\aleph$	Carnality of infinite sets that can be well ordered

Other

Name	Symbol	Comment
Real part	Re	Real part of a number
Imaginary part	Im	Imaginary part of a number

Book Constitution

Intents and Purpose

The goal of this book is to organize mathematical knowledge of topics related to the study of physics or the author's interest. It is meant to be used as a source of for future reference, not as a textbook for students new to the topics. It is a notebook of a student, thus should be treated as one and not as a textbook. At most, it could be used as a study guide along side a textbook. Definitely not as the main source for acquiring knowledge.

Layout and Organization

The book is split into parts each containing a field of study mathematics, or a topic large enough to justify giving it its own part. Each part contains chapters that focuses on a particular topic required to understand the field, with sections dedicated to describing a particular knowledge required for the topic.

As axioms, definitions, theorems, corollary, and proofs are integral and abundant to the study of mathematics, each will have a unique style. Each environment and its styles are displayed as follows:

Axiom 0.1: Axiom name
Example Axiom Axioms are the "ground rules" of the set.

Theorem 0.0.1: Theorem name or citation
Example Theorem An important logical result from the axioms, with proof.

Conjecture 0.0.1: Name of conjecture or citation
Example Conjecture A hypothesis, without proof.

Corollary 0.0.1.1:
Example Corollary An implication as a result of a theorem.

Lemma 0.0.1.1:
Example Lemma Small theorems that build up to a larger theorem.

Proposition 0.0.1.1:
Example Proposition Example proposition.

Proof: Logical deductions that results in a theorem. Proofs I've written will be in grey, which may or may not be correct. □

Definition 0.0.1: Word
Example Definition The definition of a word.

Example 0.0.1 *An example.*

Remark. *Remark A comment by the author in the textbooks used.*

Observation. *Example Observation A remark by me.*

Question. *Example Question A question from me for a mystery to be answered later.*

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1. Kenneth A. Ross - Elementary Analysis (2nd Ed.) [1]

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Resources used in part IV

Primary:

1. Brown and Churchill - Complex Variables and Applications [2]

Supplement:

1. A. David Wunsch - Complex Variables with Applications [3]

Chapter 12

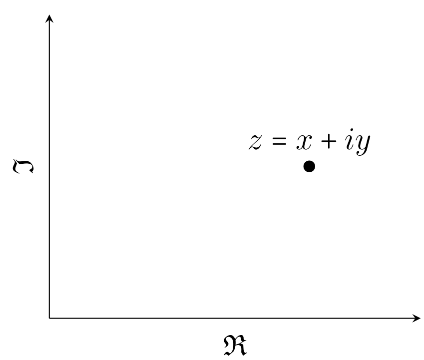
Basics

12.1 Complex Numbers

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}, i = \sqrt{-1}\}$$

Complex numbers are elements of the complex field ($\mathbb{C}$), therefore, they obey all the properties of a field.

We will denote complex numbers by $z = x + iy$ with $x, y \in \mathbb{R}$, and refer the real part as $\mathrm{Re}(z) = \mathrm{Re}(z) = x$ and imaginary part as $\mathrm{Im}(z) = \mathrm{Im}(z) = y$. Complex numbers can also be defined as an ordered pair $z = (x, y)$ which is interpreted as points in the complex plane. $(x, 0)$ are points on the real axis while $(0, y)$ are points in the imaginary axis. This expression is often called a Couple, and was presented in 1833 by mathematician William Rowan Hamilton (1805 - 1865).



Like numbers in $\mathbb{R}$, numbers in $\mathbb{C}$ obey the commutative, distributive, and associative laws. We add and multiply complex numbers in the usual way:

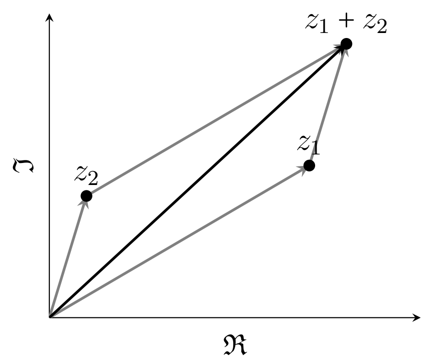
$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \end{aligned} \qquad \begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \end{aligned}$$

$\forall z \in \mathbb{C}$, there is an unique additive inverse $(-z)$ and $\forall z \in \mathbb{C} \setminus \{0\}$, there is an unique multiplicative inverse (z^{-1}) such that

$$z + (-z) = 0 \qquad\qquad zz^{-1} = 1$$
$$\implies -z = -x - iy \qquad\qquad \implies (x_1x_2 - y_1y_2) = 1 \wedge (x_1y_2 + x_2y_1) = 0$$
$$\implies z^{-1} = \frac{x_1}{x_1^2 + y_1^2} - i\frac{y_1}{x_1^2 + y_1^2}$$

The existence and uniqueness of the inverses can be easily proven.

The addition of complex numbers may also be interpreted as akin to vector addition.



Note: As a group with addition, $\mathbb{R}^2 \cong \mathbb{C}$, however this is not the case for rings. $\mathbb{C}$ is a field, but $\mathbb{R}^2$ is not. $\mathbb{R}^2$ have non-zero divisors (ie. Take any $a, b \in \mathbb{R}$, $(a, 0) \cdot (0, b) = 0$).

12.2 Triangle Inequality

It is not analysis without a section dedicated to the triangle inequality. For any given number $z_1, z_2 \in \mathbb{C}$ it makes no sense to write an inequality $z_1 = a_1 + ib_1 < a_2 + ib_2 = z_2$. Thus, we need have a different notion of size.

Definition 12.2.1: Modulus

The modulus of a complex number is a function $\mathbb{C} \to \mathbb{R}_{>0}$:

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$

It is obvious why the definition is not $|z| = \sqrt{x^2 + (iy)^2}$ as problems arise when $x = y$. The modulus is the distance of z from $(0, 0)$. $\bar{z}$ is the complex conjugate of z, which is explored in section 12.5

Theorem 12.2.1: Triangle Inequality

$\forall z_1, z_2 \in \mathbb{C}[|z_1 + z_2| \leq |z_1| + |z_2|]$

From the theorem, we can derive a similar inequality:

$$|z_1| = |z_1 + z_2 - z_2| \leq |z_1 + z_2| + |-z_2| \implies |z_1| - |z_2| \leq |z_1 + z_2|$$

An important property of polynomials is observed when theorem 12.2.1 is applied to polynomials.

Corollary 12.2.1.1:

Consider the polynomial $P(z)$ where $a_n \in \mathbb{C}$, $n \in \mathbb{N}$, $a_0 \neq 0$, and $z \in \mathbb{C}$.

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

Then $\forall z, \exists R \in \mathbb{R}_{>0}, |z| < R$ such that

$$\left|\frac{1}{P(z)}\right| < \frac{2}{|a_n| R^n}$$

Proof: Consider

$$\begin{aligned}
w = \frac{P(z)}{z_n} - a_n = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \ldots + \frac{a_{n-1}}{z} && z \neq 0 \\
\implies wz^n = a_0 + a_1 z + \ldots + a_{n-1} z^{n-1} \\
\implies |w||z|^n \leq |a_0| + |a_1||z| + \ldots + |a_{n-1}||z|^{n-1} \\
\implies |w| \leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \ldots + \frac{|a_{n-1}|}{|z|} \\
\implies |w| < n\frac{|a_n|}{2n} = \frac{|a_n|}{2} && \exists \text{ sufficiently large } R < |z| \text{ s.t.} \\
&& \forall m,\ 0 \leq m \leq n-1,\ \frac{|a_m|}{|z|^{n-m}} < \frac{|a_n|}{2n} \\
\implies |a_n + w| \geq ||a_n| - |w|| > \frac{|a_n|}{2} && R < |z| \\
\implies |P_n(z)| = |a_n + w||z|^n > \frac{|a_n|}{2}|z|^n > \frac{|a_n|}{2}R^n && R < |z| \\
\implies \left|\frac{1}{P(z)}\right| < \frac{2}{|a_n| R^n}
\end{aligned}$$

□

This tells us that if z is a solution to a polynomial $P(z)$, then the reciprocal of the polynomial $1/P(z)$ is bounded above by $R = |z|$. (i.e. It is bounded by a circle of radius $|z|$.)

12.3 Polar and Exponential Form

Definition 12.3.1: Argument of z

Consider any $z \in \mathbb{C}$ where $z \neq 0$. Let θ be the angle in radians between z and the real axis . Then $\forall n \in \mathbb{N}$, $-\pi < \theta \leq \pi$, the argument of z:

$$\arg(z) = \theta + 2n\pi$$

We know $\forall n \in \mathbb{N}$, $\theta + 2\pi n = \theta$. This leads us to the definition of the principal argument of z.

Definition 12.3.2: Principal Argument of z

Consider any $z \in \mathbb{C}$ where $z \neq 0$. Let θ be the angle in radians between z and the real axis. Then for $-pi < \theta \leq \pi$, the principal argument of z:

$$\text{Arg}(z) = \theta$$

It is clear that $\arg(z) = \text{Arg}(z) + 2n\pi$. It is common for the principal argument to be defined $-\pi < \theta \leq \pi$, although other definitions use $0 \leq \theta < 2\pi$.

Definition 12.3.3: Polar Form of z

Consider $z \in \mathbb{C}$. Let $r = |z|$, and $\theta = \arg(z)$. Then $\forall z \in \mathbb{C}, z \neq 0$:

$$z = x + iy = r(\cos(\theta) + i\sin(\theta))$$

Notice that all three definitions require that $z \neq 0$ as θ is undefined at $z = 0$.

Theorem 12.3.1: Euler's Formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

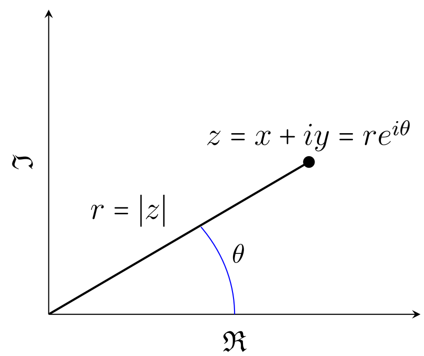
Combining definition 12.3.3 with theorem 12.3.1, we obtain the Exponential Form of z:

Definition 12.3.4: Exponential Form of z

Consider any $z \in \mathbb{C}$, and let $r = |z|$ and $\theta = \text{Arg}(z)$. Then the exponential form of z:

$$z = re^{i\theta}$$

Note: $\theta = \tan^{-1}(y/x)$ and $r = \sqrt{x^2 + y^2}$.



12.3.1 Properties of Polar and Exponential Form

It would be easier to work with the exponential form of z then convert it to the polar form later. The exponential form of a complex number is part of the exponential family of functions, thus possess all the properties of the family. Consider any complex number $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$.

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} \qquad\qquad z^n = r^n e^{in\theta} \qquad \forall n \in \mathbb{Z}$$

A special case arrives for integer exponential of z on the unit circle.

Theorem 12.3.2: de Moivre's Formula

Consider any $z = e^{i\theta} \in \mathbb{C}$ on the unit circle, and let $n \in \mathbb{Z}$.

$$\forall z \in \mathbb{C}\ \forall n \in \mathbb{Z}\,[|z| = 1 \implies (\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)]$$

Proof: Consider $z = e^{i\theta}$ and let $n \in \mathbb{Z}$.

$$z^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

□

The proof hints that theorem 12.3.2 can be generalized to $\forall n \in \mathbb{R}$, which we will see shortly in **??**. Using theorem 12.3.2, we can obtain the double angle identities.

Corollary 12.3.2.1: Double Angle Identities

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \qquad\qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

Proof: Consider any z on the unit circle, that is $z = e^{i\theta}$.

$$\begin{aligned} &(\cos(\theta) + i\sin(\theta))^2 = \cos(2\theta) + i\sin(2\theta) && \text{Theorem 12.3.2} \\ &\implies \cos^2(\theta) - \sin^2(\theta) + i2\sin(\theta)\cos(\theta) = cos(2\theta) + i\sin(2\theta) \end{aligned}$$

Equating the real and imaginary parts yield the desired results. □

12.3.2 Properties of Arguments

Recall from section 12.3.1:

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} \qquad\qquad z^n = r^n e^{in\theta} \qquad \forall n \in \mathbb{Z}$$

The arguments for the arguments of products of any $z_1, z_2 \in \mathbb{C}$ follows immediately from the properties of the exponential.

Corollary 12.3.2.2: Arguments of Products

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \qquad\qquad \operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$$
$$\arg(z^n) = n \arg(z) \qquad\qquad \operatorname{Arg}(z^n) = n \operatorname{Arg}(z)$$

Proof:

$$\begin{aligned}
z_1 z_2 &= r_1 r_2 e^{i(\theta_1+\theta_2)} \\
&\implies \arg(z_1 z_2) = \arg(z_1) + 2n_1\pi + \arg(z_2) + 2n_2\pi && n_1, n_2 \in \mathbb{Z} \\
&\implies \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \\
&\implies \operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) = \operatorname{Arg}(z_2)
\end{aligned}$$

$$\begin{aligned}
z^n &= r^n e^{in\theta} \\
&\implies \arg(z^n) = n \arg(z) + 2n\pi && n \in \mathbb{Z} \\
&\implies \arg(z^n) = n \arg(z) \\
&\implies \mathrm{z^n} = n \operatorname{Arg}(z)
\end{aligned}$$

□

It is clear that:

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) \qquad\qquad \operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \operatorname{Arg}(z_1) - \operatorname{Arg}(z_2)$$

12.4 Roots of z

In definition 12.3.4, you might be wondering why $z^n = r^n e^{in\theta}$ is not for $n \in \mathbb{R}$. That is because there is more things to consider, which we will explore in this section. Recall that $z = re^(i\theta) = re^{i(\theta+2n\pi)}$ for $n \in \mathbb{Z}$.

Definition 12.4.1: Exponential of z

Consider any $z \in \mathbb{C}$ and any $x \in \mathbb{R}$

$$z^x = \left(re^{i(\theta+2n\pi)}\right)^x = r^x e^{ix(\theta+2n\pi)}$$

For $x \notin \mathbb{Z}$, it is clear that $z^x = r^x e^{ix(\theta+2n\pi)} \neq r^x e^{ix\theta}$, since $2nx\pi = 0 \iff nx \in \mathbb{Z}$. In order to define the roots of z we must need a more general and proper definition of z.

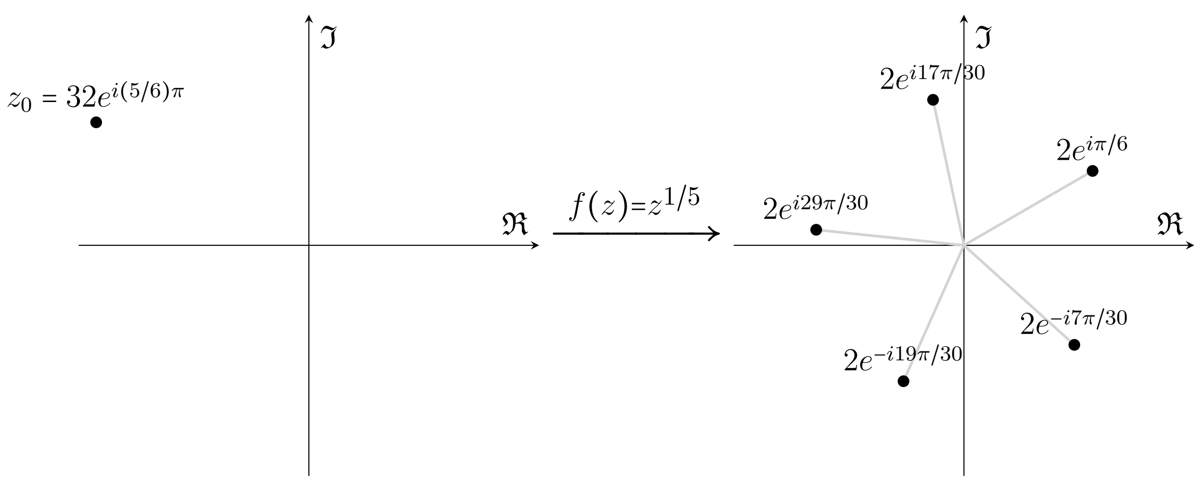
Definition 12.4.2: Roots of z_0

Consider any $z_0 \in \mathbb{C}$ and any $m \in \mathbb{N}$.

$$z_0^{\frac{1}{m}} = r_0^{\frac{1}{m}} e^{i\left(\frac{\theta_0+2n\pi}{m}\right)} = r_0^{\frac{1}{m}} e^{i\left(\frac{\theta_0}{m}+\frac{2n\pi}{m}\right)}$$

Taking the m-th root of $z_0 \in C$ scales θ_0 by $1/m$, and provides solutions at equally spaced by $2\pi/m$ on a circle of radius $r^{1/m}$. That is, the roots lie on the vertices of a regular n-sided polygon inscribed in a circle of radius $|z|^{1/m}$.

Example 12.4.1 *Consider $z_0 = 32e^{i(5/6)\pi}$, then $z_0^{(1/5)} = 3e^{i(\pi/6)+i(2/5)n\pi}$ for $n \in \mathbb{Z}$. The radius went from 35 to $35^{(1/5)} = 2$, and five roots appear equally spaced with distance of $(2/5)\pi$ on a circle with radius 2. Before and after graphs are as follows, note graph on right is zoomed in:*



We can see that the roots of z_0 form a set:

Definition 12.4.3: Set of roots of z_0

Consider the m-th root of any $z_0 \in \mathbb{C}$. Let:

$$z_0 = r_0 e^{i\theta_0} \qquad c_0 = r_0^{1/m} e^{i\theta_0/m} \qquad \omega_n = e^{\frac{i2\pi}{m}} \qquad m \in \mathbb{N}$$

Then the set of roots of z_0:

$$z_0^{1/m} = \left\{c_k = c_0\omega_m^k \mid k \in \mathbb{N},\ 0 \le k < m\right\}$$

c_0 is the principal root. The root corresponding to the principal argument of z.

Definition 12.4.4: Principal Root

Consider the m-th root of any $z_0 \in \mathbb{C}$. The principal root of z_0 is defined as:

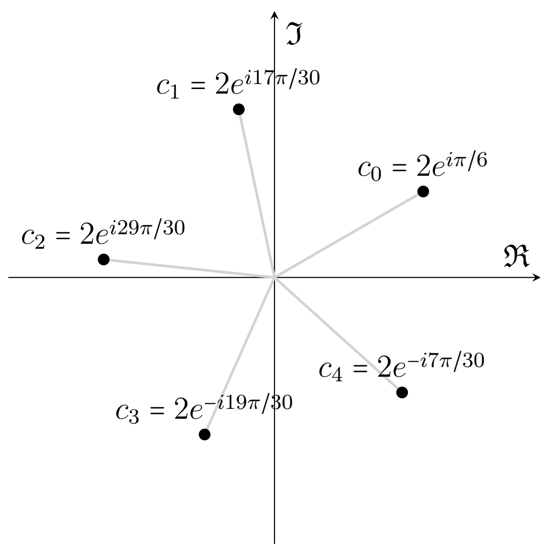
$$c_0 = r_0^{\frac{1}{m}} e^{i\frac{\theta_0}{m}}$$

Example 12.4.2 *Recall from the previous example: $z_0 = 32e^{i(5/6)\pi}$. This gives us*

$$c_0 = 32^{1/5}e^{i\pi/6} = 2e^{i\pi/6} \qquad \omega_5 = e^{i2\pi/5}$$

Then

$$\begin{aligned}
c_0 &= c_0\omega_5^0 = 2e^{i\pi/6} \\
c_1 &= c_0\omega_5^1 = 2e^{i\pi/6}e^{i2\pi/5} = 2e^{i17\pi/30} \\
c_2 &= c_0\omega_5^1 = 2e^{i\pi/6}e^{i4\pi/5} = 2e^{i29\pi/30} \\
c_3 &= c_0\omega_5^1 = 2e^{i\pi/6}e^{i6\pi/5} = 2e^{i41\pi/30} = 2e^{-i19\pi/30} \\
c_4 &= c_0\omega_5^1 = 2e^{i\pi/6}e^{i8\pi/5} = 2e^{i53\pi/30} = 2e^{-i7\pi/30}
\end{aligned}$$



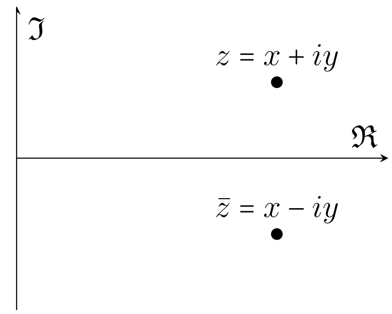
12.5 Complex Conjugate

Definition 12.5.1: Complex Conjugate

The complex conjugate of $z \in \mathbb{C}$ is denoted $\bar{z}$.

$$\bar{z} = x - iy = r(\cos(\theta) - i\sin(\theta)) = re^{-i\theta}$$

Graphically, it is the reflection of z across the real axis.



It is then easy to see

$$\mathrm{Re}(z) = \frac{z + \bar{z}}{2} \qquad \mathrm{Im}(z) = \frac{z - \bar{z}}{2i} \qquad |z|^2 = z\bar{z}$$

As $\mathrm{Re}(z) = x = r\cos(\theta)$ and $\mathrm{Im}(z) = y = r\sin(\theta)$ and using definition 12.3.4, we can obtain the complex forms of sine and cosine:

Definition 12.5.2: Complex Sine and Cosine

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

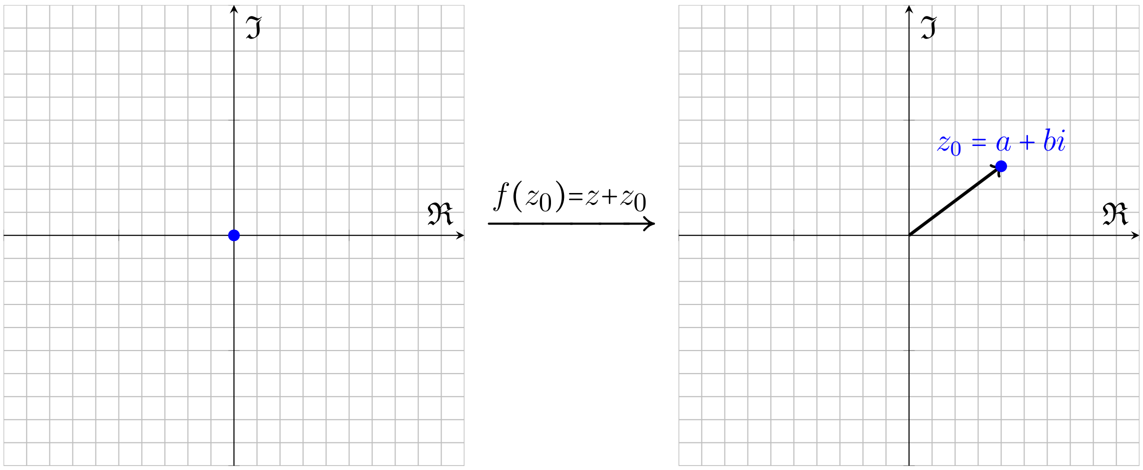
It is easy to prove $\forall z_1, z_2 \in \mathbb{C}$:

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \qquad \overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$$

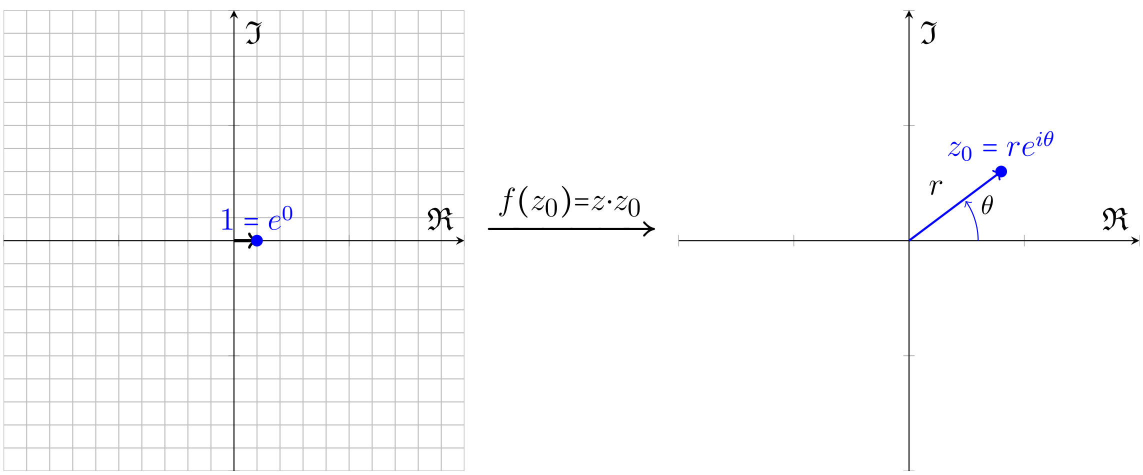
12.6 Operations as Transformations

Consider any $z \in \mathbb{C}$. A function $f : \mathbb{C} \to \mathbb{C}$ can be viewed as transformations of the complex plane.

Example 12.6.1 (Addition as translation) *Consider any $z_0 \in \mathbb{C}$, $z_0 = a + ib$ for $a, b \in \mathbb{R}$. Addition by z_0 can be seen as a shift in the complex plane by $a + bi$. (i.e. It takes the origin and shifts it by z_0.)*



Example 12.6.2 (Multiplication as scaling and rotation) *Consider any $z_0 \in \mathbb{C}$, $z_0 = re^{i\theta}$. Multiplication by z_0 scales the entire complex plane by r and rotates it by θ. (Imagine rotating and stretching out a net.)*



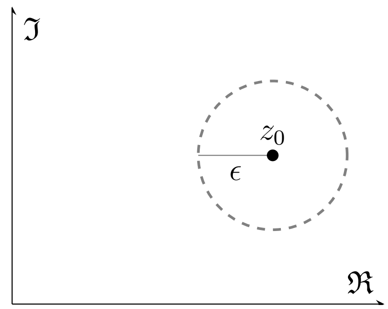
12.7 Complex Analysis Definitions

Definition 12.7.1: Neighbourhood

A neighbourhood of a point z_0 is the set of all points z with distance less than ϵ.

$$\{z : |z - z_0| < \epsilon\}$$

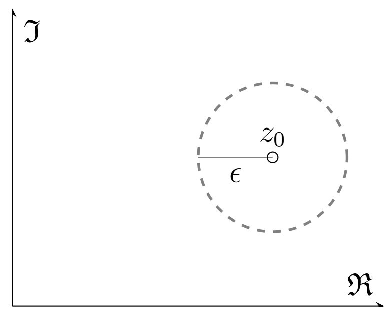
i.e. It is the set of all points that lie within a circle centred at z_0 with radius ϵ. Points on the circumference not included.



Definition 12.7.2: Deleted Neighbourhood

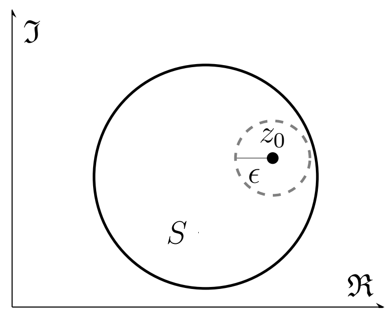
A deleted neighbourhood is the set of all points z with distance less than ϵ from a point z_0, not including z_0. That is, it is a neighbourhood of z_0 without z_0.

$$\{z : |z - z_0| < \epsilon,\ \ z \neq z_0\}$$



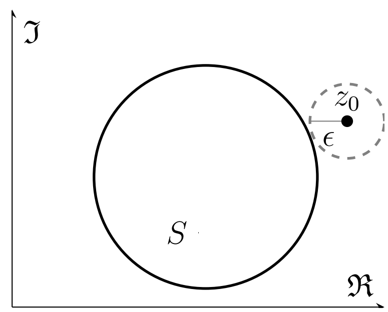
Definition 12.7.3: Interior Point

Let S be a set. A point z_0 is an interior point of S if $\exists \epsilon$ such that $\forall z$, $|z - z_0| < \epsilon \implies z \in S$. That is, z_0 is an interior point of S if it has a neighbourhood where all points in the neighbourhood is an element of S.



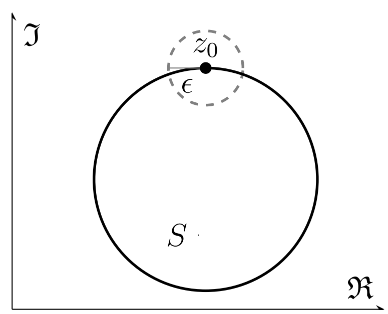
Definition 12.7.4: Exterior Point

Let S be a set. A point z_0 is an exterior point of S if $\exists \epsilon$ such that $\forall z$, $|z - z_0| < \epsilon \implies z \notin S$. That is, $z_0$is an exterior point of S if it has neighbourhood that does not contain any element of S.



Definition 12.7.5: Boundary Point

Let S be a set. A point z_0 is a boundary point of S if $\forall\epsilon$, $\exists z \in S, z' \notin S$, such that $|z - z_0|\epsilon$ and $|z' - z_0| < \epsilon$. That is, for all neighbourhoods of z_0 there exists a point that is in S and a point not in S.



Note: A boundary point of S may or may not be in S.

Definition 12.7.6: Boundary of a Set

A boundary of a set S is the set of all boundary points of S. The set containing all boundary points of S.

$$\{z_0 : \forall\epsilon\exists z \in S, z' \notin S(|z - z_0| < \epsilon \wedge |z' - z_0| < \epsilon)\}$$

Definition 12.7.7: Open Set

A set that does not contain any boundary points.

Theorem 12.7.1:

Set S is open $\iff$ $\forall s \in S$, s is an interior point of S

Proof: $\underline{\implies}$: Suppose S is open $\not\Rightarrow$ $\forall s \in S$, s is an interior point of S, for contradiction. That is, $\exists s \in S$ that is either a boundary point or an exterior point. $s \in S$ implies s is not an exterior point of S, so s has to be a boundary point of S. This contradicts that S is an open set.

$$S \text{ is open } \implies \forall s \in S(s \text{ is an interioir point of } S)$$

$\Longleftarrow$:

$$\begin{aligned}
&\forall s \in S(s \text{ is an interior point of S}) \\
&\implies \forall s' \forall \epsilon (|s' - s| < \epsilon \implies s' \in S) \\
&\implies S \text{ does not contain boundary points} \implies S \text{ is open}
\end{aligned}$$

□

A set can be neither open or closed. Consider the set $S = \{z : 0 < |z| \leq 1\}$. S is not closed since it does not contain the boundary point 0, and it is not open since it contains boundary points where $|z| = 1$. The set $\mathbb{C}$ is both open and closed since it has no boundary points.

Definition 12.7.8: Closed Set

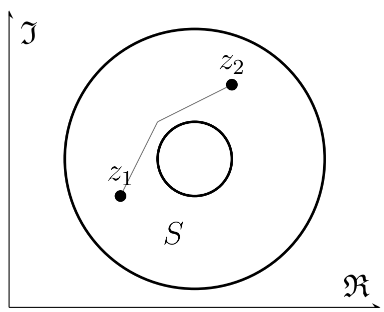
A set that contains all of its boundary points.

Definition 12.7.9: Closure of a Set

Let S be a set. The closure of S is a closed set containing all points of S and all boundary points of S.

Definition 12.7.10: Connected Set

An opens set S is connected if $\forall z_1, z_2 \in S$, z_1 and z_2 can be connected by a polygonal line lying within S.



Definition 12.7.11: Polygonal Line

A finite set of line segments joined end to end.

Definition 12.7.12: Domain

A nonempty connected set.

Note: All neighbourhoods are domains.

Definition 12.7.13: Region

A domain with none, some, or all of its boundary points.

Definition 12.7.14: Closed Region

A domain with all of its boundary points.

Definition 12.7.15: Bounded Set/Region

A set S is bounded if $\exists R = |z| > 0$ such that $\forall s \in S$, $|s| < R$. That is, S is bounded if $\forall s \in S$, s is contained in some circle of radius R centred at the origin.

Definition 12.7.16: Closed Region

A bounded and closed region.

Definition 12.7.17: Accumulation/Limit Point

A point z_0 is a accumulation point of a set S if all deleted neighbourhood of z_0 contains an element of S.

$$\forall \epsilon \exists s \in S(s \neq z_0 \wedge |z - s| < \epsilon)$$

Note: Unlike a boundary point, an accumulation point does not require that all neighbourhood of z_0 contain an element not in S.

Theorem 12.7.2:

Set S is closed $\iff$ $\forall$ accumulation points z_0 of S, $z_0 \in S$

Proof: $\underline{\implies:}$ Let S is closed and z_0 is an accumulation point of a set S where $z_0 \notin S$ for contradiction. If $\exists z_0 \notin S$, then z_0 is a boundary point of S. Contradicts closed set contains all boundary points.

$\underline{\impliedby:}$ Suppose all accumulation points of S are elements of S but S is not closed for contradiction. Then S does not contain one or more boundary points. Suppose z_0 is a boundary point of S that is not in S. Then $\forall \epsilon\ \exists s \in S$ where $|s - z_0| < \epsilon$, so by considering the deleted neighbourhood of z_0, this makes z_0 an accumulation point of S. This contradicts that all accumulation points of S is in S. $\square$

Chapter 13

Analytic Functions

13.1 Functions as mappings

A function $f : S \to S'$ is a function that maps elements from S to elements on S'. The value of f at z is denoted $f(z)$ and the set S is the domain of f while S' is the image of f. Recall section 12.6, a function can likewise be viewed as a transformation or mapping, that maps $z \in \text{dom}(f) = S$ to values $z' \in \text{img}(f) = S'$.

Definition 13.1.1: Range

Let f be a function with domain S and image S'. The range of f is the entire image of S.

Note: Image is a subset of range, and can be a single point or a set of points.

Definition 13.1.2: Inverse Range

The set of all points $s \in S$ with the value $f(s) = s'$ for some $s' \in S'$.

$$\{s : f(s) = s',\ \ s' \in S'\}$$

Note: The domain of a function is often a domain, but it does not need to be a domain.

We will consider functions $f : S \to S'$ where both $S, S' \subseteq \mathbb{C}$. For such functions we can break it into a two real valued functions:

$$\begin{aligned} f(z) &= u(x,y) + iv(x,y) \qquad\qquad \text{dom}(u) \subseteq \mathbb{R}, \text{dom}(v) \subseteq \mathbb{R} \\ &= u(r,\theta) + iv(r,\theta) \end{aligned}$$

Recall that a real-valued function is a function with a domain that is a subset of $\mathbb{R}$ (**??**). If $\forall z,\ v(x,y) = 0$, then f is called a real-valued function of a complex variable.

Definition 13.1.3: Polynomial

Let $a_i \in \mathbb{C}$, $0 \le i \le n$ where $i, n \in \mathbb{N} \cup \{0\}$. If $a_n \ne 0$, then a polynomial of degree n is

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n = \sum_{i=0}^{n} a_i z^i$$

Definition 13.1.4: Rational Functions

Let $P(z)$ and $Q(z)$ are polynomials, then rational functions are quotients:

$$\frac{P(z)}{Q(z)}$$

Defined for all z where $Q(z) \neq 0$.

Definition 13.1.5: Multiple-Valued Function

Let f be a function and $z \in \text{dom}(f)$. f is a multiple-valued function if it assigns more than one value to a point z.

"When multiple-valued functions are studied, usually just one of the possible values assigned at each point is taken, in a systematic manner and a (single-valued) function is constructed from the multiple-valued one" - Brown and Churchill [2]

What this means that for $z \in \mathbb{C}$ a function f assigns $u(z)$ and $v(z)$ to to z. By taking just u or v, we create a single-valued function from a multiple-valued function.

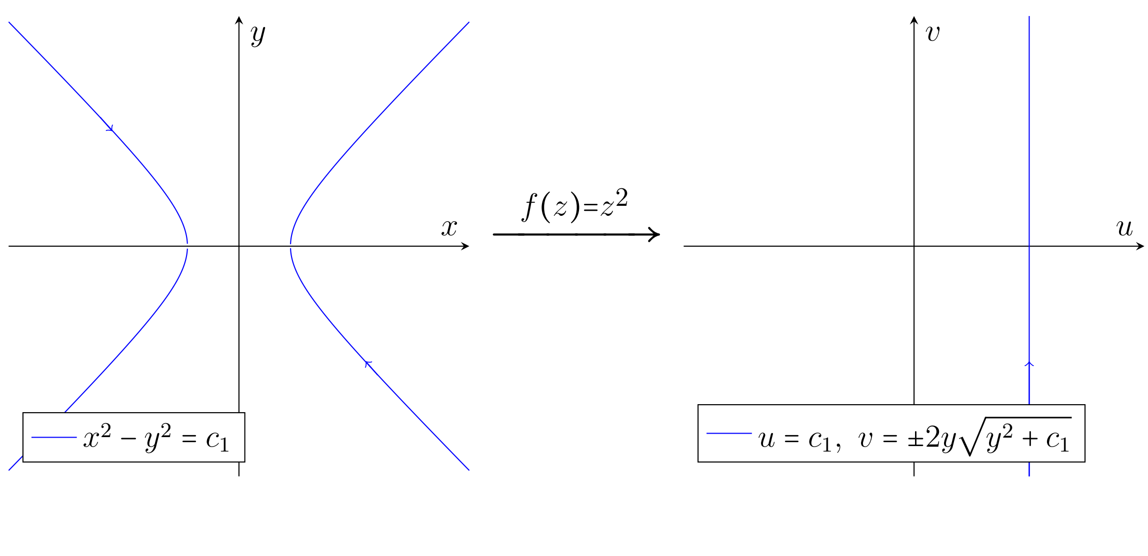
Example 13.1.1 ($f(z) = z^2$)

$$f(z) = z^2 = x^2 - y^2 + i2xy$$
$$\implies u(x,y) = x^2 - y^2 \qquad v(x,y) = 2xy$$

By setting $u = x^2 - y^2 = c_1$ where $c_1 \in \mathbb{R}_{>0}$ we can see that

$$u = x^2 - y^2 = c_1 \qquad\qquad v = 2xy = \pm 2y\sqrt{y^2 + c_1}$$

This tells us that in the complex plane of u and v, if we fix u to a constant c_1 and move along $v = \pm 2y\sqrt{y^2 + c_1}$ by incrementing y we draw out two hyperbolas in the complex plane of x and y. This means that the function $f(z) = z^2$ takes points on hyperbolas the complex plane of x and y and translates them onto a vertical line in the complex plane of u and v where u is a constant.



Likewise if we set $v = c_2$ where $c_2 \in \mathbb{R}_{>0}$, we get:

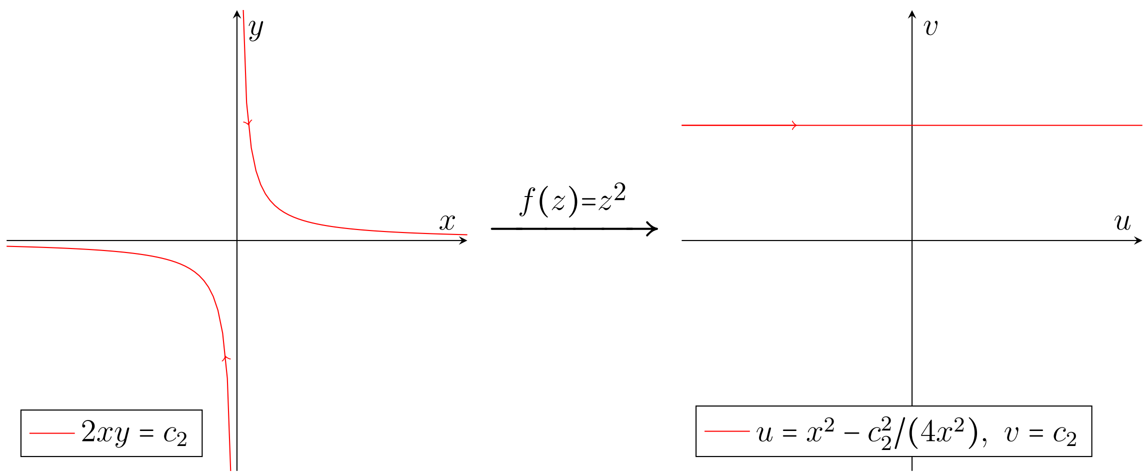
$$u = x^2 - \frac{c_2^2}{4x^2} \qquad\qquad v = 2xy = c_2$$

Taking the limits:

$$\lim_{x\to 0^+} u = -\infty \qquad\qquad \lim_{x\to\infty, x>0} u = \infty \tag{13.1}$$

$$\lim_{x\to -\infty, x<0} u = \infty \qquad\qquad \lim_{x\to 0^-} u = -\infty \tag{13.2}$$

Equation 11.1 tells us as x goes from 0 to ∞, u moves from $-\infty$ to ∞, which corresponds to the hyperbola in the first quadrant of the xy complex plane. Similarly for equations 11.2.



If we look at f using the polar representation, we get $f(z) = r^2 e^{i2\theta}$. This tells us $\forall r \geq 0$, $r \mapsto r^2 = \rho \geq 0$, and $\forall \theta$, $\theta \mapsto \phi = 2\theta$. It is worth noting that mapping of points between $0 \leq 0 < 2\pi$ is not one-to-one, since points in $0 \leq \theta < \pi$ and points in $\pi \leq \theta < 2\pi$ both get mapped to $0 \leq \phi < 2\pi$.

13.2 Limits

Definition 13.2.1: Limit

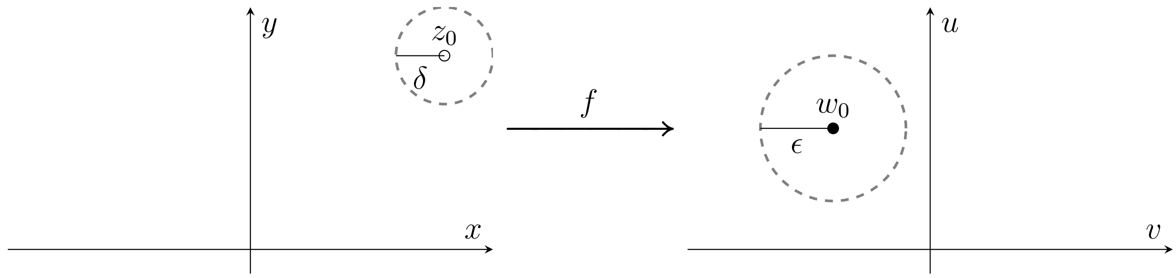
Let $z, z_0, w_0 \in \mathbb{C}$ and f be a function. We say $f(z)$ has limit w_0 as z approaches z_0 if:

$$\forall \epsilon \exists \delta [0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon]$$

We then denote: $\lim_{z\to z_0} f(z) = w_0$

This tells us that $\lim_{z\to z_0} f(z) = w_0$ if some deleted neighbourhood $|z - z_0| < \delta$ corresponds to a neighbourhood $|f(z) - w_0| < \epsilon$. Note that the mapping of all points z in $|z - z_0| < \delta$ to $|f(z) - w_0| < \epsilon$ need not be subjective. It just needs to be mapped less than distance ϵ from w_0.

Note: Definition 13.2.1 allows us to verify if a limit exists, but it is not a method for determining a limit.



Theorem 13.2.1: Uniqueness of Limits
Suppose the limit of f at z_0 exists, then it is unique.

Proof: Suppose two limits of f at z_0 exists for contradiction.

$$
\begin{aligned}
&[\lim_{z\to z_0} f(z) = w_0] \wedge [\lim_{z\to z_0} f(z) = w_1] \\
&\implies [0 < |z - z_0| < \delta_0 \implies |f(z) - w_0| < \epsilon] \wedge [0 < |z - z_0| < \delta_0 \implies |f(z) - w_0| < \epsilon]
\end{aligned}
$$

$$
\begin{aligned}
&w_1 - w_0 = [f(z) - w_0] + [w_1 - f(z)] \\
&\implies |w_1 - w_0| = |[f(z) - w_0] + [w_1 - f(z)]| \leq |f(z) - w_0| + |f(z) - w_1|
\end{aligned}
$$

Now choosing $\delta = \min\{\delta_1, \delta_2\}$, we get:

$$|w_1 - w_0| < \epsilon + \epsilon = 2\epsilon$$

Choosing ϵ to be arbitrary small, we end up with:

$$w_1 - w_0 = 0 \implies w_1 = w_0$$

□

Definition 13.2.1 requires that f be defined at all points in the deleted neighbourhood of z_0. That is, z_0 is interior to the region which f is defined. We can extend the definition by agreeing that $0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon$ also holds for z that lie in the region where f is defined and the deleted neighbourhood of z_0. That is $f(z_0)$ need not be defined for a limit at z_0 to exist.

Example 13.2.1 *Show* $(f(z) = iz/2) \wedge (|z| < 1) \implies \lim_{z\to 1} f(z) = i/2$.

We can see that we have restricted the domain of f to the region $|z| < 1$, this puts $z = 1$ right at the boundary of the domain of definition of f.

$$
\begin{aligned}
|z| < 1 &\implies \left|f(z) - \frac{i}{2}\right| = \left|\frac{iz}{2} - \frac{i}{2}\right| = \frac{|z-1|}{2} \\
&\implies \forall z \forall \epsilon \exists \delta \left[0 < |z - 1| < \delta = 2\epsilon \implies \left|f(z) - \frac{i}{2}\right| < \epsilon\right] \\
&\implies \lim_{z\to 1} f(z) = \frac{i}{2}
\end{aligned}
$$

This highlights the fact that if the limit exists, then z is allowed to approach z_0 from any arbitrary direction.

Example 13.2.2 *Limit of $f(z) = z/\bar{z}$ does not exist at $z = 0$*

Consider $\lim_{z\to 0} f(z)$. Let us approach the limit from the x-axis and the y-axis.

$$\lim_{z=(x,0)\to 0} f(z) = \frac{x + i0}{x - i0} = 1 \qquad\qquad \lim_{z=(0,y)\to 0} f(z) = \frac{0 + iy}{0 - iy} = -1$$

We end up with two different limits. As limits are unique, we conclude that $\lim_{z\to 0} f(z)$ does not exist.

13.2.1 Limit Theorems

Theorem 13.2.2:

Consider $f(z) = u(x,y) + iv(x,y)$. Let $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$.

$$\left[\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0\right] \wedge \left[\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0\right] \iff \lim_{z\to z_0} f(z) = w_0$$

Proof: $\underline{\implies:}$

By definition:

$$\begin{aligned} &\left[\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0\right] \wedge \left[\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0\right] \\ &\implies \forall\epsilon\exists\delta_1,\delta_2\left[\left(0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_1 \implies |u - u_0| < \frac{\epsilon}{2}\right)\right. \\ &\qquad\qquad \left.\wedge \left(0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_2 \implies |v - v_0| < \frac{\epsilon}{2}\right)\right] \end{aligned} \tag{13.3}$$

Triangle inequality for the distance between points:

$$\begin{aligned} |(u + iv) - (u_0 - iv_0)| &= |(u - u_0) + i(v - v_0)| \le |u - u_0| + |v - v_0| \\ \sqrt{(x-x_0)^2 + (y-y_0)^2} &= |(x - x_0) + i(v - v_0)| = |(x + iy) - (x_0 - iy_0)| \end{aligned}$$

Let $\delta = \min\{\delta_1, \delta_2\}$, it follows from eq. (13.3):

$$0 < |(x + iy) - (x_0 + iy_0)| < \delta \implies |(u + iv) - (u_0 - iv_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, $\lim_{z\to z_0} f(z) = w_0$.

$\underline{\impliedby:}$

Suppose $\lim_{z\to z_0} f(z) = w_0$.

$$\begin{aligned} &\lim_{z\to z_0} f(z) = w_0 \\ &\implies \forall\epsilon\exists\delta > 0[|(x + iy) - (x_0 - iy_0)| < \delta \implies |(u + iv) - (u_0 + iv_0)| < \epsilon] \end{aligned} \tag{13.4}$$

By the triangle inequality:

$$|u - u_0| \leq |(u - u_0) + i(v - v_0)| = |(u + iv) - (u_0 + iv_0)|$$
$$|v - v_0| \leq |(u - u_0) + i(v - v_0)| = |(u + iv) - (u_0 + iv_0)|$$

$$|(x + iy) - (x_0 + iy_0)| = |(x - x_0) + i(y - y_0)| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

Thus, it follows from the inequalities in eq. (13.4):

$$\begin{aligned} &0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \\ &\implies [|u - u_0| < \epsilon] \wedge [|v - v_0| < \epsilon] \\ &\implies \left[\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0\right] \wedge \left[\lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0\right] \end{aligned}$$

□

Theorem 13.2.3:

Suppose

$$\left[\lim_{z\to z_0} f(z) = w_0\right] \wedge \left[\lim_{z\to z_0} F(z) = W_0\right]$$

Then

$$\lim_{z\to z_0}[f(z) + F(z)] = w_0 + W_0$$
$$\lim_{z\to z_0}[f(z)F(z)] = w_0 W_0$$
$$\lim_{z\to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0} \qquad W_0 \neq 0$$

Proof: Let:

$$f(z) = u(x, y) + iv(x, y) \qquad F(z) = U(x, y) + iV(x, y)$$

$$z_0 = x_0 + iy_0 \qquad w_0 = u_0 + iv_0 \qquad W_0 = U_0 + iV_0$$

$\underline{\lim_{z\to z_0}[f(z) + F(z)] = w_0 + W_0}$

From Theorem 13.2.2:

$$\begin{aligned} &f(z) + F(z) = (u + U) + i(v + V) \\ &\implies \lim_{(x,y)\to(x_0,y_0)} f(z)F(Z) = (u_0 + U_0) + i(v_0 + V_0) = w_0 + W_0 \end{aligned}$$

$\underline{\lim_{z\to z_0}[f(z)F(z)] = w_0 W_0}$

From Theorem 13.2.2:

$$\begin{aligned}
&f(z)F(z) = (uU - vV) + i(vU + uV) \\
&\implies \lim_{(x,y)\to(x_0,y_0)} f(z)F(Z) = (u_0U_0 - v_0V_0) + i(v_0U_0 + u_0V_0) = w_0W_0
\end{aligned}$$

$\lim_{z\to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}$ if $W_0 \neq 0$

From Theorem 13.2.2:

$$\frac{f(z)}{F(z)} = \frac{u+iv}{U+iV} \implies \lim_{(x,y)\to(x_0,y_0)} \frac{f(z)}{F(z_0)} = \frac{u_0+v_0}{U_0+iV_0} = \frac{w_0}{W_0}$$

□

Corollary 13.2.3.1:

Let c be a constant, $z, z_0 \in \mathbb{C}$, and $P(z)$ be a polynomial. Then

$$\lim_{z\to z_0} c = c \qquad \lim_{z\to z_0} z = z_0 \qquad \lim_{z\to z_0} z^n = z_0^n \qquad n \in \mathbb{N}$$

$$\lim_{z\to z_0} P(z) = P(z_0)$$

Observation. *It is surprisingly quick that Brown and Churchill went from ϵ–δ proofs straight to proving with limits. This is different to the approach in Sequences of Limits Theorem for Sequences Section by Kennith A. Ross. [1]. (Section 9.1.1)*

Question. *It might be possible use a series approach to prove limit theorems for $z \in \mathbb{C}$ by having separate series for x and y (real and imaginary components of z), or a series in the form of $s_n = (x_n, y_n)$. Which would be the proper approach?*

13.2.2 Limits of Points at Infinity

Definition 13.2.2: Extended Complex Plane

The complex plane union with the points at infinity:

$$\mathbb{C} \cup \{\pm\infty, \pm i\infty\}$$

Definition 13.2.3: Riemann Sphere

A unit sphere centred at the origin of the complex plane, which is consequently bisected by the complex plane.

Definition 13.2.4: Stereographic Projection

Consider the Riemann Sphere. Let N be the northern point of the sphere (the point on the sphere above the origin of the complex plane) and z be any point in the complex plane. Let l be a line that goes through N and z, then l will intersect the Riemann Sphere. Let P be the point where l intersects the Riemann Sphere. If we let N correspond to the points at infinity, then there is a one-to-one correspondence between points on the sphere and the points on the extended complex plane. This correspondence is called the Stereographic Projection. (Figure 13.1)

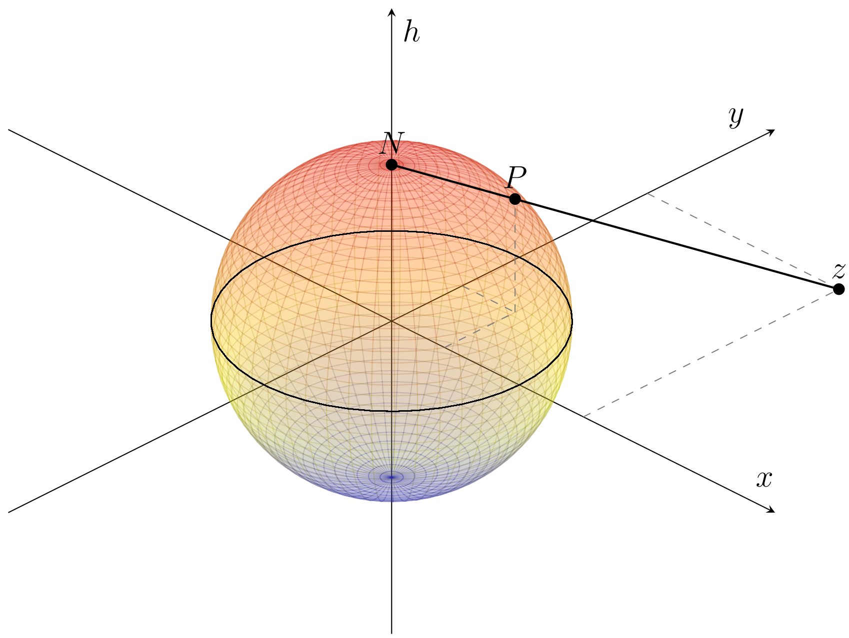


Figure 13.1: Riemann Sphere and Stereographic Projection

The region outside the unit circle enveloped by the Riemann sphere corresponds to the upper hemisphere of the Riemann sphere, with the point N deleted. N corresponds to the points at infinity, since l will be parallel to the complex plane.

Note: In some texts, the Riemann Sphere is a sphere of unit diameter (not a unit sphere, which is of unit radius) sitting on top of the Complex Plane. That is, with the south pole sitting at $(0, 0)$. The definitions for line L, and points N, P, and z remains the same. In either case, the Stereographic Projection maps to a unique point P on the sphere, and the definition of the point at infinity remains unchanged.

Definition 13.2.5: Neighbourhood of ∞

The set: $\{|z| > 1/\epsilon : \epsilon \in \mathbb{R}_{>0}\}$

Note that since ϵ is a small positive number, $|z| > 1/\epsilon$ corresponds to points far away from the unit circle, hence P is close to N.

Note: When referring to any point z, it is referring to a point in the finite plane. Points at infinity will be specifically mentioned.

Definition 13.2.6: Limit at Infinity

Let $f(z)$ be a function, and $z, z_0 \in \mathbb{C}$.

$$\forall \epsilon \in \mathbb{R}_{>0}, \exists r \in \mathbb{R}_{>0}[|z| > r \implies |f(z) - z_0| < \epsilon] \iff \lim_{z\to\infty} f(z) = z_0$$

That is, if $\forall z$ in the neighbourhood of infinity implies $|f(z) - z_0| < \epsilon$, then $\lim_{z\to\infty} f(z) = z_0$.

Theorem 13.2.4:

Let $z_0, w_0 \in \mathbb{C}$, then

$$\lim_{z\to z_0} \frac{1}{f(z)} = 0 \implies \lim_{z\to z_0} f(z) = \infty$$
$$\lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0 \implies \lim_{z\to\infty} f(z) = w_0$$
$$\lim_{z\to 0} \frac{1}{f(1/z)} = 0 \implies \lim_{z\to\infty} f(z) = \infty$$

Proof: $\underline{\lim_{z\to z_0} \frac{1}{f(z)} = 0 \implies \lim_{z\to z_0} f(z) = \infty}$

$$\begin{aligned}\lim_{z\to z_0} \frac{1}{f(z)} = 0 &\implies \forall \epsilon \exists \delta > 0\left[|z - z_0| < \delta \implies \left|\frac{1}{fz} - 0\right| < \epsilon\right] \\ &\implies \forall \epsilon \exists \delta > 0\left[|z - z_0| < \delta \implies |f(z)| > \frac{1}{\epsilon}\right] \\ &\implies \lim_{z\to z_0} f(z) = \infty\end{aligned}$$

$\underline{\lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0 \implies \lim_{z\to\infty} f(z) = w_0}$

$$\begin{aligned}\lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0 &\implies \forall \epsilon \exists \delta > 0\left[|z - 0| < \delta \implies \left|f\left(\frac{1}{z}\right) - w_0\right| < \epsilon\right] \\ &\implies \forall \epsilon \exists \delta > 0\left[|z| > \frac{1}{\delta} \implies |f(z) - w_0| < \epsilon\right] \\ &\implies \lim_{z\to\infty} f(z) = w_0\end{aligned}$$

$\underline{\lim_{z\to 0} \frac{1}{f(1/z)} = 0 \implies \lim_{z\to\infty} f(z) = \infty}$

$$\begin{aligned}\lim_{z\to 0} \frac{1}{f(1/z)} = 0 &\implies \forall \epsilon \exists \delta > 0\left[|z - 0| < \delta \implies \left|\frac{1}{f(1/z)} - 0\right| < \epsilon\right] \\ &\implies \forall \epsilon \exists \delta > 0\left[|z| > \frac{1}{\delta} \implies |f(z)| > \frac{1}{\epsilon}\right] \\ &\implies \lim_{z\to\infty} f(z) = \infty\end{aligned}$$

$\square$

Note: As δ goes to 0, $1/\delta$ goes to ∞, hence $|z|$ goes to ∞ if $|z| > 1/\delta$.

Observation. *As expected, theorem 13.2.4 is consistent if $z \in \mathbb{R}$. (Check: Section 9.1.1).*

13.3 Continuity

Definition 13.3.1: Continuous

Let f be a function. We say f is continuous at all point $z_0 \in \mathbb{C}$ if it satisfies the following:

$$\lim_{z \to z_0} f(z) \text{ exists } \wedge f(z_0) \text{ exists } \wedge \lim_{z \to z_0} f(z) = f(z_0)$$

Note:

$$\lim_{z \to z_0} f(z) = f(z_0) \implies \lim_{z \to z_0} f(z) \text{ exists } \wedge f(z_0) \text{ exists}$$
$$\forall \epsilon \exists \delta > 0 \, [|z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon] \iff \lim_{z \to z_0} f(z) = f(z_0)$$

Definition 13.3.2: Continuous at a Region

Let f be a function, $R \subset \mathbb{C}$ be a region, and $z \in R$:

$$f \text{ is continuous in } R \iff \forall z \in R (f \text{ is continuous})$$

Theorem 13.3.1:

Let $f(z)$ and $g(z)$ be continuous functions at $z_0 \in \mathbb{C}$. Then the following are also continuous at z_0:

$$f(z_0) + g(z_0) \qquad f(z_0)g(z_0) \qquad \frac{f(z_0)}{g(z_0)} \qquad g(z_0) \neq 0$$

Proof: Consequence of theorem 13.2.3. □

Corollary 13.3.1.1:

Let $P(z)$ be a polynomial, then $P(z)$ is continuous $\forall z \in \mathbb{C}$. That is $P(z)$ is continuous in the entire plane of $\mathbb{C}$.

Proof: Consequence of corollary 13.2.3.1. □

Observation. *Both theorem 13.3.1 and corollary 13.3.1.1 rely on definition 13.3.1, which state for a function f and point $z_0 \in \mathbb{C}$:*

$$\lim_{z \to z_0} f(z) \text{ exists } \implies f(z) \text{ is continuous at } z_0$$

This is why the proofs cite the results of theorem 13.2.3 and corollary 13.2.3.1.

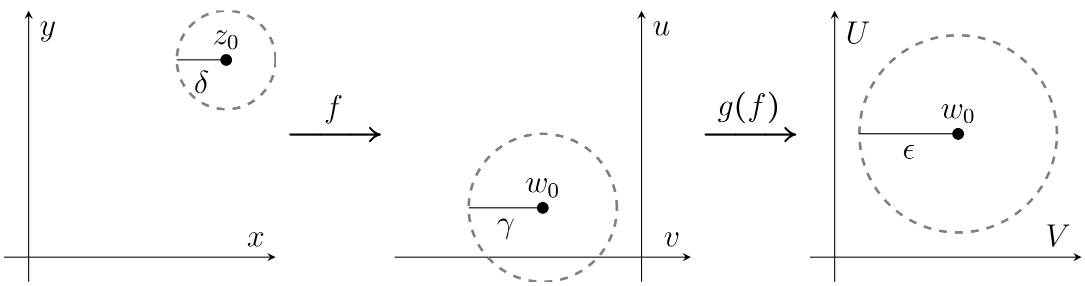
Theorem 13.3.2:
Let $f(z)$ and $g(z)$ be functions.

$$f(z) \text{ and } g(z) \text{ continuous} \implies g(f(z)) \text{ continuous}$$

Proof: Let $f(z) = w$ be defined in the neighbourhood $\forall z[|z - z - 0| < \delta]$, and $g(w) = W$ where $\text{dom}(g) = \text{img}(f)$. Suppose that f is continuous at z_0 and g is continuous at $f(z_0)$.

$$\begin{aligned} f \text{ continuous at } z_0 &\iff \forall\gamma\exists\delta > 0\,[|z - z_0| < \delta \implies |f(z) - f(z_0)| < \gamma] \\ &\implies \forall\epsilon\exists\gamma > 0\,[|f(z) - f(z_0)| < \gamma \implies |g(f(z)) - g(f(z_0))| < \epsilon] \end{aligned}$$

We can always find a small enough δ for γ to satisfy $|g(f(z)) - g(f(z_0))| < \epsilon$. □



Theorem 13.3.3:
Let $f(z)$ be a function and $f(z_0) \neq 0$.

$$f(z_0) \neq 0 \implies \exists\epsilon\forall z[|f(z) - f(z_0)| < \epsilon \implies f(z) \neq 0]$$

That is, if $f(z_0) \neq 0$ then it has a neighbourhood where $f(z) \neq 0$.

Proof: Suppose $f(z)$ is continuous and non-zero at z_0, and let $\epsilon = |f(z_0)|/2$:

$$\exists z[f(z) = 0] \wedge \forall\epsilon\exists\delta > 0\left[|z - z_0| < \delta \implies |f(z) - f(z_0)| < \frac{|f(z_0)|}{2}\right]$$

$$\implies |f(z_0)| < \frac{|f(z_0)|}{2} \qquad \text{Contradiction!}$$

□

Theorem 13.3.4:
Let $f(z) = u(x, y) + iv(x, y)$ be a function, and $z = x + iy$, $z \in \mathbb{C}$.

$$f \text{ continuous at } z_0 \iff [u \text{ continuous at } z_0] \wedge [v \text{ continuous at } z_0]$$

Proof: Direct consequence of theorem 13.2.2 □

Theorem 13.3.5:
Let f be continuous in a closed and bounded region R, then

$$\forall z \in R, \exists M \in \mathbb{R}_{>0} [|f(z) \leq M|] \wedge |\{z : f(z) = M\}| \geq 1$$

That is, for $\forall z \in R$, $|f(z)| \leq M$ and there is at least one point z where $|f(z)| = M$. $f(z)$ is bounded in R.

Proof: Let $f(z) = u(x,y) + iv(x,y)$ be continuous, then

$$\left(|f(z)| = \sqrt{[u(x,y)]^2 + [v(x,y)]^2} \text{ is continuous in } R\right) \wedge (\exists M \in \mathbb{R}_{>0}[|f(z)| < M])$$

□

13.3.1 Exercises

Example 13.3.1 *Prove:*

$$\lim_{z \to z_0} f(z) = w_0 \implies \lim_{z \to z_0} |f(z)| = |w_0|$$

Note: $||f(z_0)| - |w_0|| \leq |f(z) - w_0|$

Proof: Use definition of limit, then plug and chug. □

Example 13.3.2 *Prove: Limits involving points at infinity are unique.*

Proof: Suppose that limit of the point at infinity is not unique, that is there is two neighbourhoods of infinity. Using he definition of the limit, we will arrive at a contradiction where the two neighbourhoods are the same. □

Example 13.3.3 *Prove:*

$$S \text{ is unbounded} \iff \forall \epsilon \exists z \left[z \in S : |z| > \frac{1}{\epsilon} \right]$$

That is, S is unbounded $\iff$ every neighbourhood of the point at infinity contains at least one point in S

Proof: Proof Sketch: Recall the Riemann Sphere. (Definition 13.2.3). The set $|z| > 1/\epsilon$ corresponds to the points close to N, which is the neighbourhood of the point at infinity. If we let $\gamma = 2\epsilon$, $\exists z$ where $|z| > 1/\gamma$ holds. This along with $z \in \mathbb{C}$ (which is S in our case), implies the direction $\impliedby$ is true. That is, we can still find elements in S as we shrink the circle around N.

S is unbounded implies that for all circle with radius R centred at the origin there is at least one element of $s \in S$ where $|s| > R$. Suppose for contradiction that there is a neighbourhood of the point at infinity that does not contain any points in S. We will arrive at a contradiction, where there is $M \in \mathbb{R}_{>0}$ such that $\forall s \in S[|s| < M]$. Thus S is bounded, a contradiction. This implies that the direction $\implies$ is true. □

13.4 Differentiation

Definition 13.4.1: Derivative

Let f be a function where $|z - z_0| < \epsilon$ and $z \in \operatorname{dom}(f)$. Then the derivative of f at point z_0:

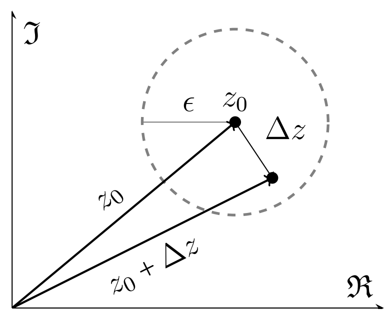
$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Definition 13.4.2: Differentiable

A function f is differentiable at $z_0 \in \mathbb{C}$ if $f'(z_0)$ exists.

If we let $\Delta z = z - z_0$ where $z \neq z_0$:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$



There's another notation by letting $\Delta w = f(z + \Delta z) - f(z)$:

$$f'(z) = \frac{\mathrm{d}w}{\mathrm{d}z} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

Observation. *The definition of a derivative in definition 13.4.1 looks similar to that of a derivative for the real numbers:*

$$F'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

However, the existence of $f'(z)$ possesses a much stronger requirement than the existence of $F'(z)$. That is, let $f(z) = u(x, y) + iv(x, y)$. The existence of $f'(z)$ at point z_0 requires the existence of both $u'(x, y)$ and $v'(x, y)$.

$$f'(z_0) = \lim_{(x,y) \to (x_0,y_0)} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{(x,y) \to (x_0,y_0)} \frac{u(z) - u(z_0)}{z - z_0} + i\frac{v(z) - v(z_0)}{z - z_0}$$

and that

$$\begin{aligned}&\lim_{(x,y_0)\to(x_0,y_0)} \frac{u(x,y_0)-u(x_0,y_0)}{x-x_0}+i\frac{v(x_0,y_0)-v(x_0,y_0)}{x-x_0}\\ &=\lim_{(x_0,y)\to(x_0,y_0)} \frac{u(x_0,y)-u(x_0,y_0)}{x-x_0}+i\frac{v(x_0,y)-v(x_0,y_0)}{x-x_0}\end{aligned}$$

That is

$$\begin{aligned}\lim_{(\Delta x,0)\to(0,0)} \frac{u(x_0+\Delta x,y_0)-u(x_0,y_0)}{\Delta x}&=\lim_{(0,\Delta y)\to(0,0)} \frac{u(x_0,y_0+\Delta y)-u(x_0,y_0)}{\Delta y}\\ \lim_{(\Delta x,0)\to(0,0)} \frac{v(x+\Delta x,y_0)-v(x_0,y_0)}{\Delta x}&=\lim_{(0,\Delta y)\to(0,0)} \frac{v(x_0,y+\Delta y)-v(x_0,y_0)}{\Delta y}\end{aligned}$$

This tells us that the existence of a derivative for a real valued function $F(x)$ does not imply the existence of a derivative for a similar function $f(z)$ in the complex plane, which we will see later. (i.e. Take $f(z)=|z|^2$ and $F(x)=|x|^2$.) We are dealing with a two-dimensional limit instead of a one dimensional limit.

Question. *Under what conditions will differentiability in$\mathbb{C}$ imply differentiability in $\mathbb{R}$, and vice versa?*

Example 13.4.1 *Let $f(z)=\bar{z}$:*

$$\frac{\Delta w}{\Delta z}=\frac{\overline{z+\Delta z}-\bar{z}}{\Delta z}=\frac{\bar{z}+\overline{\Delta z}-\bar{z}}{\Delta z}=\frac{\overline{\Delta z}}{\Delta z}$$

Consider $\Delta z=(\Delta x,\Delta y)\to(0,0)$. If we move on the real axis, that is $(\Delta x,0)$:

$$\overline{\Delta z}=\overline{\Delta x+i0}=\Delta x-i0=\Delta x+i0=\Delta z \implies \frac{\Delta w}{\Delta z}=\frac{\overline{\Delta z}}{\Delta z}=\frac{\Delta z}{\Delta z}=1$$

If we move on the imaginary axis, that is $(0,\Delta y)$:

$$\overline{\Delta z}=\overline{0+i\Delta y}=0-i\Delta y=-\Delta z \implies \frac{\Delta w}{\Delta z}=\frac{\overline{\Delta z}}{\Delta y}=\frac{-\Delta z}{\Delta z}=-1$$

Limits are unique, so the limit of $\mathrm{d}w/\mathrm{d}z$ does not exist anywhere.

Example 13.4.2 *Consider $f(z)=|z|^2$:*

$$\begin{aligned}\frac{\Delta w}{\Delta z}&=\frac{|z+\Delta z|^2-|z|^2}{\Delta z}=\frac{(z+\Delta z)(\overline{z+\Delta z})-z\bar{z}}{\Delta z}\\ &=\frac{(z+\Delta z)(\bar{z}+\overline{\Delta z})-z\bar{z}}{\Delta z}=\frac{z\bar{z}+\Delta z\bar{z}+\overline{\Delta z}z+\overline{\Delta z}\Delta z-z\bar{z}}{\Delta z}=\bar{z}+\overline{\Delta z}+z\frac{\overline{\Delta z}}{\Delta z}\end{aligned}$$

As in the previous example, as $(\Delta x,\Delta y)\to(0,0)$:

$$\begin{aligned}\overline{\Delta z}&=\Delta z && \text{From the real axis}\\ \overline{\Delta z}&=-\Delta z && \text{From the imaginary axis}\end{aligned}$$

Thus

$$\frac{\Delta w}{\Delta z} = \bar{z} + \Delta z + z \qquad\qquad \Delta z = (\Delta x, 0)$$
$$\frac{\Delta w}{\Delta z} = \bar{z} - \Delta z - z \qquad\qquad \Delta z = (0, \Delta y)$$

Therefore, by uniqueness of limits as $\Delta z \to 0$*:*

$$\lim_{\Delta z \to 0}(\bar{z} + \Delta z + z) = \lim_{\Delta z \to 0}(\bar{z} - \Delta z - z) \implies z = -z \implies z = 0$$

Hence, $\mathrm{d}w/\mathrm{d}z$ *does not exist for* $z \neq 0$*. We can also see that:*

$$\frac{\Delta w}{\Delta z} = \bar{z} + \overline{\Delta z} + z\frac{\overline{\Delta z}}{\Delta z} = \overline{\Delta z} \qquad\qquad z = 0$$

Thus, $\mathrm{d}w/\mathrm{d}z$ *only exists at* $z = 0$*:*

$$\left.\frac{\mathrm{d}w}{\mathrm{d}z}\right|_{z=0} = 0$$

Remark. *The following are facts:*

(1) A function $f(z)$ can be differentiable at a point z_0, but nowhere else in the neighbourhood of z_0.

(2) $f(z) = |z|^2 \implies u(x,y) = x^2 + y^2 \wedge v(x,y) = 0$, hence $u(x,y)$ and $v(x,y)$ can have continuous partial derivatives of all orders at a point z_0, even though f may not be differentiable at z_0.

(3) $f(z)$ differentiable at $z_0 \implies f(z)$ continuous at z_0

Proof: Assume $f'(z_0)$ exists:

$$\lim_{z \to z_0}[f(z) - f(z_0)] = \lim_{z \to z_0}\frac{f(z) - f(z_0)}{z - z_0}\lim_{z \to z_0}(z - z_0) = f'(z_0) \cdot 0 = 0$$
$$\implies \lim_{z \to z_0} f(z) = f(z_0)$$

So, f is differentiable at $z_0 \implies f$ is continuous at z_0. □

Note: Continuity of a function at $z_0 \in \mathbb{C} \nRightarrow$ existence of derivative at point z_0.

Ex: $f(z) = |z|^2$ is continuous everywhere in $\mathbb{C}$ for $z_0 \neq 0$, but $f(z_0)$ does not exist at z_0.

13.4.1 Differentiation Rules

Definition of derivative in $\mathbb{C}$ (definition 13.4.1) is the same of that in $\mathbb{R}$, so rules remain the same.

Let $c \in \mathbb{C}$ be a constant and functions f and g be differentiable at point z. Then

$$\frac{\mathrm{d}}{\mathrm{d}z}c = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}z}z = 1 \qquad \frac{\mathrm{d}}{\mathrm{d}z}[cf(z)] = cf'(z) \qquad \frac{\mathrm{d}}{\mathrm{d}z}z^n = nz^{n-1} \qquad n \in \mathbb{Z} \smallsetminus \{0\}$$

Let functions f and g be differentiable at point z. Then

$$\frac{\mathrm{d}}{\mathrm{d}z}[f(z) + g(z)] = f'(z) + g'(z) \qquad \frac{\mathrm{d}}{\mathrm{d}z}[f(z)g(z)] = f(z)g'(z) + f'(z)g(z)$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2}$$

Proof: Deriving: $\frac{\mathrm{d}}{\mathrm{d}z}[f(z)g(z)] = f(z)g'(z) + f'(z)g(z)$

Let $w = f(z)g(z)$:

$$\begin{aligned}\Delta w &= f(z+\Delta z)g(z+\Delta z) - f(z)g(z)\\ &= f(z)[g(z+\Delta z) - g(z)] + [f(z+\Delta z) - f(z)]g(z+\Delta z)\end{aligned}$$

Thus

$$\frac{\Delta w}{\Delta z} = f(z)\frac{g(z+\Delta z) - g(z)}{\Delta z} + \frac{f(z+\Delta z) - f(z)}{\Delta z}g(z+\Delta z)$$

Hence

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \lim_{\Delta z \to 0}\frac{\Delta w}{\Delta z} = f(z)g'(z) + f'(z)g(z)$$

□

Theorem 13.4.1: Chain Rule for Composite Functions

Let function f be differentiable at z_0 and function g be differentiable at $f(z_0)$. Then $F(z) = g[f(z)]$ is differentiable at z_0.

$$F'(z_0) = g'[f(z_0)]f'(z_0)$$

Proof: Suppose f is differentiable at z_0. Let $w_0 = f(z_0)$ and assume that $g'(w_0)$ exists. Then

$$\forall w \exists \epsilon[|w - w_0| < \epsilon \implies \Phi(w_0) = 0]$$

Where

$$\Phi(w) = \frac{g(w) - g(w_0)}{w - w_0} - g'(w_0) \qquad w \neq w_0$$

Note: $\lim_{w \to w_0}\Phi(w) = 0$, so Φ is continuous at w_0. Then

$$g(w) - g(w_0) = [g'(w_0) + \Phi(w)](w - w_0) \qquad |w - w_0| < \epsilon$$

Note: This is valid for $w = w_0$.

$$\begin{aligned} f'(z_0) \text{ exists } &\implies f \text{ continuous at } z_0 \\ &\implies \forall \epsilon \exists \delta > 0 [|z - z_0| < \delta \implies |w - w_0| < \epsilon] \end{aligned}$$

Hence, we can replace w by $f(z)$ when $|z - z_0| < \delta$. Subbing $w = f(z)$ and $w_0 = f(z_0)$:

$$\frac{g[f(z)] - g[f(z_0)]}{z - z_0} = \{g'[f(z_0)] + \Phi[f(z)]\} \frac{f(z) - f(z_0)}{z - z_0} \qquad 0 < |z - z_0| < \delta, \; z \neq z_0$$

Then

$$(f \text{ continous at } z_0) \wedge (\Phi \text{ continuoust at } w_0 = f(z_0)) \implies \Phi[f(z)] \text{ continuous at } z_0$$

$$\Phi(w_0) = 0 \implies \lim_{z \to z_0} \Phi[f(z) = 0]$$

Thus

$$\begin{aligned} \lim_{z \to z_0} \frac{g[f(z)] - g[f(z_0)]}{z - z_0} &= \lim_{z \to z_0} \{g'[f(z_0)] + \Phi[f(z)]\} \frac{f(z) - f(z_0)}{z - z_0} \\ &= g'[f(z_0)] f'(z_0) \end{aligned}$$

We then get

$$F'(z_0) = g'[f(z_0)] f'(z_0)$$

□

Alternatively, if we let $w = f(z)$ and $W = F(z)$, then the Chain Rule becomes:

$$\frac{\mathrm{d}W}{\mathrm{d}z} = \frac{\mathrm{d}W}{\mathrm{d}w} \frac{\mathrm{d}w}{\mathrm{d}z}$$

Note: Although this looks like a fraction, it is not a fraction and should not be treated as such! (Logical inconsistency when infinitesimals when viewed as ratios.)

Theorem 13.4.2: L'Hopital's Rule

Suppose $f(z_0) = 0$ and $g(z_0) = 0$, $f'(z_0)$ and $g(z_0)$ exists, with $g'(z_0) \neq 0$. Then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

Proof: Let $f(z_0) = 0$, $g(z_0) = 0$, and $z \neq z_0$.

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{\frac{f(z) - f(z_0)}{z - z_0}}{\frac{g(z) - g(z_0)}{z - z_0}} = \lim_{\Delta z \to 0} \frac{\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}}{\frac{g(z_0 + \Delta z) - g(z_0)}{\Delta z}} = \frac{f'(z_0)}{g'(z_0)}$$

□

13.4.2 Exercises

Example 13.4.3 *Show that $f'(z)$ does not exist for all points $z \in \mathbb{C}$ when:*

(a) $f(z) = \mathrm{Re}\{z\}$

(b) $f(z) = \mathrm{Im}\{z\}$

Proof: Let $f(z) = u(x,y) + iv(x,y)$, $\Delta w = f(x+\Delta x, y+\Delta y) - f(x,y)$.

$\underline{f(z) = \mathrm{Re}\{z\}}$

Recall $\mathrm{Re}\{z\} = x + i0$.

$$\frac{\Delta w}{\Delta z} = \frac{\mathrm{Re}\{z+\Delta z\} - \mathrm{Re}\{z\}}{\Delta z} = \frac{x + \Delta x - x}{\Delta z} = \frac{\Delta x}{\Delta x + \Delta y}$$

Now as $(\Delta x, 0) \to (0,0)$:

$$\lim_{(\Delta x,0)\to(0,0)} \frac{\Delta w}{\Delta z} = \lim_{(\Delta x,0)\to(0,0)} \frac{\Delta x}{\Delta x} = 1$$

Now as $(0, \Delta y) \to (0,0)$:

$$\lim_{(0,\Delta y)\to(0,0)} \frac{\Delta w}{\Delta z} = \lim_{(0,\Delta y)\to(0,0)} \frac{0}{\Delta y} = 0$$

Limits are unique, but this isn't the case, so we conclude that $f'(z)$ when $f(z) = \mathrm{Re}\{z\}$ does not exist.

$\underline{f(z) = \mathrm{Im}\{z\}}$

Recall $\mathrm{Im}\{z\} = 0 + iy$.

$$\frac{\Delta w}{\Delta z} = \frac{\mathrm{Im}\{z+\Delta z\} - \mathrm{Im}\{z\}}{\Delta z} = \frac{y + \Delta y - y}{\Delta z} = \frac{\Delta y}{\Delta x + \Delta y}$$

Now as $(\Delta x, 0) \to (0,0)$:

$$\lim_{(\Delta x,0)\to(0,0)} \frac{\Delta w}{\Delta z} = \lim_{(\Delta x,0)\to(0,0)} \frac{0}{\Delta x} = 0$$

Now as $(0, \Delta y) \to (0,0)$:

$$\lim_{(0,\Delta y)\to(0,0)} \frac{\Delta w}{\Delta z} = \lim_{(0,\Delta y)\to(0,0)} \frac{\Delta y}{\Delta y} = 1$$

Limits are unique, but this isn't the case, so we conclude that $f'(z)$ when $f(z) = \mathrm{Im}\{z\}$ does not exist. $\square$

13.5 Cauchy-Riemann Equations

Theorem 13.5.1: Cauchy-Riemann Equations (Cartesian)
Let $f(z) = u(x,y) + iv(x,y)$. If $f'(z)$ exists at a point $z_0 = x_0 + iy_0$, then $u'(x_0, y_0)$ and $v'(x_0, y_0)$ exists and satisfy Cauchy-Riemann equations:

$$u_x = v_y \qquad\qquad u_y = -v_x$$

Also, as a result of evaluating $f'(z)$ from the horizontal and vertical direction:

$$f'(z_0) = [u_x + iv_x]\Big|_{(x_0,y_0)} = [v_y - iu_y]\Big|_{(x_0,y_0)}$$

Proof: Let $f(z) = u(x,y) + iv(x,y)$, and suppose $f'(z)$ exists at z_0. Then

$$z_0 = x_0 + iy_0 \qquad \Delta z = \Delta x + i\Delta y \qquad \Delta w = f(z_0 + \Delta z) - f(z_0)$$

So that

$$\Delta w = [u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y)] - [u(x_0, y_0) + iv(x_0, y_0)]$$

Therefore

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta x + i\Delta y} + i\frac{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)}{\Delta x + i\Delta y}$$

Note: This equation remains valid as $(\Delta x, \Delta y) \to (0,0)$.

Horizontal Approach:

Let $(\Delta x, 0) \to (0,0)$ in the horizontal direction, then

$$\begin{aligned} f'(z_0) &= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \\ \implies f'(z_0) &= u_x(x_0, y_0) + iv_x(x_0, y_0) \end{aligned}$$

Vertical Approach:

Let $(0, \Delta y) \to (0,0)$ in the vertical direction, then

$$\begin{aligned} f'(z_0) &= \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i \lim_{\Delta x \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y} \\ &= -i \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y} + \lim_{\Delta x \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} \\ \implies f'(z_0) &= v_y(x_0, y_0) - iu_y(x_0, y_0) \end{aligned}$$

Putting it together:

For $f'(z)$ to exists at z_0, $f(z_0)$ from the horizontal approach must equal that of the vertical approach. By equating the real and imaginary parts:

$$\begin{aligned} &u_x(x_0,y_0) + iv_x(x_0,y_0) = v_y(x_0,y_0) - iu_y(x_0,y_0) \\ &\implies (u_x = v_y) \wedge (u_y = -v_x) \end{aligned}$$

□

Theorem 13.5.2: Cauchy-Riemann Equations (Polar)
Let $f(z) = u(r,\theta) + iv(r,\theta)$ be defined in some neighbourhood ϵ of $z_0 = r_0 e^{i\theta_0}$, $z_0 \neq 0$. If the first order partials derivatives of u and v with respect to r and θ exists and are continuous at z_0, and satisfies the polar form of the Cauchy-Riemann equations:

$$ru_r = v_\theta \qquad\qquad u_\theta = -rv_r$$

Then $f'(z_0)$ exists:

$$f'(z_0) = e^{-i\theta}(u_r + iv_r)\Big|_{(r_0,\theta_0)} = \frac{-i}{z_0}(u_\theta + iv_\theta)\Big|_{(r_0,\theta_0)}$$

Proof: Let $f(z) = u(r,\theta) + iv(r,\theta)$. Suppose that the first order partial derivatives of u and v exists in some neighbourhood ϵ of z_0 and is continuous at z_0. By differentiating u with respect to x and y:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \qquad\qquad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

Likewise for v. As $x = r\cos\theta$ and $y = r\sin\theta$:

$$\begin{aligned} u_r &= u_x\cos\theta + u_y\sin\theta & u_\theta &= -u_x r\sin\theta + u_y r\cos\theta \\ v_r &= v_x\cos\theta + v_y\sin\theta & v_\theta &= -v_x r\sin\theta + v_y r\cos\theta \end{aligned}$$

From theorem 13.5.1 we have:

$$u_x = v_y \qquad\qquad u_y = -v_x$$

Subbing the Cauchy-Riemann equations into v_r and v_θ:

$$\begin{aligned} u_r &= u_x\cos\theta + u_y\sin\theta & u_\theta &= -u_x r\sin\theta + u_y r\cos\theta \\ v_r &= -u_y\cos\theta + u_x\sin\theta & v_\theta &= u_y r\sin\theta + u_x r\cos\theta \end{aligned}$$

We can see that:

$$ru_r = v_\theta \qquad\qquad u_\theta = -rv_r$$

Which are the Cauchy Riemann equations in polar form. Let's verify it without relying on the Cauchy-Riemann equations in Cartesian form:

Recall:

$$u_r = u_x \cos\theta + u_y \sin\theta \qquad u_\theta = -u_x r \sin\theta + u_y r \cos\theta$$
$$v_r = v_x \cos\theta + v_y \sin\theta \qquad v_\theta = -v_x r \sin\theta + v_y r \cos\theta$$

Writing u_r and v_r in matrix notation:

$$\begin{bmatrix} u_r \\ u_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

Solving for u_x and u_y:

$$\begin{aligned} \begin{bmatrix} u_x \\ u_y \end{bmatrix} &= \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}^{-1} \begin{bmatrix} u_r \\ u_\theta \end{bmatrix} \\ &= \frac{1}{r\cos^2\theta + r\sin^2\theta} \begin{bmatrix} r\cos\theta & -\sin\theta \\ -r\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_r \\ u_\theta \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{r} \begin{bmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_r \\ u_\theta \end{bmatrix} \end{aligned}$$

It is clear that for u_x and u_y, and likewise for v_x and v_y:

$$u_x = u_r \cos\theta - \frac{1}{r} u_\theta \sin\theta \qquad u_y = u_r \sin\theta + \frac{1}{r} u_\theta \cos\theta \tag{13.5}$$
$$v_x = v_r \cos\theta - \frac{1}{r} v_\theta \sin\theta \qquad v_y = v_r \sin\theta + \frac{1}{r} v_\theta \cos\theta \tag{13.6}$$

Using the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$, we see:

$$u_r \cos\theta - \frac{1}{r} u_\theta \sin\theta = v_r \sin\theta + \frac{1}{r} v_\theta \cos\theta$$
$$u_r \sin\theta + \frac{1}{r} u_\theta \cos\theta = -v_r \cos\theta + \frac{1}{r} v_\theta \sin\theta$$

Clearly, the equations are equal only if

$$r u_r = v_\theta \qquad u_\theta = -r v_r$$

Which are the polar forms of the Cauchy-Riemann equations.

Show $f'(z_0) = e^{-i\theta}(u_r + i v_r)$:

Recall from theorem 13.5.1:

$$f'(z_0) = u_x + i v_y$$

Using eq. (13.5) and eq. (13.6) from before and substituting them into $f'(z_0)$:

$$
\begin{aligned}
f'(z_0) &= \left(u_r \cos\theta - \frac{1}{r}u_\theta \sin\theta + iv_r\cos\theta - \frac{i}{r}v_\theta\sin\theta\right)\Big|_{(r_0,\theta_0)} \\
&= \left(u_r\cos\theta + v_r\sin\theta + iv_r\cos\theta - iu_r\sin\theta\right)\Big|_{(r_0,\theta_0)} \\
&= \left[u_r(\cos\theta - i\sin\theta) + v_r(\sin\theta + i\cos\theta)\right]\Big|_{(r_0,\theta_0)} \\
&= \left[u_r(\cos\theta - i\sin\theta) + iv_r(\cos\theta - i\sin\theta)\right]\Big|_{(r_0,\theta_0)} \\
&= \left[\left(\frac{e^{i\theta}+e^{-i\theta}}{2} - \frac{e^{i\theta}-e^{-i\theta}}{2}\right)(u_r + iv_r)\right]\Big|_{(r_0,\theta_0)} \\
&= e^{-i\theta}(u_r + iv_r)\Big|_{(r_0,\theta_0)} \\
&= \frac{-i}{re^{i\theta}}(u_\theta + iv_\theta)\Big|_{(r_0,\theta_0)} = \frac{-i}{z_0}(u_\theta + iv_\theta)\Big|_{(r_0,\theta_0)} \qquad (ru_r = v_\theta)\wedge(u_\theta = -rv_r)
\end{aligned}
$$

Thus

$$f'(z_0) = e^{-i\theta}(u_r + iv_r)\Big|_{(r_0,\theta_0)} = \frac{-i}{z_0}(u_\theta + iv_\theta)\Big|_{(r_0,\theta_0)}$$

□

Question. *When comparing the Cartesian form to the polar form of the Cauchy-Riemann equations:*

$$
\begin{aligned}
&f'(z_0) \textit{ exists} \implies \forall z_0[(u_x = v_y)\wedge(u_y = -v_x)] \\
&(z_0 \neq 0)\wedge\forall z_0[(ru_r = v_\theta)\wedge(u_\theta = -rv_r)] \implies f'(z_0) \textit{ exists}
\end{aligned}
$$

Should both be $\iff$ instead of $\implies$? No, satisfying Cauchy-Riemann equations does not guarantee differentiability at a point as we will see in example 13.5.3. However, satisfying certain conditions allows allows differentiability to exist (theorem 13.5.4).

Example 13.5.1 (Solving the $f'(z)$ using the partial derivative with respect to one variable) *Recall in theorem 13.5.1:*

$$f'(z_0) = [u_x + iv_x]\Big|_{(x_0,y_0)} = [v_y - iu_y]\Big|_{(x_0,y_0)}$$

This implies we can solve $\mathrm{d}f(z)/\mathrm{d}z$ by taking the partial of $f(z)$ with respect to x or y. Consider $f(z) = z^2$:

$$f(z) = z^2 = x^2 - y^2 + i2xy$$

We then have:

$$u(x,y) = x^2 - y^2 \qquad\qquad v(x,y) = 2xy$$

Hence

$$u_x = 2x = v_y \qquad\qquad u_y = -2y = -v_x$$

Thus

$$f'(z) = 2x + i2y = 2(x+iy) = 2z$$

Example 13.5.2 (Using Cauchy-Riemann equations to find where $f(z)$ is not differentiable) *Using the contrapositive of $f'(z_0)$ exists $\implies \exists u' \exists v'[(u_x = v_y) \wedge (u_y = -v_x)]$:*

$$\exists z_0[(u_x \neq v_y) \vee (u_y \neq -v_x)] \implies f(z) \text{ not differentiable at } z_0$$

Consider $f(z) = |z|^2$:

$$u(x,y) = x^2 + y^2 \qquad\qquad v(x,y) = 0$$

By Cauchy-Riemann:

$$2x = 0 \qquad\qquad 2y = 0$$

Therefore, $f'(z)$ only exists at $(0,0)$ and does not exist elsewhere.

Note: Theorem 13.5.1 does not guarantee the existence of $f'(z)$ at z_0.

Example 13.5.3 ($f(z)$ satisfy Cauchy-Riemann equations at (0,0), but $f'(0)$ does not exist) *Consider*

$$f(z) = \begin{cases} \bar{z}^2/z & z \neq 0 \\ 0 & z = 0 \end{cases}$$

Then

$$u(x,y) = \frac{x^3 - 3xy^2}{x^2+y^2} \qquad v(x,y) = \frac{y^3 - 3x^2y}{x^2+y^2} \qquad (x,y) \neq (0,0)$$

Checking differentiability at $(0,0)$, note $u(0,0) = 0$ and $v(0,0) = 0$:

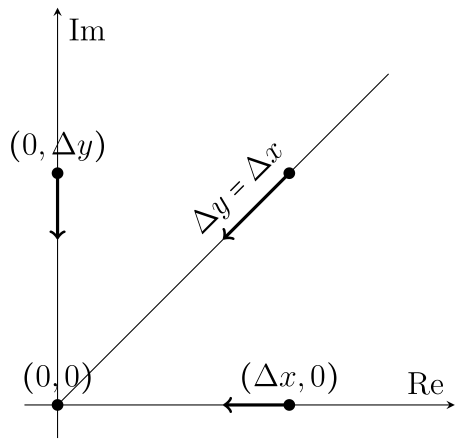
$$\begin{aligned}
u_x(0,0) &= \lim_{\Delta x \to 0} \frac{u(0+\Delta x, 0) - u(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1 \\
v_y(0,0) &= \lim_{\Delta y \to 0} \frac{v(0, 0+\Delta y) - v(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y}{\Delta y} = 1 \\
u_y(0,0) &= \lim_{\Delta y \to 0} \frac{u(0, 0+\Delta y) - u(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0/(\Delta y)^2}{\Delta y} = 0 \\
v_x(0,0) &= \lim_{\Delta x \to 0} \frac{v(0+\Delta x, 0) - v(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0/(\Delta x)^2}{\Delta x} = 0
\end{aligned}$$

We can see that the Cauchy-Riemann equations are satisfied:

$$u_x = v_y = 1 \qquad\qquad u_y = -v_x = 0$$

However, $f'(0)$ does not exist: (Brown and Churchill - Complex Variables and Applications, Section 20, Exercise 9 [2])

Let $\Delta w = f(z+\Delta z) - f(z)$. We need to show for all nonzero points on the real and imaginary axis, $\Delta w/\Delta z = -1$, but for all nonzero points on the line $\Delta x = \Delta y$, $\Delta w/\Delta z = -1$. Hence, a contradiction, so $f('0)$ does not exist.



$$\frac{\Delta w}{\Delta z} = \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{u(x+\Delta x, y+\Delta y) + v(x+\Delta x, y+\Delta y)}{\Delta x + \Delta y} - \frac{u(x,y) + v(x,y)}{\Delta x + \Delta y}$$

Along the real axis:

Evaluating along $(\Delta x, 0) \to (0,0)$.

$$\begin{aligned}\lim_{(\Delta x,0)\to(0,0)} \frac{\Delta w}{\Delta z} &= \frac{u(\Delta x,0) + v(\Delta x,0)}{\Delta x} - \frac{u(0,0)+v(0,0)}{\Delta x} \\ &= \frac{1}{\Delta x}\left[\frac{(\Delta x)^3}{(\Delta x)^2} + \frac{0}{(\Delta x)^2}\right] - 0 = \frac{\Delta x}{\Delta x} = 1\end{aligned}$$

Along the imaginary axis:

Evaluating along $(0, \Delta y) \to (0,0)$.

$$\begin{aligned}\lim_{(0,\Delta y)\to(0,0)} \frac{\Delta w}{\Delta z} &= \frac{u(0,\Delta y) + v(0,\Delta y)}{\Delta y} - \frac{u(0,0)+v(0,0)}{\Delta y} \\ &= \frac{1}{\Delta y}\left[\frac{0}{(\Delta y)^2} + \frac{(\Delta y)^3}{(\Delta y)^2}\right] - 0 = \frac{\Delta y}{\Delta y} = 1\end{aligned}$$

Along the axis $\Delta x = \Delta y$:

Evaluating along $(\Delta x, \Delta x) \to (0,0)$.

$$\lim_{(\Delta x,\Delta x)\to(0,0)} \frac{\Delta w}{\Delta z} = \frac{u(\Delta x,\Delta x)}{\Delta x + \Delta x} - \frac{u(0,0)+v(0,0)}{\Delta x+\Delta x}$$
$$= \frac{1}{2\Delta x}\left[\frac{(\Delta x)^3 - 3(\Delta x)^3}{2(\Delta x)^2} + \frac{(\Delta x)^3 - 3(\Delta x)^3}{2(\Delta x)^2}\right]$$
$$= \frac{1}{2\Delta x}\left[-\frac{2(\Delta x)^3}{2(\Delta x)^2} - \frac{2(\Delta x)^3}{2(\Delta x)^2}\right] = \frac{1}{2\Delta x}\left[-\Delta x - \Delta x\right] = -\frac{2\Delta x}{2\Delta x} = -1$$

As we can see, the limits are not unique regardless of the path we take to approach $(0,0)$, hence $f'(0)$ does not exist. Therefore, an equation can satisfy the Cauchy-Riemann equations at $0,0$, yet have a derivative that does not exist. The Cauchy-Riemann equations does not guarantee differentiability at z_0.

Example 13.5.4 *(Any branch of $f(z) = z^{1/2}$ is differentiable everywhere in domain of definition) Let*

$$f(z) = z^{1/2} = \sqrt{r}e^{i\theta} \qquad\qquad r > 0,\ \alpha < \theta < \alpha + 2\pi$$

Hence

$$u(r,\theta) = \sqrt{r}\cos\left(\frac{\theta}{2}\right) \qquad\qquad v(r,\theta) = \sqrt{r}\sin\left(\frac{\theta}{2}\right)$$

By Cauchy-Riemann:

$$ru_r = \frac{\sqrt{r}}{2}\cos\left(\frac{\theta}{2}\right) = v_\theta \qquad\qquad u_\theta = -\frac{\sqrt{r}}{2}\sin\left(\frac{\theta}{2}\right) = -rv_r$$

Thus, the derivative exists wherever $f(z)$ is defined. Also, by theorem 13.5.2:

$$f'(z) = e^{i\theta}(u_r + iv_r)\Big|_{(r_0,\theta_0)}$$
$$= e^{-i\theta}\left[\frac{1}{2\sqrt{r}}\cos\left(\frac{\theta}{2}\right) + i\frac{1}{2\sqrt{r}}\sin\left(\frac{\theta}{2}\right)\right] = \frac{1}{2\sqrt{r}}e^{-i\theta}\left[\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right]$$
$$= \frac{1}{2\sqrt{r}e^{i\theta/2}} = \frac{1}{2f(z)} = \frac{1}{2}z^{-1/2}$$

13.5.1 Complex Form of the Cauchy-Riemann Equations

Theorem 13.5.3: Cauchy-Riemann Equation (Complex Form)
Let $f(z) = u(x,y) + iv(x,y)$. If the first order partial derivatives of u and v with respect to x and y exists and satisfy the Cauchy-Riemann equations. Then

$$\frac{\partial}{\partial \bar{z}} f(z) = 0$$

Proof: Recall:

$$x = \frac{z + \bar{z}}{2} \qquad\qquad y = \frac{z - \bar{z}}{2i}$$

Let F be a real valued function, that is $x, y \in \mathbb{R}$. Then

$$\frac{\partial F}{\partial \bar{z}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \bar{z}}$$

Substituting $\frac{\partial x}{\partial \bar{z}} = 1/2$ and $\frac{\partial y}{\partial \bar{z}} = i/2$:

$$\frac{\partial F}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right)$$

Define the operator:

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Then

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right)\\ &= \frac{1}{2}\left[(u_x - v_y) + i(u_y + v_x)\right]\end{aligned}$$

We can see that if $\frac{\partial f}{\partial \bar{z}}$ satisfies the Cauchy-Riemann equations (theorem 13.5.1):

$$\frac{\partial}{\partial \bar{z}} f(z) = 0 \qquad\qquad \frac{\partial}{\partial x} f = -i\frac{\partial f}{\partial y} \implies i\frac{\partial}{\partial x} f = \frac{\partial f}{\partial y}$$

□

13.5.2 Conditions for Differentiability

Theorem 13.5.4:
Let $f(z) = u(x, y) + iv(x, y)$ be defined in some neighbourhood ϵ of point $z_0 = x_0 + iy_0$. Consider the first order partial derivatives of u and v with respect to x and y. If they

(1) Exist for all z, $|z - z_0| < \epsilon$.

(2) Are continuous at z_0.

(3) Satisfies the Cauchy-Riemann equations at z_0.

Then $f'(z_0)$ exists:

$$f'(z_0) = (u_x + iv_x)\Big|_{(x_0, y_0)}$$

Proof: Assume the first order partial derivatives of u and v with respect to x and y exists $\forall z[|z - z_0| < \epsilon]$, are continuous at z_0, and satisfies the Cauchy-Riemann equations. Let $\Delta z = \Delta x + i\Delta y$, $0 < |\Delta z| < \epsilon$, and $\Delta w = f(z_0 + \Delta z) - f(z_0)$. We then have

$$\Delta w = \Delta u + i\Delta v$$

Where

$$\begin{aligned}\Delta u &= u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)\\ \Delta v &= v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)\end{aligned}$$

Since first order partials of u and v are continuous at z_0:

$$\begin{aligned}\Delta u &= u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y\\ \Delta v &= v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \epsilon_3\Delta x + \epsilon_4\Delta y\end{aligned}$$

$$(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \to (0, 0, 0, 0) \text{ as } (\Delta x, \Delta y) \to (0, 0)$$

Substituting Δu and Δv into Δw:

$$\begin{aligned}\Delta w =& u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y\\ &+ i[v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \epsilon_3\Delta x + \epsilon_4\Delta y]\end{aligned}$$

Using the Cauchy-Riemann equations and dividing by Δz:

$$\frac{\Delta w}{\Delta z} = u_x(x_0, y_0) + iv_x(x_0, y_0) + (\epsilon_1 + i\epsilon_3)\frac{\Delta x}{\Delta z} + (\epsilon_2 + i\epsilon_4)\frac{\Delta y}{\Delta z}$$

From the inequalities $|\Delta x| \leq |\Delta z|$ and $|\Delta y| \leq |\Delta z|$:

$$\left|\frac{\Delta x}{\Delta z}\right| \leq 1 \qquad\qquad \left|\frac{\Delta y}{\Delta z}\right| \leq 1$$

So

$$\begin{aligned}\left|(\epsilon_1 + i\epsilon_3)\frac{\Delta x}{\Delta z}\right| &\leq |\epsilon_1 + i\epsilon_3| \leq |\epsilon_1| + |\epsilon_3|\\ \left|(\epsilon_2 + i\epsilon_4)\frac{\Delta y}{\Delta z}\right| &\leq |\epsilon_2 + i\epsilon_4| \leq |\epsilon_2| + |\epsilon_4|\end{aligned}$$

Then $|\epsilon_2| + |\epsilon_4| \to 0$ and $|\epsilon_1| + |\epsilon_3| \to 0$ as $\Delta z = \Delta x + i\Delta y \to 0$.

$$\implies \frac{\Delta w}{\Delta z} = u_x(x_0, y_0) + iv_x(x_0, y_0) \implies f'(z_0) \text{ exists}$$

□

Example 13.5.5 (All 3 conditions must be satisfied for $f'(z_0)$ to exist) *Do not use expression of $f'(z)$ before existence of $f'(z_0)$ is established. Consider $f(z) = x^3 + i(1 - y)^3$.*

$$u(x, y) = x^3 \qquad\qquad v(x, y) = (1 - y)^3$$

Taking the partial derivatives:

$$u_x = 3x^2 \qquad\qquad v_x = 0$$
$$u_y = 0 \qquad\qquad v_y = -3(1-y)^2$$

It would be foolish to ignore Cauchy-Riemann and directly use:

$$f'(z) = u_x + iv_x = 3x^2$$

We can see that the Cauchy-Riemann equations are satisfied only if:

$$3x^2 = -3(1-y)^2 \implies x^2 + (1-y)^2 = 0 \implies (x=0) \wedge (y=1)$$

Therefore, $f'(z)$ exists only if $z = i$, and that $f'(i) = 0$

13.6 Analytic Functions

Definition 13.6.1: Analytic/Regular/Holomorphic
Let S be an open set, $S \subset \mathbb{C}$. Let f be a function.

$$f \text{ is analytic in } S \iff \forall z \in S[f'(z) \text{ exists }]$$

We say $f(z)$ is analytic at a point z_0 if it is analytic in some neighbourhood of z_0. If we say that $f(z)$ is analytic in a closed set S' then we mean that it is analytic in an open set S where $S' \subset S$.

Definition 13.6.2: Entire
A function $f(z)$ is entire if it is analytic at all points in the plane.

Example 13.6.1

$$\text{Derivative of polynomial exists everywhere} \implies \text{All polynomials are entire functions}$$

See section 13.5.2 for conditions for a function te be differentiable, hence analytic in a set S.

Corollary 13.6.0.1:
Let $f(z)$ and $g(z)$ be analytic in a domain D. Then the following are analytic in D:

$$f(z) + g(z)$$
$$f(z)g(z)$$
$$\frac{f(z)}{g(z)} \qquad\qquad g(z) \neq 0 \forall z \in D$$

Likewise, if $P(z)$ and $Q(z)$ are polynomials, then $P(z)/Q(z)$ is analytic if $\forall z \in D[Q(z) \neq 0]$.

Corollary 13.6.0.2:
Let w be the image of D under $f(z)$ and w be the domain of g. Then $g(f(z))$ is analytic in D and

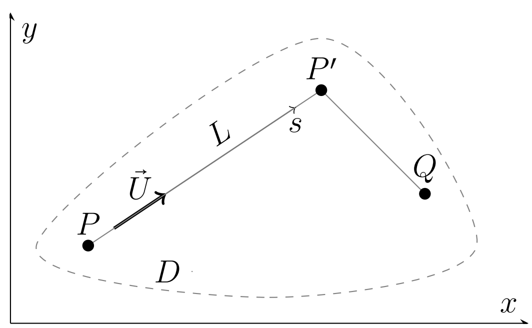
$$\frac{\mathrm{d}}{\mathrm{d}z}g[f(z)] = g'[f(z)]f'(z)$$

Theorem 13.6.1:
Let D be the domain of a function $f(z)$.

$$\forall z \in D[f'(z) = 0] \implies f(z) \text{ is constant in } D$$

Proof: Let $f(z) = u(x,y) + iv(x,y)$ with domain D, and P, P', and Q be points in D. Let $\vec{U}$ be the unit vector on the line segment L connecting P and P', and s be the distance along L.

$$f'(z) = 0 \implies \forall z \in D[u_x = u_y = v_x = v_y = 0]$$



We know that the directional derivative:

$$\frac{\mathrm{d}u}{\mathrm{d}s} = \nabla u \cdot \vec{U} \qquad\qquad \nabla u = u_x\hat{i} + u_y\hat{j}$$

Previously, $u_x = u_y = 0$, so for all points on L:

$$u_x = u_y = 0 \implies \nabla u = 0 \implies \frac{\mathrm{d}u}{\mathrm{d}s} = 0 \implies u \text{ constant on } L$$

Now, that we have established that u is constant on any given line L in D, we can see that since D is simply connected and there are finitely many lines connecting P and Q, the values of u at P and Q must be equal and constant. Hence, $\exists a \in \mathbb{R}$ such that $u(x,y) = a$ in D. Likewise, $v(x,y) = b$ in D. Thus

$$f(z) = a + bi = c \qquad\qquad c \text{ is constant}$$

□

Definition 13.6.3: Singular Point
Let ϵ be a neighbourhood of point z_0, and $f(z)$ be a function. z_0 is a singular point if $f'(z_0)$ does not exist, but $f(z)$ is differentiable in all neighbourhoods of z_0.

13.6.1 Examples

Example 13.6.2 (Determining analyticity using Cauchy-Riemann equations) *Consider* $f(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$.

$$u(x,y) = \sin(x)\cosh(y) \qquad\qquad v(x,y) = \cos(x)\sinh(y)$$

Cauchy-Riemann:

$$u_x = \cos(x)\cosh(y) = v_y \qquad\qquad u_y = \sin(x)\sinh(y) = -v_x$$

Therefore, it is clear that $f(z)$ *is entire.*

$$f'(z) = u_x + iv_x = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

Another application of Cauchy-Riemann see that $f'(z)$ *is also entire.*

Example 13.6.3 ($f(z)$ and $\overline{f(z)}$ is analytic in $D \implies f(z)$ is constant in D) *Let*

$$f(z) = u(x,y) + iv(x,y) \qquad\qquad \overline{f(z)} = u(x,y) - iv(x,y) = U(x,y) + iV(x,y)$$

Because of $f(z)$ *and* $\overline{f(z)}$ *is analytic in* D*, the Cauchy-Riemann equations hold:*

$$\begin{aligned} u_x &= v_y & u_y &= -v_x \\ U_x &= V_y & U_y &= -V_x \end{aligned}$$

We can see that:

$$u_x = -v_y = v_y \qquad\qquad u_y = v_x = -v_x$$

Hence, $u_x = 0$ *and* $v_x = 0$*, then we can conclude*

$$f'(z) = 0 \implies f(z) \text{ is constant in } D$$

Example 13.6.4 ($f(z)$ is analytic in D and $|f(z)|$ is constant in $D \implies f(z)$ is constant in D) *Let* $\forall z \in D[|f(z)| = c]$*, where* c *is a constant. It is easy to see that* $c = 0 \implies \forall z \in D[f(z) = 0]$*, so consider* $c \neq 0$*. Then*

$$f(z)\overline{f(z)} = c^2 \neq 0 \implies \forall z \in D[f(z) \neq 0]$$

Thus

$$\overline{f(z)} = \frac{c^2}{f(z)} \qquad \forall z \in D$$

Hence $\overline{f(z)}$ *is analytic everywhere in* D*, so* $f(z)$ *is constant in* D*.*

13.7 Harmonic Functions

Definition 13.7.1: Laplace's Equation

Let $F(x,y)$ be a real-valued function. That is $x, y \in \mathbb{R}$. Laplace's equation:

$$\frac{\partial^2}{\partial x^2}F + \frac{\partial^2}{\partial y^2}F = 0$$

In polar form:

$$r^2 u_{rr}(r,\theta) + r u_r(r,\theta) + u_{\theta\theta}(r,\theta) = 0$$
$$r^2 v_{rr}(r,\theta) + r v_r(r,\theta) + v_{\theta\theta}(r,\theta) = 0$$

See example 13.7.1

Definition 13.7.2: Harmonic

A real-valued function $F(x,y)$ is harmonic in the xy-plane if it satisfies Laplace's equation.

Theorem 13.7.1:

Let D be the domain of a function $f(z) = u(x,y) + iv(x,y)$.

$$f(z) \text{ is analytic in } D \implies u(x,y) \wedge v(x,y) \text{ are harmonic in } D$$

Proof: f is analytic in D, so its component functions must satisfy the Cauchy-Riemann equations:

$$(u_x = v_y) \wedge (u_y = -v_x) \implies (u_{xy} = v_{yy}) \wedge (u_{yx} = -v_{xx})$$
$$(u_x = v_y) \wedge (u_y = -v_x) \implies (u_{xx} = v_{yx}) \wedge (u_{yy} = -v_{xy})$$

Now, we know from calculus that $u_{xy} = u_{yx}$ and $v_{yx} = v_{xy}$, so we conclude

$$u_{xx} + u_{yy} = 0 \qquad\qquad v_{xx} + v_{yy} = 0$$

□

Note: The converse ($\impliedby$) is true for simply connected domains, hence, theorem 13.7.1 becomes $\iff$ in simply connected domains. (R, Boas - Invitation to Complex Analysis. (1987) Section 19.)

Corollary 13.7.1.1:

Let $F(x,y)$ is a real-valued function in a simply connected domain D. Then there exists a function $f(z)$ and $g(z)$ in D such that $f(z) = F(x,y) + iv(x,y)$ and $g(z) = u(x,y) + iF(x,y)$. That is, there exists a function where the real part equals F and a function where the imaginary part equals F.

Definition 13.7.3: Harmonic Conjugate

If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then $v(x,y)$ is the harmonic conjugate of $v(x,y)$. This is not to be confused with the complex conjugate.

Example 13.7.1 *Let $f(z) = u(r,\theta) + iv(r,\theta)$ be analytic in domain $D' = D \setminus \{0\}$. Show $u(r,\theta)$ and $v(r,\theta)$ satisfies the polar form of Laplace's equation.*

Proof: We know from the Polar form of the Cauchy-Riemann equation:

$$ru_r = v_\theta \qquad\qquad u_\theta = -rv_r$$

Operating by $r\,\partial/\partial r$ and $\partial/\partial\theta$, we obtain:

$$\begin{aligned} r\frac{\partial}{\partial r}ru_r &= ru_r + r^2u_{rr} = rv_{\theta r} \\ r\frac{\partial}{\partial r}u_\theta &= ru_{\theta r} = r\frac{\partial}{\partial r}(-rv_r) = -rv_r - r^2v_{rr} \\ \frac{\partial}{\partial\theta}ru_r &= ru_{\theta r} = v_{\theta\theta} \\ \frac{\partial}{\partial\theta}u_\theta &= u_{\theta\theta} = -rv_{r\theta} \end{aligned}$$

We can see that

$$\begin{cases} ru_r + r^2u_{rr} = -u_{\theta\theta} \\ rv_r + r^2v_{rr} = -v_{\theta\theta} \end{cases} \implies \begin{cases} r^2u_{rr} + ru_r + u_{\theta\theta} = 0 \\ r^2v_{rr} + rv_r + v_{\theta\theta} = 0 \end{cases}$$

□

Example 13.7.2 *Let $f(z) = u(x,y)+iv(x,y)$ be analytic in domain D. Consider the families of level curves $u(x,y) = c_1$ and $v(x,y) = c_2$, with $c_1, c_2 \in \mathbb{R}$ being constants. Show for $z_0 = (x_0, y_0) \in \mathbb{C}$ common to $u(x,y) = c_1$ and $v(x,y) = c_2$ and $f'(z_0) \neq 0$, then the lines tangent to $u(x,y) = c_1$ and $v(x,y) = c_2$ at z_0 are orthogonal.*

Note:

$$[u(x,y) = c_1] \wedge [v(x,y) = c_2] \implies \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0\right) \wedge \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0\right)$$

Proof: The tangent lines of $u(x,y)$ and $v(x,y)$ are

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = (u_x, u_y) \qquad\qquad \nabla v = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) = (v_x, v_y)$$

Taking the dot product, and applying the Cauchy-Riemann equations:

$$u_xv_x + u_yv_y = u_x(-u_y) + u_y(u_x) = 0$$

Hence, $u(x,y)$ and $v(x,y)$ are orthogonal.

Note:

$$\begin{aligned} f'(z_0) = 0 &\implies u_x + iv_x = 0 \implies v_y - iu_y = 0 \\ &\implies u_x = u_y = v_x = v_y = 0 \end{aligned}$$

Hence, we can see that $f'(z_0) = 0$ is required for $u(x,y)$ and $v(x,y)$ to exist and be orthogonal. □

13.8 Uniquely Determined Analytic Functions

Lemma 13.8.0.1:

Suppose a function f is analytic throughout domain D, and $f(z) = 0$ $\forall z \in D' \subset D$ or line segment contained in D. Then $f(z) \equiv 0$ throughout D.

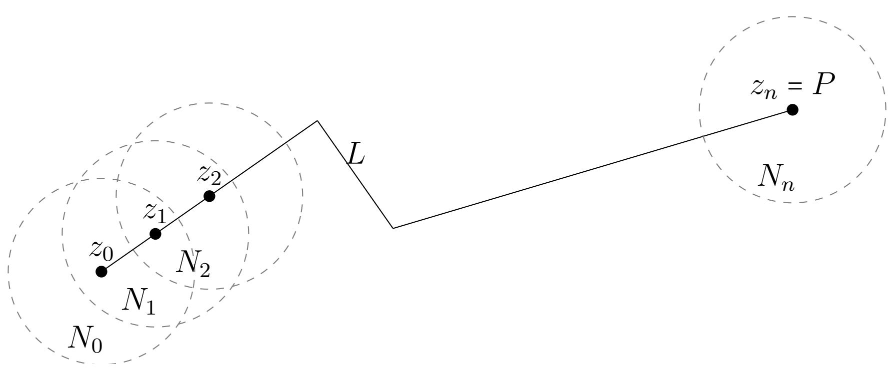
Proof: Let f be a function analytic in domain D and $f(z) = 0$ for all point or line segment in D. Let z_0 in the subdomain of D or on a line segment in D.

D is connected open set, so there is a polygonal line L jointing any point P in D to z_0 lying entirely in D. (Recall: A polygonal line consists of a finite number of lines connected end-to-end.) Let d be the shortest distance from points on L to the boundary on D, so $d > 0$, unless D is the entire plane. Then there is a sequence of points along L:

$$\{z_0, z_1, z_2, \ldots, z_{n-1}, z_n = P\} \qquad |z_k - z_{k+1}| < d \qquad k \in \mathbb{N}$$

That is, each point is sufficiently close to each other. We construct neighbourhoods of each point with radius d, all of which are in D, so points z_{k-1} and z_{k+1} lie in the neighbourhood of z_k, $k \in \mathbb{N}$:

$$\{N_0, N_1, N_2, \ldots, N_{n-1}, N_n\}$$



Now as f is analytic in N_0 and $f(z) = 0$ in a domain or line segment containing z_0, then $f(z) \equiv 0$ in N_0. z_1 is in N_0, so $f(z_0) \equiv 0$ in N_1. Continuing this we can see that $f(z_n) \equiv 0$ in N_n, hence, $f(z) \equiv 0$ in D. □

Theorem 13.8.1:

Let f be analytic in domain D. Then it's uniquely determined over D by its values in D or along a line segment in D.

Proof: Let functions f and g be analytic in some domain D, and $f(z) = g(z)\ \forall z \in D$. Then $h(z) = f(z) - g(z))$ is also analytic in D, and $h(z) = 0$ in the subdomain or along the line segment, so $h(z) \equiv 0$ throughout D. □

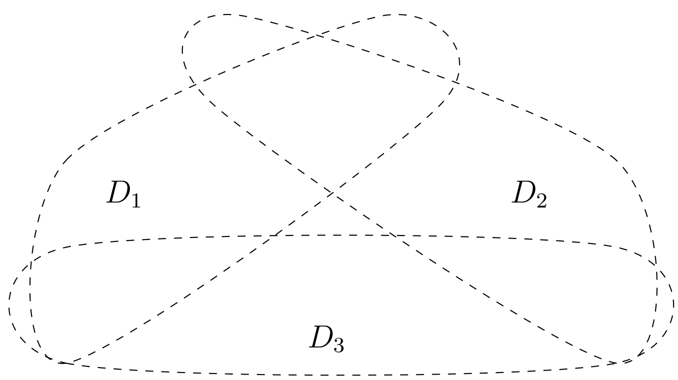
Theorem 13.8.2: Coincidence Principle

If functions f and g are analytic in D and $f(z) = g(z)$ in $D' \subset D$ with limit point $z_0 \in D$, then $f(z) = g(z)$ everywhere in D.

This is a more generalized version of theorem 13.8.1

Definition 13.8.1: Analytic continuation

Consider the domains D_1 and D_2 with intersection $D_1 \cap D_2$, and functions f_1 and f_2. If f_1 is analytic in D_1, and there exists f_2 that is analytic in D_2 such that $f_1(z) = f_2(2)$ for all $z \in D_1 \cap D_2$. Then f_2 is the analytic continuation of f_1.



Theorem 13.8.1 tells us that if such analytic continuation exists, then it is unique. Now if there exists f_3 in D_3 that is an analytic continuation of f_2, then it is not necessarily true that $f_3(z) = f_1(z)$ for all $z \in D_1 \cap D_3$. (See example 13.8.1.)

Definition 13.8.2: Elements of a function

Let f_2 be the analytic continuation of a function f_1 in D_1 into domain D_2, and let $F(z)$ be analytic in $D_1 \cup D_2$.

$$F(z) = \begin{cases} f_1(z) & z \in D_1 \\ f_2(z) & z \in D_2 \end{cases}$$

Then F is the analytic continuation of f_1 and f_2 into $D_1 \cup D_2$, and f_1 and f_2 are elements of F.

13.8.1 Reflection Principle

Generally, $\overline{f(z)} \neq f(\bar{z})$ for all z, but....

Theorem 13.8.3: Reflection Principle

Let f be a function with domain D containing a segment of the real axis $R \subset D$. Then

$$\forall z \in D[\overline{f(z)} = f(\bar{z})] \iff \forall x \in R[f(x) \in \mathbb{R}]$$

See example 13.8.2 for the case when $f(x)$ is purely imaginary.

Proof: Let $f(z)$ and $F(z)$ be analytic functions:

$$f(z) = u(x,y) + iv(x,y) \qquad\qquad F(z) = U(x,y) + iV(x,y)$$

$\underline{(\Longleftarrow):}$

Suppose $\forall x \in R[f(x) \in \mathbb{R}]$, and that $F(z) = \overline{f(\bar{z})}$.

$$f(z) = u(x,y) + iv(x,y) \qquad\qquad F(z) = U(x,y) + iV(x,y)$$

Then

$$\overline{f(\bar{z})} = u(x,-y) - iv(x,-y)$$

Therefore

$$U(x,y) = u(x,t) \qquad\qquad V(x,y) = -v(x,t) \qquad\qquad t = -y$$

$f(x,t)$ is analytic, so it satisfies the Cauchy-Riemann equations:

$$u_x = v_t \qquad\qquad u_t = -v_x$$

Hence

$$U_x = u_x \qquad\qquad V_y = -v_t \frac{\mathrm{d}t}{\mathrm{d}y} = v_t$$

$$U_y = u_t \frac{\mathrm{d}t}{\mathrm{d}y} = -u_t \qquad\qquad V_x = -v_x$$

Thus, we can see that $F(z)$ also satisfies the Cauchy Riemann equations

$$U_x = V_y \qquad\qquad U_y = -V_x$$

Since, the partial derivatives of U and V are continuous in D, we can say hat $F(z)$ is analytic in D. On the segment of the real axis $R \subset D$, $f(z)$ is real, so $v(x,0) = 0$.

$$\begin{aligned} F(x) &= U(x,0) + iV(x,0) = u(x,0) - iv(x,0) = u(x,0) \\ &\Longrightarrow \forall z \in R[F(z) = f(z)] \\ &\Longrightarrow \forall z \in D[\overline{f(\bar{z})} = f(z)] \qquad\qquad \textit{Theorem } 13.8.1 \end{aligned}$$

$(\implies)$:

Suppose $\overline{f(z)} = f(\bar{z})$. Then

$$u(x,-y) - iv(x,-y) = u(x,y) + iv(x,y)$$

Consider any point $(x,0) \in R \subset D$:

$$u(x,0) - iv(x,0) = u(x,0) + iv(x,0) \implies v(x,0) = 0$$

Hence, $f(x)$ is real $\forall x \in R \subset D$. $\square$

Theorem 13.8.3 tells us that if a complex function is real for all points on the real axis, then it will obey the Reflection Principle, and vice versa.

13.8.2 Examples

Example 13.8.1 *Consider*

$$\begin{aligned}
f_1(z) &= \sqrt{r}e^{i\theta/2} && r > 0,\ 0 < \theta < \pi \\
f_2(z) &= \sqrt{r}e^{i\theta/2} && r > 0,\ \frac{\pi}{2} < \theta < 2\pi \\
f_3(z) &= \sqrt{r}e^{i\theta/2} && r > 0,\ \pi < \theta < \frac{5\pi}{2}
\end{aligned}$$

It is clear that $f_1,\ f_2,\ f_3$ are continuous and satisfies the Cauchy-Riemann equations throughout their domain of definition, since they have a derivative everywhere in their domain of definition. Hence, they are analytic continuations of each other. Let $D_1,\ D_2$, and D_3 be the domains of $f_1,\ f_2$, and f_3, respectively. Consider f_1 and f_3 in the domain $D_1 \cap D_3$, and any z in the first quadrant of the complex plane. Then $z = re^{i\theta} = re^{i(\theta+2\pi)}$ and we have

$$\begin{aligned}
f_1(z) &= \sqrt{r}e^{i(\theta/2)} && 0 < \theta < \pi \\
f_3(z) &= \sqrt{r}e^{i(\theta/2+\pi)} && 0 < \theta < \pi
\end{aligned}$$

Hence,

$$\begin{aligned}
f_1(z) &= \sqrt{r}[\cos(\theta/2) + i\sin(\theta/2)] && 0 < \theta < \pi \\
f_3(z) &= \sqrt{r}[\cos(\theta/2+\pi) + i\sin(\theta/2+\pi)] && 0 < \theta < \pi \\
&= -\sqrt{r}[\cos(\theta/2) + i\sin(\theta/2)]
\end{aligned}$$

Thus we can see that $f_1 = -f_3$ in $D_1 \cap D_3$.

Example 13.8.2 *Consider theorem 13.8.3, but $f(x)$ is purely imaginary $\forall x \in \mathbb{R}$. We know that $\Longleftarrow$ holds, and that $F(z) = \overline{f(\bar{z})}$ satisfies the Cauchy-Riemann equations. We have*

$$F(x) = U(x,0) + iV(x,0) = u(x,0) - iv(x,0) = -iv(x,0) = -f(x)$$

Hence

$$\overline{f(\bar{z})} = -f(z) \implies \overline{f(z)} = -f(\bar{z})$$

Chapter 14

Elementary Functions

14.1 Exponential Function

Definition 14.1.1: Exponential Function

Consider $z \in \mathbb{C}$, the exponential function is defined:

$$f(z) = e^z = e^{x+iy} = e^x[\cos(y) + i\sin(y)]$$

Where y is taken in radians.

Note: This is not the same as the polar form of a complex number (definition 12.3.3).

It is clear that the set of n-th roots of e:

$$\{e^{1/n} : n \in \mathbb{N}\}$$

and

$$|e^z| = e^x \qquad \arg(e^z) = y + 2n\pi \qquad n \in \mathbb{N} \cup \{0\}$$

The exponential function follows from the usual properties of exponentials. We also know that

$$\frac{\mathrm{d}}{\mathrm{d}z}e^z = e^z \qquad \forall z \in \mathbb{C}$$

so, e^z is entire. We should also note that e^z is periodic due to e^{iy}.

14.2 Logarithmic Function

Definition 14.2.1: Logarithmic Function

Consider any $z \in \mathbb{C}$ in exponential form:

$$\log(z) = \ln(r) + i(\theta + 2n\pi) = \ln(|z|) + i\arg(z) \qquad n \in \mathbb{Z}$$

Note: This is a multi-valued function.

Definition 14.2.2: Principal Value of the Logarithmic Function

Let $z \in \mathbb{C}$, the principal value of the logarithmic function is denoted by $\mathrm{Log}(z)$*.*

$$\mathrm{Log}(z) = \ln(r) + i\theta$$

It is clear that

$$\log(z) = \mathrm{Log}(z) + 2n\pi \qquad n \in \mathbb{Z}$$

and for any z on the real axis, the logarithmic function reduces to

$$\mathrm{Log}(z) = \ln(x) \qquad x \in \mathbb{R}$$

14.2.1 Branches and Derivatives of Logarithms

$\log(z)$ is a multi-valued function. Let $\alpha \in \mathbb{R}$:

$$\log(z) = \ln(r) + i\theta = u(r,\theta) + iv(r,\theta) \qquad\qquad r > 0, \alpha < \theta < \alpha + 2\pi$$

Note: If $\log(z)$ is defined on $\theta = \alpha$, then it is not continuous there, as there is a discontinuity between points near α and $\alpha + 2\pi$.

The first order partials of u and v are continuous in the domain, and satisfies the Cauchy-Riemann equations:

$$ru_r = v_\theta \qquad\qquad u_\theta = -rv_\theta$$

So its derivative exists everywhere in the domain.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\log(z) = e^{-i\theta}(u_r + iv_r) = e^{i\theta}\left(\frac{1}{r} + i0\right) = \frac{1}{re^{i\theta}}\\ \implies \frac{\mathrm{d}}{\mathrm{d}z}\log(z) = \frac{1}{z} \qquad\qquad & |z| > 0,\ \alpha < \arg(z) < \alpha + 2\pi\\ \implies \frac{\mathrm{d}}{\mathrm{d}z}\mathrm{Log}(z) = \frac{1}{z} \qquad\qquad & |z| > 0,\ -\pi < \mathrm{Arg}(z) < \pi\end{aligned}$$

Definition 14.2.3: Branch

A branch is a single-valued function F of a multi-valued function f. F is analytic throughout some domain of f and assumes the one of the values of f.

Definition 14.2.4: Principal Branch

$$\mathrm{Log}(z) = \ln(r) + i\theta \qquad\qquad r > 0,\ -\pi < \theta < \pi$$

Definition 14.2.5: Branch Cut

A portion of a line or curved introduced to define a branch F of a multi-valued function f. Points on the branch cut of F are singular points of F.

Definition 14.2.6: Branch Point
A singular point common to all branch cuts of a multi-valued function f.

Example 14.2.1 *The branch cut for* $\text{Log}(z) = \ln(r) + i\theta$, $r > 0$, $-\pi < \theta < \pi$, *is the origin and* $\theta = \pi$.

Branch points for all branches of $\log(z)$ *is the origin.*

Different branches may result in different values.

Example 14.2.2 *Consider* $\log(i^2)$ *in the branch:*

$$\log(z) = \ln(r) + i\theta \qquad\qquad r > 0,\ \frac{\pi}{4} < \theta < \frac{9\pi}{4}$$

Then

$$\log\left(i^2\right) = \log(-1) = \ln(1) + i\pi = i\pi$$
$$2\log(i) = 2\left(\ln(1) + i\frac{\pi}{2}\right) = \pi i$$

Therefore

$$\log\left(i^2\right) = 2\log(i) \qquad\qquad r > 0,\ \frac{\pi}{4} < \theta < \frac{9\pi}{4}$$

Now consider the branch:

$$\log(z) = \ln(r) + i\theta \qquad\qquad r > 0,\ \frac{3\pi}{4} < \theta < \frac{11\pi}{4}$$

Then

$$\log\left(i^2\right) = \log(-1) = \ln(1) + i\pi = i\pi$$
$$2\log(i) = 2\left(\ln(1) + i\frac{5\pi}{2}\right) = 5\pi i$$

Therefore,

$$\log\left(i^2\right) \neq 2\log(i) \qquad\qquad r > 0,\ \frac{3\pi}{4} < \theta < \frac{11\pi}{4}$$

14.2.2 Identities of Logarithms

Let $z_1, z_2 \in \mathbb{C}$, then

$$\log(z_1 z_2) = \log(z_1) + \log(z_2)$$

can be interpreted as

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

therefore

$$\ln|z_1 z_2| + i\arg(z_1 z_2) = (\ln|z_1| + i\arg(z_1)) + (\ln|z_2| + i\arg(z_2))$$

Rest of the identities are the same as for elements in $\mathbb{R}$, but beware of branches and arguments.

Example 14.2.3 *Show* $\forall z_1, z_2 \in \mathbb{C}$

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log}(z_1) + \operatorname{Log}(z_2) + 2N\pi i \qquad N \in \{0, \pm 1\}$$

Consider:

$$\begin{aligned}\log(z_1 z_2) &= \ln|z_1 z_2| + i\arg(z_1 z_2)\\ &= \ln(r_1) + \ln(r_2) + i\arg(z_1) + i\arg(z_2)\\ &= \ln(r_1) + \ln(r_2) + i\theta_1 + i\theta_2 + 2n\pi i && n \in \mathbb{Z}\\ &= \ln(r_1) + \ln(r_2) + i\operatorname{Arg}(z_1) + i\operatorname{Arg}(z_2) + 2n\pi i && n \in \mathbb{Z}\end{aligned}$$

Then, since $-\pi < \operatorname{Arg}(z_1) < \pi$ *and* $-\pi < \operatorname{Arg}(z_2) < \pi$*:*

$$\begin{aligned}\operatorname{Log}(z_1 z_2) &= \ln(r_1) + \ln(r_2) + i\operatorname{Arg}(z_1) + i\operatorname{Arg}(z_2) + 2N\pi i && N \in \{0, \pm 1\}\\ &= \operatorname{Log}(z_1) + \operatorname{Log}(z_2) + 2N\pi i && N \in \{0, \pm 1\}\end{aligned}$$

14.2.3 Power Function

Definition 14.2.7: Power Function

Let $z, c \in \mathbb{C}$*. The Power Function:*

$$z^c = e^{c\log(z)} \qquad z \neq 0$$

Likewise

$$c^z = e^{z\log(c)} \qquad c \neq 0$$

The logarithm is multi-valued $\implies$ the power function is multi-valued.

The principle branch of the Power Function is log being replaced by Log:

$$\begin{aligned}z^c &= e^{c\operatorname{Log}(z)} && z \neq 0\\ c^z &= e^{z\operatorname{Log}(c)} && c \neq 0\end{aligned}$$

When a branch is specified, $\log(z)$ becomes single-valued and analytic. Hence the derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}z} z^c = \frac{\mathrm{d}}{\mathrm{d}z} e^{c\log(z)} = \frac{c}{z} e^{c\log(z)} = cz^{z-1} \qquad |z| > 0,\ \alpha < \arg(z) < \alpha + 2\pi,\ \alpha \in \mathbb{R}$$

When value of $\log(c)$ is specified, c^z is entire function of z and

$$\frac{\mathrm{d}}{\mathrm{d}z} c^z = \frac{\mathrm{d}}{\mathrm{d}z} e^{z\log(c)} = e^{z\log(c)}\log(c) = c^z\log(c)$$

14.3 Trigonometric Functions

Recall: Definition 12.5.2. Likewise for any $z \in \mathbb{C}$:

Definition 14.3.1: Complex Sine and Cosine Functions

For any $z \in \mathbb{C}$:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

Sine and cosine are entire functions as e^{iz} and e^{-iz} are entire.

Taking the derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}z}e^{iz} = ie^{iz} \implies \left(\frac{\mathrm{d}}{\mathrm{d}z}\sin(z) = \cos z\right) \wedge \left(\frac{\mathrm{d}}{\mathrm{d}z}\cos(z) = -\sin(z)\right)$$

It's also easy to see that:

$$\sin(-z) = \sin(z) \qquad \cos(-z) = -\cos(z) \qquad e^{iz} = \cos(z) + i\sin(z)$$

The usual trigonometric identities apply, such as:

$$\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2)$$
$$\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin(z_2)\sin(z_2)$$

Now suppose $y \in \mathbb{R}$, and take the hyperbolic functions:

$$\sinh(y) = \frac{e^{y} - e^{-y}}{2} \qquad \cosh(y) = \frac{e^{y} + e^{-y}}{2}$$

Then we get:

$$\sin(iy) = i\sinh(y) \qquad \cos(iy) = \cosh(y)$$

If we let $z = x + iy$, we can define:

$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$
$$\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

and that

$$|\sin(z)|^2 = \sin^2(x) + \sinh^2(y)$$
$$|\cos(z)|^2 = \cos^2(x) + \sinh^2(y)$$

Note: Unlike in $\mathbb{R}$ where sine and cosine are bounded by 1 and -1, it is clear that sine and cosine are not bounded in the complex plane, since sinh is unbounded for all values of y.

14.3.1 Zeros and Singularities

Definition 14.3.2: Zero (function)
Let $f(z)$ be a function. A zero of f is a point z_0 such that

$$f(z_0) = 0$$

Theorem 14.3.1:
The zeros of $\cos(z)$ and $\sin(z)$ for $z \in \mathbb{C}$ is the same as the zeros of $\cos(x)$ and $\sin(x)$ for $x \in \mathbb{R}$, that is

$$\forall x \in \mathbb{R} \forall z \in \mathbb{C} \forall n \in \mathbb{Z}\left[(\cos(x) = 0) \wedge (\cos(z) = 0) \iff z = x = \frac{\pi}{2} + n\pi\right]$$
$$\forall x \in \mathbb{R} \forall z \in \mathbb{C} \forall n \in \mathbb{Z}\left[(\sin(x) = 0) \wedge (\sin(z) = 0) \iff z = x = n\pi\right]$$

Proof: Let $z = x + iy$ and consider $\sin(z) = 0$:

$$\begin{aligned} \sin(z) = 0 &\implies \sin^2(x) + \sinh^2(y) = 0 && |\sin(z)|^2 = \sin^2(x) + \sinh^2(y) \\ &\implies [\sin(x) = 0] \wedge [\sinh(y) = 0] \\ &\implies [x = n\pi] \wedge [y = 0] && n \in \mathbb{Z} \\ &\implies z = x = n\pi && n \in \mathbb{Z} \end{aligned}$$

As for cosine, we know that:

$$\cos(z) = \sin\left(z + \frac{\pi}{2}\right)$$

Thus

$$\cos(z) = 0 \implies z = x = n\pi + \frac{\pi}{2} \qquad n \in \mathbb{Z}$$

□

Example 14.3.1 *Show $\forall z \in \mathbb{C}$:*
The Reflection Principle:

$$\forall z \in D \subset \mathbb{C}[\overline{f(z)} = f(\bar{z})] \iff \forall x \in \mathbb{R}[f(x) \in \mathbb{R}]$$

(a) $\overline{\cos(z)} = \cos(\bar{z})$

Proof: It is clear that $\forall x \in \mathbb{R}$, $\sin(x) \in \mathbb{R}$. The result follows from the Reflection Principle. Also

$$\overline{\sin(z)} = \overline{\frac{z - \bar{z}}{2i}} = \frac{\bar{z} - z}{2i} = \sin(\bar{z})$$

□

(b) $\overline{\sin(z)} = \sin(\bar{z})$

Proof: It is clear that $\forall x \in \mathbb{R}$, $\cos(x) \in \mathbb{R}$. The result follows from the Reflection Principle. Also

$$\overline{\cos(z)} = \frac{\overline{z+\bar{z}}}{2} = \frac{\bar{z}+z}{2} = \cos(\bar{z})$$

$\square$

Example 14.3.2 *Show:*

(a) $\forall z \in \mathbb{C}[\overline{\cos(iz)} = \cos(i\bar{z})]$

Proof:

$$\overline{\cos(iz)} = \frac{\overline{iz + \overline{iz}}}{2} = \frac{\overline{iz} + iz}{2} = \frac{-i\bar{z} + iz}{2} = i\frac{z-\bar{z}}{2} = i\,\mathrm{Im}\{z\}$$
$$\cos(i\bar{z}) = \frac{i\bar{z} + \overline{i\bar{z}}}{2} = \frac{i\bar{z} - iz}{2} = i\frac{\bar{z}-z}{2} = i\,\mathrm{Im}\{z\}$$

Hence $\forall z \in \mathbb{C}$

$$\overline{\cos(iz)} = \cos(i\bar{z})$$

$\square$

(b) $\forall z \in \mathbb{C}\forall n \in \mathbb{Z}[\overline{\sin(iz)} = \sin(i\bar{z}) \iff z = n\pi i]$

Proof:

$$\overline{\sin(iz)} = \frac{\overline{iz - \overline{iz}}}{2i} = \frac{-i\bar{z} - iz}{2i} = -\frac{z+\bar{z}}{2} = -\mathrm{Re}\{z\}$$
$$\sin(i\bar{z}) = \frac{i\bar{z} - \overline{i\bar{z}}}{2i} = \frac{i\bar{z} + iz}{2i} = \frac{z+\bar{z}}{2} = \mathrm{Re}\{z\}$$

We know that

$$\mathrm{Re}\{z\} = -\mathrm{Re}\{z\} \implies \mathrm{Re}\{z\} = 0 \implies \overline{\sin(iz)} = \sin(i\bar{z}) = 0 \iff z = n\pi i$$

$\square$

14.4 Hyperbolic Functions

Definition 14.4.1: Hyperbolic Sine and Cosine Functions
Let $z \in \mathbb{C}$:

$$\sinh(z) = \frac{e^z - e^{-z}}{2} \qquad \cosh(z) = \frac{e^z + e^{-z}}{2}$$

It is clear that the derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}z}\sinh(z) = \cosh(z) \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}z}\cosh(z) = \sinh(z)$$

The relationships with sine and cosine:

$$\begin{aligned} -i\sinh(iz) &= \sin(z) & \qquad \cosh(iz) &= \cos(z) \\ -i\sin(iz) &= \sinh(z) & \cos(iz) &= \cosh(z) \end{aligned}$$

Hence in the complex plane, sinh and cosh are periodic with period $2\pi i$.

Identities:

$$\sinh(-z) = -\sinh(z) \qquad \cosh(-z) = \cosh(z) \qquad \cosh^2(z) - \sinh^2(z) = 1$$

$$\begin{aligned} \sinh(z_1 + z_2) &= \sinh(z_1)\cosh(z_2) + \cosh(z_1)\sinh(z_2) \\ \cosh(z_1 + z_2) &= \cosh(z_2)\cosh(z_2) + \sinh(z_1)\sinh(z_2) \end{aligned}$$

$$\begin{aligned} \sinh(z) &= \sinh(x)\cos(y) + i\cosh(x)\sin(y) \\ \cosh(z) &= \cosh(x)\cos(y) + i\sinh(x)\sin(y) \end{aligned}$$

$$\begin{aligned} |\sinh(z)|^2 &= \sinh^2(x) + \sin^2(y) \\ |\cosh(z)|^2 &= \sinh^2(x) + \cos^2(y) \end{aligned}$$

Theorem 14.4.1:

The zeros of hyperbolic zine and cosine:

$$\begin{aligned} \sinh(z) = 0 &\iff z = n\pi i & \qquad n \in \mathbb{Z} \\ \cosh(z) = 0 &\iff z = \left(\frac{\pi}{2} + n\pi\right)i & \qquad n \in \mathbb{Z} \end{aligned}$$

Example 14.4.1 *Show:*

(a) $\sinh(z + \pi i) = -\sinh(z)$

Proof:

$$\sinh(z + \pi i) = \frac{e^{z+\pi i} - e^{-z-\pi i}}{2} = \frac{-e^{z} + e^{-z}}{2} = -\frac{e^{z} - e^{-z}}{2} = -\sinh(z)$$

□

(b) $\cosh(z + \pi i) = -\cosh(z)$

Proof:

$$\cosh(z+\pi i) = \frac{e^{z+\pi i}+e^{-z-\pi i}}{2} = -\frac{e^{z}+e^{-z}}{2} = -\cosh(z)$$

□

(c) $\tanh(z+\pi i) = \tanh(z)$

Proof:

$$\tanh(z+\pi i) = \frac{\sinh(z+\pi i)}{\cosh(z+\pi i)} = \frac{-\sinh(z)}{-\cosh(z)} = \tanh(z)$$

□

Example 14.4.2 *Show* $\forall z \in \mathbb{C}$*:*

$$\overline{\sinh(z)} = \sinh(\bar{z}) \qquad \overline{\cosh(z)} = \cosh(\bar{z}) \qquad \forall z \neq 0\left[\overline{\tanh(z)} = \tanh(\bar{z})\right]$$

Proof: We can see that $\forall x \in \mathbb{R}$, $\sinh(x) \in \mathbb{R}$ and $\cosh(x) \in \mathbb{R}$, so we can conclude from the Reflection Principle (theorem 13.8.3) that $\forall z \in \mathbb{C}$:

$$\overline{\sinh(z)} = \sinh(\bar{z}) \qquad \overline{\cosh(z)} = \cosh(\bar{z})$$

Thus it follows that

$$\forall z \neq 0\left[\overline{\tanh(z)} = \tanh(\bar{z})\right]$$

□

14.5 Inverse Trigonometric and Hyperbolic Functions

Theorem 14.5.1:
Let $z \in \mathbb{C}$*:*

$$\sin^{-1}(z) = -i\log[iz + (1-z^2)^{1/2}]$$
$$\cos^{-1}(z) = -i\log[z + i(i-z^2)^{1/2}]$$
$$\tan^{-1}(z) = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right)$$

$\cos^{-1}(z)$ and $\tan^{-1}(z)$ are multi-valued. All inverse trigonometric functions become single-valued and analytic when in specific branches of the square root and logarithmic functions.

Proof: $\underline{\sin^{-1}(z) = -i\log[iz + (1-z^2)^{1/2}]}$
Let $w = \sin^{-1}(z)$ whenever $z = \sin(w)$

$$h$$

□

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