

# The Book of Math (Notes)

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# Forward and Disclaimer

These are math notes made by a student (with a physics major and math minor) based off text books. It may contain misconceptions and misinterpretations, thus should not be viewed in the same light of a text book. Use at your own risk and mental sanity.

## Symbols

### Logic

Name	Symbol	Comment
Exists	$\exists$	There exists at least one
For all	$\forall$	
Not exists	$\nexists$	There does not exist
Exists one	$\exists!$	There only exists one and only one
And	$\wedge$	
Or	$\vee$	Inclusive or
Not	$\neg$	
Logically implies	$\implies$	If
Logically implied by	$\impliedby$	Only if
Logically equivalent	$\iff$	If and only if
Implies	$\rightarrow$	
Implied by	$\leftarrow$	
Double Implication	$\longleftrightarrow$	

### Set Notation

Name	Symbol	Comment
Empty Set	$\emptyset$	The set that is empty
Natural Numbers	$\mathbb{N}$	Set of natural numbers not containing 0, equivalent to the set of positive integers
Integers	$\mathbb{Z}$	Set of integers
Rational Numbers	$\mathbb{Q}$	
Algebraic Numbers	$\mathbb{A}$	
Real Numbers	$\mathbb{R}$	
Complex Numbers	$\mathbb{C}$	
In	$\in$	
Not in	$\notin$	
Owns	$\ni$	Has an element
Proper Subset	$\subset$	Subset that is not itself
Subset	$\subseteq$	
Superset	$\supset$	Superset that is not itself
Proper Superset	$\supsetneq$	

Power set	$\wp$
Union	$\cup$
Intersection	$\cap$
Difference	$\setminus$

## Relationships

Name	Symbol	Comment
Defined	$\doteq$	
Approximate	$\approx$	
Equivalent	$\equiv$	Isomorphic (Group Theory)
Congruent	$\cong$	Homomorphic (Group Theory)
Proportional	$\propto$	

## Operators

Name	Symbol	Comment
	$\oplus$	
	$\otimes$	
	$\odot$	
	$\circ$	Convolution
Dagger	$\dagger$	Complex conjugate transpose of a matrix

## Arrows

Name	Symbol	Comment
Maps to	$\mapsto$	

## Hebrew

Name	Symbol	Comment
Aleph	$\aleph$	Carnality of infinite sets that can be well ordered

## Other

Name	Symbol	Comment
Real part	Re	Real part of a number
Imaginary part	Im	Imaginary part of a number

# Book Constitution

## Intents and Purpose

The goal of this book is to organize mathematical knowledge of topics related to the study of physics or the author's interest. It is meant to be used as a source of for future reference, not as a textbook for students new to the topics. It is a notebook of a student, thus should be treated as one and not as a textbook. At most, it could be used as a study guide along side a textbook. Definitely not as the main source for acquiring knowledge.

## Layout and Organization

The book is split into parts each containing a field of study mathematics, or a topic large enough to justify giving it its own part. Each part contains chapters that focuses on a particular topic required to understand the field, with sections dedicated to describing a particular knowledge required for the topic.

As axioms, definitions, theorems, corollary, and proofs are integral and abundant to the study of mathematics, each will have a unique style. Each environment and its styles are displayed as follows:

### **Axiom 0.1: Axiom name**

*Example Axiom Axioms are the “ground rules” of the set.*

### **Theorem 0.0.1: Theorem name or citation**

*Example Theorem An important logical result from the axioms, with proof.*

### **Conjecture 0.0.1: Name of conjecture or citation**

*Example Conjecture A hypothesis, without proof.*

### **Corollary 0.0.1.1:**

*Example Corollary An implication as a result of a theorem.*

### **Lemma 0.0.1.1:**

*Example Lemma Small theorems that build up to a larger theorem.*

### **Proposition 0.0.1.1:**

*Example Proposition Example proposition.*

*Proof:* Logical deductions that results in a theorem. Proofs I've written will be in grey, which may or may not be correct. □

### **Definition 0.0.1: Word**

*Example Definition The definition of a word.*

**Example 0.0.1** *An example.*

**Remark.** *Remark A comment by the author in the textbooks used.*

**Observation.** *Example Observation A remark by me.*

**Question.** *Example Question A question from me for a mystery to be answered later.*



# Contents

<b>I</b>	<b>Logic [Empty]</b>	<b>1</b>
1	Proofs	3
<b>II</b>	<b>Numbers [Empty]</b>	<b>5</b>
2	Natural $\mathbb{N}$	7
3	Integers $\mathbb{Z}$	9
4	Rationals $\mathbb{Q}$	11
5	Constructible	13
6	Algebraic $\mathbb{A}$	15
7	Reals $\mathbb{R}$	17
8	Complex $\mathbb{C}$	19
<b>III</b>	<b>Real Analysis [Empty]</b>	<b>21</b>
9	Sequences	23
9.1	Limits . . . . .	23
9.1.1	Limit Theorems . . . . .	23
9.2	Monotone and Cauchy Sequences . . . . .	23
9.3	Subsequences . . . . .	23
9.4	$\limsup$ and $\liminf$ . . . . .	23

9.5	Series . . . . .	23
9.6	Alternating Series and Integral Tests . . . . .	23
<b>10</b>	<b>Continuity</b>	<b>25</b>
10.1	Continuous Functions . . . . .	25
10.1.1	Properties . . . . .	25
10.2	Uniform Continuity . . . . .	25
10.3	Limits of Functions . . . . .	25
<b>11</b>	<b>Metric Spaces</b>	<b>27</b>
<b>IV</b>	<b>Complex Analysis</b>	<b>29</b>
<b>12</b>	<b>Basics</b>	<b>31</b>
12.1	Complex Numbers . . . . .	31
12.2	Triangle Inequality . . . . .	32
12.3	Polar and Exponential Form . . . . .	33
12.3.1	Properties of Polar and Exponential Form . . . . .	35
12.3.2	Properties of Arguments . . . . .	36
12.4	Roots of $z$ . . . . .	36
12.5	Complex Conjugate . . . . .	38
12.6	Operations as Transformations . . . . .	39
12.7	Complex Analysis Definitions . . . . .	40
<b>13</b>	<b>Analytic Functions</b>	<b>45</b>
13.1	Functions as mappings . . . . .	45
13.2	Limits . . . . .	47
13.2.1	Limit Theorems . . . . .	49
13.2.2	Limits of Points at Infinity . . . . .	51
13.3	Continuity . . . . .	54
13.3.1	Exercises . . . . .	56
13.4	Differentiation . . . . .	57
13.4.1	Differentiation Rules . . . . .	59



13.4.2 Exercises . . . . .	62
13.5 Cauchy-Riemann Equations . . . . .	63
13.5.1 Complex Form of the Cauchy-Riemann Equations . . . . .	69
13.5.2 Conditions for Differentiability . . . . .	70
13.6 Analytic Functions . . . . .	72
13.6.1 Examples . . . . .	74
13.7 Harmonic Functions . . . . .	75
13.8 Uniquely Determined Analytic Functions . . . . .	77
13.8.1 Reflection Principle . . . . .	79
13.8.2 Examples . . . . .	80
<b>14 Elementary Functions</b>	<b>81</b>
14.1 Exponential Function . . . . .	81
14.2 Logarithmic Function . . . . .	81
14.2.1 Branches and Derivatives of Logarithms . . . . .	82
14.2.2 Identities of Logarithms . . . . .	83
14.2.3 Power Function . . . . .	84
14.3 Trigonometric Functions . . . . .	85
14.3.1 Zeros and Singularities . . . . .	86
14.4 Hyperbolic Functions . . . . .	87
14.5 Inverse Trigonometric and Hyperbolic Functions . . . . .	89
14.6 Phasors . . . . .	91
<b>15 Integrals</b>	<b>93</b>
15.1 Derivatives of Functions . . . . .	93
15.2 Definite Integrals of Functions . . . . .	93
15.3 Contours . . . . .	94
15.4 Contour Integrals . . . . .	96
15.4.1 Upper Bounds for the Moduli . . . . .	97
15.5 Antiderivatives . . . . .	99
15.6 Cauchy-Goursat Theorem . . . . .	102
15.6.1 Morera's Theorem . . . . .	106
15.6.2 Simply Connected Domains . . . . .	107

15.6.3	Multiply Connected Domains . . . . .	108
15.6.4	Examples . . . . .	109
15.7	Cauchy Integral Formula . . . . .	113
15.7.1	Consequences . . . . .	116
15.8	Liouville's Theorem and the Fundamental Theorem of Algebra . . . . .	117
15.9	Maximum Modulus Principle . . . . .	118
15.9.1	Examples . . . . .	121
15.10	Poisson Integral Formula . . . . .	122
<b>16</b>	<b>Series</b>	<b>125</b>
16.1	Convergence . . . . .	125
16.2	Taylor Series . . . . .	130
16.3	Laurent Series . . . . .	132
16.3.1	Examples . . . . .	135
16.4	Absolute and Uniform Convergence of Power Series . . . . .	139
16.5	Continuity of Sums of Power Series . . . . .	142
16.6	Integration and Differentiation of Power Series . . . . .	143
16.7	Uniqueness of Series Representations . . . . .	145
16.8	Multiplication and Division of Power Series . . . . .	147
16.9	$z$ -Transform . . . . .	149
16.9.1	Product of $z$ -Transforms . . . . .	150
<b>17</b>	<b>Residues and Poles</b>	<b>153</b>
17.1	Residues . . . . .	153
17.1.1	Residue at Infinity . . . . .	154
17.2	Three Types of Isolated Singular Points . . . . .	156
17.3	Residue at Poles . . . . .	157
17.4	Zeros of Analytic Functions . . . . .	160
17.5	Zeros and Poles . . . . .	162
17.6	Behaviour of Functions Near Isolated Singular Points . . . . .	164
17.6.1	Removable Singular Points . . . . .	164
17.6.2	Essential Singular Points . . . . .	165
17.6.3	Poles of Order $m$ . . . . .	166

17.7	Application of Residues . . . . .	166
17.7.1	Evaluation of Improper Integrals . . . . .	166
17.7.2	Improper Integrals from Fourier Analysis . . . . .	170
17.7.3	Jordan's Lemma . . . . .	170
17.7.4	Indented Paths . . . . .	174
17.7.5	Integration Along a Branch Cut . . . . .	177
17.7.6	Indefinite Integrals Involving Sines and Cosines . . . . .	180
17.7.7	Argument Principle . . . . .	181
17.7.8	Rouché's Theorem . . . . .	184
17.7.9	Inverse Laplace Transforms . . . . .	189
17.7.10	Hilbert Transform . . . . .	192
17.7.11	Gamma Function . . . . .	196
<b>18</b>	<b>Mapping by Elementary Functions</b>	<b>201</b>
18.1	Linear Transformations . . . . .	201
18.2	Transformation $w = 1/z$ . . . . .	201
18.2.1	Mapping by $1/z$ . . . . .	202
18.3	Linear Fractional Transformations . . . . .	202
18.3.1	Implicit Form . . . . .	204
18.4	Mappings of the Upper Half Plane . . . . .	206
18.5	Mappings by the Exponential Function . . . . .	208
18.6	Mapping by $w = \sin(z)$ . . . . .	209
18.6.1	Related Mappings . . . . .	211
18.7	Mappings by $z^2$ . . . . .	211
18.8	Mappings by Branches of $z^{1/n}$ . . . . .	212
18.9	Square Roots of Polynomials . . . . .	213
18.10	Riemann Surface . . . . .	216
18.10.1	Surfaces for Related Functions . . . . .	218
<b>19</b>	<b>Conformal Mapping</b>	<b>221</b>
19.1	Preserving Angles and Scale Factors . . . . .	221
19.2	Local Inverses . . . . .	225
19.3	Harmonic Conjugates . . . . .	227

V	Ordinary Differential Equations [Empty]	229
VI	Nonlinear Dynamics [Empty]	231
VII	Partial Differential Equations [Empty]	233
VIII	Integral Equations [Empty]	235
IX	Linear Algebra [Empty]	237
20	Markov Chains	239
X	Tensors [Empty]	241
XI	Riemann Geometry [Empty]	243
XII	Abstract Algebra [Empty]	245
21	Groups	247
22	Rings	249
22.1	Ideals . . . . .	249
23	Integral Domains	251
24	GCD Domains	253
25	Unique Factorization Domains	255
26	Principal Ideal Domains	257
27	Fields	259

<b>XIII</b>	<b>Galois Theory [Empty]</b>	<b>261</b>
<b>XIV</b>	<b>Lie Theory [Empty]</b>	<b>263</b>
28	Lie Groups	265
29	Lie Algebra	267
<b>XV</b>	<b>C-Star Algebra [Empty]</b>	<b>269</b>
<b>XVI</b>	<b>Set Theory [Empty]</b>	<b>271</b>
<b>XVII</b>	<b>Model Theory [Empty]</b>	<b>273</b>
<b>XVIII</b>	<b>Statistics [Empty]</b>	<b>275</b>
<b>XIX</b>	<b>Tips and Tricks [Empty]</b>	<b>277</b>
<b>30</b>	<b>Integration Techniques</b>	<b>279</b>
30.1	DI Method (Integration Table) . . . . .	279
30.2	Feynman Integration . . . . .	279
<b>XX</b>	<b>Index and Bibliography</b>	<b>281</b>



# Part I

Logic [Empty]





# Chapter 1

## Proofs



## Part II

Numbers [Empty]

## Resources used in part II

content...

# Chapter 2

## Natural $\mathbb{N}$



# Chapter 3

## Integers $\mathbb{Z}$





# Chapter 4

## Rationals $\mathbb{Q}$



# Chapter 5

## Constructible



# Chapter 6

## Algebraic $\mathbb{A}$



# Chapter 7

## Reals $\mathbb{R}$





# Chapter 8

## Complex $\mathbb{C}$



## Part III

### Real Analysis [Empty]

## **Resources used in part III**

1. Kenneth A. Ross - Elementary Analysis (2nd Ed.) [1]

# Chapter 9

## Sequences

**Corollary 9.0.0.1:**

*Absolutely convergent series are convergent.*

### 9.1 Limits

#### 9.1.1 Limit Theorems

### 9.2 Monotone and Cauchy Sequences

### 9.3 Subsequences

### 9.4 $\limsup$ and $\liminf$

### 9.5 Series

### 9.6 Alternating Series and Integral Tests



# Chapter 10

## Continuity

### 10.1 Continuous Functions

#### 10.1.1 Properties

### 10.2 Uniform Continuity

### 10.3 Limits of Functions





# Chapter 11

## Metric Spaces



# Part IV

## Complex Analysis

## **Resources used in part IV**

Primary:

1. Brown and Churchill - Complex Variables and Applications [2]

Supplement:

1. A. David Wunsch - Complex Variables with Applications [3]

# Chapter 12

## Basics

### 12.1 Complex Numbers

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}, i = \sqrt{-1}\}$$

Complex numbers are elements of the complex field ( $\mathbb{C}$ ), therefore, they obey all the properties of a field.

We will denote complex numbers by  $z = x + iy$  with  $x, y \in \mathbb{R}$ , and refer the real part as  $\operatorname{Re}(z) = \operatorname{Re}(z) = x$  and imaginary part as  $\operatorname{Im}(z) = \operatorname{Im}(z) = y$ . Complex numbers can also be defined as an ordered pair  $z = (x, y)$  which is interpreted as points in the complex plane.  $(x, 0)$  are points on the real axis while  $(0, y)$  are points in the imaginary axis. This expression is often called a Couple, and was presented in 1833 by mathematician William Rowan Hamilton (1805 - 1865).



Like numbers in  $\mathbb{R}$ , numbers in  $\mathbb{C}$  obey the commutative, distributive, and associative laws. We add and multiply complex numbers in the usual way:

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) & z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) & &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \end{aligned}$$

$\forall z \in \mathbb{C}$ , there is an unique additive inverse  $(-z)$  and  $\forall z \in \mathbb{C} \setminus \{0\}$ , there is an unique multiplicative inverse  $(z^{-1})$  such that

$$\begin{aligned} z + (-z) &= 0 & zz^{-1} &= 1 \\ \implies -z &= -x - iy & \implies (x_1x_2 - y_1y_2) &= 1 \wedge (x_1y_2 + x_2y_1) = 0 \\ & & \implies z^{-1} &= \frac{x_1}{x_1^2 + y_1^2} - i \frac{y_1}{x_1^2 + y_1^2} \end{aligned}$$

The existence and uniqueness of the inverses can be easily proven.

The addition of complex numbers may also be interpreted as akin to vector addition.



Note: As a group with addition,  $\mathbb{R}^2 \cong \mathbb{C}$ , however this is not the case for rings.  $\mathbb{C}$  is a field, but  $\mathbb{R}^2$  is not.  $\mathbb{R}^2$  have non-zero divisors (ie. Take any  $a, b \in \mathbb{R}^2$ ,  $(a, 0) \cdot (0, b) = 0$ ).

## 12.2 Triangle Inequality

It is not analysis without a section dedicated to the triangle inequality. For any given number  $z_1, z_2 \in \mathbb{C}$  it makes no sense to write an inequality  $z_1 = a_1 + ib_1 < a_2 + ib_2 = z_2$ . Thus, we need have a different notion of size.

### Definition 12.2.1: Modulus

*The modulus of a complex number is a function  $\mathbb{C} \rightarrow \mathbb{R}_{>0}$ :*

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$

It is obvious why the definition is not  $|z| = \sqrt{x^2 + (iy)^2}$  as problems arise when  $x = y$ . The modulus is the distance of  $z$  from  $(0, 0)$ .  $\bar{z}$  is the complex conjugate of  $z$ , which is explored in section 12.5

### Theorem 12.2.1: Triangle Inequality

$$\forall z_1, z_2 \in \mathbb{C} [|z_1 + z_2| \leq |z_1| + |z_2|]$$

From the theorem, we can derive a similar inequality:

$$|z_1| = |z_1 + z_2 - z_2| \leq |z_1 + z_2| + |-z_2| \implies |z_1| - |z_2| \leq |z_1 + z_2|$$

An important property of polynomials is observed when theorem 12.2.1 is applied to polynomials.

**Corollary 12.2.1.1:**

*Consider the polynomial  $P(z)$  where  $a_n \in \mathbb{C}$ ,  $n \in \mathbb{N}$ ,  $a_0 \neq 0$ , and  $z \in \mathbb{C}$ .*

$$P(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n$$

*Then  $\forall z, \exists R \in \mathbb{R}_{>0}, |z| < R$  such that*

$$\left| \frac{1}{P(z)} \right| < \frac{2}{|a_n|R^n}$$

*Proof:* Consider

$$w = \frac{P(z)}{z^n} - a_n = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \dots + \frac{a_{n-1}}{z} \quad z \neq 0$$

$$\implies wz^n = a_0 + a_1z + \dots + a_{n-1}z^{n-1}$$

$$\implies |w||z|^n \leq |a_0| + |a_1||z| + \dots + |a_{n-1}||z|^{n-1}$$

$$\implies |w| \leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \dots + \frac{|a_{n-1}|}{|z|}$$

$$\implies |w| < n \frac{|a_n|}{2n} = \frac{|a_n|}{2} \quad \exists \text{ sufficiently large } R < |z| \text{ s.t.}$$

$$\forall m, 0 \leq m \leq n-1, \frac{|a_m|}{|z|^{n-m}} < \frac{|a_n|}{2n}$$

$$\implies |a_n + w| \geq ||a_n| - |w|| > \frac{|a_n|}{2} \quad R < |z|$$

$$\implies |P_n(z)| = |a_n + w||z|^n > \frac{|a_n|}{2}|z|^n > \frac{|a_n|}{2}R^n \quad R < |z|$$

$$\implies \left| \frac{1}{P(z)} \right| < \frac{2}{|a_n|R^n}$$

□

This tells us that if  $z$  is a solution to a polynomial  $P(z)$ , then the reciprocal of the polynomial  $1/P(z)$  is bounded above by  $R = |z|$ . (i.e. It is bounded by a circle of radius  $|z|$ .)

## 12.3 Polar and Exponential Form

### Definition 12.3.1: Argument of $z$

Consider any  $z \in \mathbb{C}$  where  $z \neq 0$ . Let  $\theta$  be the angle in radians between  $z$  and the real axis . Then  $\forall n \in \mathbb{N}$ ,  $-\pi < \theta \leq \pi$ , the argument of  $z$ :

$$\arg(z) = \theta + 2n\pi$$

We know  $\forall n \in \mathbb{N}$ ,  $\theta + 2\pi n = \theta$ . This leads us to the definition of the principal argument of  $z$ .

**Definition 12.3.2: Principal Argument of  $z$**

Consider any  $z \in \mathbb{C}$  where  $z \neq 0$ . Let  $\theta$  be the angle in radians between  $z$  and the real axis. Then for  $-\pi < \theta \leq \pi$ , the principal argument of  $z$ :

$$\text{Arg}(z) = \theta$$

It is clear that  $\arg(z) = \text{Arg}(z) + 2n\pi$ . It is common for the principal argument to be defined  $-\pi < \theta \leq \pi$ , although other definitions use  $0 \leq \theta < 2\pi$ .

**Definition 12.3.3: Polar Form of  $z$**

Consider  $z \in \mathbb{C}$ . Let  $r = |z|$ , and  $\theta = \arg(z)$ . Then  $\forall z \in \mathbb{C}, z \neq 0$ :

$$z = x + iy = r(\cos(\theta) + i \sin(\theta))$$

Notice that all three definitions require that  $z \neq 0$  as  $\theta$  is undefined at  $z = 0$ .

**Theorem 12.3.1: Euler's Formula**

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

Combining definition 12.3.3 with theorem 12.3.1, we obtain the Exponential Form of  $z$ :

**Definition 12.3.4: Exponential Form of  $z$**

Consider any  $z \in \mathbb{C}$ , and let  $r = |z|$  and  $\theta = \text{Arg}(z)$ . Then the exponential form of  $z$ :

$$z = re^{i\theta}$$

Note:  $\theta = \tan^{-1}(y/x)$  and  $r = \sqrt{x^2 + y^2}$ .





### 12.3.1 Properties of Polar and Exponential Form

It would be easier to work with the exponential form of  $z$  then convert it to the polar form later. The exponential form of a complex number is part of the exponential family of functions, thus possess all the properties of the family. Consider any complex number  $z_1 = r_1 e^{i\theta_1}$  and  $z_2 = r_2 e^{i\theta_2}$ .

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} \quad z^n = r^n e^{in\theta} \quad \forall n \in \mathbb{Z}$$

A special case arrives for integer exponential of  $z$  on the unit circle.

#### Theorem 12.3.2: de Moivre's Formula

Consider any  $z = e^{i\theta} \in \mathbb{C}$  on the unit circle, and let  $n \in \mathbb{Z}$ .

$$\forall z \in \mathbb{C} \quad \forall n \in \mathbb{Z} \quad [|z| = 1 \implies (\cos(\theta) + i \sin(\theta))^n = \cos(n\theta) + i \sin(n\theta)]$$

*Proof:* Consider  $z = e^{i\theta}$  and let  $n \in \mathbb{Z}$ .

$$z^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i \sin(n\theta)$$

□

The proof hints that theorem 12.3.2 can be generalized to  $\forall n \in \mathbb{R}$ , which we will see shortly in section 12.4. Using theorem 12.3.2, we can obtain the double angle identities.

#### Corollary 12.3.2.1: Double Angle Identities

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \quad \sin(2\theta) = 2 \sin(\theta) \cos(\theta)$$

*Proof:* Consider any  $z$  on the unit circle, that is  $z = e^{i\theta}$ .

$$\begin{aligned} (\cos(\theta) + i \sin(\theta))^2 &= \cos(2\theta) + i \sin(2\theta) && \text{Theorem 12.3.2} \\ \implies \cos^2(\theta) - \sin^2(\theta) + i 2 \sin(\theta) \cos(\theta) &= \cos(2\theta) + i \sin(2\theta) \end{aligned}$$

Equating the real and imaginary parts yield the desired results.

□

### 12.3.2 Properties of Arguments

Recall from section 12.3.1:

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} \qquad z^n = r^n e^{in\theta} \qquad \forall n \in \mathbb{Z}$$

The arguments for the arguments of products of any  $z_1, z_2 \in \mathbb{C}$  follows immediately from the properties of the exponential.

#### Corollary 12.3.2.2: Arguments of Products

$$\begin{aligned} \arg(z_1 z_2) &= \arg(z_1) + \arg(z_2) & \text{Arg}(z_1 z_2) &= \text{Arg}(z_1) + \text{Arg}(z_2) \\ \arg(z^n) &= n \arg(z) & \text{Arg}(z^n) &= n \text{Arg}(z) \end{aligned}$$

*Proof:*

$$\begin{aligned} z_1 z_2 &= r_1 r_2 e^{i(\theta_1 + \theta_2)} \\ &\implies \arg(z_1 z_2) = \arg(z_1) + 2n_1\pi + \arg(z_2) + 2n_2\pi & n_1, n_2 \in \mathbb{Z} \\ &\implies \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \\ &\implies \text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2) \\ \\ z^n &= r^n e^{in\theta} \\ &\implies \arg(z^n) = n \arg(z) + 2n\pi & n \in \mathbb{Z} \\ &\implies \arg(z^n) = n \arg(z) \\ &\implies \text{Arg}(z^n) = n \text{Arg}(z) \end{aligned}$$

□

It is clear that:

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) \qquad \text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}(z_1) - \text{Arg}(z_2)$$

## 12.4 Roots of $z$

In definition 12.3.4, you might be wondering why  $z^n = r^n e^{in\theta}$  is not for  $n \in \mathbb{R}$ . That is because there is more things to consider, which we will explore in this section. Recall that  $z = re^{i\theta} = re^{i(\theta + 2n\pi)}$  for  $n \in \mathbb{Z}$ .

### Definition 12.4.1: Exponential of $z$

Consider any  $z \in \mathbb{C}$  and any  $x \in \mathbb{R}$

$$z^x = \left( r e^{i(\theta+2n\pi)} \right)^x = r^x e^{ix(\theta+2n\pi)}$$

For  $x \notin \mathbb{Z}$ , it is clear that  $z^x = r^x e^{ix(\theta+2n\pi)} \neq r^x e^{ix\theta}$ , since  $2nx\pi = 0 \iff nx \in \mathbb{Z}$ . In order to define the roots of  $z$  we must need a more general and proper definition of  $z$ .

### Definition 12.4.2: Roots of $z_0$

Consider any  $z_0 \in \mathbb{C}$  and any  $m \in \mathbb{N}$ .

$$z_0^{\frac{1}{m}} = r_0^{\frac{1}{m}} e^{i\left(\frac{\theta_0+2n\pi}{m}\right)} = r_0^{\frac{1}{m}} e^{i\left(\frac{\theta_0}{m} + \frac{2n\pi}{m}\right)}$$

Taking the  $m$ -th root of  $z_0 \in \mathbb{C}$  scales  $\theta_0$  by  $1/m$ , and provides solutions at equally spaced by  $2\pi/m$  on a circle of radius  $r^{1/m}$ . That is, the roots lie on the vertices of a regular  $n$ -sided polygon inscribed in a circle of radius  $|z|^{1/m}$ .

**Example 12.4.1** Consider  $z_0 = 32e^{i(5/6)\pi}$ , then  $z_0^{(1/5)} = 2e^{i(\pi/6)+i(2/5)n\pi}$  for  $n \in \mathbb{Z}$ . The radius went from 32 to  $32^{(1/5)} = 2$ , and five roots appear equally spaced with distance of  $(2/5)\pi$  on a circle with radius 2. Before and after graphs are as follows, note graph on right is zoomed in:



We can see that the roots of  $z_0$  form a set:

### Definition 12.4.3: Set of roots of $z_0$

Consider the  $m$ -th root of any  $z_0 \in \mathbb{C}$ . Let:

$$z_0 = r_0 e^{i\theta_0} \quad c_0 = r_0^{1/m} e^{i\theta_0/m} \quad \omega_n = e^{\frac{i2\pi}{m}} \quad m \in \mathbb{N}$$

Then the set of roots of  $z_0$ :

$$z_0^{1/m} = \{ c_k = c_0 \omega_m^k \mid k \in \mathbb{N}, 0 \leq k < m \}$$

$c_0$  is the principal root. The root corresponding to the principal argument of  $z$ .

#### Definition 12.4.4: Principal Root

Consider the  $m$ -th root of any  $z_0 \in \mathbb{C}$ . The principal root of  $z_0$  is defined as:

$$c_0 = r_0^{\frac{1}{m}} e^{i \frac{\theta_0}{m}}$$

**Example 12.4.2** Recall from the previous example:  $z_0 = 32e^{i(5/6)\pi}$ . This gives us

$$c_0 = 32^{1/5} e^{i\pi/6} = 2e^{i\pi/6} \qquad \omega_5 = e^{i2\pi/5}$$

Then

$$\begin{aligned} c_0 &= c_0 \omega_5^0 = 2e^{i\pi/6} \\ c_1 &= c_0 \omega_5^1 = 2e^{i\pi/6} e^{i2\pi/5} = 2e^{i17\pi/30} \\ c_2 &= c_0 \omega_5^2 = 2e^{i\pi/6} e^{i4\pi/5} = 2e^{i29\pi/30} \\ c_3 &= c_0 \omega_5^3 = 2e^{i\pi/6} e^{i6\pi/5} = 2e^{i41\pi/30} = 2e^{-i19\pi/30} \\ c_4 &= c_0 \omega_5^4 = 2e^{i\pi/6} e^{i8\pi/5} = 2e^{i53\pi/30} = 2e^{-i7\pi/30} \end{aligned}$$



## 12.5 Complex Conjugate

### Definition 12.5.1: Complex Conjugate

The complex conjugate of  $z \in \mathbb{C}$  is denoted  $\bar{z}$ .

$$\bar{z} = x - iy = r(\cos(\theta) - i \sin(\theta)) = re^{-i\theta}$$

Graphically, it is the reflection of  $z$  across the real axis.



It is then easy to see

$$\operatorname{Re}(z) = \frac{z + \bar{z}}{2} \qquad \operatorname{Im}(z) = \frac{z - \bar{z}}{2i} \qquad |z|^2 = z\bar{z}$$

As  $\operatorname{Re}(z) = x = r \cos(\theta)$  and  $\operatorname{Im}(z) = y = r \sin(\theta)$  and using definition 12.3.4, we can obtain the complex forms of sine and cosine:

**Definition 12.5.2: Complex Sine and Cosine**

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

It is easy to prove  $\forall z_1, z_2 \in \mathbb{C}$ :

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \qquad \overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$$

## 12.6 Operations as Transformations

Consider any  $z \in \mathbb{C}$ . A function  $f : \mathbb{C} \rightarrow \mathbb{C}$  can be viewed as transformations of the complex plane.

**Example 12.6.1** (Addition as translation) *Consider any  $z_0 \in \mathbb{C}$ ,  $z_0 = a + ib$  for  $a, b \in \mathbb{R}$ . Addition by  $z_0$  can be seen as a shift in the complex plane by  $a + bi$ . (i.e. It takes the origin and shifts it by  $z_0$ .)*



**Example 12.6.2** (Multiplication as scaling and rotation) Consider any  $z_0 \in \mathbb{C}$ ,  $z_0 = re^{i\theta}$ . Multiplication by  $z_0$  scales the entire complex plane by  $r$  and rotates it by  $\theta$ . (Imagine rotating and stretching out a net.)



## 12.7 Complex Analysis Definitions

### Definition 12.7.1: Neighbourhood

A neighbourhood of a point  $z_0$  is the set of all points  $z$  with distance less than  $\epsilon$ .

$$\{z : |z - z_0| < \epsilon\}$$

i.e. It is the set of all points that lie within a circle centred at  $z_0$  with radius  $\epsilon$ . Points on the circumference not included.



### Definition 12.7.2: Deleted Neighbourhood

A deleted neighbourhood is the set of all points  $z$  with distance less than  $\epsilon$  from a point  $z_0$ , not including  $z_0$ . That is, it is a neighbourhood of  $z_0$  without  $z_0$ .

$$\{z : |z - z_0| < \epsilon, z \neq z_0\}$$



### Definition 12.7.3: Interior Point

Let  $S$  be a set. A point  $z_0$  is an interior point of  $S$  if  $\exists \epsilon$  such that  $\forall z, |z - z_0| < \epsilon \implies z \in S$ . That is,  $z_0$  is an interior point of  $S$  if it has a neighbourhood where all points in the neighbourhood are an element of  $S$ .



### Definition 12.7.4: Exterior Point

Let  $S$  be a set. A point  $z_0$  is an exterior point of  $S$  if  $\exists \epsilon$  such that  $\forall z, |z - z_0| < \epsilon \implies z \notin S$ . That is,  $z_0$  is an exterior point of  $S$  if it has neighbourhood that does not contain any element of  $S$ .



### Definition 12.7.5: Boundary Point

Let  $S$  be a set. A point  $z_0$  is a boundary point of  $S$  if  $\forall \epsilon, \exists z \in S, z' \notin S$ , such that  $|z - z_0| < \epsilon$  and  $|z' - z_0| < \epsilon$ . That is, for all neighbourhoods of  $z_0$  there exists a point that is in  $S$  and a point not in  $S$ .



Note: A boundary point of  $S$  may or may not be in  $S$ .

### Definition 12.7.6: Boundary of a Set

A boundary of a set  $S$  is the set of all boundary points of  $S$ . The set containing all boundary points of  $S$ .

$$\{z_0 : \forall \epsilon \exists z \in S, z' \notin S (|z - z_0| < \epsilon \wedge |z' - z_0| < \epsilon)\}$$

### Definition 12.7.7: Open Set

A set that does not contain any boundary points.

### Theorem 12.7.1:

Set  $S$  is open  $\iff \forall s \in S, s$  is an interior point of  $S$

*Proof:*  $\implies$ : Suppose  $S$  is open  $\nRightarrow \forall s \in S, s$  is an interior point of  $S$ , for contradiction. That is,  $\exists s \in S$  that is either a boundary point or an exterior point.  $s \in S$  implies  $s$  is not an exterior point of  $S$ , so  $s$  has to be a boundary point of  $S$ . This contradicts that  $S$  is an open set.

$$S \text{ is open} \implies \forall s \in S (s \text{ is an interior point of } S)$$



$\Leftarrow$  :

$\forall s \in S (s \text{ is an interior point of } S)$   
 $\implies \forall s' \forall \epsilon (|s' - s| < \epsilon \implies s' \in S)$   
 $\implies S \text{ does not contain boundary points} \implies S \text{ is open}$

□

A set can be neither open or closed. Consider the set  $S = \{z : 0 < |z| \leq 1\}$ .  $S$  is not closed since it does not contain the boundary point 0, and it is not open since it contains boundary points where  $|z| = 1$ . The set  $\mathbb{C}$  is both open and closed since it has no boundary points.

### Definition 12.7.8: Closed Set

*A set that contains all of its boundary points.*

### Definition 12.7.9: Closure of a Set

*Let  $S$  be a set. The closure of  $S$  is a closed set containing all points of  $S$  and all boundary points of  $S$ .*

### Definition 12.7.10: Connected Set

*An opens set  $S$  is connected if  $\forall z_1, z_2 \in S$ ,  $z_1$  and  $z_2$  can be connected by a polygonal line lying within  $S$ .*



### Definition 12.7.11: Polygonal Line

*A finite set of line segments joined end to end.*

### Definition 12.7.12: Domain

*A nonempty connected set.*

Note: All neighbourhoods are domains.

### Definition 12.7.13: Region

*A domain with none, some, or all of its boundary points.*

### Definition 12.7.14: Bounded Set/Region

*A set  $S$  is bounded if  $\exists R = |z| > 0$  such that  $\forall s \in S, |s| < R$ . That is,  $S$  is bounded if  $\forall s \in S$ ,  $s$  is contained in some circle of radius  $R$  centred at the origin.*

**Definition 12.7.15: Closed Region**

*A domain with all its boundary points. A bounded and closed region.*

**Definition 12.7.16: Accumulation/Limit Point**

*A point  $z_0$  is a accumulation point of a set  $S$  if all deleted neighbourhood of  $z_0$  contains an element of  $S$ .*

$$\forall \epsilon \exists s \in S (s \neq z_0 \wedge |z - s| < \epsilon)$$

Note: Unlike a boundary point, an accumulation point does not require that all neighbourhood of  $z_0$  contain an element not in  $S$ .

**Theorem 12.7.2:**

*Set  $S$  is closed  $\iff \forall$  accumulation points  $z_0$  of  $S$ ,  $z_0 \in S$*

*Proof:*  $\implies$ : Let  $S$  is closed and  $z_0$  is an accumulation point of a set  $S$  where  $z_0 \notin S$  for contradiction. If  $\exists z_0 \notin S$ , then  $z_0$  is a boundary point of  $S$ . Contradicts closed set contains all boundary points.

$\impliedby$ : Suppose all accumulation points of  $S$  are elements of  $S$  but  $S$  is not closed for contradiction. Then  $S$  does not contain one or more boundary points. Suppose  $z_0$  is a boundary point of  $S$  that is not in  $S$ . Then  $\forall \epsilon \exists s \in S$  where  $|s - z_0| < \epsilon$ , so by considering the deleted neighbourhood of  $z_0$ , this makes  $z_0$  an accumulation point of  $S$ . This contradicts that all accumulation points of  $S$  is in  $S$ .  $\square$

# Chapter 13

## Analytic Functions

### 13.1 Functions as mappings

A function  $f : S \rightarrow S'$  is a function that maps elements from  $S$  to elements on  $S'$ . The value of  $f$  at  $z$  is denoted  $f(z)$  and the set  $S$  is the domain of  $f$  while  $S'$  is the image of  $f$ . Recall section 12.6, a function can likewise be viewed as a transformation or mapping, that maps  $z \in \text{dom}(f) = S$  to values  $z' \in \text{img}(f) = S'$ .

**Definition 13.1.1: Range**

*Let  $f$  be a function with domain  $S$  and image  $S'$ . The range of  $f$  is the entire image of  $S$ .*

Note: Image is a subset of range, and can be a single point or a set of points.

**Definition 13.1.2: Inverse Range**

*The set of all points  $s \in S$  with the value  $f(s) = s'$  for some  $s' \in S'$ .*

$$\{s : f(s) = s', s' \in S'\}$$

Note: The domain of a function is often a domain, but it does not need to be a domain.

We will consider functions  $f : S \rightarrow S'$  where both  $S, S' \subseteq \mathbb{C}$ . For such functions we can break it into a two real valued functions:

$$\begin{aligned} f(z) &= u(x, y) + iv(x, y) & \text{dom}(u) \subseteq \mathbb{R}, \text{dom}(v) \subseteq \mathbb{R} \\ &= u(r, \theta) + iv(r, \theta) \end{aligned}$$

Recall that a real-valued function is a function with a domain that is a subset of  $\mathbb{R}$  (??). If  $\forall z, v(x, y) = 0$ , then  $f$  is called a real-valued function of a complex variable.

**Definition 13.1.3: Polynomial**

*Let  $a_i \in \mathbb{C}$ ,  $0 \leq i \leq n$  where  $i, n \in \mathbb{N} \cup \{0\}$ . If  $a_n \neq 0$ , then a polynomial of degree  $n$  is*

$$P(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n = \sum_{i=0}^n a_iz^i$$

**Definition 13.1.4: Rational Functions**

Let  $P(z)$  and  $Q(z)$  are polynomials, then rational functions are quotients:

$$\frac{P(z)}{Q(z)}$$

Defined for all  $z$  where  $Q(z) \neq 0$ .

**Definition 13.1.5: Multiple-Valued Function**

Let  $f$  be a function and  $z \in \text{dom}(f)$ .  $f$  is a multiple-valued function if it assigns more than one value to a point  $z$ .

“When multiple-valued functions are studied, usually just one of the possible values assigned at each point is taken, in a systematic manner and a (single-valued) function is constructed from the multiple-valued one” - Brown and Churchill [2]

What this means that for  $z \in \mathbb{C}$  a function  $f$  assigns  $u(z)$  and  $v(z)$  to  $z$ . By taking just  $u$  or  $v$ , we create a single-valued function from a multiple-valued function.

**Example 13.1.1** ( $f(z) = z^2$ )

$$\begin{aligned} f(z) = z^2 &= x^2 - y^2 + i2xy \\ \implies u(x, y) &= x^2 - y^2 \quad v(x, y) = 2xy \end{aligned}$$

By setting  $u = x^2 - y^2 = c_1$  where  $c_1 \in \mathbb{R}_{>0}$  we can see that

$$u = x^2 - y^2 = c_1 \quad v = 2xy = \pm 2y\sqrt{y^2 + c_1}$$

This tells us that in the complex plane of  $u$  and  $v$ , if we fix  $u$  to a constant  $c_1$  and move along  $v = \pm 2y\sqrt{y^2 + c_1}$  by incrementing  $y$  we draw out two hyperbolas in the complex plane of  $x$  and  $y$ . This means that the function  $f(z) = z^2$  takes points on hyperbolas the complex plane of  $x$  and  $y$  and translates them onto a vertical line in the complex plane of  $u$  and  $v$  where  $u$  is a constant.



Likewise if we set  $v = c_2$  where  $c_2 \in \mathbb{R}_{>0}$ , we get:

$$u = x^2 - \frac{c_2^2}{4x^2} \qquad v = 2xy = c_2$$

Taking the limits:

$$\lim_{x \rightarrow 0^+} u = -\infty \qquad \lim_{x \rightarrow \infty, x > 0} u = \infty \qquad (13.1)$$

$$\lim_{x \rightarrow -\infty, x < 0} u = \infty \qquad \lim_{x \rightarrow 0^-} u = -\infty \qquad (13.2)$$

Equation 11.1 tells us as  $x$  goes from 0 to  $\infty$ ,  $u$  moves from  $-\infty$  to  $\infty$ , which corresponds to the hyperbola in the first quadrant of the  $xy$  complex plane. Similarly for equations 11.2.



If we look at  $f$  using the polar representation, we get  $f(z) = r^2 e^{i2\theta}$ . This tells us  $\forall r \geq 0$ ,  $r \mapsto r^2 = \rho \geq 0$ , and  $\forall \theta$ ,  $\theta \mapsto \phi = 2\theta$ . It is worth noting that mapping of points between  $0 \leq \theta < 2\pi$  is not one-to-one, since points in  $0 \leq \theta < \pi$  and points in  $\pi \leq \theta < 2\pi$  both get mapped to  $0 \leq \phi < 2\pi$ .

## 13.2 Limits

### Definition 13.2.1: Limit

Let  $z, z_0, w_0 \in \mathbb{C}$  and  $f$  be a function. We say  $f(z)$  has limit  $w_0$  as  $z$  approaches  $z_0$  if:

$$\forall \epsilon \exists \delta [0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon]$$

We then denote:  $\lim_{z \rightarrow z_0} f(z) = w_0$

This tells us that  $\lim_{z \rightarrow z_0} f(z) = w_0$  if some deleted neighbourhood  $|z - z_0| < \delta$  corresponds to a neighbourhood  $|f(z) - w_0| < \epsilon$ . Note that the mapping of all points  $z$  in  $|z - z_0| < \delta$  to  $|f(z) - w_0| < \epsilon$  need not be surjective. It just needs to be mapped less than distance  $\epsilon$  from  $w_0$ .

Note: Definition 13.2.1 allows us to verify if a limit exists, but it is not a method for determining a limit.



### Theorem 13.2.1: Uniqueness of Limits

*Suppose the limit of  $f$  at  $z_0$  exists, then it is unique.*

*Proof:* Suppose two limits of  $f$  at  $z_0$  exists for contradiction.

$$\begin{aligned} & [\lim_{z \rightarrow z_0} f(z) = w_0] \wedge [\lim_{z \rightarrow z_0} f(z) = w_1] \\ \implies & [0 < |z - z_0| < \delta_0 \implies |f(z) - w_0| < \epsilon] \wedge [0 < |z - z_0| < \delta_0 \implies |f(z) - w_1| < \epsilon] \end{aligned}$$

$$\begin{aligned} w_1 - w_0 &= [f(z) - w_0] + [w_1 - f(z)] \\ \implies & |w_1 - w_0| = |[f(z) - w_0] + [w_1 - f(z)]| \leq |f(z) - w_0| + |f(z) - w_1| \end{aligned}$$

Now choosing  $\delta = \min\{\delta_1, \delta_2\}$ , we get:

$$|w_1 - w_0| < \epsilon + \epsilon = 2\epsilon$$

Choosing  $\epsilon$  to be arbitrary small, we end up with:

$$w_1 - w_0 = 0 \implies w_1 = w_0$$

□

Definition 13.2.1 requires that  $f$  be defined at all points in the deleted neighbourhood of  $z_0$ . That is,  $z_0$  is interior to the region which  $f$  is defined. We can extend the definition by agreeing that  $0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon$  also holds for  $z$  that lie in the region where  $f$  is defined and the deleted neighbourhood of  $z_0$ . That is  $f(z_0)$  need not be defined for a limit at  $z_0$  to exist.

**Example 13.2.1** Show  $(f(z) = iz/2) \wedge (|z| < 1) \implies \lim_{z \rightarrow 1} f(z) = i/2$ .

We can see that we have restricted the domain of  $f$  to the region  $|z| < 1$ , this puts  $z = 1$  right at the boundary of the domain of definition of  $f$ .

$$\begin{aligned} |z| < 1 &\implies \left| f(z) - \frac{i}{2} \right| = \left| \frac{iz}{2} - \frac{i}{2} \right| = \frac{|z - 1|}{2} \\ &\implies \forall z \forall \epsilon \exists \delta \left[ 0 < |z - 1| < \delta = 2\epsilon \implies \left| f(z) - \frac{i}{2} \right| < \epsilon \right] \\ &\implies \lim_{z \rightarrow 1} f(z) = \frac{i}{2} \end{aligned}$$

This highlights the fact that if the limit exists, then  $z$  is allowed to approach  $z_0$  from any arbitrary direction.

**Example 13.2.2** Limit of  $f(z) = z/\bar{z}$  does not exist at  $z = 0$

Consider  $\lim_{z \rightarrow 0} f(z)$ . Let us approach the limit from the  $x$ -axis and the  $y$ -axis.

$$\lim_{z=(x,0) \rightarrow 0} f(z) = \frac{x+i0}{x-i0} = 1 \qquad \lim_{z=(0,y) \rightarrow 0} f(z) = \frac{0+iy}{0-iy} = -1$$

We end up with two different limits. As limits are unique, we conclude that  $\lim_{z \rightarrow 0} f(z)$  does not exist.

### 13.2.1 Limit Theorems

**Theorem 13.2.2:**

Consider  $f(z) = u(x, y) + iv(x, y)$ . Let  $z_0 = x_0 + iy_0$  and  $w_0 = u_0 + iv_0$ .

$$\left[ \lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = u_0 \right] \wedge \left[ \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = v_0 \right] \iff \lim_{z \rightarrow z_0} f(z) = w_0$$

*Proof:*  $\implies$ :

By definition:

$$\begin{aligned} & \left[ \lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = u_0 \right] \wedge \left[ \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = v_0 \right] \\ & \implies \forall \epsilon \exists \delta_1, \delta_2 \left[ \left( 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_1 \implies |u-u_0| < \frac{\epsilon}{2} \right) \right. \\ & \quad \left. \wedge \left( 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta_2 \implies |v-v_0| < \frac{\epsilon}{2} \right) \right] \end{aligned} \tag{13.3}$$

Triangle inequality for the distance between points:

$$\begin{aligned} |(u+iv) - (u_0+iv_0)| &= |(u-u_0) + i(v-v_0)| \leq |u-u_0| + |v-v_0| \\ \sqrt{(x-x_0)^2 + (y-y_0)^2} &= |(x-x_0) + i(y-y_0)| = |(x+iy) - (x_0+iy_0)| \end{aligned}$$

Let  $\delta = \min\{\delta_1, \delta_2\}$ , it follows from eq. (13.3):

$$0 < |(x+iy) - (x_0+iy_0)| < \delta \implies |(u+iv) - (u_0+iv_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus,  $\lim_{z \rightarrow z_0} f(z) = w_0$ .

$\impliedby$ :

Suppose  $\lim_{z \rightarrow z_0} f(z) = w_0$ .

$$\begin{aligned} & \lim_{z \rightarrow z_0} f(z) = w_0 \\ & \implies \forall \epsilon \exists \delta > 0 [ |(x+iy) - (x_0+iy_0)| < \delta \implies |(u+iv) - (u_0+iv_0)| < \epsilon ] \end{aligned} \tag{13.4}$$

By the triangle inequality:

$$\begin{aligned}|u - u_0| &\leq |(u - u_0) + i(v - v_0)| = |(u + iv) - (u_0 + iv_0)| \\ |v - v_0| &\leq |(u - u_0) + i(v - v_0)| = |(u + iv) - (u_0 + iv_0)|\end{aligned}$$

$$|(x + iy) - (x_0 + iy_0)| = |(x - x_0) + i(y - y_0)| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

Thus, it follows from the inequalities in eq. (13.4):

$$\begin{aligned}0 &< \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \\ \implies & [|u - u_0| < \epsilon] \wedge [|v - v_0| < \epsilon] \\ \implies & \left[ \lim_{(x,y) \rightarrow (x_0,y_0)} u(x,y) = u_0 \right] \wedge \left[ \lim_{(x,y) \rightarrow (x_0,y_0)} v(x,y) = v_0 \right]\end{aligned}$$

□

### Theorem 13.2.3:

Suppose

$$\left[ \lim_{z \rightarrow z_0} f(z) = w_0 \right] \wedge \left[ \lim_{z \rightarrow z_0} F(z) = W_0 \right]$$

Then

$$\begin{aligned}\lim_{z \rightarrow z_0} [f(z) + F(z)] &= w_0 + W_0 \\ \lim_{z \rightarrow z_0} [f(z)F(z)] &= w_0 W_0 \\ \lim_{z \rightarrow z_0} \frac{f(z)}{F(z)} &= \frac{w_0}{W_0} \quad W_0 \neq 0\end{aligned}$$

*Proof:* Let:

$$f(z) = u(x, y) + iv(x, y) \quad F(z) = U(x, y) + iV(x, y)$$

$$z_0 = x_0 + iy_0 \quad w_0 = u_0 + iv_0 \quad W_0 = U_0 + iV_0$$

$$\underline{\lim_{z \rightarrow z_0} [f(z) + F(z)] = w_0 + W_0}$$

From Theorem 13.2.2:

$$\begin{aligned}f(z) + F(z) &= (u + U) + i(v + V) \\ \implies \lim_{(x,y) \rightarrow (x_0,y_0)} f(z)F(z) &= (u_0 + U_0) + i(v_0 + V_0) = w_0 + W_0\end{aligned}$$

$$\underline{\lim_{z \rightarrow z_0} [f(z)F(z)] = w_0 W_0}$$



From Theorem 13.2.2:

$$\begin{aligned} f(z)F(z) &= (uU - vV) + i(vU + uV) \\ \implies \lim_{(x,y) \rightarrow (x_0,y_0)} f(z)F(z) &= (u_0U_0 - v_0V_0) + i(v_0U_0 + u_0V_0) = w_0W_0 \end{aligned}$$

$$\lim_{z \rightarrow z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0} \text{ if } W_0 \neq 0$$

From Theorem 13.2.2:

$$\frac{f(z)}{F(z)} = \frac{u + iv}{U + iV} \implies \lim_{(x,y) \rightarrow (x_0,y_0)} \frac{f(z)}{F(z)} = \frac{u_0 + v_0}{U_0 + iV_0} = \frac{w_0}{W_0}$$

□

### Corollary 13.2.3.1:

Let  $c$  be a constant,  $z, z_0 \in \mathbb{C}$ , and  $P(z)$  be a polynomial. Then

$$\lim_{z \rightarrow z_0} c = c \qquad \lim_{z \rightarrow z_0} z = z_0 \qquad \lim_{z \rightarrow z_0} z^n = z_0^n \qquad n \in \mathbb{N}$$

$$\lim_{z \rightarrow z_0} P(z) = P(z_0)$$

**Observation.** It is surprisingly quick that Brown and Churchill went from  $\epsilon$ - $\delta$  proofs straight to proving with limits. This is different to the approach in Sequences of Limits Theorem for Sequences Section by Kenneth A. Ross. [1]. (Section 9.1.1)

**Question.** It might be possible use a series approach to prove limit theorems for  $z \in \mathbb{C}$  by having separate series for  $x$  and  $y$  (real and imaginary components of  $z$ ), or a series in the form of  $s_n = (x_n, y_n)$ . Which would be the proper approach?

## 13.2.2 Limits of Points at Infinity

### Definition 13.2.2: Extended Complex Plane

The complex plane union with the points at infinity:

$$\mathbb{C} \cup \{\pm\infty, \pm i\infty\}$$

### Definition 13.2.3: Riemann Sphere

A unit sphere centred at the origin of the complex plane, which is consequently bisected by the complex plane.

**Definition 13.2.4: Stereographic Projection**

Consider the Riemann Sphere. Let  $N$  be the northern point of the sphere (the point on the sphere above the origin of the complex plane) and  $z$  be any point in the complex plane. Let  $l$  be a line that goes through  $N$  and  $z$ , then  $l$  will intersect the Riemann Sphere. Let  $P$  be the point where  $l$  intersects the Riemann Sphere. If we let  $N$  correspond to the points at infinity, then there is a one-to-one correspondence between points on the sphere and the points on the extended complex plane. This correspondence is called the Stereographic Projection. (Figure 13.1)



Figure 13.1: Riemann Sphere and Stereographic Projection

The region outside the unit circle enveloped by the Riemann sphere corresponds to the upper hemisphere of the Riemann sphere, with the point  $N$  deleted.  $N$  corresponds to the points at infinity, since  $l$  will be parallel to the complex plane.

Note: In some texts, the Riemann Sphere is a sphere of unit diameter (not a unit sphere, which is of unit radius) sitting on top of the Complex Plane. That is, with the south pole sitting at  $(0,0)$ . The definitions for line  $L$ , and points  $N$ ,  $P$ , and  $z$  remains the same. In either case, the Stereographic Projection maps to a unique point  $P$  on the sphere, and the definition of the point at infinity remains unchanged.

**Definition 13.2.5: Neighbourhood of  $\infty$** 

The set:  $\{|z| > 1/\epsilon : \epsilon \in \mathbb{R}_{>0}\}$

Note that since  $\epsilon$  is a small positive number,  $|z| > 1/\epsilon$  corresponds to points far away from the unit circle, hence  $P$  is close to  $N$ .

Note: When referring to any point  $z$ , it is referring to a point in the finite plane. Points at infinity will be specifically mentioned.

**Definition 13.2.6: Limit at Infinity**

Let  $f(z)$  be a function, and  $z, z_0 \in \mathbb{C}$ .

$$\forall \epsilon \in \mathbb{R}_{>0}, \exists r \in \mathbb{R}_{>0} [|z| > r \implies |f(z) - z_0| < \epsilon] \iff \lim_{z \rightarrow \infty} f(z) = z_0$$

That is, if  $\forall z$  in the neighbourhood of infinity implies  $|f(z) - z_0| < \epsilon$ , then  $\lim_{z \rightarrow \infty} f(z) = z_0$ .

**Theorem 13.2.4:**

Let  $z_0, w_0 \in \mathbb{C}$ , then

$$\begin{aligned} \lim_{z \rightarrow z_0} \frac{1}{f(z)} = 0 &\implies \lim_{z \rightarrow z_0} f(z) = \infty \\ \lim_{z \rightarrow 0} f\left(\frac{1}{z}\right) = w_0 &\implies \lim_{z \rightarrow \infty} f(z) = w_0 \\ \lim_{z \rightarrow 0} \frac{1}{f(1/z)} = 0 &\implies \lim_{z \rightarrow \infty} f(z) = \infty \end{aligned}$$

*Proof:*  $\lim_{z \rightarrow z_0} \frac{1}{f(z)} = 0 \implies \lim_{z \rightarrow z_0} f(z) = \infty$

$$\begin{aligned} \lim_{z \rightarrow z_0} \frac{1}{f(z)} = 0 &\implies \forall \epsilon \exists \delta > 0 \left[ |z - z_0| < \delta \implies \left| \frac{1}{f(z)} - 0 \right| < \epsilon \right] \\ &\implies \forall \epsilon \exists \delta > 0 \left[ |z - z_0| < \delta \implies |f(z)| > \frac{1}{\epsilon} \right] \\ &\implies \lim_{z \rightarrow z_0} f(z) = \infty \end{aligned}$$

$\lim_{z \rightarrow 0} f\left(\frac{1}{z}\right) = w_0 \implies \lim_{z \rightarrow \infty} f(z) = w_0$

$$\begin{aligned} \lim_{z \rightarrow 0} f\left(\frac{1}{z}\right) = w_0 &\implies \forall \epsilon \exists \delta > 0 \left[ |z - 0| < \delta \implies \left| f\left(\frac{1}{z}\right) - w_0 \right| < \epsilon \right] \\ &\implies \forall \epsilon \exists \delta > 0 \left[ |z| > \frac{1}{\delta} \implies |f(z) - w_0| < \epsilon \right] \\ &\implies \lim_{z \rightarrow \infty} f(z) = w_0 \end{aligned}$$

$\lim_{z \rightarrow 0} \frac{1}{f(1/z)} = 0 \implies \lim_{z \rightarrow \infty} f(z) = \infty$

$$\begin{aligned} \lim_{z \rightarrow 0} \frac{1}{f(1/z)} = 0 &\implies \forall \epsilon \exists \delta > 0 \left[ |z - 0| < \delta \implies \left| \frac{1}{f(1/z)} - 0 \right| < \epsilon \right] \\ &\implies \forall \epsilon \exists \delta > 0 \left[ |z| > \frac{1}{\delta} \implies |f(z)| > \frac{1}{\epsilon} \right] \\ &\implies \lim_{z \rightarrow \infty} f(z) = \infty \end{aligned}$$

□

Note: As  $\delta$  goes to 0,  $1/\delta$  goes to  $\infty$ , hence  $|z|$  goes to  $\infty$  if  $|z| > 1/\delta$ .

**Observation.** As expected, theorem 13.2.4 is consistent if  $z \in \mathbb{R}$ . (Check: Section 9.1.1).

## 13.3 Continuity

### Definition 13.3.1: Continuous

Let  $f$  be a function. We say  $f$  is continuous at all point  $z_0 \in \mathbb{C}$  if it satisfies the following:

$$\lim_{z \rightarrow z_0} f(z) \text{ exists} \wedge f(z_0) \text{ exists} \wedge \lim_{z \rightarrow z_0} f(z) = f(z_0)$$

Note:

$$\begin{aligned} \lim_{z \rightarrow z_0} f(z) = f(z_0) &\implies \lim_{z \rightarrow z_0} f(z) \text{ exists} \wedge f(z_0) \text{ exists} \\ \forall \epsilon \exists \delta > 0 [ |z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon ] &\iff \lim_{z \rightarrow z_0} f(z) = f(z_0) \end{aligned}$$

### Definition 13.3.2: Continuous at a Region

Let  $f$  be a function,  $R \subset \mathbb{C}$  be a region, and  $z \in R$ :

$$f \text{ is continuous in } R \iff \forall z \in R (f \text{ is continuous})$$

### Theorem 13.3.1:

Let  $f(z)$  and  $g(z)$  be continuous functions at  $z_0 \in \mathbb{C}$ . Then the following are also continuous at  $z_0$ :

$$f(z_0) + g(z_0) \quad f(z_0)g(z_0) \quad \frac{f(z_0)}{g(z_0)} \quad g(z_0) \neq 0$$

*Proof:* Consequence of theorem 13.2.3. □

### Corollary 13.3.1.1:

Let  $P(z)$  be a polynomial, then  $P(z)$  is continuous  $\forall z \in \mathbb{C}$ . That is  $P(z)$  is continuous in the entire plane of  $\mathbb{C}$ .

*Proof:* Consequence of corollary 13.2.3.1. □

**Observation.** Both theorem 13.3.1 and corollary 13.3.1.1 rely on definition 13.3.1, which state for a function  $f$  and point  $z_0 \in \mathbb{C}$ :

$$\lim_{z \rightarrow z_0} f(z) \text{ exists} \implies f(z) \text{ is continuous at } z_0$$

This is why the proofs cite the results of theorem 13.2.3 and corollary 13.2.3.1.

**Theorem 13.3.2:**

Let  $f(z)$  and  $g(z)$  be functions.

$$f(z) \text{ and } g(z) \text{ continuous} \implies g(f(z)) \text{ continuous}$$

*Proof:* Let  $f(z) = w$  be defined in the neighbourhood  $\forall z[|z - z_0| < \delta]$ , and  $g(w) = W$  where  $\text{dom}(g) = \text{img}(f)$ . Suppose that  $f$  is continuous at  $z_0$  and  $g$  is continuous at  $f(z_0)$ .

$$\begin{aligned} f \text{ continuous at } z_0 &\iff \forall \gamma \exists \delta > 0 [|z - z_0| < \delta \implies |f(z) - f(z_0)| < \gamma] \\ &\implies \forall \epsilon \exists \gamma > 0 [|f(z) - f(z_0)| < \gamma \implies |g(f(z)) - g(f(z_0))| < \epsilon] \end{aligned}$$

We can always find a small enough  $\delta$  for  $\gamma$  to satisfy  $|g(f(z)) - g(f(z_0))| < \epsilon$ .  $\square$



**Theorem 13.3.3:**

Let  $f(z)$  be a function and  $f(z_0) \neq 0$ .

$$f(z_0) \neq 0 \implies \exists \epsilon \forall z [|f(z) - f(z_0)| < \epsilon \implies f(z) \neq 0]$$

That is, if  $f(z_0) \neq 0$  then it has a neighbourhood where  $f(z) \neq 0$ .

*Proof:* Suppose  $f(z)$  is continuous and non-zero at  $z_0$ , and let  $\epsilon = |f(z_0)|/2$ :

$$\begin{aligned} &\exists z[f(z) = 0] \wedge \forall \epsilon \exists \delta > 0 [|z - z_0| < \delta \implies |f(z) - f(z_0)| < \frac{|f(z_0)|}{2}] \\ &\implies |f(z_0)| < \frac{|f(z_0)|}{2} \end{aligned} \quad \text{Contradiction!}$$

$\square$

**Theorem 13.3.4:**

Let  $f(z) = u(x, y) + iv(x, y)$  be a function, and  $z = x + iy$ ,  $z \in \mathbb{C}$ .

$$f \text{ continuous at } z_0 \iff [u \text{ continuous at } z_0] \wedge [v \text{ continuous at } z_0]$$

*Proof:* Direct consequence of theorem 13.2.2 □

**Theorem 13.3.5:**

*Let  $f$  be continuous in a closed and bounded region  $R$ , then*

$$\forall z \in R, \exists M \in \mathbb{R}_{>0} [|f(z)| \leq M] \wedge |\{z : |f(z)| = M\}| \geq 1$$

*That is, for  $\forall z \in R$ ,  $|f(z)| \leq M$  and there is at least one point  $z$  where  $|f(z)| = M$ .  $f(z)$  is bounded in  $R$ .*

*Proof:* Let  $f(z) = u(x, y) + iv(x, y)$  be continuous, then

$$|f(z)| = \sqrt{[u(x, y)]^2 + [v(x, y)]^2} \text{ is continuous in } R \implies \exists M \in \mathbb{R}_{>0} [|f(z)| \leq M]$$

□

### 13.3.1 Exercises

**Example 13.3.1** *Prove:*

$$\lim_{z \rightarrow z_0} f(z) = w_0 \implies \lim_{z \rightarrow z_0} |f(z)| = |w_0|$$

*Note:*  $||f(z_0)| - |w_0|| \leq |f(z) - w_0|$

*Proof:* Use definition of limit, then plug and chug. □

**Example 13.3.2** *Prove: Limits involving points at infinity are unique.*

*Proof:* Suppose that limit of the point at infinity is not unique, that is there is two neighbourhoods of infinity. Using the definition of the limit, we will arrive at a contradiction where the two neighbourhoods are the same. □

**Example 13.3.3** *Prove:*

$$S \text{ is unbounded} \iff \forall \epsilon \exists z \left[ z \in S : |z| > \frac{1}{\epsilon} \right]$$

*That is,  $S$  is unbounded  $\iff$  every neighbourhood of the point at infinity contains at least one point in  $S$*

*Proof:* Proof Sketch: Recall the Riemann Sphere. (Definition 13.2.3). The set  $|z| > 1/\epsilon$  corresponds to the points close to  $N$ , which is the neighbourhood of the point at infinity. If we let  $\gamma = 2\epsilon$ ,  $\exists z$  where  $|z| > 1/\gamma$  holds. This along with  $z \in \mathbb{C}$  (which is  $S$  in our case), implies the direction  $\Leftarrow$  is true. That is, we can still find elements in  $S$  as we shrink the circle around  $N$ .

$S$  is unbounded implies that for all circle with radius  $R$  centred at the origin there is at least one element of  $s \in S$  where  $|s| > R$ . Suppose for contradiction that there is a neighbourhood of the point at infinity that does not contain any points in  $S$ . We will arrive at a contradiction, where there is  $M \in \mathbb{R}_{>0}$  such that  $\forall s \in S [|s| < M]$ . Thus  $S$  is bounded, a contradiction. This implies that the direction  $\implies$  is true. □

## 13.4 Differentiation

### Definition 13.4.1: Derivative

Let  $f$  be a function where  $|z - z_0| < \epsilon$  and  $z \in \text{dom}(f)$ . Then the derivative of  $f$  at point  $z_0$ :

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

### Definition 13.4.2: Differentiable

A function  $f$  is differentiable at  $z_0 \in \mathbb{C}$  if  $f'(z_0)$  exists.

If we let  $\Delta z = z - z_0$  where  $z \neq z_0$ :

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$



There's another notation by letting  $\Delta w = f(z + \Delta z) - f(z)$ :

$$f'(z) = \frac{dw}{dz} = \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z}$$

**Observation.** The definition of a derivative in definition 13.4.1 looks similar to that of a derivative for the real numbers:

$$F'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

However, the existence of  $f'(z)$  possesses a much stronger requirement than the existence of  $F'(z)$ . That is, let  $f(z) = u(x, y) + iv(x, y)$ . The existence of  $f'(z)$  at point  $z_0$  requires the existence of both  $u'(x, y)$  and  $v'(x, y)$ .

$$f'(z_0) = \lim_{(x,y) \rightarrow (x_0,y_0)} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{(x,y) \rightarrow (x_0,y_0)} \frac{u(z) - u(z_0)}{z - z_0} + i \frac{v(z) - v(z_0)}{z - z_0}$$

and that

$$\begin{aligned} & \lim_{(x,y_0) \rightarrow (x_0,y_0)} \frac{u(x,y_0) - u(x_0,y_0)}{x - x_0} + i \frac{v(x_0,y_0) - v(x_0,y_0)}{x - x_0} \\ &= \lim_{(x_0,y) \rightarrow (x_0,y_0)} \frac{u(x_0,y) - u(x_0,y_0)}{x - x_0} + i \frac{v(x_0,y) - v(x_0,y_0)}{x - x_0} \end{aligned}$$

That is

$$\begin{aligned} \lim_{(\Delta x, 0) \rightarrow (0,0)} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} &= \lim_{(0, \Delta y) \rightarrow (0,0)} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y} \\ \lim_{(\Delta x, 0) \rightarrow (0,0)} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} &= \lim_{(0, \Delta y) \rightarrow (0,0)} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} \end{aligned}$$

This tells us that the existence of a derivative for a real valued function  $F(x)$  does not imply the existence of a derivative for a similar function  $f(z)$  in the complex plane, which we will see later. (i.e. Take  $f(z) = |z|^2$  and  $F(x) = |x|^2$ .) We are dealing with a two-dimensional limit instead of a one dimensional limit.

**Question.** Under what conditions will differentiability in  $\mathbb{C}$  imply differentiability in  $\mathbb{R}$ , and vice versa?

**Example 13.4.1** Let  $f(z) = \bar{z}$ :

$$\frac{\Delta w}{\Delta z} = \frac{\overline{z + \Delta z} - \bar{z}}{\Delta z} = \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}$$

Consider  $\Delta z = (\Delta x, \Delta y) \rightarrow (0, 0)$ . If we move on the real axis, that is  $(\Delta x, 0)$ :

$$\overline{\Delta z} = \overline{\Delta x + i0} = \Delta x - i0 = \Delta x + i0 = \Delta z \implies \frac{\Delta w}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta z}{\Delta z} = 1$$

If we move on the imaginary axis, that is  $(0, \Delta y)$ :

$$\overline{\Delta z} = \overline{0 + i\Delta y} = 0 - i\Delta y = -\Delta z \implies \frac{\Delta w}{\Delta z} = \frac{\overline{\Delta z}}{\Delta y} = \frac{-\Delta z}{\Delta z} = -1$$

Limits are unique, so the limit of  $dw/dz$  does not exist anywhere.

**Example 13.4.2** Consider  $f(z) = |z|^2$ :

$$\begin{aligned} \frac{\Delta w}{\Delta z} &= \frac{|z + \Delta z|^2 - |z|^2}{\Delta z} = \frac{(z + \Delta z)(\overline{z + \Delta z}) - z\bar{z}}{\Delta z} \\ &= \frac{(z + \Delta z)(\bar{z} + \overline{\Delta z}) - z\bar{z}}{\Delta z} = \frac{z\bar{z} + \Delta z\bar{z} + \overline{\Delta z}z + \overline{\Delta z}\Delta z - z\bar{z}}{\Delta z} = \bar{z} + \overline{\Delta z} + z \frac{\overline{\Delta z}}{\Delta z} \end{aligned}$$

As in the previous example, as  $(\Delta x, \Delta y) \rightarrow (0, 0)$ :

$$\overline{\Delta z} = \Delta z$$

From the real axis

$$\overline{\Delta z} = -\Delta z$$

From the imaginary axis



Thus

$$\begin{aligned}\frac{\Delta w}{\Delta z} &= \bar{z} + \Delta z + z & \Delta z &= (\Delta x, 0) \\ \frac{\Delta w}{\Delta z} &= \bar{z} - \Delta z - z & \Delta z &= (0, \Delta y)\end{aligned}$$

Therefore, by uniqueness of limits as  $\Delta z \rightarrow 0$ :

$$\lim_{\Delta z \rightarrow 0} (\bar{z} + \Delta z + z) = \lim_{\Delta z \rightarrow 0} (\bar{z} - \Delta z - z) \implies z = -z \implies z = 0$$

Hence,  $dw/dz$  does not exist for  $z \neq 0$ . We can also see that:

$$\frac{\Delta w}{\Delta z} = \bar{z} + \overline{\Delta z} + z \frac{\overline{\Delta z}}{\Delta z} = \overline{\Delta z} \quad z = 0$$

Thus,  $dw/dz$  only exists at  $z = 0$ :

$$\left. \frac{dw}{dz} \right|_{z=0} = 0$$

**Remark.** The following are facts:

- (1) A function  $f(z)$  can be differentiable at a point  $z_0$ , but nowhere else in the neighbourhood of  $z_0$ .
- (2)  $f(z) = |z|^2 \implies u(x, y) = x^2 + y^2 \wedge v(x, y) = 0$ , hence  $u(x, y)$  and  $v(x, y)$  can have continuous partial derivatives of all orders at a point  $z_0$ , even though  $f$  may not be differentiable at  $z_0$ .
- (3)  $f(z)$  differentiable at  $z_0 \implies f(z)$  continuous at  $z_0$

*Proof:* Assume  $f'(z_0)$  exists:

$$\begin{aligned}\lim_{z \rightarrow z_0} [f(z) - f(z_0)] &= \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \lim_{z \rightarrow z_0} (z - z_0) = f'(z_0) \cdot 0 = 0 \\ &\implies \lim_{z \rightarrow z_0} f(z) = f(z_0)\end{aligned}$$

So,  $f$  is differentiable at  $z_0 \implies f$  is continuous at  $z_0$ . □

*Note:* Continuity of a function at  $z_0 \in \mathbb{C} \not\Rightarrow$  existence of derivative at point  $z_0$ .

*Ex:*  $f(z) = |z|^2$  is continuous everywhere in  $\mathbb{C}$  for  $z_0 \neq 0$ , but  $f'(z_0)$  does not exist at  $z_0$ .

### 13.4.1 Differentiation Rules

Definition of derivative in  $\mathbb{C}$  (definition 13.4.1) is the same of that in  $\mathbb{R}$ , so rules remain the same.

Let  $c \in \mathbb{C}$  be a constant and functions  $f$  and  $g$  be differentiable at point  $z$ . Then

$$\frac{d}{dz}c = 0 \quad \frac{d}{dz}z = 1 \quad \frac{d}{dz}[cf(z)] = cf'(z) \quad \frac{d}{dz}z^n = nz^{n-1} \quad n \in \mathbb{Z} \setminus \{0\}$$

Let functions  $f$  and  $g$  be differentiable at point  $z$ . Then

$$\frac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z) \quad \frac{d}{dz}[f(z)g(z)] = f(z)g'(z) + f'(z)g(z)$$

$$\frac{d}{dz} \left[ \frac{f(z)}{g(z)} \right] = \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2}$$

*Proof:* Deriving:  $\frac{d}{dz}[f(z)g(z)] = f(z)g'(z) + f'(z)g(z)$

Let  $w = f(z)g(z)$ :

$$\begin{aligned} \Delta w &= f(z + \Delta z)g(z + \Delta z) - f(z)g(z) \\ &= f(z)[g(z + \Delta z) - g(z)] + [f(z + \Delta z) - f(z)]g(z + \Delta z) \end{aligned}$$

Thus

$$\frac{\Delta w}{\Delta z} = f(z) \frac{g(z + \Delta z) - g(z)}{\Delta z} + \frac{f(z + \Delta z) - f(z)}{\Delta z} g(z + \Delta z)$$

Hence

$$\frac{dw}{dz} = \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} = f(z)g'(z) + f'(z)g(z)$$

□

### Theorem 13.4.1: Chain Rule for Composite Functions

Let function  $f$  be differentiable at  $z_0$  and function  $g$  be differentiable at  $f(z_0)$ . Then  $F(z) = g[f(z)]$  is differentiable at  $z_0$ .

$$F'(z_0) = g'[f(z_0)]f'(z_0)$$

*Proof:* Suppose  $f$  is differentiable at  $z_0$ . Let  $w_0 = f(z_0)$  and assume that  $g'(w_0)$  exists. Then

$$\forall w \exists \epsilon [ |w - w_0| < \epsilon \implies \Phi(w_0) = 0 ]$$

Where

$$\Phi(w) = \frac{g(w) - g(w_0)}{w - w_0} - g'(w_0) \quad w \neq w_0$$

Note:  $\lim_{w \rightarrow w_0} \Phi(w) = 0$ , so  $\Phi$  is continuous at  $w_0$ . Then

$$g(w) - g(w_0) = [g'(w_0) + \Phi(w)](w - w_0) \quad |w - w_0| < \epsilon$$

Note: This is valid for  $w = w_0$ .

$$\begin{aligned} f'(z_0) \text{ exists} &\implies f \text{ continuous at } z_0 \\ &\implies \forall \epsilon \exists \delta > 0 [|z - z_0| < \delta \implies |w - w_0| < \epsilon] \end{aligned}$$

Hence, we can replace  $w$  by  $f(z)$  when  $|z - z_0| < \delta$ . Subbing  $w = f(z)$  and  $w_0 = f(z_0)$ :

$$\frac{g[f(z)] - g[f(z_0)]}{z - z_0} = \{g'[f(z_0)] + \Phi[f(z)]\} \frac{f(z) - f(z_0)}{z - z_0} \quad 0 < |z - z_0| < \delta, \quad z \neq z_0$$

Then

$$(f \text{ continuous at } z_0) \wedge (\Phi \text{ continuous at } w_0 = f(z_0)) \implies \Phi[f(z)] \text{ continuous at } z_0$$

$$\Phi(w_0) = 0 \implies \lim_{z \rightarrow z_0} \Phi[f(z)] = 0$$

Thus

$$\begin{aligned} \lim_{z \rightarrow z_0} \frac{g[f(z)] - g[f(z_0)]}{z - z_0} &= \lim_{z \rightarrow z_0} \{g'[f(z_0)] + \Phi[f(z)]\} \frac{f(z) - f(z_0)}{z - z_0} \\ &= g'[f(z_0)] f'(z_0) \end{aligned}$$

We then get

$$F'(z_0) = g'[f(z_0)] f'(z_0)$$

□

Alternatively, if we let  $w = f(z)$  and  $W = F(z)$ , then the Chain Rule becomes:

$$\frac{dW}{dz} = \frac{dW}{dw} \frac{dw}{dz}$$

Note: Although this looks like a fraction, it is not a fraction and should not be treated as such! (Logical inconsistency when infinitesimals when viewed as ratios.)

### Theorem 13.4.2: L'Hopital's Rule

Suppose  $f(z_0) = 0$  and  $g(z_0) = 0$ ,  $f'(z_0)$  and  $g'(z_0)$  exists, with  $g'(z_0) \neq 0$ . Then

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

*Proof:* Let  $f(z_0) = 0$ ,  $g(z_0) = 0$ , and  $z \neq z_0$ .

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \lim_{z \rightarrow z_0} \frac{\frac{f(z) - f(z_0)}{z - z_0}}{\frac{g(z) - g(z_0)}{z - z_0}} = \lim_{\Delta z \rightarrow 0} \frac{\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}}{\frac{g(z_0 + \Delta z) - g(z_0)}{\Delta z}} = \frac{f'(z_0)}{g'(z_0)}$$

□

### 13.4.2 Exercises

**Example 13.4.3** Show that  $f'(z)$  does not exist for all points  $z \in \mathbb{C}$  when:

(a)  $f(z) = \operatorname{Re}\{z\}$

(b)  $f(z) = \operatorname{Im}\{z\}$

*Proof:* Let  $f(z) = u(x, y) + iv(x, y)$ ,  $\Delta w = f(x + \Delta x, y + \Delta y) - f(x, y)$ .

$f(z) = \operatorname{Re}\{z\}$

Recall  $\operatorname{Re}\{z\} = x + i0$ .

$$\frac{\Delta w}{\Delta z} = \frac{\operatorname{Re}\{z + \Delta z\} - \operatorname{Re}\{z\}}{\Delta z} = \frac{x + \Delta x - x}{\Delta z} = \frac{\Delta x}{\Delta x + \Delta y}$$

Now as  $(\Delta x, 0) \rightarrow (0, 0)$ :

$$\lim_{(\Delta x, 0) \rightarrow (0, 0)} \frac{\Delta w}{\Delta z} = \lim_{(\Delta x, 0) \rightarrow (0, 0)} \frac{\Delta x}{\Delta x} = 1$$

Now as  $(0, \Delta y) \rightarrow (0, 0)$ :

$$\lim_{(0, \Delta y) \rightarrow (0, 0)} \frac{\Delta w}{\Delta z} = \lim_{(0, \Delta y) \rightarrow (0, 0)} \frac{0}{\Delta y} = 0$$

Limits are unique, but this isn't the case, so we conclude that  $f'(z)$  when  $f(z) = \operatorname{Re}\{z\}$  does not exist.

$f(z) = \operatorname{Im}\{z\}$

Recall  $\operatorname{Im}\{z\} = 0 + iy$ .

$$\frac{\Delta w}{\Delta z} = \frac{\operatorname{Im}\{z + \Delta z\} - \operatorname{Im}\{z\}}{\Delta z} = \frac{y + \Delta y - y}{\Delta z} = \frac{\Delta y}{\Delta x + \Delta y}$$

Now as  $(\Delta x, 0) \rightarrow (0, 0)$ :

$$\lim_{(\Delta x, 0) \rightarrow (0, 0)} \frac{\Delta w}{\Delta z} = \lim_{(\Delta x, 0) \rightarrow (0, 0)} \frac{0}{\Delta x} = 0$$

Now as  $(0, \Delta y) \rightarrow (0, 0)$ :

$$\lim_{(0, \Delta y) \rightarrow (0, 0)} \frac{\Delta w}{\Delta z} = \lim_{(0, \Delta y) \rightarrow (0, 0)} \frac{\Delta y}{\Delta y} = 1$$

Limits are unique, but this isn't the case, so we conclude that  $f'(z)$  when  $f(z) = \operatorname{Im}\{z\}$  does not exist.  $\square$

## 13.5 Cauchy-Riemann Equations

### Theorem 13.5.1: Cauchy-Riemann Equations (Cartesian)

Let  $f(z) = u(x, y) + iv(x, y)$ . If  $f'(z)$  exists at a point  $z_0 = x_0 + iy_0$ , then  $u'(x_0, y_0)$  and  $v'(x_0, y_0)$  exists and satisfy Cauchy-Riemann equations:

$$u_x = v_y \qquad u_y = -v_x$$

Also, as a result of evaluating  $f'(z)$  from the horizontal and vertical direction:

$$f'(z_0) = [u_x + iv_x] \Big|_{(x_0, y_0)} = [v_y - iu_y] \Big|_{(x_0, y_0)}$$

*Proof:* Let  $f(z) = u(x, y) + iv(x, y)$ , and suppose  $f'(z)$  exists at  $z_0$ . Then

$$z_0 = x_0 + iy_0 \qquad \Delta z = \Delta x + i\Delta y \qquad \Delta w = f(z_0 + \Delta z) - f(z_0)$$

So that

$$\Delta w = [u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y)] - [u(x_0, y_0) + iv(x_0, y_0)]$$

Therefore

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta x + i\Delta y} + i \frac{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)}{\Delta x + i\Delta y}$$

Note: This equation remains valid as  $(\Delta x, \Delta y) \rightarrow (0, 0)$ .

#### Horizontal Approach:

Let  $(\Delta x, 0) \rightarrow (0, 0)$  in the horizontal direction, then

$$\begin{aligned} f'(z_0) &= \lim_{\Delta x \rightarrow 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \lim_{\Delta x \rightarrow 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \\ \implies f'(z_0) &= u_x(x_0, y_0) + iv_x(x_0, y_0) \end{aligned}$$

#### Vertical Approach:

Let  $(0, \Delta y) \rightarrow (0, 0)$  in the vertical direction, then

$$\begin{aligned} f'(z_0) &= \lim_{\Delta y \rightarrow 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i \lim_{\Delta y \rightarrow 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y} \\ &= -i \lim_{\Delta y \rightarrow 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y} + \lim_{\Delta y \rightarrow 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} \\ \implies f'(z_0) &= v_y(x_0, y_0) - iu_y(x_0, y_0) \end{aligned}$$

Putting it together:

For  $f'(z)$  to exist at  $z_0$ ,  $f'(z_0)$  from the horizontal approach must equal that of the vertical approach. By equating the real and imaginary parts:

$$\begin{aligned} u_x(x_0, y_0) + iv_x(x_0, y_0) &= v_y(x_0, y_0) - iu_y(x_0, y_0) \\ \implies (u_x = v_y) \wedge (u_y = -v_x) \end{aligned}$$

□

### Theorem 13.5.2: Cauchy-Riemann Equations (Polar)

Let  $f(z) = u(r, \theta) + iv(r, \theta)$  be defined in some neighbourhood  $\epsilon$  of  $z_0 = r_0 e^{i\theta_0}$ ,  $z_0 \neq 0$ . If the first order partial derivatives of  $u$  and  $v$  with respect to  $r$  and  $\theta$  exist and are continuous at  $z_0$ , and satisfies the polar form of the Cauchy-Riemann equations:

$$ru_r = v_\theta \qquad u_\theta = -rv_r$$

Then  $f'(z_0)$  exists:

$$f'(z_0) = e^{-i\theta}(u_r + iv_r) \Big|_{(r_0, \theta_0)} = \frac{-i}{z_0}(u_\theta + iv_\theta) \Big|_{(r_0, \theta_0)}$$

*Proof:* Let  $f(z) = u(r, \theta) + iv(r, \theta)$ . Suppose that the first order partial derivatives of  $u$  and  $v$  exist in some neighbourhood  $\epsilon$  of  $z_0$  and is continuous at  $z_0$ . By differentiating  $u$  with respect to  $x$  and  $y$ :

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} \qquad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta}$$

Likewise for  $v$ . As  $x = r \cos \theta$  and  $y = r \sin \theta$ :

$$\begin{aligned} u_r &= u_x \cos \theta + u_y \sin \theta & u_\theta &= -u_x r \sin \theta + u_y r \cos \theta \\ v_r &= v_x \cos \theta + v_y \sin \theta & v_\theta &= -v_x r \sin \theta + v_y r \cos \theta \end{aligned}$$

From theorem 13.5.1 we have:

$$u_x = v_y \qquad u_y = -v_x$$

Subbing the Cauchy-Riemann equations into  $v_r$  and  $v_\theta$ :

$$\begin{aligned} u_r &= u_x \cos \theta + u_y \sin \theta & u_\theta &= -u_x r \sin \theta + u_y r \cos \theta \\ v_r &= -u_y \cos \theta + u_x \sin \theta & v_\theta &= u_y r \sin \theta + u_x r \cos \theta \end{aligned}$$

We can see that:

$$ru_r = v_\theta \qquad u_\theta = -rv_r$$

Which are the Cauchy Riemann equations in polar form. Let's verify it without relying on the Cauchy-Riemann equations in Cartesian form:

Recall:

$$\begin{aligned} u_r &= u_x \cos \theta + u_y \sin \theta & u_\theta &= -u_x r \sin \theta + u_y r \cos \theta \\ v_r &= v_x \cos \theta + v_y \sin \theta & v_\theta &= -v_x r \sin \theta + v_y r \cos \theta \end{aligned}$$

Writing  $u_r$  and  $v_r$  in matrix notation:

$$\begin{bmatrix} u_r \\ u_\theta \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

Solving for  $u_x$  and  $u_y$ :

$$\begin{aligned} \begin{bmatrix} u_x \\ u_y \end{bmatrix} &= \begin{bmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{bmatrix}^{-1} \begin{bmatrix} u_r \\ u_\theta \end{bmatrix} \\ &= \frac{1}{r \cos^2 \theta + r \sin^2 \theta} \begin{bmatrix} r \cos \theta & -\sin \theta \\ -r \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} u_r \\ u_\theta \end{bmatrix} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{r} \begin{bmatrix} r \cos \theta & -\sin \theta \\ r \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} u_r \\ u_\theta \end{bmatrix} \end{aligned}$$

It is clear that for  $u_x$  and  $u_y$ , and likewise for  $v_x$  and  $v_y$ :

$$u_x = u_r \cos \theta - \frac{1}{r} u_\theta \sin \theta \quad u_y = u_r \sin \theta + \frac{1}{r} u_\theta \cos \theta \quad (13.5)$$

$$v_x = v_r \cos \theta - \frac{1}{r} v_\theta \sin \theta \quad v_y = v_r \sin \theta + \frac{1}{r} v_\theta \cos \theta \quad (13.6)$$

Using the Cauchy-Riemann equations  $u_x = v_y$  and  $u_y = -v_x$ , we see:

$$\begin{aligned} u_r \cos \theta - \frac{1}{r} u_\theta \sin \theta &= v_r \sin \theta + \frac{1}{r} v_\theta \cos \theta \\ u_r \sin \theta + \frac{1}{r} u_\theta \cos \theta &= -v_r \cos \theta + \frac{1}{r} v_\theta \sin \theta \end{aligned}$$

Clearly, the equations are equal only if

$$r u_r = v_\theta \quad u_\theta = -r v_r$$

Which are the polar forms of the Cauchy-Riemann equations.

Show  $f'(z_0) = e^{-i\theta}(u_r + i v_r)$ :

Recall from theorem 13.5.1:

$$f'(z_0) = u_x + i v_y$$

Using eq. (13.5) and eq. (13.6) from before and substituting them into  $f'(z_0)$ :

$$\begin{aligned}
f'(z_0) &= \left( u_r \cos \theta - \frac{1}{r} u_\theta \sin \theta + i v_r \cos \theta - \frac{i}{r} v_\theta \sin \theta \right) \Big|_{(r_0, \theta_0)} \\
&= (u_r \cos \theta + v_r \sin \theta + i v_r \cos \theta - i u_r \sin \theta) \Big|_{(r_0, \theta_0)} \\
&= [u_r (\cos \theta - i \sin \theta) + v_r (\sin \theta + i \cos \theta)] \Big|_{(r_0, \theta_0)} \\
&= [u_r (\cos \theta - i \sin \theta) + i v_r (\cos \theta - i \sin \theta)] \Big|_{(r_0, \theta_0)} \\
&= \left[ \left( \frac{e^{i\theta} + e^{-i\theta}}{2} - \frac{e^{i\theta} - e^{-i\theta}}{2} \right) (u_r + i v_r) \right] \Big|_{(r_0, \theta_0)} \\
&= e^{-i\theta} (u_r + i v_r) \Big|_{(r_0, \theta_0)} \\
&= \frac{-i}{r e^{i\theta}} (u_\theta + i v_\theta) \Big|_{(r_0, \theta_0)} = \frac{-i}{z_0} (u_\theta + i v_\theta) \Big|_{(r_0, \theta_0)} \quad (ru_r = v_\theta) \wedge (u_\theta = -rv_r)
\end{aligned}$$

Thus

$$f'(z_0) = e^{-i\theta} (u_r + i v_r) \Big|_{(r_0, \theta_0)} = \frac{-i}{z_0} (u_\theta + i v_\theta) \Big|_{(r_0, \theta_0)}$$

□

**Question.** When comparing the Cartesian form to the polar form of the Cauchy-Riemann equations:

$$\begin{aligned}
f'(z_0) \text{ exists} &\implies \forall z_0 [(u_x = v_y) \wedge (u_y = -v_x)] \\
(z_0 \neq 0) \wedge \forall z_0 [(ru_r = v_\theta) \wedge (u_\theta = -rv_r)] &\implies f'(z_0) \text{ exists}
\end{aligned}$$

Should both be  $\iff$  instead of  $\implies$ ? No, satisfying Cauchy-Riemann equations does not guarantee differentiability at a point as we will see in example 13.5.3. However, satisfying certain conditions allows differentiability to exist (theorem 13.5.4).

**Example 13.5.1** (Solving the  $f'(z)$  using the partial derivative with respect to one variable)  
Recall in theorem 13.5.1:

$$f'(z_0) = [u_x + i v_x] \Big|_{(x_0, y_0)} = [v_y - i u_y] \Big|_{(x_0, y_0)}$$

This implies we can solve  $df(z)/dz$  by taking the partial of  $f(z)$  with respect to  $x$  or  $y$ . Consider  $f(z) = z^2$ :

$$f(z) = z^2 = x^2 - y^2 + i2xy$$

We then have:

$$u(x, y) = x^2 - y^2 \qquad v(x, y) = 2xy$$



Hence

$$u_x = 2x = v_y$$

$$u_y = -2y = -v_x$$

Thus

$$f'(z) = 2x + i2y = 2(x + iy) = 2z$$

**Example 13.5.2** (Using Cauchy-Riemann equations to find where  $f(z)$  is not differentiable)  
Using the contrapositive of  $f'(z_0)$  exists  $\implies \exists u' \exists v' [(u_x = v_y) \wedge (u_y = -v_x)]:$

$$\exists z_0 [(u_x \neq v_y) \vee (u_y \neq -v_x)] \implies f(z) \text{ not differentiable at } z_0$$

Consider  $f(z) = |z|^2$ :

$$u(x, y) = x^2 + y^2$$

$$v(x, y) = 0$$

By Cauchy-Riemann:

$$2x = 0$$

$$2y = 0$$

Therefore,  $f'(z)$  only exists at  $(0, 0)$  and does not exist elsewhere.

Note: Theorem 13.5.1 does not guarantee the existence of  $f'(z)$  at  $z_0$ .

**Example 13.5.3** ( $f(z)$  satisfy Cauchy-Riemann equations at  $(0, 0)$ , but  $f'(0)$  does not exist)  
Consider

$$f(z) = \begin{cases} \bar{z}^2/z & z \neq 0 \\ 0 & z = 0 \end{cases}$$

Then

$$u(x, y) = \frac{x^3 - 3xy^2}{x^2 + y^2}$$

$$v(x, y) = \frac{y^3 - 3x^2y}{x^2 + y^2}$$

$$(x, y) \neq (0, 0)$$

Checking differentiability at  $(0, 0)$ , note  $u(0, 0) = 0$  and  $v(0, 0) = 0$ :

$$u_x(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{u(0 + \Delta x, 0) - u(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x} = 1$$

$$v_y(0, 0) = \lim_{\Delta y \rightarrow 0} \frac{v(0, 0 + \Delta y) - v(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{\Delta y}{\Delta y} = 1$$

$$u_y(0, 0) = \lim_{\Delta y \rightarrow 0} \frac{u(0, 0 + \Delta y) - u(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{0/(\Delta y)^2}{\Delta y} = 0$$

$$v_x(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{v(0 + \Delta x, 0) - v(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0/(\Delta x)^2}{\Delta x} = 0$$

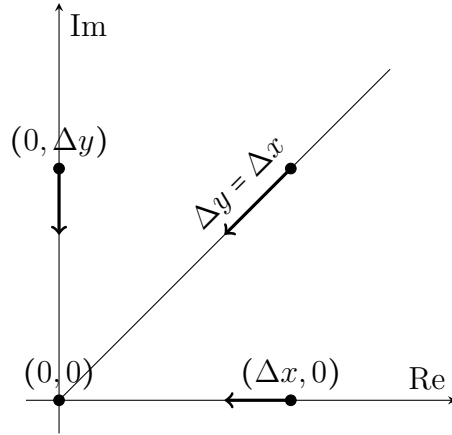
We can see that the Cauchy-Riemann equations are satisfied:

$$u_x = v_y = 1$$

$$u_y = -v_x = 0$$

However,  $f'(0)$  does not exist: (Brown and Churchill - Complex Variables and Applications, Section 20, Exercise 9 [2])

Let  $\Delta w = f(z + \Delta z) - f(z)$ . We need to show for all nonzero points on the real and imaginary axis,  $\Delta w/\Delta z = -1$ , but for all nonzero points on the line  $\Delta x = \Delta y$ ,  $\Delta w/\Delta z = 1$ . Hence, a contradiction, so  $f'(0)$  does not exist.



$$\frac{\Delta w}{\Delta z} = \frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{u(x + \Delta x, y + \Delta y) + v(x + \Delta x, y + \Delta y)}{\Delta x + \Delta y} - \frac{u(x, y) + v(x, y)}{\Delta x + \Delta y}$$

**Along the real axis:**

Evaluating along  $(\Delta x, 0) \rightarrow (0, 0)$ .

$$\begin{aligned} \lim_{(\Delta x, 0) \rightarrow (0, 0)} \frac{\Delta w}{\Delta z} &= \frac{u(\Delta x, 0) + v(\Delta x, 0)}{\Delta x} - \frac{u(0, 0) + v(0, 0)}{\Delta x} \\ &= \frac{1}{\Delta x} \left[ \frac{(\Delta x)^3}{(\Delta x)^2} + \frac{0}{(\Delta x)^2} \right] - 0 = \frac{\Delta x}{\Delta x} = 1 \end{aligned}$$

**Along the imaginary axis:**

Evaluating along  $(0, \Delta y) \rightarrow (0, 0)$ .

$$\begin{aligned} \lim_{(0, \Delta y) \rightarrow (0, 0)} \frac{\Delta w}{\Delta z} &= \frac{u(0, \Delta y) + v(0, \Delta y)}{\Delta y} - \frac{u(0, 0) + v(0, 0)}{\Delta y} \\ &= \frac{1}{\Delta y} \left[ \frac{0}{(\Delta y)^2} + \frac{(\Delta y)^3}{(\Delta y)^2} \right] - 0 = \frac{\Delta y}{\Delta y} = 1 \end{aligned}$$

**Along the axis  $\Delta x = \Delta y$ :**

Evaluating along  $(\Delta x, \Delta x) \rightarrow (0, 0)$ .

$$\begin{aligned} \lim_{(\Delta x, \Delta x) \rightarrow (0, 0)} \frac{\Delta w}{\Delta z} &= \frac{u(\Delta x, \Delta x)}{\Delta x + \Delta x} - \frac{u(0, 0) + v(0, 0)}{\Delta x + \Delta x} \\ &= \frac{1}{2\Delta x} \left[ \frac{(\Delta x)^3 - 3(\Delta x)^3}{2(\Delta x)^2} + \frac{(\Delta x)^3 - 3(\Delta x)^3}{2(\Delta x)^2} \right] \\ &= \frac{1}{2\Delta x} \left[ -\frac{2(\Delta x)^3}{2(\Delta x)^2} - \frac{2(\Delta x)^3}{2(\Delta x)^2} \right] = \frac{1}{2\Delta x} [-\Delta x - \Delta x] = -\frac{2\Delta x}{2\Delta x} = -1 \end{aligned}$$

As we can see, the limits are not unique regardless of the path we take to approach  $(0, 0)$ , hence  $f'(0)$  does not exist. Therefore, an equation can satisfy the Cauchy-Riemann equations at  $0, 0$ , yet have a derivative that does not exist. The Cauchy-Riemann equations does not guarantee differentiability at  $z_0$ .

**Example 13.5.4** (Any branch of  $f(z) = z^{1/2}$  is differentiable everywhere in domain of definition) Let

$$f(z) = z^{1/2} = \sqrt{r}e^{i\theta} \quad r > 0, \alpha < \theta < \alpha + 2\pi$$

Hence

$$u(r, \theta) = \sqrt{r} \cos\left(\frac{\theta}{2}\right) \quad v(r, \theta) = \sqrt{r} \sin\left(\frac{\theta}{2}\right)$$

By Cauchy-Riemann:

$$ru_r = \frac{\sqrt{r}}{2} \cos\left(\frac{\theta}{2}\right) = v_\theta \quad u_\theta = -\frac{\sqrt{r}}{2} \sin\left(\frac{\theta}{2}\right) = -rv_r$$

Thus, the derivative exists wherever  $f(z)$  is defined. Also, by theorem 13.5.2:

$$\begin{aligned} f'(z) &= e^{i\theta}(u_r + iv_r) \Big|_{(r_0, \theta_0)} \\ &= e^{-i\theta} \left[ \frac{1}{2\sqrt{r}} \cos\left(\frac{\theta}{2}\right) + i \frac{1}{2\sqrt{r}} \sin\left(\frac{\theta}{2}\right) \right] = \frac{1}{2\sqrt{r}} e^{-i\theta} \left[ \cos\left(\frac{\theta}{2}\right) + i \sin\left(\frac{\theta}{2}\right) \right] \\ &= \frac{1}{2\sqrt{r}e^{i\theta/2}} = \frac{1}{2f(z)} = \frac{1}{2}z^{-1/2} \end{aligned}$$

### 13.5.1 Complex Form of the Cauchy-Riemann Equations

#### Theorem 13.5.3: Cauchy-Riemann Equation (Complex Form)

Let  $f(z) = u(x, y) + iv(x, y)$ . If the first order partial derivatives of  $u$  and  $v$  with respect to  $x$  and  $y$  exists and satisfy the Cauchy-Riemann equations. Then

$$\frac{\partial}{\partial \bar{z}} f(z) = 0$$

*Proof:* Recall:

$$x = \frac{z + \bar{z}}{2} \qquad y = \frac{z - \bar{z}}{2i}$$

Let  $F$  be a real valued function, that is  $x, y \in \mathbb{R}$ . Then

$$\frac{\partial F}{\partial \bar{z}} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial \bar{z}} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial \bar{z}}$$

Substituting  $\frac{\partial x}{\partial \bar{z}} = 1/2$  and  $\frac{\partial y}{\partial \bar{z}} = i/2$ :

$$\frac{\partial F}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial F}{\partial x} + i \frac{\partial F}{\partial y} \right)$$

Define the operator:

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

Then

$$\begin{aligned} \frac{\partial f}{\partial \bar{z}} &= \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) = \frac{1}{2} \left( \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} + i \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} \right) \\ &= \frac{1}{2} [(u_x - v_y) + i(u_y + v_x)] \end{aligned}$$

We can see that if  $\frac{\partial f}{\partial \bar{z}}$  satisfies the Cauchy-Riemann equations (theorem 13.5.1):

$$\frac{\partial}{\partial \bar{z}} f(z) = 0 \qquad \frac{\partial}{\partial x} f = -i \frac{\partial f}{\partial y} \implies i \frac{\partial}{\partial x} f = \frac{\partial f}{\partial y}$$

□

## 13.5.2 Conditions for Differentiability

### Theorem 13.5.4:

Let  $f(z) = u(x, y) + iv(x, y)$  be defined in some neighbourhood  $\epsilon$  of point  $z_0 = x_0 + iy_0$ . Consider the first order partial derivatives of  $u$  and  $v$  with respect to  $x$  and  $y$ . If they

- (1) Exist for all  $z$ ,  $|z - z_0| < \epsilon$ .
- (2) Are continuous at  $z_0$ .
- (3) Satisfies the Cauchy-Riemann equations at  $z_0$ .

Then  $f'(z_0)$  exists:

$$f'(z_0) = (u_x + iv_x) \Big|_{(x_0, y_0)}$$

*Proof:* Assume the first order partial derivatives of  $u$  and  $v$  with respect to  $x$  and  $y$  exists  $\forall z[|z - z_0| < \epsilon]$ , are continuous at  $z_0$ , and satisfies the Cauchy-Riemann equations. Let  $\Delta z = \Delta x + i\Delta y$ ,  $0 < |\Delta z| < \epsilon$ , and  $\Delta w = f(z_0 + \Delta z) - f(z_0)$ . We then have

$$\Delta w = \Delta u + i\Delta v$$

Where

$$\begin{aligned}\Delta u &= u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) \\ \Delta v &= v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)\end{aligned}$$

Since first order partials of  $u$  and  $v$  are continuous at  $z_0$ :

$$\begin{aligned}\Delta u &= u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y \\ \Delta v &= v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \epsilon_3\Delta x + \epsilon_4\Delta y \\ (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) &\rightarrow (0, 0, 0, 0) \text{ as } (\Delta x, \Delta y) \rightarrow (0, 0)\end{aligned}$$

Substituting  $\Delta u$  and  $\Delta v$  into  $\Delta w$ :

$$\begin{aligned}\Delta w &= u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y \\ &\quad + i[v_x(x_0, y_0)\Delta x + v_y(x_0, y_0)\Delta y + \epsilon_3\Delta x + \epsilon_4\Delta y]\end{aligned}$$

Using the Cauchy-Riemann equations and dividing by  $\Delta z$ :

$$\frac{\Delta w}{\Delta z} = u_x(x_0, y_0) + iv_x(x_0, y_0) + (\epsilon_1 + i\epsilon_3)\frac{\Delta x}{\Delta z} + (\epsilon_2 + i\epsilon_4)\frac{\Delta y}{\Delta z}$$

From the inequalities  $|\Delta x| \leq |\Delta z|$  and  $|\Delta y| \leq |\Delta z|$ :

$$\left| \frac{\Delta x}{\Delta z} \right| \leq 1 \qquad \left| \frac{\Delta y}{\Delta z} \right| \leq 1$$

So

$$\begin{aligned}\left| (\epsilon_1 + i\epsilon_3)\frac{\Delta x}{\Delta z} \right| &\leq |\epsilon_1 + i\epsilon_3| \leq |\epsilon_1| + |\epsilon_3| \\ \left| (\epsilon_2 + i\epsilon_4)\frac{\Delta y}{\Delta z} \right| &\leq |\epsilon_2 + i\epsilon_4| \leq |\epsilon_2| + |\epsilon_4|\end{aligned}$$

Then  $|\epsilon_2| + |\epsilon_4| \rightarrow 0$  and  $|\epsilon_1| + |\epsilon_3| \rightarrow 0$  as  $\Delta z = \Delta x + i\Delta y \rightarrow 0$ .

$$\implies \frac{\Delta w}{\Delta z} = u_x(x_0, y_0) + iv_x(x_0, y_0) \implies f'(z_0) \text{ exists}$$

□

**Example 13.5.5** (All 3 conditions must be satisfied for  $f'(z_0)$  to exist) *Do not use expression of  $f'(z)$  before existence of  $f'(z_0)$  is established. Consider  $f(z) = x^3 + i(1 - y)^3$ .*

$$u(x, y) = x^3 \qquad v(x, y) = (1 - y)^3$$

Taking the partial derivatives:

$$\begin{array}{ll} u_x = 3x^2 & v_x = 0 \\ u_y = 0 & v_y = -3(1-y)^2 \end{array}$$

It would be foolish to ignore Cauchy-Riemann and directly use:

$$f'(z) = u_x + iv_x = 3x^2$$

We can see that the Cauchy-Riemann equations are satisfied only if:

$$3x^2 = -3(1-y)^2 \implies x^2 + (1-y)^2 = 0 \implies (x=0) \wedge (y=1)$$

Therefore,  $f'(z)$  exists only if  $z = i$ , and that  $f'(i) = 0$

## 13.6 Analytic Functions

### Definition 13.6.1: Analytic/Regular/Holomorphic

Let  $S$  be an open set,  $S \subset \mathbb{C}$ . Let  $f$  be a function.

$$f \text{ is analytic in } S \iff \forall z \in S [f'(z) \text{ exists}]$$

We say  $f(z)$  is analytic at a point  $z_0$  if it is analytic in some neighbourhood of  $z_0$ . If we say that  $f(z)$  is analytic in a closed set  $S'$  then we mean that it is analytic in an open set  $S$  where  $S' \subset S$ .

### Definition 13.6.2: Entire

A function  $f(z)$  is entire if it is analytic at all points in the plane.

#### Example 13.6.1

Derivative of polynomial exists everywhere  $\implies$  All polynomials are entire functions

See section 13.5.2 for conditions for a function to be differentiable, hence analytic in a set  $S$ .

### Corollary 13.6.0.1:

Let  $f(z)$  and  $g(z)$  be analytic in a domain  $D$ . Then the following are analytic in  $D$ :

$$\begin{array}{ll} f(z) + g(z) \\ f(z)g(z) \\ \frac{f(z)}{g(z)} & g(z) \neq 0 \forall z \in D \end{array}$$

Likewise, if  $P(z)$  and  $Q(z)$  are polynomials, then  $P(z)/Q(z)$  is analytic if  $\forall z \in D [Q(z) \neq 0]$ .

**Corollary 13.6.0.2:**

Let  $w$  be the image of  $D$  under  $f(z)$  and  $w$  be the domain of  $g$ . Then  $g(f(z))$  is analytic in  $D$  and

$$\frac{d}{dz}g[f(z)] = g'[f(z)]f'(z)$$

**Theorem 13.6.1:**

Let  $D$  be the domain of a function  $f(z)$ .

$$\forall z \in D [f'(z) = 0] \implies f(z) \text{ is constant in } D$$

*Proof:* Let  $f(z) = u(x, y) + iv(x, y)$  with domain  $D$ , and  $P$ ,  $P'$ , and  $Q$  be points in  $D$ . Let  $\vec{U}$  be the unit vector on the line segment  $L$  connecting  $P$  and  $P'$ , and  $s$  be the distance along  $L$ .

$$f'(z) = 0 \implies \forall z \in D [u_x = u_y = v_x = v_y = 0]$$



We know that the directional derivative:

$$\frac{du}{ds} = \nabla u \cdot \vec{U} \qquad \nabla u = u_x \hat{i} + u_y \hat{j}$$

Previously,  $u_x = u_y = 0$ , so for all points on  $L$ :

$$u_x = u_y = 0 \implies \nabla u = 0 \implies \frac{du}{ds} = 0 \implies u \text{ constant on } L$$

Now, that we have established that  $u$  is constant on any given line  $L$  in  $D$ , we can see that since  $D$  is simply connected and there are finitely many lines connecting  $P$  and  $Q$ , the values of  $u$  at  $P$  and  $Q$  must be equal and constant. Hence,  $\exists a \in \mathbb{R}$  such that  $u(x, y) = a$  in  $D$ . Likewise,  $v(x, y) = b$  in  $D$ . Thus

$$f(z) = a + bi = c \qquad c \text{ is constant}$$

□

**Definition 13.6.3: Singular Point**

Let  $\epsilon$  be a neighbourhood of point  $z_0$ , and  $f(z)$  be a function.  $z_0$  is a singular point if  $f'(z_0)$  does not exist, but  $f(z)$  is differentiable in all neighbourhoods of  $z_0$ .

### 13.6.1 Examples

**Example 13.6.2** (Determining analyticity using Cauchy-Riemann equations) Consider  $f(z) = \sin(x) \cosh(y) + i \cos(x) \sinh(y)$ .

$$u(x, y) = \sin(x) \cosh(y)$$

$$v(x, y) = \cos(x) \sinh(y)$$

Cauchy-Riemann:

$$u_x = \cos(x) \cosh(y) = v_y$$

$$u_y = \sin(x) \sinh(y) = -v_x$$

Therefore, it is clear that  $f(z)$  is entire.

$$f'(z) = u_x + iv_x = \cos(x) \cosh(y) - i \sin(x) \sinh(y)$$

Another application of Cauchy-Riemann see that  $f'(z)$  is also entire.

**Example 13.6.3** ( $f(z)$  and  $\overline{f(z)}$  is analytic in  $D \implies f(z)$  is constant in  $D$ ) Let

$$f(z) = u(x, y) + iv(x, y) \quad \overline{f(z)} = u(x, y) - iv(x, y) = U(x, y) + iV(x, y)$$

Because of  $f(z)$  and  $\overline{f(z)}$  is analytic in  $D$ , the Cauchy-Riemann equations hold:

$$u_x = v_y$$

$$u_y = -v_x$$

$$U_x = V_y$$

$$U_y = -V_x$$

We can see that:

$$u_x = -v_y = v_y$$

$$u_y = v_x = -v_x$$

Hence,  $u_x = 0$  and  $v_x = 0$ , then we can conclude

$$f'(z) = 0 \implies f(z) \text{ is constant in } D$$

**Example 13.6.4** ( $f(z)$  is analytic in  $D$  and  $|f(z)|$  is constant in  $D \implies f(z)$  is constant in  $D$ ) Let  $\forall z \in D[|f(z)| = c]$ , where  $c$  is a constant. It is easy to see that  $c = 0 \implies \forall z \in D[f(z) = 0]$ , so consider  $c \neq 0$ . Then

$$f(z)\overline{f(z)} = c^2 \neq 0 \implies \forall z \in D[f(z) \neq 0]$$

Thus

$$\overline{f(z)} = \frac{c^2}{f(z)} \quad \forall z \in D$$

Hence  $\overline{f(z)}$  is analytic everywhere in  $D$ , so  $f(z)$  is constant in  $D$ .



## 13.7 Harmonic Functions

Harmonic functions are functions where the curvature in each component direction cancels each other out.

### Definition 13.7.1: Laplace's Equation

Let  $F(x, y)$  be a real-valued function. That is  $x, y \in \mathbb{R}$ . Laplace's equation:

$$\frac{\partial^2}{\partial x^2} F + \frac{\partial^2}{\partial y^2} F = 0$$

In polar form:

$$\begin{aligned} r^2 u_{rr}(r, \theta) + r u_r(r, \theta) + u_{\theta\theta}(r, \theta) &= 0 \\ r^2 v_{rr}(r, \theta) + r v_r(r, \theta) + v_{\theta\theta}(r, \theta) &= 0 \end{aligned}$$

See example 13.7.1

### Definition 13.7.2: Harmonic

A real-valued function  $F(x, y)$  is harmonic in the  $xy$ -plane if it satisfies Laplace's equation.

### Theorem 13.7.1:

Let  $D$  be the domain of a function  $f(z) = u(x, y) + iv(x, y)$ .

$$f(z) \text{ is analytic in } D \implies u(x, y) \wedge v(x, y) \text{ are harmonic in } D$$

*Proof:*  $f$  is analytic in  $D$ , so its component functions must satisfy the Cauchy-Riemann equations:

$$\begin{aligned} (u_x = v_y) \wedge (u_y = -v_x) &\implies (u_{xy} = v_{yy}) \wedge (u_{yx} = -v_{xx}) \\ (u_x = v_y) \wedge (u_y = -v_x) &\implies (u_{xx} = v_{yx}) \wedge (u_{yy} = -v_{xy}) \end{aligned}$$

Now, we know from calculus that  $u_{xy} = u_{yx}$  and  $v_{yx} = v_{xy}$ , so we conclude

$$u_{xx} + u_{yy} = 0 \qquad v_{xx} + v_{yy} = 0$$

□

Note: The converse (  $\Leftarrow$  ) is true for simply connected domains, hence, theorem 13.7.1 becomes  $\iff$  in simply connected domains. (R, Boas - Invitation to Complex Analysis. (1987) Section 19.)

### Corollary 13.7.1.1:

Let  $F(x, y)$  is a real-valued function in a simply connected domain  $D$ . Then there exists a function  $f(z)$  and  $g(z)$  in  $D$  such that  $f(z) = F(x, y) + iv(x, y)$  and  $g(z) = u(x, y) + iF(x, y)$ . That is, there exists a function where the real part equals  $F$  and a function where the imaginary part equals  $F$ .

**Definition 13.7.3: Harmonic Conjugate**

If  $f(z) = u(x, y) + iv(x, y)$  is analytic in a domain  $D$ , then  $v(x, y)$  is the harmonic conjugate of  $u(x, y)$ . This is not to be confused with the complex conjugate.

**Example 13.7.1** Let  $f(z) = u(r, \theta) + iv(r, \theta)$  be analytic in domain  $D' = D \setminus \{0\}$ . Show  $u(r, \theta)$  and  $v(r, \theta)$  satisfies the polar form of Laplace's equation.

*Proof:* We know from the Polar form of the Cauchy-Riemann equation:

$$ru_r = v_\theta \qquad u_\theta = -rv_r$$

Operating by  $r \frac{\partial}{\partial r}$  and  $\frac{\partial}{\partial \theta}$ , we obtain:

$$\begin{aligned} r \frac{\partial}{\partial r} ru_r &= ru_r + r^2 u_{rr} = rv_{\theta r} \\ r \frac{\partial}{\partial r} u_\theta &= ru_{\theta r} = r \frac{\partial}{\partial r} (-rv_r) = -rv_r - r^2 v_{rr} \\ \frac{\partial}{\partial \theta} ru_r &= ru_{\theta r} = v_{\theta \theta} \\ \frac{\partial}{\partial \theta} u_\theta &= u_{\theta \theta} = -rv_{r\theta} \end{aligned}$$

We can see that

$$\begin{cases} ru_r + r^2 u_{rr} = -u_{\theta \theta} \\ rv_r + r^2 v_{rr} = -v_{\theta \theta} \end{cases} \implies \begin{cases} r^2 u_{rr} + ru_r + u_{\theta \theta} = 0 \\ r^2 v_{rr} + rv_r + v_{\theta \theta} = 0 \end{cases}$$

□

**Example 13.7.2** Let  $f(z) = u(x, y) + iv(x, y)$  be analytic in domain  $D$ . Consider the families of level curves  $u(x, y) = c_1$  and  $v(x, y) = c_2$ , with  $c_1, c_2 \in \mathbb{R}$  being constants. Show for  $z_0 = (x_0, y_0) \in \mathbb{C}$  common to  $u(x, y) = c_1$  and  $v(x, y) = c_2$  and  $f'(z_0) \neq 0$ , then the lines tangent to  $u(x, y) = c_1$  and  $v(x, y) = c_2$  at  $z_0$  are orthogonal.

*Note:*

$$[u(x, y) = c_1] \wedge [v(x, y) = c_2] \implies \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx} = 0 \right) \wedge \left( \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \frac{dy}{dx} = 0 \right)$$

*Proof:* The tangent lines of  $u(x, y)$  and  $v(x, y)$  are

$$\nabla u = \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) = (u_x, u_y) \qquad \nabla v = \left( \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right) = (v_x, v_y)$$

Taking the dot product, and applying the Cauchy-Riemann equations:

$$u_x v_x + u_y v_y = u_x (-u_y) + u_y (u_x) = 0$$

Hence,  $u(x, y)$  and  $v(x, y)$  are orthogonal.

Note:

$$\begin{aligned} f'(z_0) = 0 &\implies u_x + iv_x = 0 \implies v_y - iu_y = 0 \\ &\implies u_x = u_y = v_x = v_y = 0 \end{aligned}$$

Hence, we can see that  $f'(z_0) = 0$  is required for  $u(x, y)$  and  $v(x, y)$  to exist and be orthogonal.  $\square$

## 13.8 Uniquely Determined Analytic Functions

### Lemma 13.8.0.1:

*Suppose a function  $f$  is analytic throughout domain  $D$ , and  $f(z) = 0 \forall z \in D' \subset D$  or line segment contained in  $D$ . Then  $f(z) \equiv 0$  throughout  $D$ .*

*Proof:* Let  $f$  be a function analytic in domain  $D$  and  $f(z) = 0$  for all point or line segment in  $D$ . Let  $z_0$  in the subdomain of  $D$  or on a line segment in  $D$ .

$D$  is connected open set, so there is a polygonal line  $L$  jointing any point  $P$  in  $D$  to  $z_0$  lying entirely in  $D$ . (Recall: A polygonal line consists of a finite number of lines connected end-to-end.) Let  $d$  be the shortest distance from points on  $L$  to the boundary on  $D$ , so  $d > 0$ , unless  $D$  is the entire plane. Then there is a sequence of points along  $L$ :

$$\{z_0, z_1, z_2, \dots, z_{n-1}, z_n = P\} \quad |z_k - z_{k+1}| < d \quad k \in \mathbb{N}$$

That is, each point is sufficiently close to each other. We construct neighbourhoods of each point with radius  $d$ , all of which are in  $D$ , so points  $z_{k-1}$  and  $z_{k+1}$  lie in the neighbourhood of  $z_k$ ,  $k \in \mathbb{N}$ :

$$\{N_0, N_1, N_2, \dots, N_{n-1}, N_n\}$$



Now as  $f$  is analytic in  $N_0$  and  $f(z) = 0$  in a domain or line segment containing  $z_0$ , then  $f(z) \equiv 0$  in  $N_0$ .  $z_1$  is in  $N_0$ , so  $f(z_1) \equiv 0$  in  $N_1$ . Continuing this we can see that  $f(z_n) \equiv 0$  in  $N_n$ , hence,  $f(z) \equiv 0$  in  $D$ .  $\square$

**Theorem 13.8.1:**

Let  $f$  be analytic in domain  $D$ . Then it's uniquely determined over  $D$  by its values in  $D$  or along a line segment in  $D$ .

*Proof:* Let functions  $f$  and  $g$  be analytic in some domain  $D$ , and  $f(z) = g(z) \forall z \in D$ . Then  $h(z) = f(z) - g(z)$  is also analytic in  $D$ , and  $h(z) = 0$  in the subdomain or along the line segment, so  $h(z) \equiv 0$  throughout  $D$ .  $\square$

**Theorem 13.8.2: Coincidence Principle**

If functions  $f$  and  $g$  are analytic in  $D$  and  $f(z) = g(z)$  in  $D' \subset D$  with limit point  $z_0 \in D$ , then  $f(z) = g(z)$  everywhere in  $D$ .

This is a more generalized version of theorem 13.8.1

**Definition 13.8.1: Analytic continuation**

Consider the domains  $D_1$  and  $D_2$  with intersection  $D_1 \cap D_2$ , and functions  $f_1$  and  $f_2$ . If  $f_1$  is analytic in  $D_1$ , and there exists  $f_2$  that is analytic in  $D_2$  such that  $f_1(z) = f_2(z)$  for all  $z \in D_1 \cap D_2$ . Then  $f_2$  is the analytic continuation of  $f_1$ .



Theorem 13.8.1 tells us that if such analytic continuation exists, then it is unique. Now if there exists  $f_3$  in  $D_3$  that is an analytic continuation of  $f_2$ , then it is not necessarily true that  $f_3(z) = f_1(z)$  for all  $z \in D_1 \cap D_3$ . (See example 13.8.1.)

**Definition 13.8.2: Elements of a function**

Let  $f_2$  be the analytic continuation of a function  $f_1$  in  $D_1$  into domain  $D_2$ , and let  $F(z)$  be analytic in  $D_1 \cup D_2$ .

$$F(z) = \begin{cases} f_1(z) & z \in D_1 \\ f_2(z) & z \in D_2 \end{cases}$$

Then  $F$  is the analytic continuation of  $f_1$  and  $f_2$  into  $D_1 \cup D_2$ , and  $f_1$  and  $f_2$  are elements of  $F$ .

### 13.8.1 Reflection Principle

Generally,  $\overline{f(z)} \neq f(\bar{z})$  for all  $z$ , but....

#### Theorem 13.8.3: Reflection Principle

Let  $f$  be a function with domain  $D$  containing a segment of the real axis  $R \subset D$ . Then

$$\forall z \in D [\overline{f(z)} = f(\bar{z})] \iff \forall x \in R [f(x) \in \mathbb{R}]$$

See example 13.8.2 for the case when  $f(x)$  is purely imaginary.

*Proof:* Let  $f(z)$  and  $F(z)$  be analytic functions:

$$f(z) = u(x, y) + iv(x, y) \qquad F(z) = U(x, y) + iV(x, y)$$

( $\Leftarrow$ ):

Suppose  $\forall x \in R [f(x) \in \mathbb{R}]$ , and that  $F(z) = \overline{f(\bar{z})}$ .

$$f(z) = u(x, y) + iv(x, y) \qquad F(z) = U(x, y) + iV(x, y)$$

Then

$$\overline{f(\bar{z})} = u(x, -y) - iv(x, -y)$$

Therefore

$$U(x, y) = u(x, t) \qquad V(x, y) = -v(x, t) \qquad t = -y$$

$f(x, t)$  is analytic, so it satisfies the Cauchy-Riemann equations:

$$u_x = v_t \qquad u_t = -v_x$$

Hence

$$\begin{aligned} U_x &= u_x & V_y &= -v_t \frac{dt}{dy} = v_t \\ U_y &= u_t \frac{dt}{dy} = -u_t & V_x &= -v_x \end{aligned}$$

Thus, we can see that  $F(z)$  also satisfies the Cauchy Riemann equations

$$U_x = V_y \qquad U_y = -V_x$$

Since, the partial derivatives of  $U$  and  $V$  are continuous in  $D$ , we can say that  $F(z)$  is analytic in  $D$ . On the segment of the real axis  $R \subset D$ ,  $f(z)$  is real, so  $v(x, 0) = 0$ .

$$F(x) = U(x, 0) + iV(x, 0) = u(x, 0) - iv(x, 0) = u(x, 0)$$

$$\implies \forall z \in R [F(z) = f(z)]$$

$$\implies \forall z \in D [\overline{f(\bar{z})} = f(z)]$$

Theorem 13.8.1

( $\implies$ ):

Suppose  $\overline{f(z)} = f(\bar{z})$ . Then

$$u(x, -y) - iv(x, -y) = u(x, y) + iv(x, y)$$

Consider any point  $(x, 0) \in R \subset D$ :

$$u(x, 0) - iv(x, 0) = u(x, 0) + iv(x, 0) \implies v(x, 0) = 0$$

Hence,  $f(x)$  is real  $\forall x \in R \subset D$ . □

Theorem 13.8.3 tells us that if a complex function is real for all points on the real axis, then it will obey the Reflection Principle, and vice versa.

## 13.8.2 Examples

**Example 13.8.1** Consider

$$\begin{aligned} f_1(z) &= \sqrt{r}e^{i\theta/2} & r > 0, 0 < \theta < \pi \\ f_2(z) &= \sqrt{r}e^{i\theta/2} & r > 0, \frac{\pi}{2} < \theta < 2\pi \\ f_3(z) &= \sqrt{r}e^{i\theta/2} & r > 0, \pi < \theta < \frac{5\pi}{2} \end{aligned}$$

It is clear that  $f_1$ ,  $f_2$ ,  $f_3$  are continuous and satisfies the Cauchy-Riemann equations throughout their domain of definition, since they have a derivative everywhere in their domain of definition. Hence, they are analytic continuations of each other. Let  $D_1$ ,  $D_2$ , and  $D_3$  be the domains of  $f_1$ ,  $f_2$ , and  $f_3$ , respectively. Consider  $f_1$  and  $f_3$  in the domain  $D_1 \cap D_3$ , and any  $z$  in the first quadrant of the complex plane. Then  $z = re^{i\theta} = re^{i(\theta+2\pi)}$  and we have

$$\begin{aligned} f_1(z) &= \sqrt{r}e^{i(\theta/2)} & 0 < \theta < \pi \\ f_3(z) &= \sqrt{r}e^{i(\theta/2+\pi)} & 0 < \theta < \pi \end{aligned}$$

Hence,

$$\begin{aligned} f_1(z) &= \sqrt{r}[\cos(\theta/2) + i\sin(\theta/2)] & 0 < \theta < \pi \\ f_3(z) &= \sqrt{r}[\cos(\theta/2 + \pi) + i\sin(\theta/2 + \pi)] & 0 < \theta < \pi \\ &= -\sqrt{r}[\cos(\theta/2) + i\sin(\theta/2)] \end{aligned}$$

Thus we can see that  $f_1 = -f_3$  in  $D_1 \cap D_3$ .

**Example 13.8.2** Consider theorem 13.8.3, but  $f(x)$  is purely imaginary  $\forall x \in \mathbb{R}$ . We know that  $\Leftarrow$  holds, and that  $F(z) = \overline{f(\bar{z})}$  satisfies the Cauchy-Riemann equations. We have

$$F(x) = U(x, 0) + iV(x, 0) = u(x, 0) - iv(x, 0) = -iv(x, 0) = -f(x)$$

Hence

$$\overline{f(\bar{z})} = -f(z) \implies \overline{f(z)} = -f(\bar{z})$$

# Chapter 14

## Elementary Functions

### 14.1 Exponential Function

#### Definition 14.1.1: Exponential Function

Consider  $z \in \mathbb{C}$ , the exponential function is defined:

$$f(z) = e^z = e^{x+iy} = e^x [\cos(y) + i \sin(y)]$$

Where  $y$  is taken in radians.

Note: This is not the same as the polar form of a complex number (definition 12.3.3).

It is clear that the set of  $n$ -th roots of  $e$ :

$$\{e^{1/n} : n \in \mathbb{N}\}$$

and

$$|e^z| = e^x \qquad \arg(e^z) = y + 2n\pi \qquad n \in \mathbb{N} \cup \{0\}$$

The exponential function follows from the usual properties of exponentials. We also know that

$$\frac{d}{dz} e^z = e^z \qquad \forall z \in \mathbb{C}$$

so,  $e^z$  is entire. We should also note that  $e^z$  is periodic due to  $e^{iy}$ .

### 14.2 Logarithmic Function

#### Definition 14.2.1: Logarithmic Function

Consider any  $z \in \mathbb{C}$  in exponential form:

$$\log(z) = \ln(r) + i(\theta + 2n\pi) = \ln(|z|) + i \arg(z) \qquad n \in \mathbb{Z}$$

Note: This is a multi-valued function.

**Definition 14.2.2: Principal Value of the Logarithmic Function**

Let  $z \in \mathbb{C}$ , the principal value of the logarithmic function is denoted by  $\text{Log}(z)$ .

$$\text{Log}(z) = \ln(r) + i\theta$$

It is clear that

$$\log(z) = \text{Log}(z) + 2n\pi \quad n \in \mathbb{Z}$$

and for any  $z$  on the real axis, the logarithmic function reduces to

$$\text{Log}(z) = \ln(x) \quad x \in \mathbb{R}$$

**14.2.1 Branches and Derivatives of Logarithms**

$\log(z)$  is a multi-valued function. Let  $\alpha \in \mathbb{R}$ :

$$\log(z) = \ln(r) + i\theta = u(r, \theta) + iv(r, \theta) \quad r > 0, \alpha < \theta < \alpha + 2\pi$$

Note: If  $\log(z)$  is defined on  $\theta = \alpha$ , then it is not continuous there, as there is a discontinuity between points near  $\alpha$  and  $\alpha + 2\pi$ .

The first order partials of  $u$  and  $v$  are continuous in the domain, and satisfies the Cauchy-Riemann equations:

$$ru_r = v_\theta \quad u_\theta = -rv_\theta$$

So its derivative exists everywhere in the domain.

$$\begin{aligned} \frac{d}{dz} \log(z) &= e^{-i\theta} (u_r + iv_r) = e^{i\theta} \left( \frac{1}{r} + i0 \right) = \frac{1}{re^{i\theta}} \\ \implies \frac{d}{dz} \log(z) &= \frac{1}{z} & |z| > 0, \alpha < \arg(z) < \alpha + 2\pi \\ \implies \frac{d}{dz} \text{Log}(z) &= \frac{1}{z} & |z| > 0, -\pi < \text{Arg}(z) < \pi \end{aligned}$$

**Definition 14.2.3: Branch**

A branch is a single-valued function  $F$  of a multi-valued function  $f$ .  $F$  is analytic throughout some domain of  $f$  and assumes the one of the values of  $f$ .

**Definition 14.2.4: Principal Branch**

$$\text{Log}(z) = \ln(r) + i\theta \quad r > 0, -\pi < \theta < \pi$$

**Definition 14.2.5: Branch Cut**

A portion of a line or curved introduced to define a branch  $F$  of a multi-valued function  $f$ . Points on the branch cut of  $F$  are singular points of  $F$ .



**Definition 14.2.6: Branch Point**

*A singular point common to all branch cuts of a multi-valued function  $f$ .*

**Example 14.2.1** The branch cut for  $\text{Log}(z) = \ln(r) + i\theta$ ,  $r > 0$ ,  $-\pi < \theta < \pi$ , is the origin and  $\theta = \pi$ .

Branch points for all branches of  $\log(z)$  is the origin.

Different branches may result in different values.

**Example 14.2.2** Consider  $\log(i^2)$  in the branch:

$$\log(z) = \ln(r) + i\theta \quad r > 0, \quad \frac{\pi}{4} < \theta < \frac{9\pi}{4}$$

Then

$$\begin{aligned} \log(i^2) &= \log(-1) = \ln(1) + i\pi = i\pi \\ 2\log(i) &= 2\left(\ln(1) + i\frac{\pi}{2}\right) = \pi i \end{aligned}$$

Therefore

$$\log(i^2) = 2\log(i) \quad r > 0, \quad \frac{\pi}{4} < \theta < \frac{9\pi}{4}$$

Now consider the branch:

$$\log(z) = \ln(r) + i\theta \quad r > 0, \quad \frac{3\pi}{4} < \theta < \frac{11\pi}{4}$$

Then

$$\begin{aligned} \log(i^2) &= \log(-1) = \ln(1) + i\pi = i\pi \\ 2\log(i) &= 2\left(\ln(1) + i\frac{5\pi}{2}\right) = 5\pi i \end{aligned}$$

Therefore,

$$\log(i^2) \neq 2\log(i) \quad r > 0, \quad \frac{3\pi}{4} < \theta < \frac{11\pi}{4}$$

**14.2.2 Identities of Logarithms**

Let  $z_1, z_2 \in \mathbb{C}$ , then

$$\log(z_1 z_2) = \log(z_1) + \log(z_2)$$

can be interpreted as

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

therefore

$$\ln |z_1 z_2| + i \arg(z_1 z_2) = (\ln |z_1| + i \arg(z_1)) + (\ln |z_2| + i \arg(z_2))$$

Rest of the identities are the same as for elements in  $\mathbb{R}$ , but beware of branches and arguments.

**Example 14.2.3** Show  $\forall z_1, z_2 \in \mathbb{C}$

$$\text{Log}(z_1 z_2) = \text{Log}(z_1) + \text{Log}(z_2) + 2N\pi i \quad N \in \{0, \pm 1\}$$

Consider:

$$\begin{aligned} \log(z_1 z_2) &= \ln |z_1 z_2| + i \arg(z_1 z_2) \\ &= \ln(r_1) + \ln(r_2) + i \arg(z_1) + i \arg(z_2) \\ &= \ln(r_1) + \ln(r_2) + i\theta_1 + i\theta_2 + 2n\pi i & n \in \mathbb{Z} \\ &= \ln(r_1) + \ln(r_2) + i \text{Arg}(z_1) + i \text{Arg}(z_2) + 2n\pi i & n \in \mathbb{Z} \end{aligned}$$

Then, since  $-\pi < \text{Arg}(z_1) < \pi$  and  $-\pi < \text{Arg}(z_2) < \pi$ :

$$\begin{aligned} \text{Log}(z_1 z_2) &= \ln(r_1) + \ln(r_2) + i \text{Arg}(z_1) + i \text{Arg}(z_2) + 2N\pi i & N \in \{0, \pm 1\} \\ &= \text{Log}(z_1) + \text{Log}(z_2) + 2N\pi i & N \in \{0, \pm 1\} \end{aligned}$$

### 14.2.3 Power Function

#### Definition 14.2.7: Power Function

Let  $z, c \in \mathbb{C}$ . The Power Function:

$$z^c = e^{c \log(z)} \quad z \neq 0$$

Likewise

$$c^z = e^{z \log(c)} \quad c \neq 0$$

The logarithm is multi-valued  $\implies$  the power function is multi-valued.

The principle branch of the Power Function is log being replaced by Log:

$$\begin{aligned} z^c &= e^{c \text{Log}(z)} & z \neq 0 \\ c^z &= e^{z \text{Log}(c)} & c \neq 0 \end{aligned}$$

When a branch is specified,  $\log(z)$  becomes single-valued and analytic. Hence the derivatives:

$$\frac{d}{dz} z^c = \frac{d}{dz} e^{c \log(z)} = \frac{c}{z} e^{c \log(z)} = c z^{c-1} \quad |z| > 0, \alpha < \arg(z) < \alpha + 2\pi, \alpha \in \mathbb{R}$$

When value of  $\log(c)$  is specified,  $c^z$  is entire function of  $z$  and

$$\frac{d}{dz} c^z = \frac{d}{dz} e^{z \log(c)} = e^{z \log(c)} \log(c) = c^z \log(c)$$

## 14.3 Trigonometric Functions

Recall: Definition 12.5.2. Likewise for any  $z \in \mathbb{C}$ :

### Definition 14.3.1: Complex Sine and Cosine Functions

For any  $z \in \mathbb{C}$ :

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

Sine and cosine are entire functions as  $e^{iz}$  and  $e^{-iz}$  are entire.

Taking the derivatives:

$$\frac{d}{dz} e^{iz} = i e^{iz} \implies \left( \frac{d}{dz} \sin(z) = \cos(z) \right) \wedge \left( \frac{d}{dz} \cos(z) = -\sin(z) \right)$$

It's also easy to see that:

$$\sin(-z) = -\sin(z) \qquad \cos(-z) = \cos(z) \qquad e^{iz} = \cos(z) + i \sin(z)$$

The usual trigonometric identities apply, such as:

$$\begin{aligned} \sin(z_1 + z_2) &= \sin(z_1) \cos(z_2) + \cos(z_1) \sin(z_2) \\ \cos(z_1 + z_2) &= \cos(z_1) \cos(z_2) - \sin(z_1) \sin(z_2) \end{aligned}$$

Now suppose  $y \in \mathbb{R}$ , and take the hyperbolic functions:

$$\sinh(y) = \frac{e^y - e^{-y}}{2} \qquad \cosh(y) = \frac{e^y + e^{-y}}{2}$$

Then we get:

$$\sin(iy) = i \sinh(y) \qquad \cos(iy) = \cosh(y)$$

If we let  $z = x + iy$ , we can define:

$$\begin{aligned} \sin(z) &= \sin(x) \cosh(y) + i \cos(x) \sinh(y) \\ \cos(z) &= \cos(x) \cosh(y) - i \sin(x) \sinh(y) \end{aligned}$$

and that

$$\begin{aligned} |\sin(z)|^2 &= \sin^2(x) + \sinh^2(y) \\ |\cos(z)|^2 &= \cos^2(x) + \sinh^2(y) \end{aligned}$$

Note: Unlike in  $\mathbb{R}$  where sine and cosine are bounded by 1 and  $-1$ , it is clear that sine and cosine are not bounded in the complex plane, since  $\sinh$  is unbounded for all values of  $y$ .

### 14.3.1 Zeros and Singularities

#### Definition 14.3.2: Zero (function)

Let  $f(z)$  be a function. A zero of  $f$  is a point  $z_0$  such that

$$f(z_0) = 0$$

#### Theorem 14.3.1:

The zeros of  $\cos(z)$  and  $\sin(z)$  for  $z \in \mathbb{C}$  is the same as the zeros of  $\cos(x)$  and  $\sin(x)$  for  $x \in \mathbb{R}$ , that is

$$\begin{aligned} \forall x \in \mathbb{R} \forall z \in \mathbb{C} \forall n \in \mathbb{Z} \left[ (\cos(x) = 0) \wedge (\cos(z) = 0) \iff z = x = \frac{\pi}{2} + n\pi \right] \\ \forall x \in \mathbb{R} \forall z \in \mathbb{C} \forall n \in \mathbb{Z} \left[ (\sin(x) = 0) \wedge (\sin(z) = 0) \iff z = x = n\pi \right] \end{aligned}$$

*Proof:* Let  $z = x + iy$  and consider  $\sin(z) = 0$ :

$$\begin{aligned} \sin(z) = 0 &\implies \sin^2(x) + \sinh^2(y) = 0 & |\sin(z)|^2 = \sin^2(x) + \sinh^2(y) \\ &\implies [\sin(x) = 0] \wedge [\sinh(y) = 0] \\ &\implies [x = n\pi] \wedge [y = 0] & n \in \mathbb{Z} \\ &\implies z = x = n\pi & n \in \mathbb{Z} \end{aligned}$$

As for cosine, we know that:

$$\cos(z) = \sin\left(z + \frac{\pi}{2}\right)$$

Thus

$$\cos(z) = 0 \implies z = x = n\pi + \frac{\pi}{2} \quad n \in \mathbb{Z}$$

□

**Example 14.3.1** Show  $\forall z \in \mathbb{C}$ :

*The Reflection Principle:*

$$\forall z \in D \subset \mathbb{C} [\overline{f(z)} = f(\bar{z})] \iff \forall x \in \mathbb{R} [f(x) \in \mathbb{R}]$$

$$(a) \quad \overline{\cos(z)} = \cos(\bar{z})$$

*Proof:* It is clear that  $\forall x \in \mathbb{R}, \sin(x) \in \mathbb{R}$ . The result follows from the Reflection Principle. Also

$$\overline{\sin(z)} = \overline{\frac{z - \bar{z}}{2i}} = \frac{\bar{z} - z}{2i} = \sin(\bar{z})$$

□

$$(b) \overline{\sin(z)} = \sin(\bar{z})$$

*Proof:* It is clear that  $\forall x \in \mathbb{R}, \cos(x) \in \mathbb{R}$ . The result follows from the Reflection Principle. Also

$$\overline{\cos(z)} = \frac{\overline{z + \bar{z}}}{2} = \frac{\bar{z} + z}{2} = \cos(\bar{z})$$

□

**Example 14.3.2** Show:

$$(a) \forall z \in \mathbb{C} [\overline{\cos(iz)} = \cos(i\bar{z})]$$

*Proof:*

$$\begin{aligned} \overline{\cos(iz)} &= \frac{\overline{iz + i\bar{z}}}{2} = \frac{\overline{iz} + i\bar{z}}{2} = \frac{-i\bar{z} + iz}{2} = i \frac{z - \bar{z}}{2} = i \operatorname{Im}\{z\} \\ \cos(i\bar{z}) &= \frac{i\bar{z} + i\overline{\bar{z}}}{2} = \frac{i\bar{z} - iz}{2} = i \frac{\bar{z} - z}{2} = i \operatorname{Im}\{z\} \end{aligned}$$

Hence  $\forall z \in \mathbb{C}$

$$\overline{\cos(iz)} = \cos(i\bar{z})$$

□

$$(b) \forall z \in \mathbb{C} \forall n \in \mathbb{Z} [\overline{\sin(iz)} = \sin(i\bar{z}) \iff z = n\pi i]$$

*Proof:*

$$\begin{aligned} \overline{\sin(iz)} &= \frac{\overline{iz - i\bar{z}}}{2i} = \frac{-i\bar{z} - iz}{2i} = -\frac{z + \bar{z}}{2} = -\operatorname{Re}\{z\} \\ \sin(i\bar{z}) &= \frac{i\bar{z} - i\overline{\bar{z}}}{2i} = \frac{i\bar{z} + iz}{2i} = \frac{z + \bar{z}}{2} = \operatorname{Re}\{z\} \end{aligned}$$

We know that

$$\operatorname{Re}\{z\} = -\operatorname{Re}\{z\} \implies \operatorname{Re}\{z\} = 0 \implies \overline{\sin(iz)} = \sin(i\bar{z}) = 0 \iff z = n\pi i$$

□

## 14.4 Hyperbolic Functions

### Definition 14.4.1: Hyperbolic Sine and Cosine Functions

Let  $z \in \mathbb{C}$ :

$$\sinh(z) = \frac{e^z - e^{-z}}{2} \qquad \cosh(z) = \frac{e^z + e^{-z}}{2}$$

It is clear that the derivatives:

$$\frac{d}{dz} \sinh(z) = \cosh(z) \qquad \frac{d}{dz} \cosh(z) = \sinh(z)$$

The relationships with sine and cosine:

$$\begin{aligned} -i \sinh(iz) &= \sin(z) & \cosh(iz) &= \cos(z) \\ -i \sin(iz) &= \sinh(z) & \cos(iz) &= \cosh(z) \end{aligned}$$

Hence in the complex plane,  $\sinh$  and  $\cosh$  are periodic with period  $2\pi i$ .

Identities:

$$\sinh(-z) = -\sinh(z) \qquad \cosh(-z) = \cosh(z) \qquad \cosh^2(z) - \sinh^2(z) = 1$$

$$\begin{aligned} \sinh(z_1 + z_2) &= \sinh(z_1) \cosh(z_2) + \cosh(z_1) \sinh(z_2) \\ \cosh(z_1 + z_2) &= \cosh(z_1) \cosh(z_2) + \sinh(z_1) \sinh(z_2) \end{aligned}$$

$$\begin{aligned} \sinh(z) &= \sinh(x) \cos(y) + i \cosh(x) \sin(y) \\ \cosh(z) &= \cosh(x) \cos(y) + i \sinh(x) \sin(y) \end{aligned}$$

$$\begin{aligned} |\sinh(z)|^2 &= \sinh^2(x) + \sin^2(y) \\ |\cosh(z)|^2 &= \sinh^2(x) + \cos^2(y) \end{aligned}$$

#### **Theorem 14.4.1:**

*The zeros of hyperbolic sine and cosine:*

$$\begin{aligned} \sinh(z) = 0 &\iff z = n\pi i & n \in \mathbb{Z} \\ \cosh(z) = 0 &\iff z = \left(\frac{\pi}{2} + n\pi\right)i & n \in \mathbb{Z} \end{aligned}$$

**Example 14.4.1** *Show:*

$$(a) \quad \sinh(z + \pi i) = -\sinh(z)$$

*Proof:*

$$\sinh(z + \pi i) = \frac{e^{z+\pi i} - e^{-z-\pi i}}{2} = \frac{-e^z + e^{-z}}{2} = -\frac{e^z - e^{-z}}{2} = -\sinh(z)$$

□

$$(b) \quad \cosh(z + \pi i) = -\cosh(z)$$

*Proof:*

$$\cosh(z + \pi i) = \frac{e^{z+\pi i} + e^{-z-\pi i}}{2} = -\frac{e^z + e^{-z}}{2} = -\cosh(z)$$

□

(c)  $\tanh(z + \pi i) = \tanh(z)$

*Proof:*

$$\tanh(z + \pi i) = \frac{\sinh(z + \pi i)}{\cosh(z + \pi i)} = \frac{-\sinh(z)}{-\cosh(z)} = \tanh(z)$$

□

**Example 14.4.2** Show  $\forall z \in \mathbb{C}$ :

$$\overline{\sinh(z)} = \sinh(\bar{z}) \quad \overline{\cosh(z)} = \cosh(\bar{z}) \quad \forall z \neq 0 \left[ \overline{\tanh(z)} = \tanh(\bar{z}) \right]$$

*Proof:* We can see that  $\forall x \in \mathbb{R}$ ,  $\sinh(x) \in \mathbb{R}$  and  $\cosh(x) \in \mathbb{R}$ , so we can conclude from the Reflection Principle (theorem 13.8.3) that  $\forall z \in \mathbb{C}$ :

$$\overline{\sinh(z)} = \sinh(\bar{z}) \quad \overline{\cosh(z)} = \cosh(\bar{z})$$

Thus it follows that

$$\forall z \neq 0 \left[ \overline{\tanh(z)} = \tanh(\bar{z}) \right]$$

□

## 14.5 Inverse Trigonometric and Hyperbolic Functions

### Definition 14.5.1: Inverse Trigonometric Functions

Let  $z \in \mathbb{C}$ :

$$\begin{aligned} \sin^{-1}(z) &= -i \log[iz + (1 - z^2)^{1/2}] \\ \cos^{-1}(z) &= -i \log[z + i(i - z^2)^{1/2}] \\ \tan^{-1}(z) &= \frac{i}{2} \log\left(\frac{i+z}{i-z}\right) \end{aligned}$$

$\cos^{-1}(z)$  and  $\tan^{-1}(z)$  are multi-valued. All inverse trigonometric functions become single-valued and analytic when in specific branches of the square root and logarithmic functions.

*Proof:*  $\sin^{-1}(z) = -i \log[iz + (1 - z^2)^{1/2}]$

Let  $w = \sin^{-1}(z)$  whenever  $z = \sin(w)$

$$z = \sin(w) \implies z = \frac{e^{iw} - e^{-iw}}{2i} \implies (e^{iw})^2 - 2ize^{iw} - 1 = 0$$

Using the quadratic formula to solve for  $e^{iw}$ :

$$\begin{aligned} e^{iw} = iz + (1 - z^2)^{1/2} &\implies iw = \log(iz + (1 - z^2)^{1/2}) \\ &\implies \sin^{-1}(z) = -i \log[iz + (1 - z^2)^{1/2}] \end{aligned}$$

$\cos^{-1}(z) = -i \log[z + i(i - z^2)^{1/2}]$

Likewise, let  $w = \cos^{-1}(z)$  whenever  $z = \cos(w)$

$$z = \cos(w) \implies z = \frac{e^{iw} + e^{-iw}}{2} \implies (e^{iw})^2 - 2ze^{iw} + 1 = 0$$

Using the quadratic formula to solve for  $e^{iw}$ :

$$\begin{aligned} e^{iw} &= \frac{2z \pm \sqrt{4z^2 - 4}}{2} = z \pm \sqrt{z^2 - 1} = z \pm i(1 - z^2)^{1/2} \\ &\implies iw = \log[z \pm i(1 - z^2)^{1/2}] \\ &\implies w = -i \log[z \pm i(1 - z^2)^{1/2}] \end{aligned}$$

$\tan^{-1}(z) = \frac{i}{2} \log\left(\frac{i+z}{i-z}\right)$

Again, let  $w = \tan^{-1}(z)$  whenever  $z = \tan(w)$

$$\begin{aligned} z = \tan(w) &= \frac{e^{iw} - e^{-iw}}{i(e^{iw} + e^{-iw})} \\ \implies iz &= \frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} \implies iz e^{iw} + iz e^{-iw} = e^{iw} - e^{-iw} \\ \implies (iz - 1)e^{iw} + (iz + 1)e^{-iw} &= 0 \implies (iz - 1)e^{2iw} + (iz + 1) = 0 \\ \implies e^{iw} &= \left(\frac{-iz - 1}{iz - 1}\right)^{\frac{1}{2}} \implies iw = \frac{1}{2} \log\left(\frac{-iz - 1}{iz - 1}\right) \implies w = -\frac{i}{2} \log\left(\frac{-(iz + 1)}{iz - 1}\right) \\ \implies w &= \tan^{-1}(z) = \frac{i}{2} \log\left(\frac{i + z}{i - z}\right) \end{aligned}$$

□

Derivatives:

$\frac{d}{dz} \sin^{-1}(z) = \frac{1}{(1 - z^2)^{1/2}}$	Depends on value chosen for square root
$\frac{d}{dz} \cos^{-1}(z) = -\frac{1}{(1 - z^2)^{1/2}}$	Depends on value chosen for square root
$\frac{d}{dz} \tan^{-1}(z) = \frac{1}{1 + z^2}$	Independent on value chosen for square root

Using the same procedures on the hyperbolic functions, we obtain the inverse hyperbolic functions:



### Definition 14.5.2: Inverse Hyperbolic Functions

Let  $z \in \mathbb{C}$ :

$$\sinh^{-1}(z) = \log [z + (z^2 + 1)^{1/2}]$$

$$\cosh^{-1}(z) = \log [z + (z^2 - 1)^{1/2}]$$

$$\tanh^{-1}(z) = \frac{1}{2} \log \left( \frac{1+z}{1-z} \right)$$

## 14.6 Phasors

### Definition 14.6.1: Phasor

Consider the function

$$f(t) = \operatorname{Re}\{F e^{st}\}$$

$$F = F_0 e^{i\theta}$$

$$s = \sigma + i\omega, \quad \sigma, \omega \in \mathbb{R}$$

$F$  is the phasor associated with  $f(t)$ .

We can see that  $f(t)$  grows exponentially according to the value of  $\sigma$ , has a phase of  $\theta$ , and a phase frequency of  $\omega$ . (It is clear why engineers love this.)

Properties:

1.

$$f(t) = \operatorname{Re}\{F e^{st}\} \implies \begin{cases} F \text{ is unique} & \omega \neq 0 \\ \text{Only } \operatorname{Re}\{F\} \text{ is unique} & \omega = 0 \end{cases}$$

2.

$$\forall t \in \mathbb{R} [f(t) = g(t)] \implies \begin{cases} F = G & \omega \neq 0 \\ \operatorname{Re}\{F\} = \operatorname{Re}\{G\} & \omega = 0 \end{cases}$$

3. For any given  $s = \sigma + i\omega$ , there is only one function of  $t$  corresponding to a phasor.

4. Let  $f(t)$  and  $g(t)$  have the same complex frequency, that is,  $\omega_1 = \omega_2$ , then the phasor for  $f(t) + g(t)$  is  $F + G$ .

5.  $\forall M \in \mathbb{R}$ . The phasor for  $Mf(t)$  is  $MF$ .

6. “For a given complex frequency, the function of  $t$  corresponding to the sum of two or more phasors is the sum of the time functions for each.” -Wunsch

7. Let  $n \in \mathbb{N}$ .

$$(f(t) \text{ has phasor } F) \wedge (df/dt \text{ has phasor } sF) \implies d^n f/dt^n \text{ has phasor } s^n F$$

8.

$$f(t) \text{ has phasor } F \implies \int^t f(t')dt' \text{ has phasor } \frac{F}{s} \quad s \neq 0$$

These properties follow from the properties of  $e$ .

**Example 14.6.1** Consider:

$$R\mathbf{i}(t) + L\frac{d\mathbf{i}}{dt} = V_0 \cos(\omega t)$$

Suppose the complex frequency is  $s = i\omega$ . If  $I$  is the phasor for  $\mathbf{i}$ , then substituting it into the differential equation:

$$RI + i\omega LI = (R + i\omega L)I = V_0 \quad \text{Phasors on both side must equal}$$

Then solving for the phasor:

$$I = \frac{V_0}{R + i\omega L} = \frac{V_0 e^{i\theta}}{\sqrt{R^2 + \omega^2 L^2}} \quad \theta = -\tan^{-1}\left(\frac{\omega L}{R}\right)$$

Using  $s = i\omega$  to obtain  $\mathbf{i}(t)$ , we have

$$\mathbf{i}(t) = \text{Re}\left\{\frac{V_0 e^{i\theta}}{\sqrt{R^2 + (\omega L)^2}} e^{i\omega t}\right\}$$

# Chapter 15

## Integrals

### 15.1 Derivatives of Functions

#### Definition 15.1.1: Derivative of Complex-Valued Function

Consider a complex-valued function  $w(t) = u(t) + iv(t)$ , with  $u$  and  $v$  being real-valued functions. If  $w(t)$  is differentiable at  $t$ , then its derivative with respect to  $t$ :

$$w'(t) = \frac{d}{dt}w(t) = u'(t) + iv'(t)$$

The rules for calculus in  $\mathbb{R}$  still applies.

Note: The mean value theorem for derivatives no longer apply for complex-valued functions.

**Example 15.1.1** Let  $w(t) = e^{it}$  be continuous on  $[0, 2\pi]$ , so  $u(t)$  and  $v(t)$  are also continuous on  $[0, 2\pi]$ . For the mean value theorem to hold, there must exist  $a, b, c \in \mathbb{C}$ , where  $a < c < b$ , such that

$$w'(c) = \frac{w(b) - w(a)}{b - a}$$

We can see that

$$|w'(t)| = |ie^{it}| = 1 \qquad \frac{w(b) - w(a)}{b - a} = \frac{w(2\pi) - w(0)}{2\pi - 0} = \frac{e^{i2\pi} - e^{i0}}{2\pi} = \frac{1 - 1}{2\pi} = 0$$

So we can see that there does not exist a  $c \in \mathbb{C}$  such that the mean value theorem holds.

### 15.2 Definite Integrals of Functions

#### Definition 15.2.1: Definite Integral of Complex-Valued Function

Consider a complex-valued function  $w(t) = u(t) + iv(t)$ , with  $u$  and  $v$  being real-valued

functions. If  $u$  and  $v$  are piecewise continuous on interval  $[a, b]$ , then the definite integral of  $w(t)$  over interval  $[a, b]$ :

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt$$

The rules for integrals in  $\mathbb{R}$  and the Fundamental Theorem of Calculus still applies.

Likewise with derivatives of complex-valued functions, the mean value theorem does not hold for complex-valued integrals.

**Example 15.2.1** Let  $w(t) = e^{it}$  be a complex-valued function of  $t$ . For  $w(t)$  to hold on  $[a, b]$ , this must hold for some  $a < c < b$ :

$$\int_a^b w(t)dt = w(c)(b - a)$$

Consider  $w(t)$  on  $[0, 2\pi]$ . Then

$$\int_a^b w(t)dt = \int_0^{2\pi} e^{it}dt = \left. \frac{e^{it}}{i} \right|_0^{2\pi} = 0 \quad |w(c)(b - a)| = |e^{ic}|2\pi = 2\pi$$

We can see there does not exist  $c$ ,  $0 < c < 2\pi$ , such that both sides of the equations are equal.

## 15.3 Contours

### Definition 15.3.1: Arc

An arc is a set of points dependent on a parameter  $t \in \mathbb{R}$ .

$$\{z = (x(t), y(t)) : t \in [a, b]\}$$

Where  $x(t)$  and  $y(t)$  are continuous functions.

It is convenient in  $\mathbb{C}$  to use:

$$z = z(t) = x(t) + iy(t) \quad a \leq t \leq b$$

### Definition 15.3.2: Simple/Jordan Arc

An arc is simple if it does not cross itself. That is:  $t_1 \neq t_2 \implies z(t_1) \neq z(t_2)$

### Definition 15.3.3: Simple Closed Curve / Jordan Curve

A simple curve, but endpoints gets mapped to equal values. That is,  $z(a) = z(b)$  for  $a \leq t \leq b$ . It is positively oriented if it is counterclockwise.

The interval for which the arc is parameterized is not unique. Consider

$$t = \phi(\tau) \quad \alpha \leq \tau \leq \beta$$

Then  $\phi(\alpha) = a$  and  $\phi(\beta) = b$ . We have

$$z(t) = Z(\tau) = z[\phi(\tau)] \quad \alpha \leq \tau \leq \beta$$

**Definition 15.3.4: Differentiable Arc**

Suppose  $d/dt z(t) = z'(t) = x'(t) + iy'(t)$  exists and is continuous. Then  $z'(t)$  is a differentiable arc.

We can integrate over the differential arc in the interval  $[a, b]$ :

$$L = \int_a^b |z'(t)| dt \qquad |z'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

Note:  $|z'(t)|$  is a real-valued function.

Again, due to the curve being invariant under the representation for the arc:

$$L = \int_a^b |z'(t)| dt = \int_\alpha^\beta |z'[\phi(t)]| \phi'(t) dt = \int_\alpha^\beta |Z'(\tau)| d\tau \qquad Z'(\tau) = z'[\phi(\tau)] \phi'(\tau)$$

If the differentiable arc  $z'(t) \neq 0$  in the interval  $[a, b]$ , then the unit tangent vector is defined in said interval:

$$\hat{\mathbf{T}} = \frac{z'(t)}{|z'(t)|}$$

Recall: The gradient of a function is perpendicular to the function, so  $\hat{\mathbf{T}}$  is normal to  $z(t)$ , over the interval  $[a, b]$ .

**Definition 15.3.5: Smooth**

An arc  $z(t)$  is smooth in the interval  $[a, b]$  if  $z'(t)$  is continuous  $\forall t \in [a, b]$  and non-zero  $\forall t \in (a, b)$ .

**Definition 15.3.6: Contour**

A piecewise smooth arc.

**Definition 15.3.7: Simple Closed Contour**

A contour where only  $z(a) = z(b)$  in the interval  $[a, b]$ .

**Theorem 15.3.1: Jordan Curve Theorem**

All points on a simple close curve or simple closed contour  $z(t)$  are boundary points of two distinct domains. One is bounded and interior to  $z(t)$  and the other is unbounded and exterior to  $z(t)$ . That is,  $z(t)$  as a boundary line of two domains.

*Proof:* Good luck! □

i.e. If  $z(t)$  is a circle, then one domain is within the circle and contains the points on the parameter, and one is exterior to the circle.

## 15.4 Contour Integrals

Evaluating an integral over a contour. It is common to assume that a line integral represents an area under a curve. Generally, this is a far too simplistic approach, even for real-valued functions. Consider the limit definition of the line integral.

### Definition 15.4.1: Line Integral (Real)

*Let  $f$  be a real-valued function.*

$$\int_a^b f(z) ds = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k, y_k) \Delta s_k \quad \Delta s_k = (x_k - x_{k-1}) + (y_k - y_{k-1})$$

Note: The integral exists only if the limit exists.

We can see that  $\Delta s_k$  acts like a vector, and the definition of the line integral assigns a value according to the weighting function  $f$  to each  $\Delta s_k$ . The contour integral is then the weighted sum of these vectors from  $a$  to  $b$  as  $n \rightarrow \infty$ .

The complex line integral is defined similarly.

### Definition 15.4.2: Line Integral (Complex)

*Let  $f$  be a complex-valued function.*

$$\int_a^b f(z) dz = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(z_k) \Delta z_k \quad \Delta z = (x_k - x_{k-1}) + i(y_k - y_{k-1})$$

Upon expanding, we can see that:

$$\begin{aligned} \int_a^b f(z) dz &= \lim_{n \rightarrow \infty} \sum_{k=1}^n f(z_k) \Delta z_k \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n [u(x_k, y_k) + iv(x_k, y_k)] (\Delta x_k + i \Delta y_k) \\ &= \int_a^b (u + iv)(dx + i dy) = \int_a^b u dx - v dy + i \int_a^b v dx + u dy \end{aligned}$$

These are the integrals taken in each direction when evaluating from  $a$  to  $b$  for  $a, b \in \mathbb{C}$ .

Note: This reduces to a regular integral  $\int_a^b u dx$  in the reals when  $v = 0$  and  $dy = 0$ .

A contour integral is a line integral over a contour. Here we define it parameterized by  $t$ .

### Definition 15.4.3: Contour Integral

*Let  $z(t)$  in  $C = [a, b]$  be a contour, and  $f[z(t)]$  be a piecewise continuous function on  $C$ . Then the contour integral:*

$$\int_C f(z) dz = \int_a^b f[z(t)] z'(t) dt$$

*Note:  $C$  is contour  $\implies z'(t)$  is piecewise continuous on  $C \implies$  Existence of integral on  $C$*

Notation: Let  $f(z)$  be a function evaluated over the contour  $C$ .

$$\int_C f(z)dz \qquad \int_{z_1}^{z_2} f(z)dz$$

$\int_{z_1}^{z_2}$  is often used when the integral is independent of the path between end points.  $\int_{-C}$  represents the same contour, but in reverse.

Following from the contours, the integral is invariant under change in representation of the contour.

#### Definition 15.4.4: Sum (Contour)

Let  $C_1$  be contour from  $z_1$  to  $z_2$ , and  $C_2$  be from  $z_2$  to  $z_3$ , then the sum is contour  $C$  from  $z_1$  to  $z_3$ .



Some properties (which follows from integrals):

$$\int_C f(z)dz = \int_{z_1}^{z_2} f(z)dz + \int_{z_2}^{z_3} f(z)dz \qquad \int_C f(z) + g(z)dz = \int_C f(z)dz + \int_C g(z)dz$$

$$\begin{aligned} \int_{-C} f(z)dz &= \int_{-b}^{-a} f([z(-t)]) \frac{d}{dt} z(-t)dt = - \int_{-b}^{-a} f[z(-t)]z'(-t)dt \\ &= - \int_{-b}^{-a} f[z(\tau)]z'(\tau)d\tau \\ &= - \int_C f(z)dz \end{aligned}$$

$$\begin{aligned} \int_a^b f[z(t)]z'(t)dt &= \int_a^c f[z(t)]z'(t)dt + \int_c^b f[z(t)]z'(t)dt \\ \int_C f(z)dz &= \int_{C_1} f(z)dz + \int_{C_2} f(z)dz \end{aligned}$$

### 15.4.1 Upper Bounds for the Moduli

It is not analysis without inequality involving modulus.

#### Lemma 15.4.0.1:

Let  $w(t)$  be a piecewise smooth function defined on  $[a, b]$ . Then

$$\left| \int_a^b w(t)dt \right| \leq \int_a^b |w(t)|dt$$

*Proof:* Assume  $\int_a^b w(t)dt$  is non-zero, otherwise the inequality is trivial.

$$\begin{aligned}\int_a^b w(t)dt = r_0 e^{i\theta_0} &\implies r_0 = e^{-i\theta_0} \int_a^b w(t)dt \\ &\implies r_0 = \operatorname{Re}\left\{e^{-i\theta_0} \int_a^b w(t)dt\right\} \quad r_0 \in \mathbb{R} \\ &\implies r_0 = \int_a^b \operatorname{Re}\{e^{i\theta_0} w(t)\}dt\end{aligned}$$

Now

$$\begin{aligned}\operatorname{Re}\{e^{-i\theta_0} w(t)\} &\leq |e^{-i\theta_0} w(t)| = |e^{-i\theta_0}| |w(t)| = |w(t)| \\ &\implies r_0 \leq \int_a^b |w(t)|dt \\ &\implies \left|\int_a^b w(t)dt\right| \leq \int_a^b |w(t)|dt \quad r_0 = \left|\int_a^b w(t)dt\right|\end{aligned}$$

□

### Theorem 15.4.1: ML Inequality

Let  $f(z)$  be piecewise continuous function on contour  $C$  with length  $L$ .

$$\forall z \in C \exists M \in \mathbb{R} [|f(z)| \leq M] \implies \left|\int_C f(z)dz\right| \leq ML$$

That is, if  $f(z)$  is bounded on the contour, then the value of it's integral is bounded.

*Proof:* Let  $z = z(t)$  in  $[a, b]$  be a parametric representation of  $C$ . Then

$$\begin{aligned}\left|\int_C f(z)dz\right| &= \left|\int_a^b f[z(t)]z'(t)dt\right| \leq \int_a^b |f[z(t)]z'(t)|dt = \int_a^b |f[z(t)]||z'(t)|dt \\ &\leq M \int_a^b |z'(t)|dt = M|z(t)| = ML\end{aligned}$$

□

Note: According to the Extreme Value Theorem, any continuous real-valued function on a closed interval is bounded, so such  $M \in \mathbb{R}$  will always exist.

**Observation.** Let  $l$  be the length along the contour  $C$ . Graphically, what this is telling us:





This is useful in evaluating the size of the integral, and if we are lucky:

**Example 15.4.1** Let  $C_R$  be the semicircle:

$$z = Re^{i\theta} \qquad 0 \leq \theta \leq \pi, \quad R > 3$$

Consider

$$\lim_{r \rightarrow \infty} \int_{C_R} \frac{z+1}{(z^2+4)(z^2+9)} dz$$

We know that:

$$\begin{aligned} |z+1| &\leq |z|+1 = R+1 \\ |z^2+4| &\geq ||z|^2-4| = R^2-4 \\ |z^2+9| &\geq ||z|^2-9| = R^2-9 \end{aligned}$$

Then

$$|f(z)| = \left| \frac{z+1}{(z^2+4)(z^2+9)} \right| = \frac{|z+1|}{|z^2+4||z^2+9|} \leq \frac{R+1}{(R^2-4)(R^2-9)} = M_R$$

Now we have

$$M_R = \frac{R+1}{(R^2-4)(R^2-9)} \qquad L = \pi R$$

Since

$$\left| \int_{C_R} \frac{z+1}{(z^2+4)(z^2+9)} dz \right| \leq M_R L$$

Hence

$$\lim_{R \rightarrow \infty} M_R L = \lim_{R \rightarrow \infty} \frac{R^2+R}{(R^2-4)(R^2-9)} \pi = \lim_{R \rightarrow \infty} \frac{\frac{1}{R^2} + \frac{1}{R^3}}{\left(1 - \frac{4}{R^2}\right) \left(1 - \frac{9}{R^2}\right)} \pi = 0$$

Thus

$$\lim_{r \rightarrow \infty} \int_{C_R} \frac{z+1}{(z^2+4)(z^2+9)} dz = 0$$

## 15.5 Antiderivatives

### Definition 15.5.1: Antiderivative

Let  $f(z)$  be a continuous function on domain  $D$ , the antiderivative is a function  $F(z)$  such that

$$F'(z) = f(z) \qquad \forall z \in D$$

Note: By definition,  $F(z)$  is an analytic function, and an antiderivative is unique up to an additive constant. An indefinite integral is the family of functions that are the antiderivative of a particular function.

**Theorem 15.5.1:**

Let  $f(z)$  be a continuous function on domain  $D$ . TFAE:

(a) There is a function  $F(z)$  such that

$$\forall z \in D [F'(z) = f(z)]$$

( $f(z)$  has an antiderivative throughout  $D$ .)

(b) All contours of  $f(z)$  in  $D$  from any point  $z_1$  to  $z_2$  all have the same value. That is

$$\int_{z_1}^{z_2} f(z) dz = F(z) \Big|_{z_1}^{z_2} = F(z_2) - F(z_1)$$

(c) For all closed contours  $C$  lying in  $D$ :

$$\oint_C f(z) dz = 0$$

Simply:

$$\forall z \in D [F'(z) = f(z)] \iff \int_{z_1}^{z_2} f(z) dz = F(z) \Big|_{z_1}^{z_2} \iff \oint_C f(z) dz = 0$$

*Proof:* (a)  $\implies$  (b):

Suppose  $F'(z)$  exists for  $f(z)$  for all  $z \in D$ . We know:

$$\frac{d}{dt} F[z(t)] = F'(z(t)) z'(t) = f[z(t)] z'(t) \quad t \in [a, b]$$

Then

$$\int_C f(z) dz = \int_a^b f[z(t)] z'(t) dt = F[z(t)] \Big|_a^b = F[z(b)] - F[z(a)] = F(z_2) - F(z_1)$$

If  $C$  consists of a finite number of smooth arc  $C_k$ ,  $k \in \{1, 2, 3, \dots, n\}$ :

$$\int_C f(z) dz = \sum_{k=1}^n \int_{C_k} f(z) dz = \sum_{k=1}^n \int_{z_k}^{z_{k+1}} f(z) dz = \sum_{k=1}^n [F(z_{k+1}) - F(z_k)]$$

Then

$$\int_C f(z) dz = F(z_{n+1}) - F(z_1)$$

Thus

$$\forall z \in D[F'(z) = f(z)] \implies \int_{z_1}^{z_2} f(z)dz = F(z) \Big|_{z_1}^{z_2}$$

(b)  $\implies$  (c):

Let  $C_1$  and  $C_2$  be contours with endpoints  $z_1$  and  $z_2$ . Suppose that integration is path independent, then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz \implies \int_{C_1} f(z)dz + \int_{-C_2} f(z)dz = 0 \implies \int_{C=C_1-C_2} f(z)dz = 0$$

Thus

$$\int_{z_1}^{z_2} f(z)dz = F(z) \Big|_{z_1}^{z_2} \implies \oint_C f(z)dz = 0$$

(c)  $\implies$  (a):

Suppose integration around a closed contour  $C$  in  $D$  is zero. Since integration is path independent in  $D$ , we can define the function:

$$F(z) = \int_{z_0}^z f(z)ds$$

Let  $z + \Delta z$  be any point in the neighbourhood of  $z$  contained in  $D$ , then

$$F(z + \Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(z)ds - \int_{z_0}^z f(z)ds = \int_z^{z+\Delta z} f(s)ds$$

Since

$$\int_z^{z+\Delta z} ds = \Delta z \implies f(z) = \frac{1}{\Delta z} \int_z^{z+\Delta z} f(z)ds$$

Then

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z} \int_z^{z+\Delta z} f(s) - f(z)ds$$

As  $f$  is continuous at the point  $z$ :

$$\forall \epsilon \exists \delta [ |s - z| < \delta \implies |f(s) - f(z)| < \epsilon ]$$

If  $|\Delta z| < \delta$ :

$$\left| \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) \right| < \frac{1}{|\Delta z|} \epsilon |\Delta z| = \epsilon$$

Then

$$\lim_{\Delta z \rightarrow 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z) \implies F'(z) = f(z)$$

Thus

$$\oint_C f(z)dz = 0 \implies \forall z \in D[F'(z) = f(z)]$$

□

**Example 15.5.1** Let  $f(z) = z^{-2}$ . We can see that  $f$  is continuous everywhere except at the origin, and has antiderivative  $F(z) = -z^{-1}$  in  $|z| > 0$ . Thus around the unit circle:

$$\int_C z^{-2} dz = 0 \qquad z = e^{i\theta}, \theta \in [-\pi, \pi]$$

**Example 15.5.2** Consider  $f(z) = z^{-1}$ . It has an antiderivative  $F(z) = \log(z)$ , which is not differentiable or defined along its branch cut. To evaluate the integral along the unit circle, we can break it up into two domains to avoid this issue. First consider  $C_1$ :

$$z = e^{i\theta} \qquad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

Then

$$\begin{aligned} \int_{C_1} z^{-1} dz &= \int_{-i}^i z^{-1} dz = \text{Log}(z) \Big|_{-i}^i = \text{Log}(i) - \text{Log}(-i) \\ &= \left( \ln(1) + i\frac{\pi}{2} \right) - \left( \ln(1) - i\frac{\pi}{2} \right) = \pi i \end{aligned}$$

Now consider  $C_2$ :

$$z = e^{i\theta} \qquad \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$$

Then

$$\begin{aligned} \int_{C_2} z^{-1} dz &= \int_i^{-i} z^{-1} dz = \log(z) \Big|_i^{-i} = \log(-i) - \log(i) \\ &= \left( \ln(1) + i\frac{3\pi}{2} \right) - \left( \ln(1) + i\frac{\pi}{2} \right) = \pi i \end{aligned}$$

Thus around the circle  $C = C_1 + C_2$ :

$$\int_C z^{-1} dz = \int_{C_1} z^{-1} dz + \int_{C_2} z^{-1} dz = \pi i + \pi i = 2\pi i$$

## 15.6 Cauchy-Goursat Theorem

Previously: A function  $f$  that has an antiderivative in any domain  $D$ , then the integral of  $f$  around any closed contour in  $D$  is zero. (Theorem 15.5.1) Now, it's for simple closed contours.

Recall:

$$\begin{aligned} \int f(z) dz &= \int (u + iv)(dx + idy) = \int u dx - v dy + i \int v dx + u dy \\ &= \iint -v_x - u_x dx dy + i \iint u_x - v_y dx dy && \text{Theorem 19.3.3} \\ &= 0 && \text{Theorem 13.5.1} \end{aligned}$$

This result requires  $f'(z)$  be continuous, due to the requirement of Green's Theorem (theorem 19.3.3). The Cauchy-Goursat theorem eliminates this requirement.

**Theorem 15.6.1: Cauchy-Goursat Theorem**

Let  $C$  be a simple closed contour. If a function  $f$  is analytic for all set of points  $z$  on and in  $C$ , then

$$\int_C f(z)dz = 0$$

*Proof:* First, a lemma:

**Lemma 15.6.1.1:**

Let  $C$  be a closed contour,  $R$  denote the region enclosed and on the contour, and  $f$  be a function analytic in  $R$ . The region  $R$  can be covered by a finite number of squares or partial squares, indexed  $j = 1, 2, \dots, n$ , such that for some  $\epsilon > 0$ :

$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \epsilon$$

Holds for all points  $z$  other than a fixed point  $z_j$  in that square or partial square. We let a square denote a region with boundary points included with points interior to it. If a square has points not in  $R$ , then we remove those points and it becomes a partial square.

*Proof:* Suppose there does not exist a  $z_j$  where

$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \epsilon$$

holds after subdividing a square a finite number of times for contradiction. Let  $\sigma_0$  denote the original square or the entire square of the partial square,  $\sigma_1$  denote the squares after subdividing  $\sigma_0$  into four equal smaller squares, and so on. After subdividing  $\sigma_0$ , one of the  $\sigma_1$  must contain points of  $R$  but still no such  $z_j$  exists, so we continue to subdivide such  $\sigma_1$  since the inequality does not hold. We will then obtain an infinite sequence

$$\sigma_0, \sigma_1, \sigma_2, \dots, \sigma_{k-1}, \sigma_k, \dots$$

There is a point  $z_0$  that is common to each of these squares and each of these squares contain points of  $R$  other than  $z_0$ . As the size of the squares are decreasing, there exists a neighbourhood  $\delta > |z - z_0|$  containing the squares with diagonals less than  $\delta$ , so each neighbourhood  $\delta$  contains points of  $R$  distinct from  $z_0$ . Thus  $z_0$  is an accumulation point of  $R$  (definition 12.7.16), and since  $R$  is a closed set,  $z_0 \in R$ .

Now, since  $f$  is analytic in  $R$  and  $z_0$ ,  $f'(z_0)$  exists and according to definition 13.4.1:

$$\forall \epsilon \exists \delta > 0 \left[ |z - z_0| < \delta \implies \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon \right]$$

However, such neighbourhood  $|z - z_0| < \delta$  contains  $\sigma_K$  for some sufficiently large  $K$ , so  $z_0$  serves as  $z_j$  for a the subregion of  $\sigma_K$  or part of  $\sigma_K$ , thus there is no need to subdivide  $\sigma_K$ . We have reached a contradiction.  $\square$

Upper bound for modulus of an integral:

Given some  $\epsilon$  we cover region  $R$  with squares such that

$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \epsilon$$

We define a neighbourhood  $\delta_j(z)$  enclosing the  $j$ -th square or partial square by

$$\delta_j(z) = \begin{cases} \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) & z \neq z_j \\ 0 & z = z_j \end{cases}$$

Then

$$\forall z \in \sigma \subset R [|\delta_j(z)| < \epsilon]$$

As  $f(z)$  is continuous throughout subregion  $\sigma$ ,  $\delta_j(z)$  is continuous in  $\sigma$  and

$$\lim_{z \rightarrow z_j} \delta_j(z) = f'(z_j) - f'(z_j) = 0$$

Now, let  $C_j$  denote the positively oriented contours on the boundaries of the squares and partial squares covering  $R$ . Then on any  $C_j$  be definition of  $\delta_j(z)$ :

$$\begin{aligned} f(z) &= f(z_j) - z_j f'(z_j) + f'(z_j)z + (z - z_j)\delta_j(z) \\ \implies \int_{C_j} f(z)dz &= [f(z_j) - z_j f'(z_j)] \int_{C_j} dz + f'(z_j) \int_{C_j} z dz + \int_{C_j} (z - z_j)\delta_j(z)dz \end{aligned}$$

However, according to theorem 15.5.1:

$$\int_{C_j} dz = 0 \qquad \int_{C_j} z dz = 0$$

So

$$\begin{aligned} \int_{C_j} f(z)dz &= \int_{C_j} (z - z_j)\delta_j(z)dz & j = 1, 2, 3, \dots, n \\ \implies \sum_{j=1}^n \int_{C_j} f(z)dz &= \sum_{j=1}^n \int_{C_j} (z - z_j)\delta_j(z)dz \end{aligned}$$

Now as boundaries of adjacent subregions cancel each other out, since they are taken along opposite senses to each other, only those on  $C$  remain, so

$$\int_C f(z)dz = \sum_{j=1}^n \int_{C_j} (z - z_j)\delta_j(z)dz \implies \left| \int_C f(z)dz \right| \leq \sum_{j=1}^n \left| \int_{C_j} (z - z_j)\delta_j(z)dz \right|$$

Endgame:

Let  $s_j$  denote the length of the sides of the square or partial square  $\sigma_j$ , since  $C_j$  is on the boundary or part of the boundary of the square.

$$|z - z_j| \leq \sqrt{2}s_j \quad \sqrt{2}s_j \text{ is diagonal of square}$$

Then

$$|\delta_j(z)| < \epsilon \implies |(z - z_j)\delta_j(z)| = |z - z_j||\delta_j(z)| < \sqrt{2}s_j\epsilon$$

Let  $A_j$  be the area of the square. If  $C_j$  is the boundary of a square, then the length of  $C_j$  is  $4s_j$  and we have

$$\left| \int_{C_j} (z - z_j)\delta_j(z)dz \right| < \sqrt{2}s_j\epsilon 4s_j = 4\sqrt{2}A_j\epsilon$$

Now, if  $C_j$  is the boundary of a partial square, then the length of  $C_j$  is less than  $4s_j + L_j$ , where  $L_j$  is the length of  $C_j$  that is a part of  $C$ . Let  $S$  be the length of the sides of some square that entirely encloses  $C$ , so sum of  $A_j$  is less than  $S^2$ . Then we have:

$$\left| \int_{C_j} (z - z_j)\delta_j(z)dz \right| < \sqrt{2}s_j\epsilon(4s_j + L_j) < 4\sqrt{2}A_j\epsilon + \sqrt{2}SL_j\epsilon$$

If  $L$  is the length of  $C$ , then

$$\begin{aligned} \left| \int_C f(z)dz \right| &\leq \sum_{j=1}^n \left| \int_{C_j} (z - z_j)\delta_j(z)dz \right| < \sum_{j=1}^n (4\sqrt{2}A_j\epsilon + \sqrt{2}SL_j\epsilon) \\ &< (4\sqrt{2}S^2 + \sqrt{2}SL)\epsilon \end{aligned}$$

Since  $\epsilon$  is arbitrary, we can choose it to be as small as we like, so

$$\forall \epsilon > 0 \left[ \left| \int_C f(z)dz \right| < (4\sqrt{2}S^2 + \sqrt{2}SL)\epsilon \right] \implies \left| \int_C f(z)dz \right| = 0$$

Hence, if function  $f$  is analytic on all  $z \in C$  where  $C$  is a simple closed contour, then

$$\int_C f(z)dz = 0$$

**TLDR:**

We found that the upper bound for  $f$  around the contour integral  $C$  is less than or equal to the sum of all the contours around the squares covering the region bounded by  $C$ . Since  $f$  is analytic in  $R$ , the sum of the contours of the squares in  $R$  is a function of the neighbourhood  $\delta_j(z)$  surrounding  $z_j$  in each square, which is chosen to be less than some  $\epsilon$ , the error between the derivative of  $f$  and the finite difference of  $f$ . As  $\epsilon$  can be made arbitrary small and the inequality must hold for all values of  $\epsilon$ , we find  $\int_C f(z)dz = 0$ .  $\square$

### 15.6.1 Morera's Theorem

A converse to the Cauchy-Goursat theorem.

#### Lemma 15.6.1.2:

*Suppose  $P(x, y)$ ,  $Q(x, y)$ ,  $P_y$ , and  $Q_x$  are continuous in a simply connected domain  $D$ . Then for all simple closed contour  $C$  in  $D$ :*

$$\int_C P \, dx + Q \, dy = 0 \implies \frac{\partial}{\partial x} Q = \frac{\partial}{\partial y} P$$

*Proof:* Suppose  $\exists x_0, y_0 \in D$  such that  $\partial Q/\partial x - \partial P/\partial y > 0$ . Then there exists a circle  $C$  in  $D$  centred at  $(x_0, y_0)$  such that  $\partial Q/\partial x - \partial P/\partial y > 0$  in and on  $C$ . That is, there exists a neighbourhood that  $\partial Q/\partial x - \partial P/\partial y > 0$  holds. Then by Green's Theorem (theorem 19.3.3):

$$\int_C P \, dx + Q \, dy = \iint \left( \frac{\partial}{\partial x} Q - \frac{\partial}{\partial y} P \right) \, dx \, dy$$

Then

$$\iint \left( \frac{\partial}{\partial x} Q - \frac{\partial}{\partial y} P \right) \, dx \, dy > 0 \implies \int_C P \, dx + Q \, dy > 0$$

But by hypothesis

$$\int_C P \, dx + Q \, dy = 0$$

We have a contradiction, so

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \leq 0$$

Using a similar argument, we have  $\partial Q/\partial x - \partial P/\partial y \geq 0$  for  $\partial Q/\partial x - \partial P/\partial y < 0$ , so we have

$$\frac{\partial}{\partial x} Q = \frac{\partial}{\partial y} P$$

□

#### Theorem 15.6.2: Morera's Theorem

*Let  $f(x, y) = u(x, y) + iv(x, y)$  where  $u$  and  $v$  are continuous in a domain  $D$ . Then for every simple closed contour  $C$  in  $D$ :*

$$\int_C f(z) dz = 0 \implies f(z) \text{ is analytic in } D$$



*Proof:*

$$\begin{aligned}
\int_C f(z)dz = 0 &\implies \int_C u \, dx - v \, dy + i v \, dx + i u \, dy = 0 \\
&\implies \left[ \int_C u \, dx - v \, dy = 0 \right] \wedge \left[ \int_C v \, dx + u \, dy = 0 \right] \\
&\implies \left[ \frac{\partial}{\partial y} u = -\frac{\partial}{\partial x} v \right] \wedge \left[ \frac{\partial}{\partial y} v = \frac{\partial}{\partial x} u \right]
\end{aligned}$$

These are the Cauchy-Riemann equations (theorem 13.5.1), thus  $f(z)$  is analytic in  $D$ .  $\square$

Note: This proof requires that the partial derivatives to be continuous in  $D$ . However, there is a proof that eliminates this requirement.

## 15.6.2 Simply Connected Domains

### Definition 15.6.1: Simply Connected Domain

*A domain  $D$  which every simple closed contour that lies within it only encloses points in  $D$ .*

### Theorem 15.6.3:

*Let  $f$  be a function that is analytic throughout a simply connected domain  $D$ . Then for every closed contour  $C$  lying in  $D$ :*

$$\int_C f(z)dz = 0$$

*We will later learn in theorem 15.7.4 that this is  $\iff$ , due to theorem 15.5.1.*

*Proof:* Suppose  $C$  is simple and lies entirely in  $D$ . The result follows from the Cauchy-Goursat theorem (theorem 15.6.1).

Suppose that  $C$  is closed, but intersects itself a finite number of times, then it consists of a finite number of simple closed contours. Result again follows from the Cauchy-Goursat Theorem.

Note: There are subtleties for infinite number of self-intersection points.  $\square$

### Corollary 15.6.3.1:

*If  $f$  is a function analytic throughout a simply connected domain  $D$ , then it has antiderivatives everywhere in  $D$ .*

*Proof:* If  $f$  is analytic in a simply connected domain  $D$  then it is continuous in  $D$ . Then

$$\int_C f(z)dz = 0 \iff \forall z \in D \exists F(z)[F'(z) = f(z)] \quad \text{Theorem 15.5.1}$$

$\square$

### Corollary 15.6.3.2:

*Entire functions have antiderivatives everywhere in their domain of definition.*

*Proof:* Consequence of previous corollary and that finite plane is simply connected.  $\square$

## 15.6.3 Multiply Connected Domains

### Definition 15.6.2: Multiply Connected Domains

*A domain that is not simply connected.*

### Theorem 15.6.4:

*Let  $C$  be a simple closed contour in the positive direction, and  $C_k$  ( $k \in \{1, 2, 3, \dots, n\}$ ) be simple closed contours in  $C$  taken in the negative direction that are disjoint with no common interior points. If a function  $f$  is analytic on  $C$  and  $C_k$  and throughout the multiply connected domain consisting of points inside  $C$  but exterior to all  $C_k$ , then*

$$\int_C f(z)dz + \sum_{k=1}^n \int_{C_k} f(z)dz = 0$$

*Proof:* Let a polygonal path  $L_1$  connect  $C$  to the inner contour  $C_1$ ,  $L_2$  connecting  $C_1$  to  $C_2$ , and continue in this manner. Finally, let  $L_{n+1}$  connect  $C_n$  to  $C$ . Then we have two contours  $\Gamma_1$  and  $\Gamma_2$ .  $\Gamma_1$  consisting of parts of the contours  $C$ ,  $C_k$ , and  $L_k$ .  $\Gamma_2$  consisting of the remaining parts of contours  $C$ ,  $C_k$ , and  $-L_k$ . If we apply the Cauchy-Goursat theorem to  $\Gamma_1$  and  $\Gamma_2$ , then

$$\int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz = 0$$

Now, since the integrals along  $L_k$  cancel (due to being taken in the opposite direction), only integrals along  $C$  and  $C_k$  remain. Hence

$$\begin{aligned} \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz &= 0 \\ \implies \int_C f(z)dz + \sum_{k=1}^{n+1} \int_{L_k} f(z)dz - \sum_{k=1}^{n+1} \int_{L_k} f(z)dz + \sum_{k=1}^n \int_{C_k} f(z)dz &= 0 \\ \implies \int_C f(z)dz + \sum_{k=1}^n \int_{C_k} f(z)dz &= 0 \end{aligned}$$

$\square$

**Observation.** *Basically, imagine a slice of Swiss cheese. We took a knife and cut a single path through all of the holes in order. Another way of saying it is that we cut through each hole only once. This way we end up with two slices each consisting a part of the outer edge of the original slice, a part of the edge of the holes, and the edges introduced by our cut. Since we have cut through all of the holes, our two slices will not have holes so we can integrate along the outer edge of each of those slices. The Cauchy-Goursat theorem tells us that the value for the sum will be zero, since they are now simply connected domains.*

**Question.** We pretty much end up with two simply connected domains, right? If so then shouldn't it be:

$$\left( \int_{\Gamma_1} f(z)dz = 0 \right) \wedge \left( \int_{\Gamma_2} f(z)dz = 0 \right) \implies \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz = 0$$

As apposed to just

$$\int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz = 0$$

which does not imply

$$\left( \int_{\Gamma_1} f(z)dz = 0 \right) \wedge \left( \int_{\Gamma_2} f(z)dz = 0 \right)$$

### Corollary 15.6.4.1: Principle of Deformation of Paths

Let  $C_1$  and  $C_2$  be positively oriented simple closed contours, with  $C_1$  interior to  $C_2$ . If a function  $f$  is analytic on  $C_1$ ,  $C_2$ , and the regions between them, then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

*Proof:* It follows from theorem 15.6.4 that

$$\int_{C_1} f(z)dz - \int_{-C_2} f(z)dz = 0 \implies \int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

□

**Observation.** It is easy to see that for any contour  $C_3$  lying between contours  $C_1$  and  $C_2$  and on points on  $C_1$  and  $C_2$ :

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz = \int_{C_3} f(z)dz$$

In fact, this theorem more powerful then it seems. It is implying that the contour integrals of a function  $f$  over any contour is equal, given that one contour can continuously deform into the other without crossing any singular points of  $f(z)$ . It is the key to choosing a simpler path for integration.

## 15.6.4 Examples

**Example 15.6.1** (Beware of domain of definition and analyticity of function when using Cauchy-Goursat Theorem) Let  $C$  be a positively oriented curve of a semicircle from of radius  $|z| \leq 1$  from  $\theta \in [0, \pi]$ , and consider a function:

$$f(z) = z^{1/2} \begin{cases} 0 & z = 0 \\ \sqrt{r}e^{i\theta/2} & r > 0, \theta \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right) \end{cases}$$

Show:

$$\int_C f(z)dz = 0$$

*Proof:* From the domain of definition of the function, we can see that it is not defined on the negative real axis. Now, let's check the analyticity. We have the polar form of the Cauchy-Riemann Equations (theorem 13.5.2):

$$ru_r = v_\theta$$

$$u_\theta = -rv_r$$

Knowing  $f(z) = \sqrt{r}e^{i\theta/2} = \sqrt{r}[\cos(\theta/2) + i\sin(\theta/2)]$ :

$$\begin{aligned} ru_r &= r \left[ \frac{1}{2} r^{-1/2} \cos\left(\frac{\theta}{2}\right) \right] = \frac{1}{2} \sqrt{r} \cos\left(\frac{\theta}{2}\right) & v_\theta &= \sqrt{r} \cos\left(\frac{\theta}{2}\right) \frac{1}{2} = \frac{1}{2} \sqrt{r} \cos\left(\frac{\theta}{2}\right) \\ u_\theta &= -\sqrt{r} \sin\left(\frac{\theta}{2}\right) \frac{1}{2} = -\frac{1}{2} \sqrt{r} \sin\left(\frac{\theta}{2}\right) & -rv_r &= -r \left[ \frac{1}{2} r^{-1/2} \sin\left(\frac{\theta}{2}\right) \right] = -\frac{1}{2} \sqrt{r} \sin\left(\frac{\theta}{2}\right) \end{aligned}$$

While the polar form of the Cauchy-Riemann equations are satisfied, keep in mind that it does not apply at  $z = 0$ .  $f(z)$  fails to be analytic at  $z = 0$ . Due to these, we can not use the Cauchy-Goursat Theorem. We must integrate  $f(z)$  over the contour directly. Let  $l_1$  be the line from 0 to 1,  $l_2$  be a line from  $-1$  to 0, and  $C_R$  be a semicircular contour from 0 to  $\pi$  with radius  $|z| = 1$ . The parameterizations are then:

$$l_1: z = r$$

$$l_2: z = r$$

$$C_R: z = e^{i\theta}$$

The contour integral:

$$\begin{aligned} \int_C f(z) dz &= \int_0^1 \sqrt{r} dr + \int_{-1}^0 \sqrt{r} dr + \int_{C_R} f(z) dz = \int_{-1}^1 \sqrt{r} dr + \int_0^\pi e^{i\theta/2} i e^{i\theta} d\theta \\ &= \frac{1}{2} r^{-1/2} \Big|_{-1}^1 + i \int_0^\pi e^{i3\theta/2} d\theta = \frac{1}{2} (1 + i) + i \frac{2}{3} e^{i3\theta/2} \Big|_0^\pi \\ &= \frac{1}{2} (1 + i) + i \frac{2}{3} (-i - 1) = 0 \end{aligned}$$

□



**Example 15.6.2** Show

$$\int_0^\infty \exp(-x^2) dx = \frac{\sqrt{\pi}}{2}$$

*Proof:* Consider

$$\begin{aligned}
\int_0^\infty \exp(-x^2) dx \int_0^\infty \exp(-y^2) dy &= \int_0^\infty \int_0^\infty \exp[-(x^2 + y^2)] dx dy \\
&= \int_0^{\pi/2} \int_0^\infty \exp(-r^2) r dr d\theta & r^2 = x^2 + y^2 \\
&= \frac{\pi}{2} \int_0^\infty \exp(-r^2) r dr \\
&= \frac{\pi}{4} \int_0^\infty \exp(-u) du & u = r^2 \\
&= \frac{\pi}{4} e^{-u} \Big|_0^\infty = \frac{\pi}{4}
\end{aligned}$$

Now, if we let  $x = y$ , we can see that

$$\int_0^\infty \exp(-x^2) dx \int_0^\infty \exp(-y^2) dy = \left[ \int_0^\infty \exp(-x^2) dx \right]^2$$

Thus

$$\int_0^\infty \exp(-x^2) dx = \frac{\sqrt{\pi}}{2}$$

□

**Example 15.6.3** Show

$$\int_0^\infty \exp(-x^2) \cos(2bx) dx = \frac{\sqrt{\pi}}{2} \exp(-b^2) \quad b > 0$$

*Proof:* We will consider the simple closed contour formed by the lines:  $l_1$  from  $-a$  to  $a$ ,  $l_2$  from  $a$  to  $a+bi$ ,  $l_3$  from  $a+bi$  to  $-a+bi$ , and  $l_4$  from  $-a+bi$  to  $-a$ . The parameterization are given by:

$$l_1 : z = x \quad l_2 : z = a + yi \quad l_3 : z = x + bi \quad l_4 : z = -a + yi$$

Evaluating the contour, and using the Cauchy-Goursat Theorem (theorem 15.6.1):

$$\begin{aligned}
&\int_C \exp(-z^2) dz \\
&= \int_{-a}^a e^{-x^2} dx + \int_0^b e^{-(a+yi)^2} i dy + \int_a^{-a} e^{-(x+bi)^2} dx + \int_b^0 e^{-(-a+yi)^2} i dy \\
&= \int_{-a}^a e^{-x^2} - e^{-x^2+b^2-2bxi} dx + i \int_0^b e^{-a^2+y^2-2ayi} - e^{-a^2+y^2+2ayi} dy \\
&= 2 \int_{-a}^0 e^{-x^2} - e^{b^2} e^{-x^2} [\cos(-2bx) + i \sin(-2bx)] dx + i e^{-a^2} \int_0^b e^{y^2} [e^{-2ayi} - e^{2ayi}] dy \\
&= 0
\end{aligned}$$

Since we are evaluating over a symmetric integral, only even functions remain, so the sine term is zero. We also know that

$$\cos(-2bx) = \cos(2bx) \qquad e^{-2ayi} - e^{2ayi} = -2i \sin(2ay)$$

Making these substitutions and rearranging:

$$2 \int_0^a e^{b^2} e^{-x^2} \cos(2bx) dx = 2 \int_0^a e^{-x^2} dx + 2e^{-a^2} \int_0^b e^{y^2} \sin(2ay) dy$$

Dividing everything by  $e^{b^2}$  since  $b > 0$  is a constant:

$$\int_0^a e^{-x^2} \cos(2bx) dx = e^{-b^2} \int_0^a e^{-x^2} dx + e^{-(a^2+b^2)} \int_0^b e^{y^2} \sin(2ay) dy$$

Taking the limit as  $a \rightarrow \infty$ :

$$\lim_{a \rightarrow \infty} \int_0^a e^{-x^2} \cos(2bx) dx = \lim_{a \rightarrow \infty} \left[ e^{-b^2} \int_0^a e^{-x^2} dx + e^{-(a^2+b^2)} \int_0^b e^{y^2} \sin(2ay) dy \right]$$

Using the knowledge from example 15.6.2, and also:

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \qquad \left| \int_0^b e^{y^2} \sin(2ay) dy \right| \leq \int_0^b e^{y^2} dy$$

We get

$$\int_0^\infty \exp(-x^2) \cos(2bx) dx = \frac{\sqrt{\pi}}{2} \exp(-b^2) \qquad b > 0$$

□



**Example 15.6.4** Show for a positively oriented simple closed contour  $C$ , the area of the region enclosed by  $C$  is

$$\frac{1}{2i} \int_C \bar{z} dz$$

*Proof:* The parameterization of  $f(z)$ :

$$\begin{aligned} \int_C f(z) dz &= \int_a^b f[z(t)] z'(t) dt \\ &= \int_a^b [u(x, y) + iv(x, y)] [x'(t) + y'(t)] dt \\ &= \int_a^b (ux' - vy') dt + i \int_a^b (vx' + uy') dt = \int_C u \, dx - v \, dy + i \int_C v \, dx + u \, dy \end{aligned}$$

Using Green's Theorem (theorem 19.3.3):

$$\int_C f(z)dz = \iint_R (-v_x - u_y)dA + i \iint_R (u_x - v_y)dA$$

From  $\bar{z} = x - iy$ :

$$u_x = 1 \qquad v_y = -1 \qquad v_x = 0 \qquad u_y = 0$$

Substituting these in, we get:

$$\int_C \bar{z}dz = i \iint_R 2 dA = 2i \iint_R dA = 2iA$$

$A$  is the area of the region enclosed by  $C$ , thus:

$$A = \frac{1}{2i} \int_C \bar{z}dz$$

□

## 15.7 Cauchy Integral Formula

The Cauchy-Goursat theorem (theorem 15.6.1) gives us the value of a contour integral without singularities in the interior. What if there is a singularity in the interior? See below.

### Theorem 15.7.1: Cauchy Integral Formula

*Let a function  $f$  be analytic on and inside a simple closed contour  $C$  oriented positively. Then for all  $z_0$  interior to  $C$ :*

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$$

*That is, if  $f$  is analytic within and on a simple closed contour  $C$ , then values of  $f$  interior to  $C$  is determined by values of  $f$  on  $C$ .*

*Proof:* Let  $C$  be a positively oriented contour and  $z_0$  be any point interior to  $C$ . Let  $C_\rho$  be a positively oriented circular contour lying inside  $C$  centred at  $z_0$ . That is,  $C_\rho$  lies on points  $|z - z_0| = \rho$ . Then from corollary 15.6.4.1, we can write:

$$\begin{aligned} \int_C \frac{f(z)}{z - z_0} dz &= \int_{C_\rho} \frac{f(z)}{z - z_0} dz \\ \implies \int_C \frac{f(z)}{z - z_0} dz - f(z_0) \int_{C_\rho} \frac{1}{z - z_0} dz &= \int_C \frac{f(z) - f(z_0)}{z - z_0} dz \end{aligned}$$

Now

$$\begin{aligned} \int_{C_\rho} \frac{1}{z - z_0} dz &= \int_{C_\rho} \frac{1}{re^{i\theta}} ire^{i\theta} d\theta & z = z_0 + re^{i\theta} \\ &= \int_{C_\rho} i d\theta = 2\pi i \end{aligned}$$

So

$$\int_C \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) = \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} dz$$

Since  $f$  is analytic, thus continuous:

$$\forall \epsilon > 0, \exists \delta > 0 [ |z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon ]$$

Now,  $|z - z_0| = \rho < \delta$  for all  $z$  on  $C_\rho$ , so according to theorem 15.4.1:

$$\left| \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} dz \right| < \frac{\epsilon}{\rho} (2\pi\rho) = 2\pi\epsilon \qquad \left| \frac{f(z) - f(z_0)}{z - z_0} \right| < \frac{\epsilon}{\delta}$$

Then

$$\left| \int_C \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) \right| < 2\pi\epsilon$$

This inequality must hold for all values of  $\epsilon > 0$ , so

$$\int_C \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) = 0 \implies \int_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$

□

The Cauchy Integral Theorem links  $f(z_0)$  to a contour integral. The extension of the Cauchy Integral Formula links the  $n$ -th derivative of  $f$ ,  $f^{(n)}(z_0)$ , to the contour integral of  $f$  at  $z_0$ .

### **Theorem 15.7.2: Cauchy Integral Formula (Extension)**

*Let a function  $f$  be analytic on and inside a simple closed contour  $C$  oriented positively. Then for all  $z_0$  interior to  $C$ :*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \qquad n \in \mathbb{N} \cup \{0\}$$

*Proof:* Proof that is not a proof, but a verification in Brown and Churchill [2].

Taking the original Cauchy Integral formula:

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s - z} ds \implies f'(z) = \frac{1}{2\pi i} \int_C f(s) \frac{\partial}{\partial z} (s - z)^{-1} ds = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z)^2} ds$$

Continued differentiation under the integral sign yields the desired result...or does it? Verification is needed.

### **Verification**

Let  $z$  be any point interior to a simple closed contour  $C$ , and  $d$  denote the smallest distance from  $z$  to points  $s$  on  $C$ . Assume  $0 < |\Delta z| < d$ , then

$$\begin{aligned} \frac{f(z + \Delta z) - f(z)}{\Delta z} &= \frac{1}{2\pi i} \int_C \left( \frac{1}{s - z - \Delta z} - \frac{1}{s - z} \right) \frac{f(s)}{\Delta z} ds \\ &= \frac{1}{2\pi i} \int_C \frac{f(s)}{(s - z - \Delta z)(s - z)} ds \end{aligned}$$



Now

$$\frac{1}{(s-z-\Delta z)(s-z)} = \frac{1}{(s-z)^2} + \frac{\Delta z}{(s-z-\Delta z)(s-z)^2}$$

Hence

$$\frac{f(z+\Delta z) - f(s)}{\Delta z} - \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z)^2} ds = \frac{1}{2\pi i} \int_C \frac{f(s)\Delta z}{(s-z-\Delta z)(s-z)^2} ds$$

Let  $M = \max|f(s)|$  on  $C$ , since  $|s-z| \geq d$  and  $|\Delta z| < d$ :

$$|s-z-\Delta z| = |(s-z) - \Delta z| \geq ||s-z| - |\Delta z|| \geq d - |\Delta z| > 0$$

Then letting  $L$  be the length of  $C$  and using theorem 15.4.1:

$$\left| \int_C \frac{f(s)\Delta z}{(s-z-\Delta z)(s-z)^2} \right| \leq \frac{|\Delta z|M}{(d-|\Delta z|)d^2} L$$

Taking the limit  $\Delta z \rightarrow 0$ :

$$\lim_{\Delta z \rightarrow 0} \left| \int_C \frac{f(s)\Delta z}{(s-z-\Delta z)(s-z)^2} \right| \leq \lim_{\Delta z \rightarrow 0} \frac{|\Delta z|M}{(d-|\Delta z|)d^2} L = 0$$

Hence

$$\lim_{\Delta z \rightarrow 0} \frac{f(z+\Delta z) - f(s)}{\Delta z} - \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z)^2} ds = \lim_{\Delta z \rightarrow 0} \frac{1}{2\pi i} \int_C \frac{f(s)\Delta z}{(s-z-\Delta z)(s-z)^2} ds = 0$$

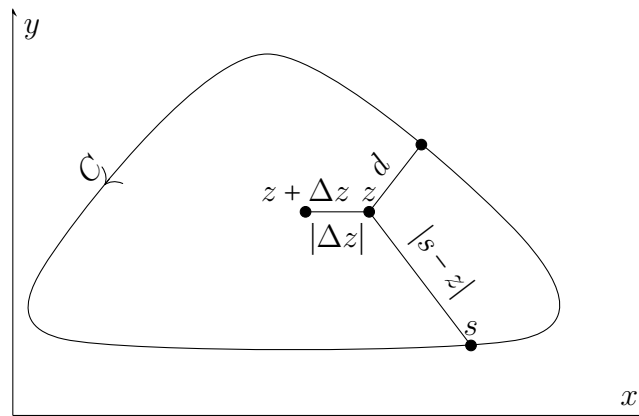
Thus

$$\lim_{\Delta z \rightarrow 0} \frac{f(z+\Delta z) - f(s)}{\Delta z} = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z)^2} ds$$

By induction, we get:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(s)}{(s-z)^{n+1}} ds \quad n \in \mathbb{N} \cup \{0\}$$

□



**Example 15.7.1** We can then rewrite the Legendre Polynomials:

$$P_n(z) = \frac{1}{n!2^n} \frac{d^n}{dz^n} (z^2 - 1)^n = \frac{1}{2^{n+1}\pi i} \int_C \frac{(s^2 - 1)^n}{(s-z)^{n+1}} ds \quad n \in \mathbb{N} \cup \{0\}$$

### 15.7.1 Consequences

#### Theorem 15.7.3:

Let a function  $f$  be analytic at a point  $z_0$ , then  $f^{(n)}$  exists at  $z_0$  for all  $n \in \mathbb{N}$ . That is, the derivative of  $f$  of all orders are analytic at  $z_0$ .

*Proof:* Suppose a function  $f$  is analytic at point  $z_0$ , then there exists a neighbourhood  $\epsilon > |z - z_0|$  where  $f$  is analytic. By extension, there is a positively oriented circular contour  $C_0$  centred at  $z_0$  with radius  $\epsilon/2$  where  $f$  is analytic on and inside  $C_0$ . Then by theorem 15.7.2:

$$f''(z) = \frac{1}{\pi i} \int_{C_0} \frac{f(s)}{(s-z)^3} ds \quad \forall z \text{ interior to } C_0$$

The existence of  $f''(z)$  in  $|z - z_0| < \epsilon \implies f'$  is analytic at  $z_0$ . The same argument on  $f'$  implies  $f''$  is also analytic.  $\square$

Note: Suppose  $f(z) = u(x, y) + iv(x, y)$  is analytic at  $z = (x, y)$ . Then it is also continuous:

$$\begin{aligned} f(z) = u(x, y) + iv(x, y) &\implies [f'(z) = u_x + iv_x = v_y - iu_y] \wedge [f' \text{ is continuous}] \\ &\implies [f''(z) = u_{xx} + iv_{xx} = v_{xy} - iu_{yx}] \wedge [f'' \text{ is continuous}] \end{aligned}$$

#### Corollary 15.7.3.1:

Let a function  $f(z) = u(x, y) + iv(x, y)$  be analytic at a point  $z_0$ . Then  $u$  and  $v$  have continuous partial derivatives of all orders at  $z_0$ .

#### Theorem 15.7.4:

Let a function  $f$  be continuous on domain  $D$ , and  $C$  be any closed contour lying in  $D$ .

$$\forall C \left[ \int_C f(z) dz = 0 \right] \implies f \text{ is analytic throughout } D$$

If  $D$  is simply connected, this is the converse of theorem 15.6.3.

*Proof:*

$$\begin{aligned} f \text{ is continuous in } D &\implies \forall z \in D, \exists F(z) [F'(z) = f(z)] && \text{Theorem 15.5.1} \\ &\implies f \text{ is analytic in } D && \text{Theorem 15.7.3} \end{aligned}$$

$\square$

#### Theorem 15.7.5: Cauchy's Inequality

Let a function  $f$  be analytic on and inside a positively oriented circular contour  $C_R$  centered at  $z_0$  with radius  $R$ .

$$M_R = \max_{C_R} |f(z)| \implies |f^{(n)}(z_0)| \leq \frac{n! M_R}{R^n} \quad n \in \mathbb{N}$$

*Proof:* From theorem 15.4.1:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_R} \frac{f(z)}{(z - z_0)^{n+1}} \quad n \in \mathbb{N}$$

$$\implies |f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \cdot \frac{M_R}{R^{n+1}} \cdot 2\pi R = \frac{n! M_R}{R^n} \quad |z - z_0| \leq R, \quad n \in \mathbb{N}$$

□

## 15.8 Liouville's Theorem and the Fundamental Theorem of Algebra

### Theorem 15.8.1: Liouville's Theorem

*Let  $f$  be a function in the complex plane*

$$f \text{ is entire and bounded in } \mathbb{C} \implies f(z) \text{ is constant in } \mathbb{C}$$

*Proof:*  $f$  is entire so  $\forall z \in \mathbb{C}$   $f'(z)$  exists. Then From theorem 15.7.5, and  $f$  being bounded:

$$|f'(z_0)| \leq \frac{M_R}{R} = \frac{M}{R} \quad n = 1, \forall z \in C, \exists M[|f(z)| \leq M]$$

This inequality must hold for all values of  $R$  ( $R$  can be arbitrarily large), so we find:

$$|f'(z_0)| = 0 \implies f \text{ is constant}$$

□

**Observation.** *Liouville's Theorem implies that any non-constant function in  $\mathbb{C}$  is either not entire or unbounded. See the Maximum Modulus Principle (theorem 15.9.2).*

**Question.** *Shouldn't Liouville's theorem be  $\iff$  since a constant function is also entire and bounded?*

Intuitively, this tells us that if  $f$  is bounded by  $f(z_0)$ , then  $f(z_0)$  must either be a maximum or a minimum. This can not be the case since it violates  $f$  being a harmonic function, where the sum of curvatures in each component direction is zero.

### Theorem 15.8.2: Fundamental Theorem of Algebra

*Let  $P(z) = \sum_{i=0}^n a_i z^i$  be any polynomial, then*

$$\forall n \in \mathbb{N}, \exists z_0 \in \mathbb{C} [P(z_0) = 0]$$

*Proof:* Suppose for contradiction  $\nexists z_0 \in \mathbb{C}$  such that  $P(z_0) = 0$  Recall from corollary 12.2.1.1 that  $\exists R \in \mathbb{R}$  such that

$$\left| \frac{1}{P(z)} \right| < \frac{2}{|a_n|R^n} \quad \forall z \in \mathbb{C}[|z| > R]$$

$1/P(z)$  is bounded on for  $|z| > R$ , but  $P(z)$  is continuous on  $|z| \leq R$ , which implies that  $1/P(z)$  is bounded for  $|z| \leq R$ . Thus  $P(z)$  is bounded in the entire complex plane. (Theorem 13.3.5) It follows from theorem 15.8.1 that  $1/P(z)$  is constant  $\implies P(z)$  is constant, but  $P(z)$  is not constant. Contradiction!  $\square$

Theorem 15.8.2 tells us that any polynomial of degree  $n \geq 1$  can be expressed as a product of linear factors:

$$P(z) = c \prod_{i=1}^{i=n} (z - z_i)$$

Since the existence of a zero  $z_1$  implies

$$P(z) = (z - z_1)Q_1(z) \quad \deg(Q_1) = n - 1$$

Result follows from induction.

## 15.9 Maximum Modulus Principle

### Theorem 15.9.1: Gauss's Mean Value Theorem

*Let  $f$  be a function analytic in and on a circular contour  $C_\rho$  centred at  $z_0$ , then  $f(z_0)$  is the arithmetic mean of the values on the circle:*

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) d\theta \quad z = z_0 + \rho e^{i\theta}, \quad 0 \leq \theta \leq 2\pi$$

*That is, the value of  $f(z_0)$  is the average of the values of  $f(z)$  in some neighbourhood with radius  $\rho$  around  $z_0$ .*

*Proof:* Let  $C_\rho$  be a circular contour centred at  $z_0$ , and  $|z - z_0| = \rho$ . Then by the Cauchy Integral formula (theorem 15.7.1):

$$f(z_0) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) d\theta \quad z = z_0 + \rho e^{i\theta}, \quad 0 \leq \theta \leq 2\pi$$

$\square$

**Lemma 15.9.1.1:**

Let  $f$  be a function analytic in some neighbourhood  $|z - z_0| < \epsilon$ :

$$\forall z \in \{z : |z - z_0| < \epsilon\} [|f(z)| \leq |f(z_0)|] \implies \forall z \in \{z : |z - z_0| < \epsilon\} [f(z) = f(z_0)]$$

That is, if  $f$  is bounded by its value at  $z_0$  in the neighbourhood of  $z_0$ , then it is constant throughout the neighbourhood with value  $f(z_0)$ .

*Proof:* Following theorem 15.9.1, we have

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) d\theta \implies |f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| d\theta$$

However, since we have condition  $|f(z)| = |f(z_0 + \rho e^{i\theta})| \leq |f(z_0)|$ :

$$\begin{aligned} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| d\theta &\leq \int_0^{2\pi} |f(z_0)| d\theta = 2\pi |f(z_0)| & 0 \leq \theta \leq 2\pi \\ \implies |f(z_0)| &\geq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| d\theta \end{aligned}$$

The inequalities tells us:

$$|f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| d\theta \implies \int_0^{2\pi} |f(z_0)| - |f(z_0 + \rho e^{i\theta})| d\theta = 0$$

Our condition  $|f(z)| = |f(z_0 + \rho e^{i\theta})| \leq |f(z_0)|$  tells us that

$$\int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| d\theta \leq \int_0^{2\pi} |f(z_0)| d\theta \implies \int_0^{2\pi} |f(z_0)| - |f(z_0 + \rho e^{i\theta})| d\theta \geq 0$$

So for  $\int_0^{2\pi} |f(z_0)| - |f(z_0 + \rho e^{i\theta})| d\theta = 0$ , the integrand must be zero:

$$\begin{aligned} |f(z_0 + \rho e^{i\theta})| - |f(z_0)| &= 0 & 0 \leq \theta \leq 2\pi \\ \implies \forall z \in \{z : |z - z_0| = \rho\} & [|f(z)| = |f(z_0)|] \end{aligned}$$

Since  $0 < |z - z_0| < \epsilon$  and  $|f(z)| = |f(z_0)|$  for all  $0 < \rho < \epsilon$ , we have  $|f(z)| = |f(z_0)|$  for  $|z - z_0| < \epsilon$ . We know that if a function is analytic in a domain and its modulus is constant in the domain, then the function is constant (example 13.6.4), thus

$$\forall z \in \{z : |z - z_0| < \epsilon\} [f(z) = f(z_0)]$$

That is  $f(z)$  is constant in the neighbourhood  $|z - z_0| < \epsilon$  with value  $f(z_0)$ . □

**Theorem 15.9.2: Maximum Modulus Principle**

Let  $f$  be an analytic function in a domain  $D$ .

$$f \text{ not constant in } D \implies \forall z \in D, \nexists z_0 \in D [|f(z)| \leq |f(z_0)| = M]$$

That is, if an analytic function is not constant in a domain  $D$ , then it is not bound.

*Proof:* Suppose for contradiction  $f$  is bounded in domain  $D$ . That is  $\forall z \in D[|f(z)| \leq M]$ . Let  $L$  be a polygonal line lying in  $D$  extending from  $z_0$  to any arbitrary  $z_n = P$  in  $D$ , and  $d$  be the shortest distance from points on  $L$  to the boundary of  $D$ . Then for each point  $z_k$  ( $k \in [0, n]$ ), we have  $|z_k - z_{k-1}| < d$  and neighbourhoods  $N_k$ .

Each neighbourhood  $N_k$  has radius  $d$  and the center of each neighbourhood  $N_k$  lies in the neighbourhood of  $N_{k-1}$ .

Since  $\max |f(z)| = |f(z_0)|$ , by lemma 15.9.1.1, all points in  $N_0$  has value  $f(z_0)$ . The neighbourhoods overlap, so by extension  $\forall k \in [0, n]$ ,  $f(z_k) = f(z_0)$ , and we have  $f(z)$  is constant in  $D$  with value  $f(z_0)$ .  $f$  is then bounded in  $D$  and we have a contradiction! Thus, if an analytic function  $f$  is not constant in domain  $D$ , then it is not bounded.  $\square$



For a closed bounded region  $R$ , the Maximum Modulus Principle may seem to contradict theorem 13.3.5. It is important to realize that we are working with a domain and the differences between a domain (definition 12.7.12) and region (definition 12.7.13).

### Corollary 15.9.2.1:

*Let  $f$  be an analytic function on a closed bounded region  $R$  that is not constant in the interior of  $R$ . Then  $\max |f(z)|$  in  $R$  is always reached and only reached at some boundary of  $R$ , never in the interior of  $R$ .*

*Proof:* Consider

$$f(x, y) = u(x, y) + iv(x, y)$$

Then as  $f$  is analytic in  $R$ ,  $u$  is harmonic in  $R$  and can not assume maximum value in the interior of  $R$ . The same logic applies to  $v$ . (See Maximum Principle.)

More precisely, consider  $g(z) = e^{f(z)}$ , then  $g$  is analytic, continuous, and non-constant in the interior of  $R$ . Hence  $|g(z)| = |e^{f(z)}|$  must assume its maximum value at the boundary of  $R$ , so  $f(z)$  must also obtain its maximum at the boundary of  $R$ .  $\square$

Note: Same is true for  $\min |f(z)|$  (Example 15.9.2).

Note: Corollary 15.9.2.1 follows from the properties of  $f(z) = u(x, y) + iv(x, y)$  being able to be expressed in terms of real valued functions and that harmonic real valued functions only have maximum and minimum values occurring at the boundaries of a closed and bounded region. We will soon see this in example 15.9.2 and example 15.9.3.

Note: There is a difference between complex-valued functions and real-valued functions. For complex valued functions  $f(z)$ , the maximums and minimums of the modulus  $|f(z)|$  occur at some boundary of  $R$ , while for some real-valued function  $u(x, y)$  (no modulus) the maximum and minimum occurs only at some boundary of  $R$ .

Note: This can be seen a result of Gauss's Mean Value Theorem (theorem 15.9.1).

### 15.9.1 Examples

**Example 15.9.1** Suppose that  $f(z)$  is entire and that the function  $u(x, y) = \operatorname{Re}\{f(z)\}$  is harmonic and has upper bound  $u_0 \geq u(x, y)$  for all  $(x, y) \in \mathbb{R}^2$ . Then  $u(x, y)$  is constant in  $\mathbb{R}^2$ .

*Proof:* Consider  $g(z) = e^{f(z)} = e^{u(x, y)} e^{iv(x, y)}$ .  $e^{iv(x, y)}$  is a phase, so we ignore it and focus on  $e^{u(x, y)}$ :

$e^{u(x, y)}$  is entire

$\implies \exists (x_0, y_0) \in \mathbb{R}^2 [ |e^{u(x, y)}| \leq u_0 = u(x_0, y_0) ]$  Cauchy's Inequality (Theorem 15.7.5)

$\implies e^{u(x, y)}$  is constant Liouville's Theorem (Theorem 15.8.1)

□

**Example 15.9.2** Let a function  $f$  be continuous on a closed bounded region  $R$ , and be analytic and not constant throughout the interior of  $R$ . Also, let  $f(z) \neq 0$  for all  $z \in R$ . Prove  $\min |f(z)|$  only occurs at the boundaries and never in the interiors of  $R$ .

*Proof:* Consider  $g(z) = 1/f(z)$ .

$f$  is continuous, analytic, non-constant, and  $\forall z \in R [f(z) \neq 0]$

$\implies g$  is continuous, analytic, non-constant

$\implies \max |g(z)|$  only occurs at boundary of  $R$  Corollary 15.9.2.1

$\implies \min |f(z)|$  occurs only at some boundary of  $R$

As for why  $f(z) \neq 0$  for all  $z \in R$  is required:

Suppose  $f(z_0) = 0$  for some  $z_0$  in the interior of  $R$ .

$\implies g(z) = 1/f(z)$  is not continuous in  $R$  Corollary 15.9.2.1

$\implies \max |g(z)|$  does not occur only some boundary of  $R$

$\implies \min |f(z)|$  exists in the interior of  $R$

□

**Question.** If  $f(z_0) = 0$  for some  $z_0$  in the interior of  $R$ , then does that mean  $\min |f(z)| = 0$  for all  $z_0 \in \{z : f(z) = 0\}$ ?

**Example 15.9.3** Let  $f(z) = u(x, y) + iv(x, y)$  be a function continuous on a closed bounded region  $R$  be analytic and not constant in the interior of  $R$ . Prove  $\min u(x, y)$  occurs only at some boundary of  $R$ .

*Proof:* Let  $f(z) = u_1(x, y) + iv_1(x, y)$  and  $g(z) = e^{1/f(z)} = e^{u_2(x, y) + iv_2(x, y)}$ , which is continuous, analytic, and not constant in the interior of  $R$ . Then  $|g(z)| = e^{u_2(x, y)}$  is continuous in  $R$  must have a maximum at the some boundary of  $R$ . Hence,  $f(z)$  has a minimum at some boundary of  $R$ .  $\square$

## 15.10 Poisson Integral Formula

We are looking to solve the Dirichlet Problem (definition 19.3.1). Finding a function in a harmonic domain that assumes preassigned values at the boundary.

### Definition 15.10.1: Poisson Integral Formula (Circle Interior)

Let  $f$  be a complex-valued function on a circular simple closed domain with radius  $R$ , and  $C$  be a circular contour on the boundary of the domain. Then the Dirichlet Problem is solved in the domain by

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{u(R, \phi)(R^2 - r^2)}{R^2 + r^2 - 2Rr \cos(\phi - \theta)} d\phi$$

*Proof:* Let  $z$  be any point inside a circular domain with radius  $R$ ,  $f$  be an analytic function throughout the interior of the domain, and  $C$  be a circular contour on the boundary of the domain. Consider the Cauchy Integral Formula (theorem 15.7.1):

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw$$

Define a point that lies outside the circle:  $z_1 = R^2/\bar{z}$

Where

$$\begin{aligned} |z_1| &= \frac{R^2}{|\bar{z}|} = \frac{R^2}{|z|} > R & |z| < R \\ \arg(z_1) &= \arg(z) \end{aligned}$$

Then by Cauchy-Goursat (theorem 15.6.1)

$$\frac{1}{2\pi i} \int_C \frac{f(w)}{w - z_1} dw = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - (R^2/\bar{z})} dw = 0$$



Subtracting the two equations:

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_C f(w) \left[ \frac{1}{w-z} - \frac{1}{w-(R^2/\bar{z})} \right] dw \\
&= \frac{1}{2\pi i} \int_C f(w) \left[ \frac{z - (R^2/\bar{z})}{(w-z)[w-(R^2/\bar{z})]} \right] dw \\
&= \frac{1}{2\pi i} \int_C f(R, \phi) \left[ \frac{re^{i\theta} - [R^2/(re^{-i\theta})]}{(Re^{i\phi} - re^{i\theta})\{Re^{i\phi} - [R^2/(re^{-i\theta})]\}} \right] Re^{i\phi} i d\phi \quad w = Re^{i\phi} \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(R, \phi)(R^2 - r^2)}{R^2 + r^2 - 2Rr \cos(\phi - \theta)} d\phi
\end{aligned}$$

Separating into real and imaginary parts:

$$u(r, \theta) + iv(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{[u(R, \phi) + iv(R, \phi)][R^2 - r^2]}{R^2 + r^2 - 2Rr \cos(\phi - \theta)} d\phi$$

Taking the real part yields the desired result.  $\square$

### Definition 15.10.2: Poisson Integral Formula (Upper Half-Plane)

Consider the Dirichlet problem for the upper half of the complex plane ( $y > 0$ ) which satisfies the boundary condition  $U(X, 0)$  on  $Y = 0$ . Then the Dirichlet Problem is solved by

$$v(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{u(X, 0)}{(X - x)^2 + y^2} dX$$

*Proof:* Let  $f(w) = U(X, Y) + iV(X, Y)$  be analytic for  $Y > 0$ , and  $C$  be the contour comprised of the semi-circular arc from  $R$  to  $-R$  on the upper half of the complex, and the line from  $-R$  to  $R$  on the real axis. Then from the Cauchy Integral Formula (theorem 15.7.1):

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z} dw$$

Then for all  $z$  with  $\text{Im}\{z\} > 0$  and by the Cauchy-Goursat theorem (theorem 15.6.1):

$$\frac{1}{2\pi i} \int_C \frac{f(w)}{(w-\bar{z})} dw = 0$$

Subtracting the two equations:

$$f(z) = \frac{1}{2\pi i} \int_C f(w) \left[ \frac{1}{w-z} - \frac{1}{w-\bar{z}} \right] dw = \frac{1}{2\pi i} \int_C \frac{z - \bar{z}}{(w-z)(w-\bar{z})} dw$$

By breaking up  $C$  into the line  $L$  and arc  $A$  components:

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_L \frac{(z - \bar{z})f(w)}{(w-z)(w-\bar{z})} dw + \frac{1}{2\pi i} \int_A \frac{(z - \bar{z})f(w)}{(w-z)(w-\bar{z})} dw \\
&= \frac{y}{\pi} \int_{-R}^R \frac{f(X)}{(X-x)^2 + y^2} dX + \frac{y}{\pi} \int_A \frac{f(w)}{(w-z)(w-\bar{z})} dw \quad w = X + iY
\end{aligned}$$

Taking the limit as  $R \rightarrow \infty$ , letting  $z = x + iy$  and  $M = \max |f(w)|$  on  $A$ , and knowing  $y \leq R$  and  $r \leq R$  (since  $z = re^{i\theta}$  is within the semi-circle):

$$\begin{aligned}
& \lim_{R \rightarrow \infty} \left| \int_A \frac{f(w)}{(w-z)(w-\bar{z})} dw \right| \\
& \leq \lim_{R \rightarrow \infty} \left| \frac{M}{(Re^{i\phi} - re^{i\theta})(Re^{i\phi} - re^{-i\theta})} \right| |\pi R| \quad \text{Theorem 15.4.1} \\
& = \lim_{R \rightarrow \infty} |\pi R| \left| \frac{M}{R^2 e^{2i\phi} - Rre^{i(\phi-\theta)} - Rre^{i(\phi+\theta)} + r^2} \right| \\
& \leq \lim_{R \rightarrow \infty} |\pi R| \frac{|M|}{|R^2 + r^2|} = 0
\end{aligned}$$

Hence, the contour integral on the arc disappears and we are left with:

$$f(z) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(X)}{(X-x)^2 + y^2} dX$$

Thus

$$U(x, y) + iV(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{U(X, 0) + iV(X, 0)}{(X-x)^2 + y^2} dX$$

Taking the real part yields the desired results. □

# Chapter 16

## Series

### 16.1 Convergence

#### Definition 16.1.1: Limit (Sequence)

Let  $z_1, z_2, z_3, \dots$  be an infinite sequence of complex numbers. We say a limit exists for the sequence if

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N} [n > n_0 \implies |z_n - z| < \epsilon]$$

#### Definition 16.1.2: Converge (Sequence)

If such limit exists, we say the sequence converges to  $z$  and that

$$\lim_{n \rightarrow \infty} z_n = z$$

#### Definition 16.1.3: Diverge (Sequence)

If a limit does not exist for a series.

#### Theorem 16.1.1:

Let  $z_n = x_n + iy_n$  for  $n \in \mathbb{N}$  and  $z = x + iy$ . Then

$$\lim_{n \rightarrow \infty} z_n = z \iff \left[ \lim_{n \rightarrow \infty} x_n = x \right] \wedge \left[ \lim_{n \rightarrow \infty} y_n = y \right]$$

That is, a complex series converges  $\iff$  real and imaginary parts of the sequence converge.

*Proof:*  $\Leftarrow$  :

$$\begin{aligned} & \left[ \lim_{n \rightarrow \infty} x_n = x \right] \wedge \left[ \lim_{n \rightarrow \infty} y_n = y \right] \\ & \implies \forall \epsilon > 0, \exists n_1, n_2 \in \mathbb{N} \left[ \left( n > n_1 \implies |x_n - x| < \frac{\epsilon}{2} \right) \wedge \left( n > n_2 \implies |y_n - y| < \frac{\epsilon}{2} \right) \right] \end{aligned}$$

Let  $n_0 = \max(n_1, n_2)$ , then:

$$n > n_0 \implies \left[ |x_n - x| < \frac{\epsilon}{2} \right] \wedge \left[ |y_n - y| < \frac{\epsilon}{2} \right]$$

Since

$$|(x_n - iy_n) - (x + iy)| = |(x_n - x) + i(y_n - y)| \leq |x_n - x| + |y_n - y|$$

Thus

$$n > n_0 \implies |z_n - z| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\implies$ :

$$\lim_{n \rightarrow \infty} z_n = z \implies [n > n_0 \implies |(x_n + iy_n) - (x + iy)| < \epsilon]$$

However,

$$\begin{aligned} |x_n - x| &\leq |(x_n - x) + i(y_n - y)| = |(x_n + iy_n) - (x + iy)| \\ |y_n - y| &\leq |(x_n - x) + i(y_n - y)| = |(x_n + iy_n) - (x + iy)| \end{aligned}$$

Hence

$$[n > n_0 \implies (|x_n - x| < \epsilon) \wedge (|y_n - y| < \epsilon)] \implies \left( \lim_{n \rightarrow \infty} x_n = x \right) \wedge \left( \lim_{n \rightarrow \infty} y_n = y \right)$$

Thus

$$\lim_{n \rightarrow \infty} z_n = z \iff \left[ \lim_{n \rightarrow \infty} x_n = x \right] \wedge \left[ \lim_{n \rightarrow \infty} y_n = y \right]$$

□

Be extra careful when converting to polar coordinates:

**Example 16.1.1** *It is easy to see that*

$$\lim_{n \rightarrow \infty} z_n = -1 + i \frac{(-1)^n}{n^2} = -1 \quad n \in \mathbb{N}$$

Converting to polar:

$$r_n = |z_n| \quad \Theta_n = \text{Arg}(z_n) \quad n \in \mathbb{N}, \quad -\pi < \Theta_n \leq \pi$$

Then

$$\lim_{n \rightarrow \infty} r_n = \lim_{n \rightarrow \infty} \sqrt{1 + \frac{1}{n^4}} = 1$$

But

$$\lim_{n \rightarrow \infty} \Theta_{2n} = \pi \quad \lim_{n \rightarrow \infty} \Theta_{2n-1} = -\pi \quad n \in \mathbb{N}$$

The limit of  $\Theta_n$  does not exist as  $n \rightarrow \infty$ .

**Definition 16.1.4: Partial Sum (Series)**

Consider a infinite series

$$\sum_{n=1}^{\infty} z_n$$

A partial sum is a finite sum of of the first  $N$  terms of the infinite series, that is:

$$S_N = \sum_{n=1}^N z_n \quad N \in \mathbb{N}$$

**Definition 16.1.5: Converge (Series)**

We say a series converge to sum  $S$  if the sequence of partial sums converges to  $S$ .

**Definition 16.1.6: Divergence (Series)**

A series diverge if its sequence of partial sums does not converge to sum  $S$ .

**Theorem 16.1.2:**

Let  $z_n = x_n + iy_n$  for  $n \in \mathbb{N}$  and  $S = X + iY$ , then

$$\sum_{n=1}^{\infty} z_n = S \iff \left[ \sum_{n=1}^{\infty} x_n = X \right] \wedge \left[ \sum_{n=1}^{\infty} y_n = Y \right]$$

*Proof:*

$$S_N = X_N + iY_N = \sum_{n=1}^N x_n + i \sum_{n=1}^N y_n$$

Then

$$\lim_{N \leftarrow \infty} S_N = S \iff \left[ \lim_{N \rightarrow \infty} X_N = X \right] \wedge \left[ \lim_{N \rightarrow \infty} Y_N = Y \right]$$

□

**Corollary 16.1.2.1:**

Let  $z_1, z_2, z_3, \dots$  be an infinite sequence of complex numbers. Then  $z_n \rightarrow 0$  as  $n \rightarrow \infty$ .

*Proof:* Follows from the  $n$ -th term of a convergent series of real number tends to zero as  $n \rightarrow \infty$ . □

**Corollary 16.1.2.2:**

Convergent series are bounded, that is:

$$\forall n \in \mathbb{N}, \exists M \in \mathbb{R}_{>0} [|z_n| \leq M]$$

**Definition 16.1.7: Absolutely Convergent (Series)**

A series is absolutely convergent if

$$\sum_{n=1}^{\infty} |z_n| = \sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2} \quad z_n = x_n + iy_n$$

converges to some real number.

**Corollary 16.1.2.3:**

*Absolute Convergence  $\implies$  Convergence*

*Proof:* Assume a complex series converges absolutely.

$$|x_n| \leq \sqrt{x_n^2 + y_n^2} \quad |y_n| \leq \sqrt{x_n^2 + y_n^2}$$

Then by comparison test, the following must converge

$$\sum_{n=1}^{\infty} |x_n| \quad \sum_{n=1}^{\infty} |y_n|$$

Result follows from the absolute convergence of real series implies convergence of real series (corollary 9.0.0.1).  $\square$

**Definition 16.1.8: Conditional Convergence**

A series is conditionally convergent if it converges, but does not converge absolutely.

**Definition 16.1.9: Remainder (Series Definition - Complex)**

Let  $S$  be an infinite series, the remainder after  $N$  terms:

$$\rho_N = S - S_N$$

**Corollary 16.1.2.4:**

*Series tend to  $S \iff \rho_N \rightarrow 0$  as  $N \rightarrow \infty$*

*Proof:*

$$S = S_N + \rho_N \implies |S_N - S| = |\rho_N - 0|$$

Therefore

$$\forall \epsilon > 0, \exists N_0 \in \mathbb{N} [N > N_0 \implies |S_N - S| = |\rho_N - 0| < \epsilon]$$

$\square$

**Example 16.1.2** Verify following using remainders:

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad |z| < 1$$

Consider the series:

$$Z = \sum_{n=0}^N z^n \quad z \neq 1$$

Then

$$Z - Zz = \sum_{n=0}^N z^n - \sum_{n=1}^{N+1} z^n = 1 - z^{N+1} \implies Z = \sum_{n=0}^N z^n = \frac{1 - z^{N+1}}{1 - z}$$

Now, we can use this to write the partial sum:

$$S_N(z) = \sum_{n=0}^{N-1} z^n = \frac{1 - z^N}{1 - z}$$

If we let:

$$\begin{aligned} S(z) = \frac{1}{1-z} \implies \rho_N(z) = S(z) - S_N(z) &= \frac{z^N}{1-z} \\ \implies |\rho_N(z)| &= \frac{|z|^N}{|1-z|} \end{aligned}$$

It is clear that  $\rho_N(z) \rightarrow 0$  for  $|z| < 1$ , so  $\sum_{n=0}^{\infty} z^n = 1/(1-z)$  is established.

**Theorem 16.1.3:**

The sum and product of two absolutely convergent series is absolutely convergent, and is independent of the order of terms taken. The value of the product/sum is equal to the value of the product/sum of the values of the original series.

**Theorem 16.1.4: Ratio Test**

Let  $\sum_{n=0}^{\infty} S_N$  be an infinite series, and

$$\Gamma = \lim_{n \rightarrow \infty} \left| \frac{S_{N+1}}{S_N} \right|$$

Then

1.  $\Gamma < 1 \implies$  Series converges absolutely
2.  $\Gamma > 1 \implies$  Series diverges
3.  $\Gamma = 1 \implies$  No info on convergence of series

## 16.2 Taylor Series

### Definition 16.2.1: Power Series

Let  $R$  be a region containing a point  $z_0$ . A power series is a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \quad z_0, a_n \in \mathbb{C}, \quad z \in R$$

### Theorem 16.2.1: Taylor's Theorem

Let  $f$  be a function analytic throughout a disk  $|z - z_0| < R_0$  with radius  $R_0$ . Then  $f(z)$  has the power series representation:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad a_n = \frac{f^{(n)}(z_0)}{n!} \quad |z - z_0| < R_0, \quad n \in \mathbb{N} \cup \{0\}$$

Note: By extension,  $f$  must also be analytic at  $z_0$ .

*Proof:* Let  $C_0$  be a circular contour with radius  $|s| = r_0$  and  $z$  be any point inside the circle, so  $|z| = r$ . Now suppose there is a bigger circle enveloping  $C_0$  with radius  $R_0$  such that  $r < r_0 < R_0$ . Let  $f$  be analytic in and on  $C_0$ .

$z_0 = 0$ :

$f$  is analytic in and on  $C_0$ , we can use the Cauchy-Integral formula (theorem 15.7.1):

$$f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{s - z} ds$$

Recall from example 16.1.2:

$$\frac{1}{1 - z} = \sum_{n=0}^{N-1} z^n + \frac{z^N}{1 - z} \implies \frac{1}{s - z} = \sum_{n=0}^{N-1} \frac{1}{s^{n+1}} z^n + \frac{z^N}{(s - z)s^N}$$

We can then write the Cauchy Integral formula:

$$f(z) = \frac{1}{2\pi i} \left[ \sum_{n=0}^{N-1} \left( \int_{C_0} \frac{f(s)}{s^{n+1}} ds \right) z^n + z^N \int_{C_0} \frac{f(s)}{(s - z)s^N} ds \right]$$

Using the theorem 15.7.2:

$$\frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{s^{n+1}} ds = \frac{f^{(n)}(0)}{n!} \quad n \in \mathbb{N} \cup \{0\}$$

We get

$$f(z) = \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} z^n + \rho_N(z) \quad \rho_N(z) = \frac{z^N}{2\pi i} \int_{C_0} \frac{f(s)}{(s - z)s^N} ds$$



This is the Maclaurin series (definition 16.2.2) if we let  $N \rightarrow \infty$ . To prove it:

$$|s - z| \geq ||s| - |z|| = r_0 - r$$

Letting  $M = \max |f(s)|$  on  $C_0$ :

$$|\rho_N(z)| \leq \frac{r^N}{2\pi} \cdot \frac{M}{(r_0 - r)r_0^N} 2\pi r_0 = \frac{Mr_0}{r_0 - r} \left(\frac{r}{r_0}\right)^N$$

Thus

$$\frac{r}{r_0} < 1 \implies \lim_{N \rightarrow \infty} \left(\frac{r}{r_0}\right)^N = 0 \implies \lim_{N \rightarrow \infty} \rho_N(z) = 0$$

$z_0 \neq 0$ :

Suppose  $f$  is analytic in  $|z - z_0| < R_0$ , then  $f(z + z_0)$  must be analytic in  $|(z + z_0) - z_0| < R_0$ . Then

$$\begin{aligned} f(z + z_0) &= g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n & |z| < R_0 \\ \implies f(z + z_0) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} z^n & |z| < R_0 \\ \implies f(z) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n & |z| < R_0 \end{aligned}$$

□



### Definition 16.2.2: Maclaurin Series

A Taylor series with  $z_0 = 0$ :

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \quad |z| < R_0$$

Useful identities:

$$\begin{aligned}
 \frac{1}{1-z} &= \sum_{n=0}^{\infty} z^n & |z| < 1 \\
 e^z &= \sum_{n=0}^{\infty} \frac{z^n}{n!} & |z| < \infty \\
 \sin(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} & |z| < \infty \\
 \sinh(z) &= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} & |z| < \infty \\
 \cos(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} & |z| < \infty \\
 \cosh(z) &= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} & |z| < \infty
 \end{aligned}$$

## 16.3 Laurent Series

In Taylor's theorem (theorem 16.2.1),  $f$  is required to be analytic at  $z_0$ . What about the case where  $f$  does not need to be analytic at  $z_0$ ? Well...

### Theorem 16.3.1: Laurent's Theorem

Let  $f$  be analytic throughout an annular domain  $R_1 < |z - z_0| < R_2$  centred at  $z_0$ , and  $C$  be any positively oriented simple closed contour in said domain. Then at any  $z$  in the domain:

$$\begin{aligned}
 f(z) &= \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} & R_1 < |z - z_0| < R_2 \\
 a_n &= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{f^{(n)}(z_0)}{n!} & n \in \mathbb{N} \cup \{0\} \\
 b_n &= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz & n \in \mathbb{N}
 \end{aligned}$$

Or (by replacing  $n$  by  $-n$  in  $b_n$ ):

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \quad c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \quad R_1 < |z - z_0| < R_2, \quad n \in \mathbb{Z}$$



*Proof:* Consider the annular regions  $R_1 < r_1 \leq |z| \leq r_2 < R_2$  containing the point  $z$  and simple closed contour  $C$ . Let  $C_1$  and  $C_2$  denote the circular contours with radius  $r_1$  and  $r_2$ , respectively. All the contours are positively oriented. Let  $f$  be an analytic function between the region enclosed by  $C_1$  and  $C_2$  and on  $C_1$  and  $C_2$ .

$z_0 = 0$ :

Let  $\gamma$  be a circular contour centred at  $z$  and small enough to fit in the annular region  $r_1 \leq |z| \leq r_2$ . Using the Cauchy-Goursat theorem (theorem 15.6.1) on multiply connected domains (section 15.6.3):

$$\int_{C_2} \frac{f(s)}{s-z} ds - \int_{C_1} \frac{f(s)}{s-z} ds - \int_{\gamma} \frac{f(s)}{s-z} ds = 0$$

By Cauchy Integral formula (theorem 15.7.1):

$$\int_{\gamma} \frac{f(z)}{s-z} ds = 2\pi i f(z)$$

Hence

$$2\pi i f(z) = \int_{C_2} \frac{f(s)}{s-z} ds - \int_{C_1} \frac{f(s)}{s-z} ds = \int_{C_2} \frac{f(s)}{s-z} ds + \int_{C_1} \frac{f(s)}{z-s} ds$$

Using example 16.1.2:

$$\begin{aligned} \frac{1}{s-z} &= \sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}} + \frac{z^N}{(s-z)s^N} \\ \frac{1}{z-s} &= \sum_{n=0}^{N-1} \frac{1}{s^{-n}} \cdot \frac{1}{z^{n+1}} + \frac{1}{z^N} \cdot \frac{s^N}{z-s} = \sum_{n=1}^N \frac{1}{s^{-n+1}} \cdot \frac{1}{z^n} + \frac{1}{z^N} \cdot \frac{s^N}{z-s} \end{aligned}$$

This implies

$$2\pi i f(z) = \int_{C_2} f(s) \left[ \sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}} + \frac{z^N}{(s-z)s^N} \right] ds + \int_{C_1} f(s) \left[ \sum_{n=1}^N \frac{1}{s^{-n+1}} \cdot \frac{1}{z^n} + \frac{1}{z^N} \cdot \frac{s^N}{z-s} \right] ds$$

Interchanging the integral and summation, and dividing by  $2\pi i$ :

$$f(z) = \sum_{n=0}^{N-1} a_n z^n + \rho_N(z) + \sum_{n=1}^N \frac{b_n}{z^n} + \sigma_N(z)$$

Where

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s^{n+1}} ds & \rho_N(z) &= \frac{z^N}{2\pi i} \int_{C_2} \frac{f(s)}{(s-z)s^N} ds \\ b_n &= \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{s^{-n+1}} ds & \sigma_N(z) &= \frac{1}{2\pi i z^N} \int_{C_1} \frac{s^N f(s)}{z-s} ds \end{aligned}$$

This gives us the Laurent series in  $R_1 \leq |z| \leq R_2$  given that

$$\lim_{N \rightarrow \infty} \rho_N(z) = 0 \qquad \lim_{N \rightarrow \infty} \sigma_N(z) = 0$$

Which we can prove by letting  $|z| = r$ ,  $r_1 < r < r_2$ , and  $M = \max|f(S)|$  on  $C_1$  and  $C_2$ . Then

$$\begin{aligned} |\rho_N(z)| &\leq \frac{Mr_2}{r_2 - r} \left(\frac{r}{r_2}\right)^N & |s - z| &\geq r_2 - r \text{ for } s \in C_2 \\ |\sigma_N(z)| &\leq \frac{Mr_1}{r - r_1} \left(\frac{r_1}{r}\right)^N & |z - s| &\geq r - r_1 \text{ for } s \in C_1 \end{aligned}$$

Since  $r_1 < r < r_2$ , we can see that

$$\lim_{N \rightarrow \infty} \left(\frac{r}{r_2}\right)^N = 0 \qquad \lim_{N \rightarrow \infty} \left(\frac{r_1}{r}\right)^N = 0$$

By corollary 15.6.4.1, we can replace  $C_1$  and  $C_2$  by a positively oriented closed contour  $C$ , giving us the desired expression for the Laurent series.

$z_0 \neq 0$ :

Let  $f$  be analytic in  $R_1 < |z - z_0| < R_2$ , then  $g(z) = f(z + z_0)$  is analytic in  $R_1 < |(z + z_0) - z_0| = |z| < R_2$ . Now let  $t$  parameterize the path  $\Gamma$  on  $C$  so that:

$$z = z(t) - z_0 \qquad R_1 < |z(t) - z_0| = |z| < R_2 \qquad a \leq t \leq b$$

Then  $g$  has the Laurent series representation:

$$\begin{aligned} g(z) &= \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n} & R_2 < |z| < R_2 \\ a_n &= \frac{1}{2\pi i} \int_{\Gamma} \frac{g(z)}{z^{n+1}} dz & n \in \mathbb{N} \cup \{0\} \\ b_n &= \frac{1}{2\pi i} \int_{\Gamma} \frac{g(z)}{z^{-n+1}} dz & n \in \mathbb{N} \end{aligned}$$

Replacing  $z$  by  $z - z_0$  and using corollary 15.6.4.1 to replace  $\Gamma$  by  $C$  yields the desired result. Note that:

$$2\pi i a_n = \int_{\Gamma} \frac{g(z)}{z^{n+1}} dz = \int_C \frac{f[z(t)]z'(t)}{[z(t) - z_0]^{n+1}} dt = \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \qquad z(t) = z$$

Similarly for  $b_n$ . □



Note: If  $f$  is analytic throughout the disk  $|z - z_0| < R_2$ ,  $b_n = (2\pi i)^{-1} \int_C f(z)(z - z_0)^{n-1} dz$  becomes analytic, so  $b_n = 0$  due to Cauchy-Goursat theorem (theorem 15.6.1). The Laurent series then becomes a Taylor series about  $z_0$ .

Note: In the case where  $f$  is not analytic at  $z_0$ , then  $R_1$  can become arbitrarily small, so the Laurent series is valid for the punctured disk  $0 < |z - z_0| < R_2$ . Likewise, if  $f$  is only analytic for points outside  $R_1$ , then the Laurent series is valid for the region  $R_1 < |z - z_0| < \infty$ .

**Question.**  $c_n$  in Laurent's Theorem strongly represents the extended Cauchy Integral formula (theorem 15.7.2). Where

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \quad n \in \mathbb{N} \cup \{0\}$$

The only difference is  $n \in \mathbb{Z}$  for  $c_n$ , while  $n \in \mathbb{N} \cup \{0\}$  for the extended Cauchy Integral formula. Is there a connection?

Ans: Generally, no. The Cauchy Integral theorem requires the function to be analytic throughout the domain enclosed by the contour. This is not necessarily true for a Laurent series since we have a deleted neighbourhood that excludes  $z_0$  so, in general,  $c_n \neq f^{(n)}(z_0)/n!$ .

### 16.3.1 Examples

**Example 16.3.1** (Finding Laurent Series via Known Series) *Find series representation of*

$$f(z) = \frac{1}{z(1+z^2)}$$

We have singularities at  $z = 0, \pi 1$ , and since  $|-z^2| < 1 \implies |z| < 1$ , we may use

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad |z| < 1$$

Substituting  $-z^2$  for  $z$ :

$$\begin{aligned} f(z) &= \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n z^{2n} = \sum_{n=0}^{\infty} (-1)^n z^{2n-1} & |z| < 1 \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n z^{2n-1} \\ &= \sum_{n=0}^{\infty} (-1)^{n+1} z^{2n+1} + \frac{1}{z} & (\text{Standard form}) \end{aligned}$$

Note: This is valid in the region  $|z| < 1$ , there is another representation for  $|z-i| < 1$  and  $|z+1| < 1$ .

**Example 16.3.2** (z-Transform) Suppose

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \quad R_1 < |z| < R_2, \quad n \in \mathbb{Z}$$

Show that if the Laurent series contains the unit circle  $|z| = 1$  then

$$X^{-1}(z) = x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{i\theta}) e^{in\theta} d\theta \quad n \in \mathbb{Z}$$

*Proof:* We can write:

$$\sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} x[n](z - z_0)^{-n} \quad z_0 = 0$$

It is clear that

$$\begin{aligned} x[n] = b_n &= \frac{1}{2\pi i} \int_C \frac{X(z)}{(z - z_0)^{-n+1}} dz & |z| < 1 \\ &= \frac{1}{2\pi i} \int_C \frac{X(z)}{z^{-n+1}} dz & z_0 = 0 \\ &= \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{X(e^{i\theta})}{e^{i\theta(-n+1)}} \frac{d}{d\theta} e^{i\theta} d\theta & z = e^{i\theta} \\ &= \frac{1}{2\pi i} \int_{-\pi}^{\pi} X(e^{i\theta}) e^{in\theta - i\theta} i e^{i\theta} d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{i\theta}) e^{in\theta} d\theta \end{aligned}$$

□

**Example 16.3.3** (Bessel Functions of the First Kind) Let  $z \in \mathbb{C}$  and  $C$  be the unit circle  $w = e^{i\phi}$ ,  $-\pi < \phi < \pi$ , in the  $w$ -plane. Show for the Laurent series about the origin in the  $w$ -plane:

$$\exp\left[\frac{z}{2}\left(w - \frac{1}{w}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z)w^n \quad 0 < |w| < \infty$$

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[-i(n\phi - z \sin(\phi))]d\phi \quad n \in \mathbb{Z}$$

And that

$$\operatorname{Re}\{J_n(z)\} = \frac{1}{\pi} \int_0^\pi \cos[n\phi - z \sin(\phi)]d\phi \quad n \in \mathbb{Z} \quad (16.1)$$

*Proof:* We know for a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z - z_0)^n \quad R_1 < |z - z_0| < R_2$$

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \quad n \in \mathbb{Z}$$

For  $z_0 = 0$  (since we are taking the series about the origin), we can write

$$\begin{aligned} J_n(z) = c_n &= \frac{1}{2\pi i} \int_C \frac{\exp[z2^{-1}(w - w^{-1})]}{w^{n+1}} dw & n \in \mathbb{Z} \\ &= \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{\exp[z2^{-1}(e^{i\phi} - e^{-i\phi})]}{e^{i\phi(n+1)}} (ie^{i\phi}) d\phi & w = e^{i\phi} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[zi \sin(\phi)](e^{-in\phi}) d\phi \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[-i(n\phi - z \sin(\phi))] d\phi \end{aligned}$$

Which is what we are looking for. As for  $\operatorname{Re}\{J_n(z)\}$ :

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos[-i(n\phi - z \sin(\phi))] + i \sin[-i(n\phi - z \sin(\phi))] d\phi$$

This implies

$$\begin{aligned} \operatorname{Re}\{J_n(z)\} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos[-i(n\phi - z \sin(\phi))] d\phi \\ &= \frac{1}{\pi} \int_0^\pi \cos[-i(n\phi - z \sin(\phi))] d\phi & \text{Cosine is an even function} \end{aligned}$$

□

**Example 16.3.4** (Fourier Series) Let  $f(z)$  be a function in some annular domain about the origin that includes the unit circle  $z = e^{i\phi}$ ,  $-\pi \leq \phi \leq \pi$ . Show that in the Laurent series representation for any  $z \in \mathbb{C}$ :

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\phi}) d\phi + \frac{1}{2\pi} \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} f(e^{i\phi}) \left[ \left(\frac{z}{e^{i\phi}}\right)^n + \left(\frac{e^{i\phi}}{z}\right)^n \right] d\phi \quad -\pi \leq \phi \leq \pi$$

and that for  $u(\theta) = \operatorname{Re}\{f(e^{i\phi})\}$ :

$$u(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\phi) d\phi + \frac{1}{\pi} \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} u(\phi) \cos[n(\theta - \phi)] d\phi \quad -\pi \leq \theta \leq \pi$$

which is the Fourier Series of  $u(\theta)$  about  $-\pi \leq \phi \leq \pi$ . Restrictions on  $u(\theta)$  is more severe than necessary in order for it to be represented by a Fourier series, because it needs to be piecewise continuous on  $[-\pi, \pi]$ , and periodic with period of  $2\pi$  and be everywhere differentiable in  $\mathbb{R} \cup \{-\infty, \infty\}$  (theorem 19.3.4).

*Proof:* We know that for the Laurent series representation of a function:

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} & R_1 < |z| < R_2 \\ a_n &= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz & n \in \mathbb{N} \cup \{0\} \\ b_n &= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz & n \in \mathbb{N} \end{aligned}$$

This tells us that

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \left( \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \right) (z - z_0)^n + \frac{1}{2\pi i} \sum_{n=1}^{\infty} \left( \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz \right) (z - z_0)^{-n}$$

Let  $z_0 = 0$  since we are in an annular domain about the origin:

$$\begin{aligned} 2\pi i f(z) &= \left[ \int_C \frac{f(z)}{z^{n+1}} dz \right] + \sum_{n=1}^{\infty} \left( \int_C \frac{f(z)}{z^{n+1}} dz \right) z^n + \sum_{n=1}^{\infty} \left( \int_C \frac{f(z)}{z^{-n+1}} dz \right) z^{-n} \\ &= \left[ \int_C \frac{f(z)}{z^{n+1}} dz \right] + \sum_{n=1}^{\infty} \left[ \left( \int_C \frac{f(z)}{z^{n+1}} dz \right) z^n + \left( \int_C \frac{f(z)}{z^{-n+1}} dz \right) z^{-n} \right] \end{aligned}$$

We know that

$$\begin{aligned} \int_C \frac{f(z)}{z^{n+1}} dz &= \int_{-\pi}^{\pi} \frac{f(e^{i\phi})}{e^{i\phi(n+1)}} (ie^{i\phi}) d\phi = i \int_{-\pi}^{\pi} f(e^{i\phi}) d\phi \\ \int_C \frac{f(z)}{z^{n+1}} dz &= \int_{-\pi}^{\pi} \frac{f(e^{i\phi})}{e^{i\phi(n+1)}} (ie^{i\phi}) d\phi = i \int_{-\pi}^{\pi} \frac{f(e^{i\phi})}{e^{in\phi}} d\phi \\ \int_C \frac{f(z)}{z^{-n+1}} dz &= \int_{-\pi}^{\pi} \frac{f(e^{i\phi})}{e^{i\phi(-n+1)}} (ie^{i\phi}) d\phi = i \int_{-\pi}^{\pi} f(e^{i\phi}) e^{in\phi} d\phi \end{aligned}$$

This gives us

$$2\pi i f(z) = i \int_{-\pi}^{\pi} f(e^{i\phi}) d\phi + \sum_{n=1}^{\infty} \left[ \left( i \int_{-\pi}^{\pi} \frac{f(e^{i\phi})}{e^{in\phi}} d\phi \right) z^n + \left( i \int_{-\pi}^{\pi} f(e^{i\phi}) e^{in\phi} d\phi \right) z^{-n} \right]$$

Bringing  $z$  into the integral (we can do this since  $z$  is any point in the domain, while the  $z$  in the integral is any on  $C$ . They represent two different sets of points. Bad



notation.)

$$\begin{aligned} 2\pi f(z) &= \int_{-\pi}^{\pi} f(e^{i\phi}) d\phi + \sum_{n=1}^{\infty} \left[ \left( \int_{-\pi}^{\pi} f(e^{i\phi}) \frac{e^{in\phi}}{z^n} d\phi \right) + \left( \int_{-\pi}^{\pi} f(e^{i\phi}) \frac{e^{in\phi}}{z^{-n}} d\phi \right) \right] \\ &= \int_{-\pi}^{\pi} f(e^{i\phi}) d\phi + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} f(e^{i\phi}) \left[ \left( \frac{z}{e^{i\phi}} \right)^n + \left( \frac{e^{i\phi}}{z} \right)^n \right] d\phi \end{aligned}$$

Giving us our desired equation. Now let  $u(\theta) = \operatorname{Re}\{f(e^{i\theta})\}$ :

$$\begin{aligned} 2\pi f(e^{i\theta}) &= \int_{-\pi}^{\pi} f(e^{i\phi}) d\phi + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} f(e^{i\phi}) \left[ (e^{in(\theta-\phi)}) + (e^{-in(\theta-\phi)}) \right] d\phi \\ &= \int_{-\pi}^{\pi} f(e^{i\phi}) d\phi + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} f(e^{i\phi}) [2 \cos[n(\theta - \phi)]] d\phi \end{aligned}$$

$\theta, \phi \in \mathbb{R} \implies$  Cosine is real-valued function:

$$\begin{aligned} \operatorname{Re}\{2\pi f(e^{i\phi})\} &= \int_{-\pi}^{\pi} u(e^{i\phi}) d\phi + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} u(e^{i\phi}) [2 \cos[n(\theta - \phi)]] d\phi \\ \operatorname{Re}\{f(e^{i\phi})\} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\phi) d\phi + \frac{1}{\pi} \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} u(\phi) \cos[n(\theta - \phi)] d\phi \end{aligned}$$

□

## 16.4 Absolute and Uniform Convergence of Power Series

### Definition 16.4.1: Circle of Convergence

The greatest circle centred at  $z_0$  for which the power series  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$  converges for all  $z$  interior to the circle. That is, if the circle has radius  $R$  and is centred at  $z_0$ , then the power series converges  $\forall z$  where  $|z - z_0| < R$ .

Note: As a result, the series can not converge at any point outside of the circle of convergence.

### Definition 16.4.2: Center of Expansion

The center of the circle of convergence  $z_0$ .

### Theorem 16.4.1: Absolute Convergence of Power Series

Consider a power series  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$

$$\begin{aligned} \sum_{n=0}^{\infty} a_n(z_1 - z_0)^n \text{ converges for } z_1 \neq z_0 \\ \implies \sum_{n=0}^{\infty} a_n(z - z_0)^n \text{ converges absolutely } \forall z \in \{z : |z - z_0| < R_1 = |z_1 - z_0|\} \end{aligned}$$

That is if  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$  converges for some  $z_1$ , then it converges absolutely for all points interior to the neighbourhood  $|z_1 - z_0|$ .

*Proof:* Assume the series converges, and is therefore bounded.

$$\left( \sum_{n=0}^{\infty} a_n(z_1 - z_0)^n \right) \wedge (z_1 \neq z_0) \implies \exists M \in \mathbb{R}_{>0} [|a_n(z_1 - z_0)^n| \leq M] \quad n \in \mathbb{N} \cup \{0\}$$

If we let

$$|z - z_0| < R_1 \qquad \rho = \frac{|z - z_0|}{|z_1 - z_0|}$$

Then we get

$$|a_n(z - z_0)^n| = |a_n(z_1 - z_0)^n| \left( \frac{|z - z_0|}{|z_1 - z_0|} \right)^n \leq M \rho^n \quad n \in \mathbb{N} \cup \{0\}$$

We get a geometric series which converges since  $\rho < 1$

$$\sum_{n=0}^{\infty} M \rho^n$$

By comparison test, this implies that

$$\sum_{n=0}^{\infty} |a_n(z - z_0)^n| \text{ converges on open disk } |z - z_0| < R_1$$

□

### Definition 16.4.3: Uniform Convergence of Power Series

Let

$$S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \qquad S_N(z) = \sum_{n=0}^{N-1} a_n(z - z_0)^n \qquad |z - z_0| < R$$

and the remainder

$$\rho_N(z) = S(z) - S_N(z) \qquad |z - z_0| < R$$

Convergence is uniform if

$$\forall \epsilon > 0, \exists N_{\epsilon} \in \mathbb{N} [N > N_{\epsilon} \implies |\rho_N(z)| < \epsilon]$$

Where  $N_{\epsilon}$  is only dependent on  $\epsilon$  and independent of  $z$  in the circle of convergence.

Uniform Convergence is telling us that given any  $\epsilon$  we can find a  $N_{\epsilon} \in \mathbb{N}$  such that the error for  $\rho_N(z)$  for all  $N > N_{\epsilon}$  is always within  $\epsilon$ . That is,  $S_N(z)$  is always within  $\epsilon$  of the value  $S_N$  converges to for all  $N_{\epsilon} > N$ . All values of  $S_N(z)$  for  $N > N_{\epsilon}$  fall within a static error box  $\epsilon$ .

**Theorem 16.4.2:**

Let  $z_1$  be any point inside the circle of convergence  $|z - z_0| < R$  of

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Then the series must uniformly converge inside the closed disk  $|z - z_0| \leq R_1 = |z_1 - z_0|$ . It is clear that  $R_1 < R$ .



*Proof:* Follows from definition 16.4.3 that the series converges

$$\sum_{n=0}^{\infty} |a_n (z_1 - z_0)^n|$$

Let  $m, N \in \mathbb{N}$  and  $m > N$ , then the remainders:

$$\rho_N = \lim_{m \rightarrow \infty} \sum_{n=N}^m a_n (z - z_0)^n \quad \sigma_N = \lim_{m \rightarrow \infty} \sum_{n=N}^m |a_n (z_1 - z_0)^n|$$

It is clear that

$$|\rho_N(z)| = \lim_{m \rightarrow \infty} \left| \sum_{n=N}^m a_n (z - z_0)^n \right| \quad |z - z_0| \leq |z_1 - z_0|$$

Then

$$\left| \sum_{n=N}^m a_n (z - z_0)^n \right| \leq \sum_{n=N}^m |a_n| |z - z_0|^n \leq \sum_{n=N}^m |a_n| |z_1 - z_0|^n = \sum_{n=N}^m |a_n (z_1 - z_0)^n|$$

This implies

$$|z - z_0| \leq R_1 \implies |\rho_N(z)| \leq \sigma_N$$

Then because  $\sigma_N$  are remainders of a convergent series,  $\sigma_N \rightarrow 0$  as  $N \rightarrow \infty$ , so

$$\begin{aligned} \forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N} [N > N_\epsilon \implies \sigma_N < \epsilon] \\ \implies \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ converges uniformly in } |z - z_0| \leq R_1 \end{aligned}$$

□

## 16.5 Continuity of Sums of Power Series

### Theorem 16.5.1:

Let

$$S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

If  $z_1$  is a point within the circle of convergence  $|z - z_0| = R$ , then

$$\forall \epsilon > 0, \exists \delta > 0 [|z - z_1| < \delta \implies |S(z) - S(z_1)| < \epsilon]$$

That is, the power series represents a continuous function  $S(z)$  for all points inside its circle of convergence.



*Proof:* Let

$$\begin{aligned} S_N(z) &= \sum_{n=0}^N a_n(z - z_0)^n \\ \rho_N(z) &= S(z) - S_N(z) & |z - z_0| < R \\ S(z) &= S_N(z) + \rho_N(z) & |z - z_0| < R \end{aligned}$$

Then

$$\begin{aligned} |S(z) - S(z_1)| &= |S_N(z) - S_N(z_1) + \rho_N(z) - \rho_N(z_1)| \\ |S(z) - S(z_1)| &\leq |S_N(z) - S_N(z_1)| + |\rho_N(z)| + |\rho_N(z_1)| \end{aligned}$$

Suppose that  $z$  is any point in a closed disk  $|z - z_0| \leq R_0$ , where  $|z_1 - z_0| < R_0 < R$  and  $R$  is the radius of the circle of convergence. By theorem 16.4.2:

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N} \left[ N > N_\epsilon \implies |\rho_N(z)| < \frac{\epsilon}{3} \right]$$

This also holds for any point  $z$  in the neighbourhood  $|z - z_1| < \delta$  that is contained in the disk  $|z - z_0| \leq R_0$ .

$S_N(z)$  is a polynomial, so it is continuous at  $z_1$  for all  $N$ . Then we can choose a  $\delta$  such that for  $N = N_\epsilon + 1$ :

$$|z - z_1| < \delta \implies |S_N(z) - S_N(z_1)| < \frac{\epsilon}{3}$$

Then

$$\begin{aligned} |S(z) - S(z_1)| &\leq |S_N(z) - S_N(z_1)| + |\rho_N(z)| + |\rho_N(z_1)| \\ \implies \left[ |z - z_1| < \delta \implies |S(z) - S(z_1)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \right] \end{aligned}$$

□

Theorem 16.4.1, theorem 16.4.2, and theorem 16.5.1 can all apply to series of the type

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

by letting  $w = 1/(z - z_0)$ . Then

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} = \sum_{n=1}^{\infty} b_n w^n$$

Which must converge absolutely to a continuous function when

$$|w| < \frac{1}{|z_1 - z_0|}$$

Now, since  $|z - z_0| > |z_1 - z_0|$ , the series must converge absolutely to a continuous function exterior to the circle  $|z - z_0| = R_2$  where  $|z_1 - z_0| = R_1$ . We also know that a Laurent Series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

is valid in an annulus  $R_1 < |z - z_0| < R_2$ , so both series must converge uniformly in the closed annulus.

## 16.6 Integration and Differentiation of Power Series

### Theorem 16.6.1: Integration of Power Series

Let  $C$  be any contour lying inside the circle of convergence for the series  $S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$  and  $g(z)$  be any function continuous on  $C$ . Then

$$\int_C g(z) S(z) dz = \sum_{n=0}^{\infty} a_n \int_C g(z) (z - z_0)^n dz$$

*Proof:* Let  $\rho_N(z)$  be the remainder of  $S(z)$ , then

$$g(z)S(z) = \sum_{n=0}^{N-1} a_n g(z)(z - z_0)^n + g(z)\rho_N(z)$$

Because the finite sum is continuous over  $C$ , their integral over  $C$  exists. Likewise with  $g(z)\rho_N(z)$ . Then

$$\int_C g(z)S(z)dz = \sum_{n=0}^{N-1} a_n \int_C g(z)(z - z_0)^n dz + \int_C g(z)\rho_N(z)dz$$

Since the power series converge uniformly

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N} [N > N_\epsilon \implies |\rho_N(z)| < \epsilon]$$

Letting  $M = \max |g(z)|$  over  $C$  and  $L$  be the length of  $C$ . Using theorem 15.4.1:

$$N > N_\epsilon \implies \left| \int_C g(z)\rho_N(z)dz \right| < M\epsilon L$$

Then

$$\lim_{N \rightarrow \infty} \int_C g(z)\rho_N(z)dz = 0$$

Which gives us

$$\begin{aligned} & \lim_{N \rightarrow \infty} \left[ \sum_{n=0}^{N-1} a_n \int_C g(z)(z - z_0)^n dz + \int_C g(z)\rho_N(z)dz \right] \\ &= \lim_{N \rightarrow \infty} \left[ \sum_{n=0}^{N-1} a_n \int_C g(z)(z - z_0)^n dz \right] = \int_C g(z)S(z)dz \end{aligned}$$

□

### **Corollary 16.6.1.1:**

*$S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$  is analytic for all points  $z$  interior to its circle of convergence.*

*Proof:* Let  $|g(z)| = 1$  for all  $z$  interior to the circle of convergence of  $S(z)$ . Then for every closed contour  $C$  lying in the circle of convergence

$$\int_C g(z)(z - z_0)^n dz = \int_C (z - z_0)^n dz = 0 \quad n \in \mathbb{N} \cup \{0\}$$

Thus by theorem 16.6.1:

$$\int_C S(z)dz = 0$$

By Morera's Theorem (theorem 15.6.2),  $S(z)$  is analytic in the circle of convergence.

□

**Theorem 16.6.2: Differentiation of Power Series**

Let  $S(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ , then for all points  $z$  interior to its circle of convergence:

$$S'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$$

*Proof:* Let  $z$  be any point interior to the circle of convergence of  $S(z)$  and  $C$  be any positively oriented simple closed contour lying in the circle of convergence surrounding  $z$ . Define  $\forall s \in C$ :

$$g(s) = \frac{1}{2\pi i} \cdot \frac{1}{(s - z)^2}$$

As  $g(s)$  is continuous on  $C$

$$\int_C g(s) S(s) ds = \sum_{n=0}^{\infty} a_n \int_C g(s) (s - z_0)^n ds \quad s \in C$$

Now as  $S(z)$  is analytic inside and on  $C$ , looking at the left-hand side of the equation, we can use the Extended Cauchy Integral formula (theorem 15.7.2):

$$\int_C g(s) S(s) ds = \frac{1}{2\pi i} \int_C \frac{S(s)}{(s - z)^2} ds = S'(z)$$

As for the right-hand side:

$$\int_C g(s) (s - z_0)^n ds = \frac{1}{2\pi i} \int_C \frac{(s - z_0)^n}{(s - z)^2} ds = \frac{d}{dz} (z - z_0)^n$$

Putting this together, we have

$$S'(z) = \sum_{n=0}^{\infty} a_n \frac{d}{dz} (z - z_0)^n = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}$$

□

Note: In order to integrate or differentiate a power series, we must first make sure we are doing it at within its circle of convergence. This makes sense since the series no longer converges to the function outside the circle of convergence, so integrating or differentiating it outside does not make much sense.

## 16.7 Uniqueness of Series Representations

**Theorem 16.7.1: Uniqueness of Taylor Series**

If  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$  converges to some function  $f(z)$   $\forall z$  interior to some circle  $|z - z_0| = R$ . Then it is the Taylor Series expansion of  $f$  for powers of  $z - z_0$ .

*Proof:* Let

$$f(z) = \sum_{m=0}^{\infty} a_m (z - z_0)^m \quad |z - z_0| < R$$

By theorem 16.6.1:

$$\begin{aligned} \int_C g(z) f(z) dz &= \sum_{m=0}^{\infty} a_m \int_C g(z) (z - z_0)^m dz \\ g(z) &= \frac{1}{2\pi i} \cdot \frac{1}{(z - z_0)^{n+1}} \quad n \in \mathbb{N} \cup \{0\} \end{aligned}$$

Where  $C$  is some circle centred at  $z_0$  with radius less than  $R$ . By the Extended Cauchy Integral formula (theorem 15.7.2):

$$\int_C g(z) f(z) dz = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} = \frac{f^{(n)}(z_0)}{n!}$$

This means that

$$\begin{aligned} \int_C g(z) (z - z_0)^m dz &= \frac{1}{2\pi i} \int_C \frac{1}{(z - z_0)^{n-m+1}} dz \\ &= \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{i R e^{i\theta}}{(R e^{i\theta})^{(n-m+1)}} d\theta \quad z = z_0 + R e^{i\theta} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{(R e^{i\theta})^{(n-m)}} d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta(m-n)}}{R^{(n-m)}} d\theta \end{aligned}$$

We can see that if  $m = n$  the integral is evaluated over 1 so it becomes 1. If  $m \neq n$ ,  $e^{i\theta(m-n)}$  is analytic, so the integral becomes zero by theorem 15.6.1. Hence we have:

$$\int_C g(z) (z - z_0)^m dz = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

Then

$$\sum_{m=0}^{\infty} a_m \int_C g(z) (z - z_0)^m dz = a_n$$

So

$$\begin{aligned} \int_C g(z) f(z) dz &= \sum_{m=0}^{\infty} a_m \int_C g(z) (z - z_0)^m dz \\ \implies \frac{f^{(n)}(z_0)}{n!} &= a_n \end{aligned}$$

This implies that  $\sum_{n=0}^{\infty} a_n (z - z_0)^n$  is the Taylor series for  $f$  at  $z_0$ . □



### Theorem 16.7.2: Uniqueness of Laurent Series

Suppose a series

$$\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

converges to  $f(z)$  for all points in an annular domain around  $z_0$ . Then it is the Laurent series expansion for  $f$  in powers of  $z - z_0$  in that domain.

*Proof:* Let

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

Then using the techniques in theorem 16.7.1, but for  $n, m \in \mathbb{Z}$ , we arrive at:

$$\int_C g(z) f(z) dz = \sum_{m=-\infty}^{\infty} \int_C g(z) (z - z_0)^m dz$$

Therefore

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz = \sum_{m=-\infty}^{\infty} c_m \int_C g(z) (z - z_0)^m dz$$

Which reduces to

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz = c_n \quad n \in \mathbb{Z}$$

Which implies the series is the Laurent series expansion for  $f$  in the annular domain.  $\square$

## 16.8 Multiplication and Division of Power Series

### Definition 16.8.1: Leibniz's Rule

The  $n$ -th derivative of the product of two differentiable functions  $f(z)$  and  $g(z)$ :

$$[f(z)g(z)]^{(n)} = \sum_{k=0}^n \binom{n}{k} f^{(k)}(z) g^{(n-k)}(z)$$
$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad n \in \mathbb{N}, \quad k \in \{0, 1, 2, \dots, n\}$$

For proof see example 16.8.1

**Definition 16.8.2: Cauchy Product**

Suppose within some circle  $|z - z_0| = R$ :

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z - z_0)^n & a_n &= \frac{f^{(n)}(z_0)}{n!} \\ g(z) &= \sum_{n=0}^{\infty} b_n (z - z_0)^n & b_n &= \frac{g^{(n)}(z_0)}{n!} \end{aligned}$$

Then the Cauchy Product of the series can be obtained by the Leibniz's Rule:

$$\begin{aligned} f(z)g(z) &= \left[ \sum_{n=0}^{\infty} a_n (z - z_0)^n \right] \cdot \left[ \sum_{n=0}^{\infty} b_n (z - z_0)^n \right] = \sum_{n=0}^{\infty} c_n (z - z_0)^n \\ c_n &= \sum_{k=0}^n \frac{f^{(k)}(z_0)}{k!} \cdot \frac{g^{(n-k)}(z_0)}{(n-k)!} = \sum_{k=0}^n a_k b_{n-k} \end{aligned}$$

For the division of power series, again suppose two series converges to functions  $f(z)$  and  $g(z)$  within some circle  $|z - z_0| = R$ . It may be best to find the series representation of the reciprocal of one function then take the product using the Cauchy Product. That is, write

$$\frac{f(z)}{g(z)} = f(z) \cdot \frac{1}{g(z)}$$

Then find the series representation of  $1/g(z)$  using long division. After that is done, use the Cauchy Product to multiply  $f(z)$  with  $1/g(z)$ .

Now, if  $f(z)$  and  $g(z)$  are both polynomial with  $\deg[f(z)] < \deg[g(z)]$ , then we can use the method of partial fractions to turn  $f(z)/g(z)$  into a sum of partial fractions. We can then substitute a geometric (or another known series) into each of the partial fractions to obtain a series representation for  $f(z)/g(z)$ . That is

$$\frac{f(z)}{g(z)} = \sum_{n=0}^N \frac{A_n}{(z - z_n)^{m_n}} \quad A_n, z_n \in \mathbb{C}, \quad m_n \in \mathbb{N}$$

**Example 16.8.1** (Deriving Leibniz's Rule) Show

$$(fg)^{(n)} = \sum_{k=0}^n \binom{n}{k} f^{(k)} g^{(n-k)} \quad n \in \mathbb{N}$$

*Proof:* In the case of  $n = 1$ , we have the usual chain rule:

$$(fg)' = f'g + fg'$$

Now, assume this is true up until some  $m \geq 1$ , we will show the case for  $n = m + 1$

$$\begin{aligned}
(fg)^{(m+1)} &= [(fg)']^{(m)} = [f'g + fg']^{(m)} = (f'g)^{(m)} + (fg')^{(m)} \\
&= \sum_{k=0}^m \binom{m}{k} f^{(k+1)} g^{(m-k)} + \sum_{k=0}^m \binom{m}{k} f^{(k)} g^{(m-k+1)} \\
&= f^{(k+1)} g + \sum_{k=0}^{m-1} \binom{m}{k} f^{(k+1)} g^{(m-k)} + \sum_{k=1}^m \binom{m}{k} f^{(k)} g^{(m-k+1)} + fg^{(m+1)} \\
&= f^{(k+1)} g + \sum_{k=1}^m \binom{m}{k-1} f^{(k)} g^{(m-k+1)} + \sum_{k=1}^m \binom{m}{k} f^{(k)} g^{(m-k+1)} + fg^{(m+1)} \\
&= f^{(m+1)} g + \sum_{k=1}^m \left[ \binom{m}{k} + \binom{m}{k-1} \right] f^{(k)} g^{(m-k+1)} + fg^{(m+1)}
\end{aligned}$$

Now

$$\begin{aligned}
\binom{m}{k} + \binom{m}{k-1} &= \frac{m!}{k!(m-k)!} + \frac{m!}{(k-1)!(m-k+1)!} \\
&= \frac{m!(m-k+1)}{k!(m-k+1)!} + \frac{m!(k)}{k!(m-k+1)!} \\
&= \frac{m!(m+1)}{k!(m-k+1)!} = \frac{(m+1)!}{k!(m+1-k)!} = \binom{m+1}{k}
\end{aligned}$$

Hence

$$\begin{aligned}
(fg)^{(m+1)} &= f^{(m+1)} g + \sum_{k=1}^{m+1} \binom{m+1}{k} f^{(k)} g^{m-k+1} + fg^{(m+1)} \\
&= \sum_{k=0}^{m+1} \binom{m+1}{k} f^{(k)} g^{m+1-k} = \sum_{k=0}^n \binom{n}{k} f^{(k)} g^{n-k}
\end{aligned}$$

□

## 16.9 z-Transform

Take a sequence of numbers and make an analytic function.

**Definition 16.9.1: z-Transform**

$$\mathbb{Z}[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n} \quad T \in \mathbb{R}_{>0}, \quad z \in \mathbb{C}$$

Notice we are taking  $n$  samples of function  $f$  at select intervals  $T$ . This is the Laurent series with no positive exponential terms with  $b_n = f(nT)$ . We can also have  $w = 1/z$ , which we then get the Taylor series with all its usual properties:

$$\mathbb{Z}[f(t)] = \sum_{n=0}^{\infty} f(nT) w^n$$

**Definition 16.9.2: Inverse z-Transform**

Let  $C$  be any circular contour with radius greater than  $R$ , where  $F(z)$  is analytic  $\forall |z| > R$ . Then the Inverse z-Transform of  $F(z)$ :

$$\mathbb{Z}^{-1}[F(z)] = f(nT) = \frac{1}{2\pi i} \int_C F(z) z^{n-1} dz \quad n \in \mathbb{N} \cup \{0\}$$

For derivation see example 16.3.2.

Since the z-Transform is linear, we have:

$$\begin{aligned} \mathbb{Z}[cf(t)] &= c\mathbb{Z}[f(t)] \\ \mathbb{Z}[f(t) + g(t)] &= \mathbb{Z}[f(t)] + \mathbb{Z}[g(t)] \\ \mathbb{Z}^{-1}[F(t) + G(t)] &= \mathbb{Z}^{-1}[F(t)] + \mathbb{Z}^{-1}[G(t)] \end{aligned}$$

**16.9.1 Product of z-Transforms****Definition 16.9.3: Product of z-Transform**

Let  $C$  be a circular contour with radius  $\rho = |w| = |1/z|$ , and

$$\mathbb{Z}[f(t)] = \sum_{n=0}^{\infty} c_n z^{-n} = F(z) \quad \mathbb{Z}[g(t)] = \sum_{n=0}^{\infty} d_n z^{-n} = G(z)$$

Then

$$\mathbb{Z}[f(t)g(t)] = \sum_{n=0}^{\infty} f(nT)g(nT)z^{-n} = \sum_{n=0}^{\infty} c_n d_n z^{-n} = \frac{1}{2\pi i} \int_C \frac{F(w)G(z/w)}{w} dw$$

*Proof:* By definition

$$\mathbb{Z}[f(t)g(t)] = \sum_{n=0}^{\infty} f(nT)g(nT)z^{-n} = \sum_{n=0}^{\infty} c_n d_n z^{-n}$$

Letting  $F(z)$  and  $G(z)$  be analytic in domain  $|z| > R$  we have

$$\begin{aligned} F(w) &= \sum_{m=0}^{\infty} c_m w^{-m} & |w| > R \\ G\left(\frac{z}{w}\right) &= \sum_{n=0}^{\infty} d_n \left(\frac{z}{w}\right)^{-n} = \sum_{n=0}^{\infty} d_n w^n z^{-n} & \left|\frac{z}{w}\right| > R \end{aligned}$$

Hence

$$F(w)G(z/w) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_m d_n w^{n-m} z^{-n} \quad |w| > R, \left|\frac{z}{w}\right| > R$$

Now consider a circle of radius  $\rho = |w|$  centred at the origin. Taking  $\rho > R$ , so that  $|z| > R\rho$ , the Laurent series expansion is then uniformly convergent in the domain containing the circle. So multiplying by  $1/(2\pi iw)$ :

$$\frac{1}{2\pi iw} F(w)G(z/w) = \frac{1}{2\pi i} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_m d_n \frac{z^{-n} w^{n-m}}{w} \quad |w| > R, \left| \frac{z}{w} \right| > R$$

Taking the contour integral  $C$  around  $|w| = \rho$ :

$$\frac{1}{2\pi i} \int_C \frac{F(w)G(z/w)}{w} dw = \frac{1}{2\pi i} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \int_C c_m d_n z^{-n} \frac{w^{n-m}}{w} dw$$

Now by Cauchy-Goursat Theorem *theorem* 15.6.1:

$$\int_C w^k dw = \begin{cases} 0 & k \neq -1 \\ 2\pi i & k = -1 \end{cases} \implies \int_C \frac{w^{n-m}}{w} dw = \begin{cases} 0 & n \neq m \\ 2\pi i & n = m \end{cases}$$

Hence taking the non-trivial solution

$$\frac{1}{2\pi i} \int_C \frac{F(w)G(z/w)}{w} dw = \sum_{n=0}^{\infty} c_n d_n z^{-n} = \mathbb{Z}[f(t)g(t)] \quad |z| > R\rho, \rho > R$$

□

#### Definition 16.9.4: Convolution

Let  $f(t)$  and  $g(t)$  be functions, then the convolution of the two functions:

$$f(t) * g(t) = \sum_{k=0}^{\infty} f(kT)g[(n-k)T] \quad n \in \mathbb{N} \cup \{0\}$$

#### Definition 16.9.5: Convolution of Products of z-Transform and Inverse z-Transform

$$\begin{aligned} \mathbb{Z}[h(t)] &= \mathbb{Z}[f(t) \circ g(t)] = \mathbb{Z}[f(t)]\mathbb{Z}[G(t)] = F(z)G(z) \\ \mathbb{Z}^{-1}[H(z)] &= \mathbb{Z}^{-1}[F(z) \circ G(z)] = \mathbb{Z}^{-1}[F(z)]\mathbb{Z}^{-1}[G(z)] = f(t)g(t) \end{aligned}$$

*Proof:* Let  $h(t) = f(t) \circ g(t)$ , then

$$\begin{aligned} \mathbb{Z}[h(t)] &= \sum_{n=0}^{\infty} \left[ \sum_{k=0}^{\infty} f(kT)g[(n-k)T] \right] z^{-n} = \sum_{n=0}^{\infty} \left[ \sum_{k=0}^{\infty} a_k b_{n-k} \right] z^{-n} \\ &= \end{aligned}$$

We know that

$$\mathbb{Z}[f(t)] = \sum_{k=0}^{\infty} a_k z^{-k} = F(z) \quad \mathbb{Z}[g(t)] = \sum_{j=0}^{\infty} b_j z^{-j} = G(z)$$

So

$$F(z)G(z) = \sum_{k=0}^{\infty} a_k z^{-k} \sum_{j=0}^{\infty} b_j z^{-j} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_k b_j z^{-(k+j)} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k b_{n-k} z^{-n} \quad n = j + k$$

This tells us that

$$\mathbb{Z}[h(t)] = \mathbb{Z}[f(t) \circ g(t)] = F(z)G(z)$$

□

# Chapter 17

## Residues and Poles

Cauchy-Goursat theorem (theorem 15.6.1) states the value of a integral over a simple closed contour is zero if the function being evaluated is analytic at all points within and on the contour. What if the function is not analytic for some finite points interior to  $C$ ? Well...it leaves some “residues” at those points which will contribute to the value of the integral.

### 17.1 Residues

#### Definition 17.1.1: Isolated Singular Points

*A singular point  $z_0$  is isolated if there exists a deleted neighbourhood  $\epsilon$ ,  $0 < |z - z_0| < \epsilon$ , where the function  $f$  is analytic for all points in the neighbourhood.*

Note: It is convenient to consider the point at infinity as an Isolated Singular Point. That is, if a function  $f$  is analytic in  $0 < R_1 \leq |z| < \infty$ . Then it has a isolated singular point at  $z_0 = \infty$ .

#### Definition 17.1.2: Residue

*Recall the Laurent series (theorem 16.3.1), and let  $C$  be any positively oriented simple closed contour around  $z_0$  on punctured disk  $0 < |z - z_0| < R_2$ .*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad 0 < |z - z_0| < R_2$$
$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz$$

*The residue is defined as  $b_n$  for  $n = 1$ :*

$$\text{Res}_{z=z_0} [f(z)] = \text{Res}[f(z), z_0] = b_1 = \frac{1}{2\pi i} \int_C f(z) dz$$

Hence, we can use the series expansion of  $f(z)$  to find the value of  $\int_C f(z) dz$  at  $z = z_0$  by finding the coefficient of  $1/(z - z_0)$  in the series. That is, series expand, find coefficient of  $1/(z - z_0)$ , multiply by  $2\pi i$  to find contour integral of  $f(z)$ .



### Theorem 17.1.1: Cauchy's Residue Theorem

Let  $C$  be a positively oriented simple closed contour, and  $f$  be a function analytic inside and on  $C$ , except at a finite number of singular points  $z_k$  ( $k \in \mathbb{N}$ ) inside  $C$ , then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^n \text{Res}[f(z), z_k]$$

*Proof:* Let  $C$  be a positively oriented simple closed contour surrounding the singular points  $z_k$ , and  $C_k$  be positively oriented simple closed contours lying completely in  $C$  surrounding each  $z_k$  that is small enough they don't share common points. This creates a multiply connected domain consisting of points inside  $C$  but exterior to each  $C_k$  (section 15.6.3), and using the Cauchy-Goursat Theorem (theorem 15.6.1):

$$\begin{aligned} \int_C f(z)dz - \sum_{k=1}^n \int_{C_k} f(z)dz &= 0 \implies \int_C f(z)dz - \sum_{k=1}^n 2\pi i \text{Res}[f(z), z_k] = 0 \\ &\implies \int_C f(z)dz = \sum_{k=1}^n 2\pi i \text{Res}[f(z), z_k] \end{aligned}$$

□

### 17.1.1 Residue at Infinity

#### Definition 17.1.3: Residue of $f$ at Infinity

Let  $f$  be analytic throughout the plane except by a finite number of singular points enclosed inside a positively oriented simple closed contour  $C$ , and  $R_1$  be the radius of a circle enclosing  $C$ . Then  $f$  is analytic in  $R_1 < |z| < \infty$ . Let  $C_0$  be a circular contour oriented **negatively** with radius  $R_0$  enclosing the previous circle with radius  $R_1$ . That is  $R_1 < R_0$ . Then the residue of  $f$  at infinity:

$$\int_{C_0} f(z)dz = 2\pi i \text{Res}[f(z), \infty]$$





### Theorem 17.1.2: Residue at Infinity

Let a function  $f$  be analytic everywhere except for a finite number of singular points enclosed by a positively oriented simple closed contour  $C$ , then

$$\int_C f(z) dz = 2\pi i \operatorname{Res} \left[ \frac{f(1/z)}{z^2}, 0 \right]$$

*Proof:* Following our definition (definition 17.1.3), the point of infinity lies outside of  $C_0$ . Since  $f$  is analytic throughout the region bounded by  $C$  and  $C_0$ , using the Principle of Deformation of Paths (corollary 15.6.4.1):

$$\int_C f(z) dz = \int_{-C_0} f(z) dz = - \int_{C_0} f(z) dz = -2\pi i \operatorname{Res}[f(z), \infty]$$

To find the residue, consider the Laurent series:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \quad R_1 < |z| < \infty$$

$$c_n = \frac{1}{2\pi i} \int_{-C_0} \frac{f(z)}{z^{n+1}} dz \quad n \in \mathbb{Z}$$

Replacing  $z$  by  $1/z$  and multiplying by  $1/z^2$ :

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} \frac{c_n}{z^{n+2}} = \sum_{n=-\infty}^{\infty} \frac{c_{n-2}}{z^n} \quad 0 < |z| < \frac{1}{R_1}$$

$$c_{-1} = \operatorname{Res} \left[ \frac{f(1/z)}{z^2}, 0 \right] = \frac{1}{2\pi i} \int_{-C_0} f(z) dz$$

Thus

$$\begin{aligned} \int_{C_0} f(z) dz &= 2\pi i \operatorname{Res}_{z=\infty} [f(z)] \implies \operatorname{Res}_{z=\infty} [f(z)] = - \operatorname{Res} \left[ \frac{f(1/z)}{z^2}, 0 \right] \\ &\implies \int_C f(z) dz = 2\pi i \operatorname{Res} \left[ \frac{f(1/z)}{z^2}, 0 \right] \end{aligned}$$

□

## 17.2 Three Types of Isolated Singular Points

### Definition 17.2.1: Principal Part of $f$

Let  $f$  be a function with an isolated singular point at  $z_0$  in the punctured disk  $0 < |z - z_0| < R_2$ . Then the Principal Part of  $f$  is this part of the Laurent Series (theorem 16.3.1):

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

### Definition 17.2.2: Analytic Part of $f$

Let  $f$  be a function with an isolated singular point at  $z_0$  in the punctured disk  $0 < |z - z_0| < R_2$ . Then the Analytic Part of  $f$  is this part of the Laurent Series (theorem 16.3.1):

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

It is clear why this is the analytic part, since this part is analytic throughout  $|z - z_0| < R_2$ , and converges to an analytic function. (It's the Taylor series (theorem 16.2.1) part!)

### Definition 17.2.3: Removable Singular Points

When the Principal Part of a Function is zero ( $\forall n \in \mathbb{N}[b_n = 0]$ ). That is,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad 0 < |z - z_0| < R_2$$

The residue of a removable singular point is always zero. If we define  $f$  at  $z_0$  such that  $f(z_0) = a_0$  in the expansion, then it becomes a Taylor series and the expansion becomes valid throughout the disk  $|z - z_0| < R_2$ . The singularity in a sense is removed, since we assigned  $f(z_0) = a_0$ .

### Definition 17.2.4: Essential Singular Points

If an infinite number of  $b_n$  in the Principle Part is non-zero,  $z_0$  is an essential singular point of  $f$ . That is, they can not be removed.

### Definition 17.2.5: Poles of Order $m$

If Principal Part of  $f$  contains more than one, but only finitely many non-zero terms, then

$$\exists m \in \mathbb{N}[(b_m \neq 0) \wedge (b_{m+1} = b_{m+2} = b_{m+3} = \dots = 0)]$$

That is

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^m \frac{b_n}{(z - z_0)^n}$$

### Definition 17.2.6: Simple Pole

Pole of order 1.

**Example 17.2.1** Consider:

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{z^n}$$

It's easy to see that  $e^{1/z}$  has an essential singularity at  $z_0 = 0$  where  $b_1 = 1$ . It follows that

$$e^{1/z} = -1 \quad z = \frac{1}{(2n+1)\pi i} \cdot \frac{i}{i} = -\frac{i}{(2n+1)n} \quad n \in \mathbb{Z}$$

Since

$$e^z = -1 \quad z = (2n+1)\pi i \quad n \in \mathbb{Z}$$

Hence,  $e^{1/z}$  assumes the value  $-1$  an infinite number of times in each neighbourhood of the origin. So for large enough  $n$ , an infinite number of such points lie in any given neighbourhood  $\epsilon$  of the origin, except for zero. This illustrates Picard's Theorem.

### Theorem 17.2.1: Picard's Theorem

In each neighbourhood of an essential singular point, a function assumes every finite value an infinite number of times, except with one possible exception.

## 17.3 Residue at Poles

### Theorem 17.3.1:

Let  $z_0$  be an isolated singular point of function  $f$ . The following are equivalent:

1.  $z_0$  is a pole of order  $m$  ( $m \in \mathbb{N}$ )
2.  $f$  can be written as

$$f(z) = \frac{\phi(z)}{(z - z_0)^m} \quad m \in \mathbb{N}$$

$\phi(z_0)$  is analytic and  $\phi(z_0) \neq 0$ .

- 3.

$$\text{Res}[f(z), z_0] = \lim_{z \rightarrow z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \quad m \in \mathbb{N}$$

*Proof:* Assume that  $z_0$  is a pole of order  $m$ , then  $f$  has the Laurent series expansion in the punctured disk  $0 < |z - z_0| < R_2$ :

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^m \frac{b_n}{(z - z_0)^n} \quad b_m \neq 0$$

Now, define a function  $\phi$  such that

$$\phi(z) = \begin{cases} (z - z_0)^m f(z) & z \neq z_0 \\ b_m & z = z_0 \end{cases}$$

Same as  $f(z)$ ,  $\phi(z)$  is analytic throughout  $|z - z_0| < R_2$ , and has the Laurent series expansion.

$$\phi(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n+m} + \sum_{n=1}^m \frac{b_n}{(z - z_0)^{m-n}}$$

Dividing  $\phi(z)$  by  $(z - z_0)^m$  gives us

$$f(z) = \frac{\phi(z)}{(z - z_0)^m}$$

Now, since  $\phi(z)$  is analytic at  $z_0$ , it has the Taylor series expansion in some neighbourhood  $|z - z_0| < \epsilon$  of  $z_0$ :

$$\phi(z) = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!} (z - z_0)^n$$

Therefore, for  $f(z)$  in the same neighbourhood:

$$f(z) = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!} (z - z_0)^{n-m}$$

The Laurent series with  $\phi(z_0) \neq 0$  tells us that  $z_0$  is a pole of order  $m$  of  $f(z)$ , and the coefficient of  $1/(z - z_0)$ , ( $n = m - 1$ ), tells us the coefficient of  $\text{Res}[f(z), z_0]$ .

$$\text{Res}[f(z), z_0] = \frac{\phi^{(m-1)}}{(m-1)!}$$

□

**Warning!** Resort to Laurent series for to obtain residues in cases where  $\phi(z_0) = 0$  or  $\phi(z_0)$  is undefined!

$\phi(z_0) \neq 0$  is essential since  $\phi(z_0) = 0$  implies  $z_0$  is a removable singular point which theorem 17.3.1 does not apply. **Always check**  $\phi(z_0) \neq 0$  for calculations!

If  $\phi(z_0)$  is undefined, then again theorem 17.3.1 does not apply.

From theorem 17.3.1, it is easy to see that if  $f$  has a pole of order  $m$  at  $z_0$ :

1.  $[\lim_{z \rightarrow z_0} (z - z_0)^n f(z) \neq 0] \vee [\lim_{z \rightarrow z_0} (z - z_0)^n f(z) \neq \infty] \implies f(z)$  has pole of order  $m$
2. If  $f$  has a pole of order  $m$  at  $z_0$ :

$$\lim_{z \rightarrow z_0} (z - z_0)^n f(z) = \frac{\phi(z)}{(z - z_0)^{m-n}} = \begin{cases} 0 & n > m \\ \infty & n < m \end{cases}$$

**On the term “pole”:**

Consider a function  $f(z)$  with a pole of order  $m$  at  $z_0$ . Then by theorem 17.3.1:

$$f(z) = \frac{\phi(z)}{(z - z_0)^m} \quad \phi(z) = \sum_{n=0}^N c_{-(m-n)}(z - z_0)^n \quad c_{-m} \neq 0$$

Therefore

$$\lim_{z \rightarrow z_0} f(z) \approx \frac{\phi(z_0)}{(z - z_0)^m} = \frac{c_{-m}}{(z - z_0)^m} \implies \lim_{z \rightarrow z_0} |f(z)| \approx \frac{|c_{-m}|}{|z - z_0|^m}$$

For higher orders of  $m$ , the points around  $z$  goes to infinity faster. Hence, a pole of higher order means a “thicker pole” in a graphical sense.

**Example 17.3.1** Consider

$$f(z) = \frac{1}{z^2 \sinh(z)}$$

$z^2 \sinh(z)$  has zeros  $z = n\pi i$ , where  $n \in \mathbb{Z}$ , so  $z = 0$  is an isolated singularity. It would be a mistake to write

$$f(z) = \frac{\phi(z)}{z^2} \quad \phi(z) = \frac{1}{\sinh(z)}$$

since  $\phi(z = 0)$  is undefined. We must use the Laurent Series expansion:

$$\frac{1}{z^2 \sinh(z)} = \frac{1}{z^3} - \frac{1}{6z} + \frac{7z}{360} + \dots \quad 0 < |z| < \pi$$

So  $\text{Res}_{z=0}[f(z)] = -1/6$ , and the singularity is a pole of order 3, not 2.

**Example 17.3.2** (Residue to Find Derivative) Due to the extended Cauchy Integral Formula (theorem 15.7.2), we can use it to find the residue of a function, or use a derivative to find the residue of a function. Let  $z_0$  be an isolated singular point of function  $f$  and suppose

$$f(z) = \frac{\phi(z)}{(z - z_0)^m} \quad m \in \mathbb{N}$$

where  $\phi(z_0)$  is analytic and non-zero. Since there is a neighbourhood  $|z - z_0| < \epsilon$  where  $\phi(z)$  is analytic throughout, the contour in the extended Cauchy Integral formula can be the positively oriented circle  $|z - z_0| < \epsilon/2$ . By the extended Cauchy Integral formula:

$$\begin{aligned} \phi^{(m-1)}(z_0) &= \frac{(m-1)!}{2\pi i} \int_C \frac{\phi(z)}{(z - z_0)^m} dz & m = n + 1 \\ \implies \frac{\phi^{(m-1)}(z_0)}{(m-1)!} &= \frac{1}{2\pi i} \int_C \frac{\phi(z)}{(z - z_0)^m} dz = \frac{1}{2\pi i} \int_C f(z) dz = \text{Res}[f(z), z_0] \end{aligned}$$

## 17.4 Zeros of Analytic Functions

### Definition 17.4.1: Zero of Order $m$

Let a function  $f$  be analytic at point  $z_0$ , then all derivative  $f^{(n)}(z)$ ,  $n \in \mathbb{N}$ , exist at  $z_0$  (theorem 15.7.3). We say that  $f$  has a zero of order  $m$  if there exists  $m \in \mathbb{N}$  such that

$$[f(z_0) = f'(z_0) = f''(z_0) = \dots = f^{(m-1)}(z_0) = 0] \wedge [f^{(m)}(z_0) \neq 0]$$

That is, it is the lowest order derivative of  $f$  at  $z_0$  where  $f^{(m)}(z_0) \neq 0$ .

### Theorem 17.4.1:

Let  $f$  be a function that is analytic at point  $z_0$ , then the following are equivalent:

1.  $f$  has a zero of order  $m$
2. There exists a function  $g$ , that is analytic and non-zero at  $z_0$ , such that

$$f(z) = (z - z_0)^m g(z)$$

*Proof:*  $\implies$ :

Suppose  $f$  has a zero of order  $m$  at  $z_0$ , then the analyticity of  $f$  at  $z_0$  tells us there is a Taylor series representation in some neighbourhood  $|z - z_0| < \epsilon$ :

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(m+n)}(z_0)}{(m+n)!} (z - z_0)^{m+n} = (z - z_0)^m \sum_{n=0}^{\infty} \frac{f^{(m+n)}(z_0)}{(m+n)!} (z - z_0)^n$$

Then

$$g(z) = \sum_{n=0}^{\infty} \frac{f^{(m+n)}(z_0)}{(m+n)!} (z - z_0)^n$$

Since  $f(z)$  converges in  $|z - z_0| < \epsilon$ ,  $g$  is analytic throughout  $|z - z_0| < \epsilon$ . Also

$$g(z_0) = \frac{f^{(m)}(z_0)}{m!} \neq 0$$

$\impliedby$ :

Assume there exists a function  $g$  that is analytic and non-zero at  $z_0$ , such that

$$f(z) = (z - z_0)^m g(z)$$

Then it has the Taylor series representation in some neighbourhood  $|z - z_0| < \epsilon$  of  $z_0$ :

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!} (z - z_0)^n \implies f(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!} (z - z_0)^{m+n}$$

Hence

$$f^{(m)}(z_0) = m!g(z_0) \neq 0$$

Therefore,  $z_0$  is a zero of order  $m$  of  $f$ . □

Note: It is clear to see this result by thinking of  $f(z)$  and  $g(z)$  as polynomials.

**Definition 17.4.2: Isolated Zeros**

*Let  $f$  be an analytic function. If there exists  $z_0$  such that  $f(z_0) = 0$ , then there exists a deleted neighbourhood  $0 < |z - z_0| < \epsilon$  where  $f(z) \neq 0$ . We then call  $z_0$  an isolated zero of  $f(z)$ .*

**Theorem 17.4.2:**

*Let  $f$  be a function, and suppose*

- 1.  $f$  is analytic at  $z_0$*
- 2.  $f(z_0) = 0$ , but  $f(z) \neq 0$  in any neighbourhood of  $z_0$*

*Then  $f(z) \neq 0$  in some deleted neighbourhood  $0 < |z - z_0| < \epsilon$  of  $z_0$ . That is,  $f(z) \neq 0 \implies f(z)$  only has isolated zeros.*

*Proof:* Not all derivatives of  $f$  are zero at  $z_0$ , otherwise all coefficients in the Taylor series of  $f$  would be zero, then  $f$  would be identically equal to zero in some neighbourhood of  $z_0$ . It is clear from definition 17.4.1, that  $f$  has a zero of order  $m$  at  $z_0$ , then by theorem 17.4.1:

$$f(z) = (z - z_0)^m g(z)$$

$g(z)$  analytic and nonzero at  $z_0$ . Then there exists some neighbourhood  $|z - z_0| < \epsilon$  where  $g(z) \neq 0$ , thus  $f(z) \neq 0$  in deleted neighbourhood  $0 < |z - z_0| < \epsilon$ .  $\square$

**Theorem 17.4.3:**

*Let  $f$  be a function, and  $D$  and  $L$  be a domain and line segment containing point  $z_0$ . Suppose*

- 1.  $f$  analytic throughout neighbourhood  $N_0$  of  $z_0$*
- 2.  $\forall z \in D \cup L [f(z) = 0]$*

*Then  $\forall z \in N_0 [f(z) \equiv 0]$ . That is,  $f(z)$  is identically zero in  $N_0$  if it does not have isolated zeros.*

*Proof:* Let  $f(z) \equiv 0$  in some neighbourhood  $N$  of  $z_0$ . Otherwise theorem 17.4.2 would imply a contradiction. Then

$$\begin{aligned} \forall z \in N [f(z) \equiv 0] &\implies a_n = \frac{f^{(n)}(z_0)}{n!} = 0 & n \in \mathbb{N} \cup \{0\} \\ &\implies \forall z \in N_0 [f(z) \equiv 0] \end{aligned}$$

Since the Taylor series represents  $f(z)$  in  $N_0$ .  $\square$

Note: We say a function is identically zero  $f(z) \equiv 0$  if it becomes the zero function and not merely zero at some point in the domain.

Note: From theorem 17.4.2 and theorem 17.4.3, either a zero is isolated, or it is zero throughout a domain.

## 17.5 Zeros and Poles

### Theorem 17.5.1:

Suppose

1. Functions  $p$  and  $q$  are analytic at  $z_0$
2.  $p(z_0) \neq 0$ , and  $q$  has zero of order  $m$  at  $z_0$

Then  $p(z)/q(z)$  has pole of order  $m$  at  $z_0$ . That is, the order of the pole of the function takes on the order of the zero of the quotient if the conditions are satisfied.

For converse, see example 17.5.1.

*Proof:* Suppose  $p$  and  $q$  are analytic at  $z_0$ ,  $p(z_0) \neq 0$ , and  $q(z_0)$  is a pole of order  $m$ .

$$\begin{aligned} q(z_0) \text{ is a pole of order } m & \\ \implies q(z) \neq 0 \text{ in some deleted neighbourhood of } z_0 & \quad \text{Theorem 17.4.2} \\ \implies z_0 \text{ isolated singular point of } \frac{p(z)}{q(z)} & \end{aligned}$$

Now, theorem 17.4.1 tells us there exists a function  $g(z)$  that is analytic and nonzero at  $z_0$  such that

$$q(z) = (z - z_0)^m g(z)$$

Then

$$\frac{p(z)}{q(z)} = \frac{\phi(z)}{(z - z_0)^m} \quad \phi(z) = \frac{p(z)}{g(z)}$$

Therefore

$$\phi(z_0) \text{ analytic and nonzero} \implies z_0 \text{ pole of order } m \text{ of } \frac{p(z)}{q(z)} \quad \text{Theorem 17.3.1}$$

□

### Theorem 17.5.2:

Let function  $p$  and  $q$  be analytic at  $z_0$ .

$$\begin{aligned} [p(z_0) \neq 0] \wedge [q(z_0) = 0] \wedge [q'(z_0) \neq 0] \\ \implies \left( z_0 \text{ simple pole of } \frac{p(z)}{q(z)} \right) \wedge \left( \text{Res} \left[ \frac{p(z)}{q(z)}, z_0 \right] = \frac{p(z_0)}{q'(z_0)} \right) \end{aligned}$$



*Proof:* Suppose  $[p(z_0) \neq 0] \wedge [q(z_0) = 0] \wedge [q'(z_0) \neq 0]$ , then

$$\begin{aligned} [q(z_0) = 0] \wedge [q'(z_0) \neq 0] &\implies z_0 \text{ zero of order 1 of } q(z) && \text{Definition 17.4.1} \\ &\implies q(z) = (z - z_0)g(z) && \text{Theorem 17.4.1} \end{aligned}$$

Where  $g(z)$  analytic and nonzero at  $z_0$ . Then

$$\frac{p(z)}{q(z)} = \frac{\phi(z)}{z - z_0} \qquad \phi(z) = \frac{p(z)}{g(z)}$$

$\phi(z)$  is analytic and nonzero at  $z_0$ , so

$$\text{Res} \left[ \frac{p(z)}{q(z)}, z_0 \right] = \frac{p(z_0)}{q'(z_0)} \qquad \text{Theorem 17.3.1}$$

We know that

$$\begin{aligned} q(z) = (z - z_0)g(z) &\implies q'(z) = g(z) + zg'(z) - z_0g'(z) \\ &\implies q'(z_0) = g'(z_0) \end{aligned}$$

Thus

$$\text{Res} \left[ \frac{p(z)}{q(z)}, z_0 \right] = \frac{p(z_0)}{q'(z_0)}$$

□

**Example 17.5.1** Let  $p$  and  $q$  be analytic functions at  $z_0$ . Show

$$\begin{aligned} [p(z_0) \neq 0] \wedge [q(z_0) = 0] \wedge \left[ \frac{p(z)}{q(z)} \text{ has pole of order } m \text{ at } z_0 \right] \\ \implies z_0 \text{ pole of order } m \text{ of } q \end{aligned}$$

*Proof:*

$$\frac{p(z)}{q(z)} = \frac{\phi(z)}{(z - z_0)^m} \implies \frac{p(z)}{\phi(z)}(z - z_0)^m = q(z) \qquad \text{Theorem 17.3.1}$$

Now, both  $p(z)$  and  $\phi(z)$  are analytic and nonzero at  $z_0$ , so let  $f(z) = p(z)/\phi(z)$  which is also analytic and nonzero at  $z_0$ . Then

$$f(z)(z - z_0)^m = q(z)$$

By theorem 17.4.1, we have  $q(z)$  has a zero of order  $m$  at  $z_0$ . □

### Theorem 17.5.3: Bolzano-Weierstrass Theorem

*A finite set of points lying in a closed bounded region  $R$  has at least one accumulation points in  $R$ .*

**Example 17.5.2** Show if function  $f$  is analytic in region  $R$  consisting of all points inside and on simple closed contour  $C$ , except for poles inside  $C$ , and all zeros of  $f$  in  $R$  are interior to  $C$  are finite in order, then the zeros must be finite in number.

*Proof:* Suppose for contradiction that zeros are infinite in number, that is, the set of zeros for  $f$  is an infinite set. Then  $\exists z_0 \in \{z : f(z) = 0\}$  where  $z_0$  is an accumulation point by Bolzano-Weierstrauss Theorem (theorem 17.5.3).

If all points in the deleted neighbourhood of  $z_0$  which is also in  $R$  is a zero of  $f$ , then that implies that  $f(z) = 0$  for all  $z \in R$ . This contradicts that we have poles of  $f$  in  $R$ , so this can not be the case.

Then there must exist a deleted neighbourhood  $0 < |z - z_0| < \epsilon$  where a point  $x \in R$  is not a zero of  $f$  ( $f(x) \neq 0$ ). This implies  $f(z) \neq 0$  in all neighbourhoods of  $z_0$ , so  $f(z) \neq 0$  throughout some deleted neighbourhood of  $z_0$  (theorem 17.4.2). We also know that  $f$  only has isolated zeros.

Now, take  $\{z : f(z) = 0\} \setminus \{z_0\}$ . This is still an infinite set, so we can apply Bolzano-Weierstrauss theorem to obtain another accumulation point  $z_1 \neq z_0$  that is exterior to the deleted neighbourhood of  $z_0$  where  $f(z) \neq 0$ . Applying the same logic as before, we get  $f(z) \neq 0$  throughout some deleted neighbourhood of  $z_1$ .

We repeat this process to get isolated zeros  $z_0, z_1, z_2, \dots \in R$ . Since the zeros are all isolated, the radius each deleted neighbourhood of each point is the distance of each isolated zero to their closest isolated zero. Eventually,  $R$  will be contained by the union of finitely many deleted neighbourhoods. This implies the zeros of  $f$  are finite in number, hence we have a contradiction to our original assumptions that zeros are infinite in number.  $\square$

**Example 17.5.3** Let  $R$  be a region consisting of all points interior and on a simple closed contour  $C$ . Show if  $f$  is analytic in region  $R$  except to poles inside  $C$ , then the poles must be finite in number.

*Proof:* Poles are isolated singular points, so  $f$  is analytic in the deleted neighbourhood of poles. Using the same logic as before,  $R$  will be contained by finitely many unions of the deleted neighbourhoods of the poles (which are inside  $R$ ), so we must have a finite number of poles in  $R$ .  $\square$

## 17.6 Behaviour of Functions Near Isolated Singular Points

### 17.6.1 Removable Singular Points

**Theorem 17.6.1:**

Let  $f$  be a function.

$z_0$  isolated singular point of  $f$

$\implies f$  bounded and analytic in some deleted neighbourhood  $0 < |z - z_0| < \epsilon$  of  $z_0$

*Proof:*

$$\begin{aligned}
& f \text{ analytic throughout } |z - z_0| < R_2 \\
& \implies f \text{ continuous in } |z - z_0| \leq \epsilon < R_2 \\
& \implies f \text{ bounded in } |z - z_0| \leq \epsilon < R_2 \quad \text{Theorem 13.3.5} \\
& \implies f \text{ bounded in deleted neighbourhood } 0 < |z - z_0| \leq \epsilon
\end{aligned}$$

□

### Theorem 17.6.2: Riemann's Theorem

*Let function  $f$  be bounded and analytic in some deleted neighbourhood  $0 < |z - z_0| < \epsilon$  of  $z_0$*

$$f \text{ not analytic at } z_0 \implies z_0 \text{ removable singularity of } f$$

*Proof:* Assume  $f$  not analytic at  $z_0$ , but bounded and analytic in some deleted neighbourhood  $0 < |z - z_0| < \epsilon$  of  $z_0$ . Then  $z_0$  is an isolated singularity of  $f$ . Consider the Laurent series:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad \text{In } 0 < |z - z_0| < \epsilon$$

Let  $C$  be a positively oriented circle  $|z - z_0| < \rho < \epsilon$ , then

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} \quad n \in \mathbb{N}$$

Since  $f$  is bounded,  $\exists M \in \mathbb{R}_{>0} [|f(z)| \leq M]$ . Hence

$$|b_n| \leq \frac{1}{2\pi} \cdot \frac{M}{\rho^{-n+1}} 2\pi\rho = M\rho^n \quad n \in \mathbb{N}$$

$\rho$  can be chosen to be arbitrarily small, therefore  $b_n = 0$ , and  $z_0$  is a removable singularity of  $f$  □

## 17.6.2 Essential Singular Points

### Theorem 17.6.3: Casorati-Weierstrass Theorem

*Let  $z_0$  be an essential singularity of function  $f$ , and  $w_0 \in \mathbb{C}$ . Then*

$$\forall \epsilon > 0 [0 < |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon]$$

*Proof:* Assume for contradiction that  $0 < |z - z_0| < \delta$  does not imply  $|f(z) - w_0| < \epsilon$ . Then  $0 < |z - z_0| < \delta \implies |f(z) - w_0| \geq \epsilon$ . Thus

$$g(z) = \frac{1}{f(z) - w_0} \quad 0 < |z - z_0| < \delta$$

is bounded and analytic. By Riemann's Theorem (theorem 17.6.2),  $z_0$  is removable singularity of  $g$ , so  $g$  is defined and analytic at  $z_0$ . If  $g(z_0) \neq 0$

$$f(z) = \frac{1}{g(z)} + w_0 \quad 0 < |z - z_0| < \delta$$

Then  $f(z)$  is analytic at  $z_0$ , so  $z_0$  is removable singularity of  $f$ . Contradicts assumption  $z_0$  is essential singularity of  $f$ .  $\square$

If  $g(z_0) = 0$  then  $g$  has zero of order  $m$  at  $z_0$  ( $g(z_0) \neq 0$  in  $|z - z_0| < \delta$ ), then  $f$  has pole of order  $m$  at  $z_0$  (theorem 17.5.1). Hence, contradiction!

### 17.6.3 Poles of Order $m$

#### Theorem 17.6.4:

*Let  $f$  be a function:*

$$z_0 \text{ pole of } f \implies \lim_{z \rightarrow z_0} f(z) = \infty$$

*That is,  $|f(z)| \rightarrow \infty$  as  $z \rightarrow \infty$  creating a “pole” in a non-mathematical sense.*

*Proof:* Suppose  $f$  has pole of order  $m$  at  $z_0$ . Then

$$f(z) = \frac{\phi(z)}{(z - z_0)^m} \quad \text{Theorem 17.3.1}$$

$\phi(z)$  is analytic and nonzero at  $z_0$ . Then

$$\lim_{z \rightarrow z_0} \frac{1}{f(z)} = \lim_{z \rightarrow z_0} \frac{(z - z_0)^m}{\phi(z)} = \frac{\lim_{z \rightarrow z_0} (z - z_0)^m}{\lim_{z \rightarrow z_0} \phi(z)} = \frac{0}{\phi(z_0)} = 0$$

By section 13.2.2,  $\lim_{z \rightarrow z_0} f(z) = \infty$ .  $\square$

## 17.7 Application of Residues

### 17.7.1 Evaluation of Improper Integrals

#### Definition 17.7.1: Converge (Infinite Integral)

A semi-infinite integral exists if the improper integral converges to a limit:

$$\int_0^{\infty} f(x)dx = \lim_{R \rightarrow \infty} \int_0^R f(x)dx$$

Likewise, for an infinite integral

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R_1 \rightarrow \infty} \int_{R_1}^0 f(x)dx + \lim_{R_2 \rightarrow \infty} \int_0^{R_2} f(x)dx$$

### Definition 17.7.2: Cauchy Principal Value

The Cauchy Principal Value exists if the given limit exists:

$$\text{P. V.} \int_{-\infty}^{\infty} f(x)dx = \lim_{R \rightarrow \infty} \int_{-R}^R f(x)dx$$

Note: Integral Converges  $\implies$  Cauchy Principal Value Exists, since

$$\lim_{R \rightarrow \infty} \int_{-R}^R f(x)dx = \lim_{R \rightarrow \infty} \int_{-R}^0 f(x)dx + \lim_{R \rightarrow \infty} \int_0^R f(x)dx$$

**Example 17.7.1** Cauchy Principal Value exists does not imply integral converges

$$\text{P. V.} \int_{-\infty}^{\infty} x \, dx = \lim_{R \rightarrow \infty} \int_{-R}^R x \, dx = \lim_{R \rightarrow \infty} \left. \frac{x^2}{2} \right|_{-R}^R = \lim_{R \rightarrow \infty} 0 = 0$$

However,

$$\int_{-\infty}^{\infty} = \lim_{R_1 \rightarrow \infty} \int_{-R_1}^0 x \, dx + \lim_{R_2 \rightarrow \infty} \int_0^{R_2} x \, dx = \lim_{R_1 \rightarrow \infty} \left. \frac{x^2}{2} \right|_{-R_1}^0 + \lim_{R_2 \rightarrow \infty} \left. \frac{x^2}{2} \right|_0^{R_2} = - \lim_{R_1 \rightarrow \infty} \frac{R_1^2}{2} + \lim_{R_2 \rightarrow \infty} \frac{R_2^2}{2}$$

Which the limits do not exist.

### Definition 17.7.3: Rational Function

$$f(z) = \frac{p(z)}{q(z)}$$

where  $p(z)$  and  $q(z)$  are polynomials with coefficients in  $\mathbb{R}$ , and no common factors.  $q(z)$  have no real zeros, but at least one above the real axis.

### Theorem 17.7.1:

Consider the rational function

$$f(z) = \frac{p(z)}{q(z)}$$

Then

$$\text{P. V.} \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{k=1}^n \text{Res}[f(z), z_k]$$

*Proof:* We consider the positively oriented contour  $C$  consisting of the line segment from  $-R$  to  $R$  on the real axis and the top half of the circle with radius  $|z| = R$  ( $C_R$ ), which is large enough where the zeros of the function  $z_1, z_2, z_3, \dots, z_n$  all lie within the region enclosed by the contour.



By Cauchy's Residue Theorem (theorem 17.1.1):

$$\int_{-R}^R f(x)dx + \int_{C_R} f(z)dz = 2\pi i \sum_{k=1}^n \text{Res}[f(z), z_k]$$

If

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z)dz = 0$$

Then

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{k=1}^n \text{Res}[f(z), z_k]$$

□

**Question.** *Is the choice of the region being above or below the real axis is arbitrary? It seems to be the case. The only thing that will change is the direction of the line integral on the real axis.*

### Theorem 17.7.2:

*Consider the function  $f(z)$  in the upper half-plane. Let  $C$  be a semicircular contour  $Re^{i\theta}$  in said plane, where  $\theta \in [0, \pi]$  with radius  $R$ .*

$$\exists k > 1, R_0 < R, \mu \in \mathbb{R}, \forall |z| \geq R_0 \left[ |f(z)| \leq \frac{\mu}{|z|^k} \implies \lim_{R \rightarrow \infty} \int_C f(z)dz = 0 \right]$$

*That is, if  $|f(z)|$  falls off faster than the reciprocal of the radius  $R$  of  $C$ , then the integral of  $f(z)$  around  $C$  vanish as  $R \rightarrow \infty$ .*

*Proof:* Assume  $R > R_0$ , then by the ML inequality (theorem 15.4.1):

$$\left| \int_C f(z)dz \right| \leq ML = M\pi R$$

Since  $|f(z)| \leq \mu/|z|^k = \mu/R^k$  on  $C$ :

$$\left| \int_C f(z) dz \right| \leq \frac{\pi R \mu}{R^k} = \frac{\mu \pi}{R^{k-1}} \quad k > 1$$

Thus

$$\lim_{R \rightarrow \infty} \left| \int_C f(z) dz \right| = 0$$

□

**Example 17.7.2** Consider the integral

$$\int_0^\infty f(x) dx = \int_0^\infty \frac{1}{x^6 + 1} dx$$

The function

$$f(z) = \frac{1}{z^6 + 1} \quad \text{Roots: } c_k = \exp \left[ i \left( \frac{\pi}{6} \right) + \left( \frac{\pi}{3} \right) \right]$$

Of these roots,  $c_0 = e^{i\pi/6}$ ,  $c_1 = i$ ,  $c_2 = e^{i5\pi/6}$  lie above the real axis. Letting  $B_0$ ,  $B_1$ , and  $B_2$  be their residues respectively and using Cauchy's Residue Theorem (theorem 17.1.1):

$$\int_{-R}^R f(x) dx + \int_{C_R} f(z) dz = 2\pi i (B_0 + B_1 + B_2)$$

Theorem 17.5.2 tells us that these are simple poles, so using said theorem:

$$\text{Res} \left[ \frac{p(z)}{q(z)}, z = z_0 \right] = \frac{p(z_0)}{q'(z_0)} \implies B_k = \text{Res} \left[ \frac{1}{z^6 + 1}, z_k \right] = \frac{1}{6c_k^5} = -\frac{c_k}{6}$$

Hence

$$\int_{-R}^R f(x) dx + \int_{C_R} f(z) dz = 2\pi i (B_0 + B_1 + B_2) = 2\pi i \cdot \frac{1}{6} (e^{i\pi/6} + i + e^{i5\pi/6}) = \frac{2\pi}{3}$$

Now, using the ML Inequality Theorem (theorem 15.4.1), we know

$$|z^6 + 1| \geq ||z|^6 - 1| = R^6 - 1$$

So

$$|f(z)| = \frac{1}{|z^6 + 1|} \leq M_R = \frac{1}{R^6 - 1} \implies \left| \int_{C_R} f(z) dz \right| \leq M_R \pi R = \frac{\pi R}{R^6 - 1}$$

It's clear that

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0$$

Hence

$$\lim_{R \rightarrow \infty} \int_{-R}^R \frac{1}{x^6 + 1} dx = \frac{2\pi}{3}$$

Since  $f(x)$  is an even function:

$$\int_0^\infty \frac{1}{x^6 + 1} dx = \frac{\pi}{3}$$

## 17.7.2 Improper Integrals from Fourier Analysis

Evaluating integrals of the form

$$\int_{-\infty}^{\infty} f(x) \sin(ax) dx \qquad \int_{-\infty}^{\infty} f(x) \cos(ax) dx \qquad a \in \mathbb{R}_{>0}$$

We can use

$$\int_{-R}^R f(x) \cos(ax) dx + i \int_{-R}^R f(x) \sin(ax) dx = \int_{-R}^R f(x) e^{iax} dx$$

and that

$$|e^{iaz}| = |e^{iax} e^{-ay}| = e^{-ay}$$

is bounded to the upper half of the complex plane to find the integrals using the method in the previous subsection (section 17.7.1).

## 17.7.3 Jordan's Lemma

### Lemma 17.7.2.1: Jordan's Inequality

$$\int_0^{\pi} e^{-R \sin(\theta)} d\theta < \frac{\pi}{R} \qquad R > 0$$

*Proof:* Consider the functions

$$f(\theta) = \sin(\theta) \qquad g(\theta) = \frac{2\theta}{\pi}$$

Now

$$\sin(\theta) \geq \frac{2\theta}{\pi} \implies e^{-R \sin(\theta)} \leq e^{-2R\theta/\pi} \qquad \theta \in \left[0, \frac{\pi}{2}\right], \quad R > 0$$

Thus

$$\int_0^{\pi/2} e^{-R \sin(\theta)} d\theta \leq \int_0^{\pi/2} e^{-2R\theta/\pi} d\theta = \frac{\pi}{2R} (1 - e^{-R}) \leq \frac{\pi}{2R}$$

Since  $\sin(\theta)$  is symmetric about  $\pi/2$  on  $\theta \in [0, \pi]$ :

$$\int_0^{\pi} e^{-R \sin(\theta)} d\theta < \frac{\pi}{R} \qquad R > 0$$

□





### Theorem 17.7.3: Jordan's Lemma

Suppose

1. Function  $f(z)$  is analytic at all points in the upper half plane exterior to circle  $|z| = R_0$
2.  $C_R$  is a semicircle  $z = Re^{i\theta}$ ,  $\theta \in [0, \pi]$  and  $R > R_0$
3.  $\forall z \in C_R, \exists M_R > 0$  such that

$$|f(z)| \leq M_R \qquad \lim_{R \rightarrow \infty} M_R = 0$$

Then  $\forall a > 0$ :

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) e^{iaz} dz = 0$$

*Proof:* Assuming the statements in the theorem are true:

$$\int_{C_R} f(z) e^{iaz} dz = \int_0^\pi f(Re^{i\theta}) \exp(iaRe^{i\theta}) Rie^{i\theta} d\theta$$

Since

$$|f(Re^{i\theta})| \leq M_R \qquad |\exp(iaRe^{i\theta})| \leq e^{-aR \sin(\theta)}$$

Using Jordan's Inequality (lemma 17.7.2.1):

$$\left| \int_{C_R} f(z) e^{iaz} dz \right| \leq M_R R \int_0^\pi e^{-aR \sin(\theta)} d\theta < \frac{M_R \pi}{a}$$

Since  $\lim_{R \rightarrow \infty} M_R = 0$

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) e^{iaz} dz = 0$$

□

Note: Jordan's lemma holds true for a quarter circle. This is easily seen by looking at the proof of Jordan's inequality where we have

$$\int_0^{\pi/2} e^{-R\sin(\theta)} d\theta \leq \frac{\pi}{2R}$$

then substituting it into a modified proof of Jordan's lemma.



**Example 17.7.3** (Fresnel Integrals) *Show*

$$\int_0^\infty \cos(x^2) dx = \int_0^\infty \sin(x^2) dx = \frac{1}{2} \sqrt{\frac{\pi}{2}}$$

Consider the function:

$$f(z) = \exp(iz^2)$$

We will consider  $f(z)$  over the contour  $C$  composed of the curves:  $l_1$  line from  $(0)$  to  $R$ ,  $C_R$  semicircular arc from  $R$  to  $Re^{i\pi/4}$ , and  $l_2$  line from  $Re^{i\pi/4}$  back to  $0$ . We will parameterize the curves:

$$l_1 : z = x$$

$$l_2 : z = re^{i\pi/4}$$

$$C_R : z = re^{i\theta}$$

The integral over the contour is the sum of the curves, which is also zero by Cauchy-Goursat Theorem (theorem 15.6.1):

$$\begin{aligned} \int_C f(z) dz &= \int_0^R \exp(ix^2) dx + \int_R^0 \exp(ir^2 e^{i\pi/2}) dz + \int_{C_R} f(z) dz \\ &= \int_0^R \exp(ix^2) dx - \int_0^R \exp(-r^2) e^{i\pi/4} dr + \int_{C_R} f(z) dz = 0 \end{aligned}$$

Rearranging, we have:

$$\begin{aligned} \int_0^R \exp(ix^2) dx &= e^{i\pi/4} \int_0^R \exp(-r^2) dr + \int_{C_R} f(z) dz \\ \implies \int_0^R \cos(x^2) + i \sin(x^2) dx &= \left( \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) \int_0^R \exp(-r^2) dr + \int_{C_R} f(z) dz \end{aligned}$$

Equating the real and imaginary parts:

$$\begin{aligned}\int_0^R \cos(x^2)dx &= \frac{1}{\sqrt{2}} \int_0^R \exp(-r^2)dr + \operatorname{Re}\left\{\int_{C_R} f(z)dz\right\} \\ \int_0^R \sin(x^2)dx &= \frac{1}{\sqrt{2}} \int_0^R \exp(-r^2)dr + \operatorname{Im}\left\{\int_{C_R} f(z)dz\right\}\end{aligned}$$

Now, using *Jordan's Inequality*:

$$\begin{aligned}\int_{C_R} \exp(iz^2)dz &= \int_{C_R} \exp(ir^2e^{i2\theta}) \frac{\partial}{\partial \theta} re^{i\theta} d\theta = \int_{C_R} rie^{i\theta} \exp(ir^2e^{i2\theta})d\theta \\ &= \int_{C_R} rie^{i\theta} \exp(ir^2 \cos(2\theta) - r^2 \sin(2\theta))d\theta \\ &\leq R \int_0^{\pi/4} \exp(-r^2 \sin(2\theta))d\theta = \frac{R}{2} \int_0^{\pi/2} \exp(-r^2 \sin(\phi))d\phi \quad \phi = 2\theta \\ &\leq \frac{R}{2} \cdot \frac{\pi}{2r^2} = \frac{\pi}{2r} \quad \text{Lemma 17.7.2.1}\end{aligned}$$

Taking the limit as  $R \rightarrow \infty$ :

$$\lim_{R \rightarrow \infty} \int_{C_R} \exp(iz^2)dz = \lim_{R \rightarrow \infty} \frac{\pi}{2R} = 0$$

Using this and the knowledge:

$$\int_0^\infty \exp(-r^2)dr = \frac{\sqrt{\pi}}{2} \quad \text{Example 15.6.2}$$

We get:

$$\begin{aligned}\lim_{R \rightarrow \infty} \int_0^R \cos(x^2)dx &= \lim_{R \rightarrow \infty} \frac{1}{\sqrt{2}} \int_0^R \exp(-r^2)dr = \frac{1}{2} \sqrt{\frac{\pi}{2}} \\ \lim_{R \rightarrow \infty} \int_0^R \sin(x^2)dx &= \lim_{R \rightarrow \infty} \frac{1}{\sqrt{2}} \int_0^R \exp(-r^2)dr = \frac{1}{2} \sqrt{\frac{\pi}{2}}\end{aligned}$$

Thus

$$\int_0^\infty \cos(x^2)dx = \int_0^\infty \sin(x^2)dx = \frac{1}{2} \sqrt{\frac{\pi}{2}}$$



### 17.7.4 Indented Paths

#### Definition 17.7.4: Cauchy Principal Value of Integral with Singularity

Consider the real-valued function  $f(x)$  in  $x \in [a, b] \setminus \{x_0\}$  where  $x_0 \in (a, b)$  is a singularity of  $f(x)$ . Let  $\epsilon$  be some neighbourhood of  $x_0$ , then

$$\text{P.V.} \int_a^b f(x)dx = \lim_{\epsilon \rightarrow 0} \left[ \int_a^{x_0-\epsilon} f(x)dx + \int_{x_0+\epsilon}^b f(x)dx \right]$$

#### Theorem 17.7.4:

Let

1. Function  $f(z)$  be a function with:
  - (a) Simple pole at  $z = x_0$  on the real axis
  - (b) Laurent series representation in  $0 < |z - x_0| < R_2$
  - (c) Residue  $B_0$
2.  $C_\rho$  be a semicircular **negatively** oriented contour with radius  $|z - x_0| < \rho$ ,  $\rho < R_2$ , in the upper half plane.

Then

$$\lim_{\rho \rightarrow 0} \int_{C_\rho} f(z)dz = -B_0\pi i$$



*Proof:* Assume the conditions of the theorem are true, since  $f(z)$  has a simple pole at  $z = x_0$ :

$$f(z) = g(z) + \frac{B_0}{z - x_0} \quad g(z) = \sum_{n=0}^{\infty} a_n(z - x_0)^n \quad |z - x_0| \in (0, R_2)$$

Integrating the Laurent series over  $C_\rho$ :

$$\int_{C_\rho} f(z)dz = \int_{C_\rho} g(z)dz + B_0 \int_{C_\rho} \frac{1}{z - x_0} dz$$

Since  $g(z)$  is continuous in  $|z - z_0| < R_2$ , it must be bounded on the closed disk  $|z - x_0| \leq \rho_0$ ,  $\rho < \rho_0 < R_2$ , by theorem 13.3.5. Then

$$\exists M \in \mathbb{R}_{>0} [|z - x_0| \leq \rho_0 \implies |g(z)| \leq M]$$

Using the ML Inequality (theorem 15.4.1):

$$\int_{C_\rho} g(z) dz \leq ML = M\pi\rho \implies \lim_{\rho \rightarrow 0} \int_{C_\rho} g(z) dz = 0$$

For the second integral,  $C_\rho$  has the parameterization  $z = x_0 + \rho e^{i\theta}$ ,  $\theta \in [0, \pi]$ . Then

$$\int_{C_\rho} \frac{1}{z - x_0} dz = - \int_{-C_\rho} \frac{1}{z - x_0} dz = - \int_0^\pi \frac{1}{\rho e^{i\theta}} \rho e^{i\theta} d\theta = -i \int_0^\pi d\theta = -\pi i$$

Thus

$$\lim_{\rho \rightarrow 0} \int_{C_\rho} f(z) dz = \lim_{\rho \rightarrow 0} \left[ \int_{C_\rho} g(z) dz + B_0 \int_{C_\rho} \frac{1}{z - x_0} dz \right] = -B_0 \pi i$$

□

Note: We can use theorem 17.7.4 to integrate around a isolated singular point on the real axis, or a branch cut involving the origin and  $\theta \in (-\pi + 2n\pi, 2\pi + 2n\pi)$  (that is, a branch cut below the real axis).

A more general result involving simple poles is as follows:

**Theorem 17.7.5:**

*Let function  $f(z)$  has a simple pole at  $z_0$ , with  $z_0$  as the center of arc  $C_\rho$  ( $\theta \in [\theta_0, \theta_0 + \theta_1]$ ) with radius  $\rho$ . If the arc subtends to angle  $\theta_0$  at  $z_0$ , then*

$$\lim_{\rho \rightarrow \infty} \int_{C_\rho} f(z) dz = 2\pi i \left( \frac{\theta_0}{2\pi} \text{Res}[f(z), z_0] \right) = i\theta_0 \text{Res}[f(z), z_0]$$



*Proof:* Consider the Laurent series of  $f(z)$  around  $z_0$ . Since  $f(z)$  has a simple pole at  $z_0$ :

$$f(z) = g(z) + \frac{B_0}{z - z_0} \qquad g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Note that  $g(z)$  is the Taylor series of  $f$  about  $z_0$ . Integrating the expansion over  $C_\rho$ :

$$\int_{C_\rho} f(z)dz = \int_{C_\rho} g(z)dz + \int_{C_\rho} \frac{B_0}{z - z_0} dz$$

$g(z)$  is continuous at  $z_0$ , so  $|g(z)|$  is bounded in some neighbourhood of  $z_0$ , that is,  $\exists M \in \mathbb{R}[|g(z)| \leq M]$  in the neighbourhood. By the ML inequality (theorem 15.4.1):

$$\left| \int_{C_\rho} g(z)dz \right| \leq M\rho\theta_0 \implies \lim_{\rho \rightarrow 0} \int_{C_\rho} g(z)dz = 0$$

Thus taking the limit as  $\rho \rightarrow 0$ :

$$\begin{aligned} \lim_{\rho \rightarrow 0} \int_{C_\rho} f(z)dz &= \int_{C_\rho} \frac{B_0}{z - z_0} dz = \int_{\theta_1}^{\theta_1 + \theta_0} \frac{B_0}{(z_0 + re^{i\theta}) - z_0} ire^{i\theta} d\theta \\ &= B_0\theta_0 i = i\theta_0 \text{Res}[f(z), z_0] \end{aligned}$$

□

**Observation.** When  $\theta = 2\pi$ , we get the Cauchy's Residue Theorem (theorem 17.1.1).

**Example 17.7.4** (Dirichlet's Integral) Show

$$\int_0^\infty \frac{\sin(x)}{x} dx = \frac{\pi}{2}$$

*Proof:* Consider integrating  $f(z) = e^{iz}/z$  over a the simple closed contour  $C$  composed of a line  $l_1$  from  $\rho$  to  $R$ , a semicircular contour  $C_R$  with radius  $R$  from  $0$  to  $\pi$ , a line  $l_2$  from  $-R$  to  $-\rho$ , and a semicircular contour  $C_\rho$  with radius  $\rho < R$  from  $-\pi$  to  $0$ . Notice  $C_\rho$  is defined to avoid the singularity of  $f(z)$ .

By Cauchy-Goursat Theorem (theorem 15.6.1):

$$\begin{aligned} \int_C f(z)dz &= \int_{l_1} \frac{e^{iz}}{z} dz + \int_{C_R} \frac{e^{iz}}{z} dz + \int_{C_\rho} \frac{e^{iz}}{z} dz + \int_{l_2} \frac{e^{iz}}{z} dz = 0 \\ \implies \int_{l_1} \frac{e^{iz}}{z} dz + \int_{l_2} \frac{e^{iz}}{z} dz &= - \int_{C_R} \frac{e^{iz}}{z} dz - \int_{C_\rho} \frac{e^{iz}}{z} dz \end{aligned}$$

Parameterizing the lines:

$$l_1: z = re^{i0} = r \qquad l_2: z = re^{-i\pi} = -r \qquad \rho \leq r \leq R$$

The integral over the lines:

$$\int_{l_1} \frac{e^{iz}}{z} dz + \int_{l_2} \frac{e^{iz}}{z} dz = \int_\rho^R \frac{e^{ir}}{r} dr - \int_\rho^R \frac{e^{-ir}}{r} dr = \int_\rho^R \frac{e^{ir} - e^{-ir}}{r} dr = 2i \int_\rho^R \frac{\sin(r)}{r} dr$$

The integral over the semicircles can be found by using the Laurent Series:

$$\frac{e^{iz}}{z} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{i^n z^{n-1}}{n!}$$

$f(z) = e^{iz}/z$  has a simple pole at  $z = 0$  with  $\text{Res}[f(z), 0] = 1$ , then:

$$\lim_{\rho \rightarrow 0} \int_{C_\rho} \frac{e^{iz}}{z} dz = -\pi i \quad \text{Theorem 17.7.4}$$

and

$$\left| \frac{1}{z} \right| = \frac{1}{|z|} = \frac{1}{R} \implies \lim_{R \rightarrow \infty} \int_{C_R} \frac{e^{iz}}{z} dz = 0 \quad \text{Theorem 17.7.3}$$

Hence

$$\begin{aligned} \int_{l_1} \frac{e^{iz}}{z} dz + \int_{l_2} \frac{e^{iz}}{z} dz &= - \int_{C_R} \frac{e^{iz}}{z} dz - \int_{C_\rho} \frac{e^{iz}}{z} dz \\ \implies 2i \int_0^\infty \frac{\sin(r)}{r} dr &= \pi i \implies \int_0^\infty \frac{\sin(r)}{r} dr = \frac{\pi}{2} \end{aligned}$$

□



### 17.7.5 Integration Along a Branch Cut

TLDR: Integration along both sides of a branch cut does not cancel out, due to different arguments ( $\theta$  and  $\theta + 2\pi$ ). We must do integration along branch cut with these different values of arguments taken into consideration.

**Example 17.7.5** Show for  $a, x \in \mathbb{R}$  with  $x > 0$  and  $a \in (0, 1)$ :

$$\int_0^\infty \frac{x^{-a}}{x+1} dx = \frac{\pi}{\sin(a\pi)}$$

*Proof:* Let  $C_\rho$  and  $C_R$  denote the circular contours with radius  $\rho$  and  $R$ , respectively, with  $\rho < 1 < R$ . We will integrate

$$f(z) = \frac{z^{-a}}{z+1} \quad |z| > 0, \arg(z) \in (0, 2\pi)$$

along the simple closed contour composed of a line from  $\rho$  to  $R$ , positively along  $C_R$ , line from  $R$  to  $\rho$ , and negatively along  $C_\rho$  back to  $\rho$ . The lines are taken along a branch cut, so we have

$$f(z) = \frac{\exp[-a \log(z)]}{z+1} = \frac{\exp[-a(\ln(r) + i\theta)]}{re^{i\theta} + 1} = \begin{cases} \frac{\exp[-a(\ln(r) + i0)]}{r+1} = \frac{r^{-a}}{r+1} & \theta = 0 \\ \frac{\exp[-a(\ln(r) + i2\pi)]}{r+1} = \frac{r^{-a}e^{-i2a\pi}}{r+1} & \theta = 2\pi \end{cases}$$

for paths above and below the branch cut. Then by Cauchy's Residue Theorem (theorem 17.1.1):

$$\int_{\rho}^R \frac{r^{-a}}{r+1} dr + \int_{C_R} f(z) dz - \int_{\rho}^R \frac{r^{-a} e^{-2a\pi}}{r+1} dr + \int_{C_{\rho}} f(z) dz = 2\pi i \operatorname{Res}[f(z), -1]$$

Note: This equation is only formal since  $f(z)$  is not analytic or defined on the branch cut, but it is valid as we will soon see later in the next example. Now, since there is an isolated singularity at  $z = -1$ , we define a function  $\phi(z)$ :

$$\begin{aligned} \phi(z) &= z^{-a} = \exp[-a \log(z)] = \exp[-a(\ln(r) + i\theta)] \quad r > 0, \theta \in (0, 2\pi) \\ \implies \phi(-1) &= \exp[-a(\ln(1) + i\pi)] = e^{-ia\pi} \neq 0 \end{aligned}$$

By theorem 17.3.1:

$$\operatorname{Res}[f(z), -1] = e^{-ia\pi}$$

We then get:

$$(1 - e^{-i2a\pi}) \int_{\rho}^R \frac{r^{-a}}{r+1} dr + 2\pi i e^{-ia\pi} - \int_{C_{\rho}} f(z) dz - \int_{C_R} f(z) dz$$

Using the ML Inequality (theorem 15.4.1):

$$\begin{aligned} \left| \int_{C_{\rho}} f(z) dz \right| &\leq \frac{\rho^{-a}}{1-\rho} \cdot 2\pi \rho = \frac{\rho}{\rho^a(1-\rho)} \implies \lim_{\rho \rightarrow \infty} \left| \int_{C_{\rho}} f(z) dz \right| = 0 \\ \left| \int_{C_R} f(z) dz \right| &\leq \frac{R^{-a}}{R-1} \cdot 2\pi R = \frac{2\pi R}{R-1} \cdot \frac{1}{R^a} \implies \lim_{\rho \rightarrow \infty} \left| \int_{C_R} f(z) dz \right| = 0 \end{aligned}$$

Hence

$$\int_0^{\infty} \frac{r^{-a}}{r+1} dr = 2\pi i e^{-ia\pi} \implies \int_0^{\infty} \frac{r^{-a}}{r+1} dr = 2\pi i \frac{e^{-ia\pi}}{1 - e^{-i2a\pi}} = 2\pi i \frac{1}{e^{ia\pi} - e^{-ia\pi}} = \frac{\pi}{\sin(a\pi)}$$

□





**Example 17.7.6** Why the integrals in the previous example is valid, despite  $f(z)$  being strictly formal and not analytic or defined on the branch cut.

*Proof:* Let  $\pi < \theta_0 < 3\pi/2$ . Consider two simple closed contours  $C_1$  and  $C_2$ .  $C_1$  is composed of line from  $\rho$  to  $R$ , semicircular contour  $\Gamma_R$  with radius  $R > 1$  from  $\theta = 0$  to  $\theta = \theta_0$ , line  $L$  on  $\theta = \theta_0$ , and semicircular contour  $\Gamma_\rho$  with radius  $\rho < 1$  from  $\theta_0$  to 0.  $C_2$  a contour with line from  $R$  to  $\rho$ , semicircular contour  $\gamma_\rho$  with radius  $\rho < 1$  from  $2\pi$  to  $\theta_0$ , line  $-L$  on  $\theta_0$ , and semicircular contour with radius  $\gamma_R$  from  $\theta_0$  to 0.

Applying  $f_1(z)$  to  $C_1$ :

$$f_1(z) = \frac{z^{-a}}{z+1} \quad |z| > 0, \quad -\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}$$

Then knowing  $z = re^{i0} = r$  on the line from  $\rho$  to  $R$ , and using Cauchy's Residue Theorem (theorem 17.1.1):

$$\begin{aligned} \int_{C_1} f_1(z) dz &= \int_{\rho}^R \frac{z^{-a}}{z+1} dz + \int_{\Gamma_1} f_1(z) dz + \int_L f_1(z) dz + \int_{\Gamma_\rho} f_1(z) dz \\ &= \int_{\rho}^R \frac{r^{-a}}{r+1} dz + \int_{\Gamma_1} f_1(z) dz + \int_L f_1(z) dz + \int_{\Gamma_\rho} f_1(z) dz = 2\pi i \operatorname{Res}[f_1(z), -1] \end{aligned}$$

Applying  $f_2(z)$  to  $C_2$ :

$$f_2(z) = \frac{z^{-a}}{z+1} \quad |z| > 0, \quad \frac{\pi}{2} < \arg(z) < \frac{5\pi}{2}$$

Then knowing  $z = re^{i2\pi}$  on the line from  $R$  to  $\rho$ , and using Cauchy-Goursat Theorem (theorem 15.6.1):

$$\begin{aligned} \int_{C_2} f_2(z) dz &= \int_{\rho}^R \frac{z^{-a}}{z+1} dz + \int_{\gamma_1} f_2(z) dz + \int_{-L} f_2(z) dz + \int_{\gamma_\rho} f_2(z) dz \\ &= \int_{\rho}^R \frac{r^{-a} e^{-i2a\pi}}{r+1} dz + \int_{\gamma_1} f_2(z) dz + \int_{-L} f_2(z) dz + \int_{\gamma_\rho} f_2(z) dz = 0 \end{aligned}$$

Since the integrals from  $\rho$  to  $R$  is defined and the other integrals are defined for  $(0, 2\pi)$ , we can replace  $f_1(z)$  and  $f_2(z)$  by

$$f(z) = \frac{z^{-a}}{z+1} \quad |z| > 0, \quad \arg(z) \in (0, 2\pi)$$

Adding the integrals over the  $C_1$  and  $C_2$ , the integrals on line  $L$  cancels out due to being opposite orientations, and we get back our original integral.

$$\int_{C_1} f_1(z) dz + \int_{C_2} f_2(z) dz = \int_C f(z) dz$$

□



**Example 17.7.7** (Beta Function) *Show that the Beta Function*

$$B(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt = \frac{\pi}{\sin(p\pi)} \quad p, q \in \mathbb{R}_{>0}$$

*Proof:* Let  $t = (x+1)^{-1}$ , then

$$\begin{aligned} B(p, q) &= \int_0^1 t^{p-1} (1-t)^{q-1} dt = \int_\infty^0 (x+1)^{1-p} [1 - (x+1)^{-1}]^{q-1} [-(x+1)^{-2}] dx \\ &= - \int_\infty^0 (x+1)^{-1-p} \left[ \frac{x}{x+1} \right]^{q-1} dx = \int_0^\infty \frac{x^{q-1}}{(x+1)^{q+p}} dx \end{aligned}$$

For  $q = 1 - p$ , and using our knowledge from before:

$$B(p, 1-p) = \int_0^\infty \frac{x^p}{x+1} dx = \frac{\pi}{\sin(p\pi)}$$

□

## 17.7.6 Indefinite Integrals Involving Sines and Cosines

Evaluating integrals of the type:

$$\int_0^{2\pi} F[\sin(\theta), \cos(\theta)] d\theta$$

Consider a positively oriented circular simple closed contour  $C$  on the unit circle. This suggests a parametric representation:

$$z = e^{i\theta} \quad \theta \in [0, 2\pi]$$

Using the complex representations of sine and cosine (definition 12.5.2):

$$\sin(\theta) = \frac{z - z^{-1}}{2i} \quad \cos(\theta) = \frac{z + z^{-1}}{2}$$

We obtain the contour integral:

$$\int_C \frac{1}{iz} F\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right) dz$$

If the integrand reduces to a rational function of  $z$ , we can use Cauchy's Residue theorem (theorem 17.1.1) to evaluate the integral, provided we know where the singularities are and conditions are satisfied. That is, substitute the trigonometric identities into the integrand and use Cauchy's Residue Theorem.



### 17.7.7 Argument Principle

#### Definition 17.7.5: Meromorphic

*Let  $f$  be a function with domain  $D$ .  $f$  is meromorphic if it is analytic throughout  $D$ , except at the poles.*

Let  $C$  be a positively oriented simple closed contour,  $w = f(z)$  be a function that is analytic and nonzero on  $C$  and meromorphic in domain interior to  $C$ , with  $\Gamma = \text{img}[f(C)]$ . That is,  $\Gamma$  is the image of  $C$  under  $f(z)$ .  $\Gamma$  is also a closed contour, but not necessarily simple.

Since  $f$  is nonzero on  $C$ ,  $\Gamma$  does not pass through the origin.

Suppose we now travel along  $C$  starting on  $z_0$ , likewise, we are travelling on  $\Gamma$  starting at  $w_0$  under the transformation. Eventually, we will come back to  $z_0$ , which is  $w_0$  under the transformation. Let  $\arg(w_0) = \phi_0$  and the new argument after travelling  $\Gamma$  back to  $w_0$  be  $\phi_1$ . It is clear that the change in argument is:

$$\Delta_C \arg[f(z)] = \phi_1 - \phi_0$$

It is clear that  $\Delta_C$  is an integer multiple of  $2\pi$ , which leads us to the winding number.

#### Definition 17.7.6: Winding Number

*Let  $C$ ,  $\Gamma$ ,  $z_0$ , and  $w_0$  be as stated above. The winding number of  $\Gamma$ :*

$$\frac{1}{2\pi} \Delta_C \arg[f(z)] \in \mathbb{Z}$$

Note:  $\Gamma$  does not enclose origin  $\implies \frac{1}{2\pi}\Delta_C \arg[f(z)] = 0$

It is clear that the zeros of  $f$  inside the domain enclosed by  $C$  are points that are mapped to the origin (and poles are those that are mapped to points at  $\infty$ ). Hence, we can use the number of zeros and poles to determine the winding number of  $\Gamma$ .

**Question.** *The number of zeros of  $f$  in  $C$  corresponds to the number of times a point in the domain interior to  $\Gamma$  gets mapped to the origin. It is clear this represents the amount of “loops” around the origin. What do the poles represent? They represent the number of times a point in the domain interior to  $\Gamma$  get mapped to points at infinity, which corresponds to loops around the point at infinity. Therefore, the contour is getting “flipped inside-out”?*

### Theorem 17.7.6: Argument Principle

Let  $C$  be a simple closed contour,  $f(z)$  be a function,  $P$  be number of poles and  $Z$  be number of zeros of  $f(z)$  inside  $C$ . If

1.  $f(z)$  meromorphic in domain interior to  $C$
2.  $f(z)$  analytic and nonzero on  $C$

Then

$$\frac{1}{2\pi}\Delta_C \arg[f(z)] = Z - P$$

*Proof:* Let  $C$  be a simple closed contour and  $z = z(t)$  ( $t \in [a, b]$ ) be a parametric representation for  $C$ . Consider:

$$\int_C \frac{f'(z)}{f(z)} dz = \int_a^b \frac{f'[z(t)]z'(t)}{f[z(t)]} dt$$

$f(z)$  is nonzero on  $C$ , so it does not pass through the origin, thus we can use the exponential form:

$$\begin{aligned} f[z(t)] &= \rho(t)e^{i\phi(t)} \\ \frac{d}{dt}\rho(t)e^{i\phi(t)} &= \rho'(t)e^{i\phi(t)} + i\rho(t)e^{i\phi(t)}\phi'(t) \end{aligned}$$

Substituting:

$$\int_C \frac{f'(z)}{f(z)} dz = \int_a^b \frac{\rho'(t)}{\rho(t)} dt + i \int_a^b \phi'(t) dt = \ln[\rho(t)] \Big|_a^b + i\phi(t) \Big|_a^b$$

Since we have a closed contour:

$$a = b \qquad \rho(a) = \rho(b) \qquad \phi(b) - \phi(a) = \Delta_C \arg[f(z)]$$

Hence

$$\int_C \frac{f'(z)}{f(z)} dz = i\Delta_C \arg[f(z)]$$

Using Cauchy's Residue Theorem (theorem 17.1.1), if  $f$  has a zero of order  $m_0$  at  $z_0$ :

$$\begin{aligned} f(z) &= (z - z_0)^{m_0} g(z) \\ f'(z) &= m_0(z - z_0)^{m_0-1} g(z) + (z - z_0)^{m_0} g'(z) \end{aligned} \quad \text{Theorem 17.4.1}$$

Thus

$$\frac{f'(z)}{f(z)} = \frac{m}{z - z_0} + \frac{g'(z)}{g(z)}$$

$g'(z)/g(z)$  is analytic at  $z_0$ , so it has a Taylor series at  $z_0$ . This implies that  $f'(z)/f(z)$  has a simple pole at  $z_0$  with residue  $m_0$ .

Likewise, if  $f$  has a pole of order  $p_0$  at  $z_0$ . We get

$$f(z) = (z - z_0)^{-p_0} \phi(z) \quad \text{Theorem 17.3.1}$$

Using the same procedure as before,  $f'(z)/f(z)$  has a simple pole at  $z_0$  with residue  $-p_0$ . We know the poles and zeros have to be finite (example 17.5.3 and example 17.7.11), therefore, Cauchy's Residue Theorem (theorem 17.1.1):

$$\int_C \frac{f'(z)}{f(z)} dz = 2\pi i \left[ \sum_{k=0}^N m_k + \sum_{j=0}^J (-p_j) \right]$$

Since  $m_0$  corresponds to the number of zeros at  $z_0$ , and  $p_0$  is the number of poles at  $z_0$ :

$$\left[ Z = \sum_{k=0}^N m_k \right] \wedge \left[ P = \sum_{j=0}^J (-p_j) \right] \implies \int_C \frac{f'(z)}{f(z)} dz = 2\pi i (Z - P)$$

Equating the two expressions for the contour integral together and rearranging:

$$\frac{1}{2\pi} \Delta_C \arg[f(z)] = Z - P$$

□

**Observation.** *The reason we choose to integrate over  $f'(z)/f(z)$  is because*

$$\frac{d}{dz} \log[f(z)] = \frac{f'(z)}{f(z)}$$

*and from the definition of the logarithm (definition 14.2.1):*

$$\log(z) = \ln(|z|) + i \arg(z)$$

*which is how we are able to “extract” the phase component.*

**Example 17.7.8** Consider  $f(z)$  over  $C$ .

$$f(z) = \frac{z^3 + 2}{z = z^2 + \frac{2}{z}}$$

The zeros of the function are  $z = \sqrt[3]{2} > 1$  and the poles are  $z = 0$ . If  $C$  is a contour over the unit circle in the positive direction, then by theorem 17.7.6:

$$\Delta_C \arg[f(z)] = 2\pi(0 - 1) = -2\pi$$

so  $\Gamma$  winds around the origin once in the negative direction.

## 17.7.8 Rouché's Theorem

### Theorem 17.7.7: Rouché's Theorem

Let  $C$  be a simple closed contour, and  $f(z)$  and  $g(z)$  be analytic functions inside and on  $C$ , then

$$\begin{aligned} \forall z \in C [ |f(z)| > |g(z)| ] \\ \implies [f(z)] \wedge [f(z) + g(z)] \text{ have same number of zeros, including multiplicities, inside } C \end{aligned}$$

*Proof:* We know  $\forall z \in C$ :

$$|f(z)| > |g(z)| \geq 0 \qquad |f(z) + g(z)| \geq ||f(z)| - |g(z)|| > 0$$

Letting  $Z_f$  and  $Z_{f+g}$  be the number of zeros, with multiplicities, of  $f(z)$  and  $f(z)+g(z)$  inside  $C$ . From (theorem 17.7.6):

$$Z_f = \frac{1}{2\pi} \Delta_C \arg[f(z)] \qquad Z_{f+g} = \frac{1}{2\pi} \Delta_C \arg[f(z) + g(z)] \qquad z \in C$$

Then

$$\begin{aligned} Z_{f+g} &= \frac{1}{2\pi} \Delta_C \arg[f(z) + g(z)] = \frac{1}{2\pi} \Delta_C \arg \left[ f(z) + \left( 1 + \frac{g(z)}{f(z)} \right) \right] \\ &= \frac{1}{2\pi} \Delta_C \arg[f(z)] + \frac{1}{2\pi} \Delta_C \arg \left[ 1 + \frac{g(z)}{f(z)} \right] \end{aligned}$$

Substituting  $Z_f$ :

$$Z_{f+g} = Z_f + \frac{1}{2\pi} \Delta_C \arg[F(z)] \qquad F(z) = 1 + \frac{g(z)}{f(z)}$$

This implies

$$|F(z) - 1| = \frac{|g(z)|}{|f(z)|} < 1$$

This implies the image of  $C$  under  $F(z)$  lies on open disk  $|F(z) - 1| < 1$ , hence, does not enclose the origin  $F(z) = 0$ . Therefore by example 17.7.10:

$$\Delta_C \arg[F(z)] = 0 \implies Z_{f+g} = Z_f$$

□

Theorem 17.7.7 allows us to locate the number of zeros of a function  $F(z)$  inside a simple closed contour by smartly breaking it up into two functions  $f(z)$  and  $g(z)$ . Then evaluating the number of zeros of either  $f(z)$  or  $g(z)$  inside said contour.

**Example 17.7.9** (Proof of Fundamental Theorem of Algebra via Rouché's Theorem) *Fundamental Theorem of Algebra: (Theorem 15.8.2).*

Consider a polynomial:

$$P(z) = \sum_{k=0}^n a_k z^k \quad a_n \neq 0$$

Let

$$f(z) = a_n z^n \quad g(z) = \sum_{k=0}^{n-1} a_k z^k \quad |z| = R > 1$$

Then

$$|f(z)| = |a_n| R^n \quad |g(z)| \leq \sum_{k=0}^{n-1} |a_k| R^k \leq \sum_{k=0}^{n-1} |a_k| R^{n-1} \quad R > 1$$

Then

$$\frac{|g(z)|}{|f(z)|} \leq \sum_{k=0}^{n-1} \frac{|a_k|}{|a_n| R} < 1 \implies \sum_{k=0}^{n-1} \frac{|a_k|}{|a_n|} < R$$

Therefore

$$R > 1 \implies |f(z)| > |g(z)|$$

Now that  $|f(z)| > |g(z)|$  is established, we can use Rouché's Theorem. Which tells us that  $f(z)$  and  $f(z) + g(z) = P(z)$  has the same number of zeros. It is clear that  $f(z) = a_n z^n$  has  $n$  zeros, which implies  $P(z)$  has  $n$  zeros.

Note: Liouville's Theorem (theorem 15.8.1) ensures the existence of at least one zero for a polynomial, while Rouché's Theorem ensures the existence of  $n$  zeros, including multiplicities.

**Example 17.7.10** Let  $\Gamma$  be a closed contour that is the image of a simple closed contour  $C$  under the transformation  $f(z)$ , and  $\Gamma$  does not enclose the origin  $w = 0$ , show

$$\Delta_C \arg[f(z)] = 0$$

*Proof:*

$\Gamma$  does not enclose  $w = 0$

$$\implies \forall z \in \Gamma [0 \leq \arg[f(z)] - \arg[f(z_0)] < 2\pi]$$

$$\implies 0 \leq \Delta_C \arg[f(z)] < 1$$

$$\implies \Delta_C \arg[f(z)] = 0 \qquad \Delta_C \arg[f(z)] \in \mathbb{Z}$$

□

**Example 17.7.11** ( $\forall z \in D, f(z) \neq 0 \implies$  zeros of  $f$  in  $D$  finite in order and number ) Let  $f$  be a meromorphic function in domain  $D$  interior to a simple closed contour  $C$  which  $f$  is analytic and nonzero. Suppose  $D_0$  consists of all points in  $D$  except for the poles. Show that if  $\forall z \in D[f(z) \neq 0]$ , then zeros of  $f$  in  $D$  are all finite in order and number.

Note:  $z_0 \in D$  is a zero of  $f$  that is not finite in order  $\implies \exists \epsilon > |z - z_0| [f(z) \equiv 0]$

*Proof:* Suppose  $\exists z_0 \in D$  where  $z_0$  is a zero of  $f$  that is not finite in order

$$\implies \exists \epsilon > |z - z_0| [f(z) \equiv 0] \qquad \text{Note}$$

$$\implies \forall z \in D [f(z) \equiv 0] \qquad \text{Lemma 13.8.0.1}$$

$$\implies \text{Contradiction with having poles in } D$$

By example 17.5.2: If  $f$  is analytic in region  $R$  and  $\forall z \in D_0 [f(z) \neq 0]$ , then zeros in  $R$  are finite in order and number. □

**Example 17.7.12** Let function  $f$  be analytic inside and on a positively oriented simple closed contour  $C$ , with no zeros in  $C$ . Show if  $f$  has  $n$  zeros  $z_k$  ( $k = 1, 2, \dots, n$ ) inside  $C$  with multiplicity  $m_k$ , then

$$\int_C \frac{zf'(z)}{f(z)} dz = 2\pi i \sum_{k=1}^n m_k z_k$$

*Proof:*  $f(z)$  has zero of order  $m_k$  at  $z_k$

$$\implies f(z) = (z - z_k)^{m_k} g(z) \qquad \text{Theorem 17.4.1}$$

$$\implies f'(z) = m_k (z - z_k)^{m_k-1} g(z) + (z - z_k)^{m_0} g'(z)$$

Then

$$z \left[ \frac{f'(z)}{f(z)} \right] = z \left[ \frac{m_k}{z - z_k} + \frac{g'(z)}{g(z)} \right] = \frac{m_k(z - z_k) + m_k z_k}{z - z_k} + \frac{zg'(z)}{g(z)} = m_k + \frac{m_k z_k}{z - z_k} + \frac{zg'(z)}{g(z)}$$

Since  $g(z)$  is analytic at  $z_k$ ,  $g'(z)/g(z)$  has a Taylor series at  $z_0$ , so it is clear that:

$$\text{Res} \left[ \frac{zf'(z)}{f(z)}, z_k \right] = m_k z_k$$



By Cauchy's Residue theorem (theorem 17.1.1):

$$\int_C \frac{zf'(z)}{f(z)} dz = 2\pi i \sum_{k=1}^n m_k z_k$$

□

**Example 17.7.13** Show if  $c \in \mathbb{C}$ ,  $|c| > e$ , then  $cz^n = e^z$  has  $n$  roots, including multiplicities, inside  $|z| = 1$ .

*Proof:* Rearranging, we have:

$$cz^n - e^z = 0$$

Then

$$|cz^n||c| > e \qquad |e^z| = |e^x e^{iy}| = |e^x| = |e^{\cos(\theta)}| < e$$

Letting  $f(z) = cz^n$  and  $g(z) = -e^z$ , by Rouché's theorem (theorem 17.7.7):

$$f(z) \text{ has } n \text{ zeros} \implies f(z) + g(z) \text{ has } n \text{ zeros} \implies e^z \text{ has } n \text{ zeros}$$

□

**Example 17.7.14** (Rouché Theorem Alternate Proof) Let  $f(z)$  and  $g(z)$  be functions with  $|f(z)| > |g(z)|$ , and  $C$  be a positively oriented simple closed contour. Consider

$$\Phi(t) = \frac{1}{2\pi i} \int_C \frac{f'(z) + tg'(z)}{f(z) + tg(z)} dz \qquad t \in [0, 1]$$

*Prove Rouché Theorem.*

*Proof:* Prove  $\Phi(t)$  exists:

We have  $|f(z)| > |g(z)|$ ,  $C$  is as positively oriented simple closed contour, and

$$\Phi(t) = \frac{1}{2\pi i} \int_C \frac{f'(z) + tg'(z)}{f(z) + tg(z)} dz \qquad t \in [0, 1]$$

Since  $\forall z \in C [|f(z)| > |g(z)|]$ :

$$\begin{aligned} \implies |f(z)| > |g(z)| &\leq |tg(z)| = |t||g(z)| & t \in [0, 1] \\ \implies \text{Integrand denominator is never zero} \\ \implies \Phi(t) \text{ exists} \end{aligned}$$

Prove  $\Phi(t)$  continuous:

Let  $t, t_0 \in [0, 1]$ :

$$\begin{aligned} &\Phi(t) - \Phi(t_0) \\ &= (2\pi i)^{-1} \int_C \frac{f' + tg'}{f + tg} - \frac{f' + t_0 g'}{f + t_0 g} dz \\ &= (2\pi i)^{-1} \int_C \frac{[f + t_0 g][f' + tg'] + [f + tg][f' + t_0 g']}{[f + tg][f + t_0 g]} dz \end{aligned}$$

Now

$$\begin{aligned}[f + t_0g][f' + tg'] &= ff' + tf'g' + t_0f'g + tt_0gg' \\ [f + tg][f' + t_0g'] &= ff' + tf'g' + t_0fg' + tt_0gg'\end{aligned}$$

So

$$[f + t_0g][f' + tg'] + [f + tg][f' + t_0g'] = (t - t_0)fg' + (t_0 - t)f'g$$

Hence

$$\begin{aligned}\Phi(t) - \Phi(t_0) &= (2\pi i)^{-1}(t - t_0) \int_C \frac{fg' - f'g}{(f + tg)(f + t_0g)} dz \\ \implies |\Phi(t) - \Phi(t_0)| &= (2\pi i)^{-1}|t - t_0| \left| \int_C \frac{fg' - f'g}{(f + tg)(f + t_0g)} dz \right|\end{aligned}$$

Now

$$\frac{fg' - f'g}{(f + tg)(f - t_0g)} = \frac{fg' - f'g}{f^2 + tt_0g^2 + tfg + t_0fg}$$

Since  $t, t_0 \in [0, 1]$ :

$$|tt_0g^2| < |g^2| \quad |tfg| < |fg| \quad |t_0fg| < |fg|$$

Therefore

$$\begin{aligned}|f^2 + tt_0g^2 + tfg + t_0fg| &\leq |f^2| + |g^2| + |2fg| = (|f| + |g|)^2 \\ \implies \left| \frac{fg' - f'g}{(f + tg)(f + t_0g)} \right| &\leq \frac{|fg' - f'g|}{(|f| + |g|)^2} \\ \implies |\Phi(t) - \Phi(t_0)| &\leq (2\pi i)^{-1}|t - t_0| \int_C \frac{|fg' - f'g|}{(|f| + |g|)^2} dz < |t - t_0| \int_C \frac{|fg' - f'g|}{(|f| + |g|)^2} dz\end{aligned}$$

Since  $C$  is closed and bounded, and  $(fg' + f'g)(|f| + |g|)^{-2}$  is continuous on  $C$ , by theorem 13.3.5,  $(fg' + f'g)(|f| + |g|)^{-2}$  is bounded on  $C$ . Therefore

$$\exists A \in \mathbb{R}_{>0} [|\Phi(t) - \Phi(t_0)| \leq A|t - t_0|] < \epsilon$$

We can choose this neighbourhood  $\epsilon$  to be arbitrarily small, so  $\Phi(t)$  is continuous by definition (chapter 10).

Prove Rouché Theorem (theorem 17.7.7):

Let  $F(z) = f(z) + tg(z)$ , in the proof of (theorem 17.7.6), we know that:

$$\int_C \frac{F'(z)}{F(z)} dz = 2\pi i(Z - P)$$

Hence

$$\begin{aligned}\Phi(t) &= (2\pi i)^{-1} \int_C \frac{f'(z) + tg(z)}{f(z) + tg(z)} dz = (2\pi i)^{-1} \int_C \frac{F'(z)}{F(z)} dz = Z - P \\ &= \frac{1}{2\pi} \Delta_C \arg[f(z) + tg(z)]\end{aligned}$$

We can then see that  $\Phi(t)$  represents the difference of the number of zeros and numbers of poles of  $f(z) + tg(z)$ . This number is independent of  $t$ , hence

$$\Phi(1) = \frac{1}{2\pi} \Delta_C \arg[f(z) + g(z)] = Z - P$$

$$\Phi(0) = \frac{1}{2\pi} \Delta_C \arg[f(z)] = Z - P$$

Thus, the number of zeros for  $f(z)$  and  $f(z) + g(z)$  are equal given  $|f(z)| > |g(z)|$ , proving Rouché Theorem.  $\square$

### 17.7.9 Inverse Laplace Transforms

#### Definition 17.7.7: Bromwich Integral

Let  $F(s)$  be a complex function, and  $L_R$  be a vertical line segment from  $s = \gamma - iR$  to  $s = \gamma + iR$  large enough that singularities of  $F$  lie to the left of  $L_R$ . Consider

$$f(t) = \frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{L_R} e^{st} F(s) ds \quad t > 0$$

If the limit exists, then the Bromwich Integral:

$$f(t) = \frac{1}{2\pi i} \text{P.V.} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s) ds \quad t > 0$$

The Bromwich integral is the inverse Laplace transform.

#### Definition 17.7.8: Laplace Transform

$$F(s) = \int_0^\infty e^{-st} f(t) dt$$



### Deriving the Inverse Laplace Transform.

Let  $F(s)$  and  $L_R$  be defined as above, and  $C_R$  be a semicircular contour from  $\gamma + iR$  to  $\gamma - iR$  enveloping the zeros of  $F(s)$  ( $s_k$ ,  $k = 1, 2, \dots, n$ ). By Cauchy's Residue Theorem (theorem 17.1.1):

$$\int_{L_R} e^{st} F(s) ds = 2\pi i \sum_{n=1}^N \text{Res}[e^{st} F(s), s_n] - \int_{C_R} e^{st} F(s) ds$$

Assuming

$$\lim_{R \rightarrow \infty} \int_{C_R} e^{st} F(s) ds = 0$$

Then

$$f(t) = \frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{L_R} e^{st} F(s) ds = \sum_{n=1}^N \text{Res}[e^{st} F(s), s_n] \quad t > 0$$

**Example 17.7.15** Let

$$F(s) = \frac{s}{s^2 + 4} = \frac{s}{(s + 2i)(s - 2i)}$$

Then

$$f(t) = \sum_{n=1}^N \text{Res}[e^{st} F(s), s_n] = \text{Res}\left[\frac{e^{st}s}{(s + 2i)(s - 2i)}, 2i\right] + \text{Res}\left[\frac{e^{st}s}{(s + 2i)(s - 2i)}, -2i\right]$$

The singularities are simple poles, so by theorem 17.3.1:

$$\begin{aligned} f(t) &= \text{Res}\left[\frac{\phi_1(s)}{s - 2i}, 2i\right] + \text{Res}\left[\frac{\phi_2(s)}{s + 2i}, -2i\right] & \phi_1(s) &= \frac{e^{st}s}{s + 2i} & \phi_2(s) &= \frac{e^{st}s}{s - 2i} \\ &= \phi_1(2i) + \phi(-2i) = \frac{e^{2it}(2i)}{4i} + \frac{e^{-2it}(-2i)}{-4i} \\ &= \frac{e^{i2t} + e^{-i2t}}{2} = \cos(2t) \end{aligned}$$

**Example 17.7.16** Find the inverse Laplace transform of

$$F(s) = \frac{1}{s^2} - \frac{1}{s \sinh(s)}$$

*Proof:* The isolated singularities of  $F(s)$ :

$$s_0 = 0 \quad s_n = n\pi i \quad \overline{s_n} = -n\pi i \quad n \in \mathbb{N}$$

Obtaining the series expansion:

$$\sinh(z) = \frac{e^z - e^{-z}}{2} = \frac{1}{2} \left[ \sum_{n=0}^N \frac{z^n}{n!} - \frac{(-z)^n}{n!} \right] = \frac{1}{2} \left[ \sum_{n=0}^N \frac{z^n}{n!} [1 - (-1)^n] \right] = \sum_{n=0}^N \frac{z^{2n+1}}{(2n+1)!}$$

Then

$$\frac{1}{z \sinh(z)} = \frac{1}{z} \left[ \sum_{n=0}^N \frac{z^{2n+1}}{(2n+1)!} \right]^{-1} = \frac{1}{z} \left( \frac{1}{z} - \frac{z}{3!} - \frac{z^4}{5! - 3!3!} - \dots \right) = \frac{1}{z^2} - \frac{1}{3!} - \dots$$

Hence

$$F(s) = \frac{1}{s^2} - \frac{1}{s \sinh(s)} = \frac{1}{3!} - \frac{z^3}{5! - 3!3!} - \dots$$

Thus, we can see that  $F(s)$  has a removable singularity at  $s_0 = 0$  with  $\text{Res}[F(s), 0] = 0$ .

Now, let  $p(s) = e^{st}$  and  $q(s) = [F(s)]^{-1}$ , then

$$\begin{aligned} q'(s) &= \frac{d}{ds} [F(s)]^{-1} = \frac{d}{ds} \frac{s^2 \sinh(s)}{\sinh(s) - s} \\ &= \frac{2s \sinh(s) + s^2 \cosh(s)}{\sinh(s) - s} - \frac{[s^2 \sinh(s)][\cosh(s) - 1]}{[\sinh(s) - s]^2} \\ &= \frac{2s \sinh^2(s) - s^2 \sinh(s) - s^3 \cosh(s)}{[\sinh(s) - s]^2} \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{q'(n\pi i)} &= \frac{[\sinh(s) - s]^2}{2s \sinh^2(s) - s^2 \sinh(s) - s^3 \cosh(s)} \\ &= \frac{[\sinh(n\pi i) - n\pi i]^2}{2(n\pi i) \sinh^2(n\pi i) - (n\pi i)^2 \sinh(n\pi i) - (n\pi i)^3 \cosh(n\pi i)} \\ &= \frac{(n\pi i)^2}{-(n\pi i)^3 \cosh(n\pi i)} = \frac{-1}{(n\pi i)(-1)^n} = \frac{(-1)^n i}{n\pi} \\ \frac{1}{q'(-n\pi i)} &= -\frac{(-1)^n i}{n\pi} \end{aligned}$$

Since

$$\sinh(n\pi i) = \sin(n\pi) = 0 \quad \cosh(n\pi i) = \cos(n\pi) = (-1)^n$$

Also

$$\begin{aligned} p(n\pi i) &= e^{n\pi i} \neq 0 & q(n\pi i) &= \frac{(n\pi i)^2 \sinh(n\pi i)}{\sinh(n\pi i) - n\pi i} = 0 \\ p(-n\pi i) &= e^{-n\pi i} \neq 0 & q(-n\pi i) &= \frac{(-n\pi i)^2 \sinh(-n\pi i)}{\sinh(-n\pi i) + n\pi i} = 0 \end{aligned}$$

Thus, we can see that  $e^{st}F(s)$  has a simple pole at  $s_n = n\pi i$  and  $\overline{s_n} = -n\pi i$  ( $n \in \mathbb{N}$ ), so by theorem 17.5.2:

$$\begin{aligned} \text{Res}[e^{st}F(s), n\pi i] &= \text{Res}\left[\frac{p(s)}{q(s)}, n\pi i\right] = \frac{p(n\pi i)}{q'(n\pi i)} = e^{n\pi i t} \frac{(-1)^n i}{n\pi} \\ \text{Res}[e^{st}F(s), -n\pi i] &= \text{Res}\left[\frac{p(s)}{q(s)}, -n\pi i\right] = \frac{p(-n\pi i)}{q'(-n\pi i)} = -e^{-n\pi i t} \frac{(-1)^n i}{n\pi} \end{aligned}$$

The inverse Laplace transform of  $F(s)$  is then

$$\begin{aligned}
 f(t) &= \sum_{n=1}^{\infty} \text{Res}[e^{st}F(s), s_n] & t > 0 \\
 &= \sum_{n=1}^{\infty} (\text{Res}[e^{st}F(s), n\pi i] + \text{Res}[e^{st}F(s), -n\pi i]) \\
 &= \sum_{n=1}^{\infty} \left( e^{n\pi i t} \frac{(-1)^n i}{n\pi} - e^{-n\pi i t} \frac{(-1)^n i}{n\pi} \right) = \sum_{n=1}^{\infty} \frac{(-1)^n i}{n\pi} (2i) \sin(n\pi t) \\
 &= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\pi t)
 \end{aligned}$$

□

## 17.7.10 Hilbert Transform

### Definition 17.7.9: Hilbert Transform

Let  $g(t)$  be a real-valued function. The Hilbert Transform:

$$\mathcal{H}[g(t)] = \hat{g}(t) = \text{P. V.} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(x)}{t-x} dx \quad x, t \in \mathbb{R}$$

### Definition 17.7.10: Hilbert Transform Pairs

The functions  $g(t) = \mathcal{H}^{-1}[\hat{g}(t)]$  and  $\hat{g}(t) = \mathcal{H}[g(t)]$  in a Hilbert Transform.

### Theorem 17.7.8: Inverse Hilbert Transform

The inverse Hilbert transform of function  $\hat{g}(t)$ :

$$\mathcal{H}^{-1}[\hat{g}(t)] = g(t) = -\frac{1}{\pi} \text{P. V.} \int_{-\infty}^{\infty} \frac{\hat{g}(x)}{t-x} dx$$

*Proof:* Let  $f(z) = u(x, y) + iv(x, y)$  be a function analytic in the upper half-plane ( $y = \text{Im}(z) \geq 0$ ), and  $x = t \in \mathbb{R}$ . Consider the simple closed contour  $C$  composed of a line  $L_1$  from point  $-R$  to  $\tau - \epsilon$  on the real axis, a semicircular contour  $C_\epsilon$  in the upper half-plane with radius  $\epsilon$  centred at  $t$ , a line  $L_2$  from  $\tau + \epsilon$  to  $R$ , and a semicircular contour centred at the origin with radius  $R$ .

From the Cauchy-Goursat Theorem (theorem 15.6.1):

$$\frac{1}{\pi} \int_{-R}^{\tau-\epsilon} \frac{f(x)}{t-x} dx + \frac{1}{\pi} \int_{C_\epsilon} \frac{f(z)}{t-z} dz + \frac{1}{\pi} \int_{\tau+\epsilon}^R \frac{f(x)}{t-x} dx + \frac{1}{\pi} \int_{C_R} \frac{f(z)}{t-z} dz = 0$$

Looking at  $C_\epsilon$ , and using theorem 17.7.5 and theorem 17.5.2:

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \int_{C_\epsilon} \frac{f(z)}{t-z} dz = -\frac{\pi i}{\pi} \text{Res} \left[ \frac{f(z)}{t-z}, t \right] = i f(t)$$

Assuming the integral over  $C_R$  disappears, that is

$$\lim_{R \rightarrow \infty} \frac{1}{\pi} \int_{C_R} \frac{f(z)}{t-z} dz = 0$$

Taking the limit as  $R \rightarrow \infty$  and  $\epsilon \rightarrow 0$  of  $C$ :

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x,0)}{t-x} dx + if(t) = 0 \implies \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(x,0) + iv(x,0)}{t-x} dx = -iu(t,0) + v(t,0)$$

Equating the real and imaginary parts:

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(x,0)}{t-x} dx = v(t,0) \qquad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{v(x,0)}{t-x} dx = -u(t,0)$$

Taking the limit as  $R \rightarrow \infty$  Setting  $g(x) = g(x,0)$  and  $\hat{g}(x) = h(t,0)$ , we obtain the Hilbert and Inverse Hilbert transforms:

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(x)}{t-x} dx = \hat{g}(t) \qquad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{g}(x)}{t-x} dx = -g(t)$$

□

Note: The inverse Hilbert Transform is valid if there exists an analytic function  $f(z) = u(x,y) + iv(x,y)$  in the upper half-plane ( $\text{Im}\{z\} \geq 0$ ) with  $u(x,0) = g(x)$  and  $v(x,0) = \hat{g}(x)$ , or in the lower half-plane ( $\text{Im}\{z\} \leq 0$ ) with  $u(x,0) = g(x)$  and  $v(x,0) = -\hat{g}(x)$  such that

$$\lim_{R \rightarrow \infty} \frac{1}{\pi} \int_{C_R} \frac{f(z)}{t-z} dz = 0$$

**Observation.** The Hilbert transform is the convolution of the function  $g(x)$  with  $h(\tau) = 1/(\pi\tau)$ . That is

$$\mathcal{H}[g(t)] = g(t) * \frac{1}{\pi t} = \int_{-\infty}^{\infty} g(\tau) h(t-\tau) d\tau \qquad h(\tau) = \frac{1}{\pi t}$$

### Definition 17.7.11: Analytic Signal

A complex-valued function which the imaginary part is the Hilbert transform of its real part. That is

$$f(t) = g(t) + i\hat{g}(t)$$

**Example 17.7.17** (Hilbert Transform of a Constant is Zero)

$$\text{P. V. } \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{c}{t-x} dx = 0 \qquad c, x, t \in \mathbb{R}$$

*Proof:*

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{c}{t-x} dx = \frac{c}{\pi} \int_{-\infty}^{\infty} \frac{1}{t-x} dx$$

Now, set up the integral so it does not become indefinite:

$$\lim_{R \rightarrow \infty} \left[ \int_{1/R}^R \frac{1}{t-x} dx + \int_{-R}^{-1/R} \frac{1}{t-x} dx \right] = 0 \implies \frac{c}{\pi} \int_{-\infty}^{\infty} \frac{1}{t-x} dx = 0$$

□

**Lemma 17.7.8.1: Fourier Transform of Hilbert Transform**

Let  $\mathcal{F}$  denote a Fourier transform, and  $\text{sgn}$  denote the signum function (function that extracts a sign).

$$\mathcal{F}[\mathcal{H}[g(t)]] = -i \text{sgn}(\omega) \mathcal{F}[g(t)]$$

*Proof:*

$$\begin{aligned} \mathcal{F}[\hat{g}(t)] &= \mathcal{F}\left[g(t) * \frac{1}{\pi t}\right] = \mathcal{F}[g(t)] \cdot \mathcal{F}\left[\frac{1}{\pi t}\right] \\ &= 2\pi \mathcal{F}[g(t)] \left[-\frac{i}{2\pi} \text{sgn}(\omega)\right] = -i \text{sgn}(\omega) \mathcal{F}[g(t)] \end{aligned}$$

□

Another way of obtaining the inverse Hilbert transform is by multiplying both sides of the Fourier transform by  $i \text{sgn}(\omega)$ :

$$\begin{aligned} \text{sgn}^2(\omega) \mathcal{F}[g(t)] &= i \mathcal{F}[\hat{g}(t)] \text{sgn}(\omega) \\ \implies \mathcal{F}[g(t)] &= i \mathcal{F}[\hat{g}(t)] \text{sgn}(\omega) & \text{sgn}^2(\omega) \in \{0, 1\} \\ \implies g(t) &= \mathcal{F}^{-1}[i \mathcal{F}[\hat{g}(t)] \text{sgn}(\omega)] \\ \implies g(t) &= \frac{1}{2\pi} \mathcal{F}^{-1}[\mathcal{F}[g(t)]] * \mathcal{F}^{-1}[i \text{sgn}(\omega)] & \mathcal{F}\left[\frac{1}{t}\right] = -\frac{i}{2} \text{sgn}(\omega) \\ \implies g(t) &= \frac{1}{2\pi} g(t) * \frac{-2}{t} = \frac{-1}{\pi} \int_{-\infty}^{\infty} \frac{\mathcal{H}[g(t)]}{t-x} dx \end{aligned}$$

**Lemma 17.7.8.2: Modulus of Fourier Transform of Hilbert Transform**

The magnitude of the Hilbert transform is preserved in a Fourier transform.

$$|\mathcal{F}[\mathcal{H}[g(t)]]| = |\mathcal{F}[g(t)]|$$

*Proof:* From the Fourier transform,  $|-i \text{sgn}(\omega)| = 1$ , therefore, the answer is obvious. □

**Theorem 17.7.9: Fourier Transform of an Analytic Signal**

Consider the Fourier transform of an analytic signal  $g_a(t) = g(t) + i\hat{g}(t)$ . Then

$$\mathcal{F}[g_a(t)] = \mathcal{F}[g(t)] + \text{sgn}(\omega) \mathcal{F}[g(t)]$$

*Proof:*

$$\begin{aligned} \mathcal{F}[g_a(t)] &= \mathcal{F}[g(t)] + \mathcal{F}[i\hat{g}(t)] \\ &= \mathcal{F}[g(t)] + \text{sgn}(\omega) \mathcal{F}[g(t)] & \mathcal{F}[\hat{g}(t)] = -i \text{sgn}(\omega) \mathcal{F}[g(t)] \end{aligned}$$

□



It is clear that:

$$\mathcal{F}[g_a(t)] = \begin{cases} 2\mathcal{F}[g(t)] & \omega > 0 \\ 0 & \omega < 0 \end{cases}$$

**Corollary 17.7.9.1:**

$$g_a(t) = g(t) + i\hat{g}(t) = \int_0^\infty 2\mathcal{F}[g(t)]e^{i\omega t}d\omega$$

## Causality

### Definition 17.7.12: Transfer/System Function

Consider the function:

$$Y(\omega) = G(\omega)X(\omega)$$

$G(\omega)$  is the transfer function (aka. excitation).  $X(\omega)$  is the excitation.  $Y(\omega)$  is the output.

### Definition 17.7.13: Green's Function / Impulse Response

Let  $G(\omega)$  be a transfer function. The impulse response of the system is the inverse Fourier transform of the transfer function:

$$g(t) = \int_{-\infty}^{\infty} G(\omega)e^{i\omega t}d\omega$$

### Definition 17.7.14: Causal System

A system with impulse response  $\forall t < 0 [g(t) = 0]$

### Theorem 17.7.10: Hilbert Transform and Causal Systems

Let  $G(\omega)$  be a transfer function and  $G_e(\omega)$  be an even transfer function. A transfer function of a causal system must be of the form:

$$G(\omega) = G_e(\omega) - iG_e(\omega) * \frac{1}{\pi\omega} = G_e(\omega) - \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{G_e(\omega')}{\omega - \omega'}d\omega'$$

*Proof:* Any real-valued function  $g$  is composed of a even  $g_e$  and odd  $g_o$  part.

$$g(t) = g_e(t) + g_o(t)$$

Then

$$g(t) = g_e(t) + g_o(t)$$

$$g_o(t) = \text{sgn}(t)g_e(t)$$

Applying the Fourier transform:

$$\begin{aligned} \mathcal{F}[g(t)] &= \mathcal{F}[g_e(t)] - i\mathcal{F}[g_e(t)] * \frac{1}{\pi i} & \mathcal{F}[\text{sgn}(t)] &= -\frac{1}{\pi\omega} \\ &= G_e(\omega) - \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{G_e(\omega')}{\omega - \omega'}d\omega' \end{aligned}$$

□

### 17.7.11 Gamma Function

#### Definition 17.7.15: Uniform Convergence of Integrals

We say the integral

$$\int_a^\infty f(s, z) ds$$

converges uniformly to  $F(z)$  in a closed region  $R$ , if

$$\forall z \in R, \forall \epsilon > 0, \exists \tau \in \mathbb{R} \left[ b \geq \tau \implies \left| F(z) - \int_a^b f(t, z) dz \right| < \epsilon \right]$$

( $\tau$  is independent of  $z$ , as in the requirement for uniform convergence (??))

#### Theorem 17.7.11: Analyticity of Function Defined by Integration to Infinity

Let  $f(z, t)$  be continuous for  $t \in [a, b]$ , for  $z$  interior a simple closed contour  $C$ .

$$\begin{aligned} \int_a^\infty f(t, z) dz \text{ converges uniformly to } F(z) \text{ for } z \text{ interior to } C \\ \implies F(z) \text{ analytic in domain with boundary } C \end{aligned}$$

also in this domain

$$F^{(n)}(z) = \int_a^\infty \frac{\partial^n}{\partial z^n} f(t, z) dt \quad n \in \mathbb{N}$$

#### Theorem 17.7.12: M-Test for Integration to Infinity

Let  $f(z, t)$  be continuous for  $t \geq a$  for all  $z \in R$ , where  $R$  is a closed region, and  $M(t)$  be a positive function of  $t$  be independent of  $z$ . Suppose  $\forall z \in R, \forall t \geq a [ |f(z, t)| \leq M(t) ]$ :

$$\begin{aligned} \forall z \in R, \forall t \geq a [ |f(z, t)| \leq M(t) ] \wedge \int_a^\infty M(t) dt \text{ Converges} \\ \implies \forall z \in R \left[ \int_a^\infty f(t, z) dt \text{ Converges uniformly} \right] \end{aligned}$$

That is, if  $|f(z, t)|$  is bounded by a function  $M(t)$  and the integral of  $M(t)$  converges, then the integral of  $f(t, z)$  in the same interval converges uniformly.

#### Definition 17.7.16: Gamma Function

The Gamma function is the analytic function:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad t \in \mathbb{R}$$

The gamma function is continuous and converges for all  $x \in \mathbb{R}_{>0}$ . If  $x \in (\mathbb{K}, \mathbb{K})$ , then it is discontinuous at  $t = 0$ .

#### Theorem 17.7.13: Gamma Function and Factorials

$$\Gamma(n+1) = n! \quad n \in \mathbb{N}$$

*Proof:* Consider  $x + 1$ , then applying integration by parts:

$$\Gamma(x + 1) = \int_0^\infty t^x e^{-t} dt \implies \Gamma(x + 1) = -e^{-t} t^x \Big|_{t=0}^\infty + x \int_0^\infty t^{x-1} e^{-t} dt$$

We know that

$$\begin{aligned} e^{-t} t^x \Big|_{t=0}^\infty &= 0 \\ e^{-t} t^x \Big|_\infty &= \lim_{R \rightarrow \infty} e^{-R} R^x = \lim_{R \rightarrow \infty} \frac{R^x}{\sum_{n=0}^\infty R^n / n!} = \lim_{R \rightarrow \infty} \left[ \sum_{n=0}^\infty \frac{R^{n-x}}{n!} \right]^{-1} = 0 \end{aligned}$$

Hence, we are left with

$$\Gamma(x + 1) = \int_0^\infty t^{x-1} e^{-t} dt = x \Gamma(x) \quad x > 0$$

Now

$$\Gamma(1) = \int_0^\infty e^{-t} dt = -e^{-t} \Big|_0^\infty = 1$$

Using  $\Gamma(1) = 1$  as the base case, and induction on  $\Gamma(x + 1) = x \Gamma(x)$ :

$$\Gamma(n + 1) = n! \quad n \in \mathbb{N}$$

□

Note: Since  $\Gamma(1) = 1$ , we can see why  $\Gamma(1) = 0! := 1$ .

### Definition 17.7.17: Gamma Function (Complex)

Consider  $z \in \mathbb{C}$ :

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt = \int_0^\infty e^{(z-1) \text{Log}(z)} e^{-t} dt \quad t \in \mathbb{C}$$

Note the principal part of the logarithm.

Consistent with the real definition, the complex Gamma function is analytic for  $t > 0$  and  $\text{Re}\{z\} > 0$ . By differentiating under the integral sign:

$$\Gamma'(z) = \frac{d}{dz} \int_0^\infty t^{z-1} e^{-t} dt = \int_0^\infty \frac{d}{dz} e^{(z-1) \text{Log}(z)} e^{-t} dt = \int_0^\infty t^{z-1} e^{-t} \text{Log}(t) dt$$

Hence,  $\Gamma(z)$  and its derivatives are continuous for  $\text{Re}\{z\} > 0$ .

Following the same procedure in the proof of theorem 17.7.13, we get:

$$\Gamma(z + 1) = z \Gamma(z)$$

Reordering allows us analytically extend the function into negative values of  $\text{Re}\{z\}$ :

$$\Gamma(z) = \frac{\Gamma(z + 1)}{z} \implies \Gamma(z) = \frac{\Gamma(z + 2)}{(z + 1)z}$$

Continuing with induction:

$$\Gamma(z) = \frac{\Gamma(z+n)}{\prod_{m=1}^n (z+n-m)} \quad n \in \mathbb{N}$$

Hence, we can see that  $\Gamma(z)$  has simple poles at  $z \in \{0, -1, -2, \dots\}$ .

**Theorem 17.7.14:**

*Let  $f$  be a function that is real on the real axis, then it displays opposite signs on the real axis on either side of the simple pole.*

*Proof:* Let  $f$  be a function with a simple pole on  $x_0 \in \mathbb{R}$ . Then from *theorem 17.3.1*:

$$f(x) = \frac{\phi(x)}{x - x_0}$$

Assume that  $f(x)$  is continuous on both sides of  $x_0$  and  $\phi(x)$  is continuous. Consider  $f(x_0 + c)$  and  $f(x_0 - c)$ :

$$\begin{aligned} f(x_0 + c) &= \frac{\phi(x_0 + c)}{c} \implies \phi(x_0 + c) = cf(x_0 + c) \\ f(x_0 - c) &= \frac{\phi(x_0 - c)}{-c} \implies \phi(x_0 - c) = -cf(x_0 - c) \end{aligned}$$

Since  $\phi(x)$  is nonzero at  $x_0$  and is continuous, we can choose  $c$  such that  $\phi(x_0 + c)$  and  $\phi(x_0 - c)$  have the same sign. Therefore

$$f(x_0 + c) = -f(x_0 - c)$$

□

From theorem 17.7.14, we can see that  $\Gamma(z)$  alternates signs across its simple poles.

**Theorem 17.7.15: Reflection Formula**

$$\Gamma(z)\Gamma(z-1) = \frac{\pi}{\sin(\pi z)}$$

*Proof:* Consider the Gamma function with the change of variable  $t = y^2$ ,  $dt = 2ydy$ :

$$\Gamma(z) = 2 \int_0^\infty y^{2z-1} e^{-y^2} dy$$

Then letting  $x = y$ :

$$\Gamma(1-z) = 2 \int_0^\infty x^{1-2z} e^{-x^2} dx$$

Multiplying the two equations together:

$$\Gamma(z)\Gamma(1-z) = 4 \int_0^\infty \int_0^\infty y^{2z-1} x^{1-2z} e^{-x^2} e^{-y^2} dx dy$$

Switching to polar coordinates:

$$\begin{aligned} \Gamma(z)\Gamma(1-z) &= 4 \int_0^{\pi/2} \int_0^\infty [r \sin(\theta)]^{2z-1} [r \cos(\theta)]^{1-2z} e^{-r^2 \cos^2(\theta)} e^{-r^2 \sin^2(\theta)} r dr d\theta \\ &= 4 \int_0^\infty e^{-r^2} r dr \int_0^{\pi/2} [\tan(\theta)]^{2z-1} d\theta \end{aligned}$$

Letting  $u = -r^2$ :

$$\int_0^\infty e^{-r^2} r dr = -\frac{1}{2} \int_0^{-\infty} e^u du = \frac{1}{2} e^u \Big|_{-\infty}^0 = \frac{1}{2}$$

Therefore

$$\Gamma(z)\Gamma(1-z) = 2 \int_0^{\pi/2} [\tan(\theta)]^{2z-1} d\theta$$

Letting  $x = \tan^2(\theta)$ , so  $dx = 2 \tan(\theta)[1 + \tan^2(\theta)]d\theta$ :

$$\begin{aligned} \Gamma(z)\Gamma(1-z) &= 2 \int_0^{\pi/2} [\tan(\theta)]^{2z-1} d\theta \\ &= 2 \int_0^\infty x^z \frac{1}{\tan(\theta)} \cdot \frac{1}{2 \tan(\theta)[1 + \tan^2(\theta)]} dx \\ &= \int_0^\infty \frac{x^z}{x(1+x)} dx = \int_0^\infty \frac{x^{z-1}}{1+x} dx \end{aligned}$$

From example 17.7.5, we know that

$$\int_0^\infty \frac{x^{-a}}{x+1} dx = \frac{\pi}{\sin(a\pi)} \quad z \in (0, 1)$$

Thus

$$\Gamma(z)\Gamma(1-z) = \int_0^\infty \frac{x^{z-1}}{1+x} dx = \int_0^\infty \frac{x^{-(1-z)}}{x+1} dx = \frac{\pi}{\sin((1-z)\pi)} = \frac{\pi}{\sin(z\pi)}$$

□

This is useful for speeding up numerical calculations.

**Lemma 17.7.15.1:**

*The Reflection Formula is not restricted to  $z \in (0, 1)$ .*

*Proof:* We know that the Reflection Formula has zeros at  $z \in \mathbb{Z}$ , which are the simple poles of either  $\Gamma(z)$  or  $\Gamma(1-z)$ . If we arrange the reflection formula such that

$$F(z) = \sin(\pi z)\Gamma(z)\Gamma(1-z) - \pi = 0$$

It has to be satisfied for all  $z \in (0, 1)$ , so it has removable singularities at  $z \in \mathbb{Z}$ , and can be made analytic everywhere using the integral definition of the Gamma Function.

Zeros on the other hand,  $\forall z \in (0, 1)[F(z) = 0]$ , hence the zeros are not isolated, and  $\forall z \in \mathbb{C}[F(z) = 0]$ .

Thus, the Reflection Formula fails to be analytic only when  $z \in \mathbb{Z}$ .  $\square$

### **Lemma 17.7.15.2:**

*$\Gamma(z)$  has non zeros in the complex plane, and  $1/\Gamma(z)$  is entire. .*

*Proof:* Poles:

$$\begin{aligned} &\Gamma(z) \text{ at } z \in \mathbb{Z} \setminus \mathbb{N} \\ &\Gamma(1-z) \text{ at } z \in \mathbb{N} \end{aligned}$$

$F(z) = \sin(\pi z)\Gamma(z)\Gamma(1-z) - \pi = 0$  tells us that if

$$\begin{aligned} \exists z_0 \in \mathbb{C}[\Gamma(z_0) = 0] &\implies \Gamma(1-z_0) \text{ must be pole to satisfy equation} \\ &\implies z_0 \in \mathbb{N} \end{aligned}$$

But

$$\Gamma(z_0) = (z_0 - 1)! \quad z_0 \in \mathbb{N}$$

Contradiction! Thus  $\Gamma(z)$  has no zeros in the complex plane and  $1/\Gamma(z)$  is entire.  $\square$

### **Computing $\Gamma(z)$ for odd half integers of $z$ :**

Consider  $z = 1/2$ , then

$$\Gamma(1/2)\Gamma(1-1/2) = \Gamma^2(1/2) = \frac{\pi}{\sin(\pi/2)} = \pi \implies \Gamma(1/2) = \pm\sqrt{\pi}$$

Integral definition of Gamma Function is positive for  $x > 0$ :

$$\Gamma(1/2) = \sqrt{\pi}$$

Then using

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$

We can quickly compute  $\Gamma(z)$  for  $z \in \{\pm 1/2, \pm 3/2, \pm 5/2, \dots\}$

# Chapter 18

## Mapping by Elementary Functions

### 18.1 Linear Transformations

#### Definition 18.1.1: Linear Transformation

Let  $A, B \in \mathbb{C}$ :

$$w = Az + B$$

We can see that  $A = |A|e^{i\arg(A)}$  which scales by  $|A|$  and rotates by  $\arg(A)$ , while  $B$  is a translation. The linear transformation scales, rotates, and translates the elements in the complex plane.

### 18.2 Transformation $w = 1/z$

$f(z) = 1/z$  is a 1-1 mapping from  $\mathbb{C} \setminus \{0\} \mapsto \mathbb{C} \setminus \{0\}$ . If we consider:

$$|z|^2 = z\bar{z} \implies \frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

We can see that  $1/z$  maps the points exterior to the unit circle onto nonzero point within the unit circle (from  $1/|z|$ ), followed by a reflection on the real axis (from  $\bar{z}$ ).

To make the transformation continuous on the extended complex plane, we can define:

$$T(z) = \frac{1}{z} \quad z \neq 0$$

Then

$$\begin{aligned} \lim_{z \rightarrow 0} \frac{1}{T(z)} &= \lim_{z \rightarrow 0} z = 0 \implies \lim_{z \rightarrow 0} T(z) = \infty \\ \lim_{z \rightarrow 0} \frac{1}{T(1/z)} &= \lim_{z \rightarrow 0} z = 0 \implies \lim_{z \rightarrow 0} T(z) = 0 \end{aligned}$$

To make it continuous, we can define:

$$T(0) = \infty \qquad T(\infty) = 0 \qquad T(z) = \frac{1}{z}$$

### 18.2.1 Mapping by $1/z$

Letting  $w = u + iv$  and  $z = x + iy = 1/w$ :

$$\begin{aligned} w = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2} &\implies \left[ u = \frac{x}{x^2 + y^2} \right] \wedge \left[ v = \frac{-y}{x^2 + y^2} \right] \\ z = \frac{1}{w} = \frac{\bar{w}}{w\bar{w}} = \frac{\bar{w}}{|w|^2} &\implies \left[ x = \frac{u}{u^2 + v^2} \right] \wedge \left[ y = \frac{-v}{u^2 + v^2} \right] \end{aligned}$$

We can see that  $w = 1/z$  transforms circles and lines into circles and lines by letting the equations represents an arbitrary circle or line:

$$A(x^2 + y^2) + Bx + Cy + D = 0 \qquad B^2 + C^2 > 4AD \qquad A, B, C, D \in \mathbb{R}$$

We need  $B^2 + C^2 > 4AD$  since completing the squares tells us:

$$\left(x + \frac{B}{2A}\right)^2 + \left(y + \frac{C}{2A}\right)^2 = \left(\frac{\sqrt{B^2 + C^2 - 4AD}}{2A}\right)^2$$

After substituting  $u$  and  $v$ , and rearranging:

$$\begin{aligned} A(x^2 + y^2) + Bx + Cy + D &= 0 & B^2 + C^2 > 4AD & \quad A, B, C, D \in \mathbb{R} \\ D(u^2 + v^2) + Bu - Cv + A &= 0 \end{aligned}$$

It is clear that transforming from  $z$  to  $w$ -plane:

1. Circle not passing through origin ( $A \neq 0, D \neq 0$ )  $\mapsto$  Circle not passing through origin
2. Circle through origin ( $A \neq 0, D = 0$ )  $\mapsto$  Line not passing through origin
3. Line not passing through origin ( $A = 0, D \neq 0$ )  $\mapsto$  Circle through origin
4. Line through origin ( $A = 0, D = 0$ )  $\mapsto$  Line through origin

See Desmos graph: [www.desmos.com/calculator/j6xj51risy](http://www.desmos.com/calculator/j6xj51risy)

## 18.3 Linear Fractional Transformations

**Definition 18.3.1: Linear Fractional / Bilinear / Möbius Transform**



The transform

$$w = \frac{az + b}{cz + d} \quad ad - bc \neq 0$$

Alternate form

$$Az + Bw + Cw + D = 0 \quad AD - BC \neq 0$$

We can see that:

$$w = \frac{az + b}{cz + d} \iff w = \frac{a}{c} + \frac{bc - ad}{c} \cdot \frac{1}{cz + d} \quad ad - bc \neq 0$$

Thus the Linear Fractional Transformation is composed of the transformations:

$$Z = cz + d \quad W = \frac{1}{Z} \quad w = \frac{a}{c} + \frac{bc - ad}{c} \cdot W$$

So any linear fractional transformation transforms circles and lines into circles and lines.

If we solve for  $z$ :

$$z = \frac{-dw + b}{cw - a} \quad ad - bc \neq 0$$

We can extend the transformation to the extended complex plane by defining:

$$\begin{aligned} T(z) &= \frac{az + d}{cz + d} & ad - bc \neq 0 \\ T(\infty) &= \infty & c = 0 \\ T(\infty) &= \frac{a}{c} & T\left(-\frac{d}{c}\right) = \infty & c \neq 0 \end{aligned}$$

This then becomes a 1-1 mapping from the extended  $z$ -plane to the extended  $w$ -plane. Hence there is an inverse transformation:

$$T^{-1}(w) = z \iff T(z) = w$$

Hence

$$\begin{aligned} T^{-1}(w) &= \frac{dw + b}{cw - a} & ad - bc \neq 0 \\ T^{-1}(\infty) &= \infty & c = 0 \\ T^{-1}\left(\frac{a}{c}\right) &= \infty & T^{-1}(\infty) = -\frac{d}{c} & c \neq 0 \end{aligned}$$

**Example 18.3.1** Suppose we need to find the transformation corresponding to the mapping:

$$z_1 = 1 \mapsto w_1 = i \quad z_2 = 0 \mapsto w_2 = \infty \quad z_3 = -1 \mapsto w_3 = 1$$

Therefore

$$z_2 = 0 \mapsto w_2 = \infty \implies (c \neq 0) \wedge (d = 0) \implies w = \frac{az + b}{cz} \quad bc \neq 0$$

Since  $z_1 = 1 \mapsto w_1 = i$  and  $z_3 = -1 \mapsto w_3 = 1$ :

$$[ic = a + b] \wedge [-c = -a + b] \implies [2a = (1 + i)c] \wedge [2b = (i - 1)c]$$

Subbing into  $w = (az + b)/(cz)$ :

$$w = \frac{(i + 1)z + (i - 1)}{2z}$$

### 18.3.1 Implicit Form

#### Definition 18.3.2: Implicit Form of Linear Fractional Transformation

A linear fractional transformation that maps  $z_1 \mapsto w_1$ ,  $z_2 \mapsto w_2$ , and  $z_3 \mapsto w_3$ :

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

We can verify the mapping by writing

$$(z - w_3)(w - w_1)(w_2 - w_3) = (z - z_1)(w - w_3)(z_2 - z_3)(w_2 - w_1)$$

It follows that  $z = z_1 \implies w = w_1$  and  $z = z_3 \implies w = w_3$ . If  $z = z_2$ , then

$$(w - w_1)(w_2 - w_3) = (w - w_3)(w_2 - w_1) \implies w = w_2$$

Since it is a linear fractional transformation, we can write it in the form:

$$Azw + Bz + Cw + D = 0 \quad AD - BC \neq 0$$

**Example 18.3.2** Consider again finding the transformation corresponding to the mapping:

$$z_1 = 1 \mapsto w_1 = i \quad z_2 = 0 \mapsto w_2 = \infty \quad z_3 = -1 \mapsto w_3 = 1$$

Substituting into the implicit form of the linear fractional transformation:

$$\frac{w - w_1}{w - w_3} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} \implies \frac{w - i}{w - 1} = \frac{(z - 1)(0 + 1)}{(z + 1)(0 - 1)} \implies w = \frac{(i - 1)z + (i - 1)}{2z}$$

**Example 18.3.3** Show every linear fractional transformation has at most 2 fixed points, unless it is the identity transform.

*Proof:* A point of transformation  $T(z)$  is fixed if  $T(z_0) = z_0$ , then

$$\begin{aligned} z = \frac{az + b}{cz + d} &\implies cz^2 + dz = az + b \implies cz^2 + (d - a)z - b = 0 \\ &\implies z = \frac{a - d \pm \sqrt{(d - a)^2 + 4cb}}{2c} \end{aligned}$$

Thus  $T(z)$  has at most 2 fixed points. □

**Example 18.3.4** Linear fractional transformation that maps 3 distinct points in the extended  $z$ -plane to the extended  $w$ -plane are unique.

*Proof:* Let  $T$  and  $S$  be two linear fractional transformations.

$$\begin{aligned} T(z) &= \frac{a_1 z + b_1}{c_1 z + d_1} & a_1 d_1 - b_1 c_1 &\neq 0 \\ S^{-1}(w) &= \frac{a_2 w + b_2}{c_2 w + d_2} & a_2 d_2 - b_2 c_2 &\neq 0 \end{aligned}$$

Then

$$\begin{aligned} S^{-1}[T(z)] &= \left[ a_2 \frac{a_1 z + b_1}{c_1 z + d_1} + b_2 \right] \left[ c_2 \frac{a_1 z + b_1}{c_1 z + d_1} + d_2 \right]^{-1} \\ &= \left[ \frac{a_1 a_2 z + a_2 b_1 + b_2 c_1 z + b_2 d_1}{c_1 z + d_1} \right] \left[ \frac{c_1 z + d_1}{a_1 c_2 z + b_1 c_2 + c_1 d_2 z + d_1 d_2} \right] \\ &= \frac{(a_1 a_2 + b_2 c_1)z + (a_2 b_1 + b_2 d_1)}{(a_1 c_2 + c_1 d_2)z + (b_1 c_2 + d_1 d_2)} = \frac{a_3 z + b_3}{c_3 z + d_3} \end{aligned}$$

Where

$$\begin{aligned} a_3 d_3 - b_3 c_3 &= (a_1 a_2 + b_2 c_1)(b_1 c_2 + d_1 d_2) - (a_2 b_1 + b_2 d_1)(a_1 c_2 + c_1 d_2) \\ &= a_1 a_2 b_1 c_2 + a_1 a_2 d_1 d_2 + b_1 b_2 c_1 c_2 + b_2 c_1 d_1 d_2 - a_1 a_2 b_1 c_2 - a_2 b_1 c_1 d_2 - a_1 b_2 c_2 d_1 - b_2 c_1 d_1 d_2 \\ &= a_1 a_2 d_1 d_2 + b_1 b_2 c_1 c_2 - a_2 b_1 c_1 d_2 - a_1 b_2 c_2 d_1 \\ &= (a_1 d_1 - b_1 c_1)(a_2 d_2 - b_2 c_2) \neq 0 \end{aligned}$$

Suppose that for the three unique points  $z_k$ ,  $k = 1, 2, 3$ :

$$S^{-1}[T(z_k)] = \frac{a_3 z_k + b_3}{c_3 z_k + d_3} = z_k$$

We know from previous example that any linear fractional transformation has at most two fixed points, unless it is the identity transformation. Since we have three fixed points, this implies  $\forall z$ ,  $S^{-1}[T(z)] = z$ , which implies  $T(z) = S(z)$ , so transformations are unique.  $\square$

**Example 18.3.5** Prove a linear fractional transformation is an automorphism of the real axis  $\iff$  coefficients in the transformation are all real, except possibly for a common complex factor.

*Proof:*

$$T(x) = \frac{ax + b}{cx + d} = u \in \mathbb{R}$$

This implies

$$ax + b = cxu + du \in \mathbb{R}$$

It is clear that  $a, b, c, d \in \mathbb{R}$ , unless they share a common complex factor.  $\square$

**Example 18.3.6** Show for

$$T(z) = \frac{az + b}{cz + d} \quad ad - bc \neq 0$$

which is any linear fractional transformation other than the identity.

$$T^{-1} = T \iff d = -a$$

*Proof:*

$$\begin{aligned} T(z) = \frac{az + b}{cz + d} = w &\implies az + b = czw + dw \implies az - czw = dw - b \\ &\implies z(a - cw) = dw - b \implies z = \frac{dw - b}{-cw + a} \end{aligned}$$

Therefore

$$\begin{aligned} T = T^{-1} &\implies \frac{az + b}{cz + d} = \frac{dz - b}{-cz + a} \\ &\implies -acz^2 + a^2z - bcz + ab = cdz^2 + d^2z - bcz - bd \\ &\implies c(a + d)z^2 + (d^2 - a^2)z - b(a + d) = 0 \\ &\implies (a + d)[cz^2 + (d - a)z - b] = 0 \\ &\implies d = -a \end{aligned}$$

□

## 18.4 Mappings of the Upper Half Plane

**Theorem 18.4.1:**

A linear fractional transformation that maps  $\text{Im}\{z\} > 0 \mapsto |w| < 1$  and  $\text{Im}\{z\} = 0 \mapsto |z| = 1$  is of the form:

$$w = e^{i\alpha} \cdot \frac{z - z_0}{z - \bar{z}_0} \quad \text{Im}\{z_0\} > 0, \alpha \in \mathbb{R}$$

*Proof:* **Mapping  $\implies$  equation:**

Consider mapping  $z = 0, 1, \infty \mapsto |w| = 1$

$$w = \frac{az + b}{cz + d} \quad ad - bc \neq 0$$

So

$$\begin{aligned} [z = 0 \mapsto |w| = 1] &\implies |b/d| = 1 \implies |b| = |d| \neq 0 \\ [z = \infty \mapsto w \in \mathbb{C} \iff c \neq 0] &\implies w = \frac{a}{c} \\ &\implies \left[ |w| = 1 \implies \left| \frac{a}{c} \right| = 1 \right] \\ &\implies |a| = |c| \neq 0 \end{aligned}$$

Then, we can write:

$$w = \frac{a}{c} \cdot \frac{z + (b/a)}{z + (d/c)} = e^{i\alpha} \cdot \frac{z - z_0}{z - z_1} \quad \left| \frac{a}{c} \right| = 1 \implies \left| \frac{b}{a} \right| = \left| \frac{d}{c} \right| \neq 0 \implies |z_1| = |z_0| \neq 0$$

Imposing  $z = 1 \mapsto |w| = 1$ :

$$|1 - z_1| = |1 - z_0| \implies (1 - z_1)(1 - \bar{z}_1) = (1 - z_0)(1 - \bar{z}_0)$$

$$|z_1| = |z_0| \implies z_1 \bar{z}_1 = z_0 \bar{z}_0:$$

$$\begin{aligned} z_1 + \bar{z}_1 &= z_0 + \bar{z}_0 \implies \operatorname{Re}\{z_1\} = \operatorname{Re}\{z_0\} \\ &\implies [z_1 = z_0] \vee [z_1 = \bar{z}_0] \end{aligned}$$

Since  $z_1 = z_0 \implies w = e^{i\alpha}$ , we choose

$$z_1 = \bar{z}_0 \implies w = e^{i\alpha} \cdot \frac{z - z_0}{z - \bar{z}_0}$$

Equation  $\implies$  mapping:

$$w = e^{i\alpha} \frac{z - z_0}{z - \bar{z}_0} \implies |w| = \frac{|z - z_0|}{|z - \bar{z}_0|}$$

If  $\operatorname{Im}\{z\} > 0$ , then  $z$  and  $z_0$  lie on the same side of the axis. Then  $|z - z_0| < |z - \bar{z}_0|$ , so  $|w| < 1$ . If  $\operatorname{Im}\{z\} < 0$ , then  $z$  and  $z_0$  lie on the opposite side of the axis. Then  $|z - z_0| > |z - \bar{z}_0|$ , so  $|w| > 1$ . If  $z \in \mathbb{R}$ . Then  $|z - z_0| = |z - \bar{z}_0|$ , so  $|w| = 1$ . The statement then follows.  $\square$

**Example 18.4.1** Consider

$$w = \frac{z - 1}{z + 1}$$

Then letting  $z = x + iy$  and  $w = u + iv$ :

$$w = \frac{z - 1}{z + 1} \cdot \frac{\bar{z} + 1}{\bar{z} - 1} = \frac{z\bar{z} + z - \bar{z} - 1}{z\bar{z} - z + \bar{z} - 1} = u + iv$$

Not that  $z \in \mathbb{R} \implies w \in \mathbb{R}$ , so  $y = 0 \mapsto v = 0$ . For any point  $w$  in the finite  $w$ -plane:

$$v = \operatorname{Im}\{w\} = \operatorname{Im}\left\{ \frac{(z - 1)(\bar{z} + 1)}{(z + 1)(\bar{z} - 1)} \right\} = \frac{2y}{|z + 1|^2} \quad z \neq -1$$

We can see  $y$  and  $v$  have the same sign, so  $\operatorname{Im}\{z\} > 0 \mapsto \operatorname{Im}\{w\} > 0$ , and  $\operatorname{Im}\{z\} < 0 \mapsto \operatorname{Im}\{w\} < 0$ . Since a linear fractional transformation is 1-1, the mapping is 1-1.

**Example 18.4.2** Consider

$$w = \text{Log} \left( \frac{z-1}{z+1} \right)$$

This is composed of the two transformations:

$$Z = \frac{z-1}{z+1} \qquad w = \text{Log}(Z)$$

The first mapping follows from the previous example. The second mapping:

$$\text{Log}(Z) = \ln(R) + i\Theta \qquad R > 0, \quad -\pi < \Theta < \pi$$

So letting  $Z = X + iY$ . The second mapping maps  $Y > 0 \mapsto v \in [0, \pi]$  and  $X, Y \mapsto \ln(R)$ , and  $Y < 0 \mapsto v \in [-\pi, 0]$  and  $X, Y \mapsto \ln(R)$ . (The upper and lower half plane is mapped to values between  $-\pi i$  and  $\pi i$ , and real values to  $\ln|z|$ ).

## 18.5 Mappings by the Exponential Function

We can see

$$w = e^z = e^x e^{iy} = \rho e^{i\phi}$$

Thus  $\text{Re}\{z\}$  gets mapped to rays emanating from the origin, while  $\text{Im}\{z\}$  determines the angle of said ray.



**Example 18.5.1** Consider the mapping of  $x \in [a, b]$  and  $y \in [c, d]$  onto the region  $\rho \in [e^a, e^b]$  and  $\phi \in [c, d]$ . Then the mapping is as follows:



## 18.6 Mapping by $w = \sin(z)$

### Mappings of Vertical Line Segments

Recall:  $\sin(z) = \sin(x) \cosh(y) + i \cos(x) \sinh(y)$ , so the transformation  $w = \sin(z)$  for  $z = x + iy$ :

$$w = u + iv = \sin(x) \cosh(y) + i \cos(x) \sinh(y)$$

Looking at vertical line segments, we set  $x = c_1$  where  $c_1 \in \mathbb{R}$ .

$$[u = \sin(c_1) \cosh(y)] \wedge [v = \cos(c_1) \sinh(y)] \implies \frac{u^2}{\sin^2(c_1)} - \frac{v^2}{\cos^2(c_1)} = 1$$

This gives a hyperbola with foci at points

$$w = \pm \sqrt{\sin^2(c_1) + \cos^2(c_1)} = \pm 1$$

If we look at the  $y$  and  $v$ -axis,  $(0, y) \mapsto (0, \sinh(y))$ , so the mapping is 1-1. As  $y$  increases, so does  $\sinh(y)$ . If we include negative values of  $y$ , then the mapping is no longer 1-1.



## Mappings of the Horizontal Line Segments

As before, we have:

$$u = \sin(x) \cosh(c_2) \qquad v = \cos(x) \sinh(c_2)$$

Which gives us the ellipse

$$\frac{u^2}{\cosh^2(c_2)} + \frac{v^2}{\sinh^2(c_2)} = 1$$

with foci

$$w = \pm \sqrt{\cosh^2(c_2) - \sinh^2(c_2)} = \pm 1$$

Since  $u = \sin(x) \cosh(c_2)$  and  $v = \cos(x) \sinh(c_2)$  we can see that as  $x$  increases, the ellipse goes around in a clockwise direction. Since Sine is a cyclic function, we only have 1-1 mappings for constant values of  $y$  in  $x = 2\pi$  intervals. Likewise, these 1-1 mappings only hold true for  $y > 0$  or  $y < 0$ .



## Summary

Basically, the mapping  $w = \sin(z)$  maps vertical line segments to hyperbolas and horizontal line segments to ellipses.

**Example 18.6.1** The following rectangular region is mapped onto the semi-elliptical region in a 1-1 manner. We have the mapping of the points:

$$\begin{aligned} A = (\pi/2, 0) &\mapsto A' = (1, 0) & B = (\pi/2, bi) &\mapsto B' = (\cosh(b), 0) \\ C = (0, bi) &\mapsto C' = (0, \sinh(b)) & D = (-\pi/2, bi) &\mapsto D' = (-\cosh(b), 0) \\ E = (-\pi/2, 0) &\mapsto E' = (-1, 0) & F = (0, 0) &\mapsto F' = (0, 0) \end{aligned}$$





## 18.6.1 Related Mappings

**Example 18.6.2** Consider

$$w = \cos(z) = \sin\left(z + \frac{\pi}{2}\right)$$

It is clear the cosine is just the sine function translated to the right by  $\pi/2$ . It is composed of the transformations:

$$Z = z + \frac{\pi}{2} \qquad w = \sin(Z)$$

**Example 18.6.3** Consider

$$w = \sinh(z) = -i \sin(iz)$$

Which is the composite of transformations

$$Z = iz \qquad W = \sin(Z) \qquad w = -iW$$

Hence, it is the sine transformation with rotations by  $\pi/2$  and  $-\pi/2$ .

**Example 18.6.4** Consider

$$w = \cosh(z) = \cos(iz)$$

Which is the a rotation by  $\pi/2$  followed by a cosine transform. Using

$$\sin\left(z + \frac{\pi}{2}\right) = \cos(z) \qquad \cos(iz) = \cosh(z)$$

We can write  $w = \cosh(z)$ :

$$w = \sin(Z) \qquad Z = iz + \frac{\pi}{2}$$

## 18.7 Mappings by $z^2$

We can write the transformation:

$$w = z^2 = x^2 - y^2 + 2xy = u + iv \qquad u = x^2 - y^2 \qquad v = 2xy$$

We can see that if  $x$  is constant, as  $y$  increases, the transformation curves to the left due to the decreasing value of  $u$ . It is clear that if we include negative values of  $y$ , then the mapping is not 1-1.



If  $y$  is held constant, the increasing value of  $x$  curves the transformation to the right. More accurately, while  $x$  is negative, it moves left until  $x = 0$ , then it curves to the left due to the parabolic nature of  $u = x^2 - y^2$ . The mapping is not 1-1, if we include negative values of  $y$ .



## 18.8 Mappings by Branches of $z^{1/n}$

Recall section 12.4.

**Square root transformation:**

For  $z \neq 0$  and in polar coordinates,

$$z^{1/2} = \sqrt{r}e^{i(\Theta+2\pi x)/2} \quad k \in \mathbb{N}$$

Looking at the Principal Branch, the transformation becomes:

$$F_0(z) = \sqrt{r}e^{i\Theta/2} \quad r > 0, \quad -\pi < \Theta < \pi$$

It is clear in the principal branch, the mapping is a 1-1 mapping of points in  $-\pi < \Theta < \pi$  to  $-\pi/2 < \Phi < \pi/2$ .

When  $\theta = \alpha$  is used to define a branch cut:

$$f_\alpha(z) = \sqrt{r} \exp\left(\frac{i\theta}{2}\right) \quad r > 0, \quad \alpha < \theta < \alpha + 2\pi$$

We can extend this to nonzero points on the branch cut by defining  $f_\alpha(0) = 0$ , however, such extensions are not continuous on the entire complex plane.

**$n$ -th roots of  $z$ :**

$$z^{1/n} = \exp\left(\frac{1}{n} \log(z)\right) = \sqrt[n]{r} e^{i(\Theta + 2k\pi)/n} \quad n \in \mathbb{N}, \quad k \in \{0, 1, 2, \dots, n-1\}$$

Then the transformation of each branch  $k$  of  $z^{1/n}$ :

$$F_k(z) = \sqrt[n]{r} \exp\left(\frac{i(\Theta + 2k\pi)}{n}\right) \quad k = \{0, 1, 2, \dots, n-1\}$$

It is a 1-1 mapping from the domain:

$$r \mapsto \rho = \sqrt[n]{r} \quad [-\pi < \Theta < \pi] \mapsto \left[ \frac{(2k-1)\pi}{n} < \phi < \frac{(2k+1)\pi}{n} \right]$$

Likewise, for the principal branch ( $k = 0$ ), we can construct transformations  $f_\alpha(z)$  branch cuts at  $\theta = \alpha$  as before.

## 18.9 Square Roots of Polynomials

**Example 18.9.1** If we consider  $Z^{1/2} = (z - z_0)^{1/2}$ :

$$Z^{1/2} = \sqrt{R} \exp\left(\frac{i\theta}{2}\right) \quad R > 0, \quad \alpha < \theta < \alpha + 2\pi$$

By writing

$$R = |z - z_0| \quad \Theta = \text{Arg}(z - z_0) \quad \theta = \arg(z - z_0)$$

We have the two branches of  $(z - z_0)^{1/2}$ :

$$\begin{aligned} G_0(z) &= \sqrt{R} \exp\left(\frac{i\Theta}{2}\right) & R > 0, \quad -\pi < \Theta < \pi \\ g_0(z) &= \sqrt{R} \exp\left(\frac{i\theta}{2}\right) & R > 0, \quad 0 < \Theta < 2\pi \end{aligned}$$

We can see that for  $w = G_0$ , there is a 1-1 mapping:

$$|z - z_0| \mapsto \sqrt{|z - z_0|} \quad [-\pi < \text{Arg}(z - z_0) < \pi] \mapsto \left[-\frac{\pi}{2} < \frac{\text{Arg}(z - z_0)}{2} < \frac{\pi}{2}\right]$$

For  $w = g_0(z)$ , the 1-1 mapping:

$$|z - z_0| \mapsto \sqrt{|z - z_0|} \quad [0 < \arg(z - z_0) < 2\pi] \mapsto \left[0 < \frac{\arg(z - z_0)}{2} < \pi\right]$$

**Example 18.9.2** Consider

$$w = (z^2 - 1)^{1/2}$$

Then we can write

$$\begin{aligned} (z^2 - 1)^{1/2} &= \exp\left(\frac{1}{2} \log(z^2 - 1)\right) = \exp\left(\frac{1}{2} \log(z - 1) + \frac{1}{2} \log(z + 1)\right) \\ &= (z - 1)^{1/2} (z + 1)^{1/2} \quad z \neq \pm 1 \end{aligned}$$

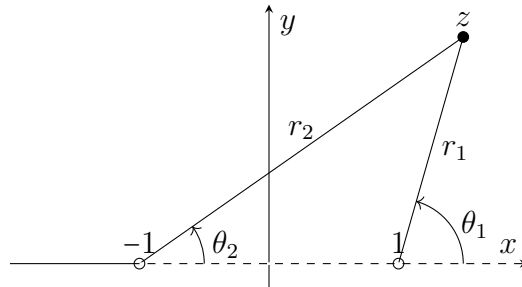
Note: If  $f_1(z)$  is a branch of  $(z - 1)^{1/2}$  defined on domain  $D_1$  and  $f_2(z)$  is a branch of  $(z + 1)^{1/2}$  defined on domain  $D_2$ , then  $f(z) = f_1(z)f_2(z)$  is a branch of  $(z^2 - 1)^{1/2}$  defined at all points in  $D_1 \cup D_2$ .

For branches of  $(z - 1)^{1/2}$  and  $(z + 1)^{1/2}$ :

$$\begin{aligned} f_1(z) &= \sqrt{r_1} \exp\left(\frac{i\theta_1}{2}\right) & r_1 &= |z - 1| & \theta_1 &= \arg(z - 1) & r_1 > 0, \theta_1 \in (0, 2\pi) \\ f_2(z) &= \sqrt{r_2} \exp\left(\frac{i\theta_2}{2}\right) & r_2 &= |z + 1| & \theta_2 &= \arg(z + 1) & r_2 > 0, \theta_2 \in (0, 2\pi) \end{aligned}$$

Then the branch  $f$  of  $(z^2 - 1)^{1/2}$ :

$$f(z) = \sqrt{r_1 r_2} \exp\left(\frac{i(\theta_1 + \theta_2)}{2}\right) \quad r_k > 0, \theta_k \in (0, 2\pi), k \in \{1, 2\}$$



We can extend this to a function that is analytic everywhere in  $\mathbb{C}$  except  $x \in [-1, 1]$ :

$$F(z) = \sqrt{r_1 r_2} \exp\left(\frac{i(\theta_1 + \theta_2)}{2}\right) \quad r_k > 0, r_1 + r_2 > 2, \theta_k \in (0, 2\pi), k \in \{1, 2\}$$

We need to show  $F$  is analytic on  $r_1 > 0$ ,  $\theta_1 = 0$ , since  $f(z) = F(z)$  everywhere except there. Consider:

$$G(z) = \sqrt{r_1 r_2} \exp\left(\frac{i(\Theta_1 \Theta_2)}{2}\right) \quad r_1 = |z-1|, \quad r_2 = |z+1|, \quad \Theta_1 = \text{Arg}(z-1), \quad \Theta_2 = \text{Arg}(z+1)$$

$$r_k > 0, \quad \Theta_k \in (-\pi, \pi), \quad k \in \{1, 2\}$$

$G$  analytic in entire  $z$ -plane except for  $r_1 \geq 0$ ,  $\Theta_1 = \pi$ .  $F(z) = G(z)$  when  $r_1 > 0$  and  $\Theta_1 \in [0, \pi)$ , so  $\theta_k = \Theta_k$ . When  $\Theta_1 \in (0, -\pi)$ ,  $\theta_k = \Theta_k + 2\pi$ . Then

$$\exp(i\theta_k/2) = -\exp(i\Theta_k/2) \implies \exp\left(\frac{i(\theta_1 + \theta_2)}{2}\right) = \exp\left(\frac{i(\Theta_2 + \Theta_2)}{2}\right)$$

So  $F(z) = G(z)$ , in the domain containing  $r_1 > 0$ ,  $\Theta_1 = 0$ , so  $G$  being analytic in the domain implies  $F$  being analytic in the domain. Thus,  $F$  is analytic everywhere except  $x \in [-1, 1]$ .

$F(z)$  cannot be extended to a function analytic at points  $x \in [-1, 1]$ , because  $F(z)$  jumps from  $i\sqrt{r_1 r_2}$  to  $-i\sqrt{r_1 r_2}$  as  $z$  moves across the line segment, thus is not continuous.

$F(z)$  is a 1-1 mapping of  $D_z \mapsto D_w$  except for  $v \in [-1, 1]$ .



Note:

$$[z = iy] \wedge [y > 0] \implies [r_1 = r_2 > 1] \wedge [\theta_1 + \theta_2 = \pi]$$

Hence the mappings:

$$\begin{array}{ll} [y > 0] \wedge [x = 0] \mapsto v > 1 & [y < 0] \wedge [x = 0] \mapsto v < -1 \\ y > 0 \mapsto v > 0 & y < 0 \mapsto v > 0 \\ [r_1 > 0] \wedge [\theta_1 = 0] \mapsto [u > 0] \wedge [v = 0] & [r_2 > 0] \wedge [\theta_2 = \pi] \mapsto [u < 0] \wedge [v = 0] \end{array}$$

To show  $w = F(z)$  is 1-1:

$$F(z_1) = F(z_2) \implies z_1^2 - 1 = z_2^2 - 1 \implies [z_1 = z_2] \vee [z_1 = -z_2]$$

Since  $F$  maps upper half plane to upper half plane and lower half planes to lower half planes, and the way portions of the real axis in  $D_z$  is mapped,  $z_1 = -z_2$  is impossible, so  $z_1 = z_2$ . Thus,  $F(z_1) = F(z_2) \implies F$  is 1-1.

We can show  $F : D_z \mapsto D_w$  by finding  $H : D_w \mapsto D_z$  which  $z = H(w) \implies w = F(z)$ . That is,  $H = F^{-1}$ . We have

$$w = (z^2 - 1)^{1/2} \implies z = (w^2 + 1)^{1/2} = (w - i)^{1/2}(w + i)^{1/2} \quad w \neq \pm i$$

Writing

$$w - i = \rho_1 e^{i\phi_1} \quad w + i = \rho_2 e^{i\phi_2} \quad \rho_k > 0, \rho_1 + \rho_2 > 2, \phi_k \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right], k \in \{1, 2\}$$

Hence

$$H(w) = \sqrt{\rho_1 \rho_2} \exp\left(\frac{i(\phi_1 + \phi_2)}{2}\right) \quad \text{Domain: } D_w$$

with mappings:

$$\begin{aligned} v > 0 &\mapsto y > 0 & v < 0 &\mapsto y < 0 \\ [u > 0] \wedge [y = 0] &\mapsto [x > 1] \wedge [y = 0] & [u < 0] \wedge [y = 0] &\mapsto [x < -1] \wedge [y = 0] \end{aligned}$$

Then

$$z = H(w) \implies z^2 = w^2 + 1 \implies w^2 = z^2 - 1$$

Since  $z \in D_z$  and  $w = \pm(z^2 - 1)^{1/2}$ , we have  $w = F(z)$  and  $w = -F(z)$ . By  $H(w)$  mapping upper half plane to upper half plane and lower half planes to lower half planes, and the mapping of segments of the real axis in the domains, we have  $w = F(z)$ .

For mappings by branches of double-valued functions:

$$w = (z^2 + Az + B)^{1/2} = [(z - z_0)^2 - z_1^2]^{1/2} \quad A = -2z_0 \quad B = z_0^2 = z_1^2 z_1 \neq 0$$

We can use methods in the above example on the successive transformations:

$$Z = \frac{z - z_0}{z_1} \quad W = (Z^2 - 1)^{1/2} \quad w = z_1 W$$

## 18.10 Riemann Surface

### Definition 18.10.1: Riemann Surface (Informal Definition)

A generalized complex plane with more than one “sheet”. It’s a one dimensional complex manifold.

On a Riemann surface, a multi-valued function is assigned a single-value. As such, the theory of single-valued function applies.

**Example 18.10.1** Consider the multi-valued function:

$$\log(z) = \ln(r) + i\theta$$

The Riemann surface of  $\log(z)$  is then the complex plane with a deleted origin, and a cut made along the positive real axis. The first sheet  $R_0$  is defined from  $\theta \in [0, 2\pi]$ , the second sheet  $R_1$  consists of  $\theta \in [2\pi, 4\pi]$ , the third sheet  $R_3$  consists of  $\theta \in [4\pi, 6\pi]$ , and so on. Likewise, the sheet  $R_{-1}$  consists of  $\theta \in [-2\pi, 0]$ .

Essentially, we get an infinite “spiral staircase” with sheets in multiples of  $2\pi$ . The radial component, of course, is also infinite.



The transformation  $w = \log(z)$  maps the Riemann surface onto the  $w$ -plane in an 1-1 manner.



$\log(z)$  on  $R_1$  is an analytic continuation of  $f(z) = \ln(r) + i\theta$ ,  $\theta \in (0, 2\pi)$ .  $\log(z)$  is then not only single-valued on all points  $z$  on the Riemann surface, but also analytic at all points.

The “spiral staircase” can be cut along any ray from the origin to for other Riemann surfaces.

**Example 18.10.2** Consider the square root function on the  $z$ -plane with a deleted origin:

$$z^2 = \sqrt{r}e^{i\theta/2}$$

The function maps  $\theta \in [0, 2\pi]$  to  $\phi \in [0, \pi]$ , and  $\theta \in [2\pi, 4\pi]$  to  $\phi \in [\pi, 2\pi]$ . Thus the Riemann surface is composed of two sheets,  $R_1$  and  $R_2$ .



The function is now single-valued. The choice of values of  $\theta$  from 0 to  $2\pi$  or  $4\pi$  to  $6\pi$  does not affect the value of  $z^{1/2}$ . Note, the value of  $z^{1/2}$  at a point where the sheet passes from  $R_0$  to  $R_1$  is different than that passing from  $R_1$  to  $R_0$ .  $R_0$  for  $\theta \in (0, 2\pi)$ , and  $R_1$  for  $\theta \in (2\pi, 4\pi)$ .

The origin is common to both  $R_0$  and  $R_1$ , thus is a branch point of the Riemann Surface.

The function is an analytic continuation of the function defined on the other sheet, thus points of  $z^{1/2}$  on the Riemann surface are all analytic except at the origin.

### 18.10.1 Surfaces for Related Functions

**Example 18.10.3** Riemann surface for

$$f(z) = (z^2 - 1)^{1/2} = \sqrt{r_1 r_2} \exp\left(\frac{i(\theta_1 + \theta_2)}{2}\right) \quad z - 1 = r_1 e^{i\theta_1} \quad z + 1 = r_2 e^{i\theta_2}$$



We have a branch cut between the points  $z = \pm 1$ , thus the Riemann surface has two sheets cut between the points. The lower edge of  $R_0$  is connected to the upper edge of  $R_1$ , and lower edge of  $R_1$  is connected to the upper edge of  $R_0$ . We have two layers where the line between  $z = \pm 1$  serves as a connection between the layers.

Any simple closed curve enclosing  $z = \pm 1$  on a sheet will return to its original position as  $\theta_1$  and  $\theta_2$  go from 0 to  $2\pi$ , and will not cross into the other sheet.

If a contour encircles  $z = 1$  twice, but does not enclose  $z = -1$ , then the contour passes from  $R_0$  to  $R_1$  and back to  $R_0$ .  $\theta_1$  then changes by  $4\pi$  while  $\theta_2$  changes by 0. Likewise for a similar case enclosing  $z = -1$ .



**Example 18.10.4** Consider the double-valued function:

$$f(z) = [z(z^2 - 1)]^{1/2} = \sqrt{rr_1r_2} \exp\left(\frac{i(\theta + \theta_1 + \theta_2)}{2}\right)$$



The branch points are  $z \in \{0, \pm 1\}$ . Since the function is double-valued, we will have two sheets,  $R_0$  and  $R_1$ . We can define a cut  $L_1$  from  $-1$  to  $0$  and a cut  $L_2$  from  $1$  to a point at infinity. The sheets  $R_0$  and  $R_1$  is then joined along  $L_1$  and  $L_2$ , with the lower edge of  $R_0$  joined to the upper edge of  $R_1$  and the lower edge of  $R_1$  joined to the lower edge of  $R_0$ .

**Question.** Is the choice of the cuts  $L_1$  and  $L_2$  arbitrary? Can we define  $L_1$  as the point at infinity to  $-1$  and  $L_2$  from  $0$  to  $1$ ?



# Chapter 19

## Conformal Mapping

A map that locally conforms to the original shape of a region.

### 19.1 Preserving Angles and Scale Factors

#### Definition 19.1.1: Conformal

*A transformation  $w = f(z)$  is conformal at point  $z_0$  if  $f$  is analytic at  $z_0$  and  $f'(z_0) \neq 0$ . That is, the orientation and magnitude of the angles between curves are preserved.*

$$[f \text{ analytic at } z_0] \wedge [f'(z_0) \neq 0] \implies f \text{ Conformal}$$

Consider an arc  $C_1$  parameterized by  $z_1(t)$  and a function  $f$  defined by all points in  $C$ :

$$z_1 = z_1(t) \qquad w_1 = f[z_1(t)] \qquad t \in [a, b]$$

Thus  $w$  is the parametric representation of image  $\Gamma_1$  of  $C_1$ . Suppose there exists  $z_0 = z_1(t_0) \in (a, b)$  such that  $f$  is analytic and  $f'(z_0) \neq 0$ . Then

$$\begin{aligned} w'_1(t_0) = f'[z_1(t_0)]z'_1(t_0) &\implies \arg[w'_1(t_0)] = \arg[f'[z_1(t_0)]] + \arg[z'_1(t_0)] \\ &\implies \arg[f'[z_1(t_0)]] = \arg[w'_1(t_0)] - \arg[z'_1(t_0)] \end{aligned}$$

Hence, we can see that  $w'(t_0)$  and  $z'(t_0)$  differs by an angle of rotation  $\varphi_0 = \arg[f'(z_0)]$ .

#### Definition 19.1.2: Angle of Rotation

*Let  $z_0 = z(t_0)$  be a point in an arc  $C$ , and  $w = f(z)$  be a conformal transformation. Then the angle of rotation is the difference between the angle of  $C$  at  $z_0$  and its image  $\Gamma$  under  $w$  at  $f(z_0)$ .*

If we consider another arc  $C_2$  passing through  $z_0 = z_2(t_0)$  with image  $\Gamma_2$  under the same transformation  $w$ . We obtain:

$$\begin{aligned} \arg[w'_2(t_0)] &= \arg[f'[z_2(t_0)]] + \arg[z'_2(t_0)] && \text{For } C_2 \\ \arg[w'_1(t_0)] &= \arg[f'[z_1(t_0)]] + \arg[z'_1(t_0)] && \text{For } C_1 \end{aligned}$$

Since we have  $z_1(t_0) = z_2(t_0) = z_0$ , if we subtract the two, we have

$$\arg[w'_1(t_0)] - \arg[w'_2(t_0)] = \arg[z'_1(t_0)] - \arg[z'_2(t_0)]$$

Hence, the angle between  $C_1$  and  $C_2$  at  $z_0$  is the same as the angle between  $\Gamma_1$  and  $\Gamma_2$  at  $f(z_0)$ . The angles between the curves at  $z_0$  and orientation of the angles are preserved in the transformation.



From theorem 17.4.2 we can see that  $w$  is also conformal in some neighbourhood of  $z_0$ . If this applies to an entire domain:

### Definition 19.1.3: Conformal Mapping/Transformation

A transformation  $w = f(z)$  is conformal if  $\forall z \in D$ ,  $f(z)$  is analytic and  $f'(z) \neq 0$ .

$$\forall z \in D [f \text{ analytic} \wedge f'(z) \neq 0] \implies f \text{ Conformal Mapping}$$

**Example 19.1.1** Consider two smooth arcs that are level curves of

$$f(z) = u(x, y) + iv(x, y)$$

Suppose  $u(x, y) = c_1$  and  $v(x, y) = c_2$  intersect at point  $z_0$ , where  $f$  is analytic and nonzero at  $z_0$ . Then  $f$  must be conformal at  $z_0$ . If the two curves are orthogonal at  $z_0$  then they are orthogonal at  $w_0 = f(z_0)$ .

### Definition 19.1.4: Isogonal Mapping

A mapping that preserves the angle between two curves, but not the orientation of the angle.

**Example 19.1.2** Consider the transformation

$$w = \bar{z}$$

It is isogonal, not conformal, due to  $w = \bar{z}$  not being an analytic function. If followed by an conformal map  $f$ , the result  $w = f(\bar{z})$  is isogonal. That is, **conformal transformations preserves isogonal transformations**.

**Definition 19.1.5: Critical Point**

A function  $f$  that is non-constant and analytic at  $z_0$ , and  $f'(z_0) = 0$ . Then  $z_0$  is a critical point of transformation  $w = f(z)$ .

**Example 19.1.3** Consider

$$w = 1 + z^2$$

Which is a composition of mappings

$$Z = z^2$$

$$w = 1 + Z$$

$z_0 = 0$  is a critical point of  $w$ . We can see that  $z_0 = 0 \mapsto w_0 = 1$ , and that  $w$  doubles any angle at  $z_0$ .

**Corollary 19.1.0.1:**

Suppose  $z_0$  is a critical point of  $w = f(z)$ . Let  $\Gamma_1$  and  $\Gamma_2$  be the image of curves  $C_1$  and  $C_2$  under the transformation respectively. If the angle between  $C_1$  and  $C_2$  at  $z_0$  is  $\alpha$ , then the angle between  $\Gamma_1$  and  $\Gamma_2$  at  $w_0 = f(z_0)$  becomes  $m\alpha$  under  $w$  for  $m \geq 2$ ,  $m \in \mathbb{N}$ . Also,  $m$  is the smallest natural number such that  $f^{(m)}(z_0) \neq 0$ .

That is, if  $z_0$  is a critical point of  $w = f(z)$ , then angles between curves  $\alpha$  at  $z_0$  becomes  $m\alpha$  under  $f(z)$  for  $m \geq 2$ ,  $m \in \mathbb{N}$ , and  $f(z)$  has a zero of order  $m$  at  $z_0$ .

*Proof:* Example 19.2.1 □

**Definition 19.1.6: Scale Factor**

The Scale Factor,  $|f'(z_0)|$ , is the amount of scaling inflicted on the distances under the transformation  $w = f(z)$ . Consider the modulus of the derivative of the transformation  $w = f(z)$ :

$$|f'(z_0)| = \left| \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \right| = \lim_{z \rightarrow z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}$$

For any  $z$  close to  $z_0$ :

$$|f'(z_0)| \approx \frac{|f(z) - f(z_0)|}{|z - z_0|}$$

which represents the ratio between the distances  $f(z) - f(z_0)$  and  $z - z_0$  under the transformation.

It is clear that  $|f'(z_0)| > 1$  represents an expansion while  $|f'(z_0)| < 1$  represents a contraction.

From the continuity of  $f'(z)$  we see that for  $z$  close to  $z_0$ :

$$\arg[f'(z)] \approx \arg[f'(z_0)]$$

$$|f'(z)| \approx |f'(z_0)|$$

Hence, the image of a conformal transformation approximates that of the original neighbourhood locally. That is, it is *conforms* to the shape of the original region. Notice, it is *locally* not *globally*.

**Example 19.1.4** Consider

$$f(z) = z^2 = x^2 - y^2 + i2xy \implies f'(z) = 2z$$

The function is entire and  $f'(z)$  is zero only at the origin. Consider the half lines

$$y = x \qquad x = 1 \qquad x, y \geq 0$$

which is denoted as the curves  $C_1$  and  $C_2$  that intersect at  $z_0 = 1 + i$ . It is clear that the angle between the curves are  $\pi/4$ .

Under the transformation,  $C_1 \mapsto \Gamma_1$ , with  $\Gamma_1$  parameterization:

$$u = 0 \qquad v = 2x \qquad 0 \leq x < \infty$$

$C_2 \mapsto \Gamma_2$ , with  $\Gamma_2$  parameterization:

$$u = 1 - y^2 \qquad v = 2y \qquad 0 \leq y < \infty$$



We then have

$$\frac{dv}{du} = \frac{dv/dy}{du/dy} = \frac{2}{-2y} = -\frac{2}{v}$$

$v = 2 \implies dv/du = -1$ , so the angle between  $\Gamma_1$  and  $\Gamma_2$  at  $w = f(1 + i) = 2i$  is  $\pi/4$ . Hence, we have conformality of the mapping.

The angle of rotation:

$$\arg[f'(1 + i)] = \arg[2(1 + i)] = \frac{\pi}{4} + 2n\pi \qquad n \in \mathbb{Z}$$

Scale factor:

$$|f'(1 + i)| = |2(1 + i)| = 2\sqrt{2}$$

## 19.2 Local Inverses

### Definition 19.2.1: Local Inverse

Suppose a transformation  $w = f(z)$  be conformal and  $w_0 = f(z_0)$ . Then a local inverse of the transformation is a unique transformation  $z = g(w)$  defined and analytic in a neighbourhood  $N$  of  $w_0$  such that  $\forall w \in N$ ,  $g(w_0) = z_0$  and  $f[g(w)] = w$ . The derivative:

$$g'(w) = \frac{1}{f'(w)}$$

*Proof:* Prove:

$$g'(w) = \frac{1}{f'(w)}$$

Let  $w = f[g(w)]$ , then

$$\frac{d}{dw}w = \frac{d}{dw}f[g(w)] \implies 1 = f'[g(w)]g'(w) \implies g'(w) = \frac{1}{f'(w)}$$

□

Note: From the definition,  $z = g(w)$  is conformal at  $w_0$ .

### Existence of the Inverse:

Conformality of the transformation  $w = f(z)$  at  $z_0$  implies there exist a neighbourhood of  $z_0$  that  $f$  is analytic. Hence

$$f(z) = u(x, y) + iv(x, y) \quad z = x + iy \quad z_0 = x_0 + iy_0$$

Then there exists some neighbourhood of  $x_0, y_0$  where  $u$  and  $v$  and their partial derivatives of all orders are continuous (theorem 15.7.3). The Jacobian is then nonzero at  $z_0$ :

$$\begin{aligned} J &= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - v_x u_y = (u_x)^2 + (v_x)^2 & u_x = v_y, \quad u_y = -v_x \\ &= |f'(z)|^2 \neq 0 & \text{Conformal at } z_0, \text{ Theorem 13.5.4} \end{aligned}$$

The condition on the Jacobian and derivatives of  $u$  and  $v$  are sufficient conditions to ensure invertibility of transform. So, if

$$u_0 = u(x_0, y_0) \quad v_0 = v(x_0, y_0)$$

then there exists a unique continuous transformation

$$x = x(u, v) \quad y = y(u, v)$$

that  $(u_0, v_0) \mapsto (x_0, y_0)$  in the neighbourhood  $N$ . In addition, the first order partial derivatives throughout  $N$  satisfy

$$x_u = \frac{v_y}{J} \quad x_v = -\frac{u_y}{J} \quad y_u = -\frac{v_x}{J} \quad y_v = \frac{u_x}{J}$$

which shows that  $g(w)$  is analytic in  $N$ .

*Proof:* Proving  $g(w)$  is analytic in  $N$ :

We know that  $f(z)$  is analytic, so it satisfies the Cauchy-Riemann equations (theorem 13.5.1). Hence, we can write the above four equations as:

$$x_u = \frac{u_x}{J} \quad x_v = -\frac{u_y}{J} \quad y_u = \frac{u_y}{J} \quad y_v = \frac{u_x}{J}$$

It is then clear that

$$x_u = y_v \quad x_v = -y_u$$

Which are the Cauchy-Riemann equations. Thus  $g(w)$  is analytic in  $N$ .  $\square$

Letting  $z = x + iy$  and  $w = u + iv$ :

$$\begin{aligned} z &= g(w) = x(u, v) + iy(u, v) \\ w &= f(z) = u(x, y) + iv(x, y) \end{aligned}$$

Thus the inverse exists.

**Example 19.2.1** Suppose that  $f$  has a zero of order  $m \in \mathbb{N}$  at  $z_0$ , that is

$$f'(z_0) = f''(z_0) = \dots = f^{(m-1)}(z_0) = 0, \quad f^{(m)}(z_0) \neq 0$$

Let  $w_0 = f(z_0)$ . Using the Taylor expansion of  $f(z)$  about  $z_0$  (theorem 16.2.1):

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n = f(z_0) + \sum_{n=m}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \\ &= f(z_0) + \frac{f^{(m)}(z_0)}{m!} (z - z_0)^m + \sum_{n=m+1}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \end{aligned}$$

This implies that

$$\begin{aligned} f(z) - f(z_0) &= f(z) - w_0 = \frac{f^{(m)}(z_0)}{m!} (z - z_0)^m \left[ 1 + \sum_{n=m+1}^{\infty} \frac{f^{(n)}(z_0)}{f^{(m)}(z_0)} \cdot \frac{m!}{n!} (z - z_0)^{n-m} \right] \\ &= \frac{f^{(m)}(z_0)}{m!} (z - z_0)^m [1 + g(z)] \end{aligned}$$

We can see that in  $g(z)$ ,  $n > m$ , so it is a Taylor series which is analytic at  $z_0$ . Also,  $g(z_0) = 0$ . Let  $C_1$  be a curve with image  $\Gamma_1$  under the transformation  $f$ , and  $\theta_0$  be the angle of inclination of  $C_1$  at  $z_0$ . Likewise with  $\phi_0$  for  $\Gamma_1$ . Then we have

$$\theta_0 = \lim_{z \rightarrow z_0} \arg(z - z_0) \quad \phi_0 = \lim_{z \rightarrow z_0} \arg[f(z) - w_0]$$

Using the results obtained previously:

$$\begin{aligned} \phi_0 &= \lim_{z \rightarrow z_0} \arg \left[ \frac{f^{(m)}(z_0)}{m!} (z - z_0)^m [1 + g(z)] \right] \\ &= \lim_{z \rightarrow z_0} \left[ m \arg(z - z_0) + \arg[f^{(m)}(z_0)] + \arg \left( \frac{1 + g(z)}{m!} \right) \right] \end{aligned}$$



Since

$$\lim_{z \rightarrow z_0} \frac{1 + g(z)}{m!} = \frac{1 + g(z_0)}{m!} = \frac{1}{m!} \in \mathbb{R} \implies \arg\left[\frac{1}{m}\right] = 0 \quad g(z_0) = 0$$

we have

$$\phi_0 = \lim_{z \rightarrow z_0} [m \arg(z - z_0)] + \arg[f^{(m)}(z_0)] = m\theta_0 + \arg[f^{(m)}(z_0)]$$

Now, let another curve  $C_2$  intersect  $C_1$  at  $z_0$ , and  $\Gamma_2$  intersect  $\Gamma_1$  at  $w_0$ . Then the angle of inclination of  $\Gamma_2$  at  $w_0$ :

$$\phi_1 = m\theta_1 + \arg[f^{(m)}(z_0)]$$

Let  $\alpha$  be the angle between  $C_1$  and  $C_2$  at  $z_0$ . Since we are still under the transformation  $f$ , the angle between  $\Gamma_1$  and  $\Gamma_2$  at  $w_0$ :

$$\phi_1 - \phi_0 = m(\theta_1 - \theta_0) + \arg[f^{(m)}(z_0)] - \arg[f^{(m)}(z_0)] = m\alpha \quad \alpha = \theta_1 - \theta_0$$

If the mapping is conformal,  $\phi_1 - \phi_0 = \theta_1 - \theta_0 = \alpha$ , so  $m = 1$ .

Note: Since  $f$  has a zero of order  $m$ ,  $f'(z_0) = 0$  for all  $m \geq 2$ , so  $z_0$  is a critical point of  $f$  for  $m \geq 2$  by definition 19.1.5. Thus the transformed angle is a natural number multiple of the original angle at a critical point.

## 19.3 Harmonic Conjugates

Review section 13.7.

Recall from the definition of a Harmonic Conjugate (definition 13.7.3). If function  $f(z) = u(x, y) + iv(x, y)$  is analytic in a domain  $D$ , then  $v(x, y)$  is a harmonic conjugate of  $u(x, y)$ .

### Theorem 19.3.1:

Let  $f(z) = u(x, y) + iv(x, y)$  be a function in domain  $D$ .

$$f \text{ analytic in } D \iff v \text{ is harmonic conjugate of } u$$

Proof:  $\Leftarrow$ :

$v$  is harmonic conjugate of  $u$

$\implies$  Cauchy-Riemann Equations Satisfied

$\implies f$  analytic in  $D$

Theorem 13.5.4

$\Rightarrow$ :

$f$  analytic in  $D$

$\implies u$  and  $v$  harmonic

Theorem 13.7.1

$\implies$  Cauchy-Riemann Equations satisfied

Theorem 13.5.1

□

**Example 19.3.1** ( $v$  harmonic conjugate of  $u$  does **not imply**  $u$  harmonic conjugate of  $v$ )  
Consider

$$u(x, y) = x^2 - y^2$$

$$v(x, y) = 2xy$$

We know that  $v$  is a harmonic conjugate of  $u$  since  $f(z) = x^2 - y^2 + i2xy = z^2$  is entire. However,  $u$  is not a harmonic conjugate of  $v$ , since  $g(z) = 2xy + i(x^2 - y^2)$  does not satisfy the Cauchy-Riemann equations and are not analytic anywhere:

$$u_x = 2y$$

$$v_y = -2y$$

$$u_y = 2x$$

$$-v_x = -2x$$

**Example 19.3.2** (Finding Harmonic Conjugates) Consider

$$u(x, y) = 2x(1 - y)$$

By Cauchy-Riemann equation, we have  $u_x = v_y = 2 - 2y$ . By integrating:

$$v(x, y) = 2y - y^2 + g(x)$$

By  $u_y = -v_x = -2x \implies g_x = 2x$ , then

$$v(x, y) = 2y - y^2 + x^2 + C$$

Therefore

$$f(z) = 2x(1 - y) + i(2y - y^2 + x^2 + C) = 2z + i(z^2 + C)$$

It is customary to write  $C = 0$ , since  $C$  is arbitrary, so

$$f(z) = 2z + iz^2$$

**Theorem 19.3.2:**

Let  $f(z) = u(x, y) + iv(x, y)$  be a harmonic function in a simply connected domain  $D$ , then  $u$  has a harmonic conjugate  $v$  in  $D$ .

*Proof:* content

□

## Part V

# Ordinary Differential Equations [Empty]

**Theorem 19.3.3: Green's Theorem**

Let  $F = P(x, y)\hat{i} + Q(x, y)\hat{j}$  be a vector field on a simple closed contour  $C$ ,  $R$  be the region enclosed and on  $C$ , and  $s$  be the path along  $C$ .

$$\int_C F \cdot ds = \iint_R \nabla \times F \, dx \, dy$$

## Part VI

### Nonlinear Dynamics [Empty]



## Part VII

### Partial Differential Equations [Empty]

**Definition 19.3.1: Dirichlet Problem**

*Finding a function in an harmonic domain that assumes preassigned values at the boundary of the domain.*

**Theorem 19.3.4: Fourier Theorem**

*Let a function  $f$ :*

- 1. Piecewise continuous on  $[-\pi, \pi]$*
- 2. Periodic with period  $2\pi \ \forall x \in \mathbb{R} \cup \{-\infty, \infty\}$*
- 3.  $\forall x \in \mathbb{R} \cup \{-\infty, \infty\}$ ,  $f'_+(x)$  and  $f'_-(x)$  both exist*

*Then the Fourier series converges to the mean value*

$$\frac{f(x+) + f(x-)}{2}$$

*of one-sided limit of  $f$  at  $x$*

**Calculus of Variations [Empty]**



## Part VIII

### Integral Equations [Empty]



## Part IX

### Linear Algebra [Empty]



# Chapter 20

## Markov Chains



## Part X

### Tensors [Empty]





## Part XI

### Riemann Geometry [Empty]



## Part XII

### Abstract Algebra [Empty]



# Chapter 21

## Groups



# Chapter 22

## Rings

### 22.1 Ideals





## Chapter 23

### Integral Domains



## Chapter 24

### GCD Domains



## Chapter 25

# Unique Factorization Domains



## Chapter 26

# Principal Ideal Domains





# Chapter 27

## Fields



## Part XIII

### Galois Theory [Empty]



## Part XIV

### Lie Theory [Empty]



# Chapter 28

## Lie Groups





# Chapter 29

## Lie Algebra



## Part XV

### C-Star Algebra [Empty]



## Part XVI

### Set Theory [Empty]



## Part XVII

### Model Theory [Empty]





**Part XVIII**

**Statistics [Empty]**



## Part XIX

### Tips and Tricks [Empty]



# Chapter 30

## Integration Techniques

### 30.1 DI Method (Integration Table)

### 30.2 Feynman Integration



## Part XX

### Index and Bibliography





# Index

- Absolute Convergence
  - Power Series, 139
  - Series, 128
- Analytic Continuation, 78
- Analytic Signal
  - Fourier Transform, 194
- Analyticity, 116
  - Function
    - Integration to Infinity, 196
- Angle of Rotation, 221
- Antiderivative, 99, 100
- Arc, 94
  - Differentiable, 95
  - Jordan, 94
  - Simple, 94
- Argument
  - Complex Number, 33
  - Principal, 34
  - Product of Complex Numbers, 36
- Argument Principle, 182
- Bessel Functions
  - First Kind, 137
- Bilinear Transform, 202
  - Implicit Form, 204
- Bolzano-Weierstrass Theorem, 163
- Branch, 82
  - Cut, 82
  - Point, 83
  - Principal, 82
- Bromwich Integral, 189
- Casorati-Weierstrass Theorem, 165
- Cauchy Integral Formula, 113
  - Extension, 114
- Cauchy Principal Value, 167
  - of Integral with Singularity, 174
- Cauchy Product, 148
- Cauchy's Inequality, 116
- Cauchy's Residue Theorem, 154
- Cauchy-Goursat Theorem, 103
- Cauchy-Riemann Equations
  - Cartesian, 63
  - Complex Form, 69
  - Polar, 64
- Causal System, 195
- Center of Expansion, 139
- Chain Rule
  - for Composite Functions, 60
- Circle of Convergence, 139
- Coincidence Principle, 78
- Complex Conjugate, 38
- Complex Number
  - Exponential Form, 34
  - Polar Form, 34
  - Roots, 37
- Complex Plane
  - Extended, 51
- Complex-Valued Function, 94
  - Definite Integral, 93
  - Derivative, 93
- Conformal, 221
- Conformal Mapping, 221, 222
- Conformal Transformation, 222
- Continuity
  - Polynomial, 54
- Contour, 95
  - Simple Closed, 95
- Contour Integral, 96, 100

- in Simply Connected Domain, 107
  - Properties, 97
- Convergence
  - Conditional, 128
  - Infinite Integral, 166
  - Uniform
    - Integrals, 196
- Convolution, 151
  - Hilbert Transform, 193
  - Inverse  $z$ -Transform, 151
  - $z$ -Transform, 151
- Cosine
  - Complex Form, 39, 85
  - Hyperbolic, 87
    - Zeros, 88
- Critical Point, 223
- Curve
  - Jordan, 94
  - Simple Closed, 94
- de Moivre's Formula, 35
- Derivative, 57
- Differentiable, 57
- Differentiability
  - Conditions, 70
- Dirichlet's Integral, 176
- Domain, 43
  - Simply Connected, 107
- Double Angle Identities, 35
- Entire Function
  - Antiderivatives, 108
- Euler's Formula, 34
- Exponential
  - Complex Number, 37
- Fourier Series, 137
- Fourier Transform
  - Analytic Signal, 194
  - Hilbert Transform, 194
- Fresnel Integrals, 172
- Function
  - Analytic, 72
  - Analytic Part, 156
  - Behaviour Near Essential Singular, 165
  - Behaviour Near Poles of Order, 166
  - Behaviour Near Removable Singular Points, 164
  - Elements, 78
  - Entire, 72
  - Exponential, 81
  - Harmonic, 75
    - Conjugate, 76, 227
  - Holomorphic, 72
  - Hyperbolic
    - Inverse, 91
  - Logarithmic, 81
  - Meromorphic, 181
  - Multiple-Valued, 46
  - Power, 84
  - Principal Part, 156
  - Rational, 46
  - Regular, 72
  - Trigonometric
    - Inverse, 89
- Fundamental Theorem of Algebra, 117
  - via Rouché's Theorem, 185
- Gamma Function, 196
  - Complex, 197
  - Factorials, 196
  - Reflection Formula, 198
- Gauss's Mean Value Theorem, 118
- Green's Function, 195
- Harmonic Conjugate, 76, 227
- High Order Derivatives, 116
- Hilbert Transform, 192
  - Causal System, 195
  - Fourier Transform, 194
  - Inverse, 192, 194
  - of a Constant, 193
  - Pairs, 192
- Impulse Response, 195
- Indented Path, 174
  - Around Simple Poles, 175
- Integration
  - Along Branch Cut, 177
- Isogonal Mapping, 222

- Jordan Curve Theorem, 95
- Jordan's Inequality, 170
- Jordan's Lemma, 171
- L'Hopital's Rule, 61
- Laplace Transform, 189
  - Inverse
    - Derivation, 190
- Laplace's Equation, 75
  - Polar Form, 76
- Laurent Series, 132
  - Uniqueness, 147
- Laurent's Theorem, 132
- Legendre Polynomials, 115
- Leibniz's Rule, 147
- Limit, 47
  - at Infinity, 53
  - Uniqueness, 48
- Line
  - Polygonal, 43
- Line Integral
  - Complex, 96
  - Real, 96
- Linear Fractional Transformation, 202
  - Implicit Form, 204
- Linear Transformation, 201
- Linear Fractional Transformation
  - Fixed Points, 204
- Liouville's Theorem, 117
- Local Inverse, 225
- Logarithm
  - Identities, 83
- M-Test
  - Integration to Infinity, 196
- Maclaurin Series, 131
- Mapping
  - $z^2$ , 211
  - $z^{1/n}$ , 212
  - Cosine, 211
  - Exponential, 208
  - Hyperbolic Cosine, 211
  - Hyperbolic Sine, 211
  - Sine, 209
- Maximum Modulus Principle, 119
- Mean Value Theorem, 94
  - Gauss's, 118
- ML Inequality, 98
- Modulus, 32
- Morera's Theorem, 106
- Möbius Transform, 202
  - Implicit Form, 204
- Neighbourhood, 40
  - Deleted, 41
  - of Infinity, 52
- Phasor, 91
  - Properties, 91
- Picard's Theorem, 157
- Point
  - Accumulation, 44
  - Boundary, 42
  - Essential Singular, 156
  - Exterior, 41
  - Function Behaviour Near Essential Singular, 165
  - Function Behaviour Near Removable Singular, 164
  - Interior, 41
  - Isolated Singular, 153
  - Limit, 44
  - Removable Singular, 156
  - Singular, 73
- Poisson Integral Formula
  - Circle Interior, 122
  - Upper Half-Plane, 123
- Pole, 159
  - and Zero Relation, 162
  - Function Behaviour Near of Order, 166
  - Simple, 156
- Poles
  - of Order, 156
- Polynomial, 45
- Power Series, 130
  - Absolute Convergence, 139
  - Continuity of Sums, 142
  - Differentiation, 145
  - Integration, 143
  - Local Uniform Convergence, 141
  - Uniform Convergence, 140

- Range, 45
  - Inverse, 45
- Ratio Test, 129
- Reflection Principle, 79
- Region, 43
  - Bounded, 43
  - Closed, 44
- Residue, 153
  - Applications, 166
  - at Infinity, 154, 155
  - at Poles, 157
  - of Simple Pole, 162
  - to Find Derivative, 159
- Riemann Sphere, 51
- Riemann Surface, 216
- Riemann's Theorem, 165
- Rouché's Theorem, 184, 187
- Scale Factor, 223
- Sequence
  - Convergence, 125
  - Divergence, 125
  - Limit, 125
- Series
  - Absolutely Convergent, 128
  - Convergence, 127
  - Divergence, 127
  - Partial Sum, 127
  - Remainder, 128
- Set
  - Boundary, 42
  - Bounded, 43
  - Closed, 43
  - Closure, 43
  - Connected, 43
  - Open, 42
- Signal
  - Analytic, 193
- Sine
  - Complex Form, 39, 85
  - Hyperbolic, 87
  - Zeros, 88
- Smooth, 95
- Stereographic Projection, 52
- Sum
  - Contour, 97
- Taylor Series
  - Uniqueness, 145
- Taylor's Theorem, 130
- Transfer Function, 195
- Triangle Inequality, 32
- Uniform Convergence
  - Power Series, 140
- Winding Number, 181
- z-Transform, 136, 149
  - Convolution, 151
  - Inverse, 150
  - Product, 150
- Zero
  - and Pole Relation, 162
  - Isolated, 161
  - of Order, 160

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