The Book of Math (Notes)

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Forward and Disclaimer

These are math notes made by a student (with a physics major and math minor) based off text books. It may contain misconceptions and misinterpretations, thus should not be viewed in the same light of a text book. Use at your own risk and mental sanity.

Symbols

Logic

Name	**Symbol**	**Comment**
Exists	$\exists$	There exists at least one
For all	$\forall$	
Not exists	$\nexists$	There does not exist
Exists one	$\exists!$	There only exists one and only one
And	$\wedge$	
Or	$\vee$	Inclusive or
Not	$\neg$	
Logically implies	$\implies$	If
Logically implied by	$\impliedby$	Only if
Logically equivalent	$\iff$	If and only if
Implies	$\longrightarrow$	
Implied by	$\longleftarrow$	
Double Implication	$\longleftrightarrow$	

Set Notation

Name	**Symbol**	**Comment**
Empty Set	$\varnothing$	The set that is empty
Natural Numbers	$\mathbb{N}$	Set of natural numbers not containing 0, equivalent to the set of positive integers
Integers	$\mathbb{Z}$	Set of integers
Rational Numbers	$\mathbb{Q}$	
Algebraic Numbers	$\mathbb{A}$	
Real Numbers	$\mathbb{R}$	
Complex Numbers	$\mathbb{C}$	
In	$\in$	
Not in	$\notin$	
Owns	$\ni$	Has an element
Proper Subset	$\subset$	Subset that is not itself
Subset	$\subseteq$	
Superset	$\supset$	Superset that is not itself
Proper Superset	$\supseteq$	

Power set	$\wp$	
Union	$\cup$	
Intersection	$\cap$	
Difference	$\setminus$	

Relationships

Name	Symbol	Comment
Defined	$\doteq$	
Approximate	$\approx$	
Equivalent	$\equiv$	Isomorphic (Group Theory)
Congruent	$\cong$	Homomorphic (Group Theory)
Proportional	$\propto$	

Operators

Name	Symbol	Comment
	$\oplus$	
	$\otimes$	
	$\odot$	
	$\circ$	Convolution
Dagger	$\dagger$	Complex conjugate transpose of a matrix

Arrows

Name	Symbol	Comment
Maps to	$\mapsto$	

Hebrew

Name	Symbol	Comment
Aleph	$\aleph$	Carnality of infinite sets that can be well ordered

Other

Name	Symbol	Comment
Real part	$\Re$	Real part of a number
Imaginary part	$\Im$	Imaginary part of a number

Book Constitution

Intents and Purpose

The goal of this book is to organize mathematical knowledge of topics related to the study of physics or the author's interest. It is meant to be used as a source of for future reference, not as a textbook for students new to the topics. It is a notebook of a student, thus should be treated as one and not as a textbook. At most, it could be used as a study guide along side a textbook. Definitely not as the main source for acquiring knowledge.

Layout and Organization

The book is split into parts each containing a field of study mathematics, or a topic large enough to justify giving it its own part. Each part contains chapters that focuses on a particular topic required to understand the field, with sections dedicated to describing a particular knowledge required for the topic.

As axioms, definitions, theorems, corollary, and proofs are integral and abundant to the study of mathematics, each will have a unique style. Each environment and its styles are displayed as follows:

Axiom 0.1: Axiom name
Example Axiom Axioms are the "ground rules" of the set.

Theorem 0.0.1: Theorem name or citation
Example Theorem An important logical result from the axioms, with proof.

Conjecture 0.0.1: Name of conjecture or citation
Example Conjecture A hypothesis, without proof.

Corollary 0.0.1.1:
Example Corollary An implication as a result of a theorem.

Lemma 0.0.1.1:
Example Lemma Small theorems that build up to a larger theorem.

Proposition 0.0.1.1:
Example Proposition Example proposition.

Proof: Logical deductions that results in a theorem. Proofs I've written will be in grey, which may or may not be correct. □

Definition 0.0.1: Word
Example Definition The definition of a word.

Example 0.0.1. *An example.*

Remark. *Remark A comment by the author in the textbooks used.*

Observation. *Example Observation A remark by me.*

Question. *Example Question A question from me for a mystery to be answered later.*

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Part III

Real Analysis

Resources used in part III

1. Kenneth A. Ross - Elementary Analysis (2nd Ed.) [1]

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Complex Analysis

Resources used in part IV

1. Brown and Churchill - Complex Variables and Applications [2]

Chapter 10

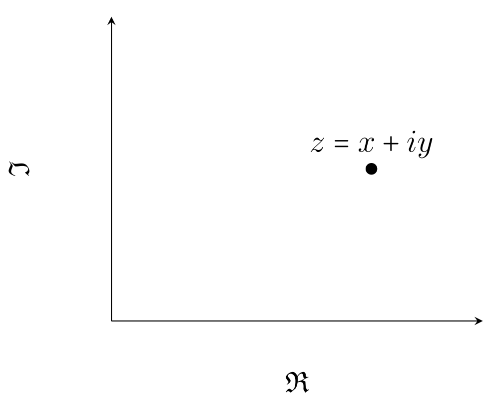
Basics

10.1 Complex Numbers

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}, i = \sqrt{-1}\}$$

Complex numbers are elements of the complex field ($\mathbb{C}$), therefore, they obey all the properties of a field.

We will denote complex numbers by $z = x + iy$ with $x, y \in \mathbb{R}$, and refer the real part as $\mathfrak{R}(z) = \mathrm{Re}(z) = x$ and imaginary part as $\mathfrak{I}(z) = \mathrm{Im}(z) = y$. Complex numbers can also be defined as an ordered pair $z = (x, y)$ which is interpreted as points in the complex plane. $(x, 0)$ are points on the real axis while $(0, y)$ are points in the imaginary axis.



We add and multiply complex numbers in the usual way:

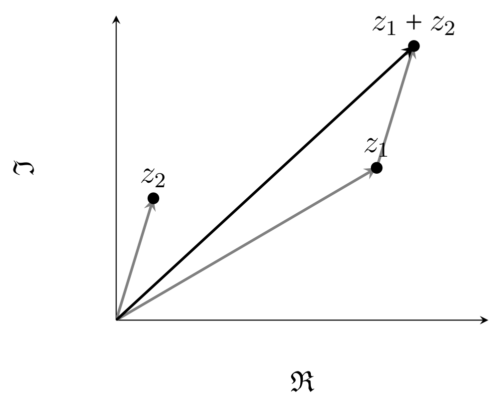
$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \end{aligned} \qquad \begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \end{aligned}$$

$\forall z \in \mathbb{C}$, there is an unique additive inverse $(-z)$ and $\forall z \in \mathbb{C} \setminus \{0\}$, there is an unique multiplicative inverse (z^{-1}) such that

$$\begin{aligned} &z + (-z) = 0 && zz^{-1} = 1 \\ &\implies -z = -x - iy && \implies (x_1x_2 - y_1y_2) = 1 \wedge (x_1y_2 + x_2y_1) = 0 \\ & && \implies z^{-1} = \frac{x_1}{x_1^2 + y_1^2} - i\frac{y_1}{x_1^2 + y_1^2} \end{aligned}$$

The existence and uniqueness of the inverses can be easily proven.

The addition of complex numbers may also be interpreted as akin to vector addition.



Likewise, this naturally extends to the definition of a modulus of a complex number.

Definition 10.1.1: Modulus

The absolute value of a real number: $|z| = \sqrt{x^2 + y^2}$

It is obvious why the definition is not $|z| = \sqrt{x^2 + (iy)^2}$ as problems arise when $x = y$. The modulus is the distance of z from $(0, 0)$.

10.2 Triangle Inequality

It is not analysis without a section dedicated to the triangle inequality.

Theorem 10.2.1: Triangle Inequality

$\forall z_1, z_2 \in \mathbb{C}[|z_1 + z_2| \leq |z_1| + |z_2|]$

From the theorem, we can derive a similar inequality:

$$|z_1| = |z_1 + z_2 - z_2| \leq |z_1 + z_2| + |-z_2| \implies |z_1| - |z_2| \leq |z_1 + z_2|$$

An important property of polynomials is observed when theorem 10.2.1 is applied to polynomials.

Corollary 10.2.1.1:

Consider the polynomial $P(z)$ where $a_n \in \mathbb{C}$, $n \in \mathbb{N}$, $a_0 \neq 0$, and $z \in \mathbb{C}$.

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

Then $\forall z, \exists R \in \mathbb{R}_{>0}, |z| < R$ such that

$$\left|\frac{1}{P(z)}\right| < \frac{2}{|a_n| R^n}$$

Proof: Consider

$$\begin{aligned}
w = \frac{P(z)}{z_n} - a_n = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \ldots + \frac{a_{n-1}}{z} & \qquad z \neq 0 \\
\implies wz^n = a_0 + a_1 z + \ldots + a_{n-1} z^{n-1} & \\
\implies |w||z|^n \leq |a_0| + |a_1||z| + \ldots + |a_{n-1}||z|^{n-1} & \\
\implies |w| \leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \ldots + \frac{|a_{n-1}|}{|z|} & \\
\implies |w| < n\frac{|a_n|}{2n} = \frac{|a_n|}{2} & \qquad \exists \text{ sufficiently large } R < |z| \text{ s.t.} \\
& \qquad \forall m,\ 0 \leq m \leq n-1,\ \frac{|a_m|}{|z|^{n-m}} < \frac{|a_n|}{2n} \\
\implies |a_n + w| \geq ||a_n| - |w|| > \frac{|a_n|}{2} & \qquad R < |z| \\
\implies |P_n(z)| = |a_n + w||z|^n > \frac{|a_n|}{2}|z|^n > \frac{|a_n|}{2}R^n & \qquad R < |z| \\
\implies \left|\frac{1}{P(z)}\right| < \frac{2}{|a_n| R^n} &
\end{aligned}$$

□

This tells us that if z is a solution to a polynomial $P(z)$, then the reciprocal of the polynomial $1/P(z)$ is bounded above by $R = |z|$. (i.e. It is bounded by a circle of radius $|z|$.)

10.3 Polar and Exponential Form

Definition 10.3.1: Argument of z

Consider any $z \in \mathbb{C}$ where $z \neq 0$. Let θ be the angle in radians between z and the real axis . Then $\forall n \in \mathbb{N}$, $0 \leq \theta < 2\pi$, the argument of z:

$$\arg(z) = \theta + 2n\pi$$

We know $\forall n \in \mathbb{N}$, $\theta + 2\pi n = \theta$. This leads us to the definition of the principal argument of z.

Definition 10.3.2: Principal Argument of z

Consider any $z \in \mathbb{C}$ where $z \neq 0$. Let θ be the angle in radians between z and the real axis. Then for $0 \leq \theta < 2\pi$, the principal argument of z:

$$\operatorname{Arg}(z) = \theta$$

It is clear that $\arg(z) = \operatorname{Arg}(z) + 2n\pi$.

Definition 10.3.3: Polar Form of z

Consider $z \in \mathbb{C}$. Let $r = |z|$, and $\theta = \arg(z)$. Then $\forall z \in \mathbb{C}, z \neq 0$:

$$z = x + iy = r(\cos(\theta) + i\sin(\theta))$$

Notice that all three definitions require that $z \neq 0$ as θ is undefined at $z = 0$.

Theorem 10.3.1: Euler's Formula

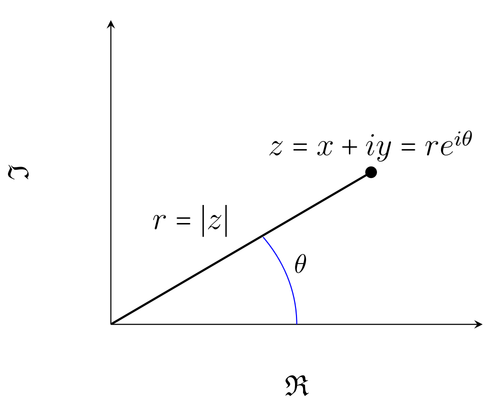
$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Combining definition 10.3.3 with theorem 10.3.1, we obtain the Exponential Form of z:

Definition 10.3.4: Exponential Form of z

Consider any $z \in \mathbb{C}$, and let $r = |z|$ and $\theta = \arg(z)$. Then the exponential form of z:

$$z = re^{i\theta}$$



10.3.1 Properties of Polar Form

10.3.2 Properties of Exponential Form

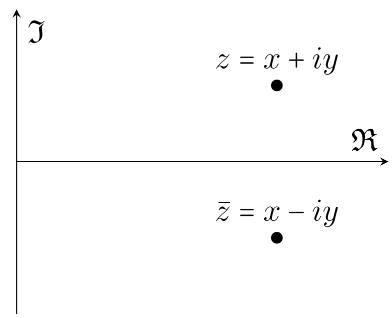
10.4 Complex Conjugates

Definition 10.4.1: Complex Conjugate

The complex conjugate of $z \in \mathbb{C}$ is denoted $\bar{z}$.

$$\bar{z} = x - iy = r(\cos(\theta) - i\sin(\theta)) = re^{-i\theta}$$

Graphically, it is the reflection of z across the real axis.



It is then easy to see

$$\mathrm{Re}(z) = \frac{z + \bar{z}}{2} \qquad \mathrm{Im}(z) = \frac{z - \bar{z}}{2i} \qquad |z|^2 = z\bar{z}$$

As $\mathrm{Re}(z) = x = r\cos(\theta)$ and $\mathrm{Im}(z) = y = r\sin(\theta)$ and using definition 10.3.4, we can obtain the complex forms of sine and cosine:

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

10.5 Operations as Transformations

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Bibliography

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[1] Kenneth A. Ross. *Elementary Analysis.* Springer, 2 edition, 2013.

[2] James Ward Brown and Ruel V. Churchill. *Complex Variables and Applications.* McGraw-Hill Education, 9 edition, 2014.