PHYS 411
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Assignment #1

1. Unbiased Estimator

$$S^2 = (N-1)^{-1} \sum_{i=1}^{N} (x - \bar{x})^2$$

$$S_b^2 = (N)^{-1} \sum_{i=1}^{N} (x - \bar{x})^2 \quad \Leftarrow \text{Clearly biased.}$$

Assuming we don't know which is biased, we need to find $E[S^2]$ and $E[S_b^2]$ and its relationship to the standard deviation σ to determine which is biased.

$$E[S^2] = E[(N-1)^{-1} \sum_{i=1}^{N} (x - \bar{x})^2]$$

We know $E[(x_i - \mu)^2] = \sigma^2$ ∴ Introduce μ

$$= (N-1)^{-1} E\left[\sum_{i=1}^{N} (x_i - \mu - (\bar{x} + \mu))^2\right]$$

$$= (N-1)^{-1} E\left[\sum_{i=1}^{N} (\alpha_i - \beta)^2\right]$$

$$= (N-1)^{-1} E\left[\sum_{i=1}^{N} (\alpha_i^2 + \beta^2 - 2\alpha_i\beta)\right]$$

$$= \sigma_\Sigma^2 (N-1)^{-1} + E\left[\sum_{i=1}^{N} (\beta^2 - 2\alpha_i\beta)\right] (N-1)^{-1}$$

$$= \sigma_\Sigma^2 (N-1)^{-1} + E\left[N\beta^2 - 2\beta \sum_{i=1}^{N} \alpha_i\right] (N-1)^{-1}$$

Detour: $\sum_{i=1}^{N} \alpha_i = \sum_{i=1}^{N} (x_i - \mu) = \sum_{i=1}^{N} x_i - N\mu.$

$$\therefore \frac{1}{N} \sum_{i=1}^{N} \alpha_i = \frac{1}{N} \sum_{i=1}^{N} x_i - \mu = \sum_{i=1}^{N} (\bar{x} - \mu) = \beta.$$

$$\therefore E[S^2] = \sigma_\Sigma^2 (N-1)^{-1} + E[N\beta^2 - 2N\beta^2] (N-1)^{-1}$$

$$= \sigma_\Sigma^2 (N-1)^{-1} + E[-N\beta^2] (N-1)^{-1}$$

We know $E[\beta] = \frac{\sigma^2}{N}$

$$= \sigma_\Sigma^2 (N-1)^{-1} - \sigma^2 (N-1)^{-1}$$

$$= (N-1)^{-1} \left[\sum_{i=1}^{N} \sigma^2 - \sigma^2\right] = (N-1)^{-1} \sigma^2 (N-1) = \sigma^2$$

For $E[S_b^2]$, it is the same procedure as finding $E[S^2]$, so skip to the last few steps.

$$\therefore E[S_b^2] = E[N^{-1} \sum_{i=1}^{N} (x - \bar{x})^2]$$

$$=$$

$$= \sigma_\Sigma^2 (N)^{-1} - \sigma^2 (N)^{-1}$$

$$= N^{-1} \sum_{i=1}^{N} \sigma^2 - \sigma^2 (N)^{-1}$$

$$= \sigma^2 - \sigma^2 (N)^{-1}$$

$$= \sigma^2 [1 - N^{-1}] = \sigma^2 \left(\frac{n-1}{n}\right) < \sigma^2$$

$\therefore$ S_b^2 is clearly biased as it underestimates σ.
S^2 is not biased as it estimates the proper value of σ.

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Assignment 1

Import required packages:

In [3]:
```
import random
import numpy as np
import scipy as sci
from scipy.stats import norm
import matplotlib.pyplot as plt
from IPython.display import Math, display
```

Question 2: Monte Carlo simulation of the Central Limit theorem

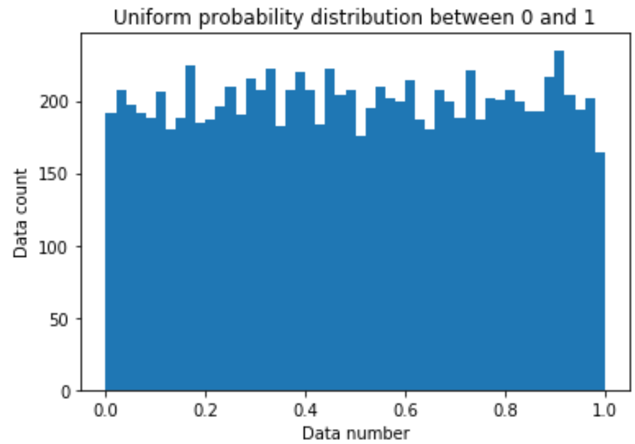
Generate Data:

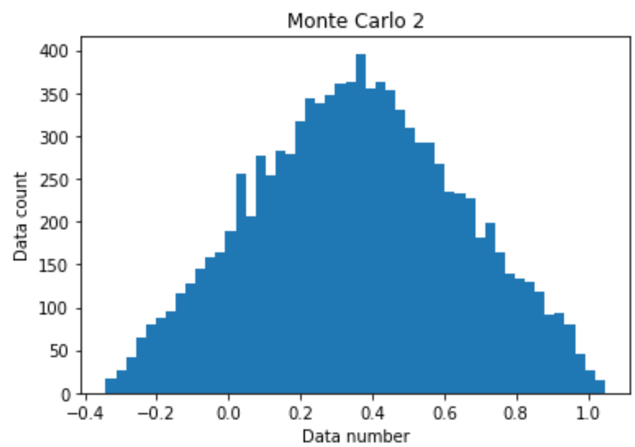
In [11]:
```
mean = 0.5
datanumber = 10000
n_bins = 50

# Generate an array A1
# Then subtract 0.5 from all the values
A0 = np.random.rand(datanumber)
A1 = np.random.rand(datanumber)
A2 = A0 + A1 - 0.5

# Doing it 4, 8, and 16 more times:
n4 = [4, 8, 16]

A3 = A1
for i in range(0, n4[0]):
    T1 = np.random.rand(datanumber)
    A3 = A3 + T1 - 0.5

A4 = A1
for j in range(0, n4[1]):
    T2 = np.random.rand(datanumber)
    A4 = A4 + T2 - 0.5

A5 = A1
for k in range(0, n4[2]):
    T3 = np.random.rand(datanumber)
    A5 = A5 + T3 - 0.5
```

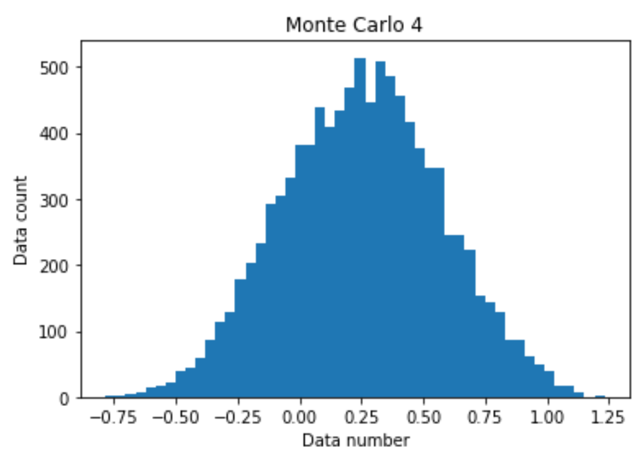
Plot Data:

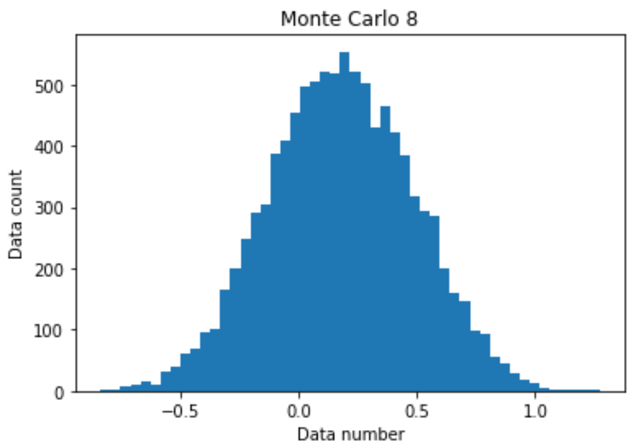
In [12]:
```
plt.figure(1)
plt.hist(A1, bins=n_bins)
plt.xlabel(r'Data number')
plt.ylabel(r'Data count')
plt.title(r'Uniform probability distribution between 0 and 1')

plt.figure(2)
plt.hist(A2/np.sqrt(2), bins=n_bins)
plt.xlabel(r'Data number')
plt.ylabel(r'Data count')
plt.title(r'Monte Carlo 2')

plt.figure(3)
plt.hist(A3/np.sqrt(n4[0]), bins=n_bins)
plt.xlabel(r'Data number')
plt.ylabel(r'Data count')
plt.title(r'Monte Carlo 4')

plt.figure(4)
plt.hist(A4/np.sqrt(n4[1]), bins=n_bins)
plt.xlabel(r'Data number')
plt.ylabel(r'Data count')
plt.title(r'Monte Carlo 8')

plt.figure(5)
plt.hist(A5/np.sqrt(n4[2]), bins=n_bins)
plt.xlabel(r'Data number')
plt.ylabel(r'Data count')
plt.title(r'Monte Carlo 16')
```

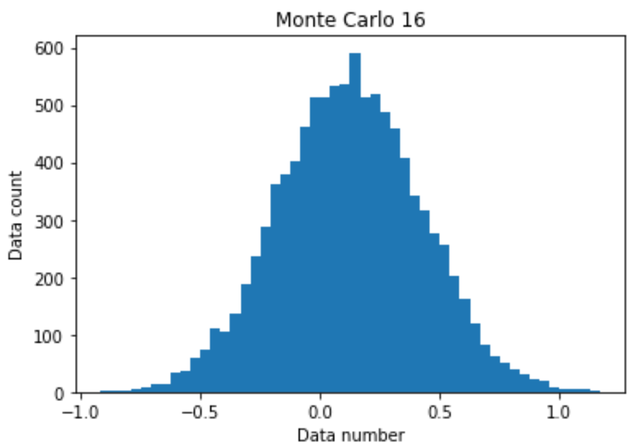
```
Out[12]: Text(0.5,1,'Monte Carlo 16')
```











As we can see, the graphs approach a Gaussian as the sample count increases. This is a result of probability. Data in this graph is generated in a range from 0 to 1, then the values are counted and plotted on a histogram. A second set of data is generated and added to the original set, this process is repeated multiple times. The histogram will approach a Gaussian because it is unlikely for a value in the data set to be repeatedly assigned an extreme value at one end of the spectrum, hence, decreasing the data count at the extremes with each successive data set. Values from the repeated addition of data sets will tend to average towards the expected value, therefore, data counts close to the expected value will increase. Thus, the data will tend towards the normal distribution (a Gaussian) as the sample size increases.

Question 3: Flipping two coins

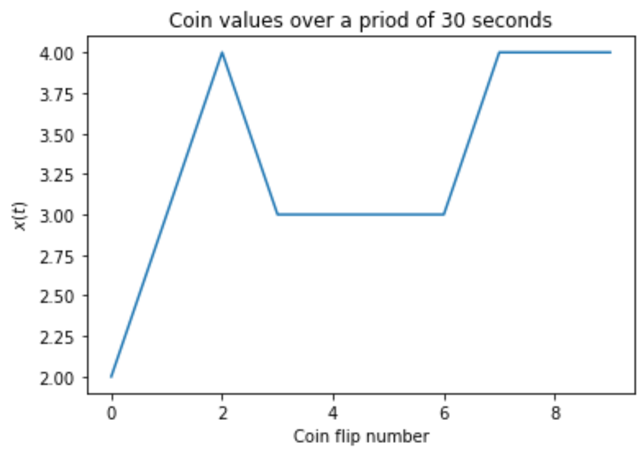
In [8]:
```
t = 60*60   # Total flipping time
p = 3       # Period between flips

# Generate coin flips
B0 = np.random.randint(1, 3, int(t/p)+2)
B1 = []

# Add the coin values
for n in range(0, len(B0)-1):
    b = B0[n] + B0[n+1]
    B1.append(b)

print(B1[:10])
```

```
[2, 3, 4, 3, 3, 3, 3, 4, 4, 4]
```

a) Possible outcome of x(t) for the first 30s:

In [9]:
```
plt.figure(6)
plt.plot(B1[:10])
plt.xlabel(r'Coin flip number')
plt.ylabel(r'$x(t)$')
plt.title(r'Coin values over a priod of 30 seconds')
print(B1[:10])
```

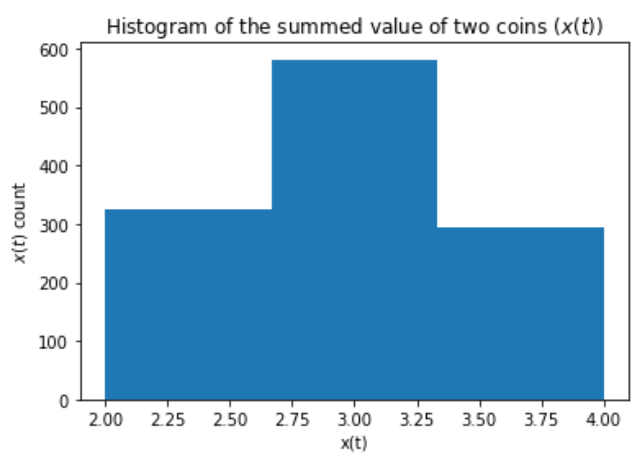
```
[2, 3, 4, 3, 3, 3, 3, 4, 4, 4]
```



b) A likely histogram of x(t)

In [10]:
```
plt.figure(7)
plt.hist(B1, bins=3)
plt.xlabel(r'x(t)')
plt.ylabel(r'$x(t)$ count')
plt.title(r'Histogram of the summed value of two coins ($x(t)$)')
```

Out[10]:
```
Text(0.5,1,'Histogram of the summed value of two coins ($x(t)$)')
```



c) Probability density function $p(x)$, and probability distribution function $P(x)$

Probability density function: $p(x)$

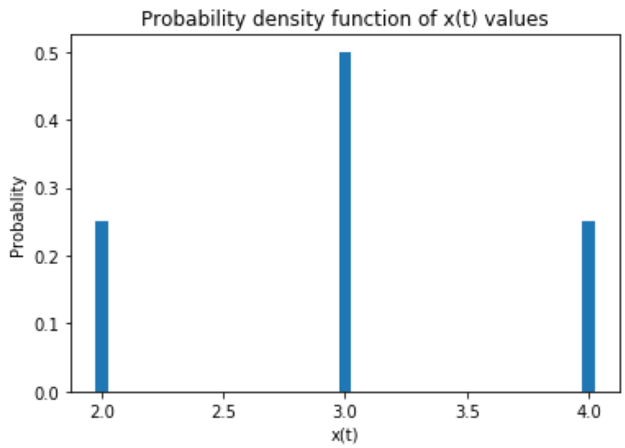
$$p(x) = \frac{1}{4}\delta(x-2) + \frac{1}{2}\delta(x-3) + \frac{1}{4}\delta(x-4)$$

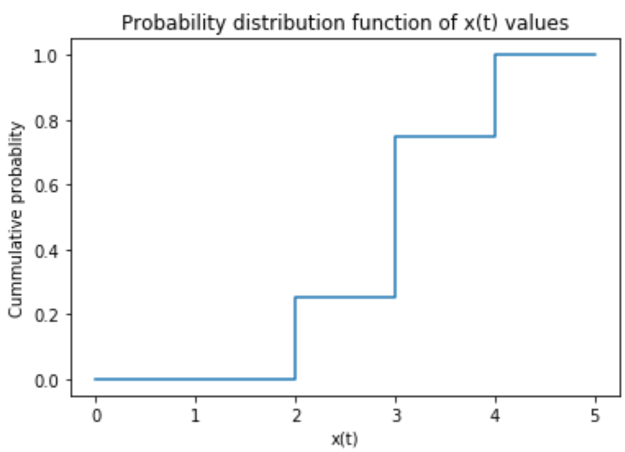
Probability distribution function: $P(x)$

$$P(x) = \begin{cases} \frac{1}{4} & x < 2 \\ \frac{3}{4} & 2 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

In [11]:
```
rang = [2, 3, 4]
RANG = [0, 2, 3, 4, 5]
pdf = [0.25, 0.5, 0.25]
PDF = [0, 0, 0.25, 0.75, 1]

# Plot the probability density function
plt.figure(8)
plt.bar(rang, pdf, width=0.05)
plt.xlabel(r'x(t)')
plt.ylabel(r'Probablity')
plt.title(r'Probability density function of x(t) values')

# Plot the probability distribution function
plt.figure(9)
plt.step(RANG, PDF)
plt.xlabel(r'x(t)')
plt.ylabel(r'Cummulative probablity')
plt.title(r'Probability distribution function of x(t) values')
```

Out[11]:
```
Text(0.5,1,'Probability distribution function of x(t) values')
```





d) The mean μ_x of $x(t)$, and the variance σ_x^2 of $x(t)$

In [12]:
```
# Find the mean
mean = round(np.mean(B1), 1)
display(Math(r'\mu_{x}=%.2f' % mean))

# Find the varience
var = round(np.var(B1), 1)
display(Math(r'\sigma_{x}^{2}=%.2f' % var))
```

$\mu_x = 3.00$

$\sigma_x^2 = 0.50$

Question 4: Comparison of minute-resolution and hourly-resolution data

In [4]:
```
# Load the data
D1 = np.loadtxt('/Users/Kev/Documents/Uvic/Python/PHYS 411 - Time Series Analysis
/Data Sets/UVicSci_temperature.dat', dtype=float)
D2 = np.loadtxt('/Users/Kev/Documents/Uvic/Python/PHYS 411 - Time Series Analysis
/Data Sets/AllStations_temperature_h_2017.dat', dtype=float)

# Check data dimensions
print(D1.shape, type(D1))
print(D2.shape, type(D2))
```

```
(2979363,) <class 'numpy.ndarray'>
(84723, 38) <class 'numpy.ndarray'>
```

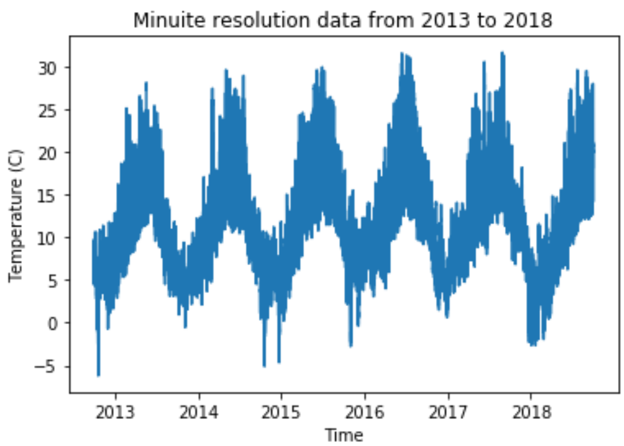
1)

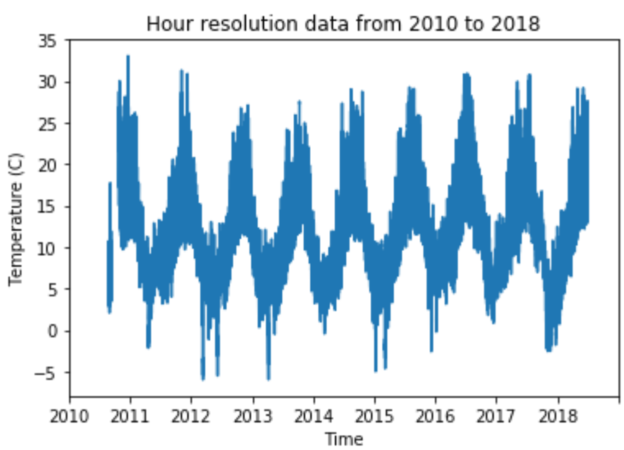
Generate graphs:

In [5]:
```
date1 = ['2013', '2014', '2015', '2016', '2017', '2018']
date2 = ['2010', '2011', '2012', '2013', '2014', '2015', '2016', '2017', '2018']

ypos1 = np.array([0, 1, 2, 3, 4, 5])*len(D1[3:])/6 + len(D1[3:])/24
ypos2 = np.array([0, 1, 2, 3, 4, 5, 6, 7, 8, 9])*((D2[84722:,0]-D2[3,0])[0]/9)+D2
[3,0]+(D2[84722:,0]-D2[3,0])[0]/18
```

In [6]:
```
# Temprature
plt.figure(10)
plt.plot(D1[3:])
plt.xlabel(r'Time')
plt.ylabel(r'Temperature (C)')
plt.xticks(ypos1, date1)
plt.title(r'Minuite resolution data from 2013 to 2018')

# Temperature all stataions
# UVic Science Building
# Longitude: 236.691
# Latitude: 48.462
# Column: 36 => Index: 25
plt.figure(11)
plt.plot(D2[3:,0],D2[3:,35])
plt.xlabel(r'Time')
plt.ylabel(r'Temperature (C)')
plt.xticks(ypos2, date2)
plt.title(r'Hour resolution data from 2010 to 2018')
```

Out[6]: `Text(0.5,1,'Hour resolution data from 2010 to 2018')`





Calculate averages and standard deviation:

In [9]:
```
# Find the mean
d1 = D1[3:]
d2 = D2[3:,35]

MnM1 = np.nanmean(d1)
display(Math(r'\mu_{M}=%.5f' % MnM1))

HrM1 = np.nanmean(d2)
display(Math(r'\mu_{H}=%.5f' % HrM1))

# Find the Standard Deviation
MnV1 = np.nanvar(d1)
display(Math(r'\sigma_{M}=%.5f' % MnV1**0.5))

HrV1 = np.nanvar(d2)
display(Math(r'\sigma_{H}=%.5f' % HrV1**0.5))

# Find the relationships
PDM = (MnM1-HrM1)/(HrM1) * 100
print(r'Percentage difference between mean values:', PDM, r'%')

PDV = (MnV1**0.5-HrV1**0.5)/(HrV1**0.5) * 100
print(r'Percentage difference between standard deviation values:', PDV, r'%')
```

$\mu_M = 11.29531$

$\mu_H = 11.19736$

$\sigma_M = 5.60895$

$\sigma_H = 5.53058$

```
Percentage difference between mean values: 0.8746927720071789 %
Percentage difference between standard deviation values: 1.4170850962461705 %
```

The average temperature for the minute resolution data is $\mu_M = 11.29530^{\circ}C$, with a standard deviation of $\sigma_M = 5.60895^{\circ}C$.

The average temperature for the hourly resolution data is $\mu_H = 11.19736^{\circ}C$, with a standard deviation of $\sigma_H = 5.53058^{\circ}C$.

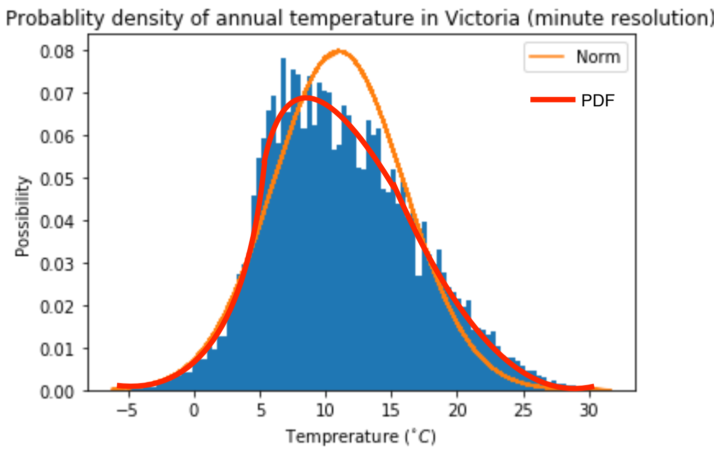
Average temperature of the minute resolution data is 0.87469% higher than the hourly resolution data, while displaying a 1.41709% higher resolution.

Generate histograms:

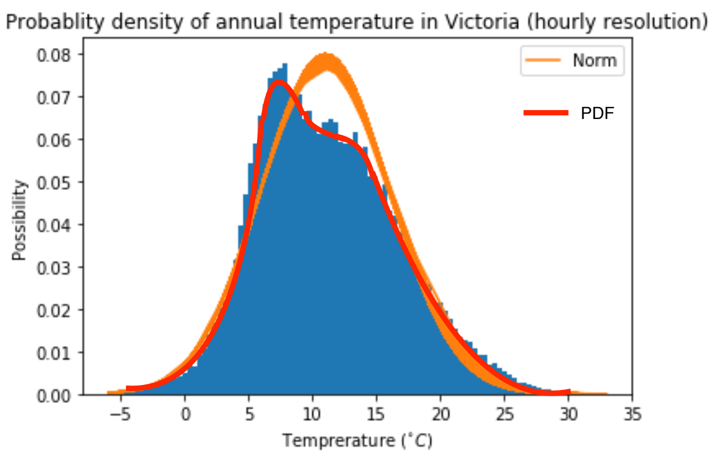
In [13]:
```
# Generate the histograms, probability density functions, and normal distribution
s
# Mn_pdf = sci.stats.norm.pdf(d1, int(MnM1), int(MnV1**0.5))
# Hr_pdf = sci.stats.norm.pdf(d2, int(HrM1), int(HrV1**0.5))
Mn_norm = norm.pdf(d1, int(MnM1), int(MnV1**0.5))
Hr_norm = norm.pdf(d2, int(HrM1), int(HrV1**0.5))

plt.figure(12)
plt.hist(d1[~np.isnan(d1)], bins=100, density=1)
# plt.plot(d1, Mn_pdf, label='PDF')
plt.plot(d1, Mn_norm, label='Norm')
plt.xlabel(r'Temprerature ($^{\circ} C$)')
plt.ylabel(r'Possibility')
plt.title(r'Probablity density of annual temperature in Victoria (minute resoluti
on)')
plt.legend()

plt.figure(13)
plt.hist(d2[~np.isnan(d2)], bins=100, density=1)
# plt.plot(d2, Hr_pdf, label='PDF')
plt.plot(d2, Hr_norm, label='Norm')
plt.xlabel(r'Temprerature ($^{\circ} C$)')
plt.ylabel(r'Possibility')
plt.title(r'Probablity density of annual temperature in Victoria (hourly resoluti
on)')
plt.legend()
```

```
/Library/Frameworks/Python.framework/Versions/3.6/lib/python3.6/site-packages/sc
ipy/stats/_distn_infrastructure.py:876: RuntimeWarning: invalid value encountere
d in greater_equal
  return (self.a <= x) & (x <= self.b)
/Library/Frameworks/Python.framework/Versions/3.6/lib/python3.6/site-packages/sc
ipy/stats/_distn_infrastructure.py:876: RuntimeWarning: invalid value encountere
d in less_equal
  return (self.a <= x) & (x <= self.b)
```

Out[13]: <matplotlib.legend.Legend at 0x118fd5048>



I could not generate a PDF due to a module error in Python, so I have hand drawn an approximate PDF (in red) .



I could not generate a PDF due to a module error in Python, so I have hand drawn an approximate PDF (in red) .

2) Data from 2012

In [33]:
```
start = int(len(D2)*3/16 + 3 + len(D2)/72)
fin = int(len(D2)*5/16 + 3 + len(D2)/72)

D3 = D2[start:fin, 35]
```

Calculate average and standard deviation:

In [34]:
```
# Find the mean
HrM2 = np.nanmean(D3)
display(Math(r'\mu_{H2012}=%.5f' % HrM2))

# Find the Standard Deviation
HrV2 = np.nanvar(D3)
display(Math(r'\sigma_{M2012}=%.5f' % HrV2**0.5))
```

$\mu_{H2012} = 9.85886$

$\sigma_{M2012} = 5.07574$

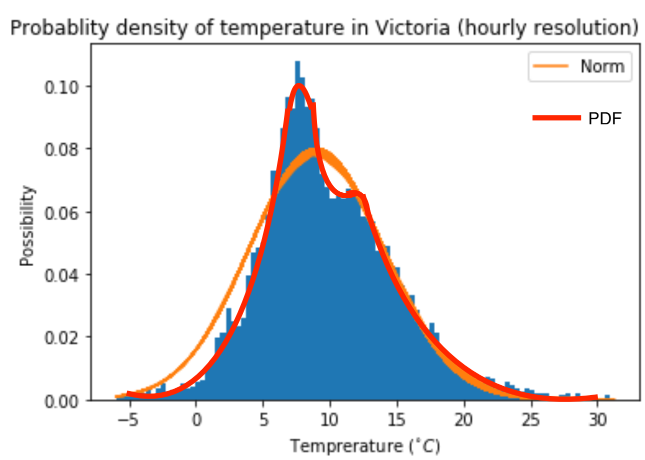
The average temperature for the minute resolution data is $\mu_M = 9.85886^{\circ}C$, with a standard deviation of $\sigma_M = 5.07574^{\circ}C$.

Generate histogram:

In [35]:
```
Hr2_norm = norm.pdf(D3, int(HrM2), int(HrV2**0.5))

plt.figure(15)
plt.hist(D3[~np.isnan(D3)], bins=100, density=1)
# plt.plot(d1, Mn_pdf, label='PDF')
plt.plot(D3, Hr2_norm, label='Norm')
plt.xlabel(r'Temprerature ($^{\circ} C$)')
plt.ylabel(r'Possibility')
plt.title(r'Probablity density of temperature in Victoria (hourly resolution) in 2012')
plt.legend()
```

```
/Library/Frameworks/Python.framework/Versions/3.6/lib/python3.6/site-packages/sc
ipy/stats/_distn_infrastructure.py:876: RuntimeWarning: invalid value encountere
d in greater_equal
  return (self.a <= x) & (x <= self.b)
/Library/Frameworks/Python.framework/Versions/3.6/lib/python3.6/site-packages/sc
ipy/stats/_distn_infrastructure.py:876: RuntimeWarning: invalid value encountere
d in less_equal
  return (self.a <= x) & (x <= self.b)
```

Out[35]: <matplotlib.legend.Legend at 0x117faef60>



I could not generate a PDF due to a module error in Python, so I have hand drawn an approximate PDF (in red) .

Discussion:

From the three probability density graphs, we can see the temperature in Victoria tends toward being cold, as the probability distribution displayed more weight towards the right side of the normal distribution. Average temperature in 2012 was colder than usual, with an average of $9.85886^{\circ}C$ compared to the averages of $11.29530^{\circ}C$ and $11.19736^{\circ}C$ from the minute and hour resolution graphs, respectively. Despite being colder, temperatures in 2012 were more stable with a standard deviation of $5.07574^{\circ}C$ compared to that of $5.60895^{\circ}C$ and $5.53058^{\circ}C$.

Probability density graphs of the minute resolution data was "hairier" compared to the hourly resolution data. This is possibly noise due to minor fluctuations in temperature probe during a short time period. The hourly resolution data is more "stable" as it contains more annual samples, and over greater periods of time.