Vertex Solutions and Extreme Points

Def Given a feasible point $\bar{x} \in S$, a constraint $a_i^T x \geq b_i$ is said to be <u>active</u> at $\bar{x}$ if $a_i^T \bar{x} = b_i$ or <u>inactive</u> at $\bar{x}$ if $a_i^T \bar{x} > b_i$.

Equality constraints (like for standard form $Ax = b, x \geq 0$) are always active.

The <u>active set</u> $A(\bar{x})$ at $\bar{x}$ is the <u>index set</u> of all active constraints at $\bar{x}$.

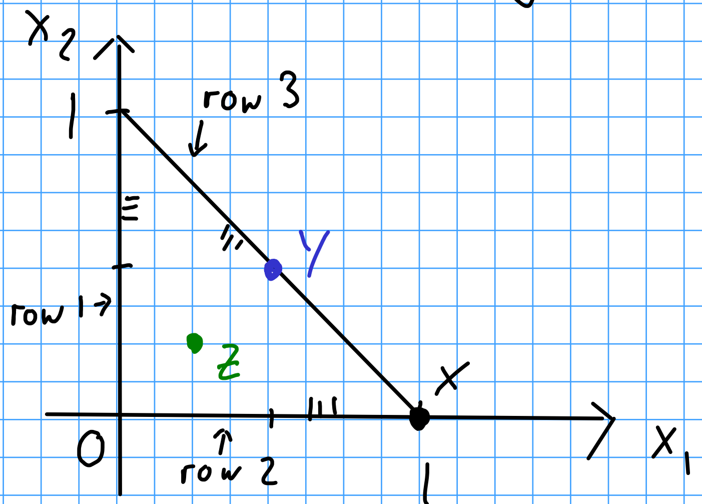
$$A(\bar{x}) = \{ i \in \{1, \dots, m\} : a_i^T \bar{x} = b_i \}$$

The <u>active constraint matrix</u> $A_{\bar{x}}$ is the row submatrix of A with only rows with indices in $A(\bar{x})$.

Example

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ $\qquad Ax \geq b$

Consider 3 feasible points: $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad y = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$ $\qquad z = \begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix}$



$A(x) = \{2, 3\}$ $\qquad A(y) = \{3\}$ $\qquad A(z) = \emptyset$

$A_x = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ $\qquad A_y = \begin{pmatrix} -1 & -1 \end{pmatrix}$ $\qquad A_z$ is empty

Def A feasible point $\bar{x} \in S$ is called a <u>vertex</u> of S if

rank $A_{\bar{x}} = n$. $\leftarrow$ dim. of space $\mathbb{R}^n$

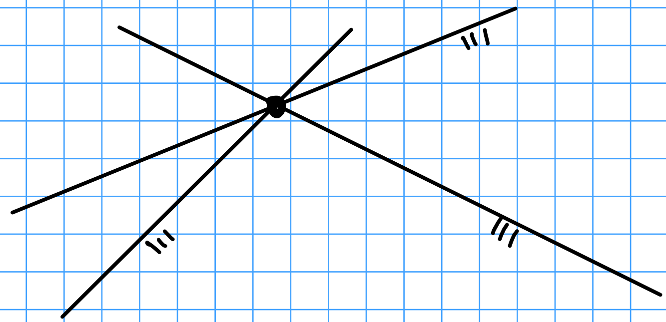
It is said to be <u>non-degenerate</u> if $|A(\bar{x})| = n$ and

<u>degenerate</u> if $|A(\bar{x})| > n$.

Degeneracy vs. Redundancy

totally different concepts for dim 3+

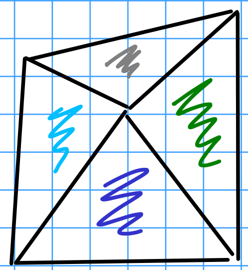
dim 2:



degenerate vertex

$\Rightarrow$ redundant constraint

dim 3:



4-sided pyramid in dim. 3

degenerate vertex at top,
but no redundant constraint

degenerate vertex

$\not\Rightarrow$ redundant constraint

Theorem A point $\bar{x}$ is a vertex of S if and only if it is an extreme point of S.

HW 13 Prove this claim.

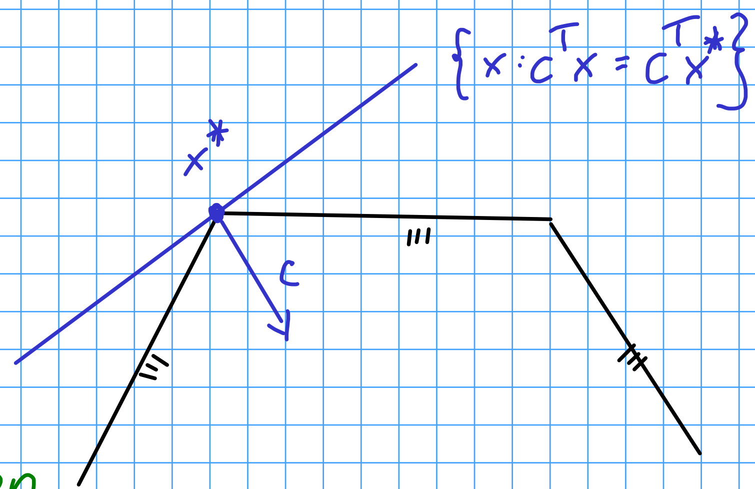
The Fundamental Theorem of Linear Programming

Def A point x^* is called optimal for (P) if it is feasible ($x^* \in S$) and if $c^T x^* \leq c^T x$ for all $x \in S$.

min $c^T x$ s.t. $Ax \geq b$

If $c = 0$, all feasible points are optimal.

If $c \neq 0$, then x^* is optimal if and only if the set $H = \{x \in \mathbb{R}^n : c^T x = c^T x^*\}$ is a supporting hyperplane at x^* with normal vector c.



$\rightarrow$ Separation $\hat{=}$ Optimization

Theorem <u>Fundamental Theorem of Linear Programming</u>

Let rank $A = n$.

- If there exists a feasible point for (P), then there exists a vertex.
  pointed polyhedron
- If there exists an optimal point for (P), then there exists an optimal vertex.

proof idea: the set of opt. solution is a polyhedron itself (specified by adding constraint $c^T x = c^T x^*$ to S), and so has a vertex (rank of constraint system is still full)

$\to$ Solving a linear program can be done by finding an optimal vertex

Optimality Conditions and Farkas Lemma

Def Given a (nonzero) vector $c \in \mathbb{R}^n$, a direction $d \in \mathbb{R}^n$ is called an <u>ascent</u> / <u>descent</u> / <u>orthogonal</u> direction with respect to c if $c^T d > 0$, $c^T d < 0$, $c^T d = 0$.

It is said to be <u>feasible</u> at $\bar{x}$ for (P) if $A_{\bar{x}}\, d \geq 0$

$\min c^T x \quad s.t. \quad Ax \geq b$

you can take a step in direction d and remain feasible

$A\bar{x} \geq b$

$$A_{\bar{x}}(\bar{x} + \varepsilon \cdot d) = b_{\bar{x}} + \varepsilon \cdot \underbrace{A_{\bar{x}}\, d}_{\geq 0} \geq b_{\bar{x}}$$

Theorem A feasible point $\bar{x}$ is optimal for (P) if and only if there exists no feasible descent direction at $\bar{x}$ with respect to c:

min $c^T x$

$Ax \geq b$

$$c^T d \geq 0 \quad \forall d \in \mathbb{R}^n \text{ with } A_{\bar{x}} d \geq 0$$

This optimality condition is highly impractical. There are infinitely many directions.