- extra credit for participation, helping others, etc. will be added to full score (by using it to "fill up" scores as needed)
- extra credit for submitting 20-4 g as part of HW collection 2: up to +0.5

Vanderbei 23.3

Facility Location Applications often have <u>Fixed Charge</u> costs, a "base-level" cost K incurred by using/selecting a location.

→ can be modeled using binary variables

Additionally, linear usage-level costs c come on top.

→ continuous variables

$$\Rightarrow \quad c(x) = \begin{cases} 0 & \text{if } x = 0 \\ K + cx & \text{if } x > 0 \end{cases}$$

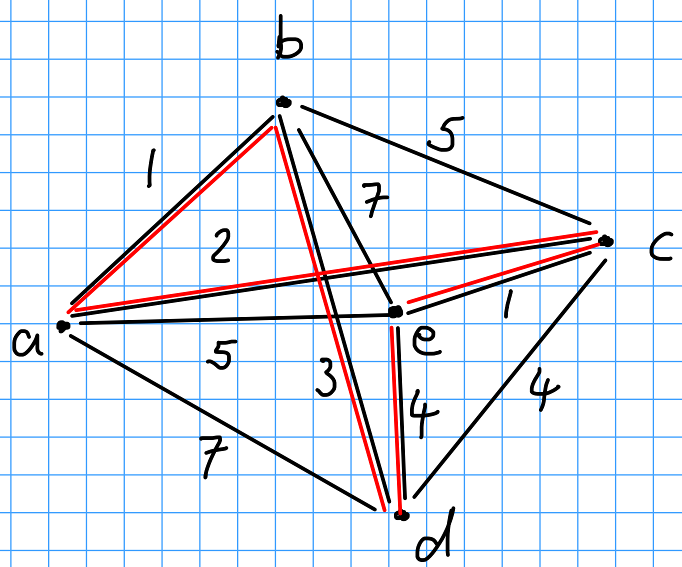
MIP →

$$c(x) = Ky + cx$$
$$x \leq uy$$
$$x \geq 0$$
$$y \in \{0, 1\}$$

← need upper bound u for construction

Traveling Salesman Problem (TSP)

Given a set of n cities with travel costs c_{ij} between city i and city j, find a closed loop that visits each city once and minimizes cost.



$\Sigma = 11$ optimal here, because this tour only uses two edges of smallest cost for each node

TSP is an NP-hard problem

Brute Force $\Theta(n!)$

possible project topic → Dynamic Programming $O(n^2 \cdot 2^n)$

Integer Programs for TSP

Model 1

- graph $G=(V,E)$ vertices, edges $V=\{1,\dots,n\}$
- $\delta(i)$: set of edges incident to vertex i
- $\delta(S)$: set of edges with exactly one vertex in $S\subset V$
- c_e : cost of edge e
- variables : $x_e\in\{0,1\}$ use of edge e

$$\min \sum_{e\in E} c_e x_e$$

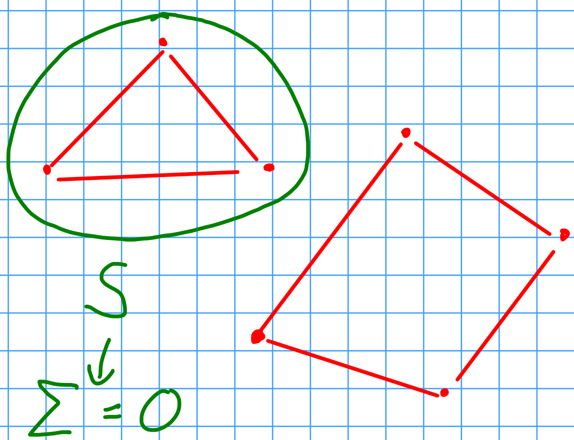
$$\sum_{e\in\delta(i)} x_e = 2 \quad \forall i\in V$$

subtour elimination constraints

$$\sum_{e\in\delta(S)} x_e \geq 2 \quad \forall S\subset V,\ S\neq\emptyset, V$$

exponentially many

$$x_e\in\{0,1\} \quad \forall e\in E$$



Model 2

- $E(S)$: set of edges with <u>both</u> vertices in S

$$\min \sum_{e\in E} c_e x_e$$

$$\sum_{e\in\delta(i)} x_e = 2 \quad \forall i \in V$$

$$\sum_{e\in E(S)} x_e \le |S| - 1 \quad \forall S \subset V,\ S \neq \emptyset, V$$

$$x_e \in \{0,1\} \quad \forall e \in E$$

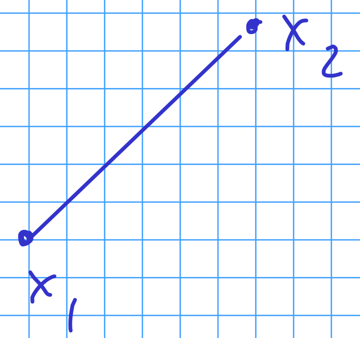
Reading: Vanderbei, Ch. 23-2, p. 393

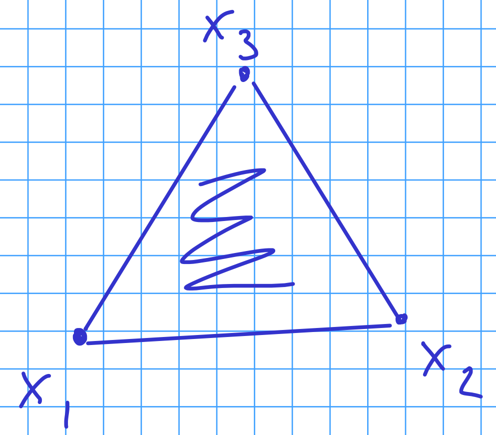
HW 10: Explain in your own words why the model in the reading is a correct TSP model! Discuss what is better and worse compared to models 1 and 2! (one typed page max)

3 Convex Analysis

Reading: Vanderbei, Chapter 10

Def A set $S \subseteq \mathbb{R}^n$ is <u>convex</u> if for every two points $x_1, x_2 \in S$ and every scalar $\lambda \in (0,1)$, the point $\lambda x_1 + (1-\lambda) x_2$ is in S.





Def Let $x_1, \dots, x_k \in \mathbb{R}^n$ be a finite collection of points. A point $x \in \mathbb{R}^n$ is called a (strict) <u>convex combination</u> of $x_1, \dots, x_k$ if $x = \sum_{i=1}^{k} \lambda_i x_i$ for some $\lambda_i \geq 0$ $(\lambda_i > 0)$ $\forall i = 1, \dots, k$ with $\sum_{i=1}^{k} \lambda_i = 1$.

Theorem A set S is convex if and only if it contains all convex combinations of points in S.

Proof Idea: „$\Leftarrow$" the set contains all convex combinations of pairs of points $(k=2)$ $\Rightarrow$ S is convex

„$\Rightarrow$" Proof by induction on number k of summands in convex combination

- ind. start $k=2$ true by definition
- general convex comb. of $z_1, \dots, z_k$:

$$z = \sum_{i=1}^{k} \lambda_i z_i \quad , \quad \sum_{i=1}^{k} \lambda_i = 1 \quad , \text{ assume } \lambda_k \neq 1$$

$$\Rightarrow z = \sum_{i=1}^{k-1} \lambda_i z_i + \lambda_k z_k = (1-\lambda_k) \underbrace{\sum_{i=1}^{k-1} \frac{\lambda_i}{1-\lambda_k} z_i}_{\in S} + \lambda_k \underset{\in S}{z_k}$$

ind. hypothesis & $\sum_{i=1}^{k-1} \frac{\lambda_i}{1-\lambda_k} = 1 \longrightarrow \in S$ $\quad \in S$ $\quad \square$