- extra credit for sharing your HW collection 1 for posting on canvas
  - should include written comments (scan, photo, annotate)
  - +0.25 for pdf and +0.25 for code. pdf + code in zip file
- score breakdown for HW collection 1 was: 1-3 : 1.5 4-7 : 3 each

<u>Theorem</u> Let S and T be convex sets in $\mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\alpha \cdot S + \beta \cdot T = \{x \in \mathbb{R}^n : x = \alpha \cdot x_s + \beta \cdot x_t,\ x_s \in S,\ x_t \in T\}$$

is convex.

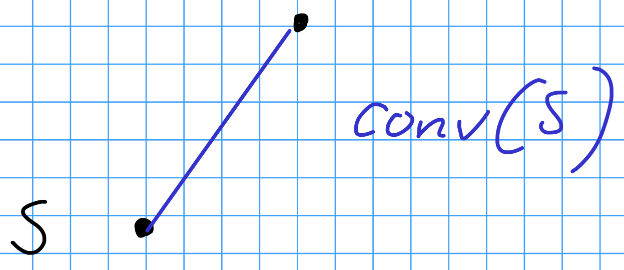
Further, the set $S \cap T$ is convex.

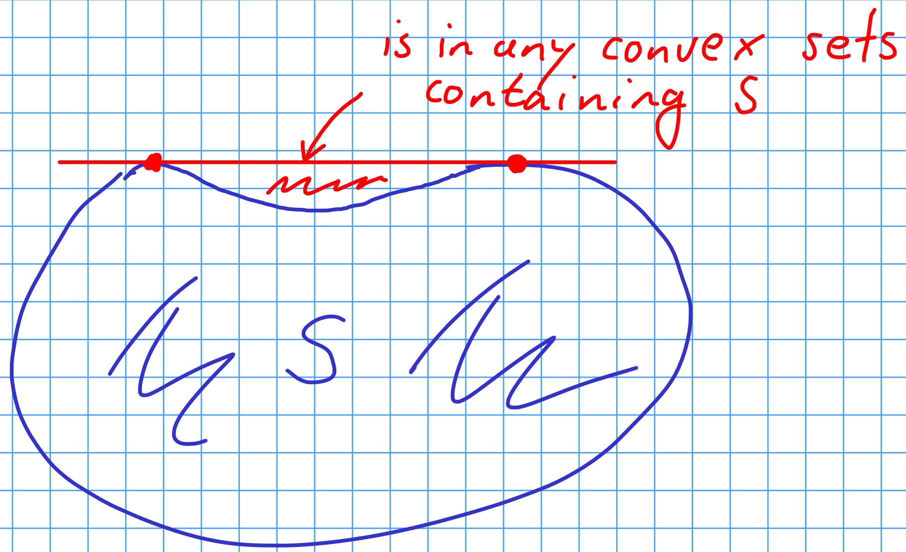
In fact, the intersection of any collection (even of infinite size) of convex sets is convex.

Note: set intersections happen when we specify constraints in our models

Def Given a set $S \subseteq \mathbb{R}^n$, the intersection of all convex sets containing S is called the <u>convex hull</u> of S.

$$\operatorname{conv}(S) = \bigcap \{ C \supseteq S : C \text{ is convex} \}$$





S together with red part gives conv(S)

<u>**Theorem**</u> The convex hull of a set S is the (smallest) set C that contains all convex combinations of points in S.

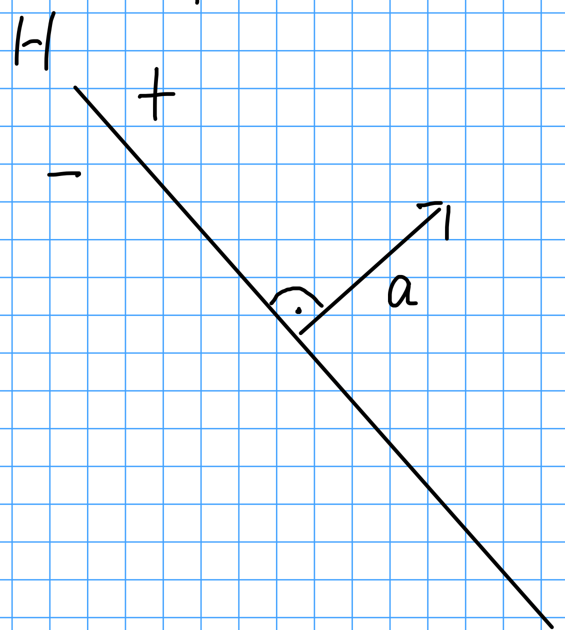
$$C = \operatorname{conv}(S) = \{ z \in \mathbb{R}^n : z = \sum_{i=1}^{k} \lambda_i z_i , \ \lambda_i \geq 0 , \ \sum_{i=1}^{k} \lambda_i = 1 , \ z_i \in S \}$$

Reading: Caratheodory's Theorem (dim + 1 points suffice)

Hyperplanes, Halfspaces, Cones

Def Given a nonzero vector $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$, the set $H = H_{a,b} = \{x \in \mathbb{R}^n : a^T x = b\}$ is called a <u>hyperplane</u>, and a is called its <u>normal vector</u>. (outer normal)

Further, the sets $H^+ = \{x \in \mathbb{R}^n : a^T x \geq b\}$ and $H^- = \{x \in \mathbb{R}^n : a^T x \leq b\}$ are called its associated <u>positive and negative halfspaces.</u>



$$H^+_>,\ H^-_< = \{x \in \mathbb{R}^n : a^T x \ {}^{>}_{<}\ b\}$$

open/strict halfspaces

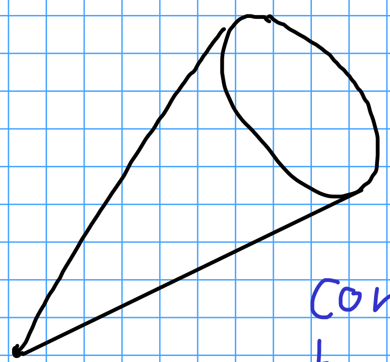
Def A set S in $\mathbb{R}^n$ is called <u>polyhedral</u> or a (convex) <u>polyhedron</u> if it can be represented as the intersection of a <u>finite</u> numbers of half spaces:

$$S = \{x \in \mathbb{R}^n : Ax \geq b\}$$

A non-empty bounded polyhedron is called a polytope.

Def A set K in $\mathbb{R}^n$ is called a <u>cone</u> if $\lambda K \subseteq K$ $\forall \lambda \geq 0$.

A <u>convex (polyhedral) cone</u> is a cone that is also convex (polyhedral).



convex cone, not polyhedral

Theorem A cone K is convex if and only if $K + K \subseteq K$.

HW 11 Prove this claim.

Separation Theorems

role of hyperplanes / halfspaces in convex analysis

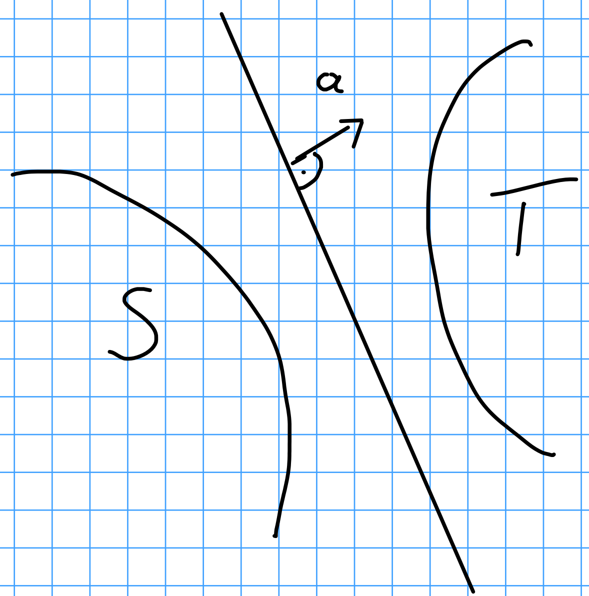
"Separation $\hat{=}$ Optimization"

finding a separating hyperplane for certain sets is essentially the same as optimizing over one of the sets

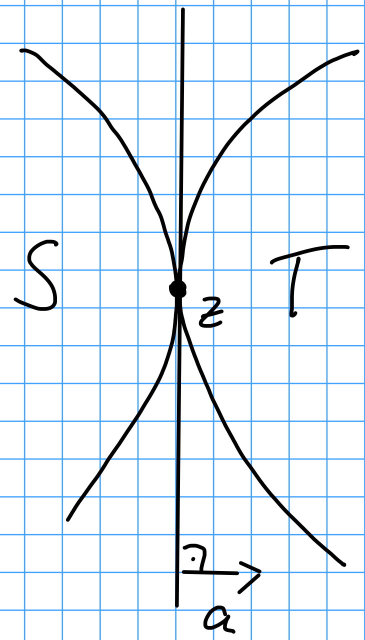
Def Given two non-empty sets $S, T \subseteq \mathbb{R}^n$, a hyperplane H in $\mathbb{R}^n$ is said to <u>separate</u> S and T if the sets are contained in the different halfspaces induced by H:

$$S \subseteq H^+ \quad \text{and} \quad T \subseteq H^- \quad \text{or vice versa}$$

The separation is <u>strict</u> if both S and T are contained in the respective strict halfspaces.



strict separation



(weak) separation

Theorem **Separating Hyperplane Theorem**

Let S and T be non-empty convex sets in $\mathbb{R}^n$ with no common interior points. Then there exists a separating hyperplane for S and T, i.e., there exists a nonzero vector a such that

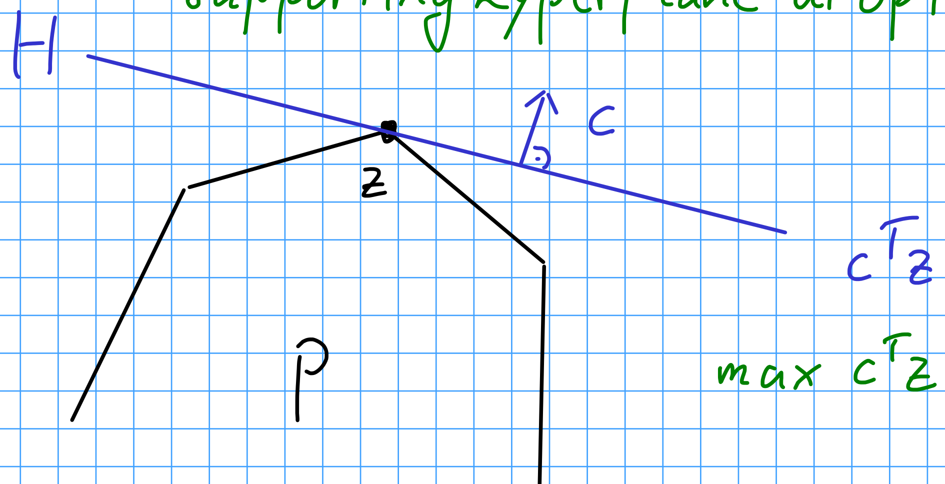
$a^T x \leq a^T y$ for all $x \in S$, $y \in T$.

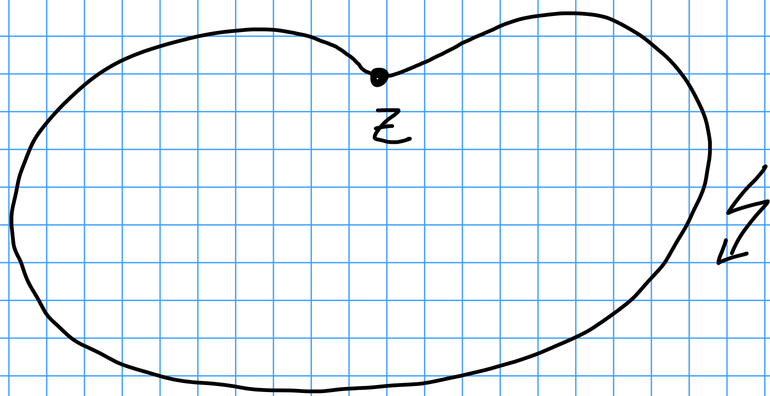
Def Given a set S in $\mathbb{R}^n$ and a point z in the closure of S, a hyperplane H in $\mathbb{R}^n$ is said to <u>support S at z</u>, if z belongs to H and S is contained in one of its associated (closed) halfspaces:

$$z \in H \quad \text{and} \quad S \subseteq H^- \quad (\text{or } S \subseteq H^+)$$

Note: Supporting hyperplanes are a special case of separating hyperplanes. $\quad S \subseteq H^- \quad \{z\} \subseteq H^+$

supporting hyperplane at optimum





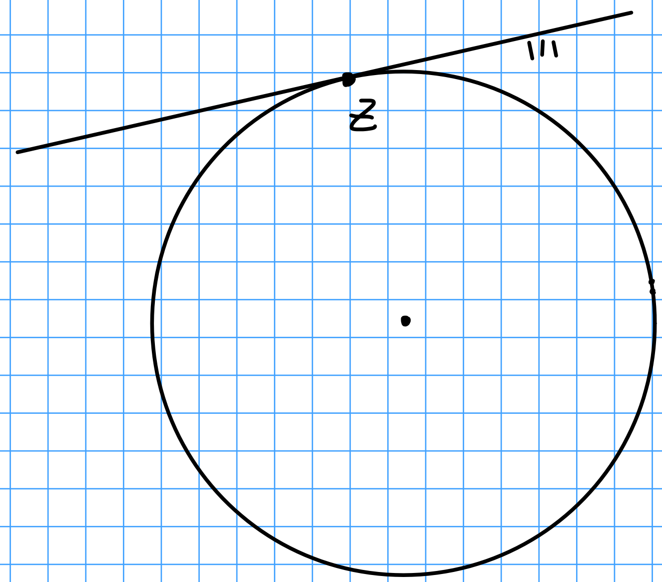
non-convex set

existence of supp. hyperplane not guaranteed

Theorem Supporting Hyperplane Theorem

Let S be a convex set in $\mathbb{R}^n$ and z be a boundary point of S. Then there exists a supporting hyperplane for S at z.

Corollary A (closed/open) set is convex if and only if it is the intersection of all its supporting (closed/open) halfspaces.



The big difference between polyhedra and general convex sets is that polyhedra can be described by finitely many supporting halfspaces.

may need infinitely many for convex, non-polyhedral sets