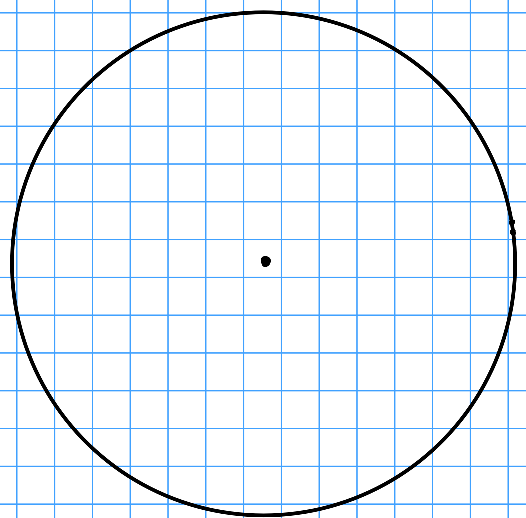
Representation Theorems

Def A point x in a convex set S is called <u>extreme point</u> of S if it cannot be written as a strict convex combination of any two other points in S.

$$x = \lambda y + (1-\lambda) z \text{ for some } 0 < \lambda < 1 \Rightarrow x = y = z$$

<u>Theorem</u> A closed bounded convex set in $\mathbb{R}^n$ is equal to the (closed) convex hull of its extreme points.

these suffice to write convex hull



← here (for a filled circle), all of the boundary points are extreme points

→ all of them are necessary

Corollary Representation Theorem for Bounded Convex Sets

Let S be a (closed) bounded convex set in $\mathbb{R}^n$.

- If S is nonempty, then S has at least one extreme point.
- A point x belongs to S if and only if x can be expressed as a convex combination of <u>finitely many</u> extreme points.

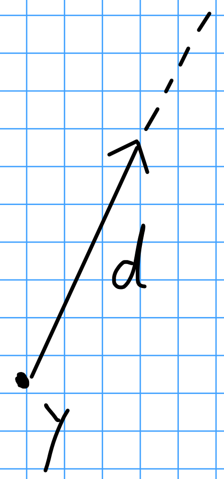
Reading: Caratheodory's Theorem

$$\operatorname{conv}(V) = \operatorname{conv}\{v_1, \dots\}$$

extreme points

Unbounded Sets and Extreme Directions

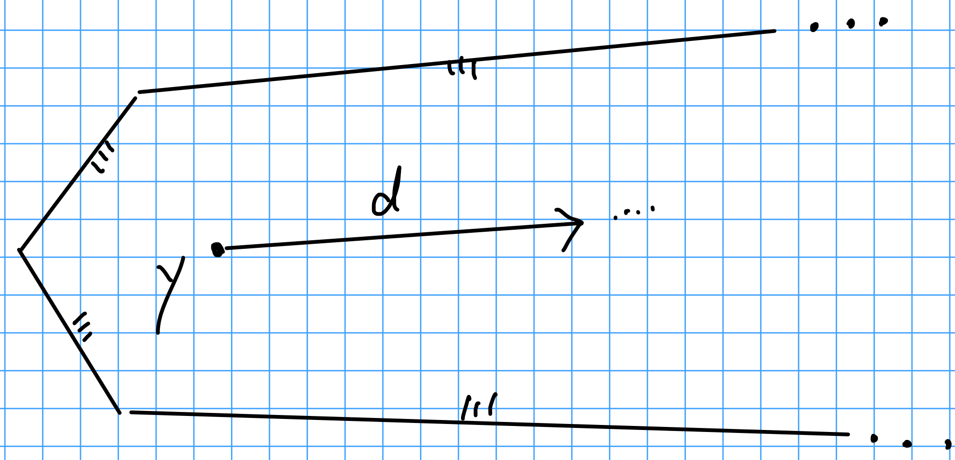
Lemma A convex set $S \subseteq \mathbb{R}^n$ is unbounded if and only if there exists a halfline in S.



$x = y + \lambda d \in S$ for some $y \in S$, $d \in \mathbb{R}^n \setminus \{0\}$, all $\lambda \geq 0$

Def A nonzero vector $d \in \mathbb{R}^n$ is called a <u>recession direction</u> of a convex sets S if

$y + \lambda d \in S$ for some (and thus all) $y \in S$ and all $\lambda \geq 0$.



Def A direction is an <u>extreme direction</u> if it cannot be written as a positive sum of two other distinct recession directions of S.

$$d = u + v \Rightarrow d = \lambda \cdot u = \mu \cdot v \quad \text{for some } \lambda, \mu \in \mathbb{R}^+$$

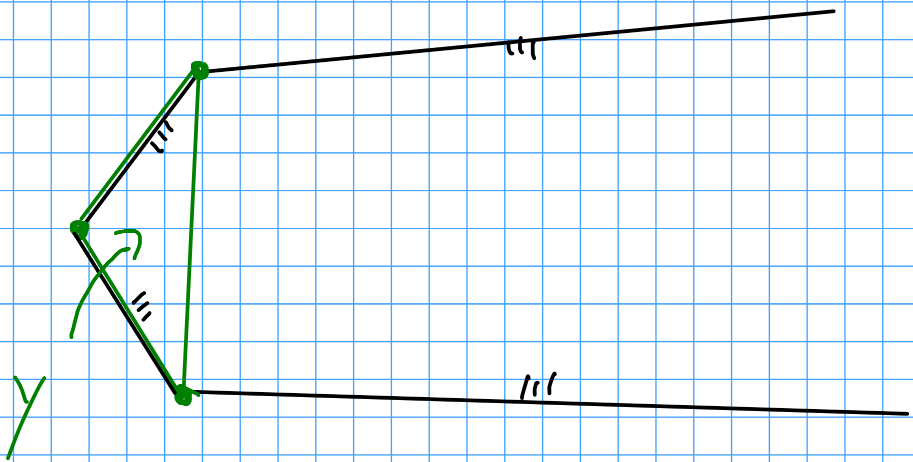
compare def. of extreme points

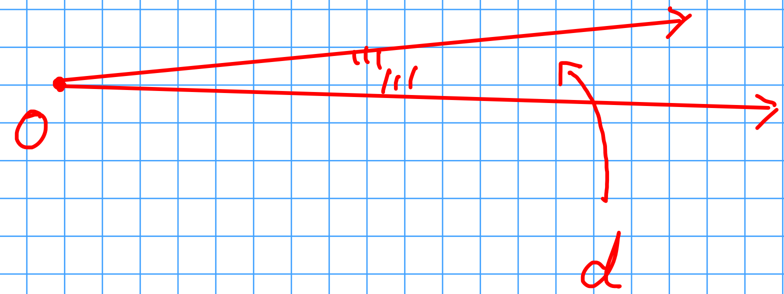
Theorem <u>Representation Theorem for Unbounded Convex Sets</u>

Let S be an unbounded convex set in $\mathbb{R}^n$ containing no lines.

- Then S has at least one extreme point and one extreme direction.
- Further, a point x belongs to S if and only if x can be expressed as a sum $x = y + d$, where

- y is a convex combination of finitely many extreme points of S, and
- d is a conic (non-negative & linear) combination of finitely many extreme directions of S





recession cone $\hat{=}$ set of all recession directions

For polyhedra (intersection of finitely many halfspaces), we will see that there can only be finitely many extreme points and extreme directions / rays.

This leads to the following notation (from introduction):

$P = \text{conv}\{v_1, \dots, v_r\}$ for polytopes (bounded polyhedra)

convex hull

vertices v_i

bounded part

unbounded part

$P = \underline{\text{conv}}\{v_1, \dots, v_r\} + \underline{\text{cone}}\{e_1, \dots, e_s\}$ for general polyhedra

extreme rays

conic combination

4 Polyhedral Theory

Reading: Vanderbei, Chapter 5
→ Duality (second half of this chapter)

We consider an LP (P) in canonical form

$$(P) \quad \min c^T x \quad \text{s.t. } Ax \geq b$$

$c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$

rows of A: a_i^T

each row $a_i^T x \geq b_i$

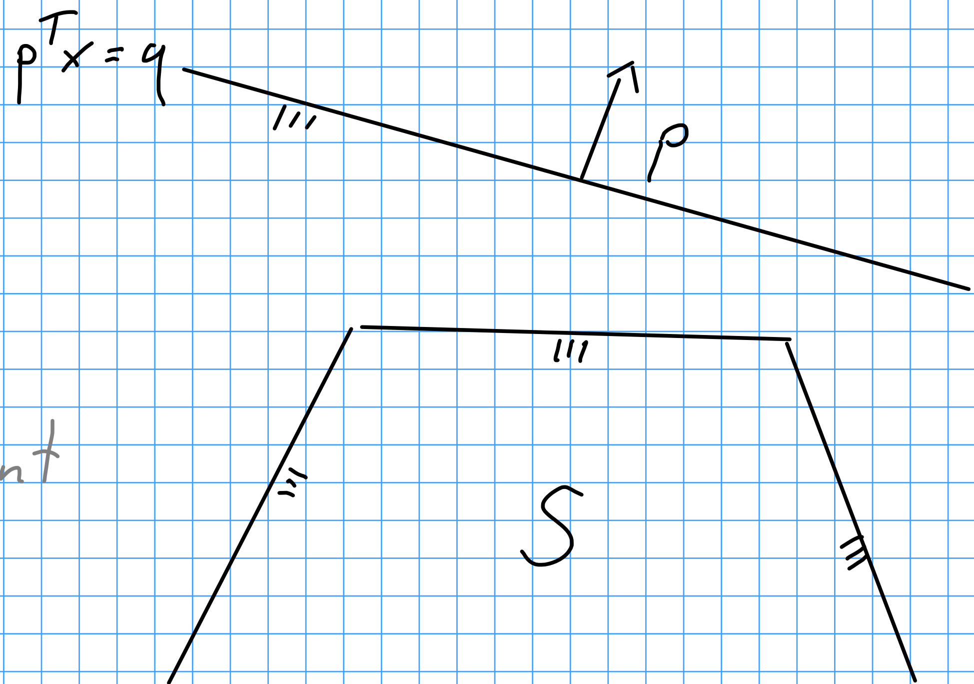
The polyhedron $S = \{x \in \mathbb{R}^n : Ax \geq b\}$ is called the <u>feasible set</u> of (P) and a point $x \in \mathbb{R}^n$ is said to be <u>feasible</u> to/for (P) if $x \in S$.

Each halfspace $a_i^T x \geq b_i$ is called a <u>constraint</u> of (P).

Without loss of generality, we assume $\underset{\#\text{rows}}{m} \geq \underset{\#\text{columns}}{n}$ for canonical form LPs. $Ax \geq b$

→ otherwise: there exist no vertices and polyhedron is guaranteed to be unbounded

A constraint is said to be <u>redundant</u> for a system of constraints $Ax \geq b$ if its addition or removal does not alter the set.



HW 12 State and explain an LP to determine if $p^Tx \geq q$ is redundant for a system $Ax \geq b$.

Vertex Solutions and Extreme Points

Def Given a feasible point $\bar{x} \in S$, a constraint $a_i^T x \geq b_i$ is said to be <u>active</u> at $\bar{x}$ if $a_i^T \bar{x} = b_i$ or <u>inactive</u> at $\bar{x}$ if $a_i^T \bar{x} > b_i$.

Equality constraints (like for standard form $Ax = b, x \geq 0$) are always active.