MATH 5718: Applied Linear Algebra, Fall 2025

Homework Assignment 3

Due: Wednesday, October 1, 11:59PM

Instructions: Answer all questions clearly and show all your work. For multi-step solutions, justify each step and cite appropriate results from the textbook or lecture notes.

Points may be deducted for incomplete explanations. Use LaTeX to typeset your solutions; alternatively, write neatly on a tablet or scan your handwritten work, and upload your solutions to Canvas.

Note that when writing proofs, the level of detail you need to show depends on how much you rely on previously proven results. Proving a result from scratch typically requires more steps than starting from an intermediate result we have already covered in class. In general, both approaches are acceptable; however, using previously proven results can shorten the time spent on each problem and reduce the likelihood of making mistakes along the way.

Question 1: (25 points) Consider the vector space $\mathcal{V} = \mathrm{span}\{e^x, xe^x, x^2e^x\}$ and let

$$B = \{e^x,\, xe^x,\, x^2e^x\}$$

be an ordered basis for $\mathcal{V}$. The derivative operator $D : \mathcal{V} \to \mathcal{V}$ maps a function to its derivative.

(a) Find the standard matrix $[D]_B$ of the derivative operator D with respect to the basis B.

(b) Let

$$f(x) = 1 \cdot e^x + 3 \cdot xe^x - 2 \cdot x^2e^x.$$

Write f as a coordinate vector $[f]_B$ relative to the basis B and use matrix multiplication

$$[D(f)]_B = [D]_B[f]_B$$

to compute the coordinate vector of $D(f)$. Finally, interpret this result by writing the explicit form of $D(f)$.

(c) Compute the fourth derivative of $x^2e^x + 2xe^x$ using standard matrices.

Question 2: (20 points) Suppose $\mathcal{V}$ and $\mathcal{W}$ are vector spaces and let

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset \mathcal{V} \quad \text{and} \quad C = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\} \subset \mathcal{W}$$

be sets of vectors.

(a) Show that if B is a basis of $\mathcal{V}$, then the function $T : \mathcal{V} \to \mathcal{W}$ defined by

$$T(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) = c_1\mathbf{w}_1 + \cdots + c_n\mathbf{w}_n$$

is a linear transformation.

(b) Show that if C is also a basis of $\mathcal{W}$, then T is invertible.

Question 3: (30 points) Suppose $\mathcal{V}$ and $\mathcal{W}$ are vector spaces and $T : \mathcal{V} \to \mathcal{W}$ is an invertible linear transformation. Let

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subseteq \mathcal{V}$$

be a set of vectors.

(a) Show that if B is linearly independent then so is

$$T(B) = \{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)\}.$$

(b) Show that if B spans $\mathcal{V}$ then $T(B)$ spans $\mathcal{W}$.

(c) Show that if B is a basis of $\mathcal{V}$ then $T(B)$ is a basis of $\mathcal{W}$.

(d) Show that $\dim(\mathcal{V}) = \dim(\mathcal{W})$.

Question 4: (25 points) Consider the transposition map $T : \mathcal{M}_2 \to \mathcal{M}_2$ defined by

$$T(A) = A^T,$$

where $\mathcal{M}_2$ is the space of all 2×2 real matrices. Let

$$E = \{E_{1,1}, E_{1,2}, E_{2,1}, E_{2,2}\}$$

be the standard basis of $\mathcal{M}_2$, where $E_{i,j}$ has a 1 in entry (i, j) and 0 elsewhere.

(a) Find the standard matrix $[T]$ of the linear transformation T with respect to the basis E.

(b) Compute $[T]^2$. *(Hint: Your answer should be a well-known named matrix.)*

(c) Explain how we could have obtained the answer to part (b) without actually computing $[T]$.

(d) Show that the standard matrix of the transpose map $T : \mathcal{M}_3 \to \mathcal{M}_3$ is the 9×9 "block" matrix

$$[T] = \begin{bmatrix} E_{1,1} & E_{2,1} & E_{3,1} \\ E_{1,2} & E_{2,2} & E_{3,2} \\ E_{1,3} & E_{2,3} & E_{3,3} \end{bmatrix},$$

where each block $E_{i,j}$ denotes the 3×3 matrix with a 1 in position (i, j) and 0 elsewhere.