- HW Collection 2 due Oct. 16, 4:30pm (in office/mailbox)
  → will return these on Oct. 23
- also okay to submit on Oct. 21, before class
  → cannot return these on Oct. 23
- on Oct. 14, office hours 10-10:45 am and 12:30-1 pm
  but not 1-1:45pm

<u>Theorem</u> A feasible point $\bar{x}$ is optimal for (P) if and only if there exists no feasible descent direction at $\bar{x}$ with respect to c:

$\min c^T x$
$Ax \geq b$

$$c^T d \geq 0 \quad \forall d \in \mathbb{R}^n \text{ with } A_{\bar{x}} d \geq 0$$

This optimality condition is highly impractical. There are infinitely many directions.

Theorem (Farkas Lemma)

Theorem of the Alternative

Let A be a matrix in $\mathbb{R}^{k \times n}$ and $c \in \mathbb{R}^n$. Then

$$c^T d \geq 0 \quad \forall d \in \mathbb{R}^n \text{ with } Ad \geq 0 \iff \exists y \geq 0 \in \mathbb{R}^k \text{ with } A^T y = c.$$

there exists a non-neg. combination of rows of A giving c

Proof: „$\Leftarrow$" Let $A^T y = c$ for some $y \geq 0$ and let d be any vector with $Ad \geq 0$. Then

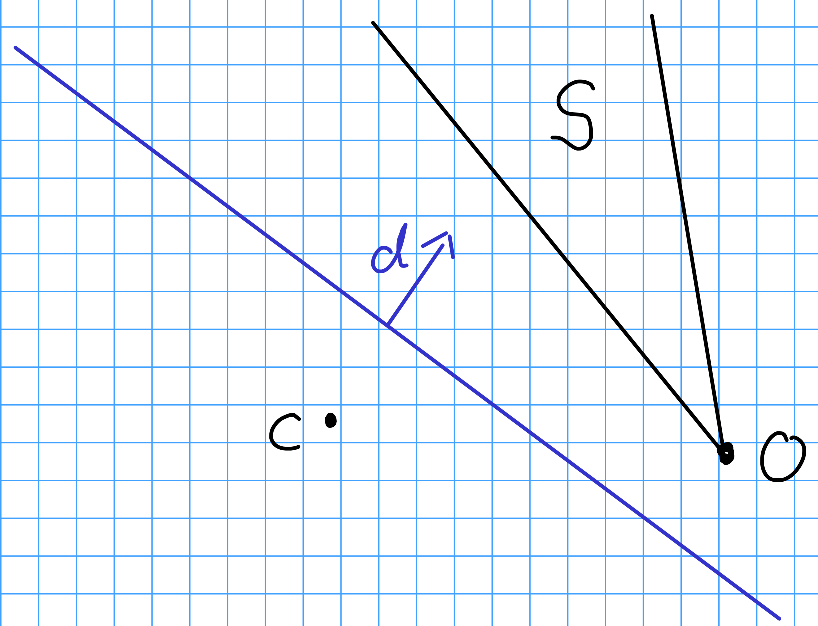
$$c^T d = (A^T y)^T d = \underbrace{y^T}_{\geq 0} \underbrace{Ad}_{\geq 0} \geq 0$$

„$\Rightarrow$" Define the set $S = \{ s \in \mathbb{R}^n : A^T y = s$ for some $y \geq 0 \}$.

S is a (closed) convex cone that contains 0 and rows a_i (from the unit vector $y = e_i$) for all $i \leq k$.

$\neg B \Rightarrow \neg A$

Suppose $c \notin S$. Then there exists a separating hyperplane with normal vector $d \in \mathbb{R}^n$ and $z \in S$ such that $d^T c < d^T z \leq d^T x \quad \forall x \in S$, so in particular $d^T c < d^T 0 = 0$. $\leftarrow$ because $0 \in S$



Further, because S is a convex cone that contains z and a_i for all $i \leq k$, we have $z + a_i \in S$. $\leftarrow K + K \subseteq K$

Thus $d^T z \leq d^T(z + a_i)$. $\Rightarrow$ $d^T a_i \geq 0$ $\forall i \leq k$

$\Rightarrow$ $Ad \geq 0$ (but at the same time $c^T d < 0$) $\square$

The Farkas Lemma comes in many variants. These are called Theorems of the Alternative. They all have great geometric interpretations.

Theorem (Farkas Lemma, variant 2)

Given a matrix $A \in \mathbb{R}^{k \times n}$ and $c \in \mathbb{R}^n$, one and only one of the following systems admits a solution:

(I) $c^T d < 0$, $Ad \geq 0$ (II) $y \geq 0$, $A^T y = c$

for $A = A_{\bar{x}}$: there exists a strictly improving direction

there exists a conic combination of the normals of the active constraints that gives c

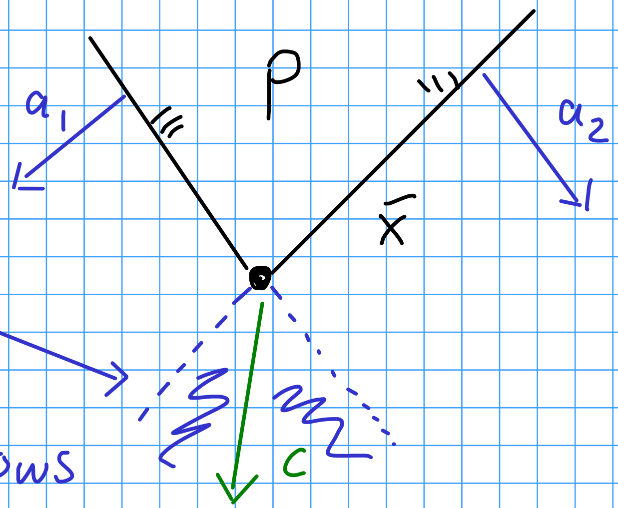
<u>Corollary</u> A feasible point $\bar{x}$ is optimal for (P) if and only if there exists a vector $y \geq 0$ such that $A_{\bar{x}}^T y = c$.

$\min c^T x,\ Ax \geq b$

nice optimality condition!

HW 14 State this corollary for all canonical form LPs ($\max/\min$, $Ax \leq b / Ax \geq b$, all 4 combinations). Visualize all 4 cases with a small sketch. (No explanation / justification.)

Visualization $\max c^T x \,,\; Ax \leq b$



$\bar{x}$ opt. vertex

normal cone N
represents all conic
combinations of active rows

$\hat{=}$ set of all objective functions for which $\bar{x}$ is optimal

Duality and Complementary Slackness

Farkas Lemma + Geometric Insight give us better optimality conditions for LPs

Theorem First-Order Optimality Conditions

$\min c^T x$
$Ax \geq b$

A point x is optimal for (P) if and only if there exists a non-negative $y \in \mathbb{R}^m$ such that

a) $Ax \geq b$ feasible

b) $A^T y = c$ construct obj. fun. from normals

c) $y_i \cdot (a_i^T x - b_i) = 0 \quad \forall i \leq m$

only use active normals (normals of active constraints) for the construction in b)

KKT conditions for LPs

Karush-Kuhn-Tucker