HW 9: Exercise 20-4 a-f (not g) from AMPL book.

Extra Credit for Class Participation for Unit 2: up to 1 credit

Logical Conditions: Either-Or

Problem: A scheduling problem in which n tasks have to be completed sequentially (i.e., for each pair of tasks $i \neq j$ either i precedes j or vice versa)

Model: Let x_i and d_i be start time and duration of task i

x_i ↑ variables; d_i ↑ parameter

$$x_j \geq x_i + d_i \quad \text{or} \quad x_i \geq x_j + d_j \qquad \forall i \neq j$$

Precedence Constraints

binary auxiliary variables

Define $y_{ij} = \begin{cases} 1 & \text{if } i \text{ precedes } j \\ 0 & \text{if } j \text{ precedes } i \end{cases}$

$x_j \geq x_i + d_i - M(1 - y_{ij})$ and $x_i \geq x_j + d_j - M y_{ij}$

big M   big M formulation

M is a sufficiently large constant, for example $M > \sum_{i=1}^{n} d_i$

In general, it is hard to identify what is sufficiently large without expert knowledge.

Logical Condition: If - then

Problem A scheduling problem with additional constraints

"if i precedes j, then k precedes m"

P $\qquad\qquad\qquad$ Q

Logic $P \Rightarrow Q$ is logically equivalent to $\neg P \vee Q$

$$x_i + d_i > x_j \qquad \text{or} \qquad x_k + d_k \leq x_m$$

$\neg P$ $\qquad\qquad\qquad$ Q

Model Let $y \in \{0,1\}$ denote whether P holds

$y = 1 \rightarrow P$ true $\qquad\qquad$ $y = 0 \rightarrow P$ false

$$x_i + d_i > x_j - My$$

$$x_k + d_k \leq x_m + M(1-y)$$

strict inequalities cannot be part of an LP or IP!

$\rightarrow$ choose a sufficiently small $\varepsilon > 0$ and write

$$x_i + d_i \geq x_j - My + \varepsilon$$

Relationship of Binary Quadratic and Linear Programs

For binary variables $(x_i \in \{0,1\})$, $x_i^2 = x_i$ and $x_i \cdot x_j = 0$ is equivalent to $x_i + x_j \leq 1$. One can linearize binary quadratic models using new variables $x_{ij} = x_i \cdot x_j$ and setting $0 \leq x_{ij} \leq x_i, x_j$ and $x_{ij} \geq x_i + x_j - 1$.

$\Rightarrow$ one can substitute x_i for all occurences of x_i^2 and replace all occurences of products $x_i \cdot x_j$ of different variables $x_i \neq x_j$ with a new variable x_{ij} and a set of linear constraints

$\Rightarrow$ binary quadratic programs are not harder than binary linear programs

Combinatorial Optimization Problems

Given a finite set $N = \{1, 2, \dots, n\}$, weights c_j for all $j \in N$, and a family $F \subseteq \mathcal{P}(N)$ of subsets of N.

$$\rightarrow \max/\min \left\{ \sum_{j \in S} c_j \; : \; S \in F \right\}$$

Example: Knapsack

- N : set of items that one can put in Knapsack
- F : subsets of item set that fit in the Knapsack

main challenge for comb. opt. problems lies in a good description of the family F (= useful for algorithms)

Integer Programs for Combinatorial Optimization Problems

Set Covering / Facility Location Problem

Given a set of regions R, a set of facility locations L, service costs c_i and subsets $R_i \subseteq R$ of regions serviced by $i \in L$, find a subset $S \subseteq L$ that cover all regions at minimal cost.

$$\min_{S \subseteq L} \left\{ \sum_{i \in S} c_i : \bigcup_{i \in S} R_i = R \right\}$$

here variables indexed by i, rows by j

IP formulation

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{else} \end{cases} \qquad a_{ji} = \begin{cases} 1 & \text{if } j \in R_i \\ 0 & \text{else} \end{cases}$$

$$\min \sum_{i \in L} c_i \cdot x_i$$

$$\sum_{i \in L} a_{ji} \cdot x_i \geq 1 \quad \forall j \in R$$

$$x_i \in \{0, 1\} \quad \forall i \in L$$

Many comb. opt. problems, like this one, have nice IP formulations and can be solved this way.