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Assignment 9

Problem 1

Consider a solution $\psi(t) = \phi(t) + z(t)$, where $\phi(t)$ is a solution satisfying $\phi^{'}(t) = F(t, \phi(t))$

$$\begin{aligned} \psi^{'}(t) &= F(t, \psi(t)) \\ &= F(t, \phi(t) + z(t)) \\ \phi^{'}(t) + z^{'}(t) &= F(t, \phi(t) + z(t)) \\ z^{'}(t) &= F(t, \phi(t) + z(t)) - F(\phi(t)) \end{aligned} \tag{1}$$

Here we use the mean value theorem (Taylor Expansion),

$$F(t, \phi(t) + z(t)) - F(\phi(t)) = F_y(t, \phi(t))z + f(t, z) \tag{2}$$

where f(t,z) is continuous in the (t, z) space, $f(t, 0) = 0$ and $\lim\limits_{|z|\to 0} \frac{|f(t,z)|}{|z|} = 0$
We shall thus consider the transformed equation

$$z^{'}(t) = F_y(t, \phi(t))z + f(t, z) \tag{3}$$

Apparently $z \equiv 0$ is an equilibrium solution. Note that $z(t) = \psi(t) - \phi(t)$
If $z \equiv 0$ is also a stable solution of the above equation, then by definition
$\forall\, \epsilon > 0$, $\exists\, \delta > 0\, s.t.\, whenever\, |z(t_0) - 0| < \delta$, $|z(t) - 0| < \epsilon\, for\, any\, t \geq t_0$ That's equivalent to $\forall\, \epsilon > 0$, $\exists\, \delta > 0\, s.t.\, whenever\, |\psi(t_0) - \phi(t_0)| < \delta$, $|\psi(t) - \phi(t)| < \epsilon\, for\, any\, t \geq t_0$
If $z \equiv 0$ is asymptotically stable, then by definition, we further have

$$\lim_{t\to\infty} |z(t)| = 0 \quad \iff \quad \lim_{t\to\infty} |\psi(t) - \phi(t)| = 0 \tag{4}$$

which leads to the asymptotic stability of the original equation

Problem 2

By the variation of constant formula,

$$y(t) = e^{(t-t_0)A}y_0 + e^{tA} \int_{t_0}^{t} e^{-sA} f(s)ds \tag{5}$$

Due to the fact that all real parts of the eigenvalues are negative, let $\rho \in (max\{Re\{\lambda_j\}, 0\})$

$$|y(t)| \leq Ke^{\rho(t-t_0)}|y_0| + Ke^{\rho t} \int_{t_0}^{t} e^{-\rho s}|f(s)|ds \tag{6}$$

Multiply both sides by $e^{-\rho t}$

$$e^{-\rho t}|y(t)| \leq Ke^{-\rho t_0}|y_0| + \int_{t_0}^{t} e^{-\rho s}|f(s)|ds \tag{7}$$

Proposition 1: As long as the initial condition of y is sufficiently small, then $|y(t)|$ will be sufficiently small, namely, $\forall\, \alpha > 0$, $\exists\, \delta > 0$ s.t. if $|y_0| < \delta$, $|y(t)| < \alpha$ for all $t \geq t_0$
Proof: Let $\delta = min\left\{\frac{\alpha}{2}, \frac{\alpha}{2K}\right\}$ If $|y(t)| < \alpha$ for some closed interval $[t_0, t_1]$, satisfying $y(t_1) = \alpha$. (Note that $|y_0| = \frac{\alpha}{2} < \alpha$. If $y(t)$ isn't bounded by α on the whole domain, then there must be a point t^* such that the equal sign holds.) On this particular interval $[t_0, t_1]$, we can apply the definition of fraction limit:

$$\lim_{y \to 0} \frac{|f(y)|}{|y|} = 0 \tag{8}$$

$\forall\, \eta > 0$, $\exists\, \alpha > 0$ s.t. If $|y| < \alpha$, $|f(y)| < \eta |y|$

$$e^{-\rho t}|y(t)| \leq K e^{-\rho t_0}|y_0| + \int_{t_0}^{t} e^{-\rho s}|f(s)| ds \leq K e^{-\rho t_0}|y_0| + K\eta \int_{t_0}^{t} e^{-\rho s}|y(s)| ds \tag{9}$$

Hence we can apply Gronwall inequality to get

$$e^{-\rho t}|y(t)| \leq K e^{-\rho t_0}|y_0| e^{K\eta(t-t_0)} \tag{10}$$

Find such η that $-\rho + K\eta < 0$, then $|y(t)| < 1$ is exponential decaying. Note that $|y_0| < \delta = min\left\{\frac{\alpha}{2}, \frac{\alpha}{2K}\right\}$

$$|y(t)| \leq K|y_0| < \frac{\alpha}{2} < \alpha \tag{11}$$

That's a contradiction, so there's no closed interval that bounds $|y(t)|$ with a constant α. Using the bootstrap argument, we finished the proof of proposition 1. $\forall\, \alpha > 0$, $\exists\, \delta > 0$ s.t. if $|y_0| < \delta$, $|y(t)| < \alpha$ for all $t \geq t_0$
Now we can apply Gronwall to the open interval (t_0, ∞)

$$e^{-\rho t}|y(t)| \leq K e^{-\rho t_0}|y_0| + K\eta \int_{t_0}^{t} e^{-\rho s}|y(s)| ds \tag{12}$$

$$e^{-\rho t}|y(t)| \leq K e^{-\rho t_0}|y_0| e^{K\eta(t-t_0)} \tag{13}$$

$$|y(t)| \leq K e^{-(\rho - K\eta)(t-t_0)}|y_0| \tag{14}$$

Choose such η that $-\rho + K\eta < 0$, so that the exponential term is decaying, we can easily find that

$$\lim_{t \to \infty} |y(t)| = 0 \tag{15}$$

and $\forall\, \epsilon > 0$, $\exists\, \delta = \epsilon/K$ s.t. If $|y_0| < \delta$, $|y(t)| < \epsilon$. Hence we proved asymptotic stability. Q.E.D.

Implications

As an analogy of the problem 1, the problem of stability of a non-linear autonomous system is identical to that of the stability of its Taylor-expansion with higher order terms (If the eigenvalues are negative or zero). In the proof above, we figured out that higher order terms are trivial. What only we care about is the linear part of the equation. So for any non-linear autonomous system, the problem can be converted to a linear system which we already know how to solve for.

Problem 3

Linearize the system and write it into matrix form

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -4 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + f(x,y) + \begin{bmatrix} 4 \\ 0 \end{bmatrix} \tag{16}$$

where $f(x,y) = \begin{bmatrix} 0 \\ xy \end{bmatrix}$
We have

$$\lim_{x\to 0, y\to 0} \frac{|f(x,y)|}{|(x,y)|} = \lim_{x\to 0, y\to 0} \frac{xy}{\sqrt{(x^2+y^2)}} = 0 \tag{17}$$

Hence the stability of equation is equivalent to the linearized homogeneous system, whose eigenvalues are solved by

$$\mathbf{det} \begin{bmatrix} \lambda+4 & 2 \\ 0 & \lambda \end{bmatrix} = 0 \tag{18}$$

$\lambda_1 = -4$, $\lambda_2 = 0$. Both of the eigenvalues are either negative or zero and simple, so the original solution is asymptotically stable.

Problem 4

Linearize the system and write it into matrix form

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ ax^2y \end{bmatrix} \tag{19}$$

We have

$$\lim_{x\to 0, y\to 0} \frac{|f(x,y)|}{|(x,y)|} = \lim_{x\to 0, y\to 0} \frac{ax^2y}{\sqrt{(x^2+y^2)}} = 0 \tag{20}$$

Hence the stability of equation is equivalent to the linearized homogeneous system, whose eigenvalues are solved by

$$\mathbf{det} \begin{bmatrix} \lambda & -1 \\ 1 & \lambda-a \end{bmatrix} = 0 \tag{21}$$

$$\lambda^2 - a\lambda + 1 = 0 \tag{22}$$

$$\begin{cases} \lambda_1 + \lambda_2 = a \\ \lambda_1\lambda_2 = 1 \end{cases} \tag{23}$$

Case 1. If $a < 0$, then both real parts of λ must be negative. Either that's because there product is positive so they're of the same sign (real number), or because they are conjugates with equal real parts (complex number) The solution is asymptotically stable.

Case 2. If $a = 0$, then $\lambda_1 = -i$, $\lambda_2 = i$. The eigenvalues are not simple with real part zero. So the solution is stable.
Case 3. If $a > 0$, then both real parts of λ must be positive. Either that's because there product is positive so they're of the same sign (real number), or because they are conjugates with equal real parts (complex number) The solution is unstable.

Problem 5

Using bootstrap argument, we want to show that if $|y_0| \leq \epsilon$, then $|y(t)| \leq 2\epsilon$ for all $t \geq t_0$
Note that $|y_0| = \epsilon < 2\epsilon$. If $y(t)$ isn't bounded by 2ϵ on the whole domain, then there must be a point t_1 such that the equal sign holds. If $|y_0| \leq \epsilon$, and $|y(t)| \leq 2\epsilon$ for a closed interval $[t_0, t_1]$, with $y(t_1) = 2\epsilon$. Then by the inequality,

$$|y(t)| \leq \sqrt{\epsilon^2 + 4\epsilon^4} \qquad t \in [t_0, t_1] \tag{24}$$

As long as $\epsilon < \frac{3}{4}$, the right hand side of the equation is always a better bound than 2ϵ
So the right hand side is sufficiently small. Also note that for autonomous system, it suffices to show a priori estimate in order to prove global existence, as $f(y)$ will always be bounded if y is bounded.