Econ C103: Game Theory and Networks

Module I (Game Theory): Lecture 5

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Bertrand Duopoly

- Two firms i and j compete by setting prices $p_i \geq 0$ and $p_j \geq 0$, resp. Consumers buy only the cheapest good; firm i faces demand function:

$$Q_i(p_i, p_j) = \begin{cases} a - bp_i & \textit{if } \; p_i < p_j \\ \frac{1}{2}(a - bp_i) & \textit{if } \; p_i = p_j \\ 0 & \textit{otherwise} \end{cases},$$

  for $a, b > 0$; marginal cost of production is $c_i > 0$. Similarly for j.
- Each firm maximizes profit "π" given the other's price. For firm i:

$$\pi_i(p_i, p_j) = \begin{cases} (p_i - c_i)(a - bp_i) & \textit{if } \; p_i < p_j \\ \frac{1}{2}(p_i - c_i)(a - bp_i) & \textit{if } \; p_i = p_j \\ 0 & \textit{otherwise} \end{cases}.$$

- Denote $p_i^m \equiv \frac{a+bc_i}{2b}$, which maximizes $\pi_i(p_i, p_j)$ when $p_j \gg 0$.

Bertrand Duopoly

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- Denote $p_i^m \equiv \frac{a+bc_i}{2b}$, which maximizes $\pi_i(p_i, p_j)$ when $p_j \gg 0$.
- Firm i's best response correspondence:

$$BR_i(p_j) = \begin{cases} (\mathrm{p}_j, \infty) & \text{if } \mathrm{p}_j < c_i \\ [\mathrm{p}_j, \infty) & \text{if } \mathrm{p}_j = c_i \\ \emptyset & \text{if } \mathrm{c}_i < p_j \leq p_i^m \\ \mathrm{p}_i^m & \text{if } \mathrm{p}_i^m < p_j \end{cases}$$

- The set of PNE is given by the intersection of $BR_i(p_j)$ and $BR_j(p_i)$...

Bertrand Duopoly

- Firm i's best response correspondence:

$$p_i^*(p_j) \equiv BR_i(p_j) = \begin{cases} (\mathrm{p}_j, \infty) & \text{if } \mathrm{p}_j < c_i \\ [\mathrm{p}_j, \infty) & \text{if } \mathrm{p}_j = c_i \\ \varnothing & \text{if } \mathrm{c}_i < p_j \leq p_i^m \\ \mathrm{p}_i^m & \text{if } \mathrm{p}_i^m < p_j \end{cases}$$

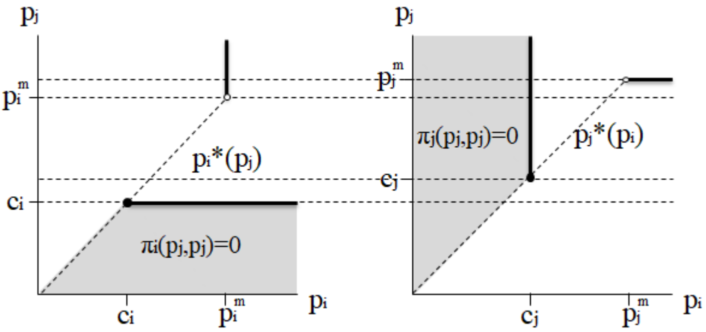

Figure: Bertrand Duopoly: $c_i < c_j$

Bertrand Duopoly

- Firm i's best response correspondence:

$$BR_i(p_j) = \begin{cases} (\mathrm{p}_j, \infty) & \text{if } \ \mathrm{p}_j < c_i \\ [\mathrm{p}_j, \infty) & \text{if } \ \mathrm{p}_j = c_i \\ \emptyset & \text{if } \mathrm{c}_i < p_j \leq p_i^m \\ \mathrm{p}_i^m & \text{if } \mathrm{p}_i^m < p_j \end{cases}$$

- PNE $= \emptyset$ if $c_i \neq c_j$, PNE $= \{c, c\}$ if $c \equiv c_i = c_j$.

Bertrand Duopoly: discrete pricing

- Assume firms can set prices equal to some multiple of the smallest denomination "cent" (e.g., 1 penny USD)?
- Redefine p_i^m to maximize $\pi_i(p_i, p_j)$ when $p_j \gg 0$ subject to p_i^m dividable by cents ($\pi_i(p_i, p_j)$ quadratic, so round p_i^m to nearest cent).
- Firm i's best response correspondence becomes:

$$BR_i(p_j) = \begin{cases} (\mathrm{p}_j, \infty) & \text{if } \mathrm{p}_j < \lceil c_i \rceil \\ \mathrm{p}_j & \text{if } \mathrm{p}_j = \lceil c_i \rceil \\ \mathrm{p}_j - 1 \text{ cent} & \text{if } \lceil c_i \rceil < p_j \leq p_i^m \\ \mathrm{p}_i^m & \text{if } \mathrm{p}_i^m < p_j \end{cases}$$

with the restriction that p_i and p_j are each dividable by cents, and $\lceil \cdot \rceil$ gives the"ceiling" operators which rounds up by one cent.

Bertrand Duopoly: discrete pricing

- Firm i's best response correspondence with discrete pricing:

$$BR_i(p_j) = \begin{cases} (\mathrm{p}_j, \infty) & \text{if } \mathrm{p}_j < \lceil c_i \rceil \\ \mathrm{p}_j & \text{if } \mathrm{p}_j = \lceil c_i \rceil \\ \mathrm{p}_j - 1 \text{ cent} & \text{if } \lceil c_i \rceil < p_j \leq p_i^m \\ \mathrm{p}_i^m & \text{if } \mathrm{p}_i^m < p_j \end{cases}$$

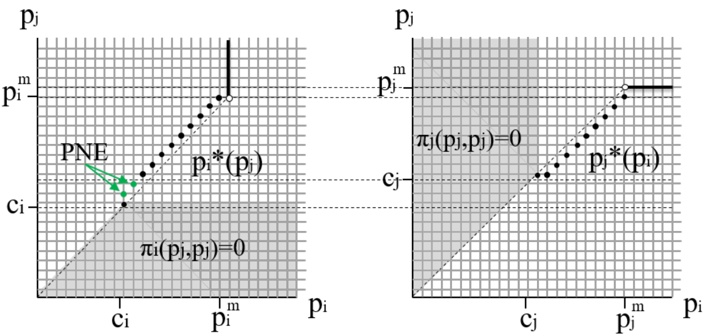

Figure: Bertrand Duopoly with discrete pricing: $c_i < c_j$

Bertrand Duopoly: discrete pricing

- Firm i's best response correspondence with discrete pricing:

$$BR_i(p_j) = \begin{cases} (\mathrm{p}_j, \infty) & \text{if } \mathrm{p}_j < \lceil c_i \rceil \\ \mathrm{p}_j & \text{if } \mathrm{p}_j = \lceil c_i \rceil \\ \mathrm{p}_j - 1 \text{ cent} & \text{if } \lceil c_i \rceil < p_j \leq p_i^m \\ \mathrm{p}_i^m & \text{if } \mathrm{p}_i^m < p_j \end{cases}$$

- If $c_i < c_j$ and $\lceil c_i \rceil < \lceil c_j \rceil$, then:

$$PNE = \left\{ (p, p + 1 \text{ cent}) : \begin{array}{c} p \in [c_i, \lceil c_j - 1 \text{ cent} \rceil] \\ \& \ p \text{ divisible by cents} \end{array} \right\};$$

  similarly if $c_i > c_j$ and $\lceil c_i \rceil > \lceil c_j \rceil$.

  If $c_i \leq c_j$ and $c \equiv \lceil c_i \rceil = \lceil c_j \rceil$, then PNE $= \{c, c\}$.

Reporting a Crime

- Players: $n > 1$ bystanders.
- Actions: $A_i = \{(R)eport, (D)on't\ Report\}$.
- Utilities:

$$u_i(a) = \begin{cases} v & if\ \ a_i = D\ \&\ a_j = R\ for\ some\ j \in N \\ v - c & if\ \ a_i = R \\ 0 & if\ \ a_j = D\ for\ all\ j \in N \end{cases},$$

  where $0 < c < v$.

- Pareto efficient outcome: exactly one bystander reports.
- Best response:
  Report if no one else reports, Don't Report if anyone else reports.
- Is existence of symmetric MNE guaranteed?

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where $0 < c < v$.

- Symmetric MNE:
Find $p^* \equiv \alpha_j(R)$, $\forall j \neq i$, leaving i indifferent between R and D.

$$\begin{aligned} v - c &= v(1 - prob(\text{no one else reports})) \\ \Leftrightarrow c/v &= prob(\text{no one else reports}) \\ &= (1 - p^*)^{n-1} \\ \Leftrightarrow p^* &= 1 - (c/v)^{1/(n-1)}. \end{aligned}$$

Reporting a Crime

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where $0 < c < v$.

- Symmetric MNE: $p^* = 1 - (c/v)^{1/(n-1)}$ for each $i \in N$.
- Probability i reports, p^*, is decreasing in n: increasing incentives to "free load" as more bystanders witness the crime (as $n \uparrow$).
- Probability no one reports, $(1 - p^*)^n = (c/v)^{n/(n-1)}$, is *also* decreasing in n(!): as more bystanders witness the crime, the "public good" of reporting the crime is more under provided.
- Equilibrium welfare? What inefficiencies are there?