Econ C103: Game Theory and Networks

Module I (Game Theory): Lecture 10

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Readings:

1. Osborne (2004) Chapter 9
2. Osborne and Rubinstein (1994) Section 2.6

Static games with incomplete information

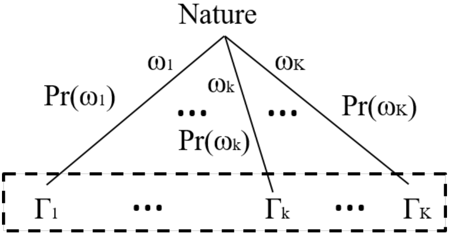
- Many (most?) games are played where players do not have complete information about the world.

- Incomplete information can be of three forms:
  1. (Symmetric information) A hidden state of the world that all players are uncertain of, but have common beliefs regarding the likelihood of different states (e.g., will it rain tomorrow?, 2020 GDP growth?).
  2. (Asymmetric/incomplete information) Players hold private information of the state of the world (e.g., used car sales, poker, blackjack).
  3. (Imperfect information) Players hold private information of players' prior actions (e.g., firm management & VC investment).

- We aim to capture each of these environments (and their mixtures!).
- Question: How??...We study (1) and (2) this week (week 4)...

Static games with incomplete information

John Harsanyi's big idea (Berkeley Nobel Laureate '94):
Start the game early, allowing Nature to move first, and capture private information using private signals.



- (For now) Assume each game Γ_k has the same action sets (for each player across games $k = 1, \ldots, K$, but different players may have different actions sets).
- If no player observes any information (a "signal") informing them of the state ω_k, then information is symmetric across the players...

Static games with incomplete and symmetric information

$\Pr(\omega_1) = p$

	L	R
T	2,1	0,0
B	0,0	1,2

state ω_1

$\Pr(\omega_2) = 1\text{-}p$

	L	R
T	2,0	0,2
B	0,1	1,0

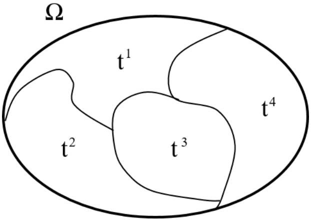
state ω_2

expected utility game

	L	R
T	2 , p	0 , (1-p)2
B	0 , 1-p	1 , p2

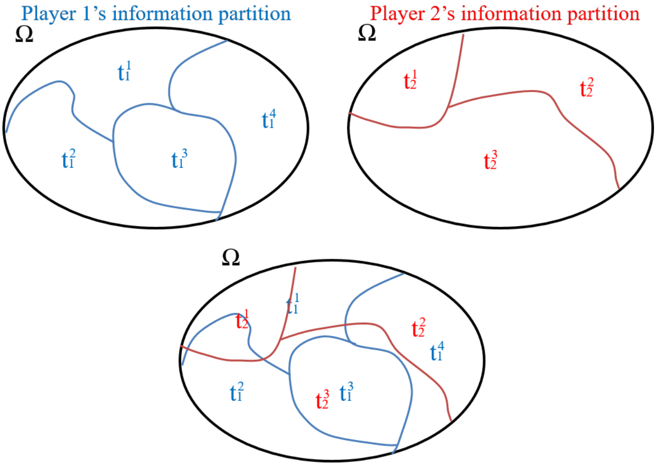
- If $p \geq (1-p)2 \Leftrightarrow p \geq 2/3$, then (T,L) is a PNE.
- If $(1-p) \leq p2 \Leftrightarrow p \geq 1/3$, then (B,R) is also a PNE.
- If $p < 1/3$, then no PNE (but, MNE like Matching Pennies).

Modeling private information with "signals"



- Define: state space $\Omega = \{\omega_1, \omega_2, \ldots\}$ (finite or infinite),
- Define: signal function $\tau : \Omega \to T \equiv \{t^1, t^2, \ldots\}$ where $|T| \leq |\Omega|$.
- Then, τ *partitions* Ω into disjoint subsets with union equal to Ω.

Modeling private information with signals



- $\tau_i : \Omega \to T_i \equiv \{t_i^1, t_i^2, \dots\}$ for $|T_i| \leq |\Omega|$ and $i \in N$ defines i's private information. Each $t_i^s \in T_i$ gives a "type" of player i.
- Notice, we've said nothing about probabilities!...

Modeling beliefs with conditional probability distributions

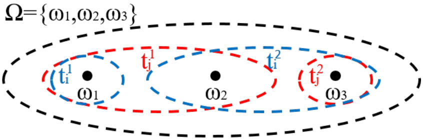
- Each signal t_i^s corresponds to an "event": $t_i^s \mapsto$ a subset of Ω.

- Recall Bayes' rule: with $\Omega = \{\omega_1, ..., \omega_K\}$, conditional likelihood of $\omega \in \Omega$ upon receiving signal $t_i^s \subseteq \Omega$:

$$\mathrm{Pr}(\omega | t_i^s) = \frac{\mathrm{Pr}(t_i^s | \omega) \, \mathrm{Pr}(\omega)}{\sum_{k=1}^{K} \mathrm{Pr}(t_i^s | \omega_k) \, \mathrm{Pr}(\omega_k)}$$

- Notice that $Pr(t_i^s | \omega_k) \in \{0, 1\}$ for all $k = 1, ..., K$.
  $\Rightarrow Pr(t_i^s | \omega_k)$ is an indicator function (with values in $\{0, 1\}$) which "picks" the states mapping to t_i^s (i.e. are in the t_i^s subset of Ω).

Modeling beliefs with conditional probability distributions



- Players $N = \{i, j\}$ hold common priors (of Nature's behavior):

$$\Pr(\omega_1) = 1/4,\ \Pr(\omega_2) = 1/4,\ \Pr(\omega_3) = 1/2.$$

- Players i's "posterior" belief after receiving each signal:

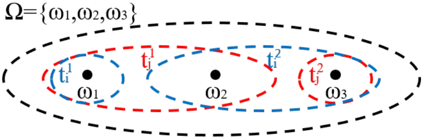
upon receiving t_i^1 :

$$\Pr(\omega_1|t_i^1) = \frac{1 \cdot 1/4}{1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + 0 \cdot \frac{1}{2}} = 1$$
$$\Pr(\omega_2|t_i^1) = \frac{0 \cdot 1/4}{1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + 0 \cdot \frac{1}{2}} = 0$$
$$\Pr(\omega_3|t_i^1) = \frac{0 \cdot 1/2}{1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + 0 \cdot \frac{1}{2}} = 0$$

upon receiving t_i^2 :

$$\Pr(\omega_1|t_i^2) = \frac{0 \cdot 1/4}{0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2}} = 0$$
$$\Pr(\omega_2|t_i^2) = \frac{1 \cdot 1/4}{0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2}} = \frac{1}{3}$$
$$\Pr(\omega_3|t_i^2) = \frac{1 \cdot 1/2}{0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2}} = \frac{2}{3}$$

Modeling beliefs with conditional probability distributions



- Players $N = \{i, j\}$ hold common priors (of Nature's behavior):

$$\Pr(\omega_1) = 1/4,\ \Pr(\omega_2) = 1/4,\ \Pr(\omega_3) = 1/2.$$

- Players j's "posterior" belief after receiving each signal:

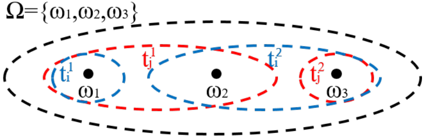
upon receiving t_j^1 :

$$\Pr(\omega_1|t_j^1) = \frac{1 \cdot 1/4}{1\cdot\frac{1}{4}+1\cdot\frac{1}{4}+0\cdot\frac{1}{2}} = \frac{1}{2}$$
$$\Pr(\omega_2|t_j^1) = \frac{1 \cdot 1/4}{1\cdot\frac{1}{4}+1\cdot\frac{1}{4}+0\cdot\frac{1}{2}} = \frac{1}{2}$$
$$\Pr(\omega_3|t_j^1) = \frac{0 \cdot 1/2}{1\cdot\frac{1}{4}+1\cdot\frac{1}{4}+0\cdot\frac{1}{2}} = 0$$

upon receiving t_j^2 :

$$\Pr(\omega_1|t_j^2) = \frac{0 \cdot 1/4}{0\cdot\frac{1}{4}+0\cdot\frac{1}{4}+1\cdot\frac{1}{2}} = 0$$
$$\Pr(\omega_2|t_j^2) = \frac{0 \cdot 1/4}{0\cdot\frac{1}{4}+0\cdot\frac{1}{4}+1\cdot\frac{1}{2}} = 0$$
$$\Pr(\omega_3|t_j^2) = \frac{1 \cdot 1/2}{0\cdot\frac{1}{4}+0\cdot\frac{1}{4}+1\cdot\frac{1}{2}} = 1$$

Modeling beliefs with conditional probability distributions



Beliefs of state, and of other player's belief in each state:

- State ω_1: Player i knows ω_1 occurred, and learns that j knows $\{\omega_1, \omega_2\}$ occurred. Player j knows $\{\omega_1, \omega_2\}$ occured, and does not know whether i knows ω_1 occurred or knows $\{\omega_2, \omega_3\}$ occurred.
- State ω_2: Player i knows $\{\omega_1, \omega_2\}$ occurred, and does not know whether j knows ω_1 occurred or knows $\{\omega_2, \omega_3\}$ occurred. Player j knows $\{\omega_2, \omega_3\}$ occurred, and does not know whether i knows ω_3 occurred or knows $\{\omega_1, \omega_2\}$ occurred.
- State ω_3: Player i knows $\{\omega_2, \omega_3\}$ occured, and does not know whether j knows ω_2 occurred or knows $\{\omega_2, \omega_3\}$ occurred. Player j knows ω_3 occurred, and learns that i knows $\{\omega_2, \omega_3\}$ occurred.

Bayesian Games

Definition (Bayesian Games)

A **Bayesian Games** *is defined as a* $\langle N, \Omega, \Pr, \{A_i, u_i, T_i, \tau_i\}_{i \in N} \rangle$*:*

- $N = \{1, ..., n\}$*: players,*
- Ω*: state space,*
- A_i*: player i's (finite or infinite) action set;* $A \equiv \times_{k \in N} A_k$*,*
- T_i*: player i's type set;* $T \equiv \times_{k \in N} T_k$*.*
- τ_i*: player i's signal function;* $\tau_i : \Omega \mapsto T_i$*.*
- u_i*: i's (vNM) utility from pair* $(\omega, \mathbf{a})$*,* $\omega \in \Omega$*,* $\mathbf{a} \in A$ *(i.e.* $u_i(\omega, \mathbf{a})$ *gives i's utility given action profile* $\mathbf{a}$ *in state* ω*).*

- In Bayesian Games, players use type-contingent (equivalently, "signal contingent") strategies:

$$s_i : T_i \mapsto A_i.$$

Bayesian Games: expected utilities

- Let $\Pr(\omega|t_i)$ denote the probability i places on state ω upon observing t_i.
- Under strategy profile $\mathbf{s} \in \times_{k \in N} S_k$, i's expected utility from playing $a_i \in A_i$ when observing t_i, $U_i(a_i|t)$, is:

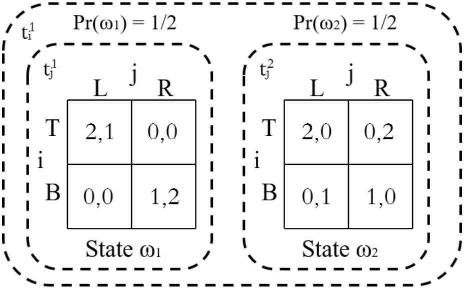
$$U_i(a_i|t_i, \mathbf{s}_{-i}) = \sum_{\omega \in \Omega} \Pr(\omega|t_i) \cdot u_i\left(\omega, a_i, (s_j(\tau_j(\omega)))_{j \neq i}\right).$$

Definition (Bayesian Nash equilibrium)

Given Bayesian game $\langle N, \Omega, \Pr, \{A_i, u_i, T_i, \tau_i\}_{i \in N}\rangle$, a strategy profile $\mathbf{s}^*$ is a **Bayesian Nash equilibrium (BNE)** *iff for each $i \in N$ and each $t_i \in T_i$:*

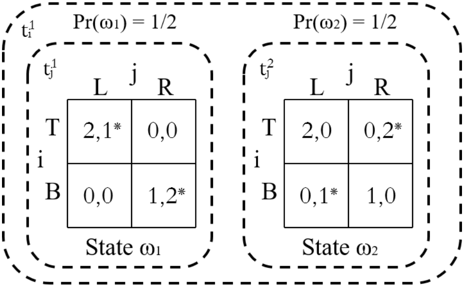
$$U_i(a_i|t_i, \mathbf{s}^*_{-i}) \geq U_i(a'_i|t_i, \mathbf{s}^*_{-i}), \ \forall a'_i \in A_i.$$

One-sided private information: example



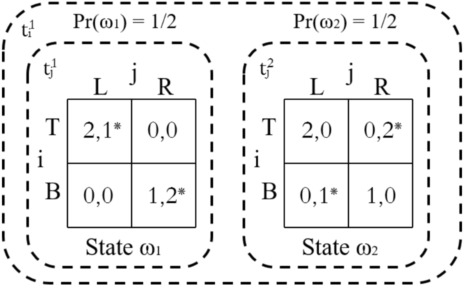
- $\Omega = \{\omega_1, \omega_2\}$.
- $t_i(\omega_1) = t_i^1 = t_i(\omega_2) = t_i^1; \qquad t_j(\omega_1) = t_j^1 \neq t_j(\omega_2) = t_j^2$.

One-sided private information: example



- Player i observes nothing (i.e. $\tau_i(\omega_1) = \tau_i(\omega_2)$), so chooses $a_i \in A_i$.
- Player j learns the state ω, and best responds to (ω, a_i).

One-sided private information: example

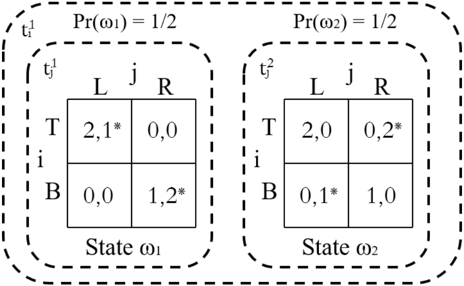


- Player i's expected utilities given $s_j(t_j^1) = L$ and $s_j(t_j^2) = R$:

$$U_i(T|t_i, s_j) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 0 = 1; \quad U_i(B|t_i, s_j) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 = \frac{1}{2}.$$

- $U_i(T|t_i, s_j) > U_i(B|t_i, s_j)$, and each type of j is best responding to T. So, $a_i^* = T$, $s_j^*(t_j^1) = L$ & $s_j^*(t_j^2) = R$ (i.e. $(T, (L, R))$) is a BNE.

One-sided private information: example

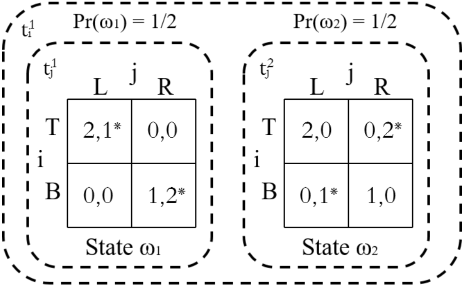


- Player i's expected utilities given $s_j(t_j^1) = R$ and $s_j(t_j^2) = L$:

$$U_i(T|t_i, s_j) = 2; \qquad U_i(B|t_i, s_j) = 0.$$

- $U_i(T|t_i, s_j) > U_i(B|t_i, s_j)$, BUT neither $s_j(t_j^1) = R$ nor $s_j(t_j^2) = L$ is optimal for j given $a_i = T$. So, $(T, (R, L))$ not a BNE.

One-sided private information: example

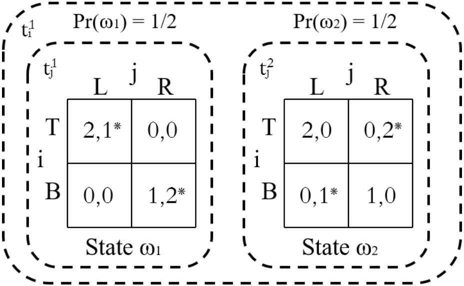


- Player i's expected utilities given $s_j(t_j^1) = L$ and $s_j(t_j^2) = L$:

$$U_i(T|t_i, s_j) = 2; \qquad U_i(B|t_i, s_j) = 0.$$

- $U_i(T|t_i, s_j) > U_i(B|t_i, s_j)$, BUT $s_j(t_j^2) = L$ is not optimal for j given $a_i = T$. So, $(T, (L, L))$ not a BNE.

One-sided private information: example



- Player i's expected utilities given $s_j(t_j^1) = R$ and $s_j(t_j^2) = R$:

$$U_i(T|t_i, s_j) = 0; \qquad U_i(B|t_i, s_j) = 1.$$

- $U_i(T|t_i, s_j) < U_i(B|t_i, s_j)$, BUT $s_j(t_j^2) = R$ is not optimal for j given $a_i = B$. So, $(T, (R, R))$ not a BNE.

Solving Bayesian games with normal forms

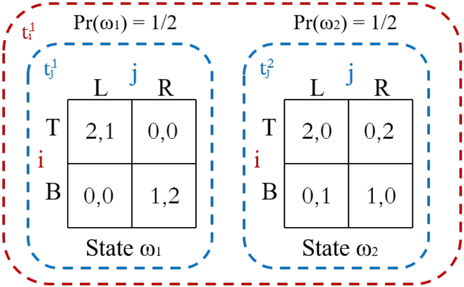
- Notice that:

$$
\begin{aligned}
& \forall t_i \in T_i: \; U_i(a_i|t_i, \mathbf{s}^*_{-i}) \geq U_i(a'_i|t_i, \mathbf{s}^*_{-i}), \; \forall a'_i \in A_i \\
\Leftrightarrow \quad & \mathbb{E}_{\{t_j\}_{j \neq i}} \left[U_i(s_i(t_i)|t_i, \mathbf{s}^*_{-i}) \right] \geq \mathbb{E}\left[U_i(s'_i(t_i)|t_i, \mathbf{s}^*_{-i}) \right], \; \forall s'_i \in S_i \\
\Leftrightarrow \quad & \mathbb{E}_{\{t_j\}_{j \in N}} \left[u_i(\omega, s_i, \mathbf{s}^*_{-i}) \right] \geq \mathbb{E}\left[u_i(\omega, s'_i, \mathbf{s}^*_{-i}) \right], \; \forall s'_i \in S_i.
\end{aligned}
$$

  Last line compares i's "ex-ante" expected utilities (i.e. without conditioning on t_i) under strategy profiles $(s_i, \mathbf{s}^*_{-i})$ and $(s'_i, \mathbf{s}^*_{-i})$.

- Thus, we can find BNE in 2-player Bayesian games as follows:
  1. construct bi-matrix of ex-ante expected utilities, with cells corresponding to each strategy profile (s_i, s_j),
  2. find PNE of the resulting normal form game: these are BNE!

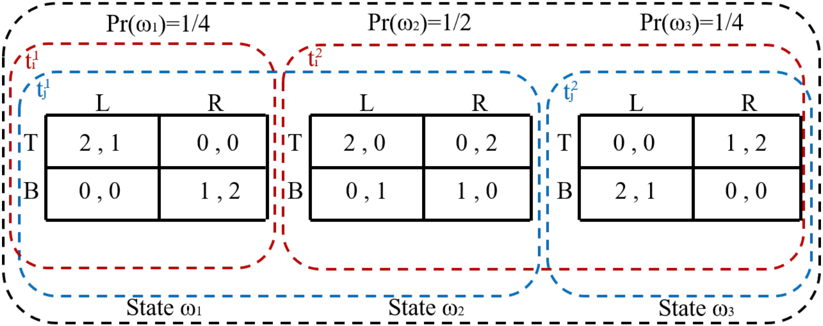
One-sided private information: example



i \ j	(L,L)	(L,R)	(R,L)	(R,R)
T	2* , 1/2	1* , 3/2*	1* , 0	0 , 1
B	0 , 1/2	1/2 , 0	1/2 , 3/2*	1* , 1

Solving Bayesian games with normal forms: example

- Player i is row player, and player j is column player:



Solving Bayesian games with normal forms: example

- Bi-matrix of expected utilities $(\mathbb{E}\left[u_i(\omega, s_i(t_i), s_j(t_j))\right] \;,\; \mathbb{E}\left[u_j(\omega, s_i(t_i), s_j(t_j))\right])$:

j

i

	LL	LR	RL	RR
TT	¼2+½2+¼0 =3/2*, ¼1+½0+¼0 =1/4	¼2+½2+¼1 =7/4*, ¼1+½0+¼2 =3/4	¼0+½0+¼0 =0, ¼0+½2+¼0 =1	¼0+½0+¼1 =1/4, ¼0+½2+¼2 =3/2*
TB	¼2+½0+¼2 =1, ¼1+½1+¼1 =1*	¼2+½0+¼0 =1/2, ¼1+½1+¼0 =3/4	¼0+½1+¼2 =1, ¼0+½0+¼1 =1/4	¼0+½1+¼0 =1/2, ¼0+½0+¼0 =0
BT	¼0+½2+¼0 =1, ¼0+½0+¼0 =0	¼0+½2+¼1 =5/4, ¼0+½0+¼2 =1/2	¼1+½0+¼0 =1/4, ¼2+½2+¼0 =3/2	¼1+½0+¼1 =1/2, ¼2+½2+¼2 =2*
BB	¼0+½0+¼2 =1/2, ¼0+½1+¼1 =3/4*	¼0+½0+¼0 =0, ¼0+½1+¼0 =1/2	¼1+½1+¼2 =**5/4***, ¼2+½0+¼1 =**3/4***	¼1+½1+¼0 =3/4*, ¼2+½0+¼0 =1/2

- $BNE = \{(BB, RL)\}$

Solving Bayesian games with normal forms: example

- Positive-probability outcomes under SPNE (BB, RL) are in **bold**: