Econ C103: Game Theory and Networks

Module I (Game Theory): Lecture 6

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Readings:

1. Osborne (2004) chapter 3.5
2. Osborne and Rubinstein (1994) Example 18.1, Exercises 18.2, 18.3

Auctions: complete information

- In this class: we define an "auction" as a market mechanism in which a single good is sold to a player ("bidder") who submits a highest bid.

- More generally: auctions may include multiple units, and more complicated actions than a single bid, e.g., "bidding fee" auctions, and "Simultaneous Multiple-Round" (SMR) auctions (more later).

- Types of auctions (we study):
  - **Second-price sealed-bid auction** (SPA): static (simultaneous move) auctions where highest bidder pays the second highest bid. SPA approximates dynamic assenting price "English" auction.
  - **First-price sealed-bid auction** (FPA): static (simultaneous move) auctions where highest bidder pays their bid. FPA approximates dynamic descending price "Dutch" auction.

- Auctions can either be under complete information (covered today), or incomplete/private information (covered in week 4).

- Both SPA and FPA under complete information yield multiple PNE...

Auctions: Second-price sealed-bid auction (SPA)

- Players: $n > 1$ bidders.
- Actions: bids $b_i \geq 0$ for each $i \in N$.
- Utilities: Each $i \in N$ endowed with valuation $v_i \geq 0$;
  Without loss of generality, order $v_1 > v_2 > \ldots > v_n$.
  $\bar{b}_{-i} \equiv \max_{j \in N \setminus \{i\}} \{\mathbf{b}_{-i}\}$ gives the highest bid (excluding b_i),
  $\bar{B}_{-i} \equiv \operatorname{argmax}_{j \in N \setminus \{i\}} \{\mathbf{b}_{-i}\}$ are the players bidding $\bar{b}_{-i}$ (excluding i).

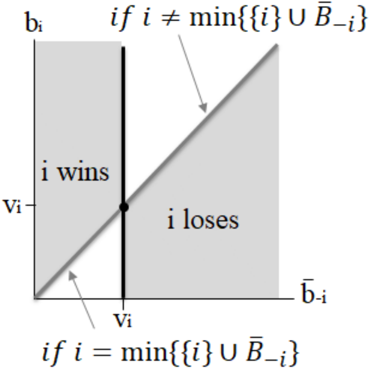
$$u_i(b_i, b_{-i}) = \begin{cases} v_i - \bar{b}_{-i} & \textit{if } b_i > \bar{b}_{-i} \\ v_i - \bar{b}_{-i} & \textit{if } b_i = \bar{b}_{-i} \ \& \ i = \min(\{i\} \cup \bar{B}_{-i}) \\ 0 & \textit{otherwise} \end{cases} .$$

note: i is lowest index with highest bid when $i = \min(\{i\} \cup \bar{B}_{-i})$, so auctioneer chooses lowest index among those bidding the most to win.

Auctions: Second-price sealed-bid auction (SPA)

- Best response in SPA:

$$BR_i(b_{-i}) = \begin{cases} [\bar{b}_{-i}, \infty) & \text{if } \bar{b}_{-i} < v_i \ \& \ i = \min(\{i\} \cup \bar{B}_{-i}) \\ (\bar{b}_{-i}, \infty) & \text{if } \bar{b}_{-i} < v_i \ \& \ i \neq \min(\{i\} \cup \bar{B}_{-i}) \\ [0, \infty) & \text{if } \bar{b}_{-i} = v_i \\ [0, \bar{b}_{-i}) & \text{if } \bar{b}_{-i} > v_i \ \& \ i = \min(\{i\} \cup \bar{B}_{-i}) \\ [0, \bar{b}_{-i}] & \text{if } \bar{b}_{-i} > v_i \ \& \ i \neq \min(\{i\} \cup \bar{B}_{-i}) \end{cases}.$$



Auctions: Second-price sealed-bid auction (SPA)

- The set of PNE in SPA include:
  - $b_i^* = v_i$ for each $i \in N$: bidder 1 wins, pays v_2 (i.e. "bid your value"),
  - $b_1^* = v_1$ and $b_i^* = 0$ for each $i > 1$: bidder 1 wins, pays 0,
  - $b_i^* = 1,000,000,000$, $b_j^* = 0$ for each $j \in N$: bidder i wins, pays 0,
  - ...an infinite number of other PNE.

- Fact: for any $i \in N$, $b_i \neq v_i$ is weakly dominated by $b_i = v_i$.
  Proof:
  - For each $b_i < v_i$, whenever $b_i < \bar{b}_{-i} < v_i$ then i loses (giving utility 0), while if $b_i = v_i$ then i wins and earns positive utility $v_i - \bar{b}_{-i}$.
  - For each $b_i > v_i$, whenever $v_i < \bar{b}_{-i} < b_i$ then i wins (giving negative utility $v_i - \bar{b}_{-i}$), while if $b_i = v_i$ then i loses and earns utility 0.

- It is convention to focus on the "bid your value" PNE.

Auctions: First-price sealed-bid auction (FPA)

- Players: $n > 1$ bidders.
- Actions: bids $b_i \geq 0$ for each $i \in N$.
- Utilities: Each $i \in N$ endowed with valuation $v_i \geq 0$;
  Without loss of generality, order $v_1 > v_2 > \ldots > v_n$.
  $\bar{b}_{-i} \equiv \max_{j \in N \setminus \{i\}} \{\mathbf{b}_{-i}\}$ gives the highest bid (excluding b_i),
  $\bar{B}_{-i} \equiv \operatorname{argmax}_{j \in N \setminus \{i\}} \{\mathbf{b}_{-i}\}$ are the players bidding $\bar{b}_{-i}$ (excluding i).

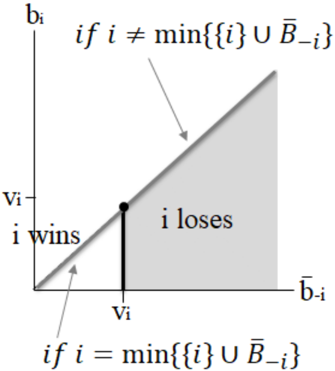
$$u_i(b_i, b_{-i}) = \begin{cases} v_i - b_i & \textit{if } \ b_i > \bar{b}_{-i} \\ v_i - b_i & \textit{if } \ b_i = \bar{b}_{-i} \ \& \ i = \min(\{i\} \cup \bar{B}_{-i}) \\ 0 & \textit{otherwise} \end{cases} .$$

note: i is lowest index with highest bid when $i = \min(\{i\} \cup \bar{B}_{-i})$, so auctioneer chooses lowest index among those bidding the most to win.

Auctions: First-price sealed-bid auction (FPA)

- Best response in FPA:

$$BR_i(b_{-i}) = \begin{cases} \bar{b}_{-i} & \text{if } \bar{b}_{-i} < v_i \;\&\; i = \min(\{i\} \cup \bar{B}_{-i}) \\ \emptyset & \text{if } \bar{b}_{-i} < v_i \;\&\; i \neq \min(\{i\} \cup \bar{B}_{-i}) \\ [0, \bar{b}_{-i}] & \text{if } \bar{b}_{-i} = v_i \\ [0, \bar{b}_{-i}) & \text{if } \bar{b}_{-i} > v_i \;\&\; i = \min(\{i\} \cup \bar{B}_{-i}) \\ [0, \bar{b}_{-i}] & \text{if } \bar{b}_{-i} > v_i \;\&\; i \neq \min(\{i\} \cup \bar{B}_{-i}) \end{cases}.$$



Auctions: First-price sealed-bid auction (FPA)

- Fact: $b_i = v_i$ is no longer a weakly dominant strategy: if $\bar{b}_{-i} < v_i$ then i can decrease ("shade") her bid keeping $b_i > \bar{b}_{-i}$, and earn a positive utility $v_i - b_i$.

- The set of PNE in FPA:

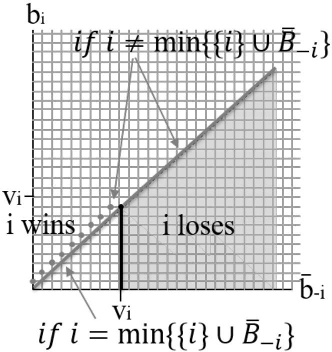
$$\left\{\mathbf{b}^* \in \mathbb{R}_+^n : b_1 \in [v_2, v_1], b_1 \geq \bar{b}_{-1}, b_1 = b_i \textit{ for some } i \in N \backslash \{1\}\right\}.$$

  In words, bidder 1 always wins in any PNE, and there must be some bid $b_i = \bar{b}_{-1}$ for $i > 1$ preventing bidder 1 from further shading her bid below $b_1 = \bar{b}_{-1}$.

Auctions: First-price sealed-bid auction (FPA)

- Under discrete bidding (i.e. bids must be multiples of "cent"), best response in FPA becomes:

$$BR_i(b_{-i}) = \begin{cases} \bar{b}_{-i} & \textit{if } \bar{b}_{-i} < v_i \;\&\; i = \min(\{i\} \cup \bar{B}_{-i}) \\ \bar{b}_{-i} + 1 \text{ cent} & \textit{if } \bar{b}_{-i} < v_i \;\&\; i \neq \min(\{i\} \cup \bar{B}_{-i}) \\ [0, \bar{b}_{-i}] & \textit{if } \bar{b}_{-i} = v_i \\ [0, \bar{b}_{-i}) & \textit{if } \bar{b}_{-i} > v_i \;\&\; i = \min(\{i\} \cup \bar{B}_{-i}) \\ [0, \bar{b}_{-i}] & \textit{if } \bar{b}_{-i} > v_i \;\&\; i \neq \min(\{i\} \cup \bar{B}_{-i}) \end{cases}.$$



Auctions: First-price sealed-bid auction (FPA)

- Fact: Under discrete bidding, $b_i = v_i$ still not weakly dominant: e.g., if $\bar{b}_{-i} < v_i - 1$ cent, then i can set $b_i = \bar{b}_{-i} + 1$ cent, and earn a positive utility $v_i - b_i$.

- The set of PNE in FPA is (essentially) unchanged:

$$\left\{ \mathbf{b}^* \in \hat{\mathbb{R}}_+^n : b_1 \in [v_2, v_1], b_1 \geq \bar{b}_{-1}, b_1 = b_i \textit{ for some } i \in N \setminus \{1\} \right\},$$

where $\hat{\mathbb{R}}_+$ gives all non-negative bids divisible by cents.

Spectrum Auctions: some facts

- Between July 1994 and February 2001, the Federal Communications Commission (FCC) conducted 33 spectrum auctions, with total revenue of $40 billion USD (P. Cramton '01).
- In January 2015, FCC raised $45 billion in one wireless spectrum auction (CNET).
- With advent of 5G, this number will only increase.
- FCC uses Simultaneous Multiple-Round (SMR) Auctions:
  - Multiple, successive rounds of bidding over licenses (dynamics).
  - After each round, bidders observe others' bids, potentially learning about others' evaluations (incomplete information and learning).
  - Bidding continues until all bidders no longer adjust their bids.
- Week 3: Extensive (dynamic) games with perfect information.
- Week 4: Static games with incomplete information.
- Weeks 5 & 6: Dynamic games with incomplete information.