Problem 1

Solve for the eigenvectors of A,

$$|\lambda I - A| = \begin{bmatrix} \lambda & -1 & 0 & 0 \\ 1 & \lambda - 2 & 0 & 0 \\ 2 & -2 & \lambda - 1 & 0 \\ 0 & -1 & 0 & \lambda + 1 \end{bmatrix} \tag{1}$$

We have $\lambda_1 = -1, \lambda_2 = \lambda_3 = \lambda_4 = 1$. For the first eigenvalue, solve

$$\begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & -3 & 0 & 0 \\ 2 & -2 & -2 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} v_1 = \mathbf{0} \tag{2}$$

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{3}$$

For the second eigenvalue (with algebraic multiplicity 3), solve for the generalized eigenvectors

$$|A - \lambda I|^3 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -4 \end{bmatrix} \tag{4}$$

$$|A - \lambda I|^3 v = 0 \tag{5}$$

The solutions are

$$v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 1 \end{bmatrix} \tag{6}$$

For the columns of a fundamental matrix,

$$y_i(t) = e^t \left\{ v_i + t(A - I)v_i + \frac{t^2}{2}(A - I)^2 v_i \right\} \tag{7}$$

Note that

$$|A - I|\, v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad |A - I|\, v_3 = \begin{bmatrix} -4 \\ -4 \\ -8 \\ -2 \end{bmatrix} \qquad |A - I|\, v_4 = \begin{bmatrix} 4 \\ 4 \\ 8 \\ -2 \end{bmatrix} \tag{8}$$

and

$$|A-I|^2 v_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (i=2,3,4) \tag{9}$$

Summing up, the fundamental matrix of A is

$$\Phi(t) = \begin{bmatrix} 0 & (4-4t)e^t & 4te^t & 0 \\ 0 & -4te^t & (4+4t)e^t & 0 \\ e^t & -8te^t & 8te^t & 0 \\ 0 & (1-2t)e^t & (1-2t)e^t & e^{-t} \end{bmatrix} \tag{10}$$

Problem 2

Recall the proof for the lemma, we apply Jordan Decomposition $A = TJT^{-1}$

$$\left|e^{tA}\right| \le |T|\left|T^{-1}\right|\left|diag\left\{e^{tJ_1}, e^{tJ_2}, ..., e^{tJ_s}\right\}\right| = \left|diag\left\{e^{tJ_1}, e^{tJ_2}, ..., e^{tJ_s}\right\}\right| \tag{11}$$

Within a Jordan block,

$$e^{tJ_i} = e^{\lambda[J_i]t} \begin{bmatrix} 1 & t & \frac{t^2}{2} & \dots & \frac{t^{k-1}}{(k-1)!} \\ 0 & 1 & t & \ddots & \vdots \\ 0 & 0 & 1 & & t \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \tag{12}$$

Generally, the norm of $\left|e^{tJ_i}\right| = p_i(t)e^{Re\lambda[J_i]t}$, where $p_i(t)$ is a polynomial, is larger than $e^{Re\lambda[J_i]t}$. The key for the proof is that when t approaches infinity, the polynomial is bounded by any exponential functions $e^{\delta t}$. Rigorously speaking, $\forall \delta > 0$, $\exists N > 0$ s.t. if $t > N$, $p_i(t) < e^{\delta t}$ However, the inequality, won't hold if $\delta = 0$. A typical counter example will be:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \left|e^{tA}\right| = e^t(2+t) \tag{13}$$

The maximum of eigenvalue is $\rho = 1$, when $k \to \infty$, however, we cannot find such k that $\left|e^{tA}\right| \le ke^t$

Problem 3

By the same procedure as Problem 2, we apply Jordan Decomposition $A = TJT^{-1}$

$$\left|e^{tA}\right| \le |T|\left|T^{-1}\right|\left|diag\left\{e^{tJ_1}, e^{tJ_2}, ..., e^{tJ_s}\right\}\right| = \left|diag\left\{e^{tJ_1}, e^{tJ_2}, ..., e^{tJ_s}\right\}\right| \tag{14}$$

Within a Jordan block,

$$e^{tJ_i} = e^{\lambda[J_i]t} \begin{bmatrix} 1 & t & \frac{t^2}{2} & \dots & \frac{t^{k-1}}{(k-1)!} \\ 0 & 1 & t & \ddots & \vdots \\ 0 & 0 & 1 & & t \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \tag{15}$$

For those with real-part-zero (non-complex) eigenvalues, $\left|e^{tJ_i}\right| = n_i$, where λ_i has multiplicity n_i. Assume the all such n_i adds up to N.
For those negative-real-part eigenvalues, apply the theory in the lemma, and choose $\rho \in (max\{Re\lambda_i \backslash \{0\}\}, 0)$, we have $\left|\sum e^{tJ_i}\right| \leq 0$, when $t \to \infty$.
The sum of them ($\left|e^{tA}\right|$) is bounded by N, hence every solution of $y^{'} = Ay$ is bounded on $(0, +\infty)$

Problem 4

If $\exists M > 0, T > 0, a \in \mathbb{R}$, s.t. $|g(t)| \leq Me^{at}$, $\forall t \geq T$
By the variation of constant formula,

$$\begin{aligned} \phi(t) &= e^{tA}y_0 + \int_0^t e^{(t-s)A}g(s)ds \\ &= e^{tA}y_0 + \int_0^T e^{(t-s)A}g(s)ds + \int_T^t e^{(t-s)A}g(s)ds \end{aligned} \tag{16}$$

$$\begin{aligned} |\phi(t)| &\leq \left|e^{tA}\right| |y_0| + \int_0^T \left|e^{(t-s)A}\right| |g(s)| \, ds + \int_T^t \left|e^{(t-s)A}\right| |g(s)| \, ds \\ &\leq ke^{\rho t} |y_0| + \int_0^T ke^{\rho(t-s)}Nds + \int_T^t ke^{\rho(t-s)}Me^{as}ds \\ &= ke^{\rho t} |y_0| + Nke^{\rho t}\frac{1}{\rho}(1 - e^{-\rho T}) + Mke^{\rho t}\frac{1}{a-\rho}\left[e^{(a-\rho)t} - e^{(a-\rho)T}\right] \end{aligned} \tag{17}$$

where N is the upper bound for $|g(t)|$, $t \in [0, T]$
We also have

$$\phi^{'}(t) = A\phi(t) + g(t) \tag{18}$$

So the explicit estimate for $\phi^{'}(t)$ is

$$\left|\phi^{'}(t)\right| \leq |A| ke^{\rho t} |y_0| + |A| Nke^{\rho t}\frac{1}{\rho}(1-e^{-\rho T}) + |A| Mke^{\rho t}\frac{1}{a-\rho}\left[e^{(a-\rho)t} - e^{(a-\rho)T}\right] + |g(t)| \tag{19}$$

which can be simplified as

$$\begin{aligned} \left|\phi^{'}(t)\right| &\leq |A| C_1e^{at} + |A| C_2e^{\rho t} + Me^{at} \qquad t \geq T \\ &\leq Ke^{max(a,\rho)t} \end{aligned} \tag{20}$$

Rmk: If $a < 0$ and $Re\lambda_i < 0$, $\lim_{t\to\infty} \phi^{'}(t) = 0$

Problem 5

Consider the partition of matrix, for the upper-left submatrix,

$$\mathbf{det}|\lambda I - A_1| = 0 \iff \lambda_1 = -1,\, \lambda_2 = -\pi \tag{21}$$

The solutions of the eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1-\pi \end{bmatrix} \tag{22}$$

For the bottom-right submatrix,

$$\mathbf{det}|\lambda I - A_1| = 0 \iff \lambda_1 = -2 + \sqrt{7}i,\, \lambda_2 = -2 - \sqrt{7}i \tag{23}$$

The solutions of the eigenvectors are

$$v_1 = \begin{bmatrix} 1-\sqrt{7}i \\ -4 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1+\sqrt{7}i \\ -4 \end{bmatrix} \tag{24}$$

The fundamental matrix for A is

$$\phi(t) = \begin{bmatrix} e^{-t} & e^{-\pi t} & 0 & 0 \\ 0 & e^{-\pi t}(1-\pi) & 0 & 0 \\ 0 & 0 & (1-\sqrt{7}i)e^{(-2+\sqrt{t}i)t} & (1+\sqrt{7}i)e^{(-2-\sqrt{7}i)t} \\ 0 & 0 & -4e^{(-2+\sqrt{t}i)t} & -4e^{(-2-\sqrt{7}i)t} \end{bmatrix} \tag{25}$$

Note that $e^{it} = cos(t) + isin(t)$, hence $\lim_{t\to\infty} e^{it} = 1$, and all real parts of the exponential parts are negative.

$$\lim_{t\to\infty} \phi(t) = \mathbf{0} \tag{26}$$

If $g(t) = e^{-t}$, using the theorem in class, we can show that g(t) grows no faster than an exponential function. (i.e. Take $M = 1, T \in \mathbb{R}^+, a = -1$, $|g(t)| \leq Me^{at}$, $\forall t \geq T$)
$a < 0$, $\rho < 0$, so the norm of the solution is bounded by $Ke^{max(a,\rho)t}$. When $t \to \infty$, $Ke^{max(a,\rho)t} \to 0$. Hence the following equation still holds.

$$\lim_{t\to\infty} \phi(t) = \mathbf{0} \tag{27}$$