Econ C103: Introduction to Mathematical Economics

UC Berkeley, Fall 2019

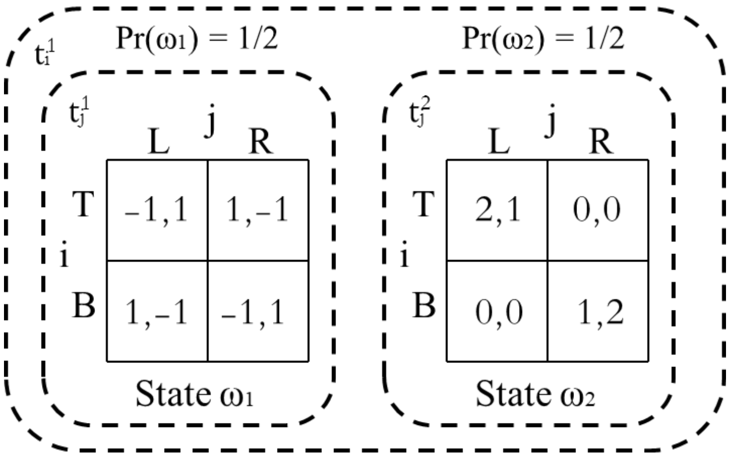
Assignment 2 Solutions *(total points: 75)*.

Problem 1: Construct the Stackelberg games (i.e. one player moves before the other) of the following static game, and determine which players have a first-mover advantage by comparing SPNE payoffs in each Stackleberg game to the set of MNE expected payoffs in the static game. *Hint: First show that L is never played by j in MNE of the static game.*

		j: L	j: C	j: R
i	T	1,-1	3,-3	-3,3
	B	2,4/5	-3,3	3,-1

Solution: We first find the set of MNE of the static game. L is never played in a MNE because L is strictly dominated by a mixed strategy which places 1/2 probability on C and R. Therefore, the unique MNE of the static game is for j to mix 2/5 on T and 3/5 on B (verify that this makes i indifferent between C and R), and for i to mix 1/2 on C and R (which makes j indifferent). The profile of equilibrium expected payoffs is (0,3/5). In the Stackelberg game in which i moves first, j will be able to play R if i played T and to play C if i played B, either of which yields j payoff 3. So, i has a first-mover disadvantage (-3<0), and j has a second-mover advantage (3>3/5). In the Stackelberg game in which j moves first, j can commit to playing L, after which i plays B. We see that j has a first-mover advantage (4/5>3/5), and i has a second mover advantage (2>0); interestingly, both players are advantaged when j moves first (as compared to the MNE of the static game).

Problem 2: Find the BNE of the following game. To do this, construct the corresponding normal-form game in which players' action sets equal their strategies in the Bayesian game, and certain payoffs equal their ex-ante expected payoffs in the Bayesian game as a function of the strategy profiles.

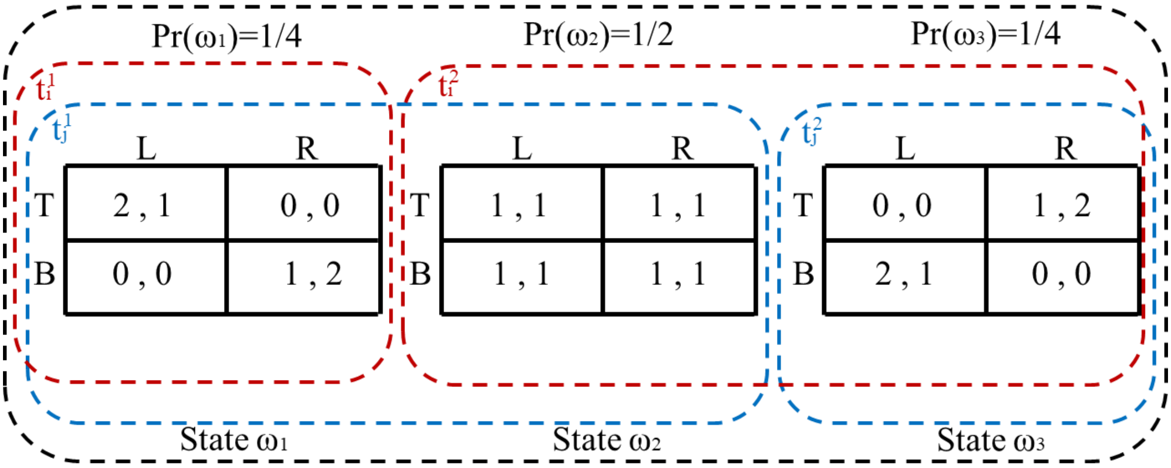


Solution:

i \ j	(L,L)	(L,R)	(R,L)	(R,R)
T	1/2*,1*	-1/2,1/2	3/2*,0	1/2*,-1/2
B	1/2*,-1/2	1*,1/2	-1/2,1/2	0,3/2*

BNE={(T,LL)}

Problem 3: Find the BNE of the following game. To do this, complete the normal-form game (below) in which players' action sets equal their strategies in the Bayesian game, and certain payoffs equal the ex-ante expected payoffs in the Bayesian game as a function of the strategy profiles.



i \ j	LL	LR	RL	RR
TT	¼2+½1+¼0 =1, ¼1+½1+¼0 =3/4	¼2+½1+¼1 =5/4, ¼1+½1+¼2 =5/4	¼0+½1+¼0 =1/2, ¼0+½1+¼0 =1/2	
TB				¼0+½1+¼0 =1/2, ¼0+½1+¼0 =1/2
BT	¼0+½1+¼0 =1/2, ¼0+½1+¼0 =1/2	¼0+½1+¼1 =3/4, ¼0+½1+¼2 =1		
BB	¼0+½1+¼2 =1, ¼0+½1+¼1 =3/4			

Solution:

j

		LL	LR	RL	RR
	TT	¼2+½1+¼0 =1, ¼1+½1+¼0 =3/4	¼2+½1+¼1 =5/4*, ¼1+½1+¼2 =5/4*	¼0+½1+¼0 =1/2, ¼0+½1+¼0 =1/2	¼0+½1+¼1 =3/4, ¼0+½1+¼2 =1
	TB	¼2+½1+¼2 =3/2*, ¼1+½1+¼1=1*	¼2+½1+¼0 =1, ¼1+½1+¼0 =3/4	¼0+½1+¼2 =1, ¼0+½1+¼1 =3/4	¼0+½1+¼0 =1/2, ¼0+½1+¼0 =1/2
i	BT	¼0+½1+¼0 =1/2, ¼0+½1+¼0 =1/2	¼0+½1+¼1 =3/4, ¼0+½1+¼2 =1	¼1+½1+¼0 =3/4, ¼2+½1+¼0 =1	¼1+½1+¼1 =1*, ¼2+½1+¼2 =3/2*
	BB	¼0+½1+¼2 =1, ¼0+½1+¼1 =3/4	¼0+½1+¼0 =1/2, ¼0+½1+¼0 =1/2	¼1+½1+¼2 =3/2*, ¼2+½1+¼1 =3/2*	¼1+½1+¼0 =1/4, ¼2+½1+¼0 =1

BNE={(TT,LR), (TB,LL), (BT,RR), (BB,RL)}

Problem 4:
Recall the Diamond-Dybvig model of lecture 11. Now consider the following variation of the model. The central bank secures consumers' deposits, as follows. If a consumer arrives at the bank at either $t = 2$ or $t = 3$ and the bank is illiquid (i.e. has no funds to offer the consumer), then the central banks will reimburse the consumer an amount $C' > 0$. What are the set off symmetric BNE of the game for each C' value (by "symmetric" I mean the BNE in which all consumers of each given type use the same strategy)?

Solution:
Case $C' > C_2^{A*}$: Because the central bank secures consumers' deposits by the amount $C' > C_2^{A*}$, if there is a bank run then the type B's will strictly prefer not to run to the bank, because if they do run they receive at most C_2^{A*}. Therefore, the bank-run equilibrium no longer exists. The unique symmetric BNE is thus for type A's to withdraw C_2^{A*} at $t = 1$ and type B's to withdraw C_2^{B*} at $t = 2$.
Case $C' < C_2^{A*}$: The insurance that the central bank promises is no longer adequate to prevent the run. Both the $\left(C_2^{A*}, C_2^{B*}\right)$ BNE and the bank run BNE exist; the only caveat is that type B's can consume C' in $t = 2$ in the latter BNE.
Case $C' = C_2^{A*}$: Both the $\left(C_2^{A*}, C_2^{B*}\right)$ BNE and the bank run BNE exist

Problem 5: This problem tests your intuition for BNE in auctions.
Part 1) Consider a sealed-bid auction with i.i.d. values distributed via cdf $F(\cdot)$. Prove that if the mechanism $P(\boldsymbol{b})$ is given by the k^{th}-highest bid (i.e. a k^{th}-price auction) then the symmetric BNE involves bidding above ones values (show this must hold for some values; this is actually true for all values). You can use all results/theorems provided in lecture. *Hint: use the fact that $F^{[k]}(v) < F^{[k+1]}(v)$ for all v and k, and that*:

$$\int_{\underline{v}}^{\overline{v}} f(v) dF^{[k]}(v) > \int_{\underline{v}}^{\overline{v}} f(v) dF^{[k+1]}(v)$$

for any monotonically increasing $f(v)$.

Part 2) Again consider a sealed-bid auction with i.i.d. values distributed via cdf $F(\cdot)$. By considering the equilibrium bidding of the 1^{st}- and 2^{nd}- price auctions, and using your intuition, postulate what equilibrium bidding in the k^{th}-price auction, for $k > 2$, will converge to as the number of bidders grows very large. Do you expect $b^*(v)$ to be increasing or decreasing in n for $k > 2$? Is expected revenue increasing or decreasing in n? Explain.

Solution:

Part 1) We use the revenue equivalence theorem. A k^{th}-price auction is a direct revelation auction that assigns the good to the highest bidder, and such that the expected payment of a bidder with value 0 is 0. Thus, it must yield the same expected revenue of the 1^{st}- and 2^{nd}- price auctions. Expected revenue is given by:

$$E[P(\boldsymbol{b}^*)] = \int_{\underline{v}}^{\overline{v}} b^*(v)dF^{[k]}(v).$$

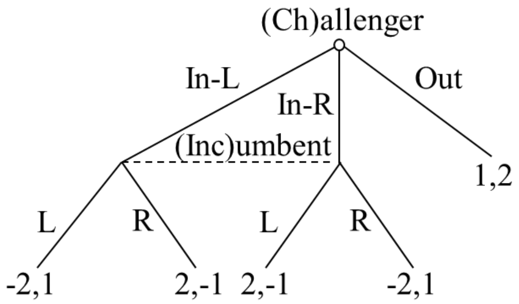
Because $F^{[k]}(v) < F^{[k+1]}(v)$ for any k, and thus that:

$$\int_{\underline{v}}^{\overline{v}} b^*(v)dF^{[k]}(v) > \int_{\underline{v}}^{\overline{v}} b^*(v)dF^{[k+1]}(v)$$

it must therefore be that $b^*(v) > v$ for at least some v if $k > 2$, because v gives the equilibrium bid in the second price auction. We can use the same logic to show that equilibrium bidding is increasing in k. Intuitively, the lower the pricing mechanism (the higher k) the more the bidders are willing to bid high to increase the probability of winning.

Part 2) As n grows large, bidding will converge on $b^*(v) = v$. To see this, as the number of competing bidders grows large, the risk in bidding above your value increases because the probability that the $k - 1^{th}$-order statistic among all others' bids ends up falling above your value is increasing in n. These events are costly, as you must pay more than your value when winning. Thus, bidders face less incentive to increase their bids above their values. As n increases, expected revenue increases in n according to the expectation of the second-order value statistic (which equals $\frac{n-1}{n+1}$ when F is uniform; see lecture 12), because bidding your value is always an equilibrium in the 2^{nd}-price auction and as a result of the revenue equivalence principal. In fact, for any k-price auction mechanism, the expected revenue to the auctioneer converges on $\bar{v}$ from below as $n \to \infty$. Thus, while over bidding (i.e. bidding above one's value) becomes less aggressive as n increases, the cost to the auctioneer in expected revenue is outweighed by the value due to more bidders competing in the auction: expected revenue increases.

Problem 6: Find the set of beliefs of the (Inc)umbent regarding the probability that (Ch)allenger choose In-L, and the set of strategies Inc that are supported in PBNE.

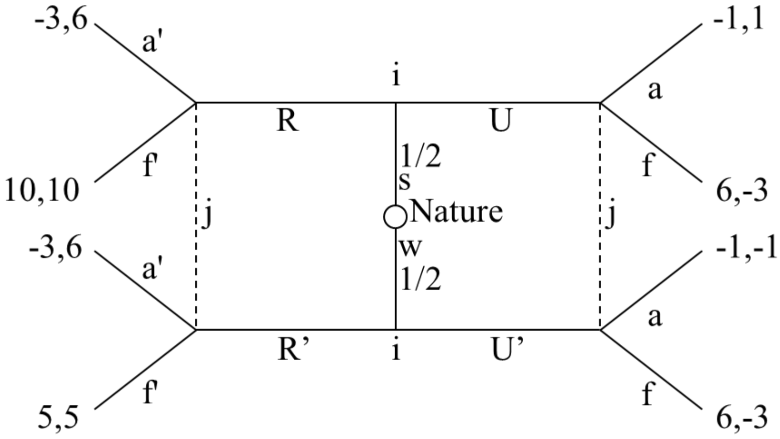


Solution: First, we argue that Ch must choose Out in any PBNE. For this, if Ch mixes on In-L and/or In-R, then Inc must use Bayes' rule to form her beliefs. If Ch mixes more on In-L then Inc has a strict preference to play L, in which case Ch strictly prefers In-R, a contradiction. Likewise, if Ch mixes more on In-R then Inc has a strict preference to play R, in which case Ch strictly prefers In-L, a contradiction. So, Ch must play Our in and PBNE. To support this, the payoff of 1 must be at least as large as that from playing In-L or In-R. Letting q give the probability Inc plays L, we need:

U_i(In-L)$= -2q + 2(1-q) \leq 1 \Leftrightarrow q \geq \frac{1}{4}$
U_i(In-R)$= 2q - 2(1-q) \leq 1 \Leftrightarrow q \leq \frac{3}{4}$

For Inc to mix on both actions L and R, she must be indifferent between L and R. For this, Inc must believe that Ch played In-L and In-R with equal probability when Inc is called upon to play.

Problem 7:



Solution:
We'll check the four pure sender (player *i*) behavioral strategies, and check if they give a PBNE by deriving consistent beliefs for the sender, sequentially rational behavioral strategies given these beliefs, and check that our assumption on the sender's behavior is sequentially rational (for both types) given the receiver's behaviors.

Separating equilibria

Player *i* type (s)trong plays R, Player *i* type (w)eak plays U':

Player j's beliefs will be updated to placing probability one on the unique type having sent the signal, given player i's behavioral strategy. Player j having received the signal “Ready” knows she is at her top node, and thus prefers f' to a' (by 10>6). Player j having received “Unready” knows she is at her bottom node. However note that regardless of her beliefs, player j always prefers a to f, and thus this will be the only sequentially rational behavior for player j. Given j's behaviors, i type strong prefers her action R (by 10>-1). However, i type weak prefers her action R' (5>-1), giving a profitable deviation.

Player i type (s)trong plays U, player i type (w)eak plays R':

Beliefs again are derived Bayes’ rules, so player i knows which type sent the signal. Player j having received the signal “Ready” knows she is at her bottom node, and thus prefers a' to f'. Given j's behaviors, player i type strong will prefer her action U (by -1>-3). However, player i type weak will prefer her action U' to R' (by -1>-3), giving a profitable deviation.

Thus, there are no separating equilibria.

Pooling equilibria

Player i plays “(R)eady” (R and R'):

Upon receiving signal “Ready”, j updates her beliefs via Bayes' rule, which will give the priors 1/2 on history sR and 1/2 on history wR'. Player j's actions a' and f' give her in expected payoff 1/2·6+1/2·6=6 and 1/2·10+1/2·5=15/2, respectively. Thus, playing f' is her unique sequentially rational behavior. Though Bayes' rule does not apply upon receiving “Unready”, player j plays action a under all beliefs. Given these behaviors, player i will be behaving optimally: 10>-1 for type strong and 5>-1 for type weak. This gives the set of PBNE:

$$\left((R, R'), \left(f', a; \mu[Ready](sR) = \frac{1}{2}, \mu[Unready](sU) \in [0,1]\right)\right).$$

Player i plays “Unready” (U and U'):

Player j receiving “Unready” updates her beliefs to her priors (1/2-1/2), but plays action a regardless. Player j's preferences over her actions upon receiving “Ready” depends on her beliefs of the likelihood player i is strong or weak (if j’s strong j prefers f', and if j's weak j prefers a'). We'll set the beliefs that make her indifferent to allow for a full range of sequentially rational behaviors. These are found by equating expected payoffs to her pure actions given beliefs, and solving for $\mu \equiv \mu_2[Ready](sR)$:

Player j plays a': $\mu \cdot 6 + (1 - \mu) \cdot 6 = 6$

Player j plays f': $\mu \cdot 10 + (1 - \mu) \cdot 5$

Equating these gives $\mu = 1/5$. Then, we need to find the range of behaviors of player j at her left information set that insures player i's pooling on “Unready” to be sequentially rational. Given $\beta_2[Ready](a') = \beta$, the conditions are 1) $\beta \cdot (-3) + (1 - \beta) \cdot 10 \leq -1$, and 2) $\beta \cdot (-3) + (1 - \beta) \cdot 5 \leq -1$. The first condition is more binding, giving $\beta \geq 11/13$. This gives the set of PBNE:

$$\left((U, U'), \left(\beta_2[Ready](a') \geq \frac{11}{13}, a; \mu[Uneady](sR) = \frac{1}{2}, \mu[Ready](sU) = \frac{1}{5}\right)\right).$$