HW10 (due Dec. 3rd Tuesday)

Instructions. You *must* declare all resources that you have used on this homework (include but not limited to anyone, any book, and any webpage). Do not skip steps.

1. Consider the following two-dimensional autonomous system

$$
\begin{aligned}
x' &= y - xf(x,y) \\
y' &= -x - yf(x,y),
\end{aligned}
$$

where $f(x,y) \geq 0$ for any $(x,y) \in \mathbb{R}^2$.
(a) Explain why the linearization method does not work.
(b) Study the stability of the zero solution using the Lyapunov's direct method.
(c) What is a sufficient condition of f to have zero solution being asymptotically stable?
(d) What is a sufficient condition of f to have zero solution being unstable?

2. Show that the zero solution of the following system is globally asymptotically stable.

$$
\begin{aligned}
x' &= y - x^3 \\
y' &= -2(x^3 + y^5).
\end{aligned}
$$

(Hint: Consider $V = x^4 + y^2$, and apply the Lyapunov theorems.)

3. *(revised form [BN Page 200 Problem 5])* Consider the simple pendulum equation

$$\theta'' + \sin(\theta) = 0$$

(a) Write the system into the vector form, and find all of the equilibrium points.
(b) Show that $\theta = \pi$ is a unstable equilibrium point. (Hint: Try linearization.)
(c) Show that $\theta = 0$ is a stable equilibrium point. (Hint: construct a Lyapunov function in the neighborhood of the equilibrium point.)
(d) Sketch the phase portrait of the system for $\theta \in (-3\pi, 3\pi)$. Are all the orbitals closed?

4. Consider $\mathbf{x} = (x_1, \cdots, x_N)$ and $\mathbf{v} = (v_1, \cdots, v_N)$ are the solutions of some differential equations (for example, the flocking model). Suppose using the structure of the equations, we find that their Euclidean norms satisfy the following inequalities

$$\frac{d\|\mathbf{x}\|}{dt} \leq \|\mathbf{v}\|, \quad \frac{d\|\mathbf{v}\|}{dt} \leq -\psi(\|\mathbf{x}\|)\|\mathbf{v}\|,$$

where $\psi(x)$ is a smooth positive function. Show that

$$V(\|\mathbf{x}\|, \|\mathbf{v}\|) := \|\mathbf{v}\| + \int_0^{\|\mathbf{x}\|} \psi(s)ds$$

is a Lyapunov function of the system – that is to show $V(\|\mathbf{x}\|, \|\mathbf{v}\|)$ is positive definite, and $\dfrac{d}{dt}V \leq 0$.