HW9 (due Nov. 21th)

Instructions. You *must* declare all resources that you have used on this homework (include but not limited to anyone, any book, and any webpage). Do not skip steps. **For problems 3 and 4, you can directly quote and apply theorems (and no need to re-prove).**

1. [B-N] Page 161 Problem 1. *(Hints: Follow a similar linearization procedure are the autonomous system.)*

2. (a) Prove the following proposition:

   Suppose all eigenvalues of A have negative real parts, $f(y)$ and $\frac{\partial f}{\partial y_j}$ are all continuous for $|y| < k$, where $k > 0$ is some constant. If

   $$\lim_{y \to 0} \frac{|f(y)|}{|y|} = 0,$$

   then the zero solution $y = 0$ of the system

   $$\frac{dy}{dt} = Ay + f(y)$$

   is asymptotically stable. *(The purpose of this exercise is to better understand the proof of the linearization theorem, aka [BN theorem 4.3]. You are expected to* ***provide a full proof*** *here mimicing the proof of that theorem, instead of directly quoting and using it.)*

   (b) Explain how this proposition could be used to study the stability of general nonlinear autonomous system

   $$y' = F(y).$$

3. Determine the stability of the equilibrium solutions of the system

   $$\begin{aligned} \frac{dx}{dt} &= -4x - 2y + 4 \\ \frac{dy}{dt} &= xy. \end{aligned}$$

4. Consider the following differential equations with some constant $a \in \mathbb{R}$

   $$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= a\left(1 - x^2\right) y - x. \end{aligned}$$

   The equilibrium solution may behaves differently for different values of a. Describe the different cases of the stability using linearization method.

5. Consider the 1-D autonomous system

   $$\frac{dy}{dt} = f(y), \ y(t_0) = y_0$$

   where $f(y)$ is a continuous function defined on $\mathbb{R}$. Suppose a solution $y(t)$ satisfies the following condition

   $$|y(t)|^2 \le |y_0|^2 + \frac{1}{4} |y(t)|^4. \tag{1}$$

   Use the bootstrapping argument to show that if $|y_0|$ is sufficiently small, then this solution $y(t)$ exist globally and stays small uniformly in time.

(Hints: 1. For autonomous system, to show global existence, it suffices to show a priori estimate. So the first claim follows directly from the second. 2. There are many ways to paraphrase the statement of "small" here – for example, you can show that if $|y_0| \le \varepsilon$, then $|y(t)| \le 2\varepsilon$ *for all* t*.)*

[Remark: This result is in fact quite interesting. It states that even if we could not show a priori estimate in the sense that $|y(t)| \le K|y_0|$ *(directly bounded by initial data), with estimate like* (1)*, there is still hope! Indeed, this kind of technique is widely applied to the study of the global existence of many nonlinear PDEs, such as the nonlinear wave equation.]*