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Assignment 6

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Problem 1

The definition of "Unstable" is as follows,
$\exists\,\epsilon > 0$ such that $\forall\,\delta > 0$, $\exists\,t_1 \geq t_0$ s.t. even if $|\phi(t_0) - y_0| < \delta$, $|\phi(t_1) - y(t_1)| \geq \epsilon$

Problem 2

c) Write the original equation into the matrix form,

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{1}$$

The eigenvalues of $A(t)$ is given by

$$det|\lambda I - A(t)| = det \begin{vmatrix} \lambda & -1 \\ 4 & \lambda \end{vmatrix} = 0 \tag{2}$$

$\lambda_1 = 2i$, $\lambda_2 = -2i$. Though the real parts are non-negative, the eigenvalues with real part zero are complex numbers. So there's still polynomial growth with respect to t. The solution is unstable.
d) As is proved in class, the stability of the same equation does not depend on initial condition. So the solution is unstable.
e) Write the equation into the matrix form,

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{3}$$

The eigenvalues of $A(t)$ is given by

$$det|\lambda I - A(t)| = det \begin{vmatrix} \lambda & -1 \\ 4 & \lambda + 4 \end{vmatrix} = 0 \tag{4}$$

$$\lambda^2 + 4\lambda + 4 = 0 \tag{5}$$

$\lambda_1 = \lambda_2 = -2$ Both real parts of the eigenvalues are negative, so the solution is asymptotically stable.
d) As is proved in class, the stability of the same equation does not depend on initial condition. So the solution is stable.

Problem 3

Write the equation into the matrix form,

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{6}$$

The eigenvalues of $A(t)$ is given by

$$det|\lambda I - A(t)| = det\begin{vmatrix}\lambda & -1 \\ 6 & \lambda+5\end{vmatrix} = 0 \tag{7}$$

$$\lambda^2 + 5\lambda + 6 = 0 \tag{8}$$

$\lambda_1 = -2$, $\lambda_2 = -3$ Both real parts of the eigenvalues are negative, so the solution is asymptotically stable.

Problem 4

Write the equation into the matrix form,

$$\begin{bmatrix}u_1 \\ u_2 \\ u_3 \\ u_4\end{bmatrix}' = \begin{bmatrix}0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 2 & 0\end{bmatrix}\begin{bmatrix}u_1 \\ u_2 \\ u_3 \\ u_4\end{bmatrix} \tag{9}$$

The eigenvalues of $A(t)$ is given by

$$det\begin{vmatrix}\lambda & -1 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ 1 & 0 & -2 & \lambda\end{vmatrix} = \lambda * det\begin{vmatrix}\lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & -2 & \lambda\end{vmatrix} - det\begin{vmatrix}-1 & 0 & 0 \\ \lambda & -1 & 0 \\ 0 & \lambda & -1\end{vmatrix} = 0 \tag{10}$$

$$\lambda^3 - 2\lambda + 1 = 0 \tag{11}$$

$\because \lambda = 1 > 0$ is an eigenvalue, $\therefore y = 0$ is unstable

Problem 5

The eigenvalues of $A(t)$ is given by

$$det\begin{vmatrix}\lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda+1 & -3 \\ 0 & 0 & 0 & \lambda+1\end{vmatrix} = \lambda^2(\lambda+1)^2 = 0 \tag{12}$$

The eigenvalues are $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \lambda_4 = -1$ All eigenvalues are non-negative and both eigenvalues with real part zero is simple. So the solution is stable (but not asymptotic stable).

Problem 6

Using the variation of constant formula, we have

$$\psi(t) = e^{A(t-t_0)}y_0 + \int_{t_0}^{t} e^{A(t-s)B(s)\psi(s)ds} \tag{13}$$

Take the norm on both side of the equation

$$\begin{aligned} |\psi(t)| &= |e^{A(t-t_0)}||y_0| + \int_{t_0}^{t} |e^{A(t-s)}||B(s)||\psi(s)|ds \\ &\leq Ke^{\rho(t-t_0)}|y_0| + Ke^{\rho t}\int_{t_0}^{t} e^{-\rho s}|B(s)||\psi(s)|ds \\ e^{-\rho t}|\psi(t)| &\leq Ke^{-\rho t_0}|y_0| + K\int_{t_0}^{t} e^{-\rho s}|B(s)||\psi(s)|ds \end{aligned} \tag{14}$$

Use Gronwall inequality,

$$e^{-\rho t}|\psi(t)| \leq Ke^{-\rho t_0}|y_0|e^{K\int_{t_0}^{t}|B(s)|ds} \tag{15}$$

$$|\psi(t)| \leq Ke^{\rho(t-t_0)}|y_0|e^{K\int_{t_0}^{t}|B(s)|ds} \tag{16}$$

Note that $\rho \in (max\{Re\lambda_j\}, 0)$ is strictly negative. $\int_{t_0}^{t}|B(s)|ds < \int_0^\infty |B(s)|ds$ is bounded. The right hand side of the inequality approaches zero when t goes to infinity. Hence the solution is asymptotically stable.

Problem 7

Solve the set of equations,

$$\begin{cases} y_1' = -y_1 + e^{2t}y_2 \\ y_2' = -y_2 \end{cases} \tag{17}$$

Apparently by separation of variables, $y_2 = C_2e^{-t}$

$$y_1' + y_1 = C_2e^t \tag{18}$$

This is a first-order non-homogeneous linear equation. Using integration factor, we have

$$y_1 = e^{-\int 1ds}\left[\int C_2e^te^{\int 1ds}dt + C_1\right] \tag{19}$$

$$y_1 = C_1e^{-t} + C_2e^t \tag{20}$$

Although the eigenvalues, calculated by $|\lambda I - A(t)| = 0$ are $\lambda_1 = \lambda_2 = -1$

$$\lim_{t\to\infty}\left\{C_1e^{-t} + C_2e^t\right\} = \infty \tag{21}$$

Hence the solution is unstable.

Problem 8

a) If the modulus of all the multipliers are smaller than 1, zero solution is asymptotically stable;
b) If the modulus of all the multipliers are less or equal to 1, and all the multipliers with modulus 1 are simple, then zero solution is stable;
c) If there exists at least one multiplier with modulus bigger than 1, then the zero solution is unstable.