Econ C103: Game Theory and Networks

Module I (Game Theory): Lecture 7

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Readings:

1. Osborne (2004) chapter 5
2. Osborne and Rubinstein (1994) chapters 6.1-6.4

Extensive games with perfect information

Nash equilibrium shortcomings (include):

- NE ignore timing / sequential structure in players' decisions.
- NE is a "weak" notion: many games yield multiple MNE.

Adding sequential structure to the game provides additional information which can be used to refine the set of MNE...

Extensive games with perfect information: definition

Definition (Extensive form game)

An **extensive game with perfect information** *is defined as a quadruplet* $\langle N, H, P, \{u_i\}_{i\in N}\rangle$.

- $N = \{1, ..., n\}$: *players,*
- H: *(finite or infinite) set of sequences ("histories"),*
- P: *player function,* $P : H\backslash Z \mapsto N$, *where player* $P(h)$ *takes action after history* $h \in H\backslash Z$ *(terminal histories* $Z \subset H$ *defined below).*
- $u_i(h)$: *i's (vNM) utility from outcome* $h \in Z$ *(defined below).*

Histories H must satisfy:

- $\varnothing \in H$ (i.e. the beginning of the game is the history $\varnothing$),
- For any $(a^k)_{k=1}^K \in H$, each $(a^k)_{k=1}^J \in H$ for $0 < J < K$,
- If $(a^k)_{k=1}^\infty$ satisfies $(a^k)_{k=1}^K \in H$ for all $K \in \mathbb{Z}$, then $(a^k)_{k=1}^\infty \in Z$.

Terminal histories Z (these are "outcomes" of the game) are given by:

$$Z \equiv \{(a^k)_{k=1}^K \in H : (a^k)_{k=1}^L \in H, L \geq K \Rightarrow L = K\}.$$

Extensive games with perfect information: strategies

- Notation: For any $h \in H \backslash Z$ of length K, denote $(h, a) \in H$ a history of length $K + 1$ given by h followed by a.
- Terminology: When Z contains no infinite histories then the game has a "finite horizon".

Strategies and outcomes:

- For each $h \in H \backslash Z$, $i = P(h)$, define $A_i(h) \equiv \{(h, a) \in H\}$.
- For each $i \in N$, define "decision nodes":

$$H_i \equiv \{h' \in H \backslash Z : P(h') = i\}.$$

- A pure strategy $s_i(h) \in A_i(h)$ for each $h \in H_i$ (*one action for each of i's decision nodes*). S_i denotes the set of all strategy functions $\{s_i\}$.
- Strategy profile $\mathbf{s} = (s_i)_{i \in N}$ maps to outcome $O(\mathbf{s}) \in Z$:

$$O(\mathbf{s}) \equiv (s_{P(\emptyset)}(\emptyset), s_{P(h^1)}(h^1), s_{P(h^2)}(h^2), \ldots),$$

where $h^{k+1} = (h^k, s_{P(h^k)}(h^k))$ for each $k \geq 0$, by setting $h^0 \equiv \emptyset$.

Extensive games with perfect information: normal form

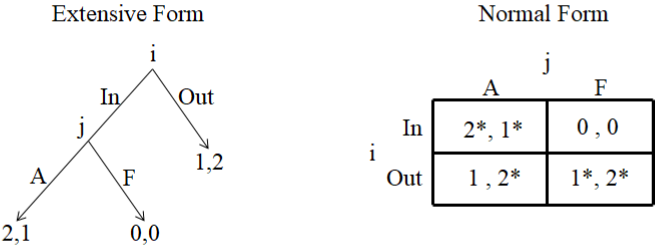
Definition (Normal form of extensive game)

Given extensive form game $\Gamma = \langle N, H, P, \{u_i\}_{i \in N} \rangle$*, the* **normal form** *of* Γ *is given by* $\Gamma' = \langle N', \{A'_i\}_{i \in N}, \{u'_i\}_{i \in N} \rangle$ *satisfying:*

- $N' = N$,
- $A'_i = S_i$ *for each* $i \in N$,
- $u'_i(\mathbf{s}) = u_i(O(s))$ *for each* $\mathbf{s} \in \times_{j \in N} S_i$.

- In words: A strategy in Γ' prescribes an action for i for each of her decision nodes, even for those that are never reached during play.
- Notice: Γ' removes the sequential information in Γ (i.e. Γ' drops the "tree structure"), but retains players' strategy spaces and preferences over outcomes in Γ.

Extensive games with perfect information: examples


Extensive Form

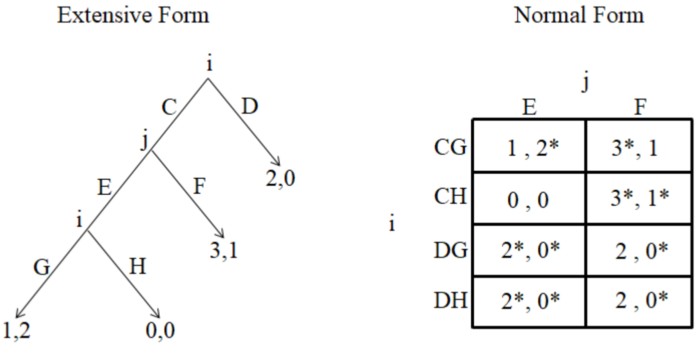
Normal Form

		j A	F
i	In	2*, 1*	0 , 0
	Out	1 , 2*	1*, 2*

Figure: Extensive game I

- $N = \{i, j\}$,
- $S_i = \{\text{In}, \text{Out}\}, \qquad S_j = \{\text{A}, \text{F}\}$,
- $PNE = \{(\text{In,A}), (\text{Out,F})\}$.

Extensive games with perfect information: examples


Extensive Form

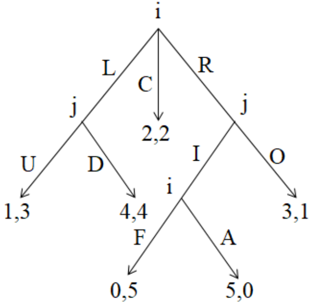
Normal Form

		j	
		E	F
i	CG	1 , 2*	3*, 1
	CH	0 , 0	3*, 1*
	DG	2*, 0*	2 , 0*
	DH	2*, 0*	2 , 0*

Figure: Extensive game II

- $N = \{i, j\}$,
- $S_i = \{\text{CG}, \text{CH}, \text{DG}, \text{DH}\}$, $\qquad S_j = \{\text{E}, \text{F}\}$
- $PNE = \{(\text{CH,F}), (\text{DG,E}), (\text{DH,E})\}$.

Extensive games with perfect information: examples

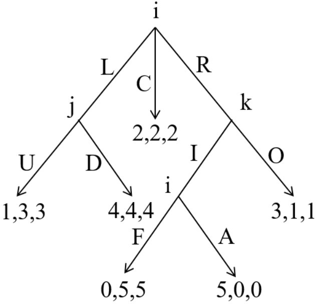


		j			
		UI	UO	DI	DO
i	LF	1 , 3	1 , 3	4 , 4*	4*, 4*
	LA	1 , 3	1 , 3	4 , 4*	4*, 4*
	CF	2 , 2*	2 , 2*	2 , 2*	2 , 2*
	CA	2 , 2*	2 , 2*	2 , 2*	2 , 2*
	RF	0 , 5*	3*, 1	0 , 5*	3 , 1
	RA	5*, 0	3*, 1*	5*, 0	3 , 1*

Figure: Extensive game III

- $N = \{i, j\}$,
- $S_i = \{\text{LF, LA, CF, CA, RF, RA}\}$, $S_j = \{\text{UI, UO, DI, DO}\}$,
- $PNE = \{(\text{LF,DO}), (\text{LA,DO}), (\text{RA,UO})\}$.

Extensive games with perfect information: examples



k

j

i	U	D
LF	1, 3, 3*	4, 4*, 4*
LA	1, 3, 3*	4, 4*, 4*
CF	2, 2*, 2*	2, 2*, 2*
CA	2, 2*, 2*	2, 2*, 2*
RF	0, 5*, 5*	0, 5*, 5*
RA	5*, 0*, 0	5*, 0*, 0

I

j

	U	D
LF	1, 3, 3*	4*, 4*, 4*
LA	1, 3, 3*	4*, 4*, 4*
CF	2, 2*, 2*	2, 2*, 2*
CA	2, 2*, 2*	2, 2*, 2*
RF	3*, 1*, 1	3, 1*, 1
RA	3*, 1*, 1*	3, 1*, 1*

O

Figure: Extensive game IV

- $N = \{i, j, k\}$,
- $S_i = \{\text{LF}, \text{LA}, \text{CF}, \text{CA}, \text{RF}, \text{RA}\}$, $S_j = \{\text{U}, \text{D}\}$, $S_k = \{\text{I}, \text{O}\}$,
- $PNE = \{(LF, D, O), (LA, D, O), (RA, U, O)\}$.

Extensive games with perfect info.: reduced normal form

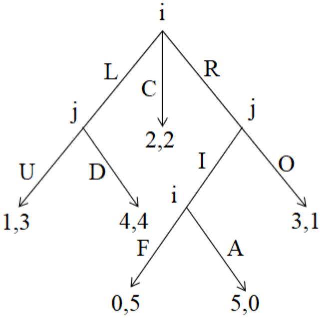
Definition (Reduced normal form of extensive game)

Given extensive form game $\Gamma = \langle N, H, P, \{u_i\}_{i \in N} \rangle$ *and corresponding normal form game* $\Gamma' = \langle N', \{A'_i\}_{i \in N}, \{u'_i\}_{i \in N} \rangle$*, the* **reduced normal form game** *is given by* $\Gamma'' = \langle N'', \{A''_i\}_{i \in N}, \{u''_i\}_{i \in N} \rangle$ *satisfying:*

- $N'' = N$,
- $A''_i \subseteq S_i$ *for each* $i \in N$ *and satisfies:*
  - (a) *for each* $s_i \in S_i \backslash A''_i$ *there is some* $s''_i \in A''_i$ *such that* $u'_j(s''_i, \mathbf{s}_{-i}) = u'_j(s_i, s_{-i})$ *for each* $\mathbf{s}_{-i} \in A'_{-i}$ *and all* $j \in N$,
  - (b) *for all* $s''_i, \tilde{s}''_i \in A''_i$, $s''_i \neq \tilde{s}''_i$, $u'_j(s''_i, \mathbf{s}_{-i}) \neq u'_j(\tilde{s}''_i, \mathbf{s}_{-i})$ *for some* $s_{-i} \in A'_{-i}$ *and some* $j \in N$.

- In Words: Γ'' combines payoff equivalent strategies into one strategy.
- Condition (a) requires that all players are indifferent between $s'' = (s''_i, \mathbf{s}_{-i})$ and $s = (s_i, \mathbf{s}_{-i})$ for s_i removed from i's strategy set.
- Condition (b) requires that $\{A''_i\}_{i \in N}$ give the most reduced action sets such that (a) is satisfied (*combine as much as possible*).

Reduced normal form: example

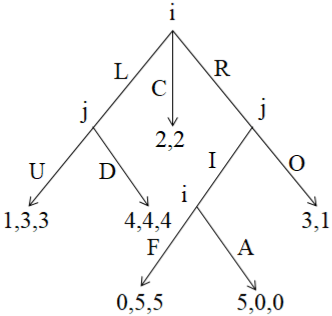


		j			
		UI	UO	DI	DO
i	LF	1 , 3	1 , 3	4 , 4	4 , 4
	LA	1 , 3	1 , 3	4 , 4	4 , 4
	CF	2 , 2	2 , 2	2 , 2	2 , 2
	CA	2 , 2	2 , 2	2 , 2	2 , 2
	RF	0 , 5	3 , 1	0 , 5	3 , 1
	RA	5 , 0	3 , 1	5 , 0	3 , 1

Figure: Extensive game III

- $N = \{i, j\}$,
- $S_i = \{\text{LF}, \text{LA}, \text{CF}, \text{CA}, \text{RF}, \text{RA}\}$, $S_j = \{\text{UI}, \text{UO}, \text{DI}, \text{DO}\}$,
- $PNE = \{(\text{LF,DO}), (\text{LA,DO})\}$.

Reduced normal form: example



		j			
		UI	UO	DI	DO
i	L	1 , 3	1 , 3	4 , 4	4 , 4
	C	2 , 2	2 , 2	2 , 2	2 , 2
	RF	0 , 5	3 , 1	0 , 5	3 , 1
	RA	5 , 0	3 , 1	5 , 0	3 , 1

Figure: Extensive game III: reduced normal form

- $N = \{i, j\}$,
- $S_i = \{\mathrm{L}, \mathrm{C}, \mathrm{RF}, \mathrm{RA}\}$, $\qquad S_j = \{\mathrm{UI}, \mathrm{UO}, \mathrm{DI}, \mathrm{DO}\}$,
- $PNE = \{(\mathrm{L{,}DO})\}$.

Behavioral strategies

Given any extensive form game Γ with strategy space S_i for i, for $s_i \in S_i$ we define i's behavioral strategy at history $h \in H \backslash Z$, such that $P(h) = i$, as the element $s_i(h) \in \{(h, a) \in H\}$.

- The set of behavioral strategies $\{s_i(h) : h \in H \backslash Z, P(h) = i\}$ gives the same information regarding i's play as does s_i.
- With behavioral strategies, we can discuss a change to i's strategy at a particular decision node of the game holding fixed her behavioral strategies at other decision nodes (we call this a "single deviation").

Subgames

Selten (1965, 1975) and Kreps and Wilson (1982): use the sequential structure to impose "credibility" in off-equilibrium behavioral strategies...

Definition (Subgame)

Given extensive form game $\Gamma = \langle N, H, P, \{u_i\}_{i \in N} \rangle$, *the* **subgame** *of* Γ *following any history* $h \in H$ *is the extensive form game* $\Gamma|_h = \langle N, H|_h, P|_h, \{u_i|_h\}_{i \in N} \rangle$ *satisfying:*

- $N|_h = N$,
- $H|_h = \{h' \in H : h' = (h, \sigma)$ *for some sequence* $\sigma\}$,
- $P|_h(h') = P(h')$,
- $u_i|_h(O|_h(\mathbf{s})) = u_i(O(\mathbf{s}))$ *(*$O|_h$ *defined below).*

- Strategy profile $\mathbf{s} = (s_i)_{i \in N}$ in $\Gamma(h)$ maps to outcome $O|_h(\mathbf{s}) \in Z$:

$$O|_h(\mathbf{s}) \equiv (h, s_{P(h)}(h), s_{P(h^1)}(h^1), s_{P(h^2)}(h^2), \ldots),$$

where $h^{k+1} = (h^k, s_{P(h^k)}(h^k))$ for each $k \geq 0$ and $h^0 \equiv h$.

Subgames

Selten (1965, 1975) and Kreps and Wilson (1982): use the sequential structure to impose "credibility" in off-equilibrium behavioral strategies.

Definition (Subgame)

Given extensive form game $\Gamma = \langle N, H, P, \{u_i\}_{i \in N} \rangle$*, the* **subgame** *of* Γ *following any history* $h \in H$ *is the extensive form game* $\Gamma|_h = \langle N, H|_h, P|_h, \{u_i|_h\}_{i \in N} \rangle$ *satisfying:*

- $N|_h = N$,
- $H|_h = \{h' \in H : h' = (h, \sigma)$ *for some sequence* $\sigma\}$,
- $P|_h(h') = P(h')$ *for each* $h' \in H|_h$,
- $u_i|_h(O|_h(\mathbf{s})) = u_i(O(\mathbf{s}))$.

- Can construct strategy space $S_i|_h$ from $H|_h$ and $P|_h$ for each $i \in N$.
- In words: $\Gamma|_h$ gives the continuation (i.e. remaining) game within the game Γ upon history $h \in H$ having been played already.

Subgames: example

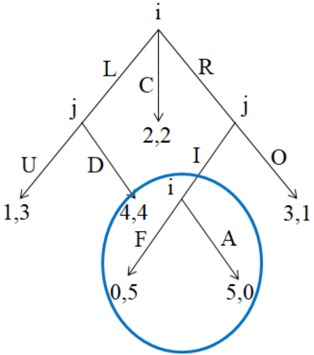


Figure: Extensive game III: 4 subgames, subgame to $h = (R, I)$ circled

- $N = \{i, j\}$,
- $S_i|_h = \{\mathsf{LF}, \mathsf{LA}\}$, $\qquad S_j|_h = \emptyset$,
- $PNE|_h = \{A\}$.

Subgames: example

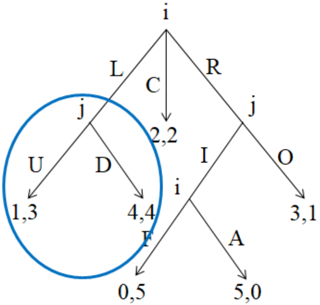


Figure: Extensive game III: 4 subgames, subgame to $h = (L)$ circled

- $N = \{i, j\}$,
- $S_i|_h = \emptyset$, $S_j|_h = \{U, D\}$,
- $PNE|_h = \{D\}$.

Subgames: example

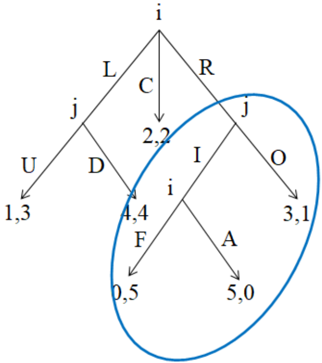


Figure: Extensive game III: 4 subgames, subgame to $h = (R)$ circled

- $N = \{i, j\}$,
- $S_i|_h = \{\mathsf{F}, \mathsf{A}\}$, $S_j|_h = \{\mathsf{I}, \mathsf{O}\}$,
- $PNE|_h = \{(\mathsf{A},\mathsf{O})\}$.

Subgames: example

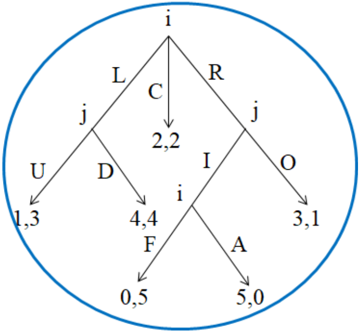


Figure: Extensive game III: 4 subgames, subgame to $h = \emptyset$ circled

- $N = \{i, j\}$,
- $S_i = \{\text{LF, LA, CF, CA, RF, RA}\}$, $S_j = \{\text{UI, UO, DI, DO}\}$,
- $PNE = \{(\text{LF,DO}), (\text{LA,DO}), (\text{RA,UO})\}$ (recall slide 8).

Subgame Perfect Nash equilibrium

Definition (Subgame Perfect Nash equilibrium)

Given extensive game $\Gamma = \langle N, H, P, \{u_i\}_{i \in N} \rangle$*, a strategy profile* $\mathbf{s}^*$ *is a* **subgame Perfect Nash equilibrium (SPNE)** *iff for each* $i \in N$ *and each subgame* $\Gamma|_h$ *for* $h \in H \backslash Z$*:*

$$u_i|_h(s_i, \mathbf{s}_{-i}) \geq u_i|_h(s_i', \mathbf{s}_{-i}), \ \forall s_i' \in S_i|_h.$$

Proposition (Kuhn)

Any extensive game with perfect information Γ *has a SPNE.*

- $\mathbf{s}^*$ a SPNE $\Rightarrow$ $\mathbf{s}^*$ is a PNE: SPNE "refines" PNE.
- Kuhn's Theorem $\Rightarrow$ there exists a PNE in any extensive game Γ: never a "Matching Pennies" scenario!
- Uniqueness is not guaranteed: can have mutiple SPNE. But, Fact: A unique SPNE obtains if for all $z, z' \in Z$ and $i \in N$, $u_i(z) \neq u_i(z')$.
- Fact: IESWS of normal form Γ' of Γ may eliminate some SPNE.

Finding SPNE with Backward Induction

Algorythm (Backward Induction)

- *Step* 1*: In all final subgames (i.e. subgames with no proper subgames), solve the subgame by setting payoff-maximizing behavioral strategy for unique player with non-empty strategy set.*
- *Step* $k > 1$*: In subgames including only subgames solved in Steps* 1 *to* $k-1$*, solve the subgame by setting payoff-maximizing behavioral strategy for player moving first, given solution in Step* $k-1$*.*
- *Continue until the subgame to empty history is solved.*

Backward Induction: example (Step 1)

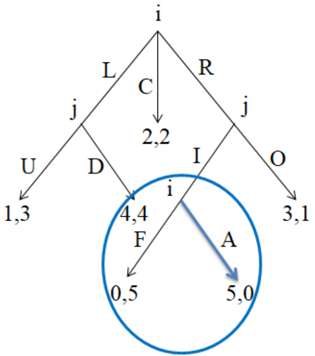


Figure: Extensive game III: 4 subgames, subgame to $h = (R, I)$ circled

- $N = \{i, j\}$,
- $S_i|_h = \{\text{LF}, \text{LA}\}$, $\quad S_j|_h = \emptyset$,
- $SPNE|_h = \{A\}$.

Backward Induction: example (Step 1)

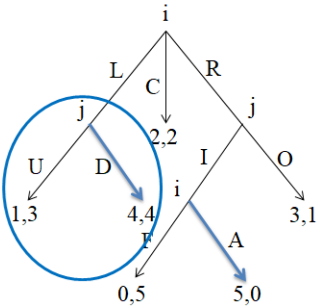


Figure: Extensive game III: 4 subgames, subgame to $h = (L)$ circled

- $N = \{i, j\}$,
- $S_i|_h = \emptyset, \qquad S_j|_h = \{U, D\}$,
- $SPNE|_h = \{D\}$.

Backward Induction: example (Step 2)

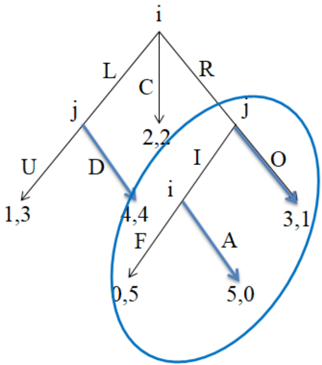


Figure: Extensive game III: 4 subgames, subgame to $h = (R)$ circled

- $N = \{i, j\}$,
- $S_i|_h = \{\mathrm{F}, \mathrm{A}\}$, $S_j|_h = \{\mathrm{I}, \mathrm{O}\}$,
- $SPNE|_h = \{(\mathrm{A,O})\}$.

Backward Induction: example (Step 3)

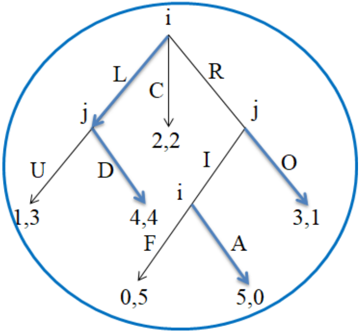
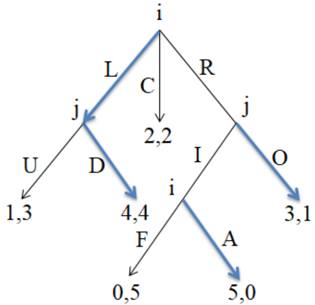


Figure: Extensive game III: 4 subgames, subgame to $h = \emptyset$ circled

- $N = \{i, j\}$,
- $S_i = \{\text{LF, LA, CF, CA, RF, RA}\}$, $S_j = \{\text{UI, UO, DI, DO}\}$,
- $SPNE = \{(\text{LA,DO})\}$ (i.e. PNE (LF,DO) and (RA,UO) excluded).

Subgame Perfect Nash equilibrium: examples

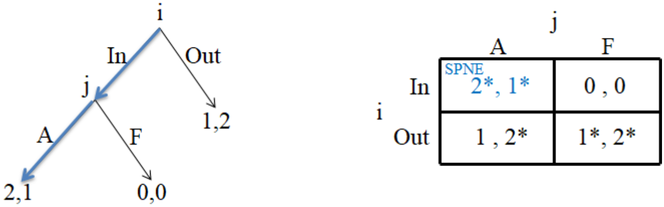


				j	
		UI	UO	DI	DO
	LF	1 , 3	1 , 3	4 , 4*	4*, 4*
	LA	1 , 3	1 , 3	4 , 4*	SPNE 4*, 4*
i	CF	2 , 2*	2 , 2*	2 , 2*	2 , 2*
	CA	2 , 2*	2 , 2*	2 , 2*	2 , 2*
	RF	0 , 5*	3*, 1	0 , 5*	3 , 1
	RA	5*, 0	3*, 1*	5*, 0	3 , 1*

Figure: Extensive game III: SPNE

- $SPNE = \{(\text{LA,DO})\} \subset PNE = \{(\text{LF,DO}), (\text{LA,DO}), (\text{RA,UO})\}$.

Subgame Perfect Nash equilibrium: examples



		j A	F
i	In	SPNE 2*, 1*	0 , 0
	Out	1 , 2*	1*, 2*

Figure: Extensive game I: SPNE

- "F" is an "incredible threat" of player j (to player i).
- $SPNE = \{(\text{In},\text{A})\} \subset PNE = \{(\text{In},\text{A}), (\text{Out},\text{F})\}$.