Week 4 HW (due Oct. 3rd)

Instructions. You *must* declare all resources that you have used on this homework (include but not limited to anyone, any book, and any webpage). Do not skip steps.

1. a) Prove the equivalence of the 2-norm and 1-norm of vectors of $\mathbb{R}^n$. (Hint: Use Cauchy-Swarts Inequality

$$\left|\sum_{i=1}^{n} u_i v_i\right|^2 \leq \sum_{i=1}^{n} |u_i|^2 \sum_{i=1}^{n} |v_i|^2)$$

   b) Consider the matrix norm on $M_n(\mathbb{R})$ as follows

$$|A| = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|,$$

   show that

$$|AB| \leq |A|\,|B|,$$

   where A and B are $n \times n$ matrices.

2. ([B-N] Page 39 Problem 3)
   Suppose $A(t)$ and $g(t)$ are continuous for $t \in (-\infty, +\infty)$ and that

$$\int_{-\infty}^{\infty} |A(t)|\,dt < \infty, \quad \int_{-\infty}^{\infty} |g(t)|\,dt < \infty.$$

   Get a priori estimate of the solution $\phi(t)$ of

$$y' = A(t)\,y + g(t)$$

   for $t \in (-\infty, \infty)$.

3. (In the proof of Abel's formula [BN] Page 46 Theorem 2.3)
   Consider

$$B_k = \begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \vdots & \vdots & & \vdots \\ \phi'_{k1} & \phi'_{k2} & \cdots & \phi'_{kn} \\ \vdots & \vdots & & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix}$$

   (defined as B_k in class, where the k-th row is differentiated), show that

$$B_k = a_{kk} \det \Phi,$$

   where Φ is the fundamental solution of the linear system.

4. ([BN] Page 49 Problem 24)
   Show that

$$\begin{bmatrix} \exp(r_1 t) & \exp(r_2 t) \\ r_1 \exp(r_1 t) & r_2 \exp(r_2 t) \end{bmatrix}$$

   is a fundamental matrix for the system $y' = Ay$, where

$$A = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix}$$

   and r_1, r_2 are the distinct roots of the quadratic equation $z^2 + a_1 z + a_2 = 0$.

5. ([B-N] Page 49 Problem 25)

   Show that $C\Phi$, where C is a constant matrix and Φ is a fundamental matrix, need not be a solution matrix of $y' = Ay$. (Try to construct a counter example of $y' = Ay$ and $C\Phi$ that does not satisfy the equation.)