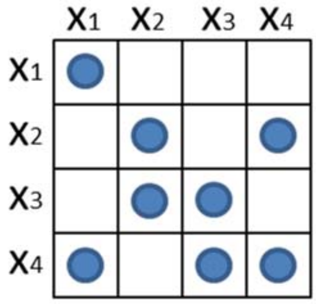
# Econ C103: Introduction to Mathematical Economics
**UC Berkeley, Fall 2019**

## Assignment 1 Solutions

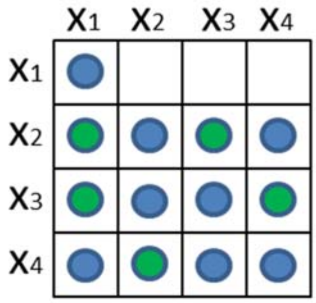
*Problem 1*: Consider the preference relationship (depicted below) over four choices $\{x_1, x_2, x_3, x_4\}$:

$$x_i \succcurlyeq x_i, \forall i = 1, \ldots, 4, x_2 \succcurlyeq x_4, x_3 \succcurlyeq x_2, x_4 \succcurlyeq x_1, x_4 \succcurlyeq x_3$$




Is the preference relationship complete? Is it transitive? Find the smallest set of additional elements that make the preference relationship rational.

*Solution*: It is neither complete nor transitive. The preference relationship is rational upon inclusion of the green elements:




Removing any element implies the preference relationship is not rational.

*Problem 2*: Formulate the following game (i.e. define the game's elements). Do so first as a static game, then as its mixed extension. Find the set of all PNE and MNE of the game. For this, graph together the best response correspondences of each player, and find their intersection. What game discussed in lecture is this game most similar to?

| | | 2: $L$ | 2: $R$ |
|---|---|---|---|
| 1 | $T$ | $2,1$ | $0,2$ |
| | $B$ | $1,2$ | $3,0$ |

Solution: To formulate this game as a strategic game in pure strategies $\Gamma = \langle N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N} \rangle$:

$$N = \{1,2\}$$

$$A_1 = \{T, B\},\ A_2 = \{L, R\}$$

$$u_1(T, L) = 2, u_1(T, R) = 0, u_1(B, L) = 1, u_1(B, R) = 3$$

$$u_2(T,L) = 1, u_2(T,R) = 2, u_2(B,L) = 2, u_2(B,R) = 0$$

To formulate this game as a strategic game in mixed strategies $\Delta\Gamma = \langle N, \{\Delta(A_i)\}_{i\in N}, \{U_i\}_{i\in N}\rangle$:

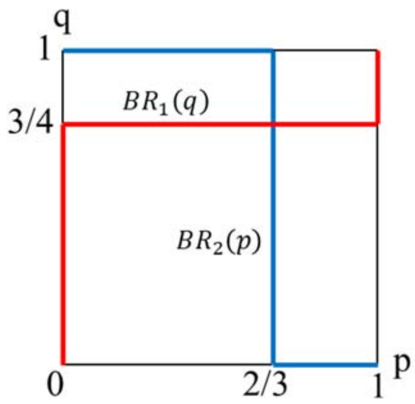
$$N = \{1,2\}$$

$$\Delta(A_1) = \{p \in [0,1]\}\; where\; p \equiv \alpha_1(T), \Delta(A_2) = \{q \in [0,1]\}\; where\; q \equiv \alpha_2(L)$$

$$U_1(p,q) = 2pq + 1(1-p)q + 3(1-p)(1-q)$$

$$U_2(p,q) = 1pq + 2p(1-q) + 2(1-p)q$$

As a static game, the game $\Gamma$ has no PNE. To find the MNE, we need to study $\Delta\Gamma$. Graphing the best responses:




Clearly, $MNE = \left\{\left(\frac{2}{3}, \frac{3}{4}\right)\right\}$. The game is most similar to matching pennies.

*Problem 3*: In the following game, what strategies survive iterated elimination of strictly dominated strategies? Find the set of MNE.

| | $L$ | $C$ | $R$ |
|---|---|---|---|
| $T$ | 2, 0 | 1, 1 | 4, 2 |
| $M$ | 3, 4 | 1, 2 | 2, 3 |
| $B$ | 1, 3 | 0, 2 | 3, 0 |

*Solution*: We begin by eliminating $B$ because it is strictly dominated by $T$, then we eliminate $C$ because it is strictly dominated by $R$ (we can only do this after eliminating $B$), leaving the 2x2 game in which the row player's action set is $\{T, M\}$ and the column player's action set is {L, R}. The remaining game is variation of "Battle of the Sexes" with the set of PNE is $\{(M,L),(T,R)\}$. For the mixed Nash, we can construct the best response correspondences. Let $p$ gives the probability the row player ($r$) plays $T$, and $q$ the probability the column player ($c$) plays $L$:

$$BR_r(q) = \begin{cases} 1 & if\; q < 2/3 \\ [0,1] & if\; q = 2/3 \\ 0 & otherwise \end{cases}$$

$$BR_r(p) = \begin{cases} 1 & if\; p < 1/3 \\ [0,1] & if\; p = 1/3 \\ 0 & otherwise \end{cases}$$

This implies that the set of MNE is $\left\{(1,0), \left(\frac{1}{3}, \frac{2}{3}\right), (0,1)\right\}$.

*Problem 4*: With the following game, verify that the order in which you eliminate strictly dominated strategies in IESDS does not carry any implication for the set of actions that survive. Note that this independence of IESDS can be proven for general finite static games.

| | L | C | R |
|---|---|---|---|
| T | 4 , 4 | 2 , 3 | 1 , 2 |
| M | 2 , 4 | 4 , 3 | 3 , 3 |
| B | 3 , 2 | 1 , 3 | 0 , 1 |

*Solution*: There are 3 sequences for IESDS:

Elimination sequence 1: R is eliminated by L, B is eliminated by T, C is eliminated by L, M is eliminated by T.

Elimination sequence 2: B is eliminated by T, R is eliminated by L, C is eliminated by L, M is eliminated by T.

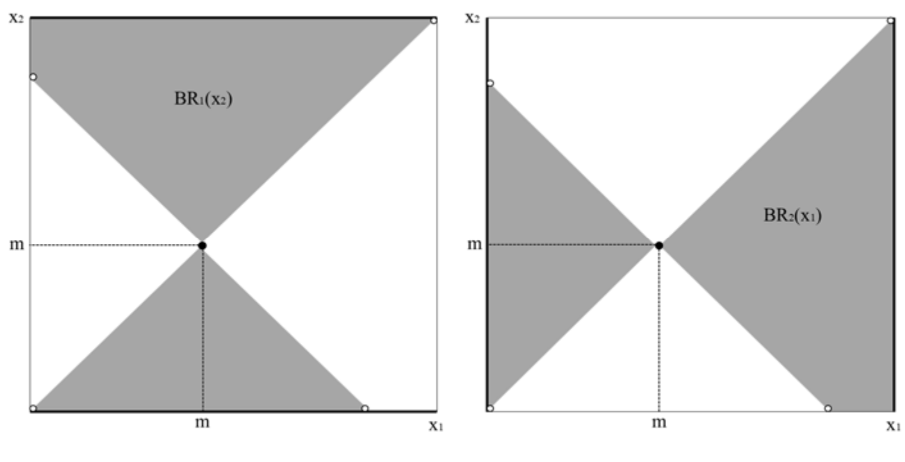
Elimination sequence 3: B is eliminated by T, C is eliminated by L, R is eliminated by L, M is eliminated by T.

All sequence result in only T and L surviving IESDS.

*Problem 5*: Consider the following “Hotelling Model” game: There is a continuum of political platforms that candidates can choose. Candidates gather the density of votes for voters that are closest to that candidate along the continuum (than any other candidate). If any number of candidates chose the same platform, these candidates split the voters that are closest to their location along the continuum. Candidates strictly prefer winning, to a tie (for 1st place) among $n$ candidates ($n > 1$), to a tie among $n + 1$ candidates, to losing. The voters are distributed via cumulative distribution function $F$ along the continuum. Let $m$ be the median of $F$. For simplicity, we can restrict platforms to the continuum $[0,1]$.
Formulate this game (i.e. define the game’s elements). Find the set of PNE when there are two candidates.

Now assume that candidates also have the option to exit the race, which is preferred to running and loosing, but tying for 1st is strictly preferred to exiting. How does the formulation of the game change? Will the set of Nash change for the two-candidate case? Find the set of PNE when there are three candidates.

*Solution*:
Two candidates case: The best response correspondences (for the case of $m < 1/2$) are graphed as follows.

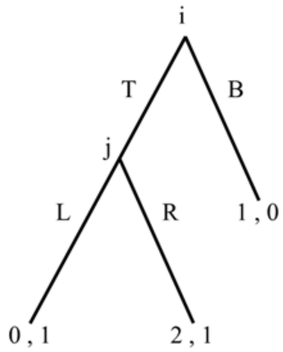



If the other player does not position herself at the median, then it is optimal to position yourself closer to the median, capturing at least half of the votes. The intersection of the best response correspondences are exactly the median, and thus the only pure Nash is (m,m).

The set of Nash does not change if the two candidates have the option to exit.

Three candidates case (with exit): The best response functions are functions of the continuum $[0,1]$union that action of exiting, crossed with itself: $([0,1] \cup \{exit\})^2$. This is hard to graph, so we'll proceed by argument. It is not Nash for no candidates to enter, because one can enter and win giving a profitable deviation. If a single candidate enters the race, it is optimal for one of the others to enter strictly closer to the median in order to win, or at the median if the first candidate positions herself at the median in order to tie. If two candidates enter, it is optimal for each to move closer to the median than the other, unless the other is at the median, in which it is optimal to position at the median (and tie). So, if two candidates enter the race, both are best responding to the other players only if they are at the median. If they both are at the median, then the third has a profitable deviation of entering and moving slightly to one side of the median, getting nearly half the votes and winning (the others will get slightly more than 1/4 of the votes). If all three enter, then there is no Nash in which all three choose the same position because if they are not at the median, each strictly prefers to move slightly closer to m in order to win, and if they are at m, each strictly prefers to move slightly off m in order to win. If all enter and do not choose the same position, then we have two cases: two are at the same position and all are at different positions. In the first case, the lone candidate either loses and prefers to exit, wins and the two losers prefer to exit, or all three tie in which case the lone candidate can move closer to m and win. In the second case, we have three sub cases: one winner, two tie, and all three tie. In the first two, the loser(s) prefer to exit. In the third, the two outside candidates could move closer to the inner candidate and win. Thus, there is always a profitable deviation, in any case. We've proven there is no Nash.

*Problem 6*: Consider the following game tree, representing an extensive form game.




Find the set of PNE and MNE of the game. Which of these equilibria are SPNE?

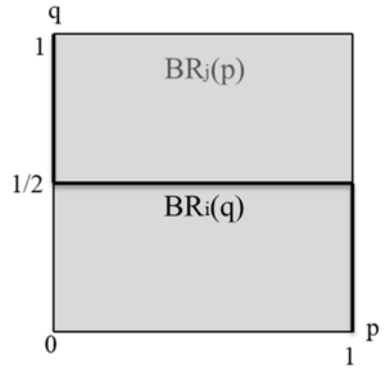
*Solution*: We can find all Nash simply by looking at the game. Player j is indifferent between L and R independent of what player j does. Thus, we can find i's best response as a function of what j does. If j puts more than probability 1/2 on R, i strictly prefers T, and if j puts less than probability 1/2 on R, i strictly prefers B. When j mixes uniformly, i is indifferent between T and B. We have described i's best response correspondence, and in doing so, found the set of Nash; let p give the probability I places on T and q the probability j places on L:

$$MNE(\Gamma) = \left\{(1,q): q \in \left[0, \frac{1}{2}\right)\right\} \cup \left\{(0,q): q \in \left(\frac{1}{2}, 1\right]\right\} \cup \left\{\left(p, \frac{1}{2}\right): p \in [0,1]\right\}$$

As seen in class, any sequential game has a strategic form. The strategic form of the above game is:

| | i \ j | q L | 1-q R |
|---|---|---|---|
| p | T | 0,1 | 2,1 |
| 1-p | B | 1,0 | 1,0 |

We can graph the best response correspondences to find all Nash (giving a second method):




Player j's best response correspondence is always her entire strategy set, so all of $A \equiv \times_{i\in N} \Delta A_i$ is gray to represent this. Player i's best response correspondence intersected with j's (which is the entire set) will thus determine the set of Nash, which we see to be $MNE(\Gamma)$ above.
Clearly, all MNE are SPNE, given player j's indifference.

*Problem 7*: Consider the following game. Two people, $i \in \{1,2\}$, work on a joint project by choosing an action $x_i \in [0,1]$ (each). The outcome of the project is worth $V(x_1, x_2)$ and each faces a cost $c(x_i)$. The value of the project is split evenly between the two, regardless of their relative effort levels. Find the PNE of the game when:

a) $V(x_1, x_2) = 3x_1x_2$ and $c(x_i) = x_i^2$, and
b) $V(x_1, x_2) = 4x_1x_2$ and $c(x_i) = x_i$.

For both cases, find a Pareto efficient outcome in which both players exert the same effort. Compare this value to each PNE value(s).

*Solution*: For a), to find the Nash equilibria of the game, we first construct the players' best response correspondences. Player 1's best response to $x_2$ is:

$$BR_1(x_2) = argmax_{y\in[0,1]} \frac{1}{2} 3yx_2 - y^2 = \frac{3}{4}x_2$$

Similarly (by the symmetry of the game), $BR_2(x_1) = \frac{3}{4}x_1$. These function have a unique intersection at $(x_1^*, x_2^*) = (0,0)$, which gives the unique PNE. Equilibrium values are thus zero. The efficient common level of effort $x^{ef}$ maximizes total utility:

$$x^{ef} = argmax_{y\in[0,1]} 3y^2 - 2y^2 = 1$$

giving a value of 1/2 to each. Thus, the equilibrium inefficiency is extreme.

For b), we again construct the players' best response correspondences. Player 1's best response to $x_2$ is:

$$BR_1(x_2) = argmax_{y\in[0,1]} 2x_1x_2 - x_1 = \begin{cases} 0 & if\ x_2 < 1/2 \\ [0,1] & if\ x_2 = 1/2 \\ 1 & if\ x_2 > 1/2 \end{cases}$$

Again, by symmetry $BR_2(x_1)$ is the same. Thus the game has three PNE: $\left\{(0,0), \left(\frac{1}{2}, \frac{1}{2}\right), (1,1)\right\}$. Respective equilibrium payoffs are 0, 0, and 1 (to each player). The efficient common level of effort $x^{ef}$ maximizes total utility:

$$x^{ef} = argmax_{y\in[0,1]} 4x_1x_2 - 2x_1 = 1$$

giving a value of 1 to each. Thus, only the equilibrium $(1,1)$ is efficient.

*Problem 8*: Consider the follows *third-price sealed-bid auction*. Bidder $N = \{1, \dots, n\}$, $n > 2$, simultaneously place non-negative bids for an indivisible good. Each player $i \in N$ has value $v_i \geq 0$ for the good; assume $v_1 > v_2 \cdots > v_n$. The winner is the bidder who submits the highest bid, and ties are broken at random. The winner must pay the third highest bid to the auctioneer.

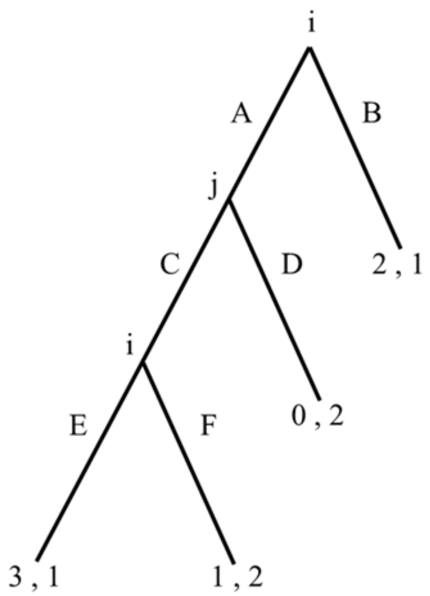
a) Show that for each $i \in N$, the bid $b_i = v_i$ weakly dominates low bids, but does not weakly dominate any higher bid.
b) Show that the action profile in which each bidder bids their value is not a PNE.
c) Find a symmetric PNE.

*Solution*: For a), the argument that a bid of $v_i$ weakly dominates any lower bid is the same as for a second-price auction: e.g., for each positive bid $b < v_i$, if all others bid between $b$ and $v_i$ then bidding $b$ yields $i$ value 0 while bidding $v_i$ gives her positive value. Now compare bids of $b_i > v_i$ and $v_i$. Suppose that one of the other players' bids is between $v_i$ and $b_i$ and all the remaining bids are less than $v_i$. If player $i$ bids $v_i$ she loses, and obtains the payoff of 0. If she bids greater than $b_i$ she wins, and pays the third highest bid, which is less than $v_i$. Thus she is better off bidding greater than $b_i$ than she is bidding $v_i$.

For b), each player's bidding her valuation is not a Nash equilibrium because player 2 can deviate and bid more than $v_1$ and obtain the object at the price $v_3$ instead of not obtaining the object.

For c), any action profile in which every player bids $b$, where $v_2 \leq b \leq v_1$ is a Nash equilibrium. (Player 1's changing her bid has no effect on her payoff. If any other player raises her bid then she wins and pays $b$, obtaining a non-positive payoff; if any other player lowers her bid the outcome does not change.)

*Problem 9*: Consider the two-player extensive game in the figure below. Formulate the following game (i.e. define the game's elements). Write the game in its strategic form and find the set of all PNE. Find the set of all SPNE of the game. Is there a unique SPNE?



*Solution*:

Players: $N = \{1,2\}$

Action sets: $\mathrm{A_i} = \{\mathrm{AE}, \mathrm{AF}, \mathrm{BE}. \mathrm{BF}\};\ \mathrm{A_j} = \{\mathrm{C}, \mathrm{D}\}$

Histories: $H = \{\emptyset, (A), (B), (A, C), (A, D), (A, C, E), (A, C, F)\}$

Terminal histories: $Z = \{(B), (A, D), (A, C, E), (A, C, F)\}$

Player function: $P(\emptyset) = P\big((A, C)\big) = i,\ P((A)) = j$

The strategic form is as follows (* gives best responses):

| i \ j | C | D |
|---|---|---|
| AE | 3* , 1 | 0 , 2* |
| AF | 1 , 2* | 0 , 2* |
| BE | 2 , 1* | 2* , 1* |
| BF | 2 , 1* | 2* , 1* |

The set of PNE is $\{(BE, D), (BF, D)\}$. The unique SPNE is determined via backwards induction, and is $(BE, D)$.

*Problem 10*: Consider the following variation of the two-player infinite horizon bargaining game studied in class. At any stage $t$, after player $i$ makes an offer $(x_i, x_j)$, if player $j$ accepts, then with probability $\lambda \in (0,1]$ they reach an agreement and receive payoffs $(\delta^{t-1}x_i, \delta^{t-1}x_j)$. However, with probability $1 - \lambda$ , player $i$ misunderstands $j$'s "accept" response, and they proceed to the next period as if the offer was rejected. A "reject" response is never misunderstood. The rest of the game remains the same. (That is, Player $i$ starts making an offer in period 1; in subsequent periods players alternate in making offers; the payoffs, the timeline, and the disagreement payoff of (0,0) remain the same; and the players have the common discount factor $\delta \in (0,1)$.)

Conjecture an SPNE. Formally, write down the strategy profile and verify that it is indeed an SPNE by using the single deviation property.

*Solution*: To conjecture equilibrium strategies, suppose there is a unique SPNE with immediate agreement whenever there is no misunderstanding. Let $V$ denote the SPNE expected utility of the player who makes the offer in period $t$. Note that because the players face the same discount factor, they each will receive expected value $V$ when offering. In the context of $x^*$ and $y^*$ of Lecture 10, which denote player $i$'s share when offers are made by $i$ and $j$, respectively, we have that $x^* = 1 - \delta V$ and $y^* = \delta V$. To see this, any responding player must be given share $\delta V$ is order for her to be indifferent between accepting and rejecting the offer. This acceptance/rejection indifference must continue to hold when $\lambda < 1$.

Assume without loss that player $i$ makes an offer in period $t$. With probability $\lambda$ player $i$'s offer is accepted, which must leave $j \neq i$ with value $\delta V$ in order for $i$'s offer to be optimal, because $j$ must be indifferent between accepting and rejecting $i$'s offer (the latter earning her expected utility $V$ in the next period $t+1$). With probability $1-\lambda$ player $i$'s offer is rejected due to misunderstanding, in which case $i$ receives continuation value $\delta^2 V$ (as above, $i$ receives value $\delta V$ in order for $j$'s offer to be optimal and for $i$ to be indifferent between accepting and rejecting $j$'s offer in period $t+1$). So, the expected utility $V$ must satisfy:

$$V = \lambda(1-\delta V) + (1-\lambda)\delta^2 V$$

Solving this gives

$$V = \frac{\lambda}{1+\lambda\delta-(1-\lambda)\delta^2}$$

Conjectured strategies: When it is your turn to offer, offer:

$$\delta V = \frac{\delta\lambda}{1+\lambda\delta-(1-\lambda)\delta^2}$$

to the other player. When you are made an offer accept if and only if you are offered at least $\delta V$.

If we let $x^*$ and $y^*$ denote player $i$'s share when offers are made by $i$ and $j$, respectively (as in the lecture), then we have that:

$$x^* = 1-\delta V = \frac{\lambda\delta^2+1-\delta^2}{1+\lambda\delta-(1-\lambda)\delta^2}, \qquad and \qquad y^* = \delta V = \frac{\delta\lambda}{1+\lambda\delta-(1-\lambda)\delta^2}$$

which implies that $x^* > V > y^*$. Note that when $\lambda = 1$ (giving the model of Lecture 9 under $\delta_i = \delta_j = \delta$), then $x^* = V$ and $x^* > y^*$ holds.

Checking that these strategies are SPNE using the single deviation property is straightforward. If a responding player rejects such an offer, she will realize value $V$ in the following period, in the equilibrium of the continuation game. This yields her continuation value $\delta V$, which is not profitable. A proposing player can offer less than $\delta V$ which is rejected, in which case she accepts $\delta V$ the following period, in the equilibrium of the continuation game. Alternatively, she can offer strictly more than $\delta V$ which is also accepting, leaving her strictly worse off.