Econ C103: Introduction to Mathematical Economics
UC Berkeley, Fall 2019

Assignment 3 Solutions

Exercise 1.1

(a) $WB_k > 0$ if and only if k has more than one link in a network and some of k's neighbors are not linked to one another:

Let us first show that if $WB_k > 0$ then k has more than one link in a network and some of k's neighbors are not linked to one another. If $WB_k > 0$ for some k, then it follows from the definition of WB_k that there exists $ij : i \neq j, k \notin \{i, j\}$ such that $W_k(ij) > 0$. From the definition of $W_k(ij)$ it follows that $P_k(ij) > 0$. Thus, at least one geodesic path between i and j passes through k. Consider such a path, denoted $i_0 i_1, ..., i_{m-1} i_m$ where the path consists of $m \geq 2$ links, with $i_0 = i$ and $i_m = j$, and $i_h = k$ where $0 < h < m$. It follows that i_{h-1} and i_{h+1} are neighbors of k. These nodes cannot be connected or else the path as above, but which eliminates links $i_{h-1} i_h$ and $i_h i_{h+1}$ and replaces them by $i_h i_{h+1}$ would lead to a shorter path connecting i to j which would contradict the fact that original path is a shortest path.

Next, let us show that if k has more than one link in a network and some of k's neighbors are not linked to one another then $WB_k > 0$. Let k be linked to i and j and these nodes not be linked to each other. Then $\ell_{ij} = 2$ and $P_k(ij) = 1$. Thus, $W_k(ij) > 0$ and then since $P(ij) > 0$ it follows that $WB_k > 0$.

(b) Let k be the center of a star that includes all nodes when $n \geq 3$. Then for each pair $ij : i \neq j, k \notin \{i, j\}$, there is a unique path from i to j and it is of length 2 and passes through k. Then $\ell_{ij} = 2$ and $P_k(ij) = 1$ and so $W_k(ij) = 1$ and then since $P(ij) = 1$ it follows that $WB_k = \sum_{ij: i \neq j, k \notin \{i,j\}} \frac{1}{(n-1)(n-2)/2}$. Since there are $(n-1)(n-2)/2$ pairs of nodes ij such that $i \neq j, k \notin \{i, j\}$, it follows that $WB_k = 1$.

(c) It is enough to show that if k is not the center of a star network, then there is some pair of nodes ij such that $i \neq j, k \notin \{i, j\}$ and for which $W_k(ij) < P(ij)$. This happens if either $P_k(ij) < P(ij)$ or if $\ell_{ij} \geq 2$. In the first case there is a shortest path between i and j that does not contain k which implies that k is not the center of a star, and in the second case there must be a node that lies on the shortest path between k and either i or j in which case k is not the center of a star.

Both nodes 4 and 5 have the same betweenness measure based on (1.1): They each have three nodes on one side of them and four on the other, so they lie on all of the paths between 12 pairs of nodes, so they each have a Freeman betweenness measure of 12/21. If we adjust these to account for the lengths of these various paths, direct calculation of the measures leads to $WB_4 = 269/(60 \cdot 21) = .2135$ and $WB_5 = 279/(60 \cdot 21) = .2214$. While 4 and 5 are equivalent according to the betweenness measure of Freeman, the weighted measure favors node 5 since both nodes lie on the same number of shortest paths, but 5 lies on some shorter shortest paths than 4 does.

Exercise 2.3

Let us show that if the degree is bounded by some K, then the diameter of the networks must be unbounded as n grows.

The number of nodes at distance 1 from some node i is at most K given the bound on i's degree. The number of nodes at distance 2 from some node i is at most $K(K-1)$, given the bound on degrees of i's neighbors. Thus, in any network, the number of nodes at distance ℓ from any given node i is at most $K(K-1)^{\ell-1}$. If we let $D = 2\ell$ be the diameter of a network, it follows that the diameter must satisfy:

$$1 + K + K(K-1) \cdots + K(K-1)^{D/2-1} = 1 + K \sum_{\ell=1}^{D/2} (K-1)^{\ell-1} \geq n.$$

If either K or D are fixed, it must be that the other (i.e. D and K, resp.) grows without bound as n grows.

Exercise 2.4

The degrees of the nodes are $(1, 2, 1)$ and so the degree centralities are $(\frac{1}{2}, 1, \frac{1}{2})$.

The betweenness centralities are $(0, 1, 0)$ as only node 2 lies on a path between the other two nodes.

As discussed in the text, in an undirected network the Katz Prestige is simply any rescaling of the degrees of the nodes, and so will be the same as the degree centralities $(\frac{1}{2}, 1, \frac{1}{2})$. You can check this by noting that

$$g = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{1}$$

and then

$$\mathbb{I} - \widehat{g} = \begin{pmatrix} 1 & -1/2 & 0 \\ -1 & 1 & -1 \\ 0 & -1/2 & 1 \end{pmatrix}. \tag{2}$$

The product of this matrix and the degrees $(1, 2, 1)$ is the 0 vector, as required in (2.5).

To calculate the Bonacich centrality measures, note that

$$\mathbb{I} - bg = \begin{pmatrix} 1 & -b & 0 \\ -b & 1 & -b \\ 0 & -b & 1 \end{pmatrix}. \tag{3}$$

From this we see that

$$(\mathbb{I} - \frac{1}{2}g)^{-1} = \begin{pmatrix} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{pmatrix}. \tag{4}$$

$$(\mathbb{I} - \frac{1}{4}g)^{-1} = \begin{pmatrix} 15/14 & 4/14 & 1/14 \\ 4/14 & 16/14 & 4/14 \\ 1/14 & 4/14 & 15/14 \end{pmatrix}. \tag{5}$$

$$(\mathbb{I} - \frac{3}{4}g)^{-1} = \begin{pmatrix} -7/2 & -6 & -9/2 \\ -6 & -8 & -6 \\ -9/2 & -6 & -7/2 \end{pmatrix}. \tag{6}$$

From this, we find that the $Ce^B(g, 1/2, 1/2) = (2, 3, 2)$; $Ce^B(g, 1/2, 1/4) = (6/7, 10/7, 6/7)$; $Ce^B(g, 1/2, 3/4) = (-10, -14, -10)$. As we increase b we increase the relative weight put on longer walks in the Bonacich centrality measure. In this network, that benefits the "peripheral" nodes 1 and 3, as their relative centrality is higher at $b = 1/2$ than at $b = 1/4$. As we continue to increase b to be too high, the sum no longer converges and the expression of $-bg$ is not the proper expression for the sum and so the centrality measure is no longer well defined and the resulting term no longer has the correct interpretation.

Exercise 2.5

Average clustering is $\sum_i Cl_i(g)\left(\frac{1}{n}\right)$.

Overall clustering is

$$Cl(g) = \frac{\sum_{i;j\neq i;k\neq j;k\neq i} g_{ij}g_{ik}g_{jk}}{\sum_{i;j\neq i;k\neq j;k\neq i} g_{ij}g_{ik}}.$$

The denominator is $\sum_i d_i(g)(d_i(g)-1)$ and the numerator is $\sum_i Cl_i(g)d_i(g)(d_i(g)-1)$. It then follows that overall clustering is

$$\sum_i Cl_i(g)\left(\frac{d_i(g)(d_i(g)-1)/2}{\sum_j d_j(g)(d_j(g)-1)/2}\right).$$

Thus, both are weighted averages of the individual clustering measures. Average clustering has an equal weighting, while overall clustering has weights that place relatively more weight on higher degree nodes. Thus, in case (a) overall clustering puts higher weight on nodes with higher clustering, and so overall clustering will be at least as high as average clustering. In case (b), this is reversed.

Exercise 4.5

(a) The conditional probability of j's degree under this scenario is d with a probability of Binom($p, d-1, n-2$) for $d \geq 1$ (and 0 probability on degree 0), where Binom($p, d-1, n-2$) is the binomial probability of d successes out of $n-2$ draws where the probability of a success is p and where d is an integer in $\{1, \cdots, n-1\}$.

(b) To show that the conditional probability of j's degree is different under this scenario than under (a), it is sufficient to consider a simple example. Suppose that there are three nodes. Under part (a), there is a p chance that node j's degree is 2 and a $1-p$ chance that j's degree 1. Let us now check the conditional distribution under the system described in (b). Here,there are seven different networks that each have one link (the same figure as that below for Exercise 6.1). If any of the one link networks arise, then j's degree under this process will be 1, and the one link networks each arise with probability $p(1-p)^2/(1-(1-p)^3)$ (conditional on there being a link in the network). If the three link networks arise, then then j's degree under this process will be 2, and this network will arise with probability $p^3/(1-(1-p)^3)$ conditional on a network having at least one link. In the two link networks case, there is a $2/3$ chance that we will end up with one of the peripheral nodes as i and then j will have degree 2, and a $1/3$ chance we end up with the center node and then j's degree will be 1. Such networks each occur with probability $p^2(1-p)/(1-(1-p)^3)$ conditional on a network having at least one link. Aggregating over these cases, we have a probability of having j's degree be 2 conditional on process (b) being:

$$\frac{p^3 + 3p^2(1-p)2/3}{1-(1-p)^3} = p\frac{2-p}{3-3p+p^2}.$$

This generally differs from p, and so the two conditional distributions differ.

In order to see what this converges to for larger networks, note that node j can only be found under the (b) procedure if any of its neighbors are found via the process of picking a node uniformly at random, and then it happens to be the one that is chosen out of that node's neighbors. Given that nodes' degrees are approximately independent in large Poisson random networks, this means that the relative probability of finding a node j is approximately proportional to its degree in a large network relative to sum of the other nodes' degrees. Thus, this calculation approaches the $\widetilde{P}(d)$ expression of $P(d)d/\langle d\rangle$. This is $\text{Binom}(p, d, n-1)d/(p(n-1))$. It is easy to see that is equal to $\text{Binom}(p, d-1, n-2)$ for $d \geq 1$ and 0 otherwise, which corresponds to that of part (a). (If $p(n) = m/(n-1)$, based on our Poisson approximation this approaches $P(d)d/m = e^{-m}m^{d-1}/((d-1)!)$ for $d \geq 1$.)

(c) Generally these two calculations will differ in the limit, while for the special case of the iid link formation that occurs under the Erdos-Renyi random graphs the limits coincide. The difference between the two calculations is that in (a), where the nodes are fixed in advance, the presence of the link ij gives us no information about the presence of other links of j. In (b), when we find a neighbor of a randomly selected node i we are more likely to find nodes that have high degree, since the chance that a node j

is found is now proportional to j's degree. We see that in the calculations in the example above, where conditional on the two-link network case, we are more likely to find the two degree node than either of the one degree nodes via the process.

Exercise 4.8 (Assume $0^0 = 1$)

(a) If we set $q = 1$, then the left hand side of (4.9) is 0 while the right hand side of (4.9) is $\sum_{d \geq 0} 0^d P(d)$. The right hand side is thus equal to $0^0 P(0) = P(0)$, and so $q = 1$ implies that $P(0) = 0$. To see the converse, note that the right hand side of (4.9) is at least $0^0 P(0) = P(0)$, and so if this is nonzero then the left hand side is also greater than zero and so $q < 1$ (which makes sense since $P(0) > 0$ implies that there isolated nodes, and so the giant component does not include all nodes). Thus, $q = 1$ is a solution to this equation if and only if $0 = P(0)$.

(b) (4.9) is then $1 - q = 1/3 + 2(1-q)^2/3$, which has a solution of $q = 1/2$.

Exercise 6.6

An efficient network is a star. Let the first link that forms be 12 (other cases are analogous). To continue on an improving path to a star, any link can form except 34 (as we know from the proof in Section 6.3.2). As there are 5 possible links, so there is a 4/5 chance that we continue on an improving path to a star. Again, without loss of generality, if we continue on a path to a star label the next link that formed 13. There are four possible remaining links, if the link 23 is identified, it will not be added given the costs outweigh the gains. Any of the other three links will be added if identified since they involve agent 4 who is not yet connected. The only one that results in a star is 14. Thus, there is a 1/3 chance of the star forming conditional on having started with a two link star. Overall, this results in a probability of 4/15 of an efficient network forming.

Exercise 6.7

Assume $n \geq 4$ is even. Proposition 6.5 shows that that any pairwise stable network can be partitioned into a collection of completely connected components, each of which consists of a different number of nodes. The payoff to a node that has no connections is 1 while the payoff to any node in a component of $m+1$ nodes ($m \geq 1$) is

$$m\left(\frac{1}{m} + \frac{1}{m} + \frac{1}{m^2}\right) = 2 + \frac{1}{m} \leq 3,$$

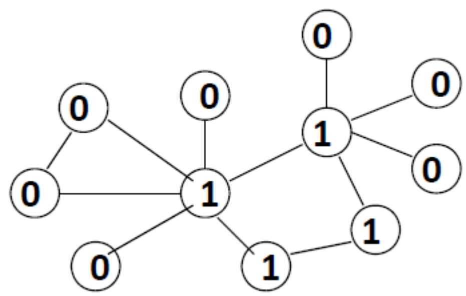

Figure 1: Exercise 9.4 - Figure

with the inequality being strict if $m > 1$. Since some component has at least 3 nodes (because $n \geq 4$ and no two components have the same size), every node in a pairwise stable network has payoff no greater than 3 and some node has a payoff strictly less than 3.

On the other hand, Proposition 6.5 shows that any efficient network consists of $n/2$ dyads, hence each each node's payoff is 3. Thus, any efficient network Pareto dominates any pairwise stable network.

Exercise 9.4

See Figure 1.

Exercise 9.13 (assume that f and c have continuous derivatives).

(a) First, note that in an equilibrium, it cannot be that all players choose 0. This follows since there is a positive level at which $f'(x^*) = c'(x^*)$, which then given the strict concavity and convexity implies that $f'(0) > c'(0)$, and so 0 is not an equilibrium since any agent can then increase payoffs by raising his or her x_i. Thus, at least one player has $x_i > 0$.

In a pure strategy equilibrium, since players are best responding it must be that

$$f'(\sum_k x_k) \leq c'(x_i)$$

for each i (or else raising x_i would increase a player's payoff), and this must hold with equality whenever $x_i \geq 0$ (or an agent could gain by lowering the action). Thus,

$$f'(\sum_k x_k) = c'(\max_k x_k).$$

Given the strict concavity and convexity of f and c, this implies that

$$f'(\sum_k x_k) > c'(x_i),$$

when $x_i < \max_k x_k$. Thus, $x_i < \max_k x_k$ cannot be part of an equilibrium, and so any pure strategy equilibrium must have all agents choose the same positive action x which then must be the solution to

$$f'(nx) = c'(x).$$

There is a solution to this since we already argued that $f'(0) > c'(0)$, and we also know that $f'(x^*) = c'(x^*)$ and that implies that $f'(nx^*) < c'(x^*)$, and so x lies between 0 and x^*. It is unique given the strict concavity of f and strict convexity of c.

(b) On the circle, order the players so that even numbered players are linked only to odd numbered players and vice versa. There is an equilibrium where even numbered players choose x^* and odd numbered players choose 0 (and there is another with the roles reversed). Note that this is a best reply for the even-numbered players since $f'(x^*) = c'(x)$. For the odd-numbered players, their neighbors produce $2x^*$ in total, and $f'(2x^*) < c'(0)$, and thus given the concavity of f and convexity of c, the derivative of $f(2x^* + x_i) - c(x_i)$ is negative for all $x_i \geq 0$ and so the best reply is 0.

Exercise 9.17

We first show the $\Rightarrow$ direction of the result. So, suppose that A is the seed ($a_i = 1$ for each $i \in A$ always), and that $B \bigcup A$ is the eventual set of nodes playing 1. For any $i \in C$, $x_i = 0$ implies that a fraction of at least $1 - q$ of i's neighbors play action 0 in all periods; these neighbors are all in C. So for any $i \in C$, a fraction of more than $1 - q$ of his/her neighbors are in C, and thus C is more than $1 - q$-cohesive. This gives the first claim of the conclusion. For the second, we prove by contradiction. If there is a nonempty subset D of B such that $D \cup C$ has a cohesiveness of more than $1 - q$, for any node in D every time there is a fraction of more than $1 - q$ of his/her neighbors play the action 0 (i.e. at some early stage in the best response dynamic) then all nodes in $D \cup C$ should stay at 0, which contradicts that fact that $D \subset B$ with B the set of nodes that change to adopt. Thus, for every nonempty subset D of B, $D \cup C$ has a cohesiveness of no more than $1-q$. This gives the second claim of the conclusion, and establishes $\Rightarrow$.

We now show the $\Leftarrow$ direction of the result. So, suppose that C is more than $1-q$-cohesive and for every nonempty subset D of B, $D \cup C$ has a cohesiveness of no more than $1-q$. From the first condition, it follows from an argument similar to that above that all nodes in C play 0 forever. Assume by contradiction there is a nonempty subset D of B such that all nodes in $D \cup C$ always play 0, so that $B \bigcup A$ is *not* the set of final adopters. Also take D to be the largest such set, that is, choose D so that $A \bigcup B \backslash D$ gives the final set of adopters. We know that $D \cup C$ has a cohesiveness of no more than $1-q$ by the second statement of the premise, so there exists at least one $i \in D$ with a fraction of no more than $1-q$ of his/her neighbors in $D \cup C$. This implies that, if there is a fraction of at least q of i's neighbors who eventually play 1, then i also prefers to play 1 eventually. (And, note that it only takes finite steps to get to a stable state.) This contradicts our choice of D, showing that D must be empty and all nodes in $B \cup A$ will play 1 eventually. We have established $\Leftarrow$.