Problem 1

Explain why linearization do not work

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} xf(x,y) \\ yf(x,y) \end{bmatrix} \tag{1}$$

To use the linearization method, we must futher impose the assumption that

$$\lim_{t\to\infty} \frac{|xf(x,y)| + |yf(x,y)|}{|x| + |y|} = 0 \tag{2}$$

which is not necessarily the case. So we can not directly use the linearization method.

Direct Method

Consider $V(x,y) = \frac{1}{2}(x^2 + y^2)$ which is positive definite. The derivative along the path is

$$\frac{dV(x,y)}{dt} = xy - x^2 f(x,y) - xy - y^2 f(x,y) = -(x^2 + y^2) f(x,y) \tag{3}$$

$\because f(x,y) \geq 0$ for all $(x,y) \in \mathbb{R}^2$ $\therefore$ the derivative is non-positive. The solution is hence stable.

Asymptotic Stability

If $f(x,y) \geq 0$ for all $(x,y) \in \mathbb{R}^2$, and $f(x,y) = 0$ if and only if $x = y = 0$, then the solution is asymptotically stable.

Unstablity

If $f(x,y) \leq 0$ for all $(x,y) \in \mathbb{R}^2$, then the solution is not stable.

Problem 2

Apparently, $V(x,y) \geq 0$, and the equal sign hold iff $x = y = 0$ (positive definite) Take the derivative along the path

$$\frac{dV(x,y)}{dt} = 4x^3x' + 2yy' = 4x^3y - 4x^3y - 4x^6 - 4y^6 = -4(x^6 + y^6) \tag{4}$$

The derivative is negative definite, ($\frac{dV(x,y)}{dt} \leq 0$, the equal sign holds iff $x = y = 0$)
Meanwhile, $V(x,y) \to \infty$ as $|y| \to \infty$ on the whole neighbourhood of $\mathbb{R}^2$. So the solution is globally asymptotically stable.

Problem 3

Vectorize the system

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} y_2 \\ -sin(y_1) \end{bmatrix} \tag{5}$$

The set of equilibrium solution is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} k\pi \\ 0 \end{bmatrix} \qquad k \in \mathbb{Z} \tag{6}$$

Linearization Method

We can linearize the whole system as

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + f(\mathbf{y}) \tag{7}$$

Note that it follows the same procedure as Taylor expansion at $y_1 = \pi$
We have $f(\mathbf{y}) = \begin{bmatrix} 0 \\ -y_1 - sin(y_1) \end{bmatrix}$ and $\lim_{y_1 \to \pi} \frac{|f(\mathbf{y})|}{|\mathbf{y}|} = 0$. The eigenvalues of the constant matrix is $\lambda_1 = 1$, $\lambda_2 = -1$. With positive eigenvalues, the solution is unstable.

Lyapunov 2nd Method

Construct Lyapunov function as follows:

$$V(y_1, y_2) = \frac{1}{2}y_2^2 + 1 - cos(y_1) \tag{8}$$

We have $V(y_1, y_2) \geq 0$ because $cos(y_1) \in [-1, 1]$ In the neighborhood of y_1 $\Omega = (-\pi, \pi)$, it is positive definite.

$$V^*(y_1, y_2) = -y_2 sin(y_1) + y_2 sin(y_1) \equiv 0 \tag{9}$$

Hence the zero solution is stable.

Problem 4

Apparently the function is positive definite

$$V(||x||, ||v||) = ||v|| + \int_0^{||x||} \psi(s)ds \tag{10}$$

because the Euclidean norm is larger or equal to zero and $\psi(t)$ is strictly positive. The equal sign may only hold when $||x|| = ||v|| = 0$

$$\begin{aligned} \frac{d}{dt}V(||x||, ||v||) &= \frac{d||v||}{dt} + \psi(||x||)\frac{d||x||}{dt} \\ &\leq \psi(||x||)||v|| - \psi(||x||)||v|| \\ &= 0 \end{aligned} \tag{11}$$

Hence the function is a Lyapunov function of the system.