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Assignment 1

Math 123
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Problem 1

a) Using seperation of variables to solve the equation

$$\begin{aligned}\frac{dy}{dt} &= y^2\\ \frac{1}{y^2}dy &= dt\end{aligned} \tag{1}$$

Note that on Step 1, we missed an equilibrium scenario $y(t) = 0$, which is also a solution of this ode.
When $y(t) \neq 0$: Calculate the integral on both sides, and we can get a general solution:

$$\begin{aligned}-\frac{1}{y} &= t + C\\ y &= -\frac{1}{t+C}\end{aligned} \tag{2}$$

For the IVP, C is determined by $f(X_0) = \eta$, Hence we got:

$$\begin{aligned}\eta &= -\frac{1}{C+t_0}\\ C &= -t_0 - \frac{1}{\eta}\\ \phi(t) &= \frac{\eta}{1-\eta(t-t_0)} \quad (\eta \neq 0)\end{aligned} \tag{3}$$

b) The interval of validity $I = (t_0 + \frac{1}{n}, +\infty)$.
Note that we shall find an interval on which

- $\phi(t)$ is continuous (to ensure the existence of $\phi'(t)$)
- $\phi(t)$ is defined in the domain for each t in I
- contains the initial value t_0

c) When $\eta = 0$, the solution of the IVP shall be $\phi(t) = 0$, which can be contained in our general solution simply be letting $\eta = 0$. So in general, $\phi(t) = \frac{\eta}{1-\eta(t-t_0)}$ is the unique general solution to the equation.

Problem 2

Verification:

$$\begin{aligned}LHS &= -\frac{1}{2}(1-t^2)^{-\frac{3}{2}}(-2t)\\ &= t(1-t^2)^{-\frac{3}{2}}\end{aligned} \tag{4}$$

$$RHS = t(1-t^2)^{-\frac{3}{2}} = LHS \tag{5}$$

Apparently, $y = 0$ when $t = 1$, so $\phi(t) = (1-t^2)^{-\frac{1}{2}}$ is a solution of the IVP on a given interval I. The interval of validity shall be $I = [-1, 1]$.

Problem 3

1. Order: 3, Autonomous: No, Linear: No
2. Order: 4, Autonomous: Yes, Linear: No
3. Order: 4, Autonomous: No, Linear: Yes

Equation 3 can be re-written as:

$$\begin{bmatrix} \dot{x} \\ u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{16}{t^3} & 0 & \frac{1}{t^3} \end{bmatrix} \begin{bmatrix} x \\ u \\ v \end{bmatrix}$$

Problem 4

Solve the equation by separating the variables:

$$\frac{dy}{dt} = \frac{y^2-1}{2}$$
$$\frac{2dy}{y^2-1} = dt$$
$$(-\frac{1}{y+1} + \frac{1}{y-1})dy = dt$$

Note that $y(t) = 1$ and $y(t) = -1$ are also equilibrium solutions of the equation.
When $y(y) \neq \pm 1$, we can integrate the equation on both sides.

$$ln\left|\frac{y-1}{y+1}\right| = t + C$$
$$\left|\frac{y-1}{y+1}\right| = Ce^t \quad (C > 0)$$
$$\frac{y-1}{y+1} = Ae^t \quad (A \neq 0)$$

Combine the general solution with $y(t) = \pm 1$, the answer is:

$$y = \frac{1+Ae^t}{1-Ae^t} \quad Or \quad y = -1$$

Note that when $A = 0$, $y = 1$. That is already contained in the former specification.
For IVP $y(t_0) = \eta$,

$$\frac{1 + Ae^{t_0}}{1 - Ae^{t_0}} = \eta$$

Hence the solution for A is

$$A = \frac{\eta - 1}{e^{t_0}(1 + \eta)}$$

there are five scenarios:

- if $\eta < -1$, then $y = \frac{1+Ae^t}{1-Ae^t}$, the interval of validity is R
- if $\eta = -1$, then $y = -1$ for all t, the interval of validity is R
- if $\eta \in [-1, 1]$, then $y = \frac{1+Ae^t}{1-Ae^t}$, the interval of validity is R
- if $\eta = 1$, then $y = 1$ for all t, the interval of validity is R
- if $\eta > 1$, then $y = \frac{1+Ae^t}{1-Ae^t}$, the interval of validity is R

Problem 5

Solve a Bernoulli Equation.

$$\frac{dy}{dt} + \frac{2y}{t} = 2ty^{\frac{1}{2}}$$
$$y^{-\frac{1}{2}}\frac{dy}{dt} + \frac{2y^{-\frac{1}{2}}}{t} = 2t$$
$$\frac{dy^{\frac{1}{2}}}{dt} + \frac{y^{\frac{1}{2}}}{t} = t$$

Note that $y(t) = 0$ is also a solution to the equation for all $t \neq 0$
If $y(t) \neq 0$, then the equation is now transformed to a "linear" version

$$y^{\frac{1}{2}} = e^{-\int \frac{1}{t}dt}[\int te^{\int \frac{1}{s}ds}dt + C]$$
$$y^{\frac{1}{2}} = \frac{1}{t}[\int t^2 dt + C]$$
$$y^{\frac{1}{2}} = \frac{t^2}{3} + \frac{C}{t}$$
$$y = \frac{t^4}{9} + \frac{2C}{3}t + \frac{C^2}{t^2}$$

That's the general solution of $y(t)$, and also remember the equilibrium solution $y(t) = 0$

Problem 6

Consider the eigenequations for the second-order o.d.e.s
1. $\lambda^2 + 12\lambda + 36 = 0$ has two equal roots -6.
The solution should be $y = (C_1 + C_2 x)e^{-6x}$
2. $\lambda^2 - 3 = 0$ has two different roots $\pm\sqrt{3}$
The solution should be $y = C_1 e^{\sqrt{3}} + C_2 e^{-\sqrt{3}}$
3. $\lambda^2 + \lambda + 1 = 0$ has two conjugate complex roots $\frac{-1+\sqrt{3}i}{2}$ and $\frac{-1-\sqrt{3}i}{2}$
The solution should be $y = e^{-\frac{1}{2}x} * (C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin -\frac{\sqrt{3}}{2}x)$