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Assignment 1

Problem 1

Assuming $p = 1$, verify that the overall dispersion can be expressed in terms of squared distances between all individuals to the centroid G.

$$\begin{aligned} LHS &= 2n(\sum_{i=1}^{n} x_i^2 - \bar{x}\sum_{i=1}^{n} x_i) \\ &= 2n\sum_{i=1}^{n} x_i^2 - 2n^2\bar{x}^2 \\ &= 2n(\sum_{i=1}^{n} x_i^2 - 2\bar{x}\sum_{i=1}^{n} x_i + n\bar{x}^2) \\ &= 2n\sum_{i=1}^{n} (x_i - \bar{x})^2 \\ &= RHS \end{aligned} \tag{1}$$

Problem 2

Given A is symmetric, we have

$$A = A^T \tag{2}$$

For eigenvalues λ_i and λ_j and their respective eigenvectors:

$$\lambda_k v_k = A v_k \quad k = i, j \tag{3}$$

In order to prove that v_i and v_j are orthogonal, we consider the value of $v_i^T v_j$

$$\begin{aligned} \lambda_i v_i^T v_j &= (A v_i)^T v_j \\ &= v_i^T A^T v_j \\ &= v_i^T A v_j \\ &= \lambda_j v_i^T v_j \end{aligned} \tag{4}$$

Given $\lambda_i \neq \lambda_j$, we have

$$v_i^T v_j = 0 \tag{5}$$

Hence the eigenvectors of different eigenvalues of a symmetric matrix are orthogonal.

Problem 3

For any covariance matrix S, S shall be symmetric and all its entries shall be defined on R. When we calculate the eigendecomposition of S, we have

$$S = QAQ^T \tag{6}$$

where A_{ii} denotes the i^{th} eigenvalue of S and Q_k denotes the k^{th} eigenvector of S
Note that the eigenvectors are unitized and transformed to be orthogonal to each other, so they're unique.

$$Q = \begin{vmatrix} v_1 & v_2 & ... & v_n \end{vmatrix} \tag{7}$$

$$A = diag\{\lambda_1, \lambda_2, ..., \lambda_n\} \tag{8}$$

$$S = \begin{vmatrix} \lambda_1 v_1 & \lambda_2 v_2 & ... & \lambda_n v_n \end{vmatrix} \begin{vmatrix} v_1 & v_2 & ... & v_n \end{vmatrix}^T = \sum_{k=1}^{p} \lambda_k v_k v_k^T \tag{9}$$

For the diagonal values of S,

$$S_{jj} = \sum_{k=1}^{p} \lambda_k v_{kj}^2 \tag{10}$$

Remark: The equation specification for us to prove might be imprecise to some extent. Only those eigenvectors that are unitized and orthogonal to each other, as those column vectors of matrix Q, satisfies eq.10. While they remain to be the eigenvectors of S if the column vectors are multiplied by a constant,(i.e. satisfies equation (3)) the original equation no longer holds.

Problem 4

a.True
b.True
c.True
d.True
e.False
f.True