Week 2 HW (due Sept. 19th)

Instructions. You *must* declare all resources that you have used on this homework (include but not limited to anyone, any book, and any webpage). **Don't be scared! The length of the assignments is long, mainly due to HINTS, not due to the length of problems.**

1. ([B-N] Page 110 Problem 2)

   Prove that the IVP
   $$y'' + g(t, y) = 0,\ y(0) = y_0,\ y'(0) = z_0. \tag{1}$$
   where g is continuous in some region D containing $(0, y_0)$, is equivalent to the integral equation
   $$y(t) = y_0 + z_0 t - \int_0^t (t-s)\, g(s, y(s))\, ds \tag{2}$$

   [Hints: a) To show that "if y is a solution of the Eq 1 on I, then it satisfies the Eq 2 on I.", we integrate Eq 1 twice and use the fact that
   $$\begin{aligned}\int_0^t \left\{\int_0^s g(\tau, y(\tau))\, d\tau\right\} ds &= \int_0^t \left\{\int_\tau^t ds\right\} g(\tau, y(\tau))\, d\tau \\ &= \int_0^t (t-\tau)\, g(\tau, y(\tau))\, d\tau.\end{aligned}$$
   Why is this equation true? Multivariable Calculus!

   b) To show that a solution of Eq 2 is a solution of Eq 1. The idea is the similar as the Lemma that we proved in class. But you will need the following formula
   $$\frac{d}{dt}\int_0^t H(t, s)\, ds = H(t, t) + \int_0^t \frac{\partial H}{\partial t}(t, s)\, ds,$$
   which could be proved by chain rule, assuming only that H and $\frac{\partial H}{\partial t}$ are continuous on some rectangle containing $s = t = 0$. Note that the more general form of this equation is the Leibniz integral rule. See https://en.wikipedia.org/wiki/Leibniz_integral_rule. You can use the formula directly.]

2. ([B-N] Page 111 Problem 4, modified a bit.)

   Prove that if ϕ is a solution of the integral equation
   $$y(t) = e^{it} + \int_t^\infty \sin(t-s)\, \frac{y(s)}{s^2} ds$$
   (assuming the existence of the integral), then ϕ satisfiews the differential equation
   $$y'' + \left(1 + \frac{1}{t^2}\right) y = 0.$$
   [Hints: similar as Hint b) in problem 1. Use the Leibniz integral rule in the link to take the derivatives!]

3. ([B-N] Page 118 Problem 13)

Consider the same integral equation as in Problem 2. Define the Picard iteration (also called successive approximation):

$$
\begin{aligned}
y_0(t) &= 0 \\
y_1(t) &= e^{it} + \int_t^\infty \sin(t-s)\,\frac{y_0(s)}{s^2}\,ds \\
&\cdots \\
y_n(t) &= e^{it} + \int_t^\infty \sin(t-s)\,\frac{y_{n-1}(s)}{s^2}\,ds \qquad \text{for } t \geq 1
\end{aligned}
$$

(a) Show by induction that

$$|y_n(t) - y_{n-1}(t)| \leq \frac{1}{(n-1)!t^{n-1}}$$

for $t \geq 1$, and $n = 1, 2, \ldots$. (Note that $0! = 1$.)

(b) Show that the following limit exists. In other words, show that the serie converges uniformly.

$$
\begin{aligned}
\lim_{n\to\infty} y_n(t) &= \lim_{n\to\infty} y_0(t) + (y_1(t) - y_0(t)) + \cdots + (y_n(t) - y_{n-1}(t)) \\
&= y_0(t) + \sum_{n=1}^{\infty} (y_n(t) - y_{n-1}(t))
\end{aligned}
$$

(Hints: Need to show $\sum_{n=1}^{\infty} \frac{1}{(n-1)!t^{n-1}}$ converges.)

(c) *(Optional)* Let us denote $y(t) = \lim_{n\to\infty} y_n(t)$. From the proofs in (a) and (b), now we know that $y(t)$ is a continuous limit function and the convergence is uniform. Show that $y(t)$ satisfies the integral equation 1.

[You can directly use the fact that

Let f_n be continuous and converge uniformly on some interval $[a, b]$, then

$$\lim_{n\to\infty} \int_a^b f_n(t)\,dt = \int_a^b \lim_{n\to\infty} f_n(t)\,dt.$$

This fact cames from real analysis. Here we directly use it. If you are interested in a proof, see http://www.math.drexel.edu/~tolya/limit%20of%20integrals.pdf.]

4. ([B-N] Page 119 Example 2)

$$y' = 3y^{2/3}, \quad y(0) = 0$$

(a) Verify that for each constant $c \geq 0$, the following function defined by

$$\phi(t) = \begin{cases} 0 & t \leq c \\ (t-c)^3 & t > c \end{cases}$$

is a solution of the given IVP. Why does this not contradicts with the uniqueness theorem?

(b) Can you prove that $y^{2/3}$ is not Lipschitz?

(c) Can we apply the local existence theorem (the more general one) to this IVP? Why?