Week 3 HW (due Sept. 26th)

Instructions. You *must* declare all resources that you have used on this homework (include but not limited to anyone, any book, and any webpage). Do not skip steps.

1. ([B-N] Page 133 Problem 6-7)

   a) Show that the solution ϕ of $y' = -y^2$, where $\phi(1) = 1$ exists for $0 < t < \infty$. (check the definition of solution); but can not be extended to the left beyond $t = 0$. (namely, explain why we can not apply the continuation theorem beyond $t = 0$.)

   b) Show that no solution, other than $\phi \equiv 0$, of the equation $y' = y^2$ can be extnded to the interval $(-\infty, \infty)$.

2. (modified from [B-N] Page 133 Problem 8)

   Consider
   $$y' = A(t)\,y + g(t)\,,\ y(t_0) = y_0$$
   where $A(t)$ and $g(t)$ are continuous functions defined on the closed interval $[a, b]$, and $t_0 \in [a, b]$, $y_0 < \infty$.

   a) Use the local existence/uniqueness theory we learned, estabilish the existence and uniquess result for the equation. (make sure to explain which theorem has been used)

   b) Denote the solution found in part (a) as ϕ, if we have
   $$|\phi(t)| \leq M$$
   for all $t \in [a, b]$. What is the interval of validity of the solution, and why?

3. [BN Page 137 Problem 1]

   Prove the following theorem.

   Let f, g be defined in a domain D, and are both bounded by the same constant
   $$|f| \leq M,\ |g| \leq M,$$
   for any $(t, y) \in D$, and are continuous satisfying Lipschitz conditions with the same Lipschitz constant L.

   Let ϕ and ψ be solutions of $y' = f(t, y)$ and $y' = g(t, y)$, respectively such that $\phi(t_0) = \psi(t_0) = y_0$ existing on a common interval $a < t < b$. Suppose
   $$|f(t, y) - g(t, y)| \leq \varepsilon$$
   for (t, y) in D. Then the solutions ϕ and ψ satisfy the estimate
   $$|\phi(t) - \psi(t)| \leq \varepsilon\,|b - a| \exp\left(L\,|t - t_0|\right).$$
   [Hint: Write things in integral form, and add/subtract a term of $\int_{t_o}^{t} f(s, \psi(s))\,ds$.]

4. Draw the bifurcation diagram of the 1-D ODE
   $$\frac{dy}{dt} = ry + y^3.$$
   This is called subcritical pitchfork bifurcation.