Econ C103: Game Theory and Networks

Module I (Game Theory): Lecture 2

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Readings:

1. Osborne (2004) chapter 2
2. Osborne and Rubinstein (1994) chapters 2.1-2.3, 2.4

Static games

Definition (Static game)

A **static game** *(or "***normal-form game***", or "***strategic game***") is defined as a triplet* $\langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$.

- $N = \{1, ..., n\}$*: players,*
- A_i*: player* i*'s (finite or infinite) action set,*
- $u_i(\mathbf{a})$*:* i*'s vNM utility from action profile* $\mathbf{a} = (a_1, ..., a_n) \in \times_{k=1}^{n} A_k$.

- Denote $\mathbf{a} \equiv (a_1, \ldots, a_n)$, and $A \equiv \times_{k=1}^{n} A_k$.
- Denote $\mathbf{a}_{-i} \equiv (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$, and $A_{-i} \equiv \times_{k=1, k \neq i}^{n} A_k$.

Strategies versus Actions:

- Action: any element from A_i for $i \in N$.
- Strategy: a "plan of action". In static games a strategy is a particular element from A_i, but notion of "strategy" is adaptable/more general.

2×2 strategic games

- 2×2 strategic games: 2 players, 2 actions each.
- $N = \{i, j\}$,
- $A_i = \{T, B\}$, $A_j = \{L, R\}$,
- $A_i \times A_j = \{(T, L), (T, R), (B, L), (B, R)\}$,

		player j	
		L	R
player i	T	$u_i(T, L), u_j(T, L)$	$u_i(T, R), u_j(T, R)$
	B	$u_i(B, L), u_j(B, L)$	$u_i(B, R), u_j(B, R)$

Classic 2×2 games

Battle of the Sexes

	L	R
T	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

Coordination Game

	L	R
T	$2, 2$	$0, 0$
B	$0, 0$	$1, 1$

Prisoner's Dilemma

	Q	T
Q	$3, 3$	$0, 4$
T	$4, 0$	$1, 1$

Hawk-Dove

	D	H
D	$3, 3$	$1, 4$
H	$4, 1$	$0, 0$

Matching Pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

A 2×3 game: Rock-paper-scissors

	(R)ock	(P)aper	(S)cissors
(R)ock	$0, 0$	$-1, 1$	$1, -1$
(P)aper	$1, -1$	$0, 0$	$-1, 1$
(S)cissors	$-1, 1$	$1, -1$	$0, 0$

A 3-player game

- Ordering utilities by $\big(u_i(a_i, a_j, a_k), u_j(a_i, a_j, a_k), u_k(a_i, a_j, a_k)\big)$:

Player k plays (I)n

		Player j L	Player j R
Player i	T	$2, 0, 1$	$0, 0, 0$
	B	$0, 0, 1$	$1, 2, 0$

Player k plays (O)ut

		Player j L	Player j R
Player i	T	$1, -1, 1$	$-1, 1, 0$
	B	$-1, 1, 0$	$1, -1, 1$

Best responses

Definition (Best response)

Given static game $\langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$, *for each* $i \in N$ *and any profile* $\mathbf{a}_{-i} \in A_{-i}$*:*

$$BR_i(a_{-i}) \equiv \{a_i \in A_i : u_i(a_i, \mathbf{a}_{-i}) \geq u_i(a_i', \mathbf{a}_{-i}),\ \forall a_i' \in A_i\}.$$

- $BR_i(\mathbf{a}_{-i})$ gives i's best action(s) when others' are conjectured to play $\mathbf{a}_{-i}$.
- We use "*" to designate a best response...

Row players i's best responses in 2×2 games

Battle of the Sexes

	L	R
T	$2^*, 1$	$0, 0$
B	$0, 0$	$1^*, 2$

$BR_i(L) = T, \; BR_i(R) = B$

Coordination Game

	L	R
T	$2^*, 2$	$0, 0$
B	$0, 0$	$1^*, 1$

$BR_i(L) = T, \; BR_i(R) = B$

Prisoner's Dilemma

	Q	T
Q	$3, 3$	$0, 4$
T	$4^*, 0$	$1^*, 1$

$BR_i(Q) = T, \; BR_i(T) = T$

Hawk-Dove

	H	D
H	$3, 3$	$1^*, 4$
D	$4^*, 1$	$0, 0$

$BR_i(H) = D, \; BR_i(D) = H$

Matching Pennies

	H	T
H	$1^*, -1$	$-1, 1$
T	$-1, 1$	$1^*, -1$

$BR_i(H) = H, \; BR_i(T) = T$

Dominance

Definition (Strict dominance)

Given static game $\langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$, an action $a_i' \in A_i$ is **strictly dominated** *by some action $a_i \in A_i \backslash \{a_i\}$ iff:*

$$u_i(a_i', \mathbf{a}_{-i}) \; < \; u_i(a_i, \mathbf{a}_{-i}),$$

for all $\mathbf{a}_{-i} \in A_{-i}$. Moreover, an action a_i is **dominant** *for player i if it strictly dominates each $a_i' \in A_i \backslash \{a_i\}$.*

Definition (Dominant solvability)

A static game $\langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ is **dominant solvable** *if each $i \in N$ has a dominant action.*

- If we use "†" to designate dominant actions...

Classic 2×2 games

Battle of the Sexes

	L	R
T	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

Coordination Game

	L	R
T	$2, 2$	$0, 0$
B	$0, 0$	$1, 1$

Prisoner's Dilemma

	Q	T
Q	$3, 3$	$0, 4^\dagger$
T	$4^\dagger, 0$	$1^\dagger, 1^\dagger$

Hawk-Dove

	H	D
H	$3, 3$	$1, 4$
D	$4, 1$	$0, 0$

Matching Pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Dominance

Definition (Weak dominance)

Given static game $\langle N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$*, an action* $a_i' \in A_i$ *is* **weakly dominated** *by some action* $a_i \in A_i \backslash \{a_i\}$ *iff:*

1. $u_i(a_i', \mathbf{a}_{-i}) \leq u_i(a_i, \mathbf{a}_{-i})$ *for all* $\mathbf{a}_{-i} \in A_{-i}$*, and*
2. $u_i(a_i', \mathbf{a}_{-i}) < u_i(a_i, \mathbf{a}_{-i})$ *for some* $\mathbf{a}_{-i} \in A_{-i}$*.*

- Clearly, strict dominance implies weak dominance.

Pure-strategy Nash equilibrium

- ...Dominant solvability is a rare and strong (interesting?) property of a game. Nash equilibrium gives a "weaker" equilibrium notion.

Definition (Pure-strategy Nash equilibrium)

Given static game $\langle N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$*, a strategy profile* $\mathbf{a}^*$ *is a* **pure-strategy Nash equilibrium (PNE)** *iff for each* $i \in N$*:*

$$u_i(a_i^*, \mathbf{a}_{-i}^*) \geq u_i(a_i', \mathbf{a}_{-i}^*),\ \forall a_i' \in A_i.$$

- $NE(\Gamma)$ denotes the set of PNE of $\Gamma = \langle N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$.
- Equivalent to the above, $a_i^* \in BR_i(\mathbf{a}_{-i}^*)$ for each $i \in N$.
- A dominant solvable game has a PNE: the profile of dominant strategies $(a_1^\dagger, \dots, a_n^\dagger)$.
- Inherent in PNE: players' hold correct beliefs of others' strategies.
- Games with PNE need not be dominant solvable. For 2×2 games...

Pure-strategy Nash equilibrium in 2×2 games (in **bold**)

Battle of the Sexes

	L	R
T	$\mathbf{2}^*, \mathbf{1}^*$	$0, 0$
B	$0, 0$	$\mathbf{1}^*, \mathbf{2}^*$

Coordination Game

	L	R
T	$\mathbf{2}^*, \mathbf{2}^*$	$0, 0$
B	$0, 0$	$\mathbf{1}^*, \mathbf{1}^*$

Prisoner's Dilemma

	Q	T
Q	$3, 3$	$0, 4^\dagger$
T	$4^\dagger, 0$	$\mathbf{1}^\dagger, \mathbf{1}^\dagger$

Hawk-Dove

	H	D
H	$3, 3$	$\mathbf{1}^*, \mathbf{4}^*$
D	$\mathbf{4}^*, \mathbf{1}^*$	$0, 0$

Matching Pennies

	H	T
H	$1^*, -1$	$-1, 1^*$
T	$-1, 1^*$	$1^*, -1$

Pure-strategy Nash equilibrium in 3-player game (in **bold**)

- Ordering utilities by $\big(u_i(a_i, a_j, a_k), u_j(a_i, a_j, a_k), u_k(a_i, a_j, a_k)\big)$:

Player k plays (I)n

		Player j: L	Player j: R
Player i	T	$\mathbf{2}^*, \mathbf{0}^*, \mathbf{1}^*$	$0, 0^*, 0^*$
	B	$0, 0, 1^*$	$1^*, 2^*, 0$

Player k plays (O)ut

		Player j: L	Player j: R
Player i	T	$1^*, -1, 1^*$	$-1, 1^*, 0^*$
	B	$-1, 1^*, 0$	$1^*, -1, 1^*$

A class of games: Zero-sum games

Definition (Zero-sum games)

A **zero-sum game** *is static game* $\langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ *which satisfies* $N = \{j, k\}$*, and for each* $\mathbf{a} \in A_j \times A_k$*,* $u_j(\mathbf{a}) + u_k(\mathbf{a}) = 0$*.*

- Matching-pennies, or Chess/Checkers/Go and two-player card games as dynamic games (later), give examples of zero-sum games.
- Fact: Zero-sum games can have PNE, e.g., (T, L) in the game:

	L	R
T	$\mathbf{0}^*, \mathbf{0}^*$	$1, -1$
B	$-1, 1^*$	$2^*, -2$

- A rich theory behind zero-sum games, which is a topic of focus in John von Neumann and Oskar Morgenstern's 1944 book "The Theory of Games and Economics Behavior".
- Many features of games not captured by zero-sum games (e.g., coordination failures, public goods).

Iterated elimination of dominated strategies

Iterated elimination of strictly-dominated strategies (IESDS):

- Take static game $\Gamma = \langle N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$ such that action $a_i \in A_i$ is strictly dominated for player i. Construct:

$$\Gamma' = \langle N, \{A_i \backslash \{a_i\}, A_k\}_{k\in N\backslash\{i\}}, \{u_i\}_{i\in N}\rangle.$$

- Continue to eliminate strictly dominated strategies in this way, until you have a game without any dominated strategies for any player.

	L	C	R
T	0,0	1,1	1,100
M	1,2	2,1	2,0
B	0,3	3,2	0,1

Figure: IESDS

Iterated elimination of dominated strategies

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	L	C	R
T	0,0	1,1*	1,100
M	1*,2*	2,1	2*,0
B	0,3*	3*,2	0,1

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	L	C	R
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T	0,0	1,1*	1,100
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$$\Gamma' = \langle N, \{A_i \setminus \{a_i\}, A_k\}_{k \in N \setminus \{i\}}, \{u_i\}_{i \in N} \rangle.$$

- Continue to eliminate strictly dominated strategies in this way, until you have a game without any dominated strategies for any player.

	L	C	R
T	0,0	1,1*	1,100
M	1*,2*	2,1	2*,0
B	0,3*	3*,2	0,1

Figure: IESDS

Iterated elimination of dominated strategies

Iterated elimination of strictly-dominated strategies (IESDS):

- Take static game $\Gamma = \langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ such that action $a_i \in A_i$ is strictly dominated for *some* player i. Construct:

$$\Gamma' = \langle N, \{A_i \backslash \{a_i\}, A_k\}_{k \in N \backslash \{i\}}, \{u_i\}_{i \in N} \rangle.$$

- Continue to eliminate strictly dominated strategies in this way, until you have a game without any dominated strategies for any player.

Proposition

Any Nash equilibrium profile $\mathbf{a}^*$ *survives IESDS.*

- Fact 1: For game Γ' obtained from Γ via IESDS:

$$\mathbf{a}^* \in NE(\Gamma') \Leftrightarrow \mathbf{a}^* \in NE(\Gamma).$$

Iterated elimination of dominated strategies

Iterated elimination of strictly-dominated strategies (IESDS):

- Take static game $\Gamma = \langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ such that action $a_i \in A_i$ is strictly dominated for *some* player i. Construct:

$$\Gamma' = \langle N, \{A_i \backslash \{a_i\}, A_k\}_{k \in N \backslash \{i\}}, \{u_i\}_{i \in N} \rangle.$$

- Continue to eliminate strictly dominated strategies in this way, until you have a game without any dominated strategies for any player.

Proposition

Any Nash equilibrium profile $(a_i^*, \dots, a_n^*)$ *survives IESDS.*

Iterated elimination of weakly-dominated strategies (IEWDS): as above but replace "strict" with "weak".

- Fact 2: For game Γ' obtained from Γ via IEWDS: $\mathbf{a}^* \in NE(\Gamma') \Rightarrow \mathbf{a}^* \in NE(\Gamma)$.
- Fact 2': However, the proposition (i.e. the converse of Fact 2) no longer holds (i.e. $\mathbf{a}^* \in NE(\Gamma)$ does not imply $\mathbf{a}^* \in NE(\Gamma')$)...

Iterated elimination of dominated strategies

Iterated elimination of strictly-dominated strategies (IESDS):

- Take static game $\Gamma = \langle N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$ such that action $a_i \in A_i$ is strictly dominated for *some* player i. Construct:

$$\Gamma' = \langle N, \{A_i \backslash \{a_i\}, A_k\}_{k\in N\backslash\{i\}}, \{u_i\}_{i\in N}\rangle.$$

- Continue to eliminate strictly dominated strategies in this way, until you have a game without any dominated strategies for any player.

Proposition

Any Nash equilibrium profile $(a_i^*, \ldots, a_n^*)$ survives IESDS.

Iterated elimination of weakly-dominated strategies (IEWDS): as above but replace “strict” with “weak”. Fact 2’ counter example:

	L	R
T	$\mathbf{2}^*, \mathbf{1}^*$	$0, 0$
B	$1, 3^*$	$\mathbf{3}^*, \mathbf{3}^*$

Best-response dynamic

Definition (Best-response dynamic)

For any static game $\Gamma = \langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$*, a* **[multilateral]/[unilateral] best-response dynamic** *starting from strategy profile* $(a_1^0, \ldots, a_n^0)$ *is the sequence* $((a_1^t, \ldots, a_n^t))_{t=1}^{\infty}$ *such that for each* $t > 0$ *and [each]/[some]* $i \in N$*:*

$$a_i^t \equiv BR_i(\mathbf{a}_{-i}^{t-1}).$$

- Unilateral best-response dynamics usually filter through players' best responses via some fixed permutation of N (e.g., player 1, then 2,...).
- Fact 3: Clearly, pure-strategy Nash equilibria give stationary points ("sinks") in multilateral and unilateral best-response dynamics.
- Fact 4: No guarantee all multilateral/unilateral best-response dynamics converge, even when a pure-strategy Nash equilibrium exists...

Multilateral best-response dynamics in 2×2 games

Battle of the Sexes

	L	R
T	$\mathbf{2}^*, \mathbf{1}^*$	$0, 0$
B	$0, 0$	$\mathbf{1}^*, \mathbf{2}^*$

$(B, L) \leftrightarrow (T, R)$

Coordination Game

	L	R
T	$\mathbf{2}^*, \mathbf{2}^*$	$0, 0$
B	$0, 0$	$\mathbf{1}^*, \mathbf{1}^*$

$(B, L) \leftrightarrow (T, R)$

Prisoner's Dilemma

	Q	T
Q	$3, 3$	$0, 4^\dagger$
T	$4^\dagger, 0$	$\mathbf{1}^\dagger, \mathbf{1}^\dagger$

$\forall (a_i, a_j) \rightarrow (T, T)$

Hawk-Dove

	H	D
H	$3, 3$	$\mathbf{1}^*, \mathbf{4}^*$
D	$\mathbf{4}^*, \mathbf{1}^*$	$0, 0$

$(T, L) \leftrightarrow (B, R)$

Matching Pennies

	H	T
H	$1^*, -1$	$-1, 1^*$
T	$-1, 1^*$	$1^*, -1$

$(T, L) \rightarrow (T, R) \rightarrow (B, R) \rightarrow (B, L) \rightarrow (T, L)$