Kaicheng Luo

Assignment 3

Math 123
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Problem 1

1.1

First, obtain our solution $\phi(t)$ using separation of variables:

$$\begin{aligned}\frac{dy}{-y^2} &= dt \\ \frac{1}{y} &= t + C\end{aligned} \tag{1}$$

Plugging in our initial condition, we have $C = 0$. (Note that $y = 0$ is also a solution)
$\phi(t) = \frac{1}{t}$ is a solution by definition because:
• For all $t \in I = (0, +\infty)$, $\phi(t)$ is well defined and continuous.
• The first-order derivative $\phi^{'}(t)$ exists on the interval.
• $\forall t \in I$, $\phi(t)$ satisfies $y^{'} = -y^2$
The solution cannot be extended to $\mathbb{R}$ because $\phi(t)$ isn't defined when $t = 0$. For the continuation theorem to hold, it is required that $f(t, y)$ is defined on the domain J ($J \subseteq \mathbb{R}^2$), among other things (Lipschitz continuous & bounded). Clearly $t = 0$ isn't contained in the domain. So we cannot apply the theorem.

1.2

For the equation $y^{'} = y^2$, we can similarly obtain its solution by separation of variables. (Either $y \equiv 0$ or $y = -\frac{1}{t+C}$) Consider the case when $t = -C$, where our solution $y = -\frac{1}{t+C}$ is clearly not defined on this point. Given our initial condition $y(t_0) = y_0$, ($y_0 \neq 0$). The domain will never extend (on the t-axis) to $\mathbb{R}$.

Problem 2

Lemma 1: For any function $f(t)$ continuous on a closed interval, there exists a minimum value and a maximum value for f.
Lemma 2: The linear function $f(t, y) = A(t)y + g(t)$ satisfies Lipschitz condition on any given interval [a,b], where $(a, b) \in \mathbb{R}^2$
Proof :
$\forall\, y_1, y_2 \in D$

$$|f(t, y_1) - f(t, y_2)| < |A(t)|\, |y_1 - y_2| \leq L\, |y_1 - y_2| \tag{2}$$

where L is the upper bound of $|A(t)|$

2.1 Existence

Using Lemma 1,2, $f(t, y)$ is Lipschitz continuous on a rectangular box D centered at (t_0, y_0). The local existence theorem ensures that there's a local solution in $(t_0 - \alpha, t_0 + \alpha)$, where $\alpha = min\{a, \frac{b}{M}\}$.

2.2 Uniqueness

Using Lemma 1,2, $f(t, y)$ is Lipschitz continuous on a rectangular box D centered at (t_0, y_0). The uniqueness theorem ensures that the solution for the ode is unique.

2.3 Solution

Using the integrated factor, we can actually solve the ode.

$$y(t) = e^{\int_{t_0}^{t} A(s)ds}[\int_{t_0}^{t} g(s)e^{-\int_{t_0}^{s} A(l)dl}ds + y_0] \tag{3}$$

2.4 Interval of Validity

The interval of validity is $[a, b]$.
Proof :
We already know $f(t, y)$ is Lipschitz continuous on domain $J = \{(t, y)|t \in [a, b], y \in \mathbb{R}\}$, Using the continuation theorem, we know that the solution can be extended to the boundary of J. Given $|\phi(t)| \leq M$, the solution can never reach the horizontal boundaries. So the solution can be extended for all $t \in [a, b]$

Problem 3

Write our conditions in the integral form:

$$\begin{aligned} \phi(t) &= y_0 + \int_{t_0}^{t} f(s, \phi(s))ds \\ \psi(t) &= y_0 + \int_{t_0}^{t} g(s, \psi(s))ds \end{aligned} \tag{4}$$

Subtraction:

$$\begin{aligned} |\phi(t) - \psi(t)| &= |\int_{t_0}^{t} f(s, \phi(s)) - g(s, \psi(s))ds| \\ &\leq \int_{t_0}^{t} |f(s, \phi(s)) - g(s, \psi(s))|ds \end{aligned} \tag{5}$$

Adding and Subtracting a term,

$$\begin{aligned}
|\phi(t) - \psi(t)| &\leq \int_{t_0}^{t} |f(s, \phi(s)) - f(s, \psi(s)) + f(s, \psi(s) - g(s, \phi(s))| ds \\
&\leq \int_{t_0}^{t} (L|\phi(s) - \psi(s)| + \epsilon) ds \\
&< \int_{t_0}^{t} L|\phi(s) - \psi(s)| ds + \int_{a}^{b} \epsilon ds \\
&= L \int_{t_0}^{t} |\phi(s) - \psi(s)| ds + \epsilon(b - a)
\end{aligned} \tag{6}$$

Using the Cronwall's Inequality,

$$|\phi(t) - \psi(t)| \leq \epsilon(b - a) e^{L|t - t_0|} \tag{7}$$

Q.E.D.

Problem 4