Econ C103: Game Theory and Networks

Module I (Game Theory): Lecture 4

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Readings:

1. Osborne (2004) chapters 2.5, 3.1, Exercise 34.1
2. Stahl, D. O. "Evolution of Smart-n Players." Games and Economic Behavior, 5, 604-617, 1993.

Welfare notions

Definition (Pareto Dominance)

Outcome $\mathbf{a} \in A$ **strongly [weakly] Pareto dominates** *outcome* $\mathbf{a}' \in A$ *iff* $u_i(\mathbf{a}) > [\geq]\ u_i(\mathbf{a}')$ *for all* $i \in N$ [*with* $u_i(\mathbf{a}) > u_i(\mathbf{a}')$ *for some* $i \in N$].

- Outcome $\mathbf{a}$ is "Pareto efficient" if it is not weak Pareto dominated.

Definition (Welfare Maximization)

Given Pareto weights $\rho_i > 0$, $\forall i \in N$, *a* **welfare-maximizing outcome** $\mathbf{a} \in A$ *satisfies:*

$$\mathbf{a} \in \operatorname*{argmax}_{\mathbf{a}' \in A} \rho_i u_i(\mathbf{a}').$$

- Setting $\rho_i = \rho_j$ for each $i, j \in N$ gives the "utilitarian solution".
- Fact 1: Welfare-maximizing outcomes are Pareto-efficient.
- Fact 2: Takashi Negishi and Hal Varian proved the converse: For Pareto-efficient $\mathbf{a}$, there exist $(\rho_i)_{i \in N}$ s.t. $\mathbf{a}$ is welfare maximizing.

Expected welfare notions for mixed extensions

Definition (Pareto Dominance)

Strategy profile $\boldsymbol{\alpha} \in \times_{i \in N} \Delta(A_i)$ **strongly [weakly] Pareto dominates** *strategy profile* $\boldsymbol{\alpha}' \in \times_{i \in N} \Delta(A_i)$ *iff* $U_i(\boldsymbol{\alpha}) > [\geq]\ U_i(\boldsymbol{\alpha}')$ *for all* $i \in N$ [*with* $U_i(\boldsymbol{\alpha}) > U_i(\boldsymbol{\alpha}')$ *for some* $i \in N$].

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- Setting $\rho_i = \rho_j$ for each $i, j \in N$ gives the "utilitarian solution".
- Facts 1 and 2 continue to hold.

Welfare in "Battle of the Sexes"

		q	1-q
		L	R
p	T	$2^*, 1^*$	$0, 0$
1-p	B	$0, 0$	$1^*, 2^*$

Welfare $(U_i^*(p^*, q^*), U_j^*(p^*, q^*))$ in Nash equilibrium (p^*, q^*):

- PNE $(1, 1)$: $(2, 1)$.
- PNE $(0, 0)$: $(1, 2)$.
- MNE $(2/3, 1/3)$:

$$\underbrace{\frac{2}{3} \cdot \frac{1}{3} \cdot (2, 1)}_{(T,L)} + \underbrace{\frac{2}{3} \cdot \frac{2}{3} \cdot (0, 0)}_{(T,R)} + \underbrace{\frac{1}{3} \cdot \frac{1}{3} \cdot (0, 0)}_{(B,L)} + \underbrace{\frac{1}{3} \cdot \frac{2}{3} \cdot (1, 2)}_{(B,R)} = \left(\frac{2}{3}, \frac{2}{3}\right).$$

- Large welfare losses to miscoordination!

Symmetric $n > 2$ player games: Symmetric MNE

Definition (Symmetric game)

A static game $\langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ is **symmetric** *if $A_i = A_j$ for all $i, j \in N$, and $u_i(a_i, a_j, \mathbf{a}_{-ij}) = u_j(a_j, a_i, \mathbf{a}_{-ij})$ for all (a_i, a_j) and $\mathbf{a}_{-ij} \equiv (a_k)_{k \neq i,j}$.*

Proposition (Symmetric MNE)

In any symmetric game with $|A_i|$ finite, there exists a symmetric MNE in which $\alpha_i^* = \alpha_j^*$ for all $i, j \in N$.

- The symmetric MNE could be a PNE.

Symmetric $n > 2$ player games: Stag Hunt

- $n > 2$ hunter-gatherers (players) coordinate to feed themselves. Each player can either participate in a (S)tag hunt, or stay home and (G)ather food: $A_i = \{S, G\}$ for each $i \in N$. A successful stag hunt, which requires $M > 1$ or more participants ($n \leq M$), yields total value $V > 2$. Denote # of hunters $\#S \equiv \#(a_i = S : i \in N)$.
- Payoffs to $i \in N$ are given by:

	$\#S \geq M$	$\#S < M$
S	V/(#S)	0
G	1	1

- PNE (1): $a_i^* = G$ for each $i \in N$.
  This PNE is Pareto inefficient...

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G	1	1

- PNE (2):
  - Case 1) $V > n$: $a_i^* = S$ for each $i \in N$.
  - Case 2) $V < n$: $a_i^* = S$ for exactly m^* players where $M/m^* > 1$ and $M/(m^* + 1) < 1$. This gives $\binom{n}{m^*}$ (i.e. *n choose* m^*) different PNE!

  Provided $V \notin \mathbb{Z}$, hunters realize value above 1: Pareto efficient.

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- Symmetric MNE: Let $p_i \equiv prob(a_i = S)$. To find symmetric $p_j = p^*$ for each $j \in N$, find p^* which leaves i indifferent between S and H:

$$f(p^*) \equiv \sum_{m=M-1}^{n-1} \binom{n-1}{m} p^{*s}(1-p^*)^{n-1-m} \frac{V}{m} = 1 \ldots$$

Symmetric $n > 2$ player games: Stag Hunt

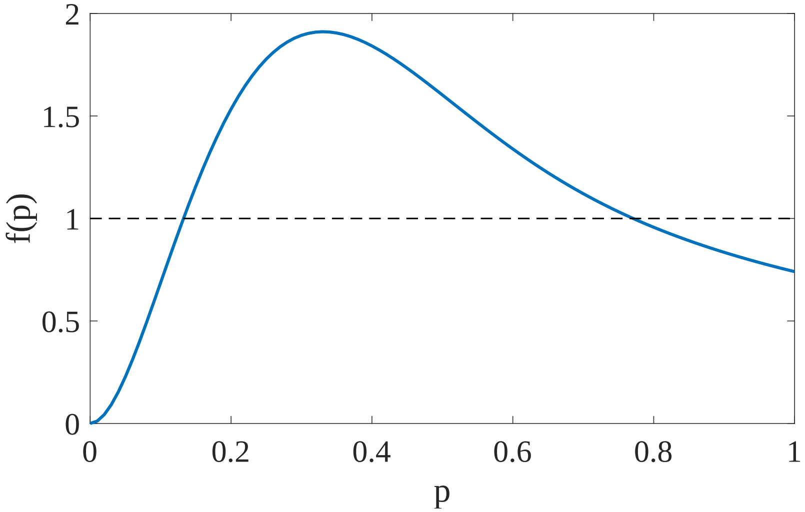


Figure: $f(p)$ for $n = 10$, $M = 3$ and $V = 20/3$.

Cournot Duopoly

- Two firms i and j compete by producing quantities $q_i \geq 0$ and $q_j \geq 0$, resp., under price function $P(q_i, q_j) = a - b(q_i + q_j)$, $a, b > 0$. Marginal cost of production is $c > 0$.
- Each firm maximizes profit given the other's production. For firm i:

$$\max_{q_i} (a - b(q_i + q_j))q_i - cq_i.$$

- First-order condition: $a - 2bq_i - bq_j - c = 0$.
- Solving for q_i gives firm i's best response:

$$q_i^*(q_j) = \begin{cases} \frac{a - bq_j - c}{2b} & \text{if } \frac{a-c}{b} > q_j \\ 0 & \text{otherwise} \end{cases}.$$

- Likewise, for firm j (finding $q_j^*(q_i)$)...

Cournot Duopoly

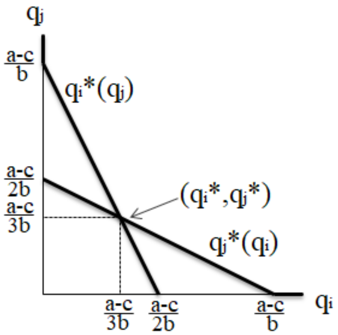


Figure: Nash equilibrium in Cournot Duopoly

- Unique PNE is $\left(\frac{a-c}{2b}, \frac{a-c}{2b}\right)$.

Cournot Duopoly

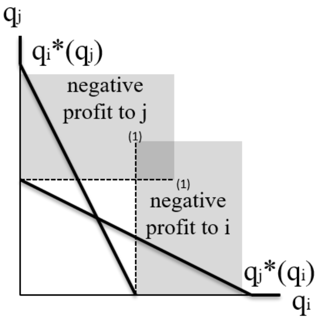
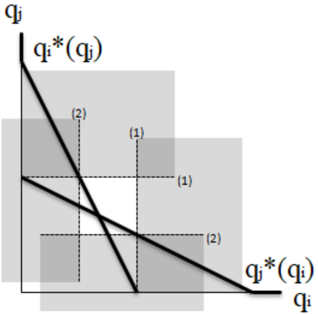


Figure: IESDS in Cournot Duopoly

Cournot Duopoly



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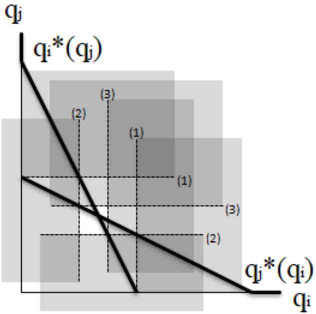

Figure: IESDS in Cournot Duopoly

Cournot Duopoly

- ...IESDS converges on the unique PNE in the Cournot duopoly.
- Consequently, no additional MNE exist (no MNE with mixing).
- Unique Nash is “robust” to dropping the PNE assumption that players’ hold correct beliefs of others’ strategies. Rather, they only need to know that all players are rational, and that this is common knowledge among the players (still a strong assumption?).

The Beauty contest revisited.

Consider the following game:

- Simultaneously, everyone choose an integer (a "guess") in the interval $[0, M]$, for some integer $M > 0$.
- The person whose guess is closest to $2/3$ of the average guess wins.
- Ties are broken uniformly at random.

IESDS in this game (allowing for mixed strategies):

- A guess of M never wins; mixing uniformly over $\{0, \dots, M-1\}$ wins with at least probability $1/(nM)$, thus a guess of M is strictly dominated.
- With M eliminated, A guess of $M-1$ never wins; mixing uniformly over $\{0, \dots, M-2\}$ wins with at least probability $1/(n(M-1))$, thus a guess of $M-1$ is strictly dominated....
- When only $\{0, 1, 2\}$ are left, a guess of 2 never wins; mixing uniformly over $\{0, 1\}$ wins with at least probability $1/(n2)$, thus a guess of 2 is strictly dominated.

$\Rightarrow$ All PNE involve all players guessing either 0 or 1.

Level-k strategy'

Definition 1 (Level-k strategy)

Take any static game Γ with best-responses $\{BR_i(\boldsymbol{\alpha}_{-i})\}_{i\in N}$. For any $i \in N$, denote $\boldsymbol{v}_{-i} = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$, where $v_j \in \Delta(A_j)$ gives the uniform distribution over actions A_j for each $j \in N \backslash \{i\}$.

- Level-0 strategy for $i \in N$: $\alpha_i^{[0]} = v_i$.
- Level-1 strategy for $i \in N$: $\alpha_i^{[1]} \in BR_i(\boldsymbol{v}_{-i})$...
- Level-k strategy for $i \in N$: $\alpha_i^{[k]} \in BR_i(\boldsymbol{\alpha}_{-i}^{[k]})$.

- Formally, Level-k strategies are given by the multilateral best-response dynamic starting from $\boldsymbol{v}$.
- Level-k strategy gives optimal play when all other players play according to level-$(k-1)$ strategy.
- Players are "bounded rational" because they believe that others believe that others believe...that players play a random strategy.

The Beauty contest and Level-k strategy

$round(x) \equiv$ nearest integer to $x \in \mathbb{R}$. In the "beauty contest" game:

- Level-0 strategy for $i \in N$:
  $\alpha_i^{[0]} = \upsilon_{-i}$, i.e. $\alpha_i^{[0]}(x) = \frac{1}{101}$ for each integer $x \in [0, 100]$.
- Level-1 strategy for $i \in N$: $\alpha_i^{[1]} = \alpha_i^D[x^{[1]}]$ where $x^{[1]}$ solves:

$$x^{[1]} = round(\frac{2}{3}\frac{50(n-1) + x^{[1]}}{n}) \ldots$$

- Level-k strategy for $i \in N$: $\alpha_i^{[1]} = \alpha_i^D[x^{[k]}]$ where $x^{[k]}$ solves:

$$x^{[k]} = round(\frac{2}{3}\frac{x^{[k-1]}(n-1) + x^{[k]}}{n}).$$

For $n = 21$:

k	1	2	3	4	5	6	7	8	9	...
$x^{[k]}$	33	22	14	9	6	4	3	2	1	1