Econ C103: Game Theory and Networks Module I (Game Theory): Lecture 2

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Readings:

- Osborne (2004) chapter 2
- Osborne and Rubinstein (1994) chapters 2.1-2.3, 2.4

Static games

Definition (Static game)

A static game (or "normal-form game", or "strategic game") is defined as a triplet $\langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$.

- $N = \{1, ..., n\}$: players,
- A_i: player i's (finite or infinite) action set,
- $u_i(\mathbf{a})$: i's vNM utility from action profile $\mathbf{a}=(a_1,...,a_n)\in \times_{k=1}^n A_k$.
- Denote $\mathbf{a} \equiv (a_1, \dots, a_n)$, and $A \equiv \times_{k=1}^n A_k$.
- Denote $\mathbf{a}_{-i} \equiv (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$, and $A_{-i} \equiv \times_{k=1, k \neq i}^n A_k$.

Strategies versus Actions:

- Action: any element from A_i for $i \in N$.
- Strategy: a "plan of action". In static games a strategy is a particular element from A_i , but notion of "strategy" is adaptable/more general.

2×2 strategic games

- 2×2 strategic games: 2 players, 2 actions each.
- $N = \{i, j\},$
- $A_i = \{T, B\}, A_j = \{L, R\},$
- $A_i \times A_j = \{(T, L), (T, R), (B, L), (B, R)\},\$

player
$$j$$

L

R

player i
 $T \parallel u_i(T, L), u_j(T, L) \mid u_i(T, R), u_j(T, R)$
 $B \parallel u_i(B, L), u_j(B, L) \mid u_i(B, R), u_j(B, R)$

Classic 2×2 games

Battle of the Sexes			
	L	R	
т	2 1	0.0	

Coordination Game			
	L	R	
Т	2, 2	0,0	
В	0,0	1, 1	

		Q	Т
•	Q	3, 3	0, 4
	Т	4, 0	1, 1

Hawk-Dove

	D	Н
D	3, 3	1,4
Н	4, 1	0, 0

Matching Pennies

	H	T
Н	1, -1	-1, 1
Т	-1, 1	1, -1

A 2×3 game: Rock-paper-scissors

	(R)ock	(P)aper	(S)cissors
(R)ock	0,0	-1, 1	1, -1
(P)aper	1, -1	0, 0	-1, 1
(S)cissors	-1,1	1, -1	0, 0

A 3-player game

• Ordering utilities by $(u_i(a_i, a_j, a_k), u_j(a_i, a_j, a_k), u_k(a_i, a_j, a_k))$:

Player k plays (I)n			
	Play	er j	
	L	R	
Player i	2, 0, 1	0, 0, 0	
	0,0,1	1, 2, 0	
Player k plays (O)ut			
Player i			

Player
$$j$$
L R

Player i
 $B \mid -1, 1, 0 \mid 1, -1, 1$

Best responses

Definition (Best response)

Given static game $\langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$, for each $i \in N$ and any profile $\mathbf{a}_{-i} \in A_{-i}$:

$$BR_i(a_{-i}) \equiv \{a_i \in A_i : u_i(a_i, \mathbf{a}_{-i}) \geq u_i(a_i', \mathbf{a}_{-i}), \ \forall a_i' \in A_i\}.$$

- $BR_i(\mathbf{a}_{-i})$ gives *i*'s best action(s) when others' are conjectured to play \mathbf{a}_{-i} .
- We use "*" to designate a best response...

Row players i's best responses in 2×2 games

Coordination Game
$$\begin{array}{c|cc} & L & R \\ \hline T & 2^*, 2 & 0, 0 \\ \hline B & 0, 0 & 1^*, 1 \\ BR_i(L) = T, \ BR_i(R) = B \end{array}$$

	Prisoner's Dilemma			
		Q	T	
	Q	3, 3	0, 4	
	Т	4*, 0	1*, 1	
٠,	(Δ) T $DD(T)$			

Matching Pennies

$$\begin{array}{c|c|c|c}
 & H & T \\
\hline
H & 1^*, -1 & -1, 1 \\
\hline
T & -1, 1 & 1^*, -1 \\
BR_i(H) = H, BR_i(T) = T
\end{array}$$

Dominance

Definition (Strict dominance)

Given static game $\langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$, an action $a_i' \in A_i$ is **strictly** dominated by some action $a_i \in A_i \setminus \{a_i\}$ iff:

$$u_i(a_i',\mathbf{a}_{-i}) < u_i(a_i,\mathbf{a}_{-i}),$$

for all $\mathbf{a}_{-i} \in A_{-i}$. Moreover, an action a_i is **dominant** for player i if it strictly dominates each $a_i' \in A_i \setminus \{a_i\}$.

Definition (Dominant solvability)

A static game $\langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ is dominant solvable if each $i \in N$ has a dominant action.

• If we use "+" to designate dominant actions...

Classic 2×2 games

Battle of the Sexes			
	L	R	
T	2 1	0.0	

Coordination Game			
	L	R	
Т	2, 2	0,0	
В	0,0	1, 1	

Prisoner's Dilemma		
	Q	T
Q	3, 3	0, 4 [†]
Т	Λ [†] Ω	1† 1†

Matching Pennies		
H		T
Н	1, -1	-1, 1
Т	-1, 1	1, -1

Dominance

Definition (Weak dominance)

Given static game $\langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$, an action $a'_i \in A_i$ is weakly dominated by some action $a_i \in A_i \setminus \{a_i\}$ iff:

- **1** $u_i(a_i', \mathbf{a}_{-i}) \le u_i(a_i, \mathbf{a}_{-i})$ for all $\mathbf{a}_{-i} \in A_{-i}$, and
- ② $u_i(a'_i, \mathbf{a}_{-i}) < u_i(a_i, \mathbf{a}_{-i})$ for some $\mathbf{a}_{-i} \in A_{-i}$.
 - Clearly, strict dominance implies weak dominance.

Pure-strategy Nash equilibrium

 ...Dominant solvability is a rare and strong (interesting?) property of a game. Nash equilibrium gives a "weaker" equilibrium notion.

Definition (Pure-strategy Nash equilibrium)

Given static game $\langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$, a strategy profile \mathbf{a}^* is a pure-strategy Nash equilibrium (PNE) iff for each $i \in N$:

$$u_i(a_i^*, \mathbf{a}_{-i}^*) \ge u_i(a_i', \mathbf{a}_{-i}^*), \ \forall a_i' \in A_i.$$

- $NE(\Gamma)$ denotes the set of PNE of $\Gamma = \langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$.
- Equivalent to the above, $a_i^* \in BR_i(\mathbf{a}_{-i}^*)$ for each $i \in N$.
- A dominant solvable game has a PNE: the profile of dominant strategies $(a_1^{\dagger}, \ldots, a_n^{\dagger})$.
- Inherent in PNE: players' hold correct beliefs of others' strategies.
- \bullet Games with PNE need not be dominant solvable. For 2 \times 2 games...

Pure-strategy Nash equilibrium in 2×2 games (in **bold**)

Bat	Battle of the Sexes		
L		R	
Т	2 *, 1 *	0,0	
В	0,0	1*, 2*	

Coordination Game		
L		R
Т	2*, 2*	0,0
В	0,0	1 *, 1 *

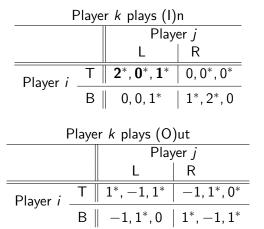
Prisoner's Dilemma			
	Q	Т	
Q	3, 3	0, 4 [†]	
T	4 [†] , 0	1 ⁺ , 1 ⁺	

Hawk-Dove		
	Н	D
Н	3, 3	1*, 4*
D	4 *, 1 *	0, 0

Matching Pennies		
	Н	T
Н	1^* , -1	$-1, 1^*$
Т	-1, 1*	1^* , -1

Pure-strategy Nash equilibrium in 3-player game (in **bold**)

• Ordering utilities by $(u_i(a_i, a_j, a_k), u_j(a_i, a_j, a_k), u_k(a_i, a_j, a_k))$:



A class of games: Zero-sum games

Definition (Zero-sum games)

A zero-sum game is static game $\langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ which satisfies $N = \{j, k\}$, and for each $\mathbf{a} \in A_i \times A_k$, $u_i(\mathbf{a}) + u_k(\mathbf{a}) = 0$.

- Matching-pennies, or Chess/Checkers/Go and two-player card games as dynamic games (later), give examples of zero-sum games.
- Fact: Zero-sum games can have PNE, e.g., (T, L) in the game:

- A rich theory behind zero-sum games, which is a topic of focus in John von Neumann and Oskar Morgenstern's 1944 book "The Theory of Games and Economics Behavior".
- Many features of games not captured by zero-sum games (e.g., coordination failures, public goods).

Iterated elimination of strictly-dominated strategies (IESDS):

• Take static game $\Gamma = \langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ such that action $a_i \in A_i$ is strictly dominated for player i. Construct:

$$\Gamma' = \langle N, \{A_i \setminus \{a_i\}, A_k\}_{k \in N \setminus \{i\}}, \{u_i\}_{i \in N} \rangle.$$

• Continue to eliminate strictly dominated strategies in this way, until you have a game without any dominated strategies for any player.

	L	C	R
T	0,0	1,1	1,100
M	1,2	2,1	2,0
В	0,3	3,2	0,1

Figure: IESDS

Iterated elimination of strictly-dominated strategies (IESDS):

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	L	C	R
T	0,0	1,1*	1,100
M	1*,2*	2,1	2*,0
В	0,3*	3*,2	0,1

Figure: IESDS

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	L	C	R	_
- T	0,0	1,1*	1,100	>
M	1*,2*	2,1	2*,0	
В	0,3*	3*,2	0,1	

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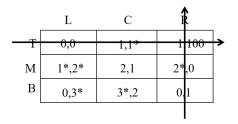


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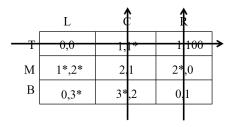


Figure: IESDS

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$$\Gamma' = \langle N, \{A_i \setminus \{a_i\}, A_k\}_{k \in N \setminus \{i\}}, \{u_i\}_{i \in N} \rangle.$$

 Continue to eliminate strictly dominated strategies in this way, until you have a game without any dominated strategies for any player.

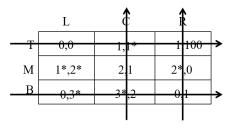


Figure: IESDS

Iterated elimination of strictly-dominated strategies (IESDS):

• Take static game $\Gamma = \langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ such that action $a_i \in A_i$ is strictly dominated for *some* player i. Construct:

$$\Gamma' = \langle N, \{A_i \setminus \{a_i\}, A_k\}_{k \in N \setminus \{i\}}, \{u_i\}_{i \in N} \rangle.$$

• Continue to eliminate strictly dominated strategies in this way, until you have a game without any dominated strategies for any player.

Proposition

Any Nash equilibrium profile a* survives IESDS.

• Fact 1: For game Γ' obtained from Γ via IESDS:

$$\mathbf{a}^* \in NE(\Gamma') \Leftrightarrow \mathbf{a}^* \in NE(\Gamma)$$
.

Iterated elimination of strictly-dominated strategies (IESDS):

• Take static game $\Gamma = \langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ such that action $a_i \in A_i$ is strictly dominated for *some* player i. Construct:

$$\Gamma' = \langle N, \{A_i \setminus \{a_i\}, A_k\}_{k \in N \setminus \{i\}}, \{u_i\}_{i \in N} \rangle.$$

• Continue to eliminate strictly dominated strategies in this way, until you have a game without any dominated strategies for any player.

Proposition

Any Nash equilibrium profile (a_i^*, \ldots, a_n^*) survives IESDS.

Iterated elimination of weakly-dominated strategies (IEWDS): as above but replace "strict" with "weak".

- Fact 2: For game Γ' obtained from Γ via IEWDS: $\mathbf{a}^* \in NE(\Gamma') \Rightarrow \mathbf{a}^* \in NE(\Gamma)$.
- Fact 2': However, the proposition (i.e. the converse of Fact 2) no longer holds (i.e. $\mathbf{a}^* \in NE(\Gamma)$ does not imply $\mathbf{a}^* \in NE(\Gamma')$)...

Iterated elimination of strictly-dominated strategies (IESDS):

• Take static game $\Gamma = \langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ such that action $a_i \in A_i$ is strictly dominated for *some* player i. Construct:

$$\Gamma' = \langle N, \{A_i \setminus \{a_i\}, A_k\}_{k \in N \setminus \{i\}}, \{u_i\}_{i \in N} \rangle.$$

 Continue to eliminate strictly dominated strategies in this way, until you have a game without any dominated strategies for any player.

Proposition

Any Nash equilibrium profile (a_i^*, \ldots, a_n^*) survives IESDS.

Iterated elimination of weakly-dominated strategies (IEWDS): as above but replace "strict" with "weak". Fact 2' counter example:

	L	R
Т	2*, 1*	0,0
В	1, 3*	3 *, 3 *

Best-response dynamic

Definition (Best-response dynamic)

For any static game $\Gamma = \langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$, a **[multilateral]/[unilateral] best-response dynamic** starting from strategy profile (a_1^0, \ldots, a_n^0) is the sequence $((a_1^t, \ldots, a_n^t))_{t=1}^{\infty}$ such that for each t > 0 and [each]/[some] $i \in N$:

$$a_i^t \equiv BR_i(\mathbf{a}_{-i}^{t-1}).$$

- Unilateral best-response dynamics usually filter through players' best responses via some fixed permutation of N (e.g., player 1, then 2,...).
- Fact 3: Clearly, pure-strategy Nash equilibria give stationary points ("sinks") in multilateral and unilateral best-response dynamics.
- Fact 4: No guarantee all multilateral/unilateral best-response dynamics converge, even when a pure-strategy Nash equilibrium exists...

Multilateral best-response dynamics in 2×2 games

Battle of the Sexes		
	L R	
Т	2 *, 1 *	0, 0
В	0,0 1*,2*	
$(B,L) \leftrightarrow (T,R)$		

Prisoner's Dilemma
$$\begin{array}{c|c|c}
 & Q & T \\
\hline
Q & 3,3 & 0,4^{\dagger} \\
\hline
T & 4^{\dagger},0 & 1^{\dagger},1^{\dagger} \\
\forall (a_i,a_j) \rightarrow (T,T)
\end{array}$$

Hawk-Dove

| H | D

| H | 3,3 |
$$\mathbf{1}^*, \mathbf{4}^*$$
| D | $\mathbf{4}^*, \mathbf{1}^*$ | 0,0 | $(T, L) \leftrightarrow (B, R)$

Matching Pennies

$$(T,L) \rightarrow (T,R) \rightarrow (B,R) \rightarrow (B,L) \rightarrow (T,L)$$