Econ C103: Game Theory and Networks

Lecture 3

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Fall 2019, UC Berkeley

Readings:

1. Osborne (2004) chapters 4.1-4.6
2. Osborne and Rubinstein (1994) chapters 2.3-2.4

Mixed strategies

Definition (Mixed strategy)

For finite set of actions A_i, a **mixed strategy** *of player i is a probability distribution $\alpha_i : A_i \mapsto [0, 1]$ satisfying $\sum_{a_i \in A_i} \alpha_i(a_i) = 1$.*

- Denote $\Delta(A_i) \equiv \{\alpha_i\}$. ΔA_i is exactly the $(|A_i| - 1)$-simplex.
- Denote $\Delta(A)$ the set of all lotteries over A, equal to the $(|A| - 1)$-simplex.
- Then, each profile of mixed strategies $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ maps to a lottery in $\Delta(A)$, precisely:

$$\boldsymbol{\alpha} \mapsto (A, P(\cdot)), \text{ where } P(\mathbf{a}) = \prod_{j \in N} \alpha_j(a_j), \ \forall \mathbf{a} \in A.$$

Mixed extensions

Definition (Mixed strategy extension)

For static game $\Gamma = \langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ *such that for each* $i \in N$ *the set of actions* A_i *is finite, a* **mixed strategy extension of** Γ *is given by* $\Delta\Gamma = \langle N, \{\Delta(A_i)\}_{i \in N}, \{U_i\}_{i \in N} \rangle$ *where for each* $i \in N$*:*

$$U_i(\boldsymbol{\alpha}) = \sum_{\mathbf{a} \in A} \left(\prod_{j \in N} \alpha_j(a_j) \right) u_i(\mathbf{a}).$$

- $\Delta\Gamma$ is itself a strategic game, with a larger action/strategy space.

Definition (Mixed-strategy Nash equilibrium)

Given mixed strategy extension $\Delta\Gamma$*, a strategy profile* $\boldsymbol{\alpha}^*$ *is a* **mixed-strategy Nash equilibrium (MNE)** *iff for each* $i \in N$*:*

$$U_i(\boldsymbol{\alpha}^*) \geq U_i(\alpha'_i, \boldsymbol{\alpha}^*_{-i}), \ \forall \alpha'_i \in \Delta(A_i).$$

Best responses in mixed extensions

- Fact: PNE are special (degenerate) cases of MNE: For any game Γ, the set of *PNE* of $\Gamma \subseteq$ the set of *MNE* of $\Delta\Gamma$.

Definition (Best response)

Given mixed extension $\langle N, \{\Delta(A_i)\}_{i\in N}, \{U_i\}_{i\in N}\rangle$, for each $i \in N$ and any profile $\boldsymbol{\alpha}_{-i} \in \times_{j\in N}\Delta(A_j)$:

$$BR_i(\boldsymbol{\alpha}_{-i}) \equiv \{\alpha_i \in \Delta(A_i) : U_i(\alpha_i, \boldsymbol{\alpha}_{-i}) \geq U_i(\alpha_i', \boldsymbol{\alpha}_{-i}),\ \forall \alpha_i' \in \Delta(A_i)\}.$$

- $BR_i(\alpha_{-i})$ gives i's best mixed-strategy when others' are conjectured to play $\boldsymbol{\alpha}_{-i}$.

Mixed-strategy Nash equilibrium (MNE)

- Fact 1: In MNE $\boldsymbol{\alpha}^*$, for each $i \in N$ and $a_i \in A_i$ s.t. $a_i \in supp(\alpha_i^*)$ (i.e. $\alpha_i^*(a_i) > 0$), a_i is a pure best response to $\boldsymbol{\alpha}_{-i}^*$.
- Equivalently, let $\alpha_i^D[a_i]$ denote the (D)egenerate strategy on a_i (i.e. $\alpha_i^D[a_i](a_i) = 1$ and $\alpha_i^D[a_i](a_i') = 0$ for $a_i' \in A_i \backslash \{a_i\}$). In MNE $\boldsymbol{\alpha}^*$, for every $i \in N$ and $a_i \in supp(\alpha_i^*)$:

$$U_i(\alpha_i^D[a_i], \boldsymbol{\alpha}_{-i}^*) = U_i(\alpha_i^*, \boldsymbol{\alpha}_{-i}^*).$$

- In words: Equilibrium strategies $\boldsymbol{\alpha}_{-i}^*$ induce i's indifference over the support of i's equilibrium strategy α_i^*.

Mixed-strategy Nash equilibrium (MNE)

- Fact 1: In MNE $\boldsymbol{\alpha}^*$, for each $i \in N$ and $a_i \in A_i$ s.t. $a_i \in supp(\alpha_i^*)$ (i.e. $\alpha_i^*(a_i) > 0$), a_i is a pure best response to $\boldsymbol{\alpha}_{-i}^*$.
- Equivalently, let $\alpha_i^D[a_i]$ denote the degenerate strategy on a_i (i.e. $\alpha_i^D[a_i](a_i) = 1$ and $\alpha_i^D[a_i](a_i') = 0$ for $a_i' \in A_i \backslash \{a_i\}$). In MNE $\boldsymbol{\alpha}^*$, for every $i \in N$ and $a_i \in supp(\alpha_i^*)$:
$$U_i(\alpha_i^D[a_i], \boldsymbol{\alpha}_{-i}^*) = U_i(\alpha_i^*, \boldsymbol{\alpha}_{-i}^*).$$
- *Proof: If $U_i(\alpha_i^D[a_i], \boldsymbol{\alpha}_{-i}^*) < U_i(\alpha_i^*, \boldsymbol{\alpha}_{-i}^*) \equiv U_i^*$, then i profitably deviates by mixing $\tilde{\alpha}_i(a_i) = 0$, and, for each $a_i' \in supp(\alpha_i^*) \backslash \{a_i\}$, mixing $\tilde{\alpha}_i(a_i') = \alpha_i^*(a_i') + \alpha_i^*(a_i)\frac{\alpha_i^*(a_i')}{1-\alpha_i^*(a_i)}$. Then, i's expected utility:*
$$U(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}^*) = U_i^* + \frac{\alpha_i^*(a_i)}{1-\alpha_i^*(a_i)}\underbrace{(U_i^* - U_i(\alpha_i^D[a_i], \boldsymbol{\alpha}_{-i}^*))}_{+} > U_i^*.$$
Thus, $\tilde{\alpha}_i$ gives a profitable deviation for i, a contradiction.

Mixed-strategy Nash equilibrium (MNE)

- Fact 1: In MNE $\boldsymbol{\alpha}^*$, for each $i \in N$ and $a_i \in A_i$ s.t. $a_i \in supp(\alpha_i^*)$ (i.e. $\alpha_i^*(a_i) > 0$), a_i is a pure best response to $\boldsymbol{\alpha}_{-i}^*$.
- Equivalently, let $\alpha_i^D[a_i]$ denote the degenerate strategy on a_i (i.e. $\alpha_i^D[a_i](a_i) = 1$ and $\alpha_i^D[a_i](a_i') = 0$ for $a_i' \in A_i \backslash \{a_i\}$). In MNE $\boldsymbol{\alpha}^*$, for every $i \in N$ and $a_i \in supp(\alpha_i^*)$:

$$U_i(\alpha_i^D[a_i], \boldsymbol{\alpha}_{-i}^*) = U_i(\alpha_i^*, \boldsymbol{\alpha}_{-i}^*).$$

- *Proof (continued): Alternatively, if $U_i(\alpha_i^D[a_i], \boldsymbol{\alpha}_{-i}^*) > U_i(\alpha_i^*, \boldsymbol{\alpha}_{-i}^*)$, then i profitably deviates by playing strategy $\alpha_i^D[a_i]$.*

Mixed-strategy Nash equilibrium in 2×2 games

- Denote $\alpha_i(T) = p$ and $\alpha_j(L) = q$. Consider "Battle of the Sexes":

		q	1-q
		L	R
p	T	2, 1	0, 0
1-p	B	0, 0	1, 2

- To find an MSN, first find p^* which makes j indifferent between L and R:

$$1p^* + 0(1-p^*) = 0p^* + 1(1-p^*)$$

$$\Leftrightarrow p^* = 2/3.$$

  Second, find q^* which makes i indifferent between T and B:

$$2q^* + 0(1-q^*) = 0q^* + 1(1-q^*)$$

$$\Leftrightarrow q^* = 1/3.$$

- This MSN $(p^*, q^*) = (2/3, 1/3)$ is an additional Nash equilibrium to $(1,1)$ (i.e. (T,L)) and $(0,0)$ (i.e. (B,R)).

Mixed-strategy Nash equilibrium in 2×2 games

- Denote $\alpha_i(T) = p$ and $\alpha_j(L) = q$.

		j	
		L	R
i	T	2 , 1	0 , 0
	B	0 , 0	1 , 2

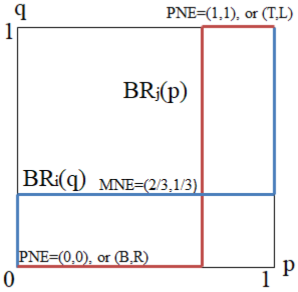

Figure: Two PNE, and one MNE in "Battle of the Sexes"

$$BR_i(q) = \begin{cases} 0 & if \ \ q < 1/3 \\ [0,1] & if \ \ q = 1/3 \\ 1 & if \ \ q > 1/3 \end{cases}, \quad BR_j(p) = \begin{cases} 0 & if \ \ p < 2/3 \\ [0,1] & if \ \ p = 2/3 \\ 1 & if \ \ p > 2/3 \end{cases}$$

- $NE(\Gamma) = \{(0,0), (2/3, 1/3), (1,1)\}$.

Mixed-strategy Nash equilibrium in 2×2 games

- Denote $\alpha_i(T) = p$ and $\alpha_j(L) = q$.

		j: L	j: R
i	T	1 , -1	-1 , 1
	B	-1 , 1	1 , -1

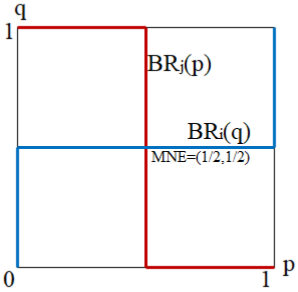

Figure: MNE in "Matching Pennies"

$$BR_i(q) = \begin{cases} 0 & \text{if } q < 1/2 \\ [0,1] & \text{if } q = 1/2 \\ 1 & \text{if } q > 1/2 \end{cases}, \quad BR_j(p) = \begin{cases} 1 & \text{if } p < 1/2 \\ [0,1] & \text{if } p = 1/2 \\ 0 & \text{if } p < 1/2 \end{cases}$$

- $NE(\Gamma) = \{(1/2, 1/2)\}$.

Mixed-strategy Nash equilibrium in 2×2 games

- Denote $\alpha_i(T) = p$ and $\alpha_j(L) = q$.

i \ j	L	R
T	2 , 2	0 , 3
B	3 , 0	1 , 1

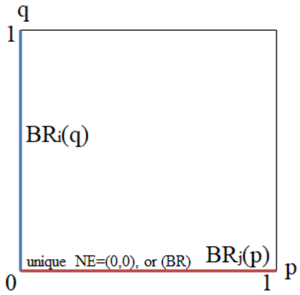

Figure: MNE in "Prisoners Dilemma"

$$BR_i(q) = 0, \quad BR_j(p) = 0.$$

- $NE(\Gamma) = \{(0, 0)\}$.

Mixed-strategy Nash equilibrium and Dominance

- Define an action $a_i \in A_i$ in $\Delta\Gamma$ as a **never-best-response** if for all $\boldsymbol{\alpha}_{-i} \in \times_{j \neq i} \Delta(A_j)$ there is some $a_i' \in A_i \backslash \{a_i\}$ such that:

$$U_i(\alpha_i^D[a_i'], \boldsymbol{\alpha}_{-i}) > U_i(\alpha_i^D[a_i], \boldsymbol{\alpha}_{-i}).$$

- Fact 2: If a_i is strictly dominated in Γ, then $a_i \in A_i$ is a never-best-response.

- Fact 3: For game Γ' obtained from Γ via IESDS:

$$\boldsymbol{\alpha}^* \in NE(\Delta\Gamma') \Leftrightarrow \boldsymbol{\alpha}^* \in NE(\Delta\Gamma).$$

- Fact 4: For game Γ' obtained from Γ via either IEWDS:

$$\boldsymbol{\alpha}^* \in NE(\Delta\Gamma') \Rightarrow \boldsymbol{\alpha}^* \in NE(\Delta\Gamma).$$

However, the converse of Fact 4 does not hold: $\boldsymbol{\alpha}^* \in NE(\Delta\Gamma)$ does not imply $\boldsymbol{\alpha}^* \in NE(\Delta\Gamma')$.

Fixed-point Theorems: Kakutani

Proposition (Kakutani)

Let $X \subseteq \mathbb{R}^n$ give a non-empty, compact (closed and bounded) and convex set. Take $f : X \mapsto X$ a set-valued function (correspondence) such that:

1. *for each $\mathbf{x} \in X$, $f(\mathbf{x})$ is non-empty and convex.*
2. *the graph of $f(\mathbf{x})$ (a subset of $X \times X$) is closed: for any $(\mathbf{x}^t, \mathbf{y}^t)_{t=1}^{\infty}$ such that $\mathbf{y}^t = f(\mathbf{x}^t)$ with $\mathbf{x}^t \to \mathbf{x}$ and $\mathbf{y}^t \to \mathbf{y}$, we have $\mathbf{y} = f(\mathbf{x})$.*

Then, there exists some $\mathbf{x}^* \in X$ such that $\mathbf{x}^* \in f(\mathbf{x}^*)$.

- Weaker necessary condition to "X bounded" and "f has closed graph": f is upper hemicontinous.

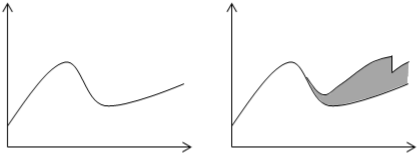

Figure: function (left), upper hemicontinous correspondence (right)

Fixed-point Theorems: Kakutani

- Nash (1950) applied Kakutani's fixed-point theorem to show equilibrium existence of MNE in finite games.
  - Construct correspondence $BR : \times_{i \in N} \Delta(A_i) \mapsto \Delta(A)$ by:

$$BR(\boldsymbol{\alpha}) \equiv (BR_i(\boldsymbol{\alpha}_{-i}))_{i \in N}.$$

  - BR satisfies the conditions of Kakutani's fixed-point theorem $\Rightarrow$ there exists some $\boldsymbol{\alpha}^* \in BR(\boldsymbol{\alpha}^*)$, giving a MNE.
- Generically, there exists an odd number of MNE in finite static games; in 2×2 games, there exists 1 or 3 MNE.

Fixed-point Theorems: Brouwer

- A special case of Kakutani's fixed-point theorem is the following.

Proposition (Brouwer)

Let $X \subseteq \mathbb{R}^n$ give a non-empty, compact (closed and bounded) and convex set. Take $f : X \mapsto X$ a single-valued continuous function. There exists some $\mathbf{x}^* \in X$ such that $\mathbf{x}^* = f(\mathbf{x}^*)$.

Fixed-point Theorems: Tarski

- A **partially ordered set** X has a binary relation $\leq$ satisfying:
  1. $\mathbf{x} \leq \mathbf{x}$, for each $\mathbf{x} \in X$ (reflexivity),
  2. $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{y} \leq \mathbf{x} \Rightarrow \mathbf{x} = \mathbf{y}$, for each $\mathbf{x}, \mathbf{y} \in X$ (antisymmetry),
  3. $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{y} \leq \mathbf{z} \Rightarrow \mathbf{x} \leq \mathbf{z}$, for each $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X$ (transitivity).
- A **complete lattice** X is a partially ordered set in which all subsets have both a "max" and a "min" in X. For $\mathbf{x}, \mathbf{x}' \in X \subseteq \mathbb{R}^n$:

$$\begin{aligned} \max(\mathbf{x}, \mathbf{x}') &= (\max(x_1, x_1'), \ldots, \max(x_n, x_n')), \\ \min(\mathbf{x}, \mathbf{x}') &= (\min(x_1, x_1'), \ldots, \min(x_n, x_n')). \end{aligned}$$

Proposition (Tarski)

Let $X \subseteq \mathbb{R}^n$ give a non-empty, compact (closed and bounded) and convex set. Take $f : X \mapsto X$ a single-valued continuous function such that $\mathbf{x} \geq \mathbf{x}' \rightarrow f(\mathbf{x}) \geq f(\mathbf{x}')$ (by vector-wise ordering). The non-empty set of $\mathbf{x}^* \in X$ satisfying $\mathbf{x}^* = f(\mathbf{x}^*)$ give a complete lattice.

- Tarski's fixed point theorem generalizes to any X a complete lattice.