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Assignment 5

Problem 1

Verify a fundamental matrix

$$det \begin{bmatrix} 0 & 1 \\ -\frac{2}{t^2} & \frac{2}{t} \end{bmatrix} = \frac{2}{t^2} > 0 \tag{1}$$

So the columns of the matrix are linearly independent.

$$\begin{aligned} LHS = \phi^{'}(t) &= \begin{bmatrix} 2t & 1 \\ 2 & 0 \end{bmatrix} \\ RHS = A(t)\phi(t) &= \begin{bmatrix} 0 & 1 \\ -\frac{2}{t^2} & \frac{2}{t} \end{bmatrix} \begin{bmatrix} t^2 & t \\ 2t & 1 \end{bmatrix} = \begin{bmatrix} 2t & 1 \\ 2 & 0 \end{bmatrix} \end{aligned} \tag{2}$$

Explain why it's a fundamental matrix even if $det[\phi(0)] = 0$

For a matrix to be the fundamental matrix of an equation on interval I, the equation shall at least be defined on every $t \in I$. In our case, A(t) is not defined when $t = 0$, so we don't have to worry about the determinant at $\phi(0)$.

Problem 2

The fundamental matrix, as given in the last question is

$$\Phi(t) = \begin{bmatrix} 2t & 1 \\ 2 & 0 \end{bmatrix} \tag{3}$$

Our solution can be written as

$$\phi(t) = \Phi(t)\Phi^{-1}(2)\phi(2) + \Phi(t)\int_2^t \Phi^{-1}(s)g(s)ds \tag{4}$$

$$\phi(t) = \begin{bmatrix} t^2 & t \\ 2t & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} t^2 & t \\ 2t & 1 \end{bmatrix} \int_2^t \begin{bmatrix} -\frac{1}{s^2} & \frac{1}{s} \\ \frac{2}{s} & -1 \end{bmatrix} \begin{bmatrix} s^4 \\ s^3 \end{bmatrix} ds \tag{5}$$

The solution matrix for the IVP is

$$\phi(t) = \begin{bmatrix} \frac{t^5}{4} + \frac{7}{2}t^2 - 7t \\ \frac{t^4}{4} + 7t - 7 \end{bmatrix} \tag{6}$$

Problem 3

Verify a fundamental matrix

Let $y^{'} = m \quad y = n$, the equation can be re-written as

$$\begin{bmatrix} n \\ m \end{bmatrix}^{'} = \begin{bmatrix} m \\ f(t) - p(t)m - q(t)n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix} \begin{bmatrix} n \\ m \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix} \tag{7}$$

Due to the fact that $\phi_i(t)$, i=1,2 are linearly independent

$$det \begin{bmatrix} \phi_i & \phi_2 \\ \phi_1^{'} & \phi_2^{'} \end{bmatrix} \neq 0 \tag{8}$$

and also $\phi_i(t)$ satisfies the equation (7), as

$$\phi^{''}(t) + p(t)\phi^{'}(t) + q(t)\phi(t) = f(t) \tag{9}$$

The matrix $\Phi = \begin{bmatrix} \phi_i & \phi_2 \\ \phi_1^{'} & \phi_2^{'} \end{bmatrix}$ is indeed a fundamental matrix

Solution for the non-homogeneous equation

For a generic ODE, the solution can be written as

$$\phi(t) = \Phi^(t)\Phi^{-1}(t_0)\phi(t_0) + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)g(s)ds \tag{10}$$

where $\Phi(t)$ is the fundamental matrix of the homogeneous equation.
In our case,

$$\phi(t_0) = 0 \qquad g(s) = \begin{bmatrix} 0 \\ f(s) \end{bmatrix} \tag{11}$$

The determinant of the fundamental matrix is Wronski $W(t)$, and the inverse of it is

$$\begin{bmatrix} \phi_2^{'} & -\phi_2 \\ -\phi_1^{'} & \phi_i \end{bmatrix} \tag{12}$$

As in our case, the initial value is a zero vector

$$\phi(t_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{13}$$

So we only have to take care about the second term.

$$\psi(t) = \int_{t_0}^{t} \frac{\begin{bmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1^{'}(t) & \phi_2^{'}(t) \end{bmatrix} \begin{bmatrix} \phi_2^{'}(s) & -\phi_2(s) \\ -\phi_1^{'}(s) & \phi_1(s) \end{bmatrix}}{W(s)} \begin{bmatrix} 0 \\ f(s) \end{bmatrix} ds \tag{14}$$

It can indeed be simplified as

$$\psi(t) = \int_{t_0}^{t} \frac{\phi_2(t)\phi_1(s) - \phi_1(t)\phi_2(s)}{W(s)} f(s)ds \tag{15}$$

Problem 4

The fundamental matrix of A can be calculated as

$$e^{tA} = \sum_{k=0}^{\infty} \frac{A^k}{k!} \tag{16}$$

Note that $A^m = 0, \forall m > n-1$.
When $m < n$, we have

$$\begin{aligned} e^{At} &= I + t\begin{bmatrix} 0 & 1 & \\ & \ddots & 1 \\ 0 & & 0 \end{bmatrix}_{n\times n} + \frac{t^2}{2!}\begin{bmatrix} 0 & 0 & 1 & \\ & & \ddots & 1 \\ & & & 0 \\ 0 & \dots & & 0 \end{bmatrix}_{n\times n} + \ldots + \frac{t^{n-1}}{(n-1)!}\begin{bmatrix} 0 & \dots & 0 & 1 \\ \vdots & \ddots & & 0 \\ & & & \\ 0 & \dots & & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \cdots & \frac{t^{n-1}}{(n-1)!} \\ \vdots & \ddots & \ddots & & & \frac{t^{n-2}}{(n-2)!} \\ & & & & & \vdots \\ & & & & & \frac{t^2}{2!} \\ \vdots & & & & & t \\ 0 & \dots & & & & 1 \end{bmatrix}_{n\times n} \end{aligned} \tag{17}$$

Problem 5

Notice that $A = \tilde{A} + 2I$, where $\tilde{A}$ is the original matrix in problem 4
We can use the properties of the fundamental matrix

$$\begin{aligned} e^{\tilde{A}+2I} &= e^{\tilde{A}}e^{2I} \\ e^{2I} &= \begin{bmatrix} e^{2t} & 0 & \dots & \\ 0 & e^{2t} & & 0 \\ \vdots & & \ddots & 0 \\ 0 & \dots & & e^{2t} \end{bmatrix} = e^{2t}I \end{aligned} \tag{18}$$

So the solution is

$$e^{At} = e^{2t}\begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \cdots & \frac{t^{n-1}}{(n-1)!} \\ \vdots & \ddots & \ddots & & & \frac{t^{n-2}}{(n-2)!} \\ & & & & & \vdots \\ & & & & & \frac{t^2}{2!} \\ \vdots & & & & & t \\ 0 & \dots & & & & 1 \end{bmatrix}_{n\times n} \tag{19}$$

Problem 6

First, carry out the eigen-decomposition of A

$$det\left|\lambda I - \begin{bmatrix} -3 & 1 & 7 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix}\right| = 0 \tag{20}$$

and get our eigen-values

$$\lambda_1 = -3 \quad \lambda_2 = 2 \quad \lambda_3 = 4 \tag{21}$$

Solve for three equations and get our eigenvectors,

$$\begin{bmatrix} 0 & -1 & -7 \\ 0 & -7 & 1 \\ 0 & 0 & -5 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{22}$$

$$\begin{bmatrix} 5 & -1 & -7 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{23}$$

$$\begin{bmatrix} 7 & -1 & -7 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{24}$$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix} \tag{25}$$

Due to the properties of the exponential operator,

$$e^{At} = exp(T\Lambda T^{-1}) = Te^{\Lambda t}T^{-1} \tag{26}$$

Also due to the fact that the non-zero linear combination of independent solutions are still independent solutions.
One of the fundamental matrix we can easily calculate is that

$$\Phi(t) = Te^{\Lambda t} = \begin{bmatrix} e^{-3t} & 3e^{2t} & e^{4t} \\ 0 & e^{2t} & 7e^{4t} \\ 0 & 2e^{2t} & 0 \end{bmatrix} \tag{27}$$