Problem 1

Proof : To prove that for all the solutions of the IVP, Eq.2 (in the question) is satisfied, we can integrate the equation on both sides for two times:

$$\int_0^t \int_0^s y''(\tau)d\tau ds = \int_0^t \int_0^s -g(\tau, y)d\tau ds \tag{1}$$

For the LHS,

$$LHS = \int_0^t \int_0^s y''(\tau)d\tau ds = \int_0^t (y'(s) - z_0)ds = y(t) - z_0 t - y_0 \tag{2}$$

For the RHS,

$$RHS = \int_0^t \int_0^s -g(\tau, y)d\tau ds = \int_0^t \left\{ \int_\tau^t ds \right\} - g(\tau, y)d\tau \tag{3}$$

(Note that the reason we can do this is that we can change the order of integration, as $g(.)$ is a function independent of s, and the domain D of S is $S \in [\tau, t]$)

$$RHS = \int_0^t -(t-\tau)g(\tau, y(\tau))d\tau \tag{4}$$

We can thus get our results:

$$y(t) = y_0 + z_0 t - \int_0^t (t-\tau)g(\tau, y(\tau))d\tau \tag{5}$$

Conversely, to prove that for all $\Pi(t)$ that satisfies Eq.(2), they're solutions of the IVP, we might differentiate the equation on both sides.

$$\begin{aligned} y'' &= [-\int_0^t (t-s)g(s, y(s))ds]'' \\ &= [-\int_0^t \frac{\partial (t-s)g(s, y(s))}{\partial t}]' \\ &= [-\int_0^t g(s, y(s))ds]' \\ &= -g(t, y(t)) \end{aligned} \tag{6}$$

Note that here we used the Leibniz integral rule.

$$\frac{d}{dt}\int_0^t H(t,s)ds = H(t,t) + \int_0^t \frac{\partial H(t,s)}{\partial t}ds \tag{7}$$

Problem 2

Proof : For the first order derivative:

$$\begin{aligned} y^{'}(t) &= ie^{it} - \frac{d}{dt}\int_{\infty}^{t} sin(t-s)\frac{y(s)}{s^2}ds \\ &= ie^{it} - \int_{\infty}^{t} cos(t-s)\frac{y(s)}{s^2}ds \end{aligned} \tag{8}$$

Then take the second order derivative,

$$y^{''} = -e^{it} - \frac{y(t)}{t^2} + \int_{\infty}^{t} sin(t-s)\frac{y(s)}{s^2}ds = -e^{it} - \frac{y(t)}{t^2} - \int_{t}^{\infty} sin(t-s)\frac{y(s)}{s^2}ds \tag{9}$$

Also note that

$$y(t) = e^{it} + \int_{t}^{\infty} sin(t-s)\frac{y(s)}{s^2}ds \tag{10}$$

It's obvious that

$$(1+\frac{1}{t^2})y + y^{''} = 0 \tag{11}$$

Problem 3.1

Proof : Apparently when $n = 1$, $|y_1(t) - y_0(t)| = |cost + isint| = 1$
When $t \geq 1$, if $\exists\, t = n$, that satisfies:

$$|y_n(t) - y_{n-1}(t)| \leq \frac{1}{(n-1)!t^{n-1}} \tag{12}$$

Consider the case when t = n+1,

$$\begin{aligned} |y_{n+1}(t) - y_n(t)| &= \left| e^{it} + \int_{t}^{\infty} sin(t-s)\frac{y_n(s) - y_{n-1}(s)}{s^2}ds - e^{it} \right| \\ &\leq \left| \int_{t}^{\infty} \frac{1}{s^2(n-1)!s^{n-1}}ds \right| \\ &= \int_{t}^{\infty} \frac{1}{(n-1)!s^{n+1}}ds \\ &= \frac{1}{n!t^n} \end{aligned} \tag{13}$$

Q.E.D.

Problem 3.2

Proof :

$$\begin{aligned}\lim_{n\to\infty} y_n(t) &= \sum_{n=1}^{\infty} (y_n(t) - y_{n-1}(t)) \\ &\leq \sum_{n=1}^{\infty} \frac{1}{(n-1)!t^n}\end{aligned} \tag{14}$$

Apparently, for all $n \geq 4$ and $t \geq 1$

$$\frac{1}{(n-1)!t^n} \leq \frac{1}{n^2} \tag{15}$$

Note that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is uniformly convergent. So the original serie converges uniformly.

Problem 3.3

Given that

$$y(t) = e^{it} + \int_t^{\infty} sin(t-s)\frac{y(s)}{s^2}ds \tag{16}$$

Take the limit on both sides,

$$\lim_{n\to\infty} y_n(t) = e^{it} + \lim_{n\to\infty} \int_t^{\infty} sin(t-s)\frac{y_{n-1}(s)}{s^2}ds \tag{17}$$

As we've proved the continuity and uniform convergence, we can use the lemma

$$\lim_{n\to\infty} y_n(t) = e^{it} + \int_t^{\infty} sin(t-s)\frac{\lim_{n\to\infty} y_{n-1}(s)}{s^2}ds \tag{18}$$

$$y(t) = e^{it} + \int_t^{\infty} sin(t-s)\frac{y(s)}{s^2}ds \tag{19}$$

Hence $y(t)$ satisfies the integral equation 1.

Problem 4

4.1 Verification

Apparently, $y' = 3y^{\frac{2}{3}} = 0$ for $\forall t \leq c$, and $y' = 3(t-c)^2 = 3y^{\frac{2}{3}}$ for $\forall t \geq c$.
As long as $c \geq 0$, the initial value $y(0) = 0$ is contained in the our domain. So it is a solution of the given IVP. (Note that there're actually infinite number of solutions given by the equation above.)

4.2 Prove that $y^{\frac{2}{3}}$ is not Lipschitz

Proof : If there's a positive value L that satisfies

$$\left|y_1^{\frac{2}{3}} - y_2^{\frac{2}{3}}\right| \leq L\left|y_1 - y_2\right| \quad \forall\, y_1, y_2 \in R \tag{20}$$

Let $y_2 = 0$. Basically we can always find y_1 where

$$0 < y_1 < L^{-3} \tag{21}$$

and that it violates the Lipschitz condition. So $y^{\frac{2}{3}}$ is not Lipschitz.

4.3 Local Existence Theorum?

Yes we can! Because for the local existence theorem to hold, we only need our original function to be continuous on our domain (which is R in this case).