Econ C103: Game Theory and Networks

Module I (Game Theory): Lecture 1

Instructor: Matt Leister

University of California, Berkeley

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Readings:

1. Osborne (2004) chapters 1-2.1, 4.12
2. Osborne and Rubinstein (1994) chapters 1.1-2.1 (*note: the authors define a "preference relation" to be complete and transitive; in this class, we call these "rational preference relations"*)

Preferences

- Binary relationship on X: set of ordered pairs (x, y) from set X.

Definition (Preference relation)

A **preference relation** $\succsim$ *is binary relation over outcomes* X*.*

- X: e.g., types of fruit, movies, bundles of fruit and movies (e.g, (3 apples, "The Other Guys")), or profiles of players' "actions" (later).

Notation:

- $x \succsim y$: $\succsim$ includes (x, y) ("x is preferred to y").
- $x \sim y$: $x \succsim y$ *and* $y \succsim x$ ("x is indifferent to y").
- $x \succ y$: $x \succsim y$ *and* not $y \succsim x$ ("x is strictly preferred to y").

Rationality

- $\succsim$ **complete**: either $x \succsim y$ OR $y \succsim x$ (*one or both*) for each $x, y \in X$.
- $\succsim$ **transitive**: for any $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.
- A preference relation is **rational** when it is transitive and complete.

- Take $X = \{$(A)pple, (O)range, (B)anana, (K)iwi$\}$.
  Consider $\succsim$ s.t. $A \succsim O$, $A \succsim K$, $K \succsim B$, $A \succsim B$, $B \succsim O$ and $K \succsim K$.

	A	O	B	K
A		●	●	●
O				
B		●		
K			●	●

Figure: A preference relation

- complete? transitive? Why?

Utility Functions

Definition (Utility function)

A **utility function** $u : X \mapsto \mathbb{R}$ *represents* $\succsim$ *when:*

$$x \succsim y \Leftrightarrow u(x) \geq u(y).$$

Gereard Debreu (1921-2004, Cal professor, and Nobel Laureate) showed:

Proposition

If $\succsim$ *is rational and continuous (see last slide for definition of continuous), then there is some continuous utility function* u *that represents* $\succsim$*.*

In the above proposition, the gray text is for the case of infinite X.

- $\Rightarrow$ when $\succsim$ satisfies these three axioms, it is sufficient to study u. That is, the shape of u gives all of the information within $\succsim$.

Utility Functions: general properties

Facts:

- If there exists some utility function (continuous or not) that represents $\succsim$, then $\succsim$ is rational.
- Monotone strictly increasing transformations preserve preferences over certain (i.e. "without risk") outcomes.

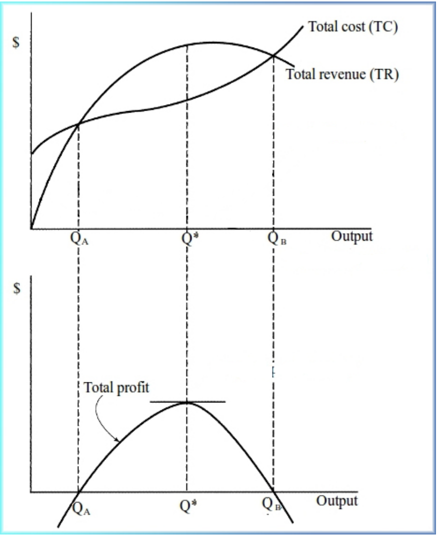
Proposition

For any u a utility representation of $\succsim$, and any increasing function $f : \mathbb{R} \mapsto \mathbb{R}$, the composition $f \circ u$ (i.e. $f(u(\cdot))$) also represents $\succsim$.

Firm profits as utility functions

- Assume: demand for output Q is decreasing in its price P, giving "inverse demand function" $P(Q)$.
- Assume: concave total revenue $TR(Q) = P(Q)Q$.
- Assume: increasing total production cost $TC(Q) \geq 0$.
- Total profit function equal to revenue minus cost: $P(Q)Q - TC(Q)$.

Firm profits as utility functions



Social preferences

- "Other-regarding preferences": $\succsim$ depends on others' payoffs/profits.
- Assume: two players 1 and 2 and two outcomes/states a and b.

  Monetary payoffs:
  outcome a: \$1 to player 1, \$0 to player 2,
  outcome b: \$0 to player 1, \$1 to player 2.
- Player 1 cares about player 2's monetary payoff according to:

$$\begin{aligned} u_1(a) &= w1 + (1-w)0 = w, \ \textit{and} \\ u_1(b) &= w0 + (1-w)1 = (1-w), \end{aligned}$$

  with $w \in [0,1]$. If $w < .5$, player 1 prefers b.
- Generally: take $\sum_{j \in \{agents\}} w_j^i = 1$, $w_j^i \geq 0$, then for outcome a:

$$u_i(a) = \sum_{j \in \{agents\}} w_j^i PO_j(a),$$

  where $PO_j(a)$ gives player j's monetary payoff in outcome a.

Probabilities, lotteries and expectations (see primer notes)

- A *probability distribution over a set* X is a function $P : X \mapsto [0, 1]$ such that $\sum_{x \in X} P(x) = 1$ if X finite; $\int_{x \in X} P(x)dx = 1$ if X infinite.
- We refer to a couple $(X, P(\cdot))$ as a "*lottery*" over X.
- $\Delta(X)$ denotes the set of all lotteries $(X, P(\cdot))$ over X.

Definition (Expectation)

For finite X*, lottery* $(X, P(\cdot))$*, and function* $f : X \mapsto \mathbb{R}$*, the* **expectation of** f *is defined as:*

$$\mathbb{E}_P[f(x)] \equiv \sum_{x \in X} P(x)f(x)$$

If instead X *is infinite, the* **expectation of** f *is defined as:*

$$\mathbb{E}_P[f(x)] \equiv \int_{x \in X} f(x)P(x)dx$$

- In this class, we (almost) always consider finite X.

vNM Expected Utilities

- Risk preferences: when $\succsim$ is defined over the set of lotteries $\Delta(X)$.

Definition (Expected utility)

For utility u, the **expected utility** *from lottery $(X, P(\cdot))$ is $\mathbb{E}_P[u(x)]$.*

John von Neumann (1903-1957) and Oskar Morgenstern (1902-1977), in "Theory of Games and Economic Behavior" showed:

Proposition (vNM utility representation)

If $\succsim$ over $\Delta(X)$ is rational, continuous (see last slide) and satisfies IIA (see last slide), then there is some measurable $u : X \mapsto \mathbb{R}$ that represents $\succsim$:

$$\mathbb{E}_P[u(x)] \geq \mathbb{E}_{P'}[u(x)] \Leftrightarrow (X, P(\cdot)) \succsim (X, P'(\cdot)).$$

In the above proposition, the gray text is for the case of infinite X.

Utility Functions: general properties

Fact:

- Affine increasing transformations preserve preferences over lotteries.

Proposition

For any vNM utility representation u of $\succsim$ over $\Delta(X)$, and any increasing function $f(z) = a + bz$ for $b > 0$, the composition $f \circ u$ (i.e. $f(u(\cdot))$) also represents $\succsim$.

Risk preferences

- We refer to u as the "Bernoulli" or "vNM" function.
- Risk preferences are captured by the concavity/convexity of u.
- *Example*: Three outcomes $x \in X = \{a, b, c\} \subseteq \mathbb{R}$, $a < b < c$, and take $(X, P(\cdot))$ s.t. $P(b) = 0$ and $\mathbb{E}[x] = P(a)a + P(c)c = b$. Receiving b with probability 1 (which yields certain utility $u(b)$) [is]/[is not] preferred to $(X, P(\cdot))$ in the [left]/[right] figure:

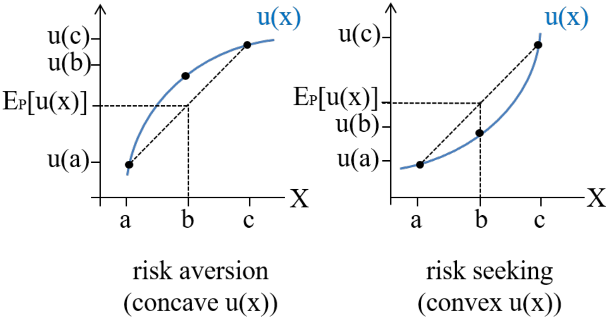


Figure: Risk preferences over three outcomes

From Decision Theory to Game Theory

- Game theory is the study of strategic interaction between players, while *taking their preferences as given*.
- We study games played by *social expected utility maximizers*.

Definition

A **static game** *(or "***normal-form game***", or "***strategic game***") is defined as the triplet* $\langle N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$.

- $N = \{1, ..., n\}$*: players,*
- A_i*: player i's (finite or infinite) action set,*
- $u_i(\mathbf{a})$*: i's vNM utility from action profile* $\mathbf{a} = (a_1, ..., a_n) \in \times_{k=1}^{n} A_k$.

- "X" now becomes the set of action profiles $\times_{k=1}^{n} A_k$.
- Players may be people, firms, even algorithms with well-defined preferences/objectives, represented by utility functions.

Formalities (for the case of infinite X)

- $\succsim$ over X is **continuous** if for any sequence of pairs $((x^t, y^t))_{t=1}$ where $x^t \succsim y^t$ for each t, $x^t \to x \in X$ and $y^t \to y \in X$, then $x \succsim y$ ("preferences do not jump").

- $\succsim$ over ΔX is **continuous** if for any $x, y, z \in \Delta X$ where $x \succsim y \succsim z$, there is some $t \in [0, 1]$ such that $tx + (1-t)z \sim y$ ("preferences over lotteries do not jump").

- $\succsim$ over ΔX satisfies **Independence of Irrelevant Alternatives** (IIA) if for any $x, y, z \in \Delta X$ where $x \succsim y$, then:

$$tx + (1-t)z \succsim ty + (1-t)z$$

for any $t \in [0, 1]$ ("preferences over lotteries remain intact when mixing-in other lotteries").