Problem 1

(a)

To demonstrate the equivalence of L_1 and L_2 norm, we aim to prove the following:

$$\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2 \tag{1}$$

where $L_1\, norm = \sum\limits_{i=1}^{n} |x_i|$, and $L_2\, norm = \sqrt{\sum\limits_{i=1}^{n} |x_i|^2}$
For the left part, square both sides and take the subtraction,

$$(\sum_{i=1}^{n} |x_i|)^2 - \sum_{i=1}^{n} |x_i|^2 = \sum_{i=1}^{n} |x_i|^2 - \sum_{i=1}^{n} |x_i|^2 + 2 \sum_{\{i,j\}\setminus(i=j)} |x_i||x_j| \geq 0 \tag{2}$$

So $\|x\|_2 \leq \|x\|_1$. For the right part, use the Cauchy-Swarts Inequality:

$$\frac{1}{n}\sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} |x_i|^2 \geq |\sum_{i=1}^{n} x_i|^2 \tag{3}$$

So $\sqrt{n}\|x\|_2 \geq \|x\|_1$, Q.E.D.

(b)

$$\begin{aligned} |AB| &= \sum_{i=1}^{n}\sum_{j=1}^{n} |\sum_{k=1}^{n} a_{ik}b_{kj}| \\ |A| &= \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}| \\ |B| &= \sum_{i=1}^{n}\sum_{j=1}^{n} |b_{ij}| \end{aligned} \tag{4}$$

To prove that $|AB| \leq |A||B|$, first note that

$$\sum_{i=1}^{n} |x_i| \sum_{j=1}^{n} |y_j| \geq \sum_{i=1}^{n} |x_i y_i| \tag{5}$$

Then rewrite $|A||B|$ as

$$\begin{aligned} |A||B| &= \sum_{i=1}^{n}\sum_{j=1}^{n}\{(|x_{i1}| + |x_{i2}| + ... + |x_{in}|)(|y_{1j} + |y_{2j}| + ... + |y_{nj}||)\} \\ &\geq \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} |x_{ik}y_{kj}| \geq \sum_{i=1}^{n}\sum_{j=1}^{n} |\sum_{k=1}^{n} x_{ik}y_{kj}| = |AB| \end{aligned} \tag{6}$$

Problem 2

Write the equation with the a priori estimate in the integral form:

$$
\begin{aligned}
|\phi(t)| &= |y_0 + \int_{t_0}^{t} [A(s)\phi(s) + g(s)]ds| \\
&\leq |y_0| + \int_{t_0}^{t} |A(s)||\phi(s)| + |g(s)|ds \\
&\leq |y_0| + \int_{t_0}^{t} C_A|\phi(s)|ds + (b-a)C_g
\end{aligned} \tag{7}
$$

where $C_A > 0$ is the upper-bound constant matrix for $|A|$, and $C_g > 0$ is the upper bound for $|g(t)|$. Also, to simplify our notations, denote $K = |y_0| + (b-a)C_g$, we have

$$
|\phi(t)| \leq K + \int_{t_0}^{t} C_A|\phi(s)|ds \tag{8}
$$

Use Gronwall inequality,

$$
|\phi(t)| \leq Ke^{\int_{t_0}^{t} C_A ds} \leq Ke^{\int_a^b C_A ds} = Ke^{(b-a)C_A} \tag{9}
$$

The RHS is independent of t, hence it's indeed bounded, thus a priori estimate.

Problem 3

We can rewrite the determinant as

$$
B_i = \begin{vmatrix}
\phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\
\vdots & \vdots & \vdots & \vdots \\
\sum_{k=1}^{n} A_{ik}\phi_{k1} & \sum_{k=1}^{n} A_{ik}\phi_{k2} & \cdots & \sum_{k=1}^{n} A_{ik}\phi_{kn} \\
\vdots & \vdots & \vdots & \vdots \\
\phi_{n1} & \phi_{n2} & \cdots & \phi_{nn}
\end{vmatrix} \tag{10}
$$

For each row j ($j \neq i$) subtract the i-th row by $j * Aij$, this won't change the value of the determinant

$$
B_i = \begin{vmatrix}
\phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\
\vdots & \vdots & \vdots & \vdots \\
A_{ii}\phi_{i1} & A_{ii}\phi_{i2} & \cdots & A_{ii}\phi_{in} \\
\vdots & \vdots & \vdots & \vdots \\
\phi_{n1} & \phi_{n2} & \cdots & \phi_{nn}
\end{vmatrix} = A_{ii}det(\Phi) \tag{11}
$$

where Φ is a fundamental solution of the linear system.

Problem 4

Expand the linear system into the normal form:

$$\begin{bmatrix} y \\ y' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} \tag{12}$$

This gives us

$$y'' + a_1 y' + a_2 y = 0 \tag{13}$$

Given that $r_1\,, r_2$ are two distinct roots of $z^2 + a_1 z + a_2 = 0$ The general solution for Eq.13 is

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t} \tag{14}$$

It'll be easy to verify that $e^{r_1 t}$ and $e^{r_2 t}$ are independent and both solutions of Eq.13 because

$$(e^{r_i t})'' + a_1 (e^{r_i t})' + a_2 e^{r_i t} = r_i^2 e^{r_1 t} + r_i a_1 e^{r_i t} + a_2 e^{r_i t} \equiv 0 \tag{15}$$

where r_i $(i = 1, 2)$ are the solution of eigen-equations.
So the original matrix is indeed a fundamental matrix.

Problem 5

If we continue to use the example in Problem 4. An example will be

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{16}$$

Note that

$$C \begin{bmatrix} exp(r_1 t) & exp(r_2 t) \\ r_1 exp(r_1 t) & r_2 exp(r_2 t) \end{bmatrix} = \begin{bmatrix} exp(r_1 t) & exp(r_2 t) \\ 0 & 0 \end{bmatrix} \tag{17}$$

Apparently, $\vec{\psi}(t) = [exp(r_1 t), 0]^T$ is not a solution to the original function, because

$$\frac{d}{dt}\{exp(r_1 t)\} \not\equiv 0 \tag{18}$$