Econ C103: Game Theory and Networks

Module I (Game Theory): Lecture 12

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Readings:

1. Osborne (2004) Section 9.6

Auctions with asymetric information: i.i.d. values

- $n > 1$ bidders in an auction have independent and identically distributed (i.i.d.) values for a single non-divisible good.
- Each v_i is distributed via continuous increasing cdf $F(\cdot)$, $\underline{v} \leq v_i \leq \overline{v}$.
- Private signals equal private valuations.
- Auction mechanism: $P(\mathbf{b})$ denotes price paid by winning bidder.
- Payoffs given i's value $v_i \geq 0$ and bids $\mathbf{b}$ is:

$$u_i(v_i, \mathbf{b}) = \begin{cases} (v_i - P(\mathbf{b}))/m & if: \begin{array}{c} b_j \leq b_i, \ \forall j \neq i, \\ \& \ b_j = b_i \ for \ m \ players \end{array} \\ 0 & otherwise \end{cases}.$$

- Strategies $S_i = \{b_i(v) : v \in [\underline{v}, \overline{v}]\}$; restrict $b_i(v)$ to be continuous increasing $\Rightarrow$ inverse b_i^{-1} exists for each $i \in N$.
- Denote the probability of winning given bid b and strategies $\mathbf{b}_{-i}$:

$$\phi(b|\mathbf{b}_{-i}) = \prod_{j \neq i} F(b_j^{-1}(b)).$$

- If $b_j(v) = b^*(v)$ for each $j \neq i$, $\phi(b|\mathbf{b}_{-i}) = (F(b^{*-1}(b))^{n-1}$.

Auctions with asymetric information: i.i.d. values

First-price auction: $P(\mathbf{b})$ = highest bid.

- Expected payoff to i given $(b, v, (b_j(\cdot))_{j\neq i})$ is:
  $$\begin{aligned} &\mathbb{E}_{\mathbf{v}_{-i}}[u_i\left(v_i, (b, (b_j(v_j))_{j\neq i}\right)] \\ &\qquad = \phi(b|\mathbf{b}_{-i})\left(v_i - \mathbb{E}_{\mathbf{v}_{-i}}[P(\mathbf{b})|b_j(v_j) < b_i, \forall j \neq i]\right) \\ &\qquad \overset{1P}{=} (F(b^{*-1}(b)))^{n-1}(v_i - b). \end{aligned}$$
- First-order condition for optimal $b^*(v_i)$ (by inverse function theorem):
  $$\frac{(n-1)(F(b^{*-1}(b)))^{n-2}F'(b^{*-1}(b))}{b^{*\prime}(v_i)}(v_i - b^*(v_i)) = (F(v_i))^{n-1}$$
  $$\Leftrightarrow (n-1)F'(b^{*-1}(b))(v_i - b^*(v_i)) = F(v_i)b^{*\prime}(v_i).$$
- If $\underline{v} = 0$, $\overline{v} = 1$, $F(v) = v$, & assume $b^*(v_i) = s^* v_i$, f.o.c. becomes:
  $$(n-1)(v_i - s^* v_i) = v_i s^* \ \Leftrightarrow\ s^* = \frac{n-1}{n}.$$
- Conclusion: each player i shades their bid by v_i/n to attempt to acquire good at a bargain (with a probability-of-winning tradeoff).

Auctions with asymmetric information: i.i.d. values

First-price auction: $P(\mathbf{b})$ = highest bid.

- $v^{[k]} \equiv k$'th order statistic (i.e. k'th highest value among n values; $v^{[1]} \geq v^{[2]} \ldots$). $v^{[k]}$ has cdf $F^{[k]}(\cdot)$ (can look up form to $F^{[k]}(\cdot)$).
- Expected revenue to auctioneer:

$$ER(P) \equiv \mathbb{E}[P(\mathbf{b}^*)] \stackrel{1P}{=} ER(1P) \equiv \int_{\underline{v}}^{\overline{v}} b^*(v) dF^{[1]}(v).$$

- If $\underline{v} = 0$, $\overline{v} = 1$ and $F(v) = v$:

$$\begin{aligned} ER_{1P} &= \int_0^1 \frac{n-1}{n} v n v^{n-1} dv \\ &= \frac{n-1}{n+1} v^{n+1} \Big|_0^1 \\ &= \frac{n-1}{n+1} \text{ (this is increasing in } n\text{).} \end{aligned}$$

- Larger $n \rightarrow$ more competitive bidding $\rightarrow$ greater expected revenue.

Auctions with asymmetric information: i.i.d. values

Second-price auction: $P(\mathbf{b})$ = second-highest bid.

- Bidding your value: $b(v_i) = v_i$ is weakly dominant (for almost all v_i):
    - Denote $\bar{b}_{-i}$ the highest bid among others' bids.
    - $\mathbf{b}_{-i}$ case 1: $\bar{b}_{-i} < v$, then $b(v) = v$ clearly optimal.
    - $\mathbf{b}_{-i}$ case 2: $\bar{b}_{-i} > v$, then any $b(v) \leq v$ optimal.
- Therefore, $b^*(v_i) = v_i$ gives BNE in weakly dominant strategies.
- Expected revenue to auctioneer:

$$ER(P) \overset{2P}{=} ER(2P) \equiv \int_{\underline{v}}^{\overline{v}} vdF^{[2]}(v).$$

- If $\underline{v} = 0$, $\overline{v} = 1$ and $F(v) = v$, then it is an easy to show Fact:

$$ER(2P) \;=\; \int_0^1 vdF^{[2]}(v) \;=\; \frac{n-1}{n+1} \;=\; ER(1P).$$

Auctions w/ asymmetric information: revenue equivalence

Other auction mechanisms:

3. Third price auction: $P(\mathbf{b})$ = third-highest bid.
4. All-pay auction: item allocated to highest bidder, and each bidder pays their bid even if they don't win.

- Direct revelation auction: no incentive to misreport your value (different to bid shading, as values not reported in 1'st-price auction).
- k'th-price ($k = 1, \ldots, n$) and all-pay are direct revelation auctions.

Theorem (Revenue Equivalence Theorem)

Suppose bidders have i.i.d. valuations, are risk neutral, and $\underline{v} = 0$. Then any symmetric and increasing equilibrium of a direct revelation auction that assigns the good to the highest bidder, and such that the expected payment of a bidder with value 0 is 0, yields the same expected revenue.

- From 1'st- to 2'nd- price auction: bidders no longer have incentive to shade, but at the revenue-cost of a more generous pricing mechanism.

Auctions with asymmetric information: 1P common value

Common-value (component) auction:

- $n > 1$ bidders in an auction have independent and identically distributed (i.i.d.) types $(t_i)_{i=1}^n$.
- Fixing $\alpha \geq \gamma > 0$, player i's value for a single non-divisible good:

$$v = \alpha t_i + \gamma \sum_{j \neq i} t_j.$$

  When $\alpha = \gamma$ then (exact) common value of the good is v.
- Each t_i is distributed i.i.d. via continuous increasing cdf $F(\cdot)$, $\underline{t} \leq t_i \leq \overline{t}$.
- Payoffs given state of the world $\mathbf{t}$ and bids $\mathbf{b}$ is:

$$u_i(\mathbf{t}, \mathbf{b}) = \begin{cases} (v - P(\mathbf{b}))/m & \textit{if}: \begin{array}{c} b_j \leq b_i, \ \forall j \neq i, \\ \& \ b_j = b_i \textit{ for } m \textit{ players} \end{array} \\ 0 & \textit{otherwise} \end{cases}.$$

Auctions with asymmetric information: 1P common value

- Strategies $S_i = \{b_i(t) : t \in [\underline{t}, \overline{t}]\}$; monotone increasing $b_i(t)$.
- If $b_j(t) = b^*(t)$ for each $j \neq i$, $\phi(b|\mathbf{b}_{-i}) = (F(b^{*-1}(b))^{n-1}$.
- Rational expectations: if i wins with bid b, she conditions on $t_j < b_j^{-1}(b) \overset{EQ}{=} b^{*-1}(b)$ when taking expectation of v:

$$
\begin{aligned}
&\mathbb{E}_{\mathbf{t}_{-i}}[u_i\left(\mathbf{t}, (b, (b_j(t_j))_{j\neq i}\right)] \\
&\qquad\qquad = \phi(b|\mathbf{b}_{-i})\left(\mathbb{E}_{\mathbf{t}_{-i}}\left[v - P(\mathbf{b}) \,\middle|\, t_j < b_j^{-1}(b), \forall j \neq i\right]\right) \\
&\overset{1P}{=} \quad \phi(b|\mathbf{b}_{-i})\left(\alpha t_i + \gamma \mathbb{E}_{\mathbf{t}_{-i}}\left[\sum_{j\neq i} t_j \,\middle|\, t_j < b_j^{-1}(b), \forall j \neq i\right] - b\right) \\
&\overset{EQ}{=} \quad (F(b^{*-1}(b)))^{n-1}\left(\alpha t_i + \gamma(n-1)\frac{\int_{\underline{t}}^{b^{*-1}(b)} tdF(t)}{F(b^{*-1}(b))} - b\right).
\end{aligned}
$$

- ...can solve for (possibly nonlinear) equilibrium bidding strategy $b^*(t)$.

Auctions with asymmetric information: 1P common value

- The "winner's curse" is due to the following inequality:

$$\begin{aligned}
\mathbb{E}_{\mathbf{t}_{-i}}\left[v \,\middle|\, t_j < b^{*-1}(b), \forall j \neq i\right] &= \alpha t_i + \gamma(n-1) \underbrace{\frac{\int_{\underline{t}}^{b^{*-1}(b)} t dF(t)}{F\left(b^{*-1}(b)\right)}}_{\mathbb{E}[t_j | t_j < b^{*-1}(b)]} \\
&< \alpha t_i + \gamma(n-1)\mathbb{E}\left[t_j\right].
\end{aligned}$$

- In words: If a bidder fails to condition on the information from winning the auction, precisely, that others' private types are lower than her own type, then she will overbid for the good.
- Example: 1P auction, $n = 2$, $\alpha = \gamma = 1$ and $F(t) = t$ for $t \in [0, 1]$: $b^*(t) = t$ in symmetric BNE (so $b^*(t) < t + \frac{1}{2}$). *Exercise: show this.*

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- Fact 1: In common-value auctions with i.i.d. types (i.e. as above; e.g., auction for management rights of collective farm land), revenue equivalence holds.
- Fact 2: However, if signals are correlated (i.e. not i.i.d.; e.g., auction for oil tract of uncertain yield), revenue equivalence can fail in common value actions: $ER_{1P} < ER_{2P}$.