Optimizing a Linear Objective Function with Constraints

Data Science

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Linear Programming (LP) Overview

LP

Linear Programming (LP) is a mathematical optimization technique used to maximize or minimize a linear objective function, subject to linear equality and inequality constraints. It is commonly used in resource allocation, production planning, transportation problems, and portfolio optimization.

Objective Function

The goal of LP is to optimize (maximize or minimize) an objective function of the form:

$$Z = c_1x_1 + c_2x_2 \cdots c_nx_n$$

where Z is the objective function, c_i are the coefficients, and x_i are the decision variables.

Constraints

Constraints are linear equations or inequalities that limit the feasible region. They can be written as:

$$Ax \leq b(\text{Inequality Constraint}$$

$$Ax = b(\text{Equality Constraint})$$

where A is a matrix of coefficients, x is the vector of decision variables, and b is the vector of constants.

Feasible Region

The feasible region represents all possible solutions that satisfy the constraints. The optimal solution lies within this region.

Optimal solution

To find the optimal solution, you typically evaluate the objective function at each corner point (vertex) of the feasible region, as the optimal solution will always occur at one of these points.

Solution Method

Python package

The scipy.optimize.linprog function in Python solves linear programming problems using the simplex method, interior-point method, or revised simplex method.

Assumptions of LP

- Proportionality: The contribution of each decision variable to the objective function or constraint is proportional.
- Additivity: The total contribution is the sum of individual contributions.
- Divisibility: Decision variables can take any fractional values (unless integer programming is specified).
- Non-negativity: Decision variables cannot be negative.

Application Areas

- Resource Allocation: Optimize production output with limited resources.
- Transportation Problems: Minimize the cost of shipping goods from warehouses to markets.
- Diet Problems: Minimize cost while meeting nutritional requirements.
- Portfolio Optimization: Maximize investment return subject to risk constraints.

Optimization Methods in linprog

- Simplex Method: Suitable for small to moderate-sized problems.
- Interior-Point Method: Suitable for large-scale problems.
- Revised Simplex Method: An efficient version of the simplex method.

Practice questions

A factory produces two types of products: Product A and Product B. The profit for Product A is \$20 per unit and for Product B is \$30 per unit (Manufacturing Problem).

- Product A requires 2 hours of labor and 1 unit of raw material.
- Product B requires 1 hour of labor and 2 units of raw material.
- Available Labor $= 100$ hours.
- Available Raw Material $= 80$ units.

Maximize the profit.

Practice questions

A company ships goods from two warehouses (A, B) to two markets (X, Y) (Transport Optimization).

1. Cost from A to X = \$4, A to Y = \$5
2. Cost from B to X = \$6, B to Y = \$3
3. Warehouse A capacity = 70 units
4. Warehouse B capacity = 50 units
5. Market X demand = 60 units
6. Market Y demand = 50 units

Minimize the transportation cost.

Practice questions

A person needs at least 2000 calories and 50g protein daily (Diet Optimization)

- Food A: 500 calories, 30g protein, \$3/unit.
- Food B: 700 calories, 20g protein, \$5/unit.

Minimize the cost. An investor has \$100,000 to invest in two stocks (Portfolio Optimization)

- Stock A: Expected return = 8%, risk = 5%.
- Stock B: Expected return = 12%, risk = 10%.

The investor wants to minimize risk while ensuring a 10% return.