Random Walk Process and Gambler's Ruin Problem

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Overview

1. Random Walk Process
   - Simple random walk
   - Basic Properties
   - Two Important Theorems
2. Gambler's Ruin Problem
   - Win Probability

Simple random walk

Let $(\varepsilon_n, n \geq 1)$ be i.i.d. (independent and identically distributed) random variables such that $P(\varepsilon_n = +1) = p$ and $P(\varepsilon_n = -1) = q = 1 - p$, where $0 < p < 1$.

Random Walk

The simple random walk on Z is the random(stochastic) process $(S_n, n \in N)$ defined as $S_0 = 0, S_n = \varepsilon_1 + \cdots + \varepsilon_n, n \geq 1$

This type of random walk could be associated with the "walk of a drunkard". The moves are independent, $S_{n+1} = S_n + \varepsilon_{n+1}$ (i.e. at each step, we take an action that has nothing to do with the previous steps). However, the position at step $n + 1$ is highly correlated with the position at step n.
If $p = \frac{1}{2}$ the random walk is symmetric.
If $p \neq \frac{1}{2} (\Rightarrow q \neq \frac{1}{2})$the random walk is asymmetric.

Basic Properties

Expectation (on average, where are we at time n?)

$$E(S_n) = E(\varepsilon_1 + \cdots + \varepsilon_n)$$

$$= E(\varepsilon_1) + \cdots + E(\varepsilon_n)$$

Identically distributed random variables

$$= nE(\varepsilon_i)$$

$$E(\varepsilon_i) = 1 \,.\, p + (-1) \,.\, q = p + (p - 1) = 2p - 1 \Rightarrow$$

$$\Rightarrow E(S_n) = n(2p - 1)$$

Variance

$$Var(S_n) = Var(\varepsilon_1 + \cdots + \varepsilon_n)$$

$$= Var(\varepsilon_1) + \cdots + Var(\varepsilon_n)$$

$$= nVar(\varepsilon_i)$$

Basic Properties

Note:

$$Var(\varepsilon_i) = E(\varepsilon_i^2) - [E(\varepsilon_i)]^2$$
$$= p + (1-p) - (2p-1)^2$$
$$= 1 - 4p2 + 4p - 1 = 4p(1-p)$$
$$\Rightarrow Var(S_n) = 4np(1-p)$$

Note: $Var(S_n)$ reaches its maximum for $p = \frac{1}{2}$, having $Var(S_n) = n$
Standard Deviation:

$$Stdev(S_n) = \sqrt{Var(S_n)} = \sqrt{4np(1-p)}$$

Two Important Theorems

The law of large numbers

For a sequence of i.i.d. random variables ε_i taking values in $\Re$,

$$\frac{S_n}{n} n \to \infty E(\varepsilon_1) = 2p - 1$$

with high probability (this can be taken to mean that the probability of this event goes to 1 as $n \to \infty$), provided that $E(\varepsilon_1) < \infty$.

The Central Limit Theorem

$$Z_n n \to \infty Z \sim N(0, 1)$$

When n goes to infinity, the distribution of Z_n will tend to a Gaussian random variable, with mean 0 and variance 1.

Gambler's Ruin Problem

Example of Gambler's Ruin Problem

Two players, A and B, play a game with independent rounds where, in each round, one of the players wins 1 USD from his opponent; A with probability p and B with probability $q = 1 - p$. A starts the game with a 100 USD and B with 50 USD. The game ends when one of the players is ruined.

- We have two gamblers, named simply A and B. There are N total dollars in the game; let A start with j dollars, which means B starts with $N - j$ dollars.
- The Gamblers play repeated, one round games. Each round, A wins with probability p, and B wins with probability $1 - p$, which we will label as q (the rounds are independent). If A wins, B gives one dollar to A, and if B wins, A gives one dollar to B.
- The game ends when one of the gamblers is ruined (they get to zero dollars), which means that the other gambler has all of the N dollars.

Discussions-Win Probability

- Naturally, some questions start to arise. Of course A had the advantage in the game above, since if $p = .6$.
- Does that mean A would win 60% of games? What about if we started A with less money, say 40 dollars (while B starts with 60)? Would A still be favored?
- Given p (the probability of A winning any one game) and N and j (the total number of dollars in the game and the total number of dollars that A starts out with) what is the probability that A wins the game and B is 'ruined?'
- We can define, for a given game (where N and p are specified), w_j as the probability that AA wins given that A has j dollars. Formally:

$$w_j = P(A_{wins} | A_{startsatjdollars})$$

The End