1. Stochastic Processes and filtrations

1. Stoch. pr., filtrations

2. BM as weak limit

3. Gaussian p.

4. Stopping times

5. Cond. expectation

6. Martingales

7. Discrete stoch. integral

8. Refl. princ., pass.times

9. Quad.var., path prop.

10. Ito integral

11. Ito's formula

A stochastic process $(X_t)_{t\in\mathbb{T}}$ is a collection of random variables on $(\Omega, \mathcal{F})$ with values in a measurable space $(S, \mathcal{S})$, i.e., for all t,

$$X_t : \Omega \to S \text{ is } \mathcal{F} - \mathcal{S}\text{-measurable.}$$

In our case

1. $\mathbb{T} = \{0, 1, \ldots, N\}$ or $\mathbb{T} = \mathbb{N} \cup \{0\}$ (discrete time)
2. or $\mathbb{T} = [0, \infty)$ (continuous time).

The state space will be $(S, \mathcal{S}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ (or $(S, \mathcal{S}) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ for some $d \geq 2$).

Example 1.1

Random walk:

Let ε_n, $n \geq 1$, be a sequence of iid random variables with

$$\varepsilon_n = \begin{cases} 1 & \textit{with probability } \frac{1}{2} \\ -1 & \textit{with probability } \frac{1}{2} \end{cases}$$

Define $\tilde{X}_0 = 0$ and, for $k = 1, 2, \ldots,$

$$\tilde{X}_k = \sum_{i=1}^{k} \varepsilon_i$$

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

For a fixed $\omega \in \Omega$ the function

$$t \mapsto X_t(\omega)$$

is called (sample) **path** (or **realization**) associated to ω of the stochastic process X.

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Let X and Y be two stochastic processes on $(\Omega, \mathcal{F})$.

Definition 1.2

(i) X and Y have the same finite-dimensional distributions if, for all $n \geq 1$, for all $t_1, \ldots, t_n \in [0, \infty)$ and $A \in \mathcal{B}(\mathbb{R}^n)$:

$$P((X_{t_1}, X_{t_2}, \ldots, X_{t_n}) \in A) = P((Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n}) \in A).$$

(ii) X and Y are modifications of each other, if, for each $t \in [0, \infty)$,

$$P(X_t = Y_t) = 1.$$

(iii) X and Y are indistinguishable, if

$$P(X_t = Y_t \text{ for every } t \in [0, \infty)) = 1.$$

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Example 1.3

1. Stoch. pr., filtrations

2. BM as weak limit

3. Gaussian p.

4. Stopping times

5. Cond. expectation

6. Martingales

7. Discrete stoch. integral

8. Refl. princ., pass.times

9. Quad.var., path prop.

10. Ito integral

11. Ito's formula

Definition 1.4

A stochastic process X is called continuous (right continuous), if almost all paths are continuous (right continuous), i.e.,

$$P(\{\omega : t \mapsto X_t(\omega) \text{ continuous (right continuous)}\}) = 1$$

.

Proposition 1.5

If X and Y are modifications of each other and if both processes are a.s. right continuous then X and Y are indistinguishable.

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Let $(\Omega, \mathcal{F})$ be given and $\{\mathcal{F}_t, t \geq 0\}$ be an increasing family of σ-algebras (i.e., for $s < t$, $\mathcal{F}_s \subseteq \mathcal{F}_t$), such that $\mathcal{F}_t \subseteq \mathcal{F}$ for all t. That's a

$$\boxed{\textbf{filtration}.}$$

For example:

Definition 1.6

The filtration $\mathcal{F}_t^X$ generated by a stochastic process X is given by

$$\mathcal{F}_t^X = \sigma\{X_s, s \leq t\}.$$

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Definition 1.7

A stochastic process is adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$ if, for each $t \geq 0$

$$X_t : (\Omega, \mathcal{F}_t) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

is measurable. (Short: X_t is $\mathcal{F}_t$-measurable.)

1. Stoch. pr., filtrations

2. BM as weak limit

3. Gaussian p.

4. Stopping times

5. Cond. expectation

6. Martingales

7. Discrete stoch. integral

8. Refl. princ., pass.times

9. Quad.var., path prop.

10. Ito integral

11. Ito's formula

Definition 1.8 ("The usual conditions")

Let $\mathcal{F}_{t^+} = \cap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon}$. We say that a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions if

1. $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous, that is, $\mathcal{F}_{t^+} = \mathcal{F}_t$, for all $t \geq 0$.
2. $(\mathcal{F}_t)_{t\geq 0}$ is complete. That is, $\mathcal{F}_0$ contains all subsets of P-nullsets.

2. Brownian motion as a weak limit of random walks

1. Stoch. pr., filtrations

2. BM as weak limit

3. Gaussian p.

4. Stopping times

5. Cond. expectation

6. Martingales

7. Discrete stoch. integral

8. Refl. princ., pass.times

9. Quad.var., path prop.

10. Ito integral

11. Ito's formula

Example 2.1

Scaled random walk: for $t_k = \frac{k}{N}$, $k = 0, 1, 2, \ldots,$ we set

$$\tilde{X}^N_{t_k} := \frac{1}{\sqrt{N}} \tilde{X}_k = \frac{1}{\sqrt{N}} \sum_{i=1}^{k} \varepsilon_i$$

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Definition 2.2 (Brownian motion)

A standard, one-dimensional Brownian motion is a continuous adapted process W on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ with $W_0 = 0$ a.s. such that, for all $0 \leq s < t$,

1. $W_t - W_s$ is independent of $\mathcal{F}_s$
2. $W_t - W_s$ is normally distributed with mean 0 and variance $t - s$.

1. Stoch. pr., filtrations

2. BM as weak limit

3. Gaussian p.

4. Stopping times

5. Cond. expectation

6. Martingales

7. Discrete stoch. integral

8. Refl. princ., pass.times

9. Quad.var., path prop.

10. Ito integral

11. Ito's formula

Recall the scaled random walk: for $t_k = \frac{k}{N}$, $k = 0, 1, 2, \ldots$, we set

$$\tilde{X}^N_{t_k} := \frac{1}{\sqrt{N}} \tilde{X}_k = \frac{1}{\sqrt{N}} \sum_{i=1}^{k} \varepsilon_i$$

By interpolation we define a continuous process on $[0, \infty)$:

$$X^N_t = \frac{1}{\sqrt{N}} \left(\sum_{i=1}^{[Nt]} \varepsilon_i + (Nt - [Nt]) \varepsilon_{[Nt]+1} \right) \tag{2.1}$$

1. Stoch. pr., filtrations

2. BM as weak limit

3. Gaussian p.

4. Stopping times

5. Cond. expectation

6. Martingales

7. Discrete stoch. integral

8. Refl. princ., pass.times

9. Quad.var., path prop.

10. Ito integral

11. Ito's formula

Theorem 2.3 (Donsker)

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a sequence of iid random variables with mean 0 and variance 1. Define $(X_t^N)_{t\geq 0}$ as in (2.1). Then $(X_t^N)_{t\geq 0}$ converges to $(W_t)_{t\geq 0}$ weakly.

1. Stoch. pr., filtrations

2. BM as weak limit

3. Gaussian p.

4. Stopping times

5. Cond. expectation

6. Martingales

7. Discrete stoch. integral

8. Refl. princ., pass.times

9. Quad.var., path prop.

10. Ito integral

11. Ito's formula

Clarification of the statement:

1. $\tilde{\Omega} = C[0, \infty)$
2. On $\tilde{\Omega} = C[0, \infty)$ one can construct a probability measure P^* (the **Wiener measure**) such that the **canonical process** $(W_t)_{t \geq 0}$ is a Brownian motion. Let $\omega = \omega(t)_{t \geq 0} \in C[0, \infty)$. Then the canonical process is given as $W_t(\omega) := \omega(t)$.
3. $(X_t^N)_{t \geq 0}$ defines a measure probability P^N on $\tilde{\Omega} = C[0, \infty)$. Indeed,
$$X^N : (\Omega, \mathcal{F}) \to (C[0, \infty), \mathcal{B}(C[0, \infty))$$
measurable, that means we consider the random variable $\omega \mapsto X^N(\omega, \cdot)$, which maps ω to a continous function in t (i.e. the path). Let $A \in \mathcal{B}(C[0, \infty))$, then
$$P^N(A) := P(\{\omega : X^N(\omega, \cdot \in A\}).$$
4. So, Donsker's theorem says that on $(\tilde{\Omega}, \tilde{\mathcal{F}}) = (C[0, \infty), \mathcal{B}(C[0, \infty)))$ we have that $P^N \xrightarrow{w} P^*$.

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Steps of the proof:

1. Show that the finite-dimensional distributions of X^N converge to the finite-dimensional distributions of W. That means, show that, for all $n \geq 1$, and all $s_1, \dots, s_n \in [0, \infty)$

$$(X^N_{s_1}, X^N_{s_2}, \dots, X^N_{s_n}) \xrightarrow{\text{w}} (W_{s_1}, W_{s_2}, \dots, W_{s_n}).$$

2. Show that $(P^N)_{N \geq 1}$ is a **tight** familiy of probability measures on $(\tilde{\Omega}, \tilde{\mathcal{F}})$.

3. (1) and (2) imply $P^N \xrightarrow{\text{w}} P^*$.

1. Stoch. pr., filtrations

2. BM as weak limit

3. Gaussian p.

4. Stopping times

5. Cond. expectation

6. Martingales

7. Discrete stoch. integral

8. Refl. princ., pass.times

9. Quad.var., path prop.

10. Ito integral

11. Ito's formula

Easy partial result of step (1) of the proof:

By (CLT), for fixed $t \in [0, \infty)$, for $N \to \infty$,

$$X_t^N \xrightarrow{\text{w}} W_t$$

Theorem 2.4 (CLT)

Let $(\xi_n)_{n \geq 1}$ be iid, mean=0, variance=σ^2. Then, for each $x \in \mathbb{R}$,

$$\lim_{n \to \infty} P\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i \leq x\right) = \Phi_\sigma(x),$$

where $\Phi_\sigma(x)$ is the distribution function of $N(0, \sigma^2)$. In other words

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i \xrightarrow{w} Z,$$

where $Z \sim N(0, \sigma^2)$.

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

3. Brownian motion as a Gaussian process

Definition 3.1

A stochastic process $(X_t)_{t\geq 0}$ is called Gaussian process, if for each finite familiy of time points $\{t_1, \dots, t_n\}$ the vector $(X_{t_1}, \dots, X_{t_n})$ has a multivariate normal distribution.
A Gaussian process is called centered if $E[X_t] = 0$ for all t.
The covariance function is given by $\gamma(s,t) = \mathrm{Cov}(X_s, X_t)$.

Theorem 3.2

A Brownian motion is a centered Gaussian process with $\gamma(s,t) = s \wedge t$ $(= \min(s,t))$. Conversely, a centered Gaussian process with continuous paths and covariance function $\gamma(s,t) = s \wedge t$ is a Brownian motion.

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

4. Stopping times

Given is a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.

A **random time** T is a measurable function

$$T : (\Omega, \mathcal{F}) \to ([0, \infty], \mathcal{B}([0, \infty])).$$

Definition 4.1

If X is a stochastic process and T a random time, we define the function X_T on $\{T < \infty\}$ by:

$$X_T(\omega) = X_{T(\omega)}(\omega)$$

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Definition 4.2

A random time T is a stopping time for $(\mathcal{F}_t)_{t\geq 0}$ if, for all $t \geq 0$,

$$\{T \leq t\} = \{\omega : T(\omega) \leq t\} \in \mathcal{F}_t.$$

Proposition 4.3

1. *If $T(\omega) = t$ a.s. for some constant $t \geq 0$, then T is a stopping time.*
2. *Every stopping time T satisfies that, for all t,*

$$\{T < t\} \in \mathcal{F}_t. \tag{4.1}$$

3. *If the filtration is right continuous and a random time T satisfies (4.1) for all t, then T is a stopping time.*

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Lemma 4.4

Let S and T be stopping times for $(\mathcal{F}_t)_{t\geq 0}$. Then $S \wedge T$, $S \vee T$ and $S+T$ are stopping times for $(\mathcal{F}_t)_{t\geq 0}$.

Definition 4.5

Let T be a stopping time for $(\mathcal{F}_t)_{t\geq 0}$. The σ-field $\mathcal{F}_T$ of events determined prior to the stopping time T is given by

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T \leq t\} \in \mathcal{F}_t, \text{ for all } t \geq 0\}.$$

Remark:

1. $\mathcal{F}_T$ is a σ-field.
2. T is $\mathcal{F}_T$-measurable.
3. If $T(\omega) = t$ a.s., then $\mathcal{F}_T = \mathcal{F}_t$.

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Example 4.6

Let $(W_t)_{t\geq 0}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.
Let T_a be the **first passage time** of the level $a \in \mathbb{R}$, i.e.,

$$T_a(\omega) = \inf\{t > 0 : W_t(\omega) = a\}. \tag{4.2}$$

Then T_a is a stopping time for $(\mathcal{F}_t)_{t\geq 0}$.

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Remark: Let S and T be stopping times for $(\mathcal{F}_t)_{t\geq 0}$.

1. For $A \in \mathcal{F}_S$ we have that $A \cap \{S \leq T\} \in \mathcal{F}_T$.
2. If, moreover, $S \leq T$ then $\mathcal{F}_S \subseteq \mathcal{F}_T$.

Definition 4.7

A stochastic process $(X_t)_{t\geq 0}$ is called progressively measurable for $(\mathcal{F}_t)_{t\geq 0}$ if, for all t and all $A \in \mathcal{B}(\mathbb{R})$, the set

$$\{(s,\omega) : 0 \leq s \leq t, \omega \in \Omega, X_s(\omega) \in A\} \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t,$$

that means

$$(s,\omega) \mapsto X_s(\omega) \text{ is } \mathcal{B}([0,t]) \otimes \mathcal{F}_t - \mathcal{B}(\mathbb{R}) - \text{measurable.}$$

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Example: If $(X_t)_{t\geq 0}$ is adapted and right-continuous then it is progressively measurable.

Proposition 4.8

Let X be progressively measurable on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ and let T be a stopping time for $(\mathcal{F}_t)_{t\geq 0}$. Then

1. *X_T defined on $\{T < \infty\}$ is an $\mathcal{F}_T$-measurable random variable and*
2. *the* **stopped** *process $(X_{T\wedge t})_{t\geq 0}$ is progressively measurable.*

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

5. Conditional expected value

Definition 5.1

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \to \mathbb{R}$ be an $\mathcal{F} - \mathcal{B}(\mathbb{R})$-measurable random variable such that $E[|X|] = \int |X| dP < \infty$. Let $\mathcal{G} \subseteq \mathcal{F}$ be a (smaller) σ-algebra. Then there exists a random variable Y with the following properties:

1. Y is $\mathcal{G} - \mathcal{B}(\mathbb{R})$-measurable.
2. Y is integrable, i.e., $E[|Y|] < \infty$.
3. For all $A \in \mathcal{G}$:
$$E[Y \mathbb{1}_A] = E[X \mathbb{1}_A].$$

The $\mathcal{G}$-measurable random variable Y is called conditional expected value of X given $\mathcal{G}$. We write $Y = E[X|\mathcal{G}]$ a.s.

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Remark:

1. $E[X|\mathcal{G}]$ is unique with respect to equality a.s.

2. Suppose that, additionally, $E[X^2] < \infty$. Then $E[X|\mathcal{G}]$ is the orthogonal projection of $X \in L^2(\Omega, \mathcal{F})$ onto the subspace $L^2(\Omega, \mathcal{G})$.

3. If $\mathcal{G} = \{\emptyset, \Omega\}$ then $E[X|\mathcal{G}] = E[X]$ a.s.

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Definition 5.2

Let Z be a function on Ω. Then $E[X|Z] := E[X|\sigma(Z)]$.

Example 5.3

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Example 5.4

$X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Suppose $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ such that $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. Let $\mathcal{G} := \sigma\{\Omega_k, k \geq 1\}$.

$$E[X|\mathcal{G}] =$$

1. Stoch. pr., filtrations

2. BM as weak limit

3. Gaussian p.

4. Stopping times

5. Cond. expectation

6. Martingales

7. Discrete stoch. integral

8. Refl. princ., pass.times

9. Quad.var., path prop.

10. Ito integral

11. Ito's formula

Example 5.5

Suppose (X, Z) has a bivariate density $f(x, z)$.

1. Stoch. pr., filtrations

2. BM as weak limit

3. Gaussian p.

4. Stopping times

5. Cond. expectation

6. Martingales

7. Discrete stoch. integral

8. Refl. princ., pass.times

9. Quad.var., path prop.

10. Ito integral

11. Ito's formula

Theorem 5.6 (A list of properties of conditional expectation)

In the following let X, Y, X_n, $n \geq 1$, be integrable random variables on $(\Omega, \mathcal{F}, P)$. The following holds

(a) *If X is $\mathcal{G}$-measurable, then $E[X|\mathcal{G}] = X$ a.s.*

(b) *Linearity: $E[aX + bY|\mathcal{G}] = aE[X|\mathcal{G}] + bE[Y|\mathcal{G}]$ a.s.*

(c) *Positivity: If $X \geq 0$ a.s. then $E[X|\mathcal{G}] \geq 0$ a.s.*

(d) *Monotone convergence (MON): If $0 \leq X_n$ and $X_n \uparrow X$ a.s. then*

$$E[X_n|\mathcal{G}] \uparrow E[X|\mathcal{G}] \quad a.s..$$

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

(e) *Fatou: If* $0 \leq X_n$, *for all* n, *then*

$$E[\liminf_n X_n|\mathcal{G}] \leq \liminf_n E[X_n|\mathcal{G}].$$

(f) *Dominated convergence (DOM): If* $|X_n| \leq Y$, *for all* $n \geq 1$ *and* $Y \in L^1$ *(i.e. integrable) and* $\lim X_n = X$ *a.s. then*

$$\lim_n E[X_n|\mathcal{G}] = E[X|\mathcal{G}] \quad a.s.$$

(g) *Jensen's inequality: Let* $\varphi : \mathbb{R} \to \mathbb{R}$ *be a convex function such that* $\varphi(X)$ *is integrable. Then*

$$\varphi\left(E[X|\mathcal{G}]\right) \leq E[\varphi(X)|\mathcal{G}]$$

1. Stoch. pr., filtrations

2. BM as weak limit

3. Gaussian p.

4. Stopping times

5. Cond. expectation

6. Martingales

7. Discrete stoch. integral

8. Refl. princ., pass.times

9. Quad.var., path prop.

10. Ito integral

11. Ito's formula

(h) *Tower Property: If $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \mathcal{F}$ then*

$$E\left[E[X|\mathcal{G}_2]|\mathcal{G}_1\right] = E[X|\mathcal{G}_1] \quad a.s.$$

(i) *Taking out what is known: Let Y be $\mathcal{G}$-measurable and let $E[|XY|] < \infty$. Then*

$$E[XY|\mathcal{G}] = YE[X|\mathcal{G}]$$

(j) *Role of independence: If X is independent of $\mathcal{G}$, then*

$$E[X|\mathcal{G}] = E[X]$$

6. Martingales and the discrete time stochastic integral

1. Stoch. pr., filtrations

2. BM as weak limit

3. Gaussian p.

4. Stopping times

5. Cond. expectation

6. Martingales

7. Discrete stoch. integral

8. Refl. princ., pass.times

9. Quad.var., path prop.

10. Ito integral

11. Ito's formula

Definition 6.1

A stochastic process $(X_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ is a martingale if X is adapted and $E[|X_t|] < \infty$, for all $t \geq 0$, and, for $s < t$,

$$E[X_t|\mathcal{F}_s] = X_s \quad \text{a.s.} \tag{6.1}$$

If the $=$ in (6.1) is replaced by $\leq$ ($\geq$) the process is called supermartingale (submartingale).

Remark:

1. (6.1) is equivalent to $E[X_t - X_s|\mathcal{F}_s] = 0$ a.s.
2. Suppose $(X_t)_{n=0}^{\infty}$ is a stochastic process in discrete time adapted to $(\mathcal{F}_n)_{n=0}^{\infty}$ then the martingale property reads as: for all $n \geq 1$

$$E[X_n|\mathcal{F}_{n-1}] = X_{n-1} \quad \text{a.s.}$$

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Example 6.2 (Random walk: sum of independent r.v.)

Let ε_k, $k \geq 1$, be independent with $E[\varepsilon_k] = 0$. Let $\mathcal{F}_n = \sigma\{\varepsilon_1, \ldots, \varepsilon_n\}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Define $X_0 = 0$ and, for $n \geq 1$,

$$X_n = \sum_{k=1}^{n} \varepsilon_k.$$

Then $(X_n)_{n \geq 0}$ is a martingale for $(\mathcal{F}_n)_{n \geq 0}$.

Example 6.3 (Product of independent r.v.)

Let ε_k, $k \geq 1$, be non-negative and independent with $E[\varepsilon_k] = 1$. Let $\mathcal{F}_n = \sigma\{\varepsilon_1, \ldots, \varepsilon_n\}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Define $X_0 = 1$ and, for $n \geq 1$,

$$X_n = \prod_{k=1}^{n} \varepsilon_k.$$

Then $(X_n)_{n \geq 0}$ is a martingale for $(\mathcal{F}_n)_{n \geq 0}$.

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Example 6.4

Let X be a r.v. on $(\Omega, \mathcal{F}, P)$ such that $E[|X|] < \infty$. Let $\mathcal{F}_t)_{t\geq 0}$ be any filtration with $\mathcal{F}_t \subseteq \mathcal{F}$, for each $t \geq 0$. Define

$$X_t = E[X|\mathcal{F}_t].$$

Then $(X_t)_{t\geq 0}$ is a martingale for $(\mathcal{F}_t)_{t\geq 0}$.

7. Stochastic integral in discrete time

1. Stoch. pr., filtrations

2. BM as weak limit

3. Gaussian p.

4. Stopping times

5. Cond. expectation

6. Martingales

7. Discrete stoch. integral

8. Refl. princ., pass.times

9. Quad.var., path prop.

10. Ito integral

11. Ito's formula

Given is a discrete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, P)$.

Definition 7.1

A discrete stochastic process $(H_n)_{n=1}^{\infty}$ is called predictable if

$$H_n : (\Omega, \mathcal{F}_{n-1}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

is measurable. (Short: H_n is $\mathcal{F}_{n-1}$-measurable.)

1. Stoch. pr., filtrations

2. BM as weak limit

3. Gaussian p.

4. Stopping times

5. Cond. expectation

6. Martingales

7. Discrete stoch. integral

8. Refl. princ., pass.times

9. Quad.var., path prop.

10. Ito integral

11. Ito's formula

Definition 7.2

Let $(X_n)_{n=0}^{\infty}$ be an adapted stochastic process and $(H_n)_{n=1}^{\infty}$ be a predictable stochastic process on $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, P)$. Then the stochastic integral in discrete time at time $n \geq 1$ is given by

$$(H \cdot X)_n = \sum_{k=1}^{n} H_k (X_k - X_{k-1}).$$

We define $(H \cdot X)_0 = 0$. The stochastic integral process is given by $((H \cdot X)_n)_{n=0}^{\infty}$.

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Theorem 7.3

Let H be bounded and predictable and X be a martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, P)$. Then the stochastic integral process $((H \cdot X)_n)_{n=0}^{\infty}$ is a martingale.

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Theorem 7.4 (Doob's Optional Stopping Theorem)

Let $(M_t)_{t\geq 0}$ be a martingale in continuous time on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. Then, for a bounded stopping time T it holds that

$$E[M_T] = M_0.$$

More generally, if $S \leq T$ are bounded stopping times, then

$$E[M_T|\mathcal{F}_S] = M_S \quad a.s.$$

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Definition 8.1 (Markov property)

Brownian motion satisfies the Markov property, that means, for all $t \geq 0$ and $s > 0$:

$$P(W_{t+s} \leq y | \mathcal{F}_t) = P(W_{t+s} \leq y | W_t) \quad \text{a.s.}$$

1. Stoch. pr., filtrations

2. BM as weak limit

3. Gaussian p.

4. Stopping times

5. Cond. expectation

6. Martingales

7. Discrete stoch. integral

8. Refl. princ., pass.times

9. Quad.var., path prop.

10. Ito integral

11. Ito's formula

Reflection principle: Recall that $T_a = \inf\{t > 0 : W_t = a\}$.

$P(T_a < t) =$

$$\boxed{P(T_a < t) = 2P(W_t > a)} \tag{8.1}$$

1. Stoch. pr., filtrations

2. BM as weak limit

3. Gaussian p.

4. Stopping times

5. Cond. expectation

6. Martingales

7. Discrete stoch. integral

8. Refl. princ., pass.times

9. Quad.var., path prop.

10. Ito integral

11. Ito's formula

In all the following $(W_t)_{t\geq 0}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.

Theorem 8.2 (Strong Markov property)

The BM $(W_t)_{t\geq 0}$ satisfies the strong Markov property: for each stopping time τ with $P(\tau < \infty) = 1$ it holds that

$$\hat{W}_t = W_{\tau+t} - W_\tau, \quad , t \geq 0$$

is a standard Brownian motion independent of $\mathcal{F}_\tau$.

We will see later that $P(T_a < \infty) = 1$ for all $a \in \mathbb{R}$.

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Theorem 8.3 (Reflection principle)

Let $a \neq 0$ and T_a be the passage time for a.

$$\tilde{W}_t = \begin{cases} W_t & \text{for } t \leq T_a \\ 2W_{T_a} - W_t = 2a - W_t & \text{for } t \geq T_a \end{cases}$$

Then $(\tilde{W}_t)_{t \geq 0}$ is again a standard Brownian motion.

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Lemma 8.4

$P(\sup_{t\geq 0} W_t = +\infty \textit{ and } \inf_{t\geq 0} W_t = -\infty) = 1$

1. Stoch. pr., filtrations

2. BM as weak limit

3. Gaussian p.

4. Stopping times

5. Cond. expectation

6. Martingales

7. Discrete stoch. integral

8. Refl. princ., pass.times

9. Quad.var., path prop.

10. Ito integral

11. Ito's formula

Theorem 8.5 (Recurrence)

$P(T_a < \infty) = 1$ for all $a \in \mathbb{R}$.

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Theorem 8.6 (Distribution of passage time)

Let $a \neq 0$. The density of T_a is given by

$$f_{T_a}(t) = \frac{|a|}{\sqrt{2\pi}} t^{-\frac{3}{2}} e^{-\frac{a^2}{2t}},$$

which is the density of an invers-gamma-distribution with parameters $\frac{1}{2}$ and $\frac{a^2}{2}$. In particular, we have that $E[T_a] = +\infty$.

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Running maximum and minimum of Brownian motion:

Let $M_t = \max_{0 \le s \le t} W_s$ and $m_t = min_{0 \le s \le t} W_s$.

Theorem 8.7

1. *For each $a > 0$:*
   $P(M_t \ge a) = P(T_a \le t) = P(T_a < t) = 2P(W_t > a)$.
2. *For each $a < 0$: $P(m_t \le a) = 2P(W_t < a)$.*

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Theorem 8.8 (Joint distribution)

Let $y \geq x \geq 0$. Then

$$P(W_t \leq x, M_t \geq y) = P(W_t \geq 2y - x).$$

This implies that the joint distribution of (W_t, M_t) has the following density: for $y \geq 0, x \leq y$

$$f_{W,M}(x,y) = \sqrt{\frac{2}{\pi}} \frac{(2y-x)}{t^{\frac{3}{2}}} e^{\frac{-(2y-x)^2}{2t}}.$$

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Let $a < 0 < b$ and let τ be the first time to leave an interval (a, b) (exit time), i.e.,

$$\tau = \inf\{t > 0 : W_t \notin (a, b)\}.$$

We have that $\tau = \min(T_a, T_b) = T_a \wedge T_b$.

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Theorem 8.9 (Exit time)

Let $a < 0 < b$ and $\tau = T_a \wedge T_b$ as above. Then $P(\tau < \infty) = 1$ and $E[\tau] = |ab|$. Moreover

$$P(W_\tau = b) = \frac{|a|}{|a| + b} = \frac{-a}{-a + b}$$
$$P(W_\tau = a) = \frac{b}{|a| + b} = \frac{b}{-a + b}$$

1. Stoch. pr., filtrations

2. BM as weak limit

3. Gaussian p.

4. Stopping times

5. Cond. expectation

6. Martingales

7. Discrete stoch. integral

8. Refl. princ., pass.times

9. Quad.var., path prop.

10. Ito integral

11. Ito's formula

Definition 9.1

A stochastic process has finite quadratic variation if there exists an a.s. finite stochastic process $([X,X]_t)_{t\geq 0}$ such that for all t and partitions $\pi_n = \{t_0^n, \ldots, t_{m_n}^n\}$ with $0 = t_0^n < t_1^n < \cdots < t_{m_n}^n = t$ and $\delta_n = \max_{i=1,\ldots,m_n}(t_i^n - t_{i-1}^n) \to 0$ we have that, for $n \to \infty$,

$$V_t^2(\pi_n) := \sum_{i=1}^{m_n} (X_{t_i^n} - X_{t_{i-1}^n})^2 \to [X,X]_t \quad \text{in probability.} \tag{9.1}$$

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Theorem 9.2

Let $(W_t)_{t\geq 0}$ be a Brownian motion. Then $V_t^2(\pi_n) \to t$ in L^2, that means

$$\lim_{n\to\infty} E[(V_t^2(\pi_n) - t)^2] = 0.$$

Therefore it follows that $V_t^2(\pi_n) \to t$ in probability. Hence $[W, W]_t = t$ a.s.

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Corollary 9.3

$(W_t)_{t\geq 0}$ *has a.s. paths of infinite variation on each interval.*

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Corollary 9.4

For almost all ω the path $t \mapsto W_t(\omega)$ is not monotone in each interval.

1. Stoch. pr., filtrations

2. BM as weak limit

3. Gaussian p.

4. Stopping times

5. Cond. expectation

6. Martingales

7. Discrete stoch. integral

8. Refl. princ., pass.times

9. Quad.var., path prop.

10. Ito integral

11. Ito's formula

Remark 9.5

Almost all paths of $(W_t)_{t\geq 0}$ *are not differentiable on each interval.*

For the whole chapter let $(W_t)_{t\geq 0}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. Everything will be based on this filtered probability space.

Recall:

Riemann integral:

$$\int_a^b f(t)dt = \lim_{\delta_n \to 0} \sum_{i=1}^{n} f(\xi_i^n)(t_i^n - t_{i-1}^n)$$

where $a = t_0^n < t_1^n < \cdots < t_n^n = b$, $t_{i-1}^n \leq \xi_i^n \leq t_i^n$ and $\delta_n = \max_{i=1,\ldots,n}(t_i^n - t_{i-1}^n)$.

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

1. Stoch. pr., filtrations

2. BM as weak limit

3. Gaussian p.

4. Stopping times

5. Cond. expectation

6. Martingales

7. Discrete stoch. integral

8. Refl. princ., pass.times

9. Quad.var., path prop.

10. Ito integral

11. Ito's formula

Riemann-Stieltjes-integral: Let f be a continuous function and g be a function of bounded variation. Then the following limit exists and is called R-S-integral of f with respect to g:

$$\int_a^b f(t)dg(t) := \lim_{\delta_n \to 0} \sum_{i=1}^{n} f(t_{i-1}^n)(g(t_i^n) - g(t_{i-1}^n)),$$

where $a = t_0^n < t_1^n < \cdots < t_n^n = b$ and $\delta_n = \max_{i=1,\dots,n}(t_i^n - t_{i-1}^n)$.

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Theorem 10.1

Fix $[0, t]$ and let, for each n, $0 = t_0^n < t_1^n < \cdots < t_n^n = t$. If g is a function such that

$$\lim_{\delta_n \to 0} \sum_{i=1}^{n} f(t_{i-1}^n)(g(t_i^n) - g(t_{i-1}^n))$$

exists for each continuous function $f : [0, t] \to \mathbb{R}$, then g is of bounded variation on $[0, t]$.

Remark 10.2

No pathwise R-S-integral for Brownian motion!!!

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Ito integral with simple integrands:

Definition 10.3

A stochastic process $(X_t)_{0\leq t\leq T}$ is called **simple predictable integrand** if

$$X_t = \xi_0 \mathbb{1}_{[t_0,t_1]}(t) + \sum_{i=1}^{n-1} \xi_i \mathbb{1}_{(t_i,t_{i+1}]}(t),$$

where $0 = t_0 < t_1 < \cdots < t_n = T$ and $\xi_0, \ldots, \xi_{n-1}$ are random variables such that ξ_i is $\mathcal{F}_{t_i}$-measurable and $E[\xi_i^2] < \infty$, for $i = 1, \ldots, n-1$.

For each $t \leq T$ the Ito integral at time t for a simple integrand X is given by:

$$\int_0^t X_s dW_s =: (X \cdot W)_t = \sum_{i=0}^{n-1} \xi_i (W_{t_{i+1}\wedge t} - W_{t_i \wedge t}).$$

1. Stoch. pr., filtrations

2. BM as weak limit

3. Gaussian p.

4. Stopping times

5. Cond. expectation

6. Martingales

7. Discrete stoch. integral

8. Refl. princ., pass.times

9. Quad.var., path prop.

10. Ito integral

11. Ito's formula

Remark 10.4

If $t < T$ is such that $t_k \leq t < t_{k+1} \leq t_n = T$, this means

$$(X \cdot W)_t = \sum_{i=0}^{k-1} \xi_i (W_{t_{i+1}} - W_{t_i}) + \xi_k (W_t - W_{t_k}).$$

For T this means

$$(X \cdot W)_T = \sum_{i=0}^{n-1} \xi_i (W_{t_{i+1}} - W_{t_i})$$

1. Stoch. pr., filtrations

2. BM as weak limit

3. Gaussian p.

4. Stopping times

5. Cond. expectation

6. Martingales

7. Discrete stoch. integral

8. Refl. princ., pass.times

9. Quad.var., path prop.

10. Ito integral

11. Ito's formula

Theorem 10.5 (Properties of Ito integral for simple integrands)

1. *Linearity: X, Y simple, α, β constants, then*

$$\int_0^T (\alpha X_t + \beta Y_t)\, dW_t = \alpha \int_0^T X_t dW_t + \beta \int_0^T Y_t dW_t.$$

2. *Let $0 \le a < b$. Then $\int_0^T \mathbb{1}_{(a,b]}(t) dW_t = W_b - W_a$ and $\int_0^T \mathbb{1}_{(a,b]}(t) X_t dW_t = \int_a^b X_t dW_t$.*
3. *Zero mean: $E[\int_0^T X_t dW_t] = 0$*
4. *Isometry: $E\left[\left(\int_0^T X_t dW_t\right)^2\right] = \int_0^T E[X_t^2] dt$*

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

(More) general integrands:

Lemma 10.6

Let $(X_t)_{0 \le t \le T}$ be a bounded and progressively measurable stochastic process. Then there exists a sequence of simple predictable processes $(X_t^m)_{0 \le t \le T}$, $m \ge 1$, such that

$$\lim_{m \to \infty} E\left[\int_0^T |X_t^m - X_t|^2 dt\right] = 0. \tag{10.1}$$

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Lemma 10.7

Let $(X_t)_{0\le t\le T}$ be a progressively measurable stochastic process such that

$$E\left[\int_0^T X_t^2\,dt\right] < \infty. \tag{10.2}$$

Then there exists a sequence of simple predictable processes $(X_t^m)_{0\le t\le T}$, $m \ge 1$, such that (10.1) holds, i.e.,

$$\lim_{m\to\infty} E\left[\int_0^T |X_t^m - X_t|^2\,dt\right] = 0.$$

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Theorem 10.8 (Ito integral for general integrands with (10.2))

Let $(X_t)_{0\leq t\leq T}$ be a progressively measurable stochastic process such that (10.2) holds, i.e., $E[\int_0^T X_t^2 dt] < \infty$. Then the stochastic integral $\int_0^T X_t dW_t$ is defined and has the following properties:

1. *Linearity: X, Y as above, then*
$$\int_0^T (\alpha X_t + \beta Y_t)\, dW_t = \alpha \int_0^T X_t dW_t + \beta \int_0^T Y_t dW_t.$$
2. *Let $0 \leq a < b$. Then $\int_0^T \mathbb{1}_{(a,b]}(t) dW_t = W_b - W_a$ and $\int_0^T \mathbb{1}_{(a,b]}(t) X_t dW_t = \int_a^b X_t dW_t$.*
3. *Zero mean: $E[\int_0^T X_t dW_t] = 0$*
4. *Isometry: $E\left[\left(\int_0^T X_t dW_t\right)^2\right] = \int_0^T E[X_t^2] dt$*

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Theorem 10.9 (Ito integral for even more general integrands)

Let $(X_t)_{0\leq t\leq T}$ be a progressively measurable stochastic process such that

$$P\left(\int_0^T X_t^2 dt < \infty\right) = 1. \tag{10.3}$$

Then the stochastic integral $\int_0^T X_t dW_t$ is defined and has satisfies (1) and (2) of Theorem 10.8.

Remark 10.10

Attention: *in this case (3) Zero-Mean and (4) Isometry are not satisfied in general!!! If, additionally the stronger condition (10.2) holds, then we are in the case of Theorem 10.8 and Zero-Mean and Isometry hold.*

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Corollary 10.11

Let X be continuous and adapted. Then $\int_0^T X_t dW_t$ exists. In particular, for any continuous function $f : \mathbb{R} \to \mathbb{R}$ the stochastic integral $\int_0^T f(W_t) dW_t$ exists.

Remark 10.12

The Ito integral is not monotone.

1. Stoch. pr., filtrations
2. BM as weak limit
3. Gaussian p.
4. Stopping times
5. Cond. expectation
6. Martingales
7. Discrete stoch. integral
8. Refl. princ., pass.times
9. Quad.var., path prop.
10. Ito integral
11. Ito's formula

Ito integral process:

Let X be progressively measurable such that (10.3) holds, i.e.,

$$P(\int_0^T X_s^2 ds < \infty) = 1.$$

Then $\int_0^t X_s dW_s$ is defined for all $t \leq T$. This gives a stochastic process $(\mathrm{I_t})_{0\leq t\leq T}$ with

$$\mathrm{I_t} = \int_0^{\mathrm{t}} \mathrm{X_s dW_s}.$$

There exists a modification with continuous paths: we always take this modification.

1. Stoch. pr., filtrations

2. BM as weak limit

3. Gaussian p.

4. Stopping times

5. Cond. expectation

6. Martingales

7. Discrete stoch. integral

8. Refl. princ., pass.times

9. Quad.var., path prop.

10. Ito integral

11. Ito's formula

Definition 10.13

A martingale $(M_t)_{0\leq t\leq T}$ is called quadratic integrable on $[0, T]$ if

$$\sup_{t\in[0,T]} E[M_t^2] < \infty.$$

Theorem 10.14

Let X be progressively measurable such that (10.2) holds, i.e., $E[\int_0^T X_s^2 ds] < \infty$. Then $(\mathrm{I_t})_{0\leq \mathrm{t}\leq \mathrm{T}}$, where $\mathrm{I_t} = \int_0^{\mathrm{t}} \mathrm{X_s dW_s}$, is a quadratic integrable martingale.

1. Stoch. pr., filtrations

2. BM as weak limit

3. Gaussian p.

4. Stopping times

5. Cond. expectation

6. Martingales

7. Discrete stoch. integral

8. Refl. princ., pass.times

9. Quad.var., path prop.

10. Ito integral

11. Ito's formula

Corollary 10.15

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and bounded function. Then $(\mathrm{I_t})_{0\leq \mathrm{t} \leq \mathrm{T}}$, where $\mathrm{I_t} = \int_0^{\mathrm{t}} \mathrm{f}(\mathrm{W_s})\mathrm{dW_s}$, is a quadratic integrable martingale.

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4. Stopping times
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11. Ito's formula

Quadratic variation and covariation of Ito integrals:

Theorem 10.16

Let X be progressively measurable such that (10.3 holds. The quadratic variation of $\int_0^t X_s dW_s$ satisfies

$$\left[\int_0^t X_s dW_s, \int_0^t X_s dW_s\right] = \int_0^t X_s^2 ds \quad a.s.$$

for all $0 \leq t \leq T$.

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Covariation: Let $\mathrm{I_t} = \int_0^{\mathrm{t}} \mathrm{X_s dW_s}$ and $\tilde{\mathrm{I}}_t = \int_0^t \tilde{X}_s dW_s$, for $0 \leq t \leq T$, and progressively measurable processes X, $\tilde{X}$ such that (10.3) holds.

The covariation of I and $\tilde{\mathrm{I}}$ is given by

$$[\mathrm{I}, \tilde{\mathrm{I}}]_{\mathrm{t}} := \frac{1}{2}\left([\mathrm{I}+\tilde{\mathrm{I}}, \mathrm{I}+\tilde{\mathrm{I}}]_{\mathrm{t}} - [\mathrm{I}, \mathrm{I}]_{\mathrm{t}} - [\tilde{\mathrm{I}}, \tilde{\mathrm{I}}]_{\mathrm{t}}\right).$$

Corollary 10.17

It holds that $[\mathrm{I}, \tilde{\mathrm{I}}]_{\mathrm{t}} = \int_0^{\mathrm{t}} \mathrm{X_s \tilde{X}_s ds}$ *a.s.*

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2. BM as weak limit
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4. Stopping times
5. Cond. expectation
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11. Ito's formula

Remark 10.18

Analogously to the quadratic variation the covariation of two stochastic processes Y and X can be defined as follows:

$$[Y,X]_t = \lim_{n\to\infty} \sum_{i=1}^{m_n} (Y_{t_i^n} - Y_{t_{i-1}^n})(X_{t_i^n} - X_{t_{i-1}^n}),$$

in probability where $0 = t_0^n < t_1^n < \dots t_{m_n}^n = t$ and $\max_{i=1,\dots,m_n}(t_i^n - t_{i-1}^n) \to 0$ for $n \to \infty$.

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9. Quad.var., path prop.
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11. Ito's formula

Integration by parts:

Theorem 10.19 (Integration by parts)

Let X, Y be continuous and adapted processes such that (10.3) holds for both processes. Then the following holds, for each $0 \leq t \leq T$,

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + [X, Y]_t.$$

Notation: $d(X_t Y_t) = X_t dY_t + Y_t dX_t + d[X, Y]_t$

11. Ito's formula: change of variables

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2. BM as weak limit

3. Gaussian p.

4. Stopping times

5. Cond. expectation

6. Martingales

7. Discrete stoch. integral

8. Refl. princ., pass.times

9. Quad.var., path prop.

10. Ito integral

11. Ito's formula

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Theorem 11.1

Let $(W_t)_{t\geq 0}$ be a Brownian motion and let $f:\mathbb{R}\to\mathbb{R}$ be a twice continuously differentiable function ($f\in C^2(\mathbb{R})$). Then, for each t,

$$f(W_t)-f(W_0)=\int_0^t f'(W_s)dW_s+\frac{1}{2}\int_0^t f''(W_s)ds. \qquad (11.1)$$

In differential notation (11.1) reads as follows:

$$df(W_t)=f'(W_t)dW_t+\frac{1}{2}f''(W_t)dt$$