Name:______________________________

1. Define two real functions f and g by

$$f(x) = \begin{cases} x^p \sin\left(\dfrac{1}{x^q}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} p^2 x^p + x^p \sin\left(\dfrac{1}{x^q}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

with $p \geq 2$ even, q a positive integer and $p - q > 1$.

(a) Prove that f and g are continuously differentiable on $\mathbb{R}$.

(b) Prove that $x = 0$ is a critical number for both f and g and that the sign of f' and g' change infinitely often around $x = 0$.

(c) Prove that f has neither a local maximum nor a local minimum at $x = 0$, but g has a local minimum at $x = 0$. Why doesn't this contradict Theorem 2.4.2?

2. Prove that for $x < 1$, we have the following inequality: $e^x \leq \dfrac{1}{1-x}$.

3. Let $f : (0, \infty) \to \mathbb{R}$ be differentiable on $(0, \infty)$ and suppose that $\lim\limits_{x\to\infty} (f(x) + f'(x)) = L$. Show that $\lim\limits_{x\to\infty} f(x) = L$ and $\lim\limits_{x\to\infty} f'(x) = 0$.

4. Let $f : [a, b] \to \mathbb{R}$ be a continuous nonnegative function such that

$$\int_a^b f = 0\,.$$

Prove or disprove that $f = 0$ on $[a, b]$.

5. Suppose $f : [0, 1] \to \mathbb{R}$ is a Lipschitz function with constant M. Prove that

$$\left| \int_0^1 f(x)\,dx - \frac{1}{n}\sum_{j=1}^{n} f\left(\frac{j}{n}\right) \right| \leq \frac{M}{n}\,.$$