1. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, then $f^{-1}(A)$ is closed for every closed set $A \subseteq \mathbb{R}$. Is the converse true? (Be sure to justify.) Prove or disprove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then $f(A)$ is closed for every closed set $A \subseteq \mathbb{R}$.

2. Let $\{K_n\}_{n=1}^{\infty}$ a nested decreasing sequence of compact sets in $\mathbb{R}$. If $f : \mathbb{R} \to \mathbb{R}$ is continuous, prove that

$$f\left(\bigcap_{n=1}^{\infty} K_n\right) = \bigcap_{n=1}^{\infty} f(K_n)\,.$$

3. Define the real function f by

$$f(x) = \begin{cases} x^p \sin\left(\dfrac{1}{x^q}\right) & \text{if } x \neq 0 \\ a & \text{if } x = 0\,. \end{cases}$$

For what real values of p, q, and a is f continuous on $\mathbb{R}$? Be sure to justify.

4. Let f be a real function that has the following property: Given a $c \in \mathbb{R}$ such that $f(a) < c < f(b)$, then there exists an x between a and b such that $f(x) = c$. Given a $r \in \mathbb{Q}$, define the following set:

$$A_r = \{x \in \mathbb{R} : f(x) = r\}\,.$$

Consider the following statement: "If for every $r \in \mathbb{Q}$ the set A_r is closed, then f is continuous." Prove or disprove this statement.

5. Let X be a set of real numbers and let $a \in \overline{X}$. Suppose that f, g, and h are real-valued functions on X. Prove that if $f(x) \leq g(x) \leq h(x)$ for all $x \in X$ and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L\,,$$

then $\lim_{x \to a} g(x) = L$. (This problem is known as the Squeeze Theorem.)