Functional Analysis (MA 8673) Homework 1

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1. Suppose A is self-adjoint and B is any operator such that $\|B - z_0\| \leq r$ for some $z_0 \in \mathbb{C}$ and $r > 0$. Show that $\sigma(A + B) \subseteq \sigma(A) + \overline{B_r(z_0)}$ where $\overline{B_r(z_0)}$ is the ball centered at z_0 with radius r.

   Proof. Let being by observing the properties of the operator $A + B - \lambda I$ with $\lambda \in \sigma(A + B)$. Note that this operator is not invertible due to the λ term. So let's rearrange the operator.

$$\begin{aligned} A + B - \lambda I &= A + B - \lambda I + z_0 I - z_0 I \\ &= (A - \lambda I + z_0 I) + (B - z_0 I) \\ &= (A - (\lambda - z_0)I) + (B - z_0 I) \end{aligned}$$

   With this rearrangement, let's now observe the spectral radius of both of the operators. First let's look at the A operator and for simplicity sake, let $\lambda - z_0 = \mu$. So we need to look at two cases of whether μ is in the spectrum of A or not.

   Case 1: $(\mu \in \sigma(A))$ So this case is trivial since if μ is in the spectrum of A, then the distance to the spectrum is 0.

   Case 2: $(\mu \notin \sigma(A))$ If μ is not in the spectrum of A, then then we know the operator $A - \mu I$ is an invertible operator. With that in mind, we go back to the original operator to factor as shown below.

$$A + B - \lambda I = (A - \mu I)[I + (A - \mu I)^{-1}(B - z_0 I)]$$

   From this we find that the operator $[I + (A - \mu I)^{-1}(B - z_0 I)]$ is not invertible. Since it is not invertible, then we also know that the norm (Von Neumann's Series Theorem) is greater than 1. With taht we get the following.

$$\|(A - \mu I)^{-1}(B - z_0 I)\| \geq 1$$

With our stated fact about the operator B being $\|B - z_0\| \leq r$ for some $z_0 \in \mathbb{C}$, we have the following inequality.

$$\begin{aligned} \|(A-\mu I)^{-1}(B - z_0 I)\| \geq 1 &\to \|(A-\mu I)^{-1}\| r \geq 1 \\ &\to \|(A-\mu I)^{-1}\| \geq \frac{1}{r} \end{aligned}$$

So now I want to the rewrite the norm on the LHS.

Claim: $\|(A-\mu I)^{-1}\| = \frac{1}{dist(\mu, \sigma(A))}$.

So first, note that A is a self-adjoint operator. With that in mind, we can rewrite the norm as below.

$$\|(A-\mu I)^{-1}\| = \sup_{\lambda \in \sigma(A)} |\frac{1}{\lambda - \mu}|$$

by spectral mapping theorem, we can do the following.

$$\sup_{\lambda \in \sigma(A)} |\frac{1}{\lambda - \mu}| = \lambda \in \sigma(A) \frac{1}{\inf_{\lambda \in \sigma(A)} |\lambda - \mu|} = \frac{1}{dist(\mu, \sigma(A))}$$

With this we have shown our claim and now we plug that into our earlier inequality to get

$$\begin{aligned} \|(A-\mu I)^{-1}\| \geq \frac{1}{r} &\to \frac{1}{dist(\mu, \sigma(A))} \geq \frac{1}{r} \\ &\to dist(\mu, \sigma(A)) \leq r \end{aligned}$$

This implies that there exists an $a \in \sigma(A)$ s.t. $|\mu - a| \leq r$. Let $w = z_0 + (\mu - a)$. Then $|w - z_0| = |\mu - a| \leq r$ and so $w \in \overline{B_r(z_0)}$. And

$$\lambda = z_0 + \mu = a + (z_0(\mu - a)) = a + w \in \sigma(A) + \overline{B_r(z_0)}$$

□

2. Let A be an self-adjoint operator and let P_A be its corresponding projection valued measures. Prove that:

$$\sigma(A) = \{\lambda \in \mathbb{R} : P_A(\lambda - \epsilon, \lambda + \epsilon) \neq 0, \forall \epsilon > 0\}.$$

Proof. ($\Rightarrow$) Assume there exists $\varepsilon > 0$ such that

$$P_A\big((\lambda - \varepsilon, \lambda + \varepsilon)\big) = 0.$$

Fix $\psi \in \mathcal{D}(A)$. By the spectral theorem,

$$\|(A-\lambda I)\psi\|^2 = \int_{\mathbb{R}} |t-\lambda|^2 \, d\mu_\psi(t), \qquad \mu_\psi(E) := \langle \psi, P_A(E)\psi \rangle.$$

Since $P_A((\lambda-\varepsilon, \lambda+\varepsilon)) = 0$, we have $\mu_\psi((\lambda-\varepsilon, \lambda+\varepsilon)) = 0$, hence the integral is supported on $\{t : \ |t-\lambda| \geq \varepsilon\}$. Therefore

$$\begin{aligned} \|(A-\lambda I)\psi\|^2 &= \int_{|t-\lambda|\geq\varepsilon} |t-\lambda|^2 \, d\mu_\psi(t) \\ &\geq \int_{|t-\lambda|\geq\varepsilon} \varepsilon^2 \, d\mu_\psi(t) \\ &= \varepsilon^2 \int_{\mathbb{R}} 1 \, d\mu_\psi(t) \\ &= \varepsilon^2 \|\psi\|^2, \end{aligned}$$

so

$$\|(A-\lambda I)\psi\| \geq \varepsilon \|\psi\| \qquad (\forall\, \psi \in \mathcal{D}(A)). \tag{1}$$

To conclude $\lambda \in \rho(A)$, we construct the inverse using functional calculus. Define a bounded Borel function

$$g(t) = \begin{cases} \dfrac{1}{t-\lambda}, & |t-\lambda| \geq \varepsilon, \\ 0, & |t-\lambda| < \varepsilon. \end{cases}$$

Then $\|g\|_\infty \leq 1/\varepsilon$, hence $B := g(A)$ is a bounded operator and $\|B\| \leq 1/\varepsilon$. Let $f(t) = t - \lambda$, so $f(A) = A - \lambda I$. Since $P_A((\lambda-\varepsilon, \lambda+\varepsilon)) = 0$, we have $g(t) = 1/(t-\lambda)$ P_A-a.e., and thus $f(t)g(t) = 1$ P_A-a.e. By the multiplicative property of the functional calculus,

$$(A-\lambda I)B = f(A)g(A) = (fg)(A) = I.$$

Similarly,

$$B(A-\lambda I) = I$$

(on $\mathcal{D}(A)$, the natural domain of $A - \lambda I$). Hence $B = (A-\lambda I)^{-1}$ is a bounded everywhere-defined inverse, so $\lambda \in \rho(A)$.

($\Leftarrow$) Conversely, assume $\lambda \in \rho(A)$. Then $\sigma(A)$ is closed, so $\rho(A) = \mathbb{C}\backslash\sigma(A)$ is open. Because $\lambda \in \mathbb{R} \cap \rho(A)$, there exists $\varepsilon > 0$ such that

$$(\lambda-\varepsilon, \lambda+\varepsilon) \cap \sigma(A) = \varnothing.$$

For a self-adjoint operator, the spectral measure P_A is supported on $\sigma(A)$, i.e.,

$$E \cap \sigma(A) = \varnothing \quad \Longrightarrow \quad P_A(E) = 0.$$

Applying this with $E = (\lambda - \varepsilon, \lambda + \varepsilon)$ yields

$$P_A\big((\lambda - \varepsilon, \lambda + \varepsilon)\big) = 0.$$

Combining the two directions proves the equivalence, and thus

$$\sigma(A) = \Big\{\lambda \in \mathbb{R} :\ P_A\big((\lambda - \epsilon, \lambda + \epsilon)\big) \neq 0,\ \forall\, \epsilon > 0\Big\}.$$

$\square$

3. Let A be a closed operator and set $|A| = \sqrt{A^*A}$. Prove that $\||A|f\| = \|Af\|$. Deduce that $\mathrm{Ker}(A) = \mathrm{Ker}(|A|) = \mathrm{Ran}(|A|)^\perp$ and that

$$U = \begin{cases} g = |A|f \to Af & \text{if } g \in \mathrm{Ran}(|A|) \\ g \to 0 & \text{if } g \in \mathrm{Ker}(|A|) \end{cases}$$

extends to a well-defined partial isometry. Conclude that $A = U|A|$. (This is an extension of the polar decomposition for not necessarily bounded operators)

Proof. **($\||A|f\| = \|Af\|$.):** So let's derive it be decomposing the norms. Also note that the abs operator is self-adjoint as well.

$$\begin{aligned} \||A|f\|^2 &= \langle |A|f, |A|f\rangle \\ &= \langle f, |A|^2 f\rangle \\ &= \langle f, A^*Af\rangle \\ &= \langle Af, Af\rangle \\ &= \|Af\|^2 \end{aligned}$$

Thus we have shown the equality.

($\mathrm{Ker}(A) = \mathrm{Ker}(|A|) = \mathrm{Ran}(|A|)^\perp$): So for the first equality, note that if $f \in \mathrm{Ker}(|A|)$, then $\||A|f\| = 0$. Since from the previous statement of the two norms being equal, then $f \in \mathrm{Ker}(|A|)$. We also get a similar result if we assume if $f \in \mathrm{Ker}(A)$.

For the second part of the equality, we use the fact that A is a self-adjoint operator. So a property for any operator is that $\mathrm{Ran}(|A|)^\perp = \mathrm{Ker}(|A^*|)$. Since A is s.a., we have $\mathrm{Ran}(|A|)^\perp = \mathrm{Ker}(|A|)$

(**Showing U extends to a well-defined partial isometry**):

Well-Defined: let f_1, f_2 be vector such that $|A|f_1 = |A|f_2$. Then

$$|A|(f_1 - f_2) = 0$$

Implying that $f_1, f_2 \in \mathrm{Ker}(|A|)$. Since $\mathrm{Ker}(|A|) = \mathrm{Ker}(A)$, then $(f_1 - f_2) \in \mathrm{Ker}(A)$. Therefore, $A(f_1 - f_2) = 0 \implies Af_1 = Af_2$. So the mapping $g \mapsto Af$ (where $g = |A|f$) is consistent.

(Partial Isometry:) For any vector $g \in \mathrm{Ran}(|A|)$, let $g = |A|f$.

$$\|Ug\| = \|U(|A|f)\| = \|Af\| = \||A|f\| = \|g\|$$

Since $\|Ug\| = \|g\|$, the operator U preserves lengths on the range of $|A|$. Since U is an isometry on $\mathrm{Ran}(|A|)$, it extends continuously to the closure $\overline{\mathrm{Ran}(|A|)}$. On the orthogonal complement $(\mathrm{Ran}(|A|)^{\perp} = \mathrm{Ker}(|A|))$, we defined U to be 0. An operator that is an isometry on a subspace and 0 on its orthogonal complement is the definition of a Partial Isometry.

($A = U|A|$.): We first analyze the RHS by letting $g \in \mathrm{Ran}(|A|)$. Then by our definition of U, $Ug = Af$. Thus showing the equality.

$\square$

4. Let A be a self-adjoint operator. Prove that the resolvent operator can be realized as the following representation:

$$R_A(z) = \int_{\mathbb{R}} \frac{1}{\lambda - z} dP_\lambda.$$

Deduce that the quadratic form $F_\psi(z) = \langle \psi, R_A(z)\psi \rangle$ is a holomorphic function from the upper half plane to itself. (These types of functions are called **Herglotz** functions and this process is called the **Borel** transforms of measures.)

Proof. (Resolvent Representation): Our goal is to show that the usual definition of the resolvent operator being $R_A(z) = (A - zI)^{-1}$, with $z \in \rho(A)$, can be rewritten as the intergal in the statement. So to do this, note that A is a s.a. operator. Then by the Spectral Theorem for self adjoint operators, we can rewrite A as

$$A = \int_{\mathbb{R}} \lambda dP_\lambda.$$

Since A is self-adjoint, its spectrum $\sigma(A)$ is real, so any non-real z belongs to $\rho(A)$. Applying the Borel Functional Calculus, we can define the operator $f(A)$ to be

$$f(A) = \int_{\mathbb{R}} f(\lambda) dP_\lambda.$$

To find the integral representation, we look for the function $f(\lambda)$ that corresponds to this operator, We define it to be

$$f(\lambda) = \frac{1}{\lambda - z}$$

Therefore, by the functional calculus:

$$R_A(z) = (A - zI)^{-1} = \int_{\mathbb{R}} \frac{1}{\lambda - z}\, dP_\lambda$$

(Herglotz): Using the result from Part 1, we substitute the integral representation into the inner product:

$$F_\psi(z) = \left\langle \psi, \left(\int_{\mathbb{R}} \frac{1}{\lambda - z} dP_\lambda \right) \psi \right\rangle$$

The scalar measure associated with the vector ψ is defined as $d\mu_\psi(\lambda) = d\langle \psi, P_\lambda \psi \rangle$. This is a positive measure.

$$F_\psi(z) = \int_{\mathbb{R}} \frac{1}{\lambda - z} d\mu_\psi(\lambda)$$

Let z be in the upper half-plane, so $z = x + iy$ with $y > 0$. We want to see if the imaginary part of $F_\psi(z)$ is also positive.

Substitute z into the integrand.

$$\frac{1}{\lambda - (x + iy)} = \frac{1}{(\lambda - x) - iy}$$

Multiply numerator and denominator by the conjugate $((\lambda - x) + iy)$ to get

$$\frac{(\lambda - x) + iy}{(\lambda - x)^2 + y^2} = \frac{\lambda - x}{(\lambda - x)^2 + y^2} + i \frac{y}{(\lambda - x)^2 + y^2}$$

Now, look at the imaginary part of the integral.

$$\mathrm{Im}(F_\psi(z)) = \int_{\mathbb{R}} \mathrm{Im}\left(\frac{1}{\lambda - z} \right) d\mu_\psi(\lambda)$$

$$\mathrm{Im}(F_\psi(z)) = \int_{\mathbb{R}} \frac{y}{(\lambda - x)^2 + y^2} d\mu_\psi(\lambda)$$

From this we have that $y > 0$ (because z is in the upper half-plane), the denominator $(\lambda - x)^2 + y^2$ is always positive, and the measure $d\mu_\psi$ is a positive measure (since $\langle \psi, P_\lambda \psi \rangle$ is a norm squared). Therefore, the integral is positive.

$$\mathrm{Im}(F_\psi(z)) > 0$$

Since $\mathrm{Im}(z) > 0$ implies $\mathrm{Im}(F_\psi(z)) > 0$, $F_\psi(z)$ maps the upper half plane to itself. $\square$