Functional Analysis Homework 8

Kevin Ho

October 3rd

1. Prove that a linear operator $T : X \to Y$ is bounded iff it maps sequences that converge to zero to bounded sequences.

   Proof. ($\Rightarrow$) Suppose T is bounded. Then there exists $M > 0$ such that $\|Tx\| \leq M\|x\|$ for all $x \in X$. Let $(x_n) \subset X$ with $x_n \to 0$. Then

   $$\|Tx_n\| \leq M\|x_n\| \longrightarrow 0,$$

   so in particular (Tx_n) is bounded.

   ($\Leftarrow$) Now assume that for every sequence $(x_n) \subset X$ with $x_n \to 0$, the image sequence (Tx_n) is bounded in Y. We will show T is bounded by contradiction. Suppose T were unbounded. Then for each $n \in \mathbb{N}$ there exists $u_n \in X$ with $\|u_n\| = 1$ and $\|Tu_n\| \geq n^3$ (since $\sup_{\|x\|=1} \|Tx\| = \infty$ when T is unbounded). Define

   $$x_n := \frac{1}{n^2} u_n.$$

   Then $\|x_n\| = \frac{1}{n^2} \to 0$, but

   $$\|Tx_n\| = \frac{1}{n^2}\|Tu_n\| \ \geq \ \frac{1}{n^2} \cdot n^3 \ = \ n,$$

   so (Tx_n) is unbounded. This contradicts the assumption. Hence T must be bounded. □

2. Compute the norm of Volterra operator. (tldr the test question) Let $T : L_2[0,1] \to L_2[0,1]$ is defined as

   $$(Tf)(t) = \int_0^t f(s)ds$$

   Proof.

   $$\|Tf(t)\|^2 = \left|\int_0^1 \int_0^t f(s)dsdt\right|^2 \leq \int_0^1 \left|\int_0^t f(s)ds\right|^2 dt$$

$$\to \int_0^1 \left| \int_0^t \frac{f(s)}{\sqrt{\cos\left(\frac{\pi}{2}s\right)}} \sqrt{\cos\left(\frac{\pi}{2}s\right)} \right|^2 dsdt$$

$$\leq \int_0^1 \left(\left| \int_0^t \frac{f(s)}{\sqrt{\cos\left(\frac{\pi}{2}s\right)}} \right|^2 ds \right) \left(\left| \int_0^t \sqrt{\cos\left(\frac{\pi}{2}s\right)} \right|^2 ds \right) dt$$

$$= \int_0^1 \int_0^t \frac{|f(s)|^2}{\cos\left(\frac{\pi}{2}s\right)} \int_0^t \cos\left(\frac{\pi}{2}s\right) ds = \frac{2}{\pi} \int_0^1 \sin\left(\frac{\pi}{2}t\right) \int_0^t \frac{|f(s)|^2}{\cos\left(\frac{\pi}{2}s\right)}$$

By Fubini's Theorem

$$= \frac{2}{\pi} \int_0^1 \int_s^1 \left(\sin\left(\frac{\pi}{2}t\right) dt\right) \frac{|f(s)|^2}{\cos\left(\frac{\pi}{2}s\right)} ds = \frac{2}{\pi} \int_0^1 \frac{2}{\pi} \frac{|f(s)|^2}{\cos\left(\frac{\pi}{2}s\right)} \cos\left(\frac{\pi}{2}s\right) ds$$

$$= \left(\frac{2}{\pi}\right)^2 \int_0^1 |f(s)|^2 ds = \left(\frac{2}{\pi}\right)^2 \|f\|_2^2 \geq \|Tf(t)\|^2$$

We can show that this ineqality is an equality by letting $f(s) = \cos\left(\frac{\pi}{2}s\right)$

$$\|Tf(t)\|^2 = \left| \int_0^1 \int_0^t f(s) dsdt \right|^2 = \left| \int_0^1 \int_0^t \cos\left(\frac{\pi}{2}s\right) dsdt \right|^2$$

$$= \left| \int_0^1 \frac{2}{\pi} \left(\sin\left(\frac{\pi}{2}t\right) dt\right) \right|^2 = \left(\frac{2}{\pi}\right)^2 \int_0^1 \sin^2\left(\frac{\pi}{2}t\right) dt$$

$$= \left(\frac{2}{\pi}\right)^2 \int_0^1 \frac{1 - \cos\left(\frac{\pi}{2}t\right)}{2} dt = \left(\frac{2}{\pi}\right)^2 \frac{1}{2}$$

$$\|f\|^2 = \int_0^1 \cos^2\left(\frac{\pi}{2}t\right) dt = \int_0^1 \frac{1 + \cos\left(\frac{\pi}{2}t\right)}{2} dt = \frac{1}{2}$$

And so

$$\|T\| = \frac{\|Tf\|}{\|f\|} = \frac{2}{\pi}$$

$\square$

3. Show that a surjective linear map $T : X \to Y$ between normed spaces is an isomorphism iff $\exists c, C > 0$ s.t.

$$c\|x\| \leqslant \|Tx\| \leqslant C\|x\|, \forall x \in X.$$

Proof. ($\Rightarrow$): So with T being a isomorphism, we know that T is continuous and so giving us that T is bounded by C s.t. $\|Tx\| \leq C\|x\|$. So we need to showing a bounded below by a constant c. By definition of T being an isomorphism, we have that T^{-1} is continuous. With that in mind, we have $\|T^{-1}y\| \leq M\|y\|$ as $Tx = y, T^{-1}y = x$ and so this lead to

$$\|T^{-1}y\| \leq c\|y\| \rightarrow \|x\| \leq M\|Tx\|$$

giving us $c\|x\| \leq \|Tx\|$

($\Leftarrow$): Now let's assume the inequality is true. To show that T is an isomorphism given the inequality and surjectivity of T. This mean we need to show that T is continuous, injective, and that T^{-1} is continuous. From the inequality of $\|Tx\| \leq C\|x\|$ we have that T is bounded and so thus is continuous. Now let's show injectivity. Let $Tx = 0$ and $x \in ker(T)$. If we plug that into our inequality, we have:

$$c\|x\| \leq \|Tx\| \rightarrow c\|x\| \leq 0$$

and so x must equal 0. Giving us injectivity. So now we want to show that T^{-1} is continuous/bounded. So we know that T^{-1} exists as there is a bijection and so let $y \in Y, T^{-1}y = x, Tx = y$. If we look at our inequality of $c\|x\| \leq \|Tx\| \rightarrow c\|T^{-1}y\| \leq \|y\|$ which gives us boundedness and so continuous.

$\square$

4. Let $T \in L(X,Y)$ and $S \in L(Y,Z)$. Show that $(ST)^* = T^*S^*$.

   Proof. LetX, Y, Z be Hilbert spaces and $(T \in L(X,Y))$, $(S \in L(Y,Z))$. By definition of the adjoint, for all $x \in X$ and $z \in Z$,

$$\langle STx, z\rangle_Z = \langle Tx, S^*z\rangle_Y = \langle x, T^*(S^*z)\rangle_X = \langle x, (T^*S^*)z\rangle_X$$

   Thus the operator $(T^*S^* : Z \rightarrow X)$ satisfies the adjoint identity for (ST). By uniqueness of adjoints, $((ST)^* = T^*S^*)$.

   $\square$

5. Let $S, T \in L(X,Y)$, and $a, b \in \mathbb{C}$. Show that $(aS + bT)^* = \bar{a}S^* + \bar{b}T^*$.

   Proof. By the definition of the adjoint, for any $x \in X, y \in Y$, we have:

$$\langle (aS + bT)x, y\rangle_Y = \langle aSx + bTx, y\rangle_Y = a, \langle Sx, y\rangle_Y + b, \langle Tx, y\rangle_Y$$
$$= a, \langle x, S^*y\rangle_X + b, \langle x, T^*y\rangle_X = \langle x, \overline{a}, S^*y + \overline{b}, T^*y\rangle_X.$$

   Since this holds for all x, y, the uniqueness of the adjoint implies $(aS + bT)^* = \bar{a}S^* + \bar{b}T^*$. $\square$

6. Let $T \in L(X,Y)$ be such that $T^{-1} \in L(Y,X)$. Show that $(T^{-1})^* = (T^*)^{-1}$

Proof.

$$(TT^{-1})^* = I_Y \ \Rightarrow\ (T^{-1})^* T^* = I_Y,$$
$$(T^{-1}T)^* = I_X \ \Rightarrow\ T^*(T^{-1})^* = I_X.$$

□

7. Assume that a kernel function $k(t,s)$ satisfies

$$\sup_{t\in[0,1]} \int_0^1 |k(t,s)|ds =: M_1 < \infty,$$

$$\sup_{s\in[0,1]} \int_0^1 |k(t,s)|ds =: M_2 < \infty,$$

Show that the integral operator (2.11) $T : L_2[0,1] \to L_2[0,1]$with kernel $k(t,s)$ is bounded and

$$\|T\| \leqslant \sqrt{M_1 M_2}.$$

Proof.

$$\|Tx\|^2 = \int_0^1 |(Tx)(t)|^2 dt = \int_0^1 \left| \int_0^1 k(t,s)x(s)ds \right|^2 dt$$

$$\leq \int_0^1 \left(\int_0^1 |k(t,s)||x(s)|ds \right)^2 dt \leq \int_0^1 \left(\int_0^1 |k(t,s)|ds \right) \left(\int_0^1 |k(t,s)||x(s)|^2 ds \right) dt$$

With Fubini's and definition of our kernel we get

$$\leq M_1 \int_0^1 \left(\int_0^1 |k(t,s)||x(s)|^2 ds \right) dt = M_1 \int_0^1 \left(\int_0^1 |k(t,s)||x(s)|^2 dt \right) ds$$

$$= M_1 \int_0^1 |x(s)|^2 \left(\int_0^1 |k(t,s)|dt \right) ds \leq M_1 M_2 \int_0^1 |x(s)|^2 ds$$

$$= M_1 M_2 \|x\|^2 \implies \|Tx\| \leq \sqrt{M_1 M_2} \|x\|$$

□

8. Let $A, B \subseteq X$ normed space. Prove the following:

   (a) $A^\perp$ is a closed linear subspace of X^*.

   (b) If $A \subseteq B$ then $A^\perp \supseteq B^\perp$.

   (c) $(A \cup B)^\perp = A^\perp \cap B^\perp$. Give an example where $(A \cap B)^\perp \neq A^\perp \cup B^\perp$.

   (d) $A^\perp = (\text{Span } A)^\perp$.

   (e) $(\bar{A})^\perp = A^\perp$.

   (f) Suppose X_0 is a closed linear subspace of X. Then $X_0^\perp = \{0\}$ is equivalent to $X_0 = X$.

Proof. (a) let $f, g \in A^\perp$ and α, β be a scalar. For any $a \in A$.

$$(\alpha f + \beta g)(a) = \alpha f(a) + \beta g(a) = \alpha * 0 + \beta * 0 = 0$$

so thus is a linear subspace of X^*. Now to show closed, let $\{f_n\} \in A^\perp$ and $f_n \to f \in X^*$ With continuity of f, $a_n \to x \implies f(a_n) \to f(x)$ and so

$$f(a) = \lim_{n\to\infty} f(a_n) = 0$$

This holds for all $a \in A$, so $f \in A^\perp$. $A^\perp$ contains all its limit points.

(b) let $f \in B^\perp$. This gives us that $f(b) = 0, b \in B$. Since $A \subseteq B \implies a \in A, a \in B$. This gives us that $f(a) = 0, \forall a \in A \implies f \in A^\perp$. Thus $A^\perp \supseteq B^\perp$.

(c)

$$f \in (A \cup B)^\perp \iff f(x) = 0, \forall x \in A \cup B$$
$$\iff f(a) = 0, \forall a \in A, f(b) = 0, \forall b \in B$$
$$\iff f \in A^\perp, f \in B^\perp \iff f \in A^\perp \cap B^\perp.$$

Example: Let $X = \mathbb{R}^2$, $A = \text{Span}\{(1, 0)\}$, $B = \text{Span}\{(0, 1)\}$.

LHS: $A \cap B = \{0\}$, so $(A \cap B)^\perp = \{0\}^\perp = X^* \cong \mathbb{R}^2$.

RHS: $A^\perp = \{f(x, y) = c_2 y\}$ (functionals zero on x-axis). $B^\perp = \{f(x, y) = c_1 x\}$ (functionals zero on y-axis).

$A^\perp \cup B^\perp$ is only the set of functionals that depend on just one axis, which is not all of X^*. For example, $f(x, y) = x + y$ is in the LHS but not the RHS.

(d) ($\supseteq$): $A \subseteq \text{Span } A$. By property (b), $A^\perp \supseteq (\text{Span } A)^\perp$.

($\subseteq$): Let $f \in A^\perp$. Let $x \in \text{Span } A$, so $x = \sum_{i=1}^n \alpha_i a_i$ for $a_i \in A$. By linearity, $f(x) = f(\sum \alpha_i a_i) = \sum \alpha_i f(a_i)$. Since $f \in A^\perp$, $f(a_i) = 0, \forall i$. $f(x) = \sum \alpha_i(0) = 0$. Thus $f \in (\text{Span } A)^\perp$.

(e) ($\supseteq$): $A \subseteq \bar{A}$. By property (b), $A^\perp \supseteq (\bar{A})^\perp$.

($\subseteq$): Let $f \in A^\perp$. Let $x \in \bar{A}$. Then $\exists \{a_n\} \in A$ s.t. $a_n \to x$. Since $f \in X^*$, f is continuous,

$$f(x) = f(\lim_{n\to\infty} a_n) = \lim_{n\to\infty} f(a_n).$$

Since $a_n \in A$, $f(a_n) = 0, \forall n$. $f(x) = \lim_{n\to\infty} 0 = 0$. Thus $f \in (\bar{A})^\perp$.

(f) ($\Longleftarrow$): If $X_0 = X$, then $X_0^\perp = X^\perp$. The only functional $f \in X^*$ that annihilates all of X is the zero functional. So $X^\perp = \{0\}$.

($\Longrightarrow$): Assume $X_0^\perp = \{0\}$. Suppose for contradiction $X_0 \neq X$. Then there exists some $x_0 \in X \setminus X_0$. By Hahn-Banach, since X_0 is a closed subspace and $x_0 \notin X_0$, $\exists f \in X^*$ such that $f(X_0) = 0$ and $f(x_0) \neq 0$. $f(X_0) = 0$ means $f \in X_0^\perp$. $f(x_0) \neq 0$ means $f \neq 0$. This f is a non-zero element of $X_0^\perp$, which contradicts the assumption $X_0^\perp = \{0\}$. Therefore, $X_0 = X$. $\square$

9. Let H be a Hilbert space. A sequlinear form on H is a function $B : H \times H \to \mathbb{C}$ which is linear in the first argument and conjugate-linear in the second argument, i.e.

$$B(a_1x_1 + a_2x_2, y) = a_1B(x_1, y) + a_2B(x_2, y),$$

$$B(x, b_1y_1 + b_2y_2) = \overline{b_1}B(x, y_1) + \overline{b_2}B(x, y_2).$$

An example of a sesquilinear form is $B(X, y) = \langle Tx, y\rangle$ where $T \in L(H, H)$. Consider a sesquilinear form $B(x, y)$ which satisfies

$$|B(x, y)| \leqslant M\|x\|\|y\|, \ x, y \in H$$

for some number M. Show that $\exists T \in L(H, H)$ with $\|T\| \leqslant M$ and such that

$$B(x, y) = \langle Tx, y\rangle, \forall x, y \in H.$$

Proof. Let H be a Hilbert space and $B : H \times H \to \mathbb{C}$ be sesquilinear (linear in the first argument and conjugate-linear in the second) with

$$|B(x, y)| \leq M\|x\|\|y\| \qquad (x, y \in H).$$

Fix $x \in H$ and define $\psi_x : H \to \mathbb{C}$ by $\psi_x(y) := \overline{B(x, y)}$. Since B is conjugate-linear in y, ψ_x is linear; moreover,

$$|\psi_x(y)| = |B(x, y)| \leq M\|x\|\,\|y\|,$$

so ψ_x is a bounded linear functional with $\|\psi_x\| \leq M\|x\|$.

By the Riesz representation theorem, there exists a unique $z \in H$ such that

$$\psi_x(y) = \langle y, z\rangle \qquad \text{for all } y \in H.$$

Define $T : H \to H$ by $T(x) := z$. Then for all $x, y \in H$,

$$B(x, y) = \overline{\psi_x(y)} = \overline{\langle y, T(x)\rangle} = \langle T(x), y\rangle,$$

which is the desired representation.

Linearity of T follows from the linearity of B in its first argument: for $a_1, a_2 \in \mathbb{C}$ and $x_1, x_2, y \in H$,

$$\langle T(a_1x_1+a_2x_2), y\rangle = B(a_1x_1+a_2x_2, y) = a_1B(x_1, y)+a_2B(x_2, y) = \langle a_1T(x_1)+a_2T(x_2), y\rangle,$$

hence $T(a_1x_1 + a_2x_2) = a_1T(x_1) + a_2T(x_2)$.

Finally, for each $x \in H$,

$$\|T(x)\| = \sup_{\|y\|=1} |\langle T(x), y\rangle| = \sup_{\|y\|=1} |B(x, y)| \leq M\|x\|,$$

so T is bounded and $\|T\| \leq M$. □

10. Let X, Y be normed spaces and $p \in [1, \infty]$. Define the direct sim of $X \oplus_p Y$ as the Cartesian product $X \times Y$ equipped with the norm

$$\|(x,y)\| := (\|x\|^p + \|y\|^p)^{\frac{1}{p}} \text{ if } p < \infty, \ \|(x,y)\| := (\max(\|x\|, \|y\|) \text{ if } p = \infty.$$

Show that $X \oplus_p Y$ is a normed space, and all norms $\|(x,y)\|_p, \ p \in [1, \infty]$, are equivalent to each other.

For this reason, the index p is usually omitted from notional and the space $X \oplus Y$ is called the direct sum of X, Y.

Proof. We first show $X \oplus_p Y$ is a normed space.

Positive definiteness and homogeneity are immediate from the corresponding properties on X and Y.

Triangle inequality. For $p = \infty$,

$$\|(x_1+x_2, y_1+y_2)\|_\infty = \max\{\|x_1+x_2\|, \|y_1+y_2\|\} \le \max\{\|x_1\|+\|x_2\|, \|y_1\|+\|y_2\|\}$$

$$\le \max\{\|x_1\|, \|y_1\|\} + \max\{\|x_2\|, \|y_2\|\} = \|(x_1, y_1)\|_\infty + \|(x_2, y_2)\|_\infty.$$

For $1 \le p < \infty$, using $\|x_1+x_2\| \le \|x_1\|+\|x_2\|$ and $\|y_1+y_2\| \le \|y_1\|+\|y_2\|$, we get

$$\|(x_1+x_2, y_1+y_2)\|_p = \big(\|x_1+x_2\|^p + \|y_1+y_2\|^p\big)^{1/p} \le \big((\|x_1\|+\|x_2\|)^p + (\|y_1\|+\|y_2\|)^p\big)^{1/p}.$$

Applying Minkowski's inequality in $\mathbb{R}^2$ with the ℓ_p norm to $u = (\|x_1\|, \|y_1\|)$ and $v = (\|x_2\|, \|y_2\|)$ gives

$$\big((\|x_1\|+\|x_2\|)^p + (\|y_1\|+\|y_2\|)^p\big)^{1/p} \le (\|x_1\|^p + \|y_1\|^p)^{1/p} + (\|x_2\|^p + \|y_2\|^p)^{1/p},$$

i.e.,

$$\|(x_1 + x_2, y_1 + y_2)\|_p \le \|(x_1, y_1)\|_p + \|(x_2, y_2)\|_p.$$

Thus $X \oplus_p Y$ is a normed space.

We now prove the norms are equivalent. Fix $1 \le p < \infty$ and set $M := \|(x,y)\|_\infty = \max\{\|x\|, \|y\|\}$. Then

$$\|(x,y)\|_\infty = M \le (\|x\|^p + \|y\|^p)^{1/p} = \|(x,y)\|_p,$$

since $M^p \le \|x\|^p + \|y\|^p$. Also,

$$\|(x,y)\|_p = (\|x\|^p + \|y\|^p)^{1/p} \le (M^p + M^p)^{1/p} = 2^{1/p} M = 2^{1/p} \|(x,y)\|_\infty.$$

Hence for all $(x,y) \in X \oplus Y$,

$$\|(x,y)\|_\infty \ \le \ \|(x,y)\|_p \ \le \ 2^{1/p} \, \|(x,y)\|_\infty.$$

Therefore $\|\cdot\|_p$ and $\|\cdot\|_\infty$ are equivalent. By transitivity of norm equivalence, all $\|\cdot\|_p$ for $p \in [1, \infty]$ are pairwise equivalent.

$\square$

11. Let X, Y be normed spaces, and X be finite dimensional. Show that every linear operator $T : X \to Y$ is bounded.

Proof. Since X is finite dim, let $dim(X) = n$ and let $\{e_n\}$ be the canonical basis for X. With this we can write each $x \in X$ to be $x = \sum_{i=1}^{n} \alpha_i e_i$ for α_i being scalars. So now let's observe the op norm of T.

$$\|Tx\| = \|T\left(\sum_{i=1}^{n} \alpha_i e_i\right)\| \leq \sum_{i=1}^{n} \|\alpha_i T e_i\| = \sum_{i=1}^{n} |\alpha_i| \|T e_i\| = C \sum_{i=1}^{n} |\alpha_i|$$

with C = $\|Te_i\|$.Now, define a new norm on X by $\|x\|_1 = \sum_{i=1}^{n} |\alpha_i|$. Since X is finite-dimensional, all norms on X are equivalent. Thus, there exists a constant $K > 0$ such that $\|x\|_1 \leq K\|x\|_X$ for all $x \in X$.

Combining these inequalities:

$$\|T(x)\|_Y \leq C\|x\|_1 \leq C(K\|x\|_X)$$

Let $M = CK$. Since C and K are finite, M is a finite constant. We have found an M such that $\|T(x)\|_Y \leq M\|x\|_X$ for all $x \in X$.

Therefore, T is bounded.

$\square$