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MA 4643/6643 Section 01 — *Homework Set 3* — *Due: 2/14/2025*

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1. Let (S, d) be a metric space and A and B are two sets in S whose closures are disjoint. Prove there exists a continuous function $f : S \to [0, 1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$. Prove or disprove there are two disjoint open sets O_1 and O_2 such that $A \subsetneqq O_1$ and $B \subsetneqq O_2$.

2. Prove that $c_0(\mathbb{N})^*$ and $\ell^1(\mathbb{N})$ are isometrically isomorphic.

3. Prove that each half-open interval of the form $[a, b)$ is homeomorphic to each closed half-interval of the form $(-\infty, c]$.

4. Let $\delta : C[0, 1] \to \mathbb{R}$ be given by $\delta(f) = f(0)$. Prove that δ is linear. Show that δ is continuous when $C[0, 1]$ is equipped with $\| \cdot \|_\infty$ and compute $\|\delta\|_{\text{op}}$. Prove or disprove that δ is continuous when $C[0, 1]$ is equipped with $\| \cdot \|_1$.

5. Let $(X, \| \cdot \|)$ be a Banach space and let $x = \{x_k\}$ be a bounded sequence in X. Let $t = \{t_k\} \in \ell^1(\mathbb{N})$ and define a map $T : \ell^1(\mathbb{N}) \to X$ by

$$T(t) = \sum_{k=1}^{\infty} t_k x_k \,.$$

Prove that T defines a continuous linear operator from $\ell^1(\mathbb{N})$ into X and compute $\|T\|_{\text{op}}$.