

Functional Analysis Homework 4

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1. Show that the orthogonal projection P_Y is a linear map. Check that $Im(P_Y) = Y$ and $ker(P_Y) = Y^\perp$. Also check that the identity map I_X on X can be decomposed as

$$I_X = P_Y + P_{Y^\perp}.$$

Proof: So first let's show the linearity of the orthogonal Projection P_Y . so P_Y is defines as

$$P_Y : X \rightarrow X, P_Y x = y$$

where it is the projection of X onto Y

- (1) let α be a scalar. Write x as the orthogonal decomposition $x = y + z$ where $y \in Y$ and $z \in Y^\perp$. By the uniqueness of the projection, we just need to check two conditions for the vector αy of whether $\alpha y \in Y$ and $(\alpha x - \alpha y) \in Y^\perp$. So it is obvious that $\alpha y \in Y$ as we know $y \in Y$. For the second condition we get

$$(\alpha x - \alpha y) = \alpha(x - y) \in Y^\perp$$

so Homogeneity is there.

- (2) Let $x_0, x_1 \in X$. We then get two different decompositions of $x_0 = y_0 + z_0$ and $x_1 = y_1 + z_1$ very similarly with homogeneity, we get a similar result of $y_0 + y_1 \in Y$ and $(x_0 - y_0) + (x_1 - y_1) \in Y^\perp$. as each term is in Y^\perp .

Thus we have shown that P_Y is a linear map. So now let's check $Im(P_Y) = Y$ and $ker(P_Y) = Y^\perp$. So since $P_Y = y \in Y$ we know the first part is true. So let's show this by showing each space is in each other. So let $y \in Y$. We want to find a vector x s.t. $P_Y x = y$ so we can choose $x=y$ to where $x = y + 0$ is the orthogonal decomposition. Since $P_Y x \in Y$ we can conclude that $Y \subseteq Im(P_Y)$. So now let's see the other direction. Let $v \in Im(P_Y)$ be any vector. This means that $v = P_Y x = y$ so $Im(P_Y) \subseteq Y$ so $Im(P_Y) = Y$.

For the $ker(P_Y) = Y^\perp$, let $z \in Y^\perp$. With this, if we take the orthogonal decomposition of v , we get that $z = 0 + z$. Because $0 \in Y$ and $z \in Y^\perp$, we get that the Y component of the decomposition

is 0 leading to $P_Y z = 0$ meaning $z \in \ker(P_Y)$ so $Y^\perp \subseteq \ker(P_Y)$. Now let $v \in \ker(P_Y)$. Then we get the decomposition of v to be $v = 0 + z$. This leads to $v = z \in Y^\perp$ thus $\ker(P_Y) \subseteq Y^\perp$ so then $\ker(P_Y) = Y^\perp$.

Now let us check the identity map. So let $x \in X$ be any vector. Then we get the equation $I_x(x) = x = y + z$ where $y \in Y, z \in Y^\perp$. When we apply $P_Y + P_{Y^\perp}$ we get the following:

$$(P_Y + P_{Y^\perp})(x) = (P_Y)(x) + (P_{Y^\perp})(x) = y + z.$$

□

2. Let A be a subset of a Hilbert space. Show that

$$A^\perp = \bar{A}^\perp$$

where \bar{A} denotes the closure of A .

($A^\perp \subseteq \bar{A}^\perp$) Let $x \in A^\perp$. By definition, $\langle x, a \rangle = 0, a \in A$. Let $v \in \bar{A}$ arbitrary. By definition of the closure, there is a sequence $\{a_n\}$ s.t. $a_n \rightarrow v$. So with $x \in A^\perp$, we get that $\langle x, a_n \rangle = 0, \forall n \in \mathbb{N}$. Because the inner product on the Hilbert Space is a continuous function, we can take the limit as $n \rightarrow \infty$.

$$\lim_{n \rightarrow \infty} \langle x, a_n \rangle = \langle x, \lim_{n \rightarrow \infty} a_n \rangle = \langle x, v \rangle = 0$$

Thus we have shown $A^\perp \subseteq \bar{A}^\perp$.

($\bar{A}^\perp \subseteq A^\perp$) Let $x \in \bar{A}^\perp$. By definition, $\langle x, y \rangle = 0, y \in \bar{A}$. So with $A \subseteq \bar{A}$, this means that $\langle x, v \rangle = 0, v \in A$ so thus $\bar{A}^\perp \subseteq A^\perp$. □

3. Let Y denote the subspace of constant functions in $L_2 = L_2(\Omega, \Sigma, \mu)$. Compute $P_Y f$ for an arbitrary function $f \in L_2$

Proof: So let $c \in Y$ to where $P_Y f = c$. This means that there must be an orthogonal decomposition of f where $f = c + g$ where $f - c \in Y^\perp$. So then we must find where $\langle f - c, g \rangle = 0$. So wlog, let $g = 1$. Then we get

$$\begin{aligned} \langle f - c, 1 \rangle &= \int_{\Omega} (f - c) * (1) d\mu = 0 \\ &\Rightarrow \int_{\Omega} f d\mu - \int_{\Omega} c(1) d\mu = 0 \\ &\Rightarrow \int_{\Omega} f d\mu - c \int_{\Omega} 1 d\mu = 0 \\ &\Rightarrow \int_{\Omega} f d\mu = c * \mu(\Omega) \\ &\Rightarrow \frac{\int_{\Omega} f d\mu}{\mu(\Omega)} = c = P_Y f \end{aligned}$$

□