Name:______________________________

MA 4633/6633 Section 01 *Homework Set 10* *Due: 11/22/2024*

1. Let $E \subseteq \mathbb{R}$ and assume $\{f_n\}$ and $\{g_n\}$ are two sequences that converge uniformly on E. Prove that $\{f_n + g_n\}$ converges uniformly on E. We know from the limit laws that $\{f_n g_n\}$ will converge pointwise on E, but show that $\{f_n g_n\}$ need not converge uniformly on E. What additional assumptions on $\{f_n\}$ and $\{g_n\}$ are needed to make the convergence uniform? Be sure to justify.

2. Suppose for every $n \geq 1$ that $\{f_n\}$ are twice differentiable functions on $[a, b]$ such that both $\{f_n\}$ and $\{f_n'\}$ converge uniformly on $[a, b]$. Prove or disprove that $\{f_n''\}$ converges uniformly on $[a, b]$.

3. Let $g : (0, \infty) \to \mathbb{R}$ and for every $n \geq 1$, $f_n : (0, \infty) \to \mathbb{R}$. For every $0 < t < T < \infty$, assume that $g \in \mathcal{R}_{\text{loc}}[t, T]$ and for every $n \geq 1$, $f_n \in \mathcal{R}_{\text{loc}}[t, T]$. If for every $n \geq 1$, $|f_n| \leq g$, $f_n \to f$ uniformly on every compact subset of $(0, \infty)$ and $g \in \mathcal{R}(0, \infty)$, prove that

$$\lim_{n\to\infty} \int_0^\infty f_n = \int_0^\infty f \,.$$

4. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

converges uniformly on every bounded interval, but does not converge absolutely for any value of x. (This shows the necessity to have both uniform convergence and absolute convergence and one doesn't necessarily imply the other.)

5. Define the function χ on $\mathbb{R}$ by, $\chi(x) = 1$ if $x > 0$ and $\chi(x) = 0$ if $x \leq 0$. Suppose the series $\sum c_n$ converges absolutely and let $\{x_n\}$ be a sequence of distinct points in (a, b). Formally define the function f on $[a, b]$ by:

$$f(x) = \sum_{n=1}^{\infty} c_n \chi(x - x_n)\,.$$

Prove that f is well-defined and real-valued on $[a, b]$ and is continuous for every $x \neq x_n$.