# Functional Analysis Homework 10

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1. Show that all finite-dimensional normed spaces $X$ have Schur property, so the weak and strong convergence in $X$ coincide.

   *Proof.* Let $X$ be finite dimensional and $x_n \rightharpoonup x$. Weak convergence implies boundedness, so $x_n$ lies in a closed ball. In finite dimensions, closed and bounded sets are compact, hence $x_n$ has a norm-convergent subsequence $x_{n_k} \to y$. Norm convergence $\Rightarrow$ weak convergence, so $x_{n_k} \rightharpoonup y$. Weak limits are unique, thus $y = x$.

   In a metric space, if every subsequence has a further subsequence converging to $x$, then the whole sequence converges to $x$. Therefore $x_n \to x$ in norm. Hence weak and strong convergence coincide on $X$. $\square$

2. Prove that in an infinite dimensional normed space $X$, weak topology is stricktly weaker than the strong topology. Why does this not contradict Schur property of $X = \ell_1$ mentioned in Remark 3.5.14?

   *Proof.* (*Strictness*) Consider the open unit ball $B = x : |x| < 1$, which is norm-open. We show $B$ is *not* weakly open. If it were, some basic weak neighborhood of 0,

   $$U = x \in X : \ |f_i(x)| < \varepsilon, \ i = 1, \dots, m,$$

   would satisfy $U \subset B$ for some $f_1, \dots, f_m \in X^*$ and $\varepsilon > 0$. But $V := \bigcap_{i=1}^m \ker f_i$ has finite codimension and is infinite dimensional. Thus we can choose $x \in V$ with $|x| > 1$. Then $x \in U$ but $x \notin B$, a contradiction. Hence $B$ is not weakly open, so the weak topology is strictly coarser.

   (No contradiction with $\ell_1$) The Schur property is a sequential statement: if $x_n \rightharpoonup x$ in $\ell_1$, then $|x_n - x| \to 0$. Topological equality is stronger: it requires the collections of all open sets to match. In infinite-dimensional spaces (including $\ell_1$) the weak topology is not first countable, so agreeing on convergent sequences does *not* force the topologies to be identical. Thus $\ell_1$ can have Schur while weak $\neq$ norm topologies. $\square$

3. Show that $\ell_\infty$ is a universal space for all separable Banach spaces. In other words, show that every separable Banach space $X$ isometrically embeds into $\ell_\infty$.

*Proof.* Let $X$ be separable. By Banach–Alaoglu, the closed unit ball $B_{X^*}$ is weak$^*$-compact; since $X$ is separable, $B_{X^*}$ is metrizable in the weak$^*$ topology, hence has a countable dense subset $\{f_n\}_{n\geq 1} \subset B_{X^*}$.

Define $T : X \to \ell_\infty$ by

$$T(x) := \big(f_1(x), f_2(x), \dots\big).$$

Linearity is clear. For each fixed $x \in X$, the map $\Phi_x : B_{X^*} \to \mathbb{K}$ given by $\Phi_x(f) = |f(x)|$ is weak$^*$-continuous, so

$$\sup_{n\geq 1} |f_n(x)| = \sup_{f \in \overline{\{f_n\}}^{w^*}} |f(x)| = \sup_{f \in B_{X^*}} |f(x)|.$$

By Hahn–Banach,

$$\|x\| = \sup_{f \in B_{X^*}} |f(x)|.$$

Therefore

$$\|T(x)\| = \sup_{n\geq 1} |f_n(x)| = \|x\|,$$

so $T$ is an isometry (hence injective). Thus $X$ embeds isometrically into $\ell_\infty$. $\square$