ST 8533 Applied Probability

Kevin Ho

Spring 2026

Contents

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

THEME 1

INTRODUCTION OF PROBABILITY THEORY

1.0.1 January 15

So for this class, we mainly discuss the topics of first being just a general review from The normal Math stats course at the undergraduate level. Later we move onto 3 full modules on Markov Chains.

The topics in Markov chains that we discuss are the definitions, transistion matricies, Chapman-Kolmogorov equations, classification, recurrence, communitating classes, stationary distributions, and long-run behavior. We then move onto topics of Poisson Processes, nonhomogeneous Poisson processes, Renewel Theory and and introduction to Brownian motion. Make sure to check the syllabus to this.

1.1 Introduction

We are first gonna go through the basics of probability theory just to make sure you remember the topcis that will be used in this Stochastics Class.

Definition 1.1. Consider a *random experiment* and let S to be the set of all possible outcomes. S is called the **sample space**.

Definition 1.2. An **event** E us a subset of the sample space S. The event E occurs if the outcomes lies in E.

So for a better visuallization of these definitions, suppose we flip two coins. The sample space is $S = \{HH, HT, TH, TT\}$ and the event of getting only one head is $E = \{HT, TH\}$.

Definition 1.3. Let A and B be two events. We define the following operations.

- (Union): $A \cup B = \{x : x \in A \text{ or } x \in B\}$, the event that either A or B occurs.
- (Intersection): $A \cap B = \{x : x \in A \text{ and } x \in B\}$, the event that both A and B occur.
- (Complement): $A^c = \{x : x \notin A\}$, the event that A does not occur.
- (Subset): $A \subseteq B$ if $x \in A$ implies $x \in B$, the event that if A occurs then B also occurs.

Most of these definitions are pretty intuitive. So now let's move onto some of the properties of these operations.

Proposition 1.4.
- (Commutative Law): $A \cup B = B \cup A$ and $A \cap B = B \cap A$.
- (Associative Law): $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$.
- (Distributive Law): $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proposition 1.5. (De Morgan's Laws)
- $(A \cup B)^c = A^c \cap B^c$.
- $(A \cap B)^c = A^c \cup B^c$.

Definition 1.6. Two events are called **disjoint** or **mutually exclusive** if $A \cap B = \emptyset$. This means that the two events cannot occur at the same time.

So far we have only discussed the basic set operations on events. Now we want to move onto the notion of probability and how it is defined on these events.

Definition 1.7. Let Ω denote the collection of all possible events. A **probability function** $P : \Omega \to [0, 1]$ is a function that satisfies the following properties:
- (Non-negativity): $P(A) \geq 0$ for any event A.
- (Normalization): $P(S) = 1$.
- (Countable Additivity): If $A_1, A_2, \ldots, A_n$ are mutually exclusive events, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i).$$

Remark 1.8. With this definition, we can derive some properties of the probability function. $\emptyset \in \Omega$ and $S \in \Omega$ since they are both events. For example, a coin is flipped. The sample space is $S = \{H, T\}$ and Ω is the power set of S, which is $\{\emptyset, \{H\}, \{T\}, S\}$.

Remark 1.9. $P(A)$ can be interpreted as the long-run relative frequency of the event A occurring in repeated independent trials of the random experiment. If the experiment is repeated n times, and $n(A)$ is the number of times event A occurs, then

$$P(A) \approx \frac{n(A)}{n} \text{ as } n \to \infty.$$

Now that we have defined the probability function, we can derive some properties of it.

Proposition 1.10.
- $P(\emptyset) = 0$.
- $P(A^c) = 1 - P(A)$.
- If $A \subseteq B$, then $P(A) \leq P(B)$.
- (Inclusion-Exclusion Principle): For any two events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Remark 1.11. The Inclusion-Exclusion Principle can be extended to n events. For events $A_1, A_2, \ldots, A_n$, we have

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \le i < j \le n} P(A_i \cap A_j) + \sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots \cap A_n).$$

1.2 Conditional Probabilities

1.2.1 January 20

BIBLIOGRAPHY

[Shu16] Neal Shusterman. *Scythe*. Arc of a Scythe. Simon & Schuster, 2016.