# Functional Analysis (MA 8673) Homework 4

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1. Consider the one-parameter unitary group given by $U(t)f(x) = f(x - t \bmod 2\pi)$ for $f \in L^2(0, 2\pi)$. What is the generator of $U$?

2. Suppose $M$ is the Riemann surface of $\sqrt{z}$ and $\mathcal{H} \in L^2(M)$ with Lebesgue measure locally. Define the following two operators:

$$A = -i\frac{\partial}{\partial x} \text{ and } -i\frac{\partial}{\partial x}$$

both with domain $\mathcal{D} = \{f \in C^\infty(M) : \text{supp}(f) \text{ is compact}, 0 \notin \text{supp}(f)\}$. Prove that both $A$ and $B$ are essentially self-adjoint, both $A, B : \mathcal{D} \to \mathcal{D}$, and $(AB)f = (BA)f,\ \forall f \in \mathcal{D}$. However show that $\exp(it\overline{A})$ and $\exp(it\overline{B})$ do not commute.

3. Let $\mu$ be a finite complex Borel measure on $\mathbb{R}$ and let

$$\hat{\mu}(t) = \int_{\mathbb{R}} e^{-it\lambda} d\mu(\lambda).$$

Prove that

$$\lim_{c\to\infty} \frac{1}{c} \int_0^c |\hat{\mu}(t)|^2 dt = \sum_{\lambda\in\mathbb{R}} |\mu(\{\lambda\})|^2$$

where the sum on the right hand side is finite.

*Proof.* So let's start by decomposing the LHS. □

4. Let $A$ be a self-adjoint operator. Using the previous problem, prove that $\lim_{t\to\infty} U(t) = 0$ where $U(t) = exp$(-itA)

5. Let $X$ be a infinite dimensional locally convex space and $X^*$ its dual space. Prove that no weakly continuous semi-norm is actually a norm. (This shows semi-norms are essential tools.)

6. Let $\{\rho_\alpha\}_{\alpha\in A}$ be a family of semi-norms so that some finite sum $\rho_{\alpha_1} + \cdots + \rho_{\alpha_n}$ is actually a norm. Prove that $\{\rho_\alpha\}_{\alpha\in A}$ is equivalent to a family of norms.