

18.786: Automorphic Forms

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Spring 2026

CONTENTS

How strange to actually have to see the path of your journey in order to make it.

—Neal Shusterman, [Shu16]

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THEME 1

INTRODUCTION

1.1 February 2

Here we go.

1.1.1 Logistic Notes

Here are some logistic notes.

- The book *Automorphic Forms and Representations* by Bump [Bum97] will be our main reference. We will focus on the third chapter.
- There are two quizzes, which will count as about 30% of the final grade. The rest of the grade will come from the homework.
- The course requires knowledge of number theory and some representation theory. Most notably, we need control of Lie groups and Lie algebras.

The story of automorphic forms begins with modular forms. Roughly speaking, a modular form is a function on the upper-half plane which is symmetric for $SL_2(\mathbb{C})$. There is an exposition in the last chapter of [Ser12]. However, our story will start with GL_1 instead of GL_2 . The perspective we take is Tate's thesis, who used Fourier analysis to reprove the analytic properties of the relevant (automorphic) L -functions.

1.1.2 Places of Global Fields

To do our Fourier analysis, we need to decompose our number field at each place, for which we will need the ring of adèles.

Definition 1.1 (global field). A *global field* F is either a number field or a function field. Here, a number field is a finite extension of \mathbb{Q} , and a function field refers to the function field of a smooth, projective, geometrically connected curve X over a finite field \mathbb{F}_q .

Example 1.2. The field $\mathbb{F}_q(t)$ is a global field.

Definition 1.3 (place). Fix a global field F . Then a *place* is an equivalence class of multiplicative absolute values of F .

- If F is a number field, then the *finite* (or *nonarchimedean*) places are those in bijection with \mathcal{O}_F , and the *infinite places* (or *archimedean*) are those in bijection with the embeddings $F \hookrightarrow \mathbb{C}$ (up to conjugation).
- If F is the function field of a curve X , then the places are in bijection with the closed points of the curve X .

We let $V(F)$ be the set of all places, and we let $V(F)_\infty$ denote the set of infinite places.

Example 1.4. For $F = \mathbb{Q}$, the place at a finite prime p is represented by $|q|_p := p^{-\nu_p(q)}$, where $\nu_p(q)$ is the number of p 's appearing in the prime factorization of q .

Example 1.5. For $F = \mathbb{F}_q(t)$, we have the curve $X = \mathbb{P}^1$, so there is one place at infinity, and the rest of the points come from \mathbb{A}^1 . The places in \mathbb{A}^1 are parameterized by the monic irreducible polynomials of $\mathbb{F}_q[t]$.

Notation 1.6. Fix a global field F . For each place v , we let F_v be the completion of F along a norm represented by v . We let \mathcal{O}_v denote the elements with norm at most 1; we let \mathcal{O}_v^\times denote the elements with norm 1, and we let \mathfrak{p}_v denote the elements with norm less than 1.

Remark 1.7. If v is nonarchimedean, then it turns out that \mathcal{O}_v is a discrete valuation ring with maximal ideal \mathfrak{p}_v . It also turns out that there is an exact sequence

$$1 \rightarrow \mathcal{O}_v^\times \rightarrow F_v^\times \rightarrow \mathbb{Z} \rightarrow 0,$$

where the map $F_v^\times \rightarrow \mathbb{Z}$ is the valuation map.

It is helpful to normalize our absolute values. Let's start with the global fields.

Notation 1.8. Fix a place v of a global field F . We normalize a choice of absolute value $|\cdot|_v$ as follows.

- For $F = \mathbb{Q}$, each prime p produces the absolute value $|q|_p := p^{-\nu_p(q)}$. The infinite place ∞ produces the absolute value $|x|_\infty$ which is the usual one (in \mathbb{R}).
- For a finite extension F of \mathbb{Q} , say that v lies over v_0 of \mathbb{Q} , and we define

$$|x|_v := \left| N_{F_v/\mathbb{Q}_{v_0}}(x) \right|_{v_0}.$$

Example 1.9. For $F_v = \mathbb{C}$, we see that $|x|_v = |x\bar{x}|_\mathbb{R}$ is the square of the usual absolute value on \mathbb{C} . Note that this norm does not obey the triangle inequality.

For a function field $\mathbb{F}_q(X)$, there is not a canonical embedding $\mathbb{F}_q(t)$ into $\mathbb{F}_q(X)$, so it does not seem suitable to proceed as above by taking norms. Instead, we normalize directly.

Notation 1.10. Fix a function field F of a smooth, projective, geometrically connected curve X over \mathbb{F}_q , and choose a place $v \in X$. Then the completion F_v is isomorphic to $k_v((t))$, where $t \in \mathcal{O}_v$ is a choice of uniformizer and k_v/\mathbb{F}_q is a finite extension. Then we normalize our norm $|\cdot|_v$ by $|t|_v := (\#k_v)^{-1}$.

These choices of normalization obey a product formula.

Proposition 1.11. Fix a global field F . For each $x \in F$,

$$\prod_{v \in V(F)} |x|_v = 1.$$

Sketch. This is included in a standard first course in number theory, so we will be brief. For number fields, this is checked directly by passing to \mathbb{Q} , where it is a consequence of unique prime factorization. For function fields $\mathbb{F}_q(X)$, we may think of $f \in \mathbb{F}_q(X)$ as a rational function X , and $|f|_v = q^{\deg(v) \cdot \text{ord}_v(f)}$, where ord_v is the order of vanishing. Thus, the product formula more or less amounts to the statement that the sum of the zeroes and poles of f all cancel out (over the algebraic closure). ■

1.1.3 Adèles

We now define the adèles by gluing together our localizations.

Definition 1.12 (adèles). Fix a global field F . Then the ring of adèles \mathbb{A}_F is defined as the restricted product

$$\mathbb{A}_F := \prod_{v \in V(F)} (F_v, \mathcal{O}_v),$$

meaning that \mathbb{A}_F consists of sequences of elements in F_v which are in \mathcal{O}_v for all but finitely many v .

Remark 1.13. By construction, we see that

$$\mathbb{A}_F = \bigcup_{\substack{\text{finite } S \subseteq V(F) \\ S \supseteq V(F)_\infty}} \left(\prod_{v \notin S} \mathcal{O}_v \times \prod_{v \in S} F_v \right).$$

Thus, \mathbb{A}_F is a colimit of (product) topological rings, so \mathbb{A}_F is a topological ring.

Remark 1.14. A basis neighborhood basis of $0 \in \mathbb{A}_F$ is given as follows: for any choice of finite $S \subseteq V(F)$ containing $V(F)_\infty$, choose open neighborhoods $U_v \subseteq \mathcal{O}_v$ of 0, and then we have the open subset

$$\prod_{v \notin S} \mathcal{O}_v \times \prod_{v \in S} U_v.$$

One can further require that the open subsets U_v take the form $\mathfrak{p}_v^{m_v}$, where m_v is some integer.

Tate's thesis is about $\text{GL}_1(\mathbb{A}_F)$, so the following group will be important to us.

Definition 1.15. Fix a global field F . Then the group of idèles \mathbb{A}_F^\times is defined as the restricted product

$$\mathbb{A}_F^\times := \prod_{v \in V(F)} (F_v^\times, \mathcal{O}_v^\times),$$

meaning that \mathbb{A}_F^\times consists of sequences of elements in F_v^\times which are in \mathcal{O}_v^\times for all but finitely many v .

Notably, $\text{GL}_1(\mathbb{A}_F) = \mathbb{A}_F^\times$.

Remark 1.16. This is not the set of nonzero elements in \mathbb{A}_F because we require the inverse to also be an adèle!

Remark 1.17. One should not give the subset $\mathbb{A}_F^\times \subseteq \mathbb{A}_F$ the subspace topology. Instead, the topology should be given by the restricted product, whose open subsets can be smaller. Thus, an element of the neighborhood basis of $1 \in \mathbb{A}_F^\times$ can be described as follows: for any choice of finite $S \subseteq V(F)$ containing $V(F)_\infty$, choose open neighborhoods $U_v \subseteq \mathcal{O}_v$ of 0, and then we have the open subset

$$\prod_{v \notin S} \mathcal{O}_v^\times \times \prod_{v \in S} U_v.$$

One can further require that the open subsets U_v take the form $\mathfrak{p}_v^{m_v}$, where m_v is some integer.

Later in the course, we will even want to study groups like $\mathrm{GL}_2(\mathbb{A}_F)$ or $\mathrm{GL}_n(\mathbb{A}_F)$. Let's be explicit about what this notation means.

Definition 1.18 (general linear group). Fix a ring R . Then we define $\mathrm{GL}_n(R)$ to be the group of invertible $n \times n$ matrices. Explicitly, this can be described as the group of $n \times n$ matrices with entries in R whose determinant is invertible.

Remark 1.19. Fix a global field F . One can check that

$$\mathrm{GL}_n(\mathbb{A}_F) = \prod_{v \in V(F)} (\mathrm{GL}_n(F_v), \mathrm{GL}_n(\mathcal{O}_v)),$$

which also tells us what the topology should be.

Remark 1.20. Here is another way to construct the topology: GL_n can embed (as a scheme) as a closed subspace of $(n^2 + 1)$ -dimensional space A , where the embedding sends $g \in \mathrm{GL}_n(R)$ to the tuple of coordinates follows by the inverse of the determinant. This is a closed embedding, essentially by definition of GL_n . Then we can give $\mathrm{GL}_n(\mathbb{A}_F)$ the natural topology given as a closed subspace of $A(\mathbb{A}_F)$. It is not too hard (but rather annoying) to check that these definitions agree.

Example 1.21. The determinant map $\det: \mathrm{GL}_n(\mathbb{A}_F) \rightarrow \mathbb{A}_F^\times$ is continuous. One can see this via Remark 1.20 because the determinant and its inverse are both continuous maps to \mathbb{A}_F . But the topology on \mathbb{A}_F^\times is given as a closed subspace of $\mathbb{A}_F \times \mathbb{A}_F$ (where the embedding is given by $x \mapsto (x, 1/x)$).

This course is interested in the representation theory of $\mathrm{GL}_n(\mathbb{A}_F)$, focusing on the cases $n \in \{1, 2\}$. If we think about such representations appropriately, it turns out that such a representation π will decompose into a tensor product $\bigotimes'_v \pi_v$, where π_v is a representation of $\mathrm{GL}_n(F_v)$. More than half of the course will thus be interested in the representation theory of $\mathrm{GL}_n(F_v)$ because we will want to study the finite and infinite places separately.

1.1.4 Characters on the Adèles

We will need more structure theory of the adèles.

Proposition 1.22. Fix a global field F . The diagonal embedding $F \hookrightarrow \mathbb{A}_F$ embeds F as a discrete subgroup.

Proof. Fix distinct $a, b \in F$. By examining the open subsets we have access to, we need to show that $|a - b|_v \geq 1$ for some v , which follows from Proposition 1.11. ■

Corollary 1.23. Fix a global field F . The diagonal embedding $F^\times \hookrightarrow \mathbb{A}_F^\times$ embeds F as a discrete subgroup.

Proof. For each $a \in F^\times$, we need to know that there is an open subset U of \mathbb{A}_F^\times for which $U \cap F^\times = \{a\}$. But there is such an open subset of \mathbb{A}_F , which continues to be open in \mathbb{A}_F^\times . ■

We will be interested in characters on \mathbb{A}_F^\times .

Notation 1.24. Fix a place v of a global field F . Given a continuous character $\chi: \mathbb{A}_F^\times \rightarrow \mathbb{C}^\times$, we let $\chi_v: F_v^\times \rightarrow \mathbb{C}^\times$ denote the induced character.

Remark 1.25. The continuity of χ forces $\chi_v|_{\mathcal{O}_v^\times} = 1$ for all but finitely many v . Conversely, given a family $\{\chi_v\}_{v \in V(F)}$ of continuous characters for which $\chi_v|_{\mathcal{O}_v^\times} = 1$ for all but finitely many v , one can check that there is a unique continuous character χ on \mathbb{A}_F^\times gluing them together.

The previous remark motivates the following definition.

Definition 1.26 (unramified). Fix a place v of a global field F . Then a character $\chi_v: F_v^\times \rightarrow \mathbb{C}^\times$ is *unramified* if and only if $\chi_v|_{\mathcal{O}_v^\times} = 1$.

Example 1.27. By definition, χ_v factors through $F_v^\times / \mathcal{O}_v^\times \cong \mathbb{Z}$. Thus, χ_v can be described as $\chi_v = |\cdot|_v^s$ for some $s \in \mathbb{C}$.

Not all characters are interesting to us because we want our characters χ_v to talk to each other.

Definition 1.28 (Hecke character). Fix a global field F . A *Hecke character* is a continuous character $\chi: \mathbb{A}_F^\times \rightarrow \mathbb{C}^\times$ which vanishes on F^\times .

Remark 1.29. It is equivalent to ask for χ to be continuous on $F^\times \backslash \mathbb{A}_F^\times$ by Corollary 1.23.

1.2 February 4

Here we go.

1.2.1 Adelic Quotients

Thus, it will be worthwhile to know something about the quotient $F^\times \backslash \mathbb{A}_F^\times$. Let's start with the additive group.

Theorem 1.30 (approximation). Fix a number field F . Then

$$\mathbb{A}_F = F + \prod_{v \notin V(F)_\infty} \mathcal{O}_v + \prod_{v \in V(F)_\infty} F_v.$$

Proof. Given an adèle $(a_v)_v \in \mathbb{A}_F$, we see that we may ignore the infinite places. Then we are asked to find $a \in F$ for which $a \equiv a_v \pmod{\mathcal{O}_v}$ for all v . After multiplying out some denominators, this amounts to the Chinese remainder theorem for \mathcal{O}_F . ■

Here is an analog for function fields.

Proposition 1.31. Fix a function field $F = \mathbb{F}_q(X)$. Then one has

$$F \backslash \mathbb{A}_F \bigg/ \prod_{v \in V(F)} \mathcal{O}_v \cong H^1(X; \mathcal{O}_X).$$

Proof. The idea is to use the “two-step complex” $F \rightarrow \mathbb{A}_F / \prod_v \mathcal{O}_v$ to compute the cohomology of \mathcal{O}_X . Note that $\mathbb{A}_F / \prod_v \mathcal{O}_v$ is the restricted product

$$\prod_{v \in V(F)} \left(\frac{F_v}{\mathcal{O}_v}, \frac{\mathcal{O}_v}{\mathcal{O}_v} \right) = \bigoplus_{v \in V(F)} F_v / \mathcal{O}_v.$$

Now, there is a short exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{K} \rightarrow \bigoplus_{v \in V(F)} i_{v*}(F_v / \mathcal{O}_v) \rightarrow 0,$$

where \mathcal{K} is the constant sheaf of rational functions. Taking the long exact sequence in cohomology produces an exact sequence

$$F \rightarrow \bigoplus_{v \in V(F)} F_v / \mathcal{O}_v \rightarrow H^1(X; \mathcal{O}_X) \rightarrow H^1(X; \mathcal{K}).$$

Because X is irreducible, the constant sheaf \mathcal{K} is flasque, so $H^1(X; \mathcal{K}) = 0$. The result now follows. ■

Remark 1.32. As an application, the right-hand side will frequently have more than one element: it has dimension over \mathbb{F}_q equal to the genus of X , so the cohomology group has one element if and only if X is $\mathbb{P}_{\mathbb{F}_q}^1$!

Remark 1.33. If one expands one of the \mathcal{O}_v s to F_v s, then it turns out that the quotient is trivial.

Remark 1.34. One can check that the stabilizer of a double coset of the F -action on a double coset is exactly $\mathbb{F}_q = H^0(X; \mathcal{O}_X)$.

Returning to number fields, we see that Theorem 1.30 grants us a surjection

$$F \otimes_{\mathbb{Q}} \mathbb{R} \twoheadrightarrow F \backslash \mathbb{A}_F \bigg/ \prod_{v \notin V(F)} \mathcal{O}_v.$$

The kernel is exactly given by the elements $x \in F$ for which $x \in \mathcal{O}_v$ for all v , which is exactly \mathcal{O}_F . Thus, there is an isomorphism

$$\frac{F \otimes_{\mathbb{Q}} \mathbb{R}}{\mathcal{O}_F} \rightarrow F \backslash \mathbb{A}_F \bigg/ \prod_{v \notin V(F)} \mathcal{O}_v$$

of topological groups. Here, the left-hand side is a torus of dimension $[F : \mathbb{Q}]$: it is isomorphic as a topological group to $\mathbb{R}^d / \mathbb{Z}^d$.

1.2.2 Idelic Quotients

Of course, we are more interested in \mathbb{A}_F^\times , so let's turn our attention there. As usual, arguing with function fields is easier.

Proposition 1.35. Fix a function field $F := \mathbb{F}_q(X)$. Then one has

$$F^\times \backslash \mathbb{A}_F^\times / \prod_{v \in V(F)} \mathcal{O}_v^\times \cong \text{Pic } X.$$

Proof. Once again, we use the “two-step complex” $F^\times \rightarrow \mathbb{A}_F^\times / \prod_v \mathcal{O}_v^\times$. Here, $\mathbb{A}_F^\times / \prod_{v \in V(F)} \mathcal{O}_v^\times$ is the restricted product

$$\prod_{v \in V(F)} \left(\frac{F_v^\times}{\mathcal{O}_v^\times}, \frac{\mathcal{O}_v^\times}{\mathcal{O}_v^\times} \right) \cong \bigoplus_{v \in V(F)} \mathbb{Z}.$$

Here, the last isomorphism occurs by taking valuations. Now, this latter group is isomorphic to $\text{Div}(X)$, and we can see that F^\times embeds via these isomorphisms as the principal divisors. The result follows. ■

Remark 1.36. It turns out that Pic upgrades into a group scheme Pic_X with a connected component Pic_X^n for each degree. The Jacobian $\text{Jac } X$ is exactly Pic_X^0 . Thus, $\text{Pic}(X)$ is infinite, but the degree-zero part $\text{Jac } X(\mathbb{F}_q)$ is some group, which has about q^g points by the Weil conjectures.

Remark 1.37. The kernel of the map $F^\times \rightarrow \text{Div } X$ is exactly the constant functions \mathbb{F}_q^\times . This reflects the fact that line bundles have some action by \mathbb{F}_q^\times .

And now we move to number fields. Here is a starting result.

Lemma 1.38. Fix a number field F . The map

$$\prod_{v|\infty} F_v^\times \rightarrow F^\times \backslash \mathbb{A}_F^\times / \prod_{v \nmid \infty} \mathcal{O}_v^\times$$

has cokernel isomorphic to the class group of F .

Proof. The cokernel is

$$F^\times \backslash \mathbb{A}_{F,f}^\times / \prod_{v \nmid \infty} \mathcal{O}_v^\times,$$

where $\mathbb{A}_{F,f}^\times$ is the ring of finite adèles. The right-hand quotient is $\bigoplus_{v \nmid \infty} F_v^\times / \mathcal{O}_v^\times$, which is isomorphic to the group of fractional ideals (or equivalently, $\text{Div}(\text{Spec } \mathcal{O}_F)$). Taking a further quotient by F^\times shows that the cokernel is the class group. ■

Remark 1.39. The kernel of the map is exactly the elements of F^\times which are units at every place, which is exactly \mathcal{O}_F^\times . It follows that we have an exact sequence

$$1 \rightarrow \mathcal{O}_F^\times \rightarrow (F \otimes_{\mathbb{Q}} \mathbb{R})^\times \rightarrow F^\times \backslash \mathbb{A}_F^\times / \prod_{v \nmid \infty} \mathcal{O}_v^\times \rightarrow \text{Cl } F \rightarrow 0.$$

To continue cutting down the size of the quotient, note that both F^\times and $\prod_v \mathcal{O}_v^\times$ have global norm in $\mathbb{A}_F^\times \rightarrow \mathbb{R}^+$ equal to 1.

Notation 1.40. Fix a number field F . Then we define $\mathbb{A}_F^{\times,1}$ to be the subset of elements with global norm 1.

Remark 1.41. The map

$$F^\times \backslash \mathbb{A}_F^\times / \prod_{v \nmid \infty} \mathcal{O}_v^\times \rightarrow F^\times \backslash \mathbb{A}_{F,f}^\times / \prod_{v \nmid \infty} \mathcal{O}_v^\times$$

continues to be surjective because we can always choose the archimedean part of an adèle so that the adèle has norm 1.

Thus, our exact sequence now looks like

$$1 \rightarrow \mathcal{O}_F^\times \rightarrow (F \otimes_{\mathbb{Q}} \mathbb{R})^{\times,1} \rightarrow F^\times \backslash \mathbb{A}_F^{1,\times} / \prod_{v \nmid \infty} \mathcal{O}_v^\times \rightarrow \text{Cl } F \rightarrow 0,$$

where $(F \otimes_{\mathbb{Q}} \mathbb{R})_1$ refers to the subgroup whose product is 1. Taking $\log |\cdot|_v$ (with $|\cdot|_v$ chosen as before) maps $(F \otimes_{\mathbb{Q}} \mathbb{R})^{\times,1}$ into a Euclidean space isomorphic to $\mathbb{R}^{r_1+r_2-1}$, where (r_1, r_2) is the signature of F . Note that the kernel of $\log |\cdot|_v$ is given by the elements of archimedean norm 1, which when restricted to \mathcal{O}_F^\times is exactly the group $\mu(F)$ of roots of unity.

Now, by Dirichlet's unit theorem, we see that \mathcal{O}_F^\times embeds as a lattice of full rank into $\mathbb{R}^{r_1+r_2-1}$, so the quotient is a compact torus. We have thus proven the following result.

Theorem 1.42. Fix a number field F with signature (r_1, r_2) . The double quotient

$$F^\times \backslash \mathbb{A}_F^{\times,1} / \prod_{v \nmid \infty} \mathcal{O}_v^\times$$

is isomorphic to an extension of $(\mathbb{R}/\mathbb{Z})^{r_1+r_2-1}$ by the class group $\text{Cl } F$. In particular, it is compact.

Remark 1.43. Thus, we see that

$$F^\times \backslash \mathbb{A}_F^\times / \prod_{v \nmid \infty} \mathcal{O}_v^\times$$

is not compact, but it is an extension of a compact abelian group by \mathbb{R}^+ .

1.2.3 Pontryagin Duality

Our next task is to do some Fourier analysis on \mathbb{A}_F and \mathbb{A}_F^\times . Let's first recall generalities of Fourier analysis on locally compact abelian topological groups.

Definition 1.44 (Pontryagin dual). Fix a locally compact abelian group X . Then its *Pontryagin dual* X^* is the set of homomorphisms $X \rightarrow S^1$, equipped with the compact open topology.

Remark 1.45. There is a functoriality as follows: for any homomorphism $f: X \rightarrow Y$, we have a homomorphism $f^*: Y^* \rightarrow X^*$ given by pre-composition.

Here are some theorems about this construction.

Theorem 1.46 (Duality). There is a natural isomorphism $\text{id} \Rightarrow (-)^{**}$. For a given group G , it is given by sending $g \in G$ to the character $\text{ev}_g: G^* \rightarrow S^1$ defined by $\text{ev}_g: \chi \mapsto \chi(g)$.

Theorem 1.47 (Exact). The functor $(-)^*$ is exact.

Let's see some examples.

Example 1.48. If $X = \mathbb{Z}$, then its Pontryagin dual is just S^1 . On the other hand, all continuous homomorphisms $S^1 \rightarrow S^1$ take the form $z \mapsto z^n$, so $(S^1)^* = \mathbb{Z}$.

Example 1.49. Homomorphisms $\mathbb{R} \rightarrow S^1$ all take the form $\chi_\xi: t \mapsto e^{i\xi t}$, where $\xi \in \mathbb{R}$ is some real number. Thus, $\mathbb{R}^* = \mathbb{R}$.

Example 1.50. In general, given a finite-dimensional real vector space V , we may identify the dual V^* with the Pontryagin dual, where one sends $\varphi: V \rightarrow \mathbb{R}$ to the character $v \mapsto e^{i\varphi(v)}$.

Example 1.51. Homomorphisms $\mathbb{Z}/n\mathbb{Z} \rightarrow S^1$ are uniquely determined by where they send 1, so

$$(\mathbb{Z}/n\mathbb{Z})^* \cong \mu_n.$$

Conversely, all homomorphisms $\mu_n \rightarrow \mu_n$ are given by $z \mapsto z^k$ for some k , so $\mu_n^* \cong \mathbb{Z}/n\mathbb{Z}$. In particular, we see that $(\mathbb{Z}/n\mathbb{Z})^*$ is non-canonically isomorphic to $\mathbb{Z}/n\mathbb{Z}$, but the isomorphism $\mathbb{Z}/n\mathbb{Z} \rightarrow (\mathbb{Z}/n\mathbb{Z})^{**}$ is canonical! In general, for any finite abelian group G , we see that G^* is non-canonically isomorphic to G .

Example 1.52. Exactness of the functor $(-)^*$ implies that

$$\mathbb{Z}_p^* = (\lim \mathbb{Z}/p^n\mathbb{Z})^* = \operatorname{colim} \mu_{p^n} = \mu_{p^\infty}.$$

Example 1.53. Once again, exactness of the functor $(-)^*$ implies that

$$\mathbb{Q}_p^* = \left(\operatorname{colim} \left(\mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p \xrightarrow{p} \cdots \right) \right)^* = \lim \left(\mu_{p^\infty} \xrightarrow{p} \mu_{p^\infty} \xrightarrow{p} \cdots \right).$$

Thus, this limit is some kind of coherent sequence of taking p th roots, which is then isomorphic to \mathbb{Q}_p . Indeed, μ_{p^∞} is isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$, which we see by sending $a/p^n \in \mathbb{Q}_p/\mathbb{Z}_p$ to $\exp(2\pi i a/p^n)$. In fact, it turns out that the exact sequence

$$0 \rightarrow \mathbb{Z}_p \rightarrow \mathbb{Q}_p \rightarrow \mathbb{Q}_p/\mathbb{Z}_p \rightarrow 0$$

dualizes to an isomorphism

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}_p & \longrightarrow & \mathbb{Q}_p & \longrightarrow & \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \lim \mu_{p^n} & \longrightarrow & \mathbb{Q}_p^* & \longrightarrow & \mu_{p^\infty} \longrightarrow 1 \end{array}$$

sending $x \in \mathbb{Z}_p$ to the sequence $\{\zeta_{p^n}^x\}_n$.

Thus, we see that all local fields are identified with their Pontryagin duals. In fact, all of our constructions amount to identifying a space with its dual upon choosing a single character.

Remark 1.54. Explicitly, given a choice of nontrivial character $\psi_p \in \mathbb{Q}_p^*$, there is a map $\mathbb{Q}_p \rightarrow \mathbb{Q}_p^*$ given by taking x to the character $y \mapsto \psi_p(xy)$. It turns out that this map is an isomorphism, so we have more or less defined a non-degenerate bilinear form $\mathbb{Q}_p \times \mathbb{Q}_p \rightarrow S^1$. This procedure also works for \mathbb{R} !

In light of the previous remark, it is useful to fix some characters.

Notation 1.55. Fix a place v of \mathbb{Q} .

- If $v = p$ is finite, then we define the character $\psi_p: \mathbb{Q}_p \rightarrow S^1$ by the composite $\mathbb{Q}_p \rightarrow \mathbb{Q}_p/\mathbb{Z}_p \cong \mu_{p^\infty}$, where the second isomorphism sends a/p^n to $e^{2\pi i a/p^n}$.
- If $v = \infty$ is infinite, then we define the character $\psi_\infty: \mathbb{R} \rightarrow S^1$ by $\psi_\infty(x) := e^{-2\pi i x}$.

Remark 1.56. The choice of ψ_∞ is done so that the assembled character $\psi: \mathbb{A}_{\mathbb{Q}} \rightarrow S^1$ vanishes on \mathbb{Q} .

1.2.4 Fourier Theory

To do Fourier analysis, we need a notion of measure.

Theorem 1.57 (Haar). Fix a locally compact group X . Then there is a left-invariant Radon measure dx on X which is unique up to scalar.

Now, here is our Fourier transform.

Definition 1.58 (Fourier transform). Fix a locally compact abelian group X . The *Fourier transform* sends a function $f \in L^1(X)$ to the function $\hat{f}: X^* \rightarrow \mathbb{C}$ given by

$$\hat{f}(\xi) = \int_X f(x) \bar{\xi}(x) dx.$$

It is a large theorem that there is an inversion.

Theorem 1.59 (Fourier inversion). Fix a locally compact abelian group X , and let dx be a Haar measure on X . Then there is a Haar measure $d\chi$ on X^* such that

$$f(x) = \int_{X^*} \hat{f}(\chi) \chi(x) d\chi$$

for any $f \in L^1(X)$ for which $\hat{f} \in L^1(X^*)$.

Remark 1.60. It turns out that the Fourier transform extends to an isomorphism $L^2(G) \rightarrow L^2(G^*)$.

Remark 1.61. Equivalently, we see that the double Fourier transform of f is $f(-x)$.

Remark 1.62. If X admits an isomorphism $X \cong X^*$, then the Haar measure $d\chi$ is not necessarily equal to the Haar measure dx because it might be off by a scalar: indeed, replacing dx with $c dx$ replaces \hat{f} with $c\hat{f}$, and so we see that we end up replacing $d\chi$ with $c^{-1} d\chi$. Thus, there is a unique measure dx on X (even up to scalar!) which is “Fourier self-dual.”

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