Functional Analysis Homework 14

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1. Show that if $f(t) \geqslant 0, \forall t$ then $f(T) \geqslant 0$.

   Proof. So note that spectral Mapping Theorem gives that $\sigma(f(T)) = f(\sigma(T)) = \{f(t) : t \in \sigma(T)\}$. Since $f(t) \geqslant 0, \forall t$, it implies that $f(\sigma(T)) \geq 0$. And so the spectrum of $f(T)$ must be positive so $f(T)$ must also be positive. $\square$

2. Compute the spectral measure for the diagonal matrix $T = diag(\lambda_1, ..., \lambda_n)$ acting as an operator on $\mathbb{C}^n$.

   Proof. So by Borel Functional Calculus, the spectral measure must satisfy the following of

   $$\langle f(T)x, y\rangle = \int_{\sigma(T)} f(\lambda) d\mu_{x,y}(\lambda)$$

   for all continuous functions f. So for a diagonal matrix T, we have $f(T) = diag(f(\lambda_1), \dots, f(\lambda_n))$. And so expanding the earlier inner product, $\langle f(T)x, y\rangle$, with $x, y \in \mathbb{C}^n$ we have

   $$\langle f(T)x, y\rangle = \sum_{j=1}^{n} f(\lambda_j) x_j \bar{y_j}.$$

   So using a property of the Dirac Delta Measure where we can rewrite the function as the following of

   $$\int f(\lambda) d\delta_c(\lambda) = f(c),$$

   We can rewrite our inner product to be

   $$\sum_{j=1}^{n} f(\lambda_j) x_j \bar{y_j} = \int_{\mathbb{C}} f(\lambda) d \left(\sum_{j=1}^{n} f(\lambda_j) x_j \bar{y_j} \delta_{\lambda_j} \right) (\lambda)$$

Leading to the spectral measure being

$$\mu_{x,y} = \sum_{j=1}^{n} x_j \bar{y}_j \delta_{\lambda_j}$$

□

3. Let T be a multiplication operator in $L_2[0,1]$ by a function $g \in L_\infty[0,1]$. Show that for $f \in \mathcal{B}[0,1]$, the operator $f(T)$ if the multiplication operator in $L_2[0,1]$ by the function $f(g(t))$.

   Proof. Let $p(x)$ be a polynomial defined by $p(x) = \sum_{k=0}^{n} c_k x^k$. By the definition of the polynomial functional calculus, the operator $p(T)$ is the linear combination $\sum_{k=0}^{n} c_k T^k$. Since T is the multiplication operator by $g(t)$, the operator T^k corresponds to multiplication by $(g(t))^k$. Therefore, for any vector $h \in L_2[0,1]$, the action of the operator is:

   $$(p(T)h)(t) = \sum_{k=0}^{n} c_k (g(t))^k h(t) = p(g(t))h(t)$$

   This confirms that for any polynomial p, the operator $p(T)$ acts strictly as the multiplication operator by the composite function $p(g(t))$.

   We next extend this property to the space of continuous functions $C[0,1]$. By the Weierstrass Approximation Theorem, any continuous function f can be uniformly approximated by a sequence of polynomials $\{p_n\}$. As $n \to \infty$, the sequence p_n converges to f uniformly on the spectrum of the operator. This implies convergence in the operator norm topology:

   $$\lim_{n\to\infty} \|p_n(T) - f(T)\| = 0$$

   Simultaneously, because the convergence is uniform, the sequence of functions $p_n(g(t))$ converges uniformly to $f(g(t))$ in $L_\infty[0,1]$. Consequently, the multiplication operators corresponding to $p_n(g(t))$ converge in the operator norm to the multiplication operator denoted by $M_{f(g)}$. Since limits in the operator norm topology are unique, we conclude:

   $$f(T) = M_{f(g)}$$

   Finally, we generalize the result to all bounded Borel functions $f \in \mathcal{B}[0,1]$. Let $\mathcal{K}$ be the class of all bounded Borel functions u such that $u(T)$ is the multiplication operator by $u(g(t))$. We have already shown that $C[0,1] \subset \mathcal{K}$. Consider a uniformly bounded sequence $\{f_n\} \subset \mathcal{K}$ such that $f_n(t) \to f(t)$ pointwise. By the properties of the functional calculus,

$f_n(T)$ converges to $f(T)$ in the strong operator topology. That is, for any $h \in L_2[0,1]$:

$$\lim_{n\to\infty} \|(f_n(T) - f(T))h\|_2 = 0$$

On the other hand, since $\{f_n(g(t))\}$ is uniformly bounded and converges pointwise to $f(g(t))$, the Dominated Convergence Theorem ensures that the associated multiplication operators converge strongly to the multiplication operator by $f(g(t))$. Because the strong limit is unique, f must essentially belong to $\mathcal{K}$. Since $\mathcal{B}[0,1]$ is the smallest class containing continuous functions that is closed under pointwise limits of uniformly bounded sequences, it follows that for all $f \in \mathcal{B}[0,1]$, $f(T)$ is the multiplication operator by $f(g(t))$.

□

4. Show that if $E_1 \subseteq E_2$ then $P_{E_1} \leqslant P_{E_2}$ and $\mathrm{Im}(E_1) \subseteq \mathrm{Im}(E_2)$.

   Proof. Since $E_1 \subseteq E_2$, the indicator functions satisfy $1_{E_1}(t) \leq 1_{E_2}(t)$ for all t. By the order-preserving property of the Borel functional calculus, it follows immediately that $P_{E_1} \leqslant P_{E_2}$.

   The functional calculus is a homomorphism, so $P_{E_2}P_{E_1} = 1_{E_2}(T)1_{E_1}(T) = (1_{E_2} \cdot 1_{E_1})(T)$. Since $E_1 \subseteq E_2$, $1_{E_2} \cdot 1_{E_1} = 1_{E_1}$. Thus, $P_{E_2}P_{E_1} = P_{E_1}$. If $y \in \mathrm{Im}(P_{E_1})$, then $y = P_{E_1}y = P_{E_2}(P_{E_1}y)$, which implies $y \in \mathrm{Im}(P_{E_2})$. Therefore, $\mathrm{Im}(P_{E_1}) \subseteq \mathrm{Im}(P_{E_2})$.

   □