

# Functional Analysis Homework 4

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1. Show that the orthogonal projection  $P_Y$  is a linear map. Check that  $\text{Im}(P_Y) = Y$  and  $\ker(P_Y) = Y^\perp$ . Also check that the identity map  $I_X$  on  $X$  can be decomposed as

$$I_X = P_Y + P_{Y^\perp}.$$

Proof: So first let's show the linearity of the orthogonal Projection  $P_Y$ . so  $P_Y$  is defines as

$$P_Y : X \rightarrow X, P_Y x = y$$

where it is the projection of  $X$  onto  $Y$

- (1) let  $\alpha$  be a scalar. Write  $x$  as the orthogonal decomposition  $x = y + z$  where  $y \in Y$  and  $z \in Y^\perp$ . By the uniqueness of the projection, we just need to check two conditions for the vector  $\alpha y$  of whether  $\alpha y \in Y$  and  $(\alpha x - \alpha y) \in Y^\perp$ . So it is obvious that  $\alpha y \in Y$  as we know  $y \in Y$ . For the second condition we get

$$(\alpha x - \alpha y) = \alpha(x - y) \in Y^\perp$$

so Homogeneity is there.

- (2) Let  $x_0, x_1 \in X$ . We then get two different decompositions of  $x_0 = y_0 + z_0$  and  $x_1 = y_1 + z_1$  very similarly with homogeneity, we get a similar result of  $y_0 + y_1 \in Y$  and  $(x_0 - y_0) + (x_1 - y_1) \in Y^\perp$ . as each term is in  $Y^\perp$ .

Thus we have shown that  $P_Y$  is a linear map. So now let's check  $\text{Im}(P_Y) = Y$  and  $\ker(P_Y) = Y^\perp$ . So since  $P_Y y = y \in Y$  we know the first part is true. So let's show this by showing each space is in each other. So let  $y \in Y$ . We want to find a vector  $x$  s.t.  $P_Y x = y$  so we can choose  $x=y$  to where  $x = y + 0$  is the orthogonal decomposition. Since  $P_Y x \in Y$  we can conclude that  $Y \subseteq \text{Im}(P_Y)$ . So now let's see the other direction. Let  $v \in \text{Im}(P_Y)$  be any vector. This means that  $v = P_Y x = y$  so  $\text{Im}(P_Y) \subseteq Y$  so  $\text{Im}(P_Y) = Y$ .

For the  $\ker(P_Y) = Y^\perp$ , let  $z \in Y^\perp$ . With this, if we take the orthogonal decomposition of  $v$ , we get that  $z = 0 + z$ . Because  $0 \in Y$  and  $z \in Y^\perp$ , we get that the  $Y$  component of the decomposition

is 0 leading to  $P_Y z = 0$  meaning  $z \in \ker(P_Y)$  so  $Y^\perp \subseteq \ker(P_Y)$ . Now let  $v \in \ker(P_Y)$ . Then we get the decomposition of  $v$  to be  $v = 0 + z$ . This leads to  $v = z \in Y^\perp$  thus  $\ker(P_Y) \subseteq Y^\perp$  so then  $\ker(P_Y) = Y^\perp$ .

Now let us check the identity map. So let  $x \in X$  be any vector. Then we get the equation  $I_x(x) = x = y + z$  where  $y \in Y, z \in Y^\perp$ . When we apply  $P_Y + P_{Y^\perp}$  we get the following:

$$(P_Y + P_{Y^\perp})(x) = (P_Y)(x) + (P_{Y^\perp})(x) = y + z.$$

□

2. Let  $A$  be a subset of a Hilbert space. Show that

$$A^\perp = \bar{A}^\perp$$

where  $\bar{A}$  denotes the closure of  $A$ .

( $A^\perp \subseteq \bar{A}^\perp$ ) Let  $x \in A^\perp$ . By definition,  $\langle x, a \rangle = 0, a \in A$ . Let  $v \in \bar{A}$  arbitrary. By definition of the closure, there is a sequence  $\{a_n\}$  s.t.  $a_n \rightarrow v$ . So with  $x \in A^\perp$ , we get that  $\langle x, a_n \rangle = 0, \forall n \in \mathbb{N}$ . Because the inner product on the Hilbert Space is a continuous function, we can take the limit as  $n \rightarrow \infty$ .

$$\lim_{n \rightarrow \infty} \langle x, a_n \rangle = \langle x, \lim_{n \rightarrow \infty} a_n \rangle = \langle x, v \rangle = 0$$

Thus we have shown  $A^\perp \subseteq \bar{A}^\perp$ .

( $\bar{A}^\perp \subseteq A^\perp$ ) Let  $x \in \bar{A}^\perp$ . By definition,  $\langle x, y \rangle = 0, y \in \bar{A}$ . So with  $A \subseteq \bar{A}$ , this means that  $\langle x, v \rangle = 0, v \in A$  so thus  $\bar{A}^\perp \subseteq A^\perp$ . □

3. Let  $Y$  denote the subspace of constant functions in  $L_2 = L_2(\Omega, \Sigma, \mu)$ . Compute  $P_Y f$  for an arbitrary function  $f \in L_2$

Proof: So let  $c \in Y$  to where  $P_Y f = c$ . This means that there must be an orthogonal decomposition of  $f$  where  $f = c + g$  where  $f - c \in Y^\perp$ . So then we must find where  $\langle f - c, g \rangle = 0$ . So wlog, let  $g = 1$ . Then we get

$$\begin{aligned} \langle f - c, 1 \rangle &= \int_{\Omega} (f - c) * (1) d\mu = 0 \\ \Rightarrow \int_{\Omega} f d\mu - \int_{\Omega} c(1) d\mu &= 0 \\ \Rightarrow \int_{\Omega} f d\mu - c \int_{\Omega} 1 d\mu &= 0 \\ \Rightarrow \int_{\Omega} f d\mu &= c * \mu(\Omega) \\ \Rightarrow \frac{\int_{\Omega} f d\mu}{\mu(\Omega)} &= c = P_Y f \end{aligned}$$

□