1. Let X be an uncountable set and Y a countable set. Let $f : X \to Y$ be a function. Prove that there exists some $y \in Y$ such that $f^{-1}(y)$ is uncountable.

2. Consider the three restrictions (i), (ii), and (iii) placed on the sets $\{F_n\}$ in Cantor's Theorem. (a) Find a sequence of sets $\{F_n\}$ that satisfies (i) and (ii), but $\cap_{n=1}^{\infty} F_n = \emptyset$. (b) Find a sequence of sets $\{F_n\}$ that satisfies (i) and (iii), but $\cap_{n=1}^{\infty} F_n = \emptyset$. (c) Find a sequence of sets $\{F_n\}$ that satisfies (ii) and (iii), but $\cap_{n=1}^{\infty} F_n = \emptyset$.

3. Let $\{a_n\}$ be a sequence in $\mathbb{R}$ and define the set

$$A = \{a \in \mathbb{R} : \text{ there is a subsequence } \{a_{n_k}\} \text{ of } \{a_n\} \text{ that converges to } a\}.$$

Prove that A is closed.

4. Let $\{A_n\}_{n\in\mathbb{N}}$ be a collection of sets in $\mathbb{R}$. Prove that for every $n \in \mathbb{N}$ that $\bigcup_{n=1}^{N} \overline{A_n} = \overline{\bigcup_{n=1}^{N} A_n}$.

Prove or disprove that $\bigcup_{n=1}^{\infty} \overline{A_n} = \overline{\bigcup_{n=1}^{\infty} A_n}$.

5. Define the countable collection of sets $\{A_k\}_{k=1}^{\infty}$ as follows:

$$\begin{aligned}
A_1 &= \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \\
A_2 &= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right] \\
A_3 &= \dots
\end{aligned}$$

Define the set A by:

$$A = \bigcap_{k=1}^{\infty} A_k .$$

Prove that A is an uncountable compact set. Moreover show that A does not contain any non-empty open interval. Does it contain any non-empty closed interval? (Be sure to prove your claim.)