Name:____________________________

Math 4633/6633 Section 01 | *Practice Final Exam* | *November 11, 2024*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work, answer or there is no justification to a solution, you will receive little or no credit!

1. Define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = nx(1-x)^n$. Study the following limit:

$$\lim_{n\to\infty} f_n(x)\,.$$

Does this limit exists for each x? Is it pointwise convergence?, uniform convergence? Prove or disprove that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx\,.$$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Define a sequence of functions $\{f_n\}$ by

$$f_n(x) = n\int_x^{x+\frac{1}{n}} f\,.$$

Prove that $f_n \to f$ pointwise on $\mathbb{R}$. If f is now assumed to be uniformly continuous, prove or disprove that $f_n \to f$ uniformly on $\mathbb{R}$.

3. Formally define the function f by

$$f(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}\,.$$

Prove that f converges absolutely and uniformly on $\mathbb{R}$. Prove that $f'' + f = 0$ on $\mathbb{R}$.

4. For a real number x, let $[x]$ denote the integer closest to x. For example, $[\pi] = 3$ or $[1.75] = 2$, etc. Formally define the function f by

$$f(x) = \sum_{n=1}^{\infty} \frac{nx - [nx]}{n^2}\,.$$

Prove that f is well-defined on all of $\mathbb{R}$. Prove that $f \in \mathcal{R}[a,b]$ for every compact interval $[a,b]$.

5. On $C^1[a,b]$ define the following:

$$\|f\|_{1,\infty} = \|f\|_\infty + \|f'\|_\infty := \sup_{x\in[a,b]} |f(x)| + \sup_{x\in[a,b]} |f'(x)| \,.$$

Prove that $(C^1[a,b], \|\cdot\|_{1,\infty})$ is a normed liner space.

6. Let S be the set of all real-valued sequences. Prove that

$$d(x,y) = \sum_{j=1}^{\infty} \frac{|x_j - y_j|}{2^j(1+|x_j-y_j|)}$$

defines a metric on S. Moreover, prove that convergence w.r.t. d in (S,d) is equivalent to pointwise convergence in (S,d).

7. For $p \geq 1$, let $x \in \ell^p(\mathbb{N})$. Prove that

$$\|x\|_\infty \leq \lim_{p\to\infty} \|x\|_p \,.$$

8. Let (X,d) be a metric space. If $x_n \to x$ and $y_n \to y$ w.r.t. d in X, prove that $d(x_n,y_n) \to d(x,y)$ in $\mathbb{R}$.

9. Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. Prove that the parallelogram law always holds, that is

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right) \,.$$