Functional Analysis Homework 7

Kevin Ho

October 3rd

1. Construct a bounded linear functional on $C[0,1]$ which does not attain its norm.

   Proof. Define the functional φ on $C[0,1]$ as:

   $$\varphi(f) = \int_0^{1/2} f(t)dt - \int_{1/2}^1 f(t)dt = \int_0^1 f(t)g(t)dt$$

   where $g(t)$ is the step function:

   $$g(t) = \begin{cases} 1 & \text{if } t \in [0, 1/2] \\ -1 & \text{if } t \in (1/2, 1] \end{cases}$$

   By the Riesz Representation Theorem, the norm of φ is the total variation of its representing measure, which is calculated as $\|\varphi\| = \int_0^1 |g(t)|dt$.

   Since $|g(t)| = 1$ everywhere on $[0,1]$, the norm is:

   $$\|\varphi\| = \int_0^1 1dt = 1$$

   The norm is attained if there exists a function $f_0 \in C[0,1]$ with $\|f_0\|_\infty = 1$ such that $|\varphi(f_0)| = 1$. This occurs only if $f_0(t)$ "aligns" perfectly with $g(t)$, meaning $f_0(t) = \text{sgn}(g(t))$ almost everywhere.

   This requires $f_0(t)$ to be the step function $g(t)$. However, $g(t)$ has a jump discontinuity at $t = 1/2$ and is therefore not in $C[0,1]$.

   Since the only function that could attain the norm is discontinuous, we conclude that the norm is not attained by any function in $C[0,1]$. □

2. Let K be a subset of a normed space s.t. $0 \in \mathring{K}$. Then K is an absorbing set.

Proof. So with $0 \in \mathring{K}$, With this let $\epsilon > 0,\ B(0,\epsilon) \in K$. We need to show that for $x \in X,\ \alpha > 0$ be a scalar, $\alpha x \in K$. So we are already given the origin point , $x = 0$, is already in K so we only need to consider the case to where $x \neq 0$. So we need to choose an α such that it is in the $B(0,\epsilon)$. So for this to be true, $\|y\| < \epsilon$. So we need to choose $\alpha > 0$ so $\|\alpha x\| < \epsilon$. With this we can modify this ineqality as follows:

$$\|\alpha x\| < \epsilon \to a\|x\| < \epsilon \to \alpha < \frac{\epsilon}{\|x\|}$$

With this we can choose $\alpha = \frac{\epsilon}{2\|x\|}$. This get is the following:

$$\|\alpha x\| = \frac{\epsilon}{2\|x\|}\|x\| = \frac{\epsilon}{2} < \epsilon$$

Since $\frac{\epsilon}{2} < \epsilon$, the vector αx is in the open ball $B(0,\epsilon)$. And because $B(0,\epsilon) \subseteq K$, it follows that $\alpha x \in K$.Since we have found a suitable α for any $x \in X$, the set K is absorbing by definition.

□

3. Show that the openness assumption in Theorem 2.3.25 is essential.

To this end, consider the linear space P of all polynomials in one variable and with real coefficients. Let the subset A consist of polynomials with negative leading coefficient, and let the subset B consists of polynomials with all non-negative coefficients. Show that A and B are disjoint convex subsets of P, and that there does not exist a nonzero linear functional f on P such that

Proof. Disjoint: Assume $p(t) \in A \cap B$. If $p(t) = 0$, then $p \in B$ (all coeffs are $0 \geq 0$). But $p \notin A$, as it does not have a negative leading coefficient. This is a contradiction. If $p(t) \neq 0$, let $\deg(p) = n$ with leading coefficient p_n. Since $p \in B$, all coefficients are non-negative, so $p_n \geq 0$. Since $p \in A$, the leading coefficient is negative, so $p_n < 0$. This is a contradiction ($p_n \geq 0$ and $p_n < 0$). Thus, $A \cap B = \emptyset$.

Convexity: B is convex: Let $p(t), q(t) \in B$ and $\lambda \in [0,1]$. Let $r(t) = \lambda p(t) + (1-\lambda)q(t)$. The k-th coefficient of r is $r_k = \lambda p_k + (1-\lambda)q_k$. Since $p_k, q_k \geq 0$ and $\lambda, (1-\lambda) \geq 0$, we have $r_k \geq 0$ for all k. Thus $r(t) \in B$.

A is convex: Let $p(t), q(t) \in A$ with $\deg(p) = n, \deg(q) = m$ and leading coefficients $p_n < 0, q_m < 0$. Let $\lambda \in [0,1]$ and $r(t) = \lambda p(t) + (1-\lambda)q(t)$.

- If $n > m$, $\deg(r) = n$ and its leading coeff is $\lambda p_n < 0$.
- If $m > n$, $\deg(r) = m$ and its leading coeff is $(1-\lambda)q_m < 0$.
- If $n = m$, the n-th coeff is $r_n = \lambda p_n + (1-\lambda)q_m$. As a convex combination of two negative numbers, $r_n < 0$. This is the leading coefficient.

In all cases, $r(t) \in A$.

Seperation: Assume for contradiction there exists a nonzero functional $f \in P^*$ and $C \in \mathbb{R}$ s.t. $f(a) \leq C \leq f(b)$ for all $a \in A, b \in B$. A functional f on P is defined by the sequence $c_k = f(t^k)$, so $f(\sum p_i t^i) = \sum p_i c_i$.

Since $0 \in B$, we have $f(0) \geq C$, which implies $0 \geq C$. For any $k \geq 0$, $t^k \in B$, so $f(t^k) = c_k \geq C$. Also, $Mt^k \in B$ for any $M > 0$. Thus $f(Mt^k) = Mc_k \geq C$. If c_k were negative, we could choose a large M s.t. $Mc_k < C$. This is a contradiction, so we must have $c_k \geq 0$ for all $k \geq 0$.

Since f is nonzero, there must exist at least one k_0 such that $c_{k_0} > 0$. Choose an integer $n > k_0$. For any $M > 0$, construct the polynomial $a(t) = -t^n + Mt^{k_0}$. The leading coefficient of $a(t)$ is -1, so $a(t) \in A$. By our assumption, $f(a) \leq C$.

$$f(a) = f(-t^n + Mt^{k_0}) = -f(t^n) + Mf(t^{k_0}) = -c_n + Mc_{k_0}$$

So, we must have $-c_n + Mc_{k_0} \leq C$, or $Mc_{k_0} \leq C + c_n$ for all $M > 0$. But $C + c_n$ is a fixed constant, and c_{k_0} is a fixed positive constant. The left side Mc_{k_0} can be made arbitrarily large by increasing M, and will eventually exceed $C + c_n$. This is a contradiction.

The only way to avoid this contradiction is if all $c_k = 0$, which means f is the zero functional. This contradicts our assumption that f was nonzero. Thus, no such functional exists. $\square$

4. Let X_0 be a closed subspace of a normed space X. Prove that there exists a functional $f \in X^*$ s.t.

$$f(x) = 0, \forall x \in X_0$$

You may deduce this from Hanh-Banach theorem directly or from a seperation theorem.

Proof. Assume X_0 is a proper closed subspace of X. Let $y \in X \setminus X_0$. Define the set $K = \{y\}$.

- X_0 is a closed convex set (as it is a subspace).
- K is a compact convex set (as it is a singleton).
- $X_0 \cap K = \emptyset$ since $y \notin X_0$.

By the Hahn-Banach, there exists a continuous linear functional $f \in X^*$ and a scalar $C \in \mathbb{R}$ such that:

$$f(x) < C < f(y) \quad \text{for all } x \in X_0$$

We must show that this f vanishes on X_0. Let $x \in X_0$. Since X_0 is a subspace, $tx \in X_0$ for all scalars $t \in \mathbb{R}$. Therefore, the separation inequality must hold for tx:

$$f(tx) < C \implies t \cdot f(x) < C, \quad \forall t \in \mathbb{R}$$

This inequality must hold for all t. Assume for contradiction that $f(x) \neq 0$.

- **Case 1: $f(x) > 0$.** Choose a large positive t, for example $t = \frac{|C|+1}{f(x)}$. Then $t \cdot f(x) = |C| + 1$. This is $\geq C$, which contradicts $t \cdot f(x) < C$.
- **Case 2: $f(x) < 0$.** Choose a large negative t, for example $t = \frac{|C|+1}{f(x)}$. Then $t \cdot f(x) = |C| + 1$. This again contradicts $t \cdot f(x) < C$.

Both cases lead to a contradiction. The only remaining possibility is $f(x) = 0$. Since x was an arbitrary element of X_0, $f(x) = 0, \forall x \in X_0$. $\square$