Name:______________________________

1. Let X be a set and $f : X \to X$ be a function. Suppose that $f \circ f \circ f$ is an injection. Prove that f must be an injection.

2. let f be a real function that is strictly increasing. Prove that for every $b \in \mathbb{R}$, $f^{-1}(b)$ is empty or consists of a single element. Must f be an injection? A surjection? Be sure to justify.

3. Let X and Y be sets and $f : X \to Y$. Let $\{P_\alpha\}_{\alpha\in A}$ be a family of sets in X and $\{Q_\beta\}_{\beta\in B}$ be a family of sets in Y. Prove that

$$(a)\ f^{-1}\left(\bigcap_{\beta\in B} Q_\beta\right) = \bigcap_{\beta\in B} f^{-1}(Q_\beta) \quad \text{and} \quad (b)\ f\left(\bigcap_{\alpha\in A} P_\alpha\right) \subseteq \bigcap_{\alpha\in A} f(P_\alpha)\,.$$

Is it true set equality holds in (b) as well? Prove or disprove it. If it is false, what extra assumption is necessary to prove set equality? Be sure to justify.

4. Let A, B, C, and D be sets. Prove or disprove that:

$$(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)\,.$$

5. Let A, B, and C be sets. Prove or disprove that if $A \cup B \neq A \cap C$, then $A \not\subseteq C$ or $B \not\subseteq A$.