Functional Analysis Homework 1

Kevin Ho

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1. Show that the intersection of an arbitrary collection of subspaces of a linear vector space E is again a subspace of E.

Proof: Let E be a vector space, and let $\{E_k\}_{k=1}^n$,for n being total subspaces in E, be an arbitrary colection of subspaces of E. Let $v = \bigcap_{k=1}^n E_k$. So to show that v is a subspace of E, we need to satisfy three different conditions of making sure that w is non-empty, closed under addition and closed under scalar multiplication.

So for the first condition, by definition, every subspace must contain the zero vector, lets denote as 0. This implies that $0 \in \bigcap_{k=1}^n E_k$.

For the second condition, let $x, y \in v$. If that is the case then that would mean $x \in E_k$ and $y \in E_k$ for each k and since E_k is a subspace, it is closed under vector addition and so $x + y \in v$ thus condition is satisfied.

For the last condition, let $x \in v$ and c be a scalar. Similar to the second condition, E_k is a subspace so is closed under scalar multiplication so $cx \in v$ so last condition is satisfied. □

2. Consider a linear operator $T : E \to F$ acting between linear spaces E and F. The operator T may not be injective; we would like to make it into an injective operator. To this end, we consider the map $\tilde{T} : X/kerT \to Y$ which sends every coset [x] into a vector Tx, i.e. $\tilde{T}[x] = Tx$

   (a) Prove that $\tilde{T}$ is well defined, i.e. $[x_1] = [x_2]$ implies $Tx_1 = Tx_2$.

Proof: Let $[x_1]$ and $[x_2]$ be two cosets in the quotient space $E/kerT$ such that $[x_1] = [x_2]$. By the definition of coset equality, $[x_1] = [x_2]$ if and only if the difference between their representatives, $x_1 - x_2$, is an element of the kernel of T.

$$[x_1] = [x_2] \iff x_1 - x_2 \in kerT$$

Let $z = x_1 - x_2$. Since $z \in kerT$, by the definition of the kernel, we have $T(z) = 0$. Now, we apply the linear operator T to the expression $x_1 - x_2$:

$$T(x_1 - x_2) = T(z)$$

Since T is a linear operator, it respects vector addition and scalar multiplication. Thus, we can write:

$$T(x_1) - T(x_2) = T(z)$$

We know that $T(z) = 0$, so we substitute this into the equation:

$$T(x_1) - T(x_2) = 0$$

Rearranging the equation, we get:

$$T(x_1) = T(x_2)$$

This shows that if $[x_1] = [x_2]$, then $T(x_1) = T(x_2)$. Since $\tilde{T}[x_1] = T(x_1)$ and $\tilde{T}[x_2] = T(x_2)$, we have $\tilde{T}[x_1] = \tilde{T}[x_2]$. Therefore, the map $\tilde{T}$ is well-defined. $\square$

(b) Check that $\tilde{T}$ is a linear and injective operator.

Proof: To show that $\tilde{T}$ is a linear operator, we must prove it preserves vector addition and scalar multiplication.

Let $[x_1]$ and $[x_2]$ be two cosets in the domain $E/kerT$.

$$\tilde{T}([x_1] + [x_2]) = \tilde{T}[x_1 + x_2] = T(x_1 + x_2)$$

Since T is a linear operator, we know $T(x_1 + x_2) = T(x_1) + T(x_2)$.

$$T(x_1) + T(x_2) = \tilde{T}[x_1] + \tilde{T}[x_2]$$

Thus, $\tilde{T}$ preserves vector addition.

Let $[x]$ be a coset in $E/kerT$ and c be a scalar.

$$\tilde{T}(c[x]) = \tilde{T}[cx] = T(cx)$$

Since T is a linear operator, we know $T(cx) = cT(x)$.

$$cT(x) = c\tilde{T}[x]$$

Thus, $\tilde{T}$ preserves scalar multiplication.

Since both conditions are satisfied, $\tilde{T}$ is linear.

To show $\tilde{T}$ is injective, we must show $ker(\tilde{T}) = \{[0]\}$.

Let $[x] \in ker(\tilde{T})$. By definition, $\tilde{T}[x] = 0$. By the definition of $\tilde{T}$, this implies $T(x) = 0$. By the definition of the kernel of T, this means $x \in kerT$. By the definition of a coset, if $x \in kerT$, then the coset $[x] = [0]$. Therefore, $\tilde{T}$ is injective. $\square$

(c) Check that T is surjective then $\tilde{T}$ is also surjective, and thus $\tilde{T}$ is a linear isomorphism between $X/kerT$ and Y.

Proof: Let $y \in F$. Since T is surjective,there exists a vector $x \in E$ such that $T(x) = y$. Consider the coset $[x] \in E/kerT$. By the definition of $\tilde{T}$, we have $\tilde{T}[x] = T(x)$.Since we know $T(x) = y$, we see that $\tilde{T}[x] = y$. Therefore, $\tilde{T}$ is surjective.

Since $\tilde{T}$ is both linear, injective, and surjective, it is a linear isomorphism between $E/kerT$ and F. □

(d) Show that $T = \tilde{T} \circ q$ where $q : X \to X/kerT$ is the quotient map. In other words every linear operator can be represented as a composition of a surjective and injective operator.

Proof: So let's observe what results from the composition of $\tilde{T} \circ g$. This is equivalent to $\tilde{T}(g(x))$ with $x \in X$. $g(x) = [x]$ so we get $\tilde{T}[x]$. and then $\tilde{T}[x] = Tx$ so thus $T = \tilde{T} \circ q$. □