Functional Analysis Homework 8

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February 9, 2026

1. Let $\{x_k\}_{k=1}^{\infty}$ be a sequence of nonzero vectors in a Banach space X. Define the space of coefficients E by:

$$E := \left\{ a = \{a_k\}_{k=1}^{\infty} : \sum_{k=1}^{\infty} a_k x_k \text{ converges in } X \right\}$$

with the norm

$$\|a\| := \sup_{n\in\mathbb{N}} \left\| \sum_{k=1}^{n} a_k x_k \right\|.$$

Prove that E is a Banach space.

Proof. (Norm axioms) If $a \in E$ then $\sum_{k=1}^{\infty} a_k x_k$ converges in X, so its partial sums $S_n(a) := \sum_{k=1}^{n} a_k x_k$ are Cauchy and hence bounded; thus $\|a\| < \infty$. Homogeneity and the triangle inequality follow from those in X: for each n,

$$\left\|\sum_{k=1}^{n}(\lambda a_k)x_k\right\| = |\lambda|\left\|\sum_{k=1}^{n} a_k x_k\right\|, \quad \left\|\sum_{k=1}^{n}(a_k+b_k)x_k\right\| \le \left\|\sum_{k=1}^{n} a_k x_k\right\| + \left\|\sum_{k=1}^{n} b_k x_k\right\|.$$

Taking sups in n gives $\|\lambda a\| = |\lambda|\,\|a\|$ and $\|a+b\| \le \|a\|+\|b\|$. For positive definiteness, if $\|a\| = 0$ then $S_n(a) = 0\ \forall n$. In particular $a_1 x_1 = 0$, so $a_1 = 0$ since $x_1 \neq 0$. Inductively, $a_1 = \cdots = a_{m-1} = 0$ implies $S_m(a) = a_m x_m = 0$, hence $a_m = 0$ (as $x_m \neq 0$). Thus $a = 0$.

(Completeness) Let $\{a^{(p)}\}_{p\ge 1}$ be Cauchy in $(E, \|\cdot\|)$ (Note: (p) and (q) refers to the index of the element not power). Write $S_n^{(p)} = \sum_{k=1}^{n} a_k^{(p)} x_k$. Then $\forall m, n, p, q$,

$$\|S_n^{(p)} - S_n^{(q)}\| \le \|a^{(p)} - a^{(q)}\|,$$

so for each fixed n, $(S_n^{(p)})_p$ is Cauchy in X and converges to some $s_n \in X$. Next, for each fixed k we have

$$\|(a_k^{(p)} - a_k^{(q)})x_k\| = \|(S_k^{(p)} - S_{k-1}^{(p)}) - (S_k^{(q)} - S_{k-1}^{(q)})\| \le 2\|a^{(p)} - a^{(q)}\|,$$

so $(a_k^{(p)})_p$ is Cauchy and converges to some a_k. For each n, continuity of finite sums gives

$$\sum_{k=1}^{n} a_k x_k = \lim_{p\to\infty} \sum_{k=1}^{n} a_k^{(p)} x_k = \lim_{p\to\infty} S_n^{(p)} = s_n.$$

We now show (s_n) is Cauchy. Fix $\varepsilon > 0$. Since $(a^{(p)})$ is Cauchy in E, choose p_0 so that $\|a^{(p)} - a^{(p_0)}\| \le \varepsilon, \forall p \ge p_0$. Then $\forall n$,

$$\|s_n - S_n^{(p_0)}\| = \lim_{p\to\infty} \|S_n^{(p)} - S_n^{(p_0)}\| \le \varepsilon.$$

Because $a^{(p_0)} \in E$, the partial sums $(S_n^{(p_0)})_n$ converge, so $\exists N$ with $\|S_n^{(p_0)} - S_m^{(p_0)}\| \le \varepsilon, \forall n, m \ge N$. Hence for $n, m \ge N$,

$$\|s_n - s_m\| \le \|s_n - S_n^{(p_0)}\| + \|S_n^{(p_0)} - S_m^{(p_0)}\| + \|S_m^{(p_0)} - s_m\| \le 3\varepsilon,$$

so (s_n) is Cauchy. Let $s = \lim_{n\to\infty} s_n$. Then $\sum_{k=1}^{\infty} a_k x_k$ converges to s.

Finally, convergence in E: for $p \ge p_0$,

$$\|a^{(p)} - a\| = \sup_n \|S_n^{(p)} - s_n\| \le \sup_n \|S_n^{(p)} - S_n^{(p_0)}\| + \sup_n \|S_n^{(p_0)} - s_n\| \le \|a^{(p)} - a^{(p_0)}\| + \varepsilon,$$

and letting $p \to \infty$ gives $\|a^{(p)} - a\| \to 0$. Thus E is complete.

Therefore E is a Banach space. $\square$

2. Show that a Hamel basis of an infinite-dimensional Banach space X is always uncountable.

Proof. Assume X is an infinite-dimensional Banach space but has a countable Hamel basis, $H = \{h_1, h_2, \dots\}$.

Let's define a sequence of nested subspaces $X_n = \text{span}\{h_1, \dots, h_n\}$. Since every vector $x \in X$ is a finite linear combination of basis vectors, every x must belong to some X_n. This allows us to write the entire space X as the countable union $X = \bigcup_{n=1}^{\infty} X_n$.

Now, let's analyze these subspaces. Each X_n is finite-dimensional, and we know that any finite-dimensional subspace of a normed space is closed. Furthermore, since X is infinite-dimensional, each X_n is a proper subspace, which means its interior is empty. A closed set with an empty interior is, by definition, a nowhere dense set.

We have just shown that the space X is a countable union of nowhere dense sets.

However, X is a Banach space, meaning it is a complete metric space. By Baire Category Theorem, a complete metric space cannot be expressed as a countable union of nowhere dense sets. Thus a contradiction and so the Hamel basis H must be uncountable. $\square$

3. Prove that a subset $A \subseteq c_0$ is precompact iff $\exists b \in c_0$ vector that majorizes all vectors $a \in A$, i.e.
$$|a_k| \leqslant b_k, \forall k \in \mathbb{N}.$$

Proof. ($\Rightarrow$)First, assume A is precompact. By pointwise boundedness, we can define a vector b by $b_k = \sup_{a\in A} |a_k|$, which is finite for each k and clearly majorizes A. To show $b \in c_0$, we use the uniform equiconvergence of A. For any $\epsilon > 0$, there exists an N such that $|a_k| < \epsilon\ \forall a \in A$ and all $k \geq N$. Taking the supremum over $a \in A$ gives $b_k = \sup |a_k| \leq \epsilon\ \forall k \geq N$. This shows that $b_k \to 0$, so the majorizing vector b is in c_0.

($\Leftarrow$)Conversely, assume $\exists b \in c_0$ such that $|a_k| \leq b_k\ \forall a \in A$ and all k. We must show A is precompact. First, A is pointwise bounded because $\sup_{a\in A} |a_k| \leq b_k < \infty$ for each k. Second, A is uniformly equiconvergent to 0. Since $b \in c_0$, we know $b_k \to 0$. Thus, for any $\epsilon > 0$, there exists an N such that $k \geq N$ implies $b_k < \epsilon$. This gives $|a_k| \leq b_k < \epsilon\ \forall a \in A$ and all $k \geq N$. Since A satisfies both conditions, it is precompact. □

4. Prove that compactness in normed spaces is stable under linear operations:

   (a) If A, B are precompact sets in a normed space, then Minkowski sum $A + B$ is precompact;

   (b) If A is a precompact subset of X and $T \in L(X, Y)$ then $T(A)$ is a precompact set in Y.

Proof. (a) Let A and B be precompact, and let $\epsilon > 0$. Since A and B are totally bounded, we can cover them with finite $\epsilon/2$-nets.

There exists a finite set $\{a_1, \dots, a_n\} \subseteq A$ such that $A \subseteq \bigcup_{i=1}^n B(a_i, \epsilon/2)$. Similarly, there exists a finite set $\{b_1, \dots, b_m\} \subseteq B$ such that $B \subseteq \bigcup_{j=1}^m B(b_j, \epsilon/2)$.

Now, consider an arbitrary element $z = a + b$ in $A + B$. Then there exist i and j such that $\|a - a_i\| < \epsilon/2$ and $\|b - b_j\| < \epsilon/2$.

We can bound the distance from z to the point $a_i + b_j$ using the triangle inequality:

$$\|z - (a_i + b_j)\| = \|(a - a_i) + (b - b_j)\| \leq \|a - a_i\| + \|b - b_j\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

This shows that any $z \in A+B$ is contained within an ϵ-ball centered at one of the points in the set $C = \{a_i + b_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$. This set C is finite, containing at most $n \times m$ elements. Thus, $A + B \subseteq \bigcup_{c\in C} B(c, \epsilon)$. Since we found a finite ϵ-net for an arbitrary ϵ, $A + B$ is totally bounded and therefore precompact.

(b) Let $\{y_n\}$ be an arbitrary sequence in $T(A)$. By definition, for each n, there exists an $x_n \in A$ such that $y_n = T(x_n)$.

Since A is precompact, its closure $\overline{A}$ is compact. This means the sequence $\{x_n\} \subseteq A$ must have a subsequence, $\{x_{n_k}\}$, that converges to some limit $x \in \overline{A}$.

The operator $T \in L(X, Y)$ is a bounded linear operator, which implies it is continuous. Applying the continuous map T to our convergent subsequence $\{x_{n_k}\}$, we find:

$$\lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} T(x_{n_k}) = T\left(\lim_{k\to\infty} x_{n_k}\right) = T(x)$$

This shows that $\{y_{n_k}\}$ is a convergent subsequence of $\{y_n\}$, and it converges to $T(x) \in Y$. Since every sequence in $T(A)$ has a convergent subsequence, $T(A)$ is relatively compact, and therefore precompact. $\square$