1. Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ such that $f(a) = 0$ and $\displaystyle\int_a^b f = 0$. Prove there exists a $c \in (a, b)$ such that $f'(c) = 0$.

2. Let $f : [a, b] \to \mathbb{R}$ be continuously differentiable on $[a, b]$ with $f(a) = f(b) = 0$ and

$$\int_a^b f^2 = 1\,.$$

Prove that

$$\int_a^b x f(x) f'(x)\,dx = -\frac{1}{2} \quad \text{and} \quad \left(\int_a^b [f'(x)]^2\,dx\right)\left(\int_a^b x^2 [f(x)]^2\,dx\right) > \frac{1}{4}\,.$$

3. Let $a \in \mathbb{R}$ and $f : (0, \infty) \to \mathbb{R}$ be given by $f(x) = x^a$. Compute $f'(x)$. This not a matter of stating the power rule but proving it, starting with $a \in \mathbb{N}$ and working your way up to $a \in \mathbb{R}$.

4. Let $f : (0, \infty) \to \mathbb{R}$ be a continuous function such that $f(x) - f(y) = f(x/y)$ for all $x, y \in (0, \infty)$. Prove that if f is differentiable at $x = 1$ with $f'(1) = 1$, then f is differentiable on $(0, \infty)$ and $f(x) = \log x$ on $(0, \infty)$.

5. Determine all real values of p such that $\displaystyle\lim_{x \to 0^+} x^p \log x$ exists and compute its value. Be sure to justify.