Name:______________________________



Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work, answer or there is no justification to a solution you will receive little or no credit!

**1**. Let $f : [0, \infty) \to \mathbb{R}$ be continuously differentiable such that $f(0) > 0$ and there exists a constant $C < 1$ such that $|f'(x)| \le C$ for all $x \in [0, \infty)$.

(a) Prove that $f(x) \le f(0) + Cx$ for all $x \ge 0$. (HINT: Use Mean Value Theorem.)

(b) Prove that $\lim_{x\to\infty}(f(x) - x) = -\infty$.

(c) Prove there exists a unique $x_0 > 0$ such that $f(x_0) = x_0$.

**2**. Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable function such that

$$\lim_{x\to\infty} f'(x) = 0\,.$$

Prove or disprove that

$$\lim_{x\to\infty}[f(x + 1) - f(x)] = 0\,.$$

**3**. Let $a \ge 0$ and let $\varphi, \psi : [a, \infty) \to (0, \infty)$ be two differentiable functions that are increasing and decreasing respectively. Define $f$ and $g$ by:

$$f(x) = \int_a^x \varphi(t)\,dt \quad \text{and} \quad g(x) = \int_a^x \psi(t)\,dt\,.$$

Prove that $\lim_{x\to\infty} f(x) = \infty$. Now assume that $\lim_{x\to\infty} \varphi(x) = L$ and $\lim_{x\to\infty} \psi(x) = M$ with both $L$ and $M$ finite and positive, prove that $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists and is finite.

**4**. Let $f(x) = x^2 \sin(\frac{1}{x})$ and $g(x) = \sin x$. Prove that

$$\lim_{x\to 0} \frac{f(x)}{g(x)} \quad \text{exists.}$$

Is it true that

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}\,?$$

**5**. For $a, b > 0$ define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \displaystyle\int_a^b t^x \; dt & x \neq -1 \\ \log b - \log a & x = -1 \, . \end{cases}$$

Prove or disprove that $f$ is continuous at $x = -1$.

**6**. Let $f : [0,1] \to \mathbb{R}$ be continuous and set $f_0 = f$. For each $n \geq 1$, define recursively $f_n$ by

$$f_n(x) = \int_0^x f_{n-1} \, .$$

Prove that if $f_n = 0$ for some $n$, then $f = 0$.

**7**. Let $f : [a,b] \to \mathbb{R}$ be a monotonic function with no jump discontinuities and $|h|$ a Riemann integrable function on $[a,b]$. Prove there is a $c \in (a,b)$ such that

$$\int_a^b f(x)|h(x)| \, dx = f(c) \int_a^b |h(x)| \, dx \, .$$

**8**. Let $f : [a,b] \to \mathbb{R}$ be continuously differentiable. Prove that

$$\lim_{n \to \infty} \int_a^b f(x) \sin(nx) \, dx = 0 \, .$$

(HINT: Use Integration by Parts somehow.)

**9**. Let $f$ be a monotonic function on $[0,1]$ with a finite number of discontinuities. Suppose that

$$\int_a^{\frac{a+b}{2}} f(x) \, dx = \int_{\frac{a+b}{2}}^b f(x) \, dx$$

for all $0 \leq a < b \leq 1$. Prove that $f$ is a constant function on $[0,1]$. (HINT: Start with $f$ having no discontinuities.)