Name:______________________________

MA 4643/6643 Section 01 *Homework Set 2* *Due: 2/7/2025*

1. Let (S_1, d_1) and (S_2, d_2) be two metric spaces. Let $E \subseteq S_1$ and let $f : E \to S_2$ be a function. Prove that f is continuous at $x \in E$ if and only if given any sequence $\{x_k\} \subseteq E$ such that if $x_k \to x$, then $f(x_k) \to f(x)$.

2. Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be two normed linear spaces. Let $f : X_1 \to X_2$ be a continuous function. Prove that $\{x \in X_1 : f(x) = 0\}$ is closed.

3. Let (S_1, d_1) and (S_2, d_2) be two metric spaces. Prove that $f : S_1 \to S_2$ is continuous if and only if $f^{-1}(E^o) \subseteq (f^{-1}(E))^o$ for every $E \subseteq S_2$.

4. Let (S, d) be a metric space and let $f, g : S \to S$ be two uniformly continuous functions. Prove or disprove that $f \circ g : S \to S$ is uniformly continuous.

5. Let $(X, \|\cdot\|)$ be a normed space and $L \subseteq X$ a subspace. Given $x \in X$, define the class $[x] := x + L$ in the quotient space X/L induced by x. Define $\|[x]\|_L = \inf\{\|x + l\| : l \in L\}$. Recall the previous homework showed $\|\cdot\|_L$ defines a semi-norm on X/L. Moreover, it showed that $\|\cdot\|_L$ is a norm if and only if L is closed. Show that the quotient map $q : X \to X/L$ given by $q(x) = [x]$ is continuous. Finally, prove that X is a Banach space if and only if X/L is a Banach space.