Functional Analysis (MA 8673) Homework 1

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1. Define

$$A = i\frac{d}{dx}$$

on L^2[0,1]. Define two domains for $A : \mathcal{D}_1(A) = AC[0,1]$ and $\mathcal{D}_2(A) = \{f \in AC[0,1] : f(0) = 0\}$. Prove that A is closed on $\mathcal{D}_1(A)$. Prove or disprove A is closed on $\mathcal{D}_2(A)$

Proof. Let $\{f_n\} \in AC[0,1]$ such that $f_n \to f \in L^2[0,1]$ and $Af_n \to g \in L^2$. Note that $Af_n = if_n'$ and so $if_n' \to g = f_n' \to -ig$.

For simplicity let $-ig = h$. Another thing to note, since we are in a finite space of $[0,1]$, by Holder's inequality, $f_n' \to h \in L^1[0,1]$. So now let's observe the integration of the sequence of f_n. By FTOC we get,

$$\int_0^x f_n'(t)dt = f_n(x) - f_n(0) \tag{1}$$

$$\to f_n(0) = f_n(x) - \int_0^x f_n'(t)dt$$

Since we know this converges in L^1, we integrate both sides w.r.t. x to get

$$\int_0^1 f_n(0)dx = \int_0^1 f_n(x)dx - \int_0^1 \int_0^x f_n'(t)dtdx$$

$$\to f_n(0) = \int_0^1 f_n(x)dx - \int_0^1 \int_0^x f_n'(t)dtdx$$

Let $c = \lim_{n\to\infty} f_n(0)$. (We know this limit exists because the RHS terms converge).

Now let's go back to (1) to get the relation

$$f_n(x) = f_n(0) + \int_0^x f_n'(t)dt$$

Taking the limit as $n \to \infty$:

$$f(x) = c + \int_0^x h(t)dt$$

This equation tells us two things:

(a) The function f is absolutely continuous (since it is an integral of an L^1 function), so $f \in \mathcal{D}_1(A)$.

(b) Differentiating both sides gives $f'(x) = h(x)$.

Recall that $h = -ig$. Therefore:

$$f' = -ig \implies if' = g \implies Af = g$$

Thus, A is closed on $\mathcal{D}_1(A)$.

So now let's observe $\{f_n\} \in D_2(A)$. Assume all the earlier assumptions as shown for $D_1(A)$. Since $f_n \in D_2(A)$, we have $f_n(0) = 0$. By FTOC we have

$$f_n(x) = \int_0^x f_n'(t)dt.$$

Now for any $x \in [0,1]$, we bound the difference by

$$|f_n(x) - \int_0^x h(t)dt| \leq \int_0^x |f_n'(t) - h(t)|dt \leq \|f_n' - h\|_{L^1}$$

Now assume $f_n \in \mathcal{D}_2(A)$, so $f_n(0) = 0$ for all n. Hence $c = \lim_{n\to\infty} f_n(0) = 0$, and so the function

$$F(x) = c + \int_0^x h(t)\, dt$$

satisfies $F(0) = 0$, i.e. $F \in \mathcal{D}_2(A)$. From the estimate above we already have $f_n \to F$ in L^2, hence $f = F$ a.e. Also $F' = h = -ig$ a.e., so $Af = g$. Therefore A is closed on $\mathcal{D}_2(A)$.

□

2. Let A and B be operators defined on a Hilbert space $\mathcal{H}$. Show that $(\alpha A)^* = \bar{\alpha} A^*$ for scalar α. Moreover show $A^* + B^* \subseteq (A+B)^*$ where $\mathcal{D}(A^* + B^*) = \mathcal{D}(A^*) \cap \mathcal{D}(B^*)$. Show that $(A+B)^* = A^* + B^*$ only if one of the operators is bounded. Give an example where equality doesn't hold.

Proof. Define $x, y \in \mathcal{H}$ s.t. $x \in \mathcal{D}(A)$ and $y \in \mathcal{D}(A^*)$ and α be a scalar. We can manipulate $\langle (\alpha A)x, y\rangle$ as follows.

$$\alpha\langle x, A^*y\rangle = \langle x, \bar{\alpha}A^*y\rangle$$

This implies that $y \in ((\alpha A)^*)$ and so $(\alpha A)^*y = \bar{\alpha}A^*y$

Now let's show that $A^* + B^* \subseteq (A+B)^*$.

$$\begin{aligned}\langle (A+B)x, y\rangle &= \langle Ax, y\rangle + \langle Bx, y\rangle \\ &= \langle x, A^*y\rangle + \langle x, B^*y\rangle \\ &= \langle x, (A+B)^*y\rangle\end{aligned}$$

This holds $\forall x \in \mathcal{D}(A+B)$, it follows that $y \in ((A+B)^*)$ and $(A+B)^*y = (A^* + B^*)y$.

Assume A is bounded. Then $\mathcal{D}(A) = \mathcal{H}$, which implies $\mathcal{D}(A^*) = \mathcal{H}$. Consequently, the domain of the sum simplifies to $\mathcal{D}(A+B) = \mathcal{D}(B)$.

From Part 2, we already know $A^* + B^* \subseteq (A+B)^*$. We must now show the reverse inclusion $(A+B)^* \subseteq A^* + B^*$.

Let $y \in \mathcal{D}((A+B)^*)$. By definition, there exists a vector $z \in \mathcal{H}$ such that for all $x \in \mathcal{D}(A+B) = \mathcal{D}(B)$:

$$\langle (A+B)x, y\rangle = \langle x, z\rangle$$

Using the linearity of the inner product, we expand the left side:

$$\langle Ax, y\rangle + \langle Bx, y\rangle = \langle x, z\rangle$$

Since A is bounded and defined on all of $\mathcal{H}$, we know that $y \in \mathcal{D}(A^*)$ so $\langle Ax, y\rangle = \langle x, A^*y\rangle$. Substituting this into the equation, we get

$$\langle x, A^*y\rangle + \langle Bx, y\rangle = \langle x, z\rangle$$

$$\langle Bx, y\rangle = \langle x, z\rangle - \langle x, A^*y\rangle$$

$$\langle Bx, y\rangle = \langle x, z - A^*y\rangle$$

This equation holds for all $x \in \mathcal{D}(B)$. By the definition of the adjoint, this implies that $y \in \mathcal{D}(B^*)$ and $B^*y = z - A^*y$.

Since $y \in \mathcal{D}(B^*)$ and $y \in \mathcal{D}(A^*)$, we have $y \in \mathcal{D}(A^*) \cap \mathcal{D}(B^*) = \mathcal{D}(A^* + B^*)$.

Thus, $(A+B)^* \subseteq A^* + B^*$ and so we have $(A+B)^* = A^* + B^*$.

Counter Example: An example of where the inequality doesn't hold is let $A = -B$. Also let A be an unbounded operator. We have the following to occur of $A + B = 0$ on $D(A)$, so $(A+B)^* = 0$ on the entire space $\mathcal{H}$. Now observing the adjoint of $(A+B)^*$, we get $A^* + B^* = A^* - A^* = 0$. This implies that the domain is restricted to $\mathcal{D}(A^*)$. This implies that the domain of $A^* + B^*$ is restricted to $\mathcal{D}(A^*)$. However, the domain of

$(A+B)^*$ is the entire space $\mathcal{H}$. This mismatch between $\mathcal{D}(A^*+B^*)$ and $\mathcal{D}((A+B)^*)$ shows that equality does not hold.

□

3. Let A and B be operators defined on a Hilbert space $\mathcal{H}$ such that AB is densely defined. Prove that $(AB)^* \supset B^*A^*$. Moreover if B is bounded then show $(BA)^* = A^*B^*$.

   Proof. Let $y \in \mathcal{D}(A^*)$ and $A^*y \in \mathcal{D}(B^*)$. Let $x \in \mathcal{D}(AB)$ so that we have $Bx \in \mathcal{D}(A)$. We then have the following

   $$\langle (AB)x, y\rangle = \langle A(Bx), y\rangle = \langle Bx, A^*y\rangle$$

   $$\langle x, B^*A^*y\rangle$$

   This shows that $y \in \mathcal{D}((AB)^*)$ and that $(AB)^* \supset B^*A^*$. Now we then want to show the other direction of $(AB)^* \subset B^*A^*$.

   Let $y \in (BA)^*$, then $\exists z \in \mathcal{H}$ such that

   $$\langle BAx, y\rangle = \langle x, z\rangle$$

   We can rewrite as follows:

   $$\langle Ax, B^*y\rangle = \langle x, z\rangle, \forall x \in \mathcal{D}(A).$$

   This means that $B^*y \in \mathcal{D}(A^*)$ and $A^*(B^*y) = z = (BA)^*y$. Leading to the equality of $(BA)^* = A^*B^*$.

   □

4. An alternative way to define a normal operator (to allow for unboundedness) is the following: A is called **normal** if $\|Af\| = \|A^*f\|$ for all $f \in \mathcal{D}(A) = \mathcal{D}(A^*)$. Prove that if A is normal then so is $A+z$ for all $z \in \mathbb{C}$

   Proof. So to satisfy this, we must show that $\mathcal{D}(A+z) = \mathcal{D}((A+z)^*)$ and $\|(A+z)f\| = \|(A+z)^*f\|, \forall f \in \mathcal{D}$.

   So first let's show the domain equality.

   $$(A+z)^* = A^* + \bar{z}$$

   Thus, the domain is given as follows.

   $$\mathcal{D}((A+z)^*) = \mathcal{D}(A^*+\bar{z}) = \mathcal{D}(A^*)$$

   Since A is normal, $\mathcal{D}(A+z) = \mathcal{D}(A) = \mathcal{D}(A^*) = \mathcal{D}((A+z)^*)$

Now we want to show the Norm equality. Let $f \in \mathcal{D}(A)$. Let's expand out both sides of the equality

$$\|(A+z)f\| = \|(A+z)^* f\|$$

LHS:

$$\begin{aligned}
|(A+z)f\|^2 &= \langle (A+z)f, (A+z)f \rangle \\
&= \langle Af + zf, Af + zf \rangle \\
&= \langle Af, Af \rangle + \langle Af, zf \rangle + \langle zf, Af \rangle + \langle zf, zf \rangle \\
&= \|Af\|^2 + \bar{z}\langle Af, f \rangle + z\langle Af, f \rangle + |z|^2 \|f\|^2
\end{aligned}$$

RHS:

$$\begin{aligned}
|(A+z)^* f\|^2 &= \langle (A+z)^* f, (A+z)^* f \rangle \\
&= \langle A^* f + \bar{z}f, A^* f + \bar{z}f \rangle \\
&= \langle A^* f, A^* f \rangle + \langle A^* f, \bar{z}f \rangle + \langle \bar{z}f, A^* f \rangle + \langle \bar{z}f, \bar{z}f \rangle \\
&= \|A^* f\|^2 + z\langle A^* f, f \rangle + \bar{z}\langle A^* f, f \rangle + |z|^2 \|f\|^2
\end{aligned}$$

Note that both sides expand to the same term except the first term of both. Note that A is normal. With that $\|Af\|^2 = \|A^* f\|^2$. Thus $A + z$ is normal.

□

5. Using the above definition, prove that normal operators are always closed.

Proof. Let $\{x_n\}$ be a sequence in $\mathcal{D}(A)$ such that $x_n \to x$ and $Ax_n \to y$ in $\mathcal{H}$. To prove that A is closed, we must show that $x \in \mathcal{D}(A)$ and $Ax = y$. Since $\{Ax_n\}$ is a convergent sequence, it is a Cauchy sequence. By the normality condition $\|Ah\| = \|A^* h\|$, we have for any n, m:

$$\|A^* x_n - A^* x_m\| = \|A^*(x_n - x_m)\| = \|A(x_n - x_m)\| = \|Ax_n - Ax_m\|$$

Since $\{Ax_n\}$ is Cauchy, $\|Ax_n - Ax_m\| \to 0$ as $n, m \to \infty$. Therefore, $\{A^* x_n\}$ is also a Cauchy sequence in $\mathcal{H}$ and must converge to some limit z.

We now have:

$$x_n \to x \quad \text{and} \quad A^* x_n \to z$$

Recall that the adjoint operator A^* is always closed. By the definition of closedness applied to A^*, this implies that $x \in \mathcal{D}(A^*)$ and $A^* x = z$.

Since A is normal, we are given that $\mathcal{D}(A) = \mathcal{D}(A^*)$. Therefore, $x \in \mathcal{D}(A)$.

Finally, we must show that $Ax = y$. We apply the norm equality to the vector $x_n - x$ to get

$$\|Ax_n - Ax\| = \|A^*(x_n - x)\| = \|A^* x_n - A^* x\|.$$

When we take the limit as $n \to \infty$, on the right side, since $A^*x_n \to z$ and we found $A^*x = z$, the term $\|A^*x_n - A^*x\| \to 0$. Thus the left side $\|Ax_n - Ax\| \to 0$, which implies $Ax_n \to Ax$.

Since we originally assumed $Ax_n \to y$, by the uniqueness of limits, we must have $Ax = y$.

Thus, $x \in \mathcal{D}(A)$ and $Ax = y$, proving that A is closed. $\square$