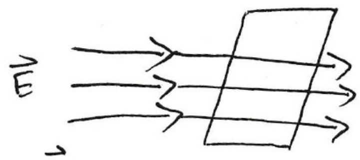
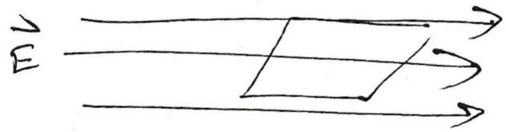
Chapter 22: Gauss's Law

Section 22-1: Electric Flux

- Flux is the "amount" of electric field that goes through an area



$\vec{E}$ goes through area:
Lots of flux



$\vec{E}$ in same plane as area:
No flux (no field lines go through area)

- Area in physics is a vector, $\vec{A}$

  Magnitude of $\vec{A}$ is the area

  Direction is normal vector

- Normal vector is a vector perpendicular to the area

  $\vec{E}$ parallel to normal: full flux

  $\vec{E}$ perpendicular to normal (in plane of area): no flux

  $\vec{E}$ antiparallel to normal: negative flux

- Flux is therefore a dot product: $\boxed{\Phi = \vec{E} \cdot \vec{A}}$

Example: A rectangular loop 3 cm by 5 cm oriented in the yz plane has a normal vector in the positive x direction and is immersed in an electric field $300\,\text{N/C}\,\hat{\imath} - 600\,\text{N/C}\,\hat{\jmath} + 200\,\text{N/C}\,\hat{k}$. Find the flux through the surface.

$$A = 0.03\,m \times 0.05\,m = 0.0015\,m^2$$

$$\vec{A} = 0.0015\,m^2\,\hat{i}$$

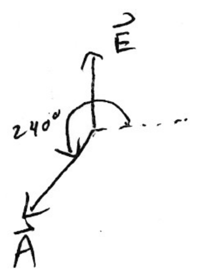
$$\vec{E}\cdot\vec{A} = (300\,N/C\,\hat{i} \cancel{- 600\,N/C\,\hat{j}} \cancel{+ 200\,N/C\,\hat{k}}) \cdot 0.0015\,\hat{i}$$

$$= \boxed{0.45\ Nm^2/C}$$

<u>Example:</u> A circular surface has a diameter of 20 cm. An electric field $E = 1000\,N/C\,\hat{j}$ passes through the surface. The surface rotates such that its normal rotates in the xy plane. What is the flux when the normal vector is at 240° c.c.w. from x?

$$A = \pi r^2 = \pi(0.1\,m)^2 = 0.0314\,m^2$$

$r = d/2$

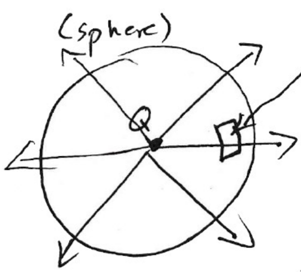


$$240° - 90° = 150°$$

$$\Phi = EA\cos 150°$$

$$= \boxed{-27.2\ Nm^2/C}$$

- For curved surface ($\vec{A}$ not constant) or when $\vec{E}$ not constant:



Small area, $d\vec{A}$

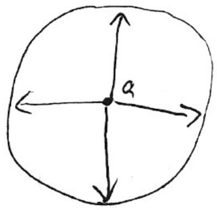
Flux through it:

$$d\Phi = \vec{E}\cdot d\vec{A}$$

Sum up (integrate): $\Phi = \int d\Phi$

$$\boxed{\Phi = \int \vec{E}\cdot d\vec{A}}$$

Example: Find the flux through a sphere of radius r with charge Q at the center.



$\vec{E}$ points out, and so does $d\vec{A}$

Thus $\vec{E} \cdot d\vec{A} = E\,dA$

$E = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2}$

$$\Phi = \int E\,dA = \int \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2}\,dA = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \int dA$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} A = \frac{1}{\cancel{4\pi}\varepsilon_0} \frac{Q}{\cancel{r^2}} \cdot \cancel{4\pi r^2} = \boxed{\frac{Q}{\varepsilon_0}}$$

- Whenever there is <u>symmetry</u>, $\int \vec{E} \cdot d\vec{A} = EA$

    Without symmetry, E changes so there's no one value to use

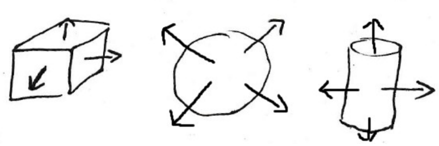
- Types of surfaces

    Open surface: 2D shapes (circle, rectangle, triangle, etc.)

        Normal vector points "up" or "down" (your choice)

    Closed surface: Outer surface of a 3D shape

        Normal vector <u>always</u> points <u>out</u> of the shape (no choice)



<u>Section 22-2: Gauss's Law</u>

- <u>Gauss's Law</u>: $\boxed{\oint \vec{E} \cdot d\vec{A} = \frac{Q_{in}}{\epsilon_0}}$

    In above example, flux is the <u>same</u> if charge not in center

    E would have different values but total flux stays the same

- Circle on integral sign means <u>closed surfaces only</u>

    <u>Do not</u> use on open surfaces!

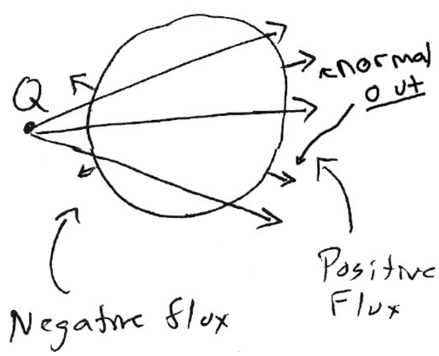
    Open surface, with constant E: flux proportional to area

    But with Gauss/closed surface, E is <u>not</u> constant

    Larger area means further from point charge, so <u>smaller</u> E with <u>same</u> flux

- Only charge <u>inside</u> closed surface causes flux

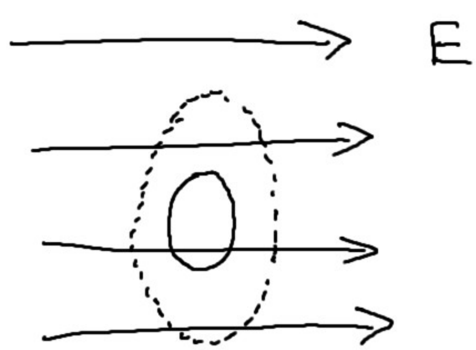
    Charges outside <u>cause zero net flux:</u>



Gauss's Law says negative and positive flux cancel for charges outside of a surface.

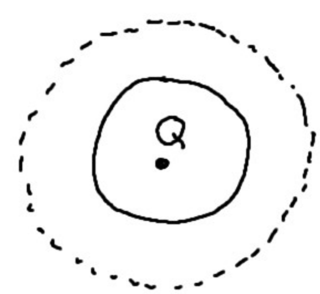
E is <u>not</u> zero but $\oint \vec{E} \cdot d\vec{A}$ <u>is</u>

Open surface, $\vec{E}$ = const



Bigger surface, <u>more</u> flux

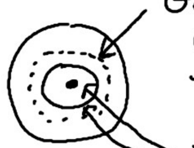
Closed surface, charge inside



Bigger area, smaller E, <u>same</u> flux

Example: A spherical metal shell has a charge of $30\mu C$. A $-20\mu C$ charge sits at its center. (a) How much charge is on the shell's inner surface? (b) How much on the outer?

(a)



Gaussian surface: $E=0$ since inside metal (conductor)

$\int \vec{E}\cdot d\vec{A} = 0$ (integral of zero is zero)

$\int \vec{E}\cdot d\vec{A} = \frac{Q_{in}}{\varepsilon_0}$ so $Q_{in} = 0$

Point in center is $-20\mu C$

Q_{inner} must be $\boxed{+20\mu C}$ since Q_{in} must be 0

(b) $Q_{inner} + Q_{outer} = Q_{total}$

$20\mu C + Q_{outer} = 30\mu C$ so $Q_{outer} = 10\mu C$

Section 22-3: Applications of Gauss's Law

- Outline of typical Gauss's Law problem:

Left-hand side $\oint \vec{E}\cdot d\vec{A}$

This is the *easy* side

Due to symmetry, the integral is always $\oint \vec{E}\cdot d\vec{A} = EA$

Need proper area for shape (sphere or cylinder)

Right-hand side $\frac{Q_{in}}{\epsilon_0}$

This is the *hard* side that requires thought

Add up charges for all objects at a *smaller* radius

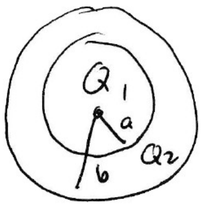
Point charge: add its charge

Line charge: $\boxed{Q = \lambda\ell}$, ℓ is its length

Surface charge: $\boxed{Q = \sigma A}$, A is area

Charge density: $\boxed{Q = \rho V}$, V is volume

Example: A charge Q_1 sits in the center of a spherical, metal shell with inner radius a and outer radius b and charge Q_2. Find E for $r<a$, $a<r<b$, and $r>b$, and find the charge on the inner and outer surfaces.



$r<a$: $\oint \vec{E}\cdot d\vec{A} = \frac{Q_{en}}{\varepsilon_0}$

$$E \cdot 4\pi r^2 = \frac{Q_1}{\varepsilon_0}$$

$$\boxed{E = \frac{Q_1}{4\pi\varepsilon_0 r^2}}$$

$a<r<b$: Conductor!

$\boxed{E = 0}$ in a conductor.

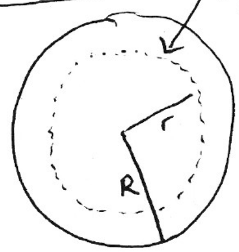
$r>b$: $\oint \vec{E}\cdot d\vec{A} = \frac{Q_{en}}{\varepsilon_0}$

$$E \cdot 4\pi r^2 = \frac{Q_1 + Q_2}{\varepsilon_0}$$

$$\boxed{E = \frac{Q_1 + Q_2}{4\pi\varepsilon_0 r^2}}$$

Example: A sphere of radius R has constant charge per unit volume ρ. Find the magnitude of the electric field inside and outside.

Inside: Spherical "Gaussian surface"



Symmetry: $\vec{E}$ is radial, so is $d\vec{A}$, so

$\vec{E} \cdot d\vec{A} = E\,dA$

Also: E depends only on r due to symmetry, so $\oint E\,dA = EA$

Facts about spheres: $\boxed{V = \frac{4}{3}\pi r^3, \quad A = 4\pi r^2 \text{ Know this!}}$

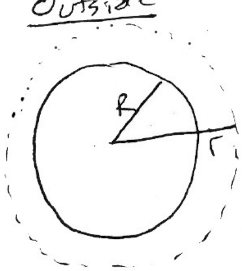
$$\Phi = EA = \frac{Q_{encl}}{\varepsilon_0}$$

$$Q_{encl} = \rho V = \frac{1}{\varepsilon_0}\rho \cdot \frac{4}{3}\pi r^3$$

$$E \cdot \cancel{4\pi r^2} = \frac{\rho}{\cancel{\varepsilon_0}} \cdot \frac{\cancel{4}}{3}\cancel{\pi} r^{\cancel{3}}$$

$$\boxed{E = \frac{\rho r}{3\varepsilon_0}}$$

Outside



$$EA = \frac{Q_{encl}}{\varepsilon_0}$$

$$Q_{encl} = \frac{\rho}{\cancel{\varepsilon_0}} \cdot \frac{4}{3}\pi R^3 \quad (\underline{\text{not}} \;\; \rho \cdot \tfrac{4}{3}\pi \underline{\underline{r}}^3)$$

$$E \cdot \cancel{4\pi} r^2 = \frac{\rho}{\varepsilon_0} \cdot \frac{\cancel{4}}{3}\cancel{\pi} R^3$$

$$\boxed{E = \frac{\rho R^3}{3\varepsilon_0 r^2}}$$

- Compare last two examples:

  One had a charge density ρ. This <u>can't</u> happen in a conductor

  Other had Q on a metal object. Q will be on <u>surface</u>

- These examples are <u>not</u> derivations!

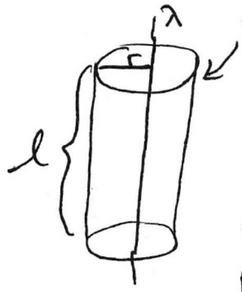
  Although I am calculating formulas, these are not derivations!

  You will be expected to come up with formulas like these

  Learn the <u>method</u>, do not memorize these formulas!

[*May skip the following example if time is short.*]

Example: Find the magnitude of the electric field a distance r from an infinite thin wire of charge per unit length λ.



Cylindrical gaussian surface

By symmetry: $\vec{E}$ points away or towards wire; no $\vec{E}$-component along its length

Therefore, no flux out of top or bottom.

All flux through curved side

$\vec{E}$ points out, same as normal, so $\vec{E} \cdot d\vec{A} = E\,dA$

E constant on surface, so $\int E\,dA = E\int dA = EA$

Cut surface to make a plane which has height l and width equal to circumference $2\pi r$

Conclusion:

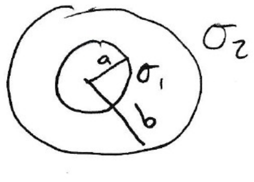
$$\oint \vec{E} \cdot d\vec{A} = 2\pi r l E$$

$$Q_{encl} = \lambda l$$

$$2\pi r \cancel{l} E = \frac{\lambda \cancel{l}}{\varepsilon_0}$$ (Length always cancels out)

$$\boxed{E = \frac{\lambda}{2\pi \varepsilon_0 r}}$$

Example: A thin shell of radius a has a surface density σ_1. It is surrounded by a thin shell with radius b with surface density σ_2. The shells are very long cylinders



$r<a$: $\oint \vec{E}\cdot d\vec{A} = \frac{Q_{en}}{\varepsilon_0}$

$E\cdot 2\pi r \ell = 0$

$$\boxed{E=0}$$

E parallel to ends, so $\oint \vec{E}\cdot d\vec{A} = 0$ on ends

For side: A = circumference × height = $2\pi r \ell$

$a<r<b$: $\oint \vec{E}\cdot d\vec{A} = \frac{Q_{en}}{\varepsilon_0}$

$E\cdot 2\pi r\ell = \frac{\sigma_1 \cdot 2\pi a \ell}{\varepsilon_0}$ ← Area of inner cylinder, radius a

$$\boxed{E = \frac{a\sigma_1}{\varepsilon_0 r}}$$

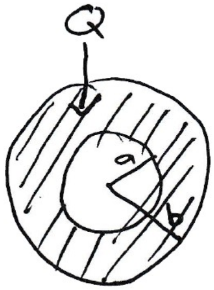
$r>b$: $\oint \vec{E}\cdot d\vec{A} = \frac{Q_{en}}{\varepsilon_0}$

$E\cdot 2\pi r\ell = \frac{\sigma_1\cdot 2\pi a\ell + \sigma_2 \cdot 2\pi b \ell}{\varepsilon_0}$

$$\boxed{E = \frac{a\sigma_1 + b\sigma_2}{\varepsilon_0 r}}$$

Example: A non-conducting cylindrical shell of length ℓ has inner radius a, outer radius b, and constant charge density with total charge Q. Find E everywhere.

First, find ρ (in terms of Q)



$$\rho = \frac{Q_{tot}}{V_{tot}} \leftarrow Q_{tot} = Q \text{ (given in problem)}$$

$$V_{tot} = \pi b^2 \ell - \pi a^2 \ell$$

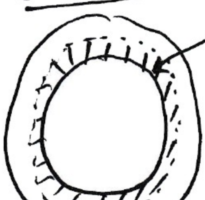
$$\rho = \frac{Q}{\pi b^2 \ell - \pi a^2 \ell} = \frac{Q}{\pi \ell (b^2 - a^2)}$$

$r < a$:

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{in}}{\varepsilon_0} = 0$$

$$\boxed{E = 0}$$

$a < r < b$



Volume inside of r : $\pi r^2 \ell - \pi a^2 \ell$

$$Q_{in} = \rho V_{in} = \frac{Q}{\cancel{\pi \ell}(b^2 - a^2)} (\cancel{\pi} r^2 \cancel{\ell} - \cancel{\pi} a^2 \cancel{\ell})$$

$$= \frac{Q(r^2 - a^2)}{b^2 - a^2}$$

$$\oint \vec{E} \cdot d\vec{A} = E \cdot 2\pi r \ell = \frac{Q_{in}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \cdot \frac{Q(r^2 - a^2)}{b^2 - a^2}$$

$$\boxed{E = \frac{Q(r^2 - a^2)}{2\pi r \ell (b^2 - a^2)}}$$

$r > b$

$$Q_{in} = Q$$

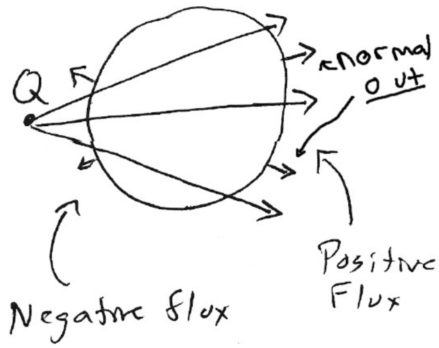
$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{in}}{\varepsilon_0}$$

$$E \cdot 2\pi r \ell = \frac{Q}{\varepsilon_0}$$

$$\boxed{E = \frac{Q}{2\pi \varepsilon_0 r \ell}}$$

Final points

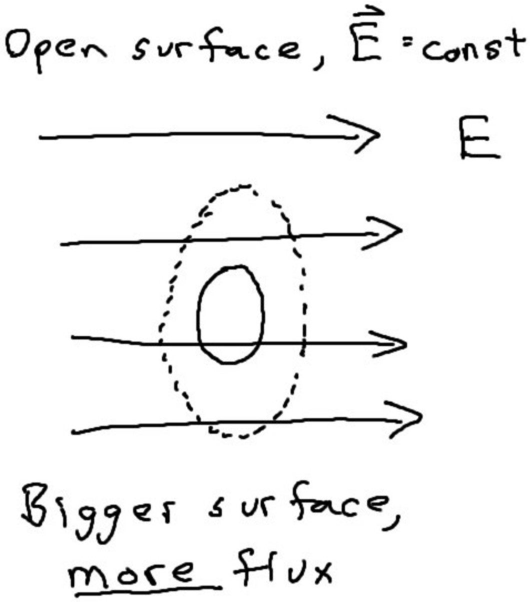
- $Q_{in} = 0$ does not mean E=0 for non-symmetric situations! Recall:

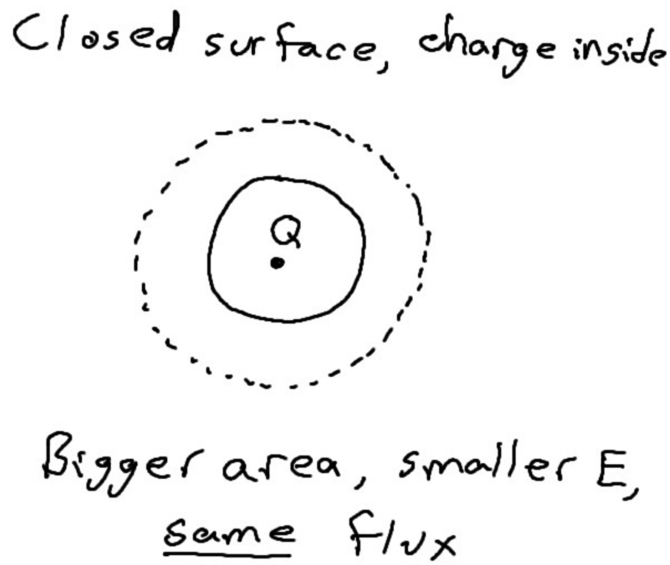


Gauss's Law says negative and positive flux cancel for charges outside of a surface.

E is not zero but $\oint \vec{E} \cdot d\vec{A}$ is

- Flux is constant only for closed surfaces! Recall:





Homework: Do Chapter 22 in Mastering Physics