

# Functional Analysis Homework 5

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1. Prove that Parseval's identity holds for an orthonormal system  $(x_k)$  iff  $(x_k)$  is complete. Therefore, the inequality cases of Bessel's inequality hold exactly when the system is complete.

$$\sum_k |\langle x, x_k \rangle|^2 = \|x\|^2$$

( $\Rightarrow$ ) So suppose the inequality above is true for every vector. Then, we want to show that the orthonormal system  $x_k$  is complete or  $\overline{\text{span}\{x_k\}} = X$ . So we can show this by showing that the only vector in the space that is orthogonal to every  $x_k$  is the zero vector. So let  $y$  be a vector orthogonal to every single basis vector. This means that  $\forall k, \langle y, x_k \rangle = 0$ . So when we plug that into Parseval's inequality, we get

$$\sum_k |\langle y, x_k \rangle|^2 = 0 = \|y\|^2 \implies \|0\|^2$$

Meaning that  $y$  must be the zero vector itself for the identity to be true, showing that the system is complete. (I guess this also is proved in question 3 as well)

( $\Leftarrow$ ) So suppose the orthonormal system  $x_k$  is complete. Then we know that  $\overline{\text{span}\{x_k\}} = X$ . With the completeness, the orthogonal projection of  $x$  onto the span is the identity map so when we observe the Fourier series for  $x$ . We get

$$S_n(x) = \sum_{k=1}^n \langle x, x_k \rangle x_k$$

and when we take the square we get a result of the Pythagorean Theorem of

$$\rightarrow \|S_n(x)\|^2 = \left\| \sum_{k=1}^n \langle x, x_k \rangle x_k \right\|^2 = \sum_{k=1}^n |\langle x, x_k \rangle|^2$$

So now let's look at the partial sums  $S_n(x)$  (left side of the inequality). So with the completeness of  $X$ , we get

$$\lim_{n \rightarrow \infty} \|x - S_n(x)\| = 0$$

and so we apply the Pythagorean Theorem to get

$$\|x\|^2 = \|S_n(x)\|^2 + \|x - S_n(x)\|^2 = \|S_n(x)\|^2 + 0$$

\*equality is true as  $n \rightarrow \infty$  Which then leads to the equality being true of Parseval's inequality.  $\square$

2. Prove that Gram-Schmidt orthogonalization of the monomials  $(t^k)_{k=0}^{\infty}$  in the space  $L_2[-1, 1]$  gives the system of Legendre orthogonal polynomials

$$P_k(t) = \frac{1}{2^k k!} \frac{d^k}{dt^k} (t^2 - 1)^k,$$

up to normalization constants. More precisely,  $P_k(t)$  form an orthogonal basis in  $L_2[-1, 1]$  and  $\|P_k\|_2^2 = \frac{2}{2k+1}$ .

Proof: So with the properties of Gram-Schmidt orthogonalization. We require two things about  $P_k(t)$  in which  $P_k(t)$  must be a polynomial of degree  $k$  and that each polynomial is orthogonal to each in the space. So the second fact is obvious but the first property comes from the Gram-Schmidt construction itself. The formula for the  $k$ -th resulting polynomial

$$P_k(t) = t^k - \sum_{j=0}^{k-1} \frac{\langle t^k, v_j(t) \rangle}{|v_j(t)|^2} v_j(t)$$

This formula shows that the polynomial is created by starting with  $t_k$  and subtracting a sum of lower-degree polynomials, which proves the resulting polynomial has a degree of  $k$ . So now lets show the two properties. So with the first properties of the polynomial being at most degree  $k$ , let's observe the system.

$$(t^2 - 1)^k \rightarrow t^{2k}$$

$$\Rightarrow \frac{d^k}{dt^k} (t^{2k})$$

leading to a polynomial degree of  $k$  after differentiating so it then satisfies the first property and now we need to show orthogonality to  $t^j$  for  $j < k$ . So we do the inner product, we get

$$\langle P_k(t), t^j \rangle = \int_{-1}^1 P_k(t)(t^j) dt = 0$$

$$\text{let } c_k = \frac{1}{2^k k!}$$

$$\rightarrow c_k \int_{-1}^1 \left( \frac{d^k}{dt^k} (t^2 - 1)^k \right) (t^j) dt$$

now we apply integration by parts to where we let  $u = t^j, dv = \frac{d^k}{dt^k}(t^2 - 1)^k dt$ . Getting us  $du = jt^{j-1}, v = \frac{d^{k-1}}{dt^{k-1}}(t^2 - 1)^k dt$ . The uv term gets us the following

$$t^j \frac{d^{k-1}}{dt^{k-1}}(t^2 - 1)^k dt \Big|_{-1}^1 = 0$$

and so we then get the remaining integral of

$$\langle P_k, t^j \rangle = -c_k \int_{-1}^1 \left( \frac{d^{k-1}}{dt^{k-1}}(t^2 - 1)^k \right) (j \cdot t^{j-1}) dt$$

Now we repeat this process for  $j+1$  times to have it being differentiated  $j+1$  times getting that

$$\frac{d^{j+1}}{dt^{j+1}}(t^j) = 0$$

This will make the entire integrand zero, and so the integral becomes zero.

$$\langle P_k, t^j \rangle = 0$$

This only works because we are given that  $j < k$ . This ensures that we can always perform the integration by parts  $j+1$  times without running out of derivatives to remove from the  $(t^2 - 1)^k$  term. Giving is the orthogonality of each polynomial.  $\square$

3. Consider a system of vectors  $(x_k)$  (not necessarily orthogonal) in a Hilbert space  $X$ . Prove that  $(x_k)$  is complete iff the only vector orthogonal to all of  $x_k$  is zero.

$(\Rightarrow)$  So let the system  $x_k$  be complete. This gives us that  $\overline{\text{span}\{x_k\}} = X$ . So let's choose an  $z \in \text{span}\{x_k\}$  and  $y$  be a vector that is orthogonal to every  $x_k$ . So let's rewrite  $z$  as

$$z = \sum_{i=1}^n c_i x_i$$

and so we get

$$\langle z, y \rangle = \langle \sum_{i=1}^n c_i x_i, y \rangle = \sum_{i=1}^n c_i \langle x_i, y \rangle = 0$$

And with the closure of the span, then there is a sequence of vectors  $z_n$  such that it converges to  $y$  as  $n \rightarrow \infty$ . So as we observe  $\lim_{n \rightarrow \infty} \langle z_n, y \rangle \rightarrow \langle y, y \rangle = \|y\|^2 = 0$  so then  $y$  must be the 0 vector.

- ( $\Leftarrow$ ) So assume that the system  $x_k$  is not complete and that the only vector orthogonal to all of  $x_k$  is the zero vector. Then  $\overline{\text{span}\{x_k\}}$  does not represent the space of  $X$ . With this in mind, let  $y \notin Y = \overline{\text{span}\{x_k\}}$  but  $y \in X$ . By the Projection Theorem, we can write  $y$  to be a unique decomposition of  $y = x_0 + z$  where  $z \in Y^\perp$  and  $x_0 \in Y$ .  $z$  cannot be the zero vector as if  $z = 0$ , then  $y = x_0$ . So then  $z$  must be a non-zero vector which is a contradiction as there should only be the zero vector to all of  $x_k$  that is orthogonal.  $\square$
4. Suppose  $w(t)$  is a continuous weight function  $\mathbb{R} \rightarrow \mathbb{R}_+$ . Consider the Hilbert space  $L_2(\mathbb{R}, w(t)dt)$ , i.e. the measure on  $\mathbb{R}$  is given by  $w(t)dt$ . The Gram Schmidt orthogonalization of monomials  $(t^k)_{k=0}^\infty$  produces a system orthogonal polynomials  $P_k(t)$  with respect to the weight  $w(t)$ , i.e.

$$\int_{\mathbb{R}} P_k(t) P_l(t) w(t) dt = \delta_{kl}.$$

Now consider the weight  $w(t) = \frac{1}{\sqrt{2\pi}} e^{\frac{t^2}{2}}$ , i.e. the standard normal density. Prove that the orthogonal polynomials with respect to this weight is the system of Hermite polynomials

$$P_k(t) = (-1)^k e^{\frac{t^2}{2}} \frac{d^k}{dt^k} e^{\frac{-t^2}{2}},$$

up to normalization constants. More precisely,  $P_k(t)$  form an orthogonal basis in  $L_2(\mathbb{R}, w(t)dt)$  and  $\|P_k\|_2^2 = k!$ .

Proof: So we approach this very similarly as question 3 where we show the two properties of Gram Schmidt to show then following is true.

So for the first property, I'm not going to write the full derivation but describe what occurs for the polynomial degree. So with  $k=0$ , the exponentials cancel out leading to a polynomial of degree 0 so the base case is accounted for. Now as the degree  $k$  increases, each derivative of  $e^{\frac{-t^2}{2}}$  gains a polynomial degree in front of it leading to the next polynomial being degree 1 for  $k = 1$  and this continues for higher levels of  $k$  leading to the first property being satisfied.

Now for the second property, we need to show that  $P_k(t)$  is orthogonal to  $t^j, j < k$ . So now let's look at the inner product space for it.

$$\begin{aligned} \langle P_k, t^j \rangle &= \int_{-\infty}^{\infty} \left[ (-1)^k e^{t^2/2} \frac{d^k}{dt^k} e^{-t^2/2} \right] t^j \left[ \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \right] dt \\ &\rightarrow \frac{(-1)^k}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( \frac{d^k}{dt^k} e^{-t^2/2} \right) t^j dt \end{aligned}$$

Now with integration by parts, we let  $u = t^j, dv = \frac{d^k}{dt^k} e^{-t^2/2} dt$  doing this  $j+1$  times. The boundary term  $uv$  will be zero at both  $\pm\infty$ . This

is because  $v$  will always be a polynomial times  $e^{-t^2/2}$ , and the exponential decay of  $e^{-t^2/2}$  goes to zero much faster than any polynomial grows.

After repeating integration by parts  $j + 1$  times, we get

$$\frac{d^{j+1}}{dt^{j+1}}(t^j) = 0$$

This will make the entire integrand zero, and so the integral becomes zero.

$$\langle P_k, t^j \rangle = 0, j < k$$

And so showing the orthogonality of all polynomials of lower degree w.r.t. the weighted inner product. Since these are the unique properties of the polynomials generated by the Gram-Schmidt process, the Hermite polynomials must be the same as the ones generated by Gram-Schmidt, up to normalization constants.  $\square$

5. Show that every linear functional  $f$  on  $\mathbb{C}^n$  has the form

$$f(x) = \sum_{k=1}^n x_k y_k = \langle x, y \rangle, x = (x_1, \dots, x_n),$$

for some  $y = (y_1, \dots, y_n) \in \mathbb{C}^n$ .

Proof: : Let  $e_k$  be the canonical basis. Then we can rewrite  $x$  as the following

$$x = \sum_{k=1}^n x_k e_k$$

and so we can rewrite  $f$  to be the following (note linearity of  $f$ ):

$$f(x) = f\left(\sum_{k=1}^n x_k e_k\right) = \sum_{k=1}^n x_k f(e_k)$$

and so let  $y_k = f(e_k)$  leading to the equality.  $\square$

6. Compute the norms of the integration and the point evaluation functionals considered in Examples 2.1.2 and 2.1.3.

2.1.2:

$$\begin{aligned} F(g) &= \int_0^1 g(t)w(t)dt \\ |F| &= \left| \int_0^1 g(t)w(t)dt \right| \leq \int_0^1 |g(t)w(t)|dt = \int_0^1 |g(t)||w(t)|dt \\ &\leq \int_0^1 \|g\|_\infty |w(t)|dt = \|g\|_\infty \int_0^1 |w(t)|dt \end{aligned}$$

$$\rightarrow |F| \leq \|g\|_\infty \int_0^1 |w(t)|dt$$

showing that  $\|F\| \leq \|g\|_\infty \int_0^1 |w(t)|dt$ , implying that the norm of the functional is bounded by the constant part  $\|F\| \leq \int_0^1 |w(t)|dt$ . So let's try to show equality now, and let

$$g(t) = \begin{cases} 1 & \text{if } w(t) > 0 \\ -1 & \text{if } w(t) < 0 \\ 0 & \text{if } w(t) = 0 \end{cases}$$

Then  $\|g\|_\infty = 1$  and the product for the integral becomes the following.

$$F(g) = \int_0^1 g(t)w(t)dt = \int_0^1 |w(t)|dt$$

Thus finding that  $|F| = \int_0^1 |w(t)|dt$  and so we get

$$\|F\| = \int_0^1 |w(t)|dt$$

2.1.3:

$$F(g) = g(t_0), t_0 \in [0, 1]$$

$$|F(g)| = |g(t_0)| \leq \max_{t \in [0, 1]} |g(t)|$$

Substituting our definitions back in, this is:

$$|F(g)| \leq 1 \cdot \|g\|_\infty$$

This gives us that  $\|F\| \leq 1$ . And now we need to show that  $\|F\| = 1$ , so let  $g(t) = 1$ . It is in the space  $C[0, 1]$ ,  $\|g\|_\infty = 1$ , and  $F(g) = g(t_0) = 1$  so  $|F(g)| = 1 \cdot \|g\|_\infty$  is true. Since the norm can't be larger than 1, and we found a function for which the ratio is exactly 1, the norm of the point evaluation function must be 1.  $\square$