Functional Analysis Homework 6

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1. Prove that $c_0^* = \ell_1$. The meaning of this is the same as in Corollary 2.2.6, i.e. the functionals on c_0 are given by summation with weight from ℓ_1

   Proof. Let $y = (y_k) \in \ell_1$ and define $T(y) \in (c_0)^*$ by

   $$(T(y))(x) = \sum_{k=1}^{\infty} x_k y_k, \qquad x = (x_k) \in c_0.$$

   This is well defined since

   $$\big|(T(y))(x)\big| \le \sum_{k=1}^{\infty} |x_k||y_k| \le \|x\|_\infty \sum_{k=1}^{\infty} |y_k| = \|x\|_\infty \|y\|_1.$$

   Hence $T : \ell_1 \to (c_0)^*$ is linear and bounded with $\|T(y)\| \le \|y\|_1$.

   We claim $\|T(y)\| = \|y\|_1$. For $N \in \mathbb{N}$ set $x^{(N)} = (x_k^{(N)})$ by $x_k^{(N)} = \operatorname{sgn}(y_k)$ for $k \le N$ and 0 otherwise. Then $x^{(N)} \in c_0$, $\|x^{(N)}\|_\infty = 1$, and

   $$(T(y))(x^{(N)}) = \sum_{k=1}^{N} |y_k|.$$

   Therefore

   $$\|T(y)\| \ge \sup_N \big|(T(y))(x^{(N)})\big| = \sup_N \sum_{k=1}^{N} |y_k| = \|y\|_1.$$

   Combined with $\|T(y)\| \le \|y\|_1$, this gives $\|T(y)\| = \|y\|_1$, so T is an isometry.

   It remains to show T is surjective. Let $f \in (c_0)^*$. Define $y_k := f(e_k)$, where e_k is the canonical basis vector. For $N \in \mathbb{N}$, set $x^{(N)} = \sum_{k=1}^{N} \operatorname{sgn}(y_k) e_k \in c_0$. Then

   $$\sum_{k=1}^{N} |y_k| = \sum_{k=1}^{N} \operatorname{sgn}(y_k) f(e_k) = f(x^{(N)}) \le \|f\| \, \|x^{(N)}\|_\infty = \|f\|.$$

Thus the partial sums are uniformly bounded, so $\sum_{k=1}^{\infty} |y_k| < \infty$ and $y \in \ell_1$.

For $x \in c_{00}$ (finite support), by linearity,

$$f(x) = \sum_{k=1}^{\infty} x_k f(e_k) = \sum_{k=1}^{\infty} x_k y_k = (T(y))(x).$$

Since c_{00} is dense in c_0 under $\|\cdot\|_\infty$ and both f and $T(y)$ are continuous on $(c_0, \|\cdot\|_\infty)$, it follows that $f(x) = (T(y))(x)$ for all $x \in c_0$. Hence $f = T(y)$ with $y \in \ell_1$, proving T is surjective.

Therefore $T : \ell_1 \to (c_0)^*$ is a surjective isometry, i.e. $(c_0)^* = \ell_1$. $\square$