Functional Analysis Homework 11

Kevin Ho

February 9, 2026

1. Show that T is compact iff it maps B_X to a precompact set in Y.

   Proof. Let X, Y be normed spaces and $T : X \to Y$ linear and bounded. Write
   $$B_X := \{x \in X : \|x\| \leq 1\}.$$
   Recall that a set $M \subset Y$ is *precompact* if its closure $\overline{M}$ is compact in Y.

   We show the two implications separately.

   ($\Rightarrow$): Suppose T is compact in the sense that it maps bounded subsets of X into precompact subsets of Y. Since B_X is bounded in X, it follows immediately that $T(B_X)$ is precompact in Y.

   ($\Leftarrow$): Now assume that $T(B_X)$ is precompact in Y. We must show that T maps any bounded subset of X to a precompact subset of Y.

   Let $A \subset X$ be bounded. Then there exists $R > 0$ such that $\|x\| \leq R$ for all $x \in A$, i.e.
   $$A \subset RB_X := \{x \in X : \|x\| \leq R\}.$$
   By linearity,
   $$T(RB_X) = \{T(Rx) : \|x\| \leq 1\} = \{RT(x) : \|x\| \leq 1\} = RT(B_X).$$
   Since $T(B_X)$ is precompact, its closure $K := \overline{T(B_X)}$ is compact in Y. Consider the map $S_R : Y \to Y$, $S_R(y) = Ry$ for the fixed scalar $R > 0$. This is a homeomorphism with continuous inverse $S_R^{-1}(y) = \frac{1}{R}y$, so it sends compact sets to compact sets. Hence
   $$\overline{RT(B_X)} = \overline{S_R(T(B_X))} \subset S_R(\overline{T(B_X)}) = RK$$
   is compact as a closed subset of the compact set RK. Therefore $RT(B_X)$ is precompact in Y.

   Now $T(A) \subset T(RB_X) = RT(B_X)$, and any subset of a precompact set is again precompact (its closure is contained in the compact closure of the larger set). Thus $T(A)$ is precompact in Y.

Since $A \subset X$ was an arbitrary bounded set, we have shown that T maps every bounded subset of X to a precompact subset of Y. By definition, this means T is compact.

$\square$

2. Show that Volterra operator is compact on $C[0,1]$, even though its kernel is discontinous.

Proof. First, $T : C[0,1] \to C[0,1]$ is linear by properties of the integral. If $\|f\|_\infty \le M$, then for all $x \in [0,1]$,

$$|(Tf)(x)| = \left| \int_0^x f(t)\,dt \right| \le \int_0^x |f(t)|\,dt \le Mx \le M,$$

so $\|Tf\|_\infty \le M$ and therefore $\|T\| \le 1$; in particular, T is bounded.

To prove compactness, it suffices (by the previous exercise) to show that T maps the unit ball

$$B := \{f \in C[0,1] : \|f\|_\infty \le 1\}$$

into a precompact subset of $C[0,1]$. By Arzelà–Ascoli, it is enough to show $\{Tf : f \in B\}$ is uniformly bounded and equicontinuous.

Uniform boundedness: for $f \in B$ and $x \in [0,1]$,

$$|(Tf)(x)| \le \int_0^x |f(t)|\,dt \le \int_0^x 1\,dt = x \le 1,$$

so $\|Tf\|_\infty \le 1$ for all $f \in B$.

Equicontinuity: let $f \in B$ and $0 \le x < y \le 1$. Then

$$|(Tf)(y)-(Tf)(x)| = \left| \int_0^y f(t)\,dt - \int_0^x f(t)\,dt \right| = \left| \int_x^y f(t)\,dt \right| \le \int_x^y |f(t)|\,dt \le \int_x^y 1\,dt = |y-x|.$$

This bound does not depend on $f \in B$, so the family $\{Tf : f \in B\}$ is equicontinuous.

Thus $T(B)$ is uniformly bounded and equicontinuous, hence its closure is compact in $C[0,1]$ by Arzelà–Ascoli. Therefore T maps B to a precompact set, so T is compact.

$\square$

3. Fix a sequence of real numbers $\{\lambda_k\}_{k=1}^\infty$, and define the linear operator $T : \ell_2 \to \ell_2$ by

$$Tx = \{\lambda_k x_k\}_{k=1}^\infty.$$

For what multiplier sequences $\{\lambda_k\}_{k=1}^\infty$ is the operator T, (a) well defined? (b) bounded? (c) compact?

Proof. (a) *Well defined.* First assume (λ_k) is bounded, say $M := \sup_k |\lambda_k| < \infty$. Then for $x \in \ell_2$,

$$\|Tx\|_2^2 = \sum_{k=1}^{\infty} |\lambda_k x_k|^2 \le M^2 \sum_{k=1}^{\infty} |x_k|^2 = M^2 \|x\|_2^2 < \infty,$$

so $Tx \in \ell_2$ and T is well defined.

Conversely, suppose T is well defined but (λ_k) is unbounded. Then we can pick indices k_j with $|\lambda_{k_j}| \ge j$. Define $x \in \ell_2$ by

$$x_{k_j} = \frac{1}{j}, \qquad x_k = 0 \text{ otherwise.}$$

Then $\|x\|_2^2 = \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty$, so $x \in \ell_2$. But

$$|\lambda_{k_j} x_{k_j}|^2 \ge j^2 \cdot \frac{1}{j^2} = 1,$$

hence

$$\sum_{k=1}^{\infty} |\lambda_k x_k|^2 \ge \sum_{j=1}^{\infty} 1 = \infty,$$

so $Tx \notin \ell_2$, contradicting that T is well defined. Thus (λ_k) must be bounded. Therefore

$$T \text{ is well defined} \iff (\lambda_k) \in \ell_\infty.$$

(b) *Bounded.* If (λ_k) is bounded with $M = \sup_k |\lambda_k|$, the estimate above gives

$$\|Tx\|_2 \le M\|x\|_2, \qquad x \in \ell_2,$$

so T is bounded and $\|T\| \le M$. For the reverse inequality, let e_k be the standard basis of ℓ_2. Then

$$\|Te_k\|_2 = |\lambda_k|$$

and $\|e_k\|_2 = 1$, so

$$|\lambda_k| = \|Te_k\|_2 \le \|T\| \quad \text{for all } k.$$

Taking the supremum in k gives $\sup_k |\lambda_k| \le \|T\|$. Hence

$$\|T\| = \sup_k |\lambda_k|,$$

and in particular T is bounded iff (λ_k) is bounded. (c) *Compact.* Suppose first that $\lambda_k \to 0$. For $N \in \mathbb{N}$ define the finite-rank operator

$$T_N x := (\lambda_1 x_1, \dots, \lambda_N x_N, 0, 0, \dots).$$

Then for $\|x\|_2 \leq 1$,

$$\|(T - T_N)x\|_2^2 = \sum_{k>N} |\lambda_k x_k|^2 \leq \big(\sup_{k>N} |\lambda_k|\big)^2 \sum_{k>N} |x_k|^2 \leq \big(\sup_{k>N} |\lambda_k|\big)^2,$$

so $\|T - T_N\| \leq \sup_{k>N} |\lambda_k| \to 0$ as $N \to \infty$. Thus T is a norm limit of finite-rank operators, hence compact.

Conversely, assume T is compact but $\lambda_k \not\to 0$. Then there exists $\varepsilon > 0$ and a subsequence (λ_{k_j}) with $|\lambda_{k_j}| \geq \varepsilon$ for all j. Consider the bounded sequence (e_{k_j}) in ℓ_2. Compactness of T implies that (Te_{k_j}) has a convergent subsequence in ℓ_2, but

$$Te_{k_j} = \lambda_{k_j} e_{k_j}, \qquad \|Te_{k_j}\|_2 = |\lambda_{k_j}| \geq \varepsilon,$$

and these vectors are pairwise orthogonal. An orthogonal sequence with norms bounded away from 0 cannot have a convergent subsequence, a contradiction. Hence we must have $\lambda_k \to 0$.

□

4. Consider an intergal operator T with kernel $k(t, s) : [0, 1]^2 \to \mathbb{R}$ which satisfies the following:

   (a) for each $s \in [0, 1]$, the function $k_s(t) = k(t, s)$ is intergrable in t;

   (b) the map $s \to k_s$ is a continuous map from $[0, 1]$ to $L_1[0, 1]$.

   Show that the integral operator T is compact in $C[0, 1]$.

   Proof. First note $s \mapsto k_s \in L_1[0, 1]$ is continuous on a compact set, so

$$M := \sup_{s\in[0,1]} \|k_s\|_{L_1} < \infty.$$

   Let $f \in C[0, 1]$. Then for each $s \in [0, 1]$,

$$|(Tf)(s)| = \left| \int_0^1 k(t, s)f(t)\, dt \right| \leq \|f\|_\infty \int_0^1 |k(t, s)|\, dt = \|f\|_\infty \|k_s\|_{L_1} \leq M\|f\|_\infty.$$

   Hence Tf is bounded and $\|T\| \leq M$.

   We also get continuity: if $s_n \to s$, then

$$|(Tf)(s_n)-(Tf)(s)| = \left| \int_0^1 \big(k(t, s_n) - k(t, s)\big) f(t)\, dt \right| \leq \|f\|_\infty \|k_{s_n} - k_s\|_{L_1} \to 0,$$

   since $s \mapsto k_s$ is continuous in L_1. Thus $Tf \in C[0, 1]$ and $T : C[0, 1] \to C[0, 1]$ is bounded.

   To prove compactness, it suffices to show T maps the unit ball

$$B := \{f \in C[0, 1] : \|f\|_\infty \leq 1\}$$

into a precompact subset of $C[0,1]$. By Arzelà–Ascoli, it is enough to prove that $\{Tf : f \in B\}$ is uniformly bounded and equicontinuous.

Uniform boundedness: for $f \in B$ and $s \in [0,1]$,

$$|(Tf)(s)| \le \|f\|_\infty \|k_s\|_{L_1} \le \|k_s\|_{L_1} \le M,$$

so $\sup_{f\in B} \|Tf\|_\infty \le M$.

Equicontinuity: let $f \in B$ and $s, s_0 \in [0,1]$. Then

$$|(Tf)(s)-(Tf)(s_0)| = \left|\int_0^1 \big(k(t,s) - k(t,s_0)\big) f(t)\,dt\right| \le \|f\|_\infty \, \|k_s - k_{s_0}\|_{L_1} \le \|k_s - k_{s_0}\|_{L_1}.$$

Since $s \mapsto k_s$ is continuous into L_1 and $[0,1]$ is compact, this map is uniformly continuous. Thus for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|s - s_0| < \delta \implies \|k_s - k_{s_0}\|_{L_1} < \varepsilon,$$

and hence

$$|(Tf)(s) - (Tf)(s_0)| < \varepsilon$$

for all $f \in B$ whenever $|s - s_0| < \delta$. This shows $\{Tf : f \in B\}$ is equicontinuous.

By Arzelà–Ascoli, the closure of $T(B)$ is compact in $C[0,1]$, so T is compact. $\square$

5. Let X be a Banach space and $T \in K(X,X)$. Show that operator $A = I - T$ satisfies

$$dim(ker(A)) = dim(ker(A^*)) = codim(Im(A)) = codim(Im(A^*)).$$

Proof. We work in several steps.

1. $\ker(A)$ **and** $\ker(A^*)$ **are finite dimensional.**

Note that

$$Ax = 0 \iff x - Tx = 0 \iff Tx = x,$$

so $\ker(A)$ is the eigenspace of T for eigenvalue 1.

If $\ker(A)$ were infinite dimensional, we could choose a sequence $(x_n) \subset \ker(A)$ with $\|x_n\| = 1$ and no convergent subsequence (possible because the unit ball of an infinite dimensional Banach space is not compact). But then

$$Tx_n = x_n \quad \text{for all } n,$$

so (Tx_n) has no convergent subsequence either. This contradicts the compactness of T. Hence $\ker(A)$ is finite dimensional.

The adjoint $T^* : X^* \to X^*$ is also compact (image of the unit ball under T^* is norm-compact in X^* because T is compact). Applying the same argument to $A^* = I - T^*$ on X^* shows $\ker(A^*)$ is finite dimensional.

2. $\mathrm{Im}(A)$ **and** $\mathrm{Im}(A^*)$ **are closed.**

Let $N := \ker(A)$, which is finite dimensional. There exists a closed complement $M \subset X$ with

$$X = N \oplus M.$$

On M the restriction $A|_M : M \to X$ is injective. We claim there is $c > 0$ such that

$$\|Ax\| \geq c\|x\| \qquad \forall x \in M. \tag{1}$$

If not, we can find $(x_n) \subset M$ with $\|x_n\| = 1$ and $\|Ax_n\| \to 0$. Then

$$Ax_n = x_n - Tx_n \to 0 \quad \Rightarrow \quad x_n - Tx_n \to 0.$$

Since (x_n) is bounded and T is compact, (Tx_n) has a convergent subsequence $Tx_{n_k} \to y$. Then

$$x_{n_k} = Ax_{n_k} + Tx_{n_k} \to 0 + y = y,$$

so $x_{n_k} \to y \in M$. Passing to the limit in $Ax_{n_k} \to 0$ gives $Ay = 0$, i.e. $y \in N \cap M = \{0\}$. Thus $y = 0$, but $\|x_{n_k}\| = 1$ forces $\|y\| = 1$, a contradiction. Hence (1) holds.

Now let (Ax_n) be a convergent sequence in $A(M)$. Then for m, n,

$$\|x_n - x_m\| \leq \frac{1}{c}\|Ax_n - Ax_m\|,$$

so (x_n) is Cauchy in M and converges to some $x \in M$. Continuity of A gives $Ax_n \to Ax$, so $\lim Ax_n \in A(M)$. Thus $A(M)$ is closed.

Since $X = N \oplus M$ and $A(N) = \{0\}$, we have

$$\mathrm{Im}(A) = A(X) = A(M).$$

Therefore $\mathrm{Im}(A)$ is closed.

The same argument, applied to $A^* = I - T^*$ on the Banach space X^*, shows that $\mathrm{Im}(A^*)$ is closed.

3. Relating codimension and adjoint kernels.

We use a standard duality fact: if Y is Banach and $M \subset Y$ is closed, then

$$(Y/M)^* \cong M^\perp := \{f \in Y^* : f|_M = 0\}.$$

Apply this with $Y = X$, $M = \mathrm{Im}(A)$. Then

$$(X/\,\mathrm{Im}(A))^* \cong \mathrm{Im}(A)^\perp.$$

But

$$\mathrm{Im}(A)^\perp = \{f \in X^* : f(Ax) = 0 \ \forall x\} = \ker(A^*).$$

Hence

$$(X/\operatorname{Im}(A))^* \cong \ker(A^*).$$

So $(X/\operatorname{Im}(A))^*$ is finite dimensional, and therefore $X/\operatorname{Im}(A)$ is finite dimensional with

$$\dim(X/\operatorname{Im}(A)) = \dim\left((X/\operatorname{Im}(A))^*\right) = \dim\ker(A^*).$$

That is,

$$\operatorname{codim}\operatorname{Im}(A) = \dim\ker(A^*). \tag{2}$$

Now apply the same reasoning to $Y = X^*$ and $M = \operatorname{Im}(A^*)$. We get

$$(X^*/\operatorname{Im}(A^*))^* \cong \operatorname{Im}(A^*)^\perp = \ker\left((A^*)^*\right) = \ker(A^{**}).$$

The canonical embedding $J : X \to X^{**}$ satisfies $A^{**}J = JA$, so $J(\ker(A)) \subset \ker(A^{**})$, and conversely if $A^{**}u = 0$ with $u = Jx$, then $J(Ax) = 0$, hence $Ax = 0$ and $x \in \ker(A)$. Thus $\ker(A^{**}) \cong \ker(A)$ and

$$\dim\ker(A^{**}) = \dim\ker(A).$$

Therefore

$$\operatorname{codim}\operatorname{Im}(A^*) = \dim\ker(A^{**}) = \dim\ker(A). \tag{3}$$

So far we have

$$\operatorname{codim}\operatorname{Im}(A) = \dim\ker(A^*), \qquad \operatorname{codim}\operatorname{Im}(A^*) = \dim\ker(A).$$

4. Equality of $\dim\ker(A)$ and $\dim\ker(A^*)$.

We have

$$\ker(A) = \ker(I - T), \qquad \ker(A^*) = \ker(I - T^*),$$

which are the eigenspaces of T and T^* for the (nonzero) eigenvalue $\lambda = 1$.

For compact operators it is a standard fact that for every nonzero λ, the eigenspaces $\ker(\lambda I - T)$ and $\ker(\lambda I - T^*)$ have the same (finite) dimension.

Applying this with $\lambda = 1$ gives

$$\dim\ker(A) = \dim\ker(A^*).$$

Combining this with (2) and (3), we obtain

$$\dim\ker(A) = \dim\ker(A^*) = \operatorname{codim}\operatorname{Im}(A) = \operatorname{codim}\operatorname{Im}(A^*),$$

as required. $\square$

6. Prove the claims about the spectra of shirt operators made in Example 4.3.8

$$\sigma_p(R) = \varnothing, \quad \sigma_c(R) = \{\lambda : |\lambda| = 1\}, \quad \sigma_r(R) = \{\lambda : |\lambda| < 1\};$$

$$\sigma_p(L) = \{\lambda : |\lambda| < 1\}, \quad \sigma_c(L) = \{\lambda : |\lambda| = 1\}, \quad \sigma_r(L) = \varnothing.$$

Proof. Let $(e_n)_{n\geq 1}$ be the standard orthonormal basis of ℓ_2.

1. Basic facts. For $x \in \ell_2$,

$$\|Rx\|_2^2 = \sum_{n\geq 1} |(Rx)_n|^2 = \sum_{n\geq 1} |x_n|^2 = \|x\|_2^2,$$

so R is an isometry and $\|R\| = 1$. Similarly

$$\|Lx\|_2^2 = \sum_{n\geq 1} |x_{n+1}|^2 \leq \sum_{n\geq 1} |x_n|^2 = \|x\|_2^2,$$

and taking $x_1 = 0$ shows $\|L\| = 1$. Hence for both operators the spectral radius satisfies $r(\cdot) \leq 1$, so

$$\sigma(R), \sigma(L) \subset \{\lambda : |\lambda| \leq 1\}.$$

For $|\lambda| > 1$ we have $\|R/\lambda\| < 1$ and $\|L/\lambda\| < 1$, so

$$\lambda I - R = \lambda\Big(I - \tfrac{1}{\lambda}R\Big), \quad \lambda I - L = \lambda\Big(I - \tfrac{1}{\lambda}L\Big)$$

are invertible with inverses given by the Neumann series $\sum_{n\geq 0}(\frac{1}{\lambda}R)^n$ and $\sum_{n\geq 0}(\frac{1}{\lambda}L)^n$. Thus $|\lambda| > 1$ lies in the resolvent set of both R and L, and so

$$\sigma(R), \sigma(L) \subset \{\lambda : |\lambda| \leq 1\}.$$

It is easy to check that $R^* = L$ and $L^* = R$, so $\sigma(R) = \sigma(L)$.

2. Eigenvalues of R and L (point spectrum).

Right shift. If $Rx = \lambda x$, then

$$(0, x_1, x_2, \dots) = (\lambda x_1, \lambda x_2, \dots).$$

From the first coordinate $\lambda x_1 = 0$, so for any λ we get $x_1 = 0$. Then $x_1 = \lambda x_2$ gives $x_2 = 0$, and inductively $x_n = 0$ for all n. Hence $\ker(R - \lambda I) = \{0\}$ for every $\lambda \in \mathbb{C}$, so

$$\sigma_p(R) = \varnothing.$$

Left shift. If $Lx = \lambda x$, we have

$$(x_2, x_3, \dots) = (\lambda x_1, \lambda x_2, \dots),$$

so $x_{n+1} = \lambda x_n$ for all $n \geq 1$. Thus

$$x_n = \lambda^{n-1} x_1, \qquad n \geq 1.$$

Then

$$\|x\|_2^2 = |x_1|^2 \sum_{n \geq 1} |\lambda|^{2(n-1)}.$$

This is finite iff $|\lambda| < 1$. So for $|\lambda| < 1$ we obtain a one-dimensional eigenspace spanned by

$$v^{(\lambda)} = (1, \lambda, \lambda^2, \dots),$$

while for $|\lambda| \geq 1$ the only solution is $x = 0$. Therefore

$$\sigma_p(L) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}.$$

3. 0 in the residual spectrum of R.

$\operatorname{Im} R$ consists of all sequences whose first coordinate is 0. This is a closed subspace of codimension 1, hence not dense in ℓ_2. Since R is injective, 0 belongs to the residual spectrum of R.

4. Residual / continuous spectrum via the adjoint.

For any bounded operator T on a Hilbert space and any $\lambda \in \mathbb{C}$,

$$\overline{\operatorname{Im}(T - \lambda I)}^{\perp} = \ker\big((T - \lambda I)^*\big) = \ker(T^* - \overline{\lambda} I).$$

In particular, if $T - \lambda I$ is injective, then

$$\lambda \in \sigma_r(T) \iff \overline{\operatorname{Im}(T - \lambda I)} \neq H \iff \ker(T^* - \overline{\lambda} I) \neq \{0\}.$$

We apply this with $T = R$ and $T = L$, using $R^* = L$, $L^* = R$.

5. Spectra for R.

We already know $R - \lambda I$ is injective for all λ (no eigenvalues). For $|\lambda| < 1$ we have $\overline{\lambda} \in \sigma_p(L)$ by Step 2, so

$$\ker(L - \overline{\lambda} I) \neq \{0\}.$$

Hence

$$\overline{\operatorname{Im}(R - \lambda I)}^{\perp} = \ker(L - \overline{\lambda} I) \neq \{0\},$$

and $\operatorname{Im}(R - \lambda I)$ is not dense. Thus for $|\lambda| < 1$, λ lies in the residual spectrum of R:

$$\sigma_r(R) = \{\lambda : |\lambda| < 1\}.$$

For $|\lambda| = 1$, we still have $\ker(R - \lambda I) = \{0\}$, but now $\overline{\lambda}$ is not in $\sigma_p(L)$, so

$$\ker(L - \overline{\lambda} I) = \{0\} \quad \Rightarrow \quad \overline{\operatorname{Im}(R - \lambda I)}^{\perp} = \{0\},$$

so $\operatorname{Im}(R - \lambda I)$ is dense. On the other hand, for $|\lambda| = 1$ the vector e_1 is not in the range of $R - \lambda I$: if $(R - \lambda I)x = e_1$, then coordinatewise

$$-\lambda x_1 = 1, \quad x_1 - \lambda x_2 = 0, \quad x_2 - \lambda x_3 = 0, \ldots$$

which forces $x_n = -\lambda^{-n}$ and thus $x \notin \ell_2$. So $R - \lambda I$ is not surjective. Therefore

$$|\lambda| = 1 \implies \lambda \in \sigma_c(R).$$

Combining with $\sigma_p(R) = \varnothing$ and $\sigma(R) \subset \{|\lambda| \le 1\}$, we obtain

$$\sigma_p(R) = \varnothing, \quad \sigma_c(R) = \{\lambda : |\lambda| = 1\}, \quad \sigma_r(R) = \{\lambda : |\lambda| < 1\}.$$

6. Spectra for L.

We already have $\sigma_p(L) = \{\lambda : |\lambda| < 1\}$. For residual spectrum, note that if $\lambda \in \sigma_r(L)$ then $L - \lambda I$ is injective but $\operatorname{Im}(L - \lambda I)$ is not dense, so

$$\ker\big((L - \lambda I)^*\big) = \ker(R - \overline{\lambda} I) \ne \{0\},$$

which would make $\overline{\lambda}$ an eigenvalue of R. But R has no eigenvalues, so this is impossible; hence

$$\sigma_r(L) = \varnothing.$$

It remains to analyze $|\lambda| = 1$. First, for $|\lambda| = 1$ we saw in Step 2 that $\ker(L - \lambda I) = \{0\}$ (no eigenvectors). We now show $\lambda \in \sigma(L)$ by constructing approximate eigenvectors.

Fix λ with $|\lambda| = 1$. For each $N \in \mathbb{N}$ define $x^{(N)} \in \ell_2$ by

$$x_k^{(N)} = \begin{cases} \lambda^{k-1}, & 1 \le k \le N, \\ 0, & k > N. \end{cases}$$

Then

$$(Lx^{(N)})_k = x_{k+1}^{(N)} = \begin{cases} \lambda^k, & 1 \le k \le N-1, \\ 0, & k \ge N, \end{cases}$$

so

$$\big((L - \lambda I)x^{(N)}\big)_k = \begin{cases} 0, & 1 \le k \le N-1, \\ -\lambda^N, & k = N, \\ 0, & k > N. \end{cases}$$

Hence $\|(L - \lambda I)x^{(N)}\|_2 = 1$. Meanwhile

$$\|x^{(N)}\|_2^2 = \sum_{k=1}^{N} |\lambda|^{2(k-1)} = N,$$

so for the unit vectors $u^{(N)} := x^{(N)}/\sqrt{N}$ we get

$$\|u^{(N)}\|_2 = 1, \qquad \|(L - \lambda I)u^{(N)}\|_2 = \frac{1}{\sqrt{N}} \to 0.$$

If λ were in the resolvent of L, then $(L - \lambda I)^{-1}$ would be bounded, say $\|(L - \lambda I)^{-1}\| \leq C$, which would imply

$$1 = \|u^{(N)}\| \leq C\,\|(L - \lambda I)u^{(N)}\| \xrightarrow[N\to\infty]{} 0,$$

a contradiction. Thus $|\lambda| = 1 \Rightarrow \lambda \in \sigma(L)$.

Finally, for such λ,

$$\overline{\mathrm{Im}(L - \lambda I)}^{\perp} = \ker((L - \lambda I)^*) = \ker(R - \overline{\lambda} I) = \{0\},$$

since R has no eigenvalues. So $\mathrm{Im}(L - \lambda I)$ is dense. We already know $L - \lambda I$ is not invertible (since $\lambda \in \sigma(L)$) and has trivial kernel, so it cannot be surjective. Therefore

$$|\lambda| = 1 \implies \lambda \in \sigma_c(L).$$

□