CO 367, F23 ASSIGNMENT 0 (optional)[1] Due Thurs. Sept. 14, 9:30AM, EST on crowdmark

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Please submit a pdf file to CROWDMARK. No other form of submission is acceptable; late assignments will not be accepted. It is the responsibility of the students to make sure that the pdf file they submit is clearly readable.

Important information:
You are expected to justify your answers, e.g., a yes or no, a number, a property, etc ... is not enough.
You are expected to do the assignments on your own. The only sources allowed for doing the assignments are:

- **the lectures presented in class;**
- **all of the material on the CO367 Fall 2023 LEARN website;**
- **the two textbooks as listed on the syllabus (and on library reserve);**
- **all of the material on the CO367 Fall 2023 Piazza website;**
- **your Instructor and TAs.**

Usage of any other sources in doing the homework assignments is against the academic integrity rules for CO367 in the Fall 2023 term.

For example, copying or paraphrasing a solution from some fellow student or from old solutions from previous offerings of related courses or various sources on the web, qualifies as cheating and the TAs have been instructed to actively look for evidence of academic offences when evaluating assignment papers. By Univ. of Waterloo policy, academic integrity violations by a student in assignments will lead to a mark of zero in that assignment and potentially a 5% deduction from the Final Course Grade. All academic offences are reported to the Associate Dean for Undergraduate Studies.

[This assignment is a *Calculus, Analysis, Topology, Linear Algebra Review* for topics most relevant to CO367. If you have trouble completing any part of this assignment it is a strong indication that you need to review the relevant material.]

The University of Waterloo subscribes to the strictest interpretation of academic integrity.

Faculty members and students bear joint responsibility in assuring that cheating on assignments or any examinations is not tolerated.

Students who engage in academic dishonesty will be subject to disciplinary action under Policy 71 Student Discipline, uwaterloo.ca/secretariat/policies-procedures-guidelines/policy-71

I confirm that I have not violated the instructions for this Assignment and I will not violate the instructions for this Assignment. I confirm that I have not received any unauthorized assistance in preparing for or writing this Assignment.

Student Name (please print)
Student Signature

Contents

1 GEOMETRY (score 20 = 10+10)

1.1 Preliminaries

For a vector $x = \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix}^T \in \mathbb{R}^n$, we define the (Euclidean) *norm* and *inner (dot) product*

$$\|x\| := \sqrt{x_1^2 + \cdots + x_n^2}$$

$$\langle x, y \rangle \cong x \cdot y := x^T y = x_1 y_1 + \cdots + x_n y_n.$$

Theorem 1.1. *Let $x, y \in \mathbb{R}^n$. Then the Cauchy-Schwarz inequality is*

$$|x \cdot y| \leq ||x|| ||y||,$$

with equality if, and only if, $x = \alpha y$, for some $\alpha \in \mathbb{R}$.

1.2 EXERCISES

1. (10) Explain the geometrical significance for the vectors x and y if:

   (a)
   $$x \cdot y < 0, \; x \cdot y > 0, \; x \cdot y = 0.$$
   (Justify/prove the explanations.)

   (b)
   $$x \cdot y \geq 0.$$

2. (10) Let $a \in \mathbb{R}^n, Q \in \mathbb{S}^n$ and define the (quadratic) function $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) = \frac{1}{2} x^T Q x + a \cdot x$. Prove that f is a continuous differentiable function, i.e., that the gradient $\nabla f(x)$ is a continuous function of x. (See **??** below.)

2 CALCULUS (score 30 = (5+10+5)+(5+5))

2.1 EXERCISES

1. For a differentiable function $g : \mathbb{R}^n \to \mathbb{R}$, the *gradient* is the vector of partial derivatives

$$\nabla g := \begin{pmatrix} \frac{\partial g}{\partial x_1} \\ \frac{\partial g}{\partial x_2} \\ \cdots \\ \frac{\partial g}{\partial x_n} \end{pmatrix}. \tag{2.1}$$

(a) (5) If $g(x) = ||x||$, $x \neq 0$, calculate $\nabla g(x)$.

(b) (10) Let g be defined as in **??** above. Let $a, b \in \mathbb{R}^n, (a + tb) \neq 0$, and define $f : \mathbb{R} \to \mathbb{R}$, by $f(t) := g(a + tb)$.
Calculate $f'(t)$.

(c) (5) Let g be as above and let $x, d \in Rn$. Calculate the directional derivative of g in the direction d at the point x.

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable at $x = a$. The *Taylor Series* of f about a is:

$$f(a) + f'(a)(x - a) + \frac{1}{2!} f''(a)(x - a)^2 + \frac{1}{3!} f'''(a)(x - a)^3 + \cdots$$

Write down the Taylor series of:

(a) (5)

$$f(x) = x^3, \text{ about } x = 1.$$

(b) (5)

$$f(x) = \log(1 + x), \text{ about } x = 0.$$

3 REAL ANALYSIS (score 35 = 15+(10+10))

3.1 Preliminaries

The *open ball* $B(x; r) := \{y \in \mathbb{R}^n : ||x - y|| < r\}$. Suppose that D is a subset of $\mathbb{R}^n$.

Interior: $x \in \operatorname{int} D$ if there exists $r > 0$ with $B(x; r) \subset D$.

Closure: $x \in \operatorname{cl} D$ if there exists a sequence $x^k \in D$ with $x^k \to x$.

Boundary: $x \in \partial D$ if $x \in \operatorname{cl} D \backslash \operatorname{int} D$.

D is *open* if $D = \operatorname{int} D$. D is *closed* if $D = \operatorname{cl} D$.

3.2 EXERCISES

1. (15) For each of the following sets, find the interior, the closure, and the boundary. Then determine which of the sets are open, closed, neither, or both.

(a)
$$\{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}.$$
(b)
$$\{(x_1, x_2) : x_1 > 0, x_2 > 0\}.$$
(c)
$$\{(x_1, x_2) : x_1 > 0, x_2 \geq 0\}.$$
(d)
$$\mathbb{R}^n$$
(e)
$$\left\{(x_1, x_2) : x_1^2 + x_2^2 < 0\right\}.$$
(f)
$$\emptyset.$$

2. (a) (10) Prove that D is closed if and only if the complement D^c is open.
   (b) (10) Prove that $x \in \partial D$ if and only if for any $r > 0$ there exists a $y \in B(x;r) \cap D$ and a $z \in B(x;r) \cap D^c$.

4 MATRICES (score 25 = 15+10)

4.1 EXERCISES

1. (15) Let
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$
   (a) Calculate the determinant of A.
   (b) Calculate the rank of A.
   (c) What is the rank of A^T.
   (d) Find the singular values and singular vectors of A.
   (e) Expand the quadratic form $q(x) = x^T A x = \sum_{ij} x_i x_j B_{ij}$ so that $B = (B_{ij}) = B^T$, i.e., so that the form is using a symmetric matrix.

2. (10) Let

$$C = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 30 & -18 & 9 \end{bmatrix}.$$

Calculate the eigenvalues and eigenvectors of C. Show your calculations.
Hint: Let $v = \begin{pmatrix} 1 & 3 & 3 \end{pmatrix}^T$ and evaluate Cv.