CO 367 Practice Problems for Final

December 6, 2023

1 Unconstrained Optimality Conditions

Find an example showing:

1. The second order necessary optimality conditions for unconstrained optimization are not sufficient.

   Solution. We find a function $f(x)$ such that its gradient at $\bar{x}$ is 0 and hessian at $\bar{x}$ is positive semidefinite, but it is not a local minimizer. Let $f(x, y) = x^3 + y^3$. Then

   $$\nabla f(x, y) = \begin{bmatrix} 3x^2 \\ 3y^2 \end{bmatrix}, \quad \nabla^2 f(x, y) = \begin{bmatrix} 6x & 0 \\ 0 & 6y \end{bmatrix}$$

   At (0,0), $\nabla f(0, 0) = (0, 0)^T$ and $\nabla^2 f(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. So this satisfies the necessary optimality conditions, but it is a saddle point.

2. The second order sufficient optimality conditions for unconstrained optimization are not necessary.

   Solution. We find a function $f(x)$ such that $\bar{x}$ is a strict local minimizer, but the hessian at $\bar{x}$ is not positive definite. Let $f(x) = x^4 + y^4$. Then

   $$\nabla f(x, y) = \begin{bmatrix} 4x^3 \\ 4y^3 \end{bmatrix}, \quad \nabla^2 f(x, y) = \begin{bmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{bmatrix}$$

   At (0,0), $\nabla f(0, 0) = (0, 0)^T$ and $\nabla^2 f(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \succeq 0$. So the hessian is not positive definite.

2 Constrained Optimality Conditions

Find an explicit optimal solution to the following problems:

1. $\min\{c^T x : Ax = b, x \in \mathbb{R}\}$

   Solution. We use second order necessary condition: Let x^* be a local minimizer and LICQ holds at x^*. Then x^* is a KTT point with unique lagrange multipliers λ^*, μ^* such that

   $$\langle \nabla^2_{x,x} L(x^*, \lambda^*, \mu^*) d, d \rangle \geq 0, \quad \forall d \in C(x^*, \lambda^*, \mu^*)$$

   Suppose we have a local minimizer $\bar{x}$ that is a KTT point and KTT pair $(\bar{x}, \mu)$. We derive it using the necessary conditions. The lagrangian is

   $$L(\bar{x}, \mu) = c^T \bar{x} + \mu^T (A\bar{x} - b)$$

   Then

   $$\nabla_x L(\bar{x}, \mu) = c + A^T \mu, \quad \nabla^2_{x,x} L(\bar{x}, \mu) = 0$$

The KTT conditions are

$$\begin{aligned} \nabla_x L(\bar{x}, \mu) = c + A^T \mu = 0 \\ A\bar{x} - b = 0 \end{aligned}$$

$\langle \nabla^2_{x,x} L(x^*, \lambda^*, \mu^*)d, d\rangle = 0$ for all d, so we just need $\bar{x}$ that satisfies the KTT conditions. So the local minimizer is $\bar{x} = A^{-1}b$ along with $\mu = -A^{-T}c$. idk

2. $\min\{c^T x : e^T x = 1, x \geq 0\}$

   Solution.

3 Constrained Equivalent Optimality Conditions

Is the following claim true or false? Justify your answer.

Consider the following constrained optimization problem:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & c_i(x) = 0, \quad i \in \mathcal{E} \end{aligned}$$

Assume LICQ holds for this problem. We consider the following equivalent problem (equivalent geometrically)

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & c_i^2(x) = 0, \quad i \in \mathcal{E} \end{aligned}$$

Let x^* be a KTT point of the above problem, then x^* satisfies

$$\begin{aligned} & 0 = \nabla f(x^*) + 2\sum_{i \in \mathcal{E}} \lambda^* c_i(x^*) \nabla c_i(x^*) \\ & 0 = c_i(x^*) \quad \forall i \in \mathcal{E} \end{aligned}$$

where λ_i^* are the corresponding Lagrangian multipliers. By rearrangement, we have $\nabla f(x^*) = 0$. This implies for equality constrained optimization problems, we can get the equivalent first order necessary optimal condition as for the unconstrained optimization problem.
Solution.

4 Duality

Find the dual problem of the following problem and the dual of the dual problem.

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & x^T A x + 2b^T x \\ \text{s.t.} \quad & \|x\|_2 \leq 1 \end{aligned}$$

where $A \in \mathcal{S}^n, b \in \mathbb{R}^n$ (Use the generalized inverse to derive the dual).
Solution.

5 Question 5

Given the following optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}, y > 0} \quad & e^{-x} \\ \text{s.t.} \quad & \frac{x^2}{y} \leq 0 \end{aligned}$$

1. Show that this is a convex optimization problem, find its global minimizer, and verify if Slater condition holds.

   Solution. First, $f(x) = e^{-x}$ is a convex function since

   $$f(\lambda x + (1-\lambda)y) = e^{-\lambda x-(1-\lambda)y} \leq \lambda e^{-x} + (1-\lambda)e^{-y} = \lambda f(x) + (1-\lambda)f(y)$$

   Then, the feasible region is

   $$\Omega = \left\{(x, y) : \frac{x^2}{y} \leq 0, y > 0\right\}$$

   Since $x^2 \geq 0$ and $y > 0$, the only way that $\frac{x^2}{y} \leq 0$ is when $x = 0$. So the feasible region is

   $$\Omega = \{(0, y) : y > 0\}$$

   This is a convex set since

   $$\lambda(0, y_1) + (1-\lambda)(0, y_2) = (0, \lambda y_1 + (1-\lambda)y_2)$$

   Since $y_1, y_2 > 0$, we get $\lambda y_1 + (1-\lambda)y_2 > 0$. So $(0, \lambda y_1 + (1-\lambda)y_2) \in \Omega$. So this is a convex optimization problem.

   Now we find its global minimizer. Since the feasible region is $\Omega = \{(0, y) : y > 0\}$, the global minimizer is $(0, y)$ for any $y > 0$ (the objective function does not depend on y). For a point to satisfy the Slater condition, we need $\frac{\bar{x}^2}{\bar{y}} \leq 0$, however, $\bar{x} = 0$, so $\frac{\bar{x}^2}{\bar{y}} = 0$. So the Slater condition does not hold.

2. Find the dual problem of this problem; find the optimal solution of the dual problem; compute the duality gap.

   Solution. The Lagrangian is

   $$L(x, y, \lambda) = e^{-x} + \lambda\left(\frac{x^2}{y}\right)$$

   The dual function is

   $$\begin{aligned} q(\lambda) &= \min_{x\in\mathbb{R}, y>0} L(x, y, \lambda) \\ &= \min_{x\in\mathbb{R}, y>0} e^{-x} + \lambda\left(\frac{x^2}{y}\right) \\ &= 1 && \text{since } x \text{ is always } 0 \end{aligned}$$

   So the dual problem is

   $$\max_{\lambda\geq 0} \quad 1$$

   The optimal solution for the dual is any $\lambda \geq 0$. The optimal value is 1 for the dual. The optimal value is $e^0 = 1$ for the primal. So the duality gap is $1 - 1 = 0$.