

CO 367 Practice Problems for Final

December 6, 2023

1 Unconstrained Optimality Conditions

Find an example showing:

1. The second order necessary optimality conditions for unconstrained optimization are not sufficient.

Solution. We find a function $f(x)$ such that its gradient at \bar{x} is 0 and hessian at \bar{x} is positive semidefinite, but it is not a local minimizer. Let $f(x, y) = x^3 + y^3$. Then

$$\nabla f(x, y) = \begin{bmatrix} 3x^2 \\ 3y^2 \end{bmatrix}, \quad \nabla^2 f(x, y) = \begin{bmatrix} 6x & 0 \\ 0 & 6y \end{bmatrix}$$

At $(0,0)$, $\nabla f(0,0) = (0,0)^T$ and $\nabla^2 f(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. So this satisfies the necessary optimality conditions, but it is a saddle point.

2. The second order sufficient optimality conditions for unconstrained optimization are not necessary.

Solution. We find a function $f(x)$ such that \bar{x} is a strict local minimizer, but the hessian at \bar{x} is not positive definite. Let $f(x) = x^4 + y^4$. Then

$$\nabla f(x, y) = \begin{bmatrix} 4x^3 \\ 4y^3 \end{bmatrix}, \quad \nabla^2 f(x, y) = \begin{bmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{bmatrix}$$

At $(0,0)$, $\nabla f(0,0) = (0,0)^T$ and $\nabla^2 f(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \succeq 0$. So the hessian is not positive definite.

2 Constrained Optimality Conditions

Find an explicit optimal solution to the following problems:

1. $\min\{c^T x : Ax = b, x \in \mathbb{R}\}$

Solution. We use second order necessary condition: Let x^* be a local minimizer and LICQ holds at x^* . Then x^* is a KKT point with unique lagrange multipliers λ^*, μ^* such that

$$\langle \nabla_{x,x}^2 L(x^*, \lambda^*, \mu^*) d, d \rangle \geq 0, \quad \forall d \in C(x^*, \lambda^*, \mu^*)$$

Suppose we have a local minimizer \bar{x} that is a KKT point and KKT pair (\bar{x}, μ) . We derive it using the necessary conditions. The lagrangian is

$$L(\bar{x}, \mu) = c^T \bar{x} + \mu^T (A\bar{x} - b)$$

Then

$$\nabla_x L(\bar{x}, \mu) = c + A^T \mu, \quad \nabla_{x,x}^2 L(\bar{x}, \mu) = 0$$

The KKT conditions are

$$\begin{aligned}\nabla_x L(\bar{x}, \mu) &= c + A^T \mu = 0 \\ A\bar{x} - b &= 0\end{aligned}$$

$\langle \nabla_{x,x}^2 L(x^*, \lambda^*, \mu^*) d, d \rangle = 0$ for all d , so we just need \bar{x} that satisfies the KKT conditions. So the local minimizer is $\bar{x} = A^{-1}b$ along with $\mu = -A^{-T}c$. **idk**

2. $\min\{c^T x : e^T x = 1, x \geq 0\}$