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1 Introduction

1.1 Lecture 1-Preliminaries

Definition 1.1 – Quadratic Form

Let A be a symmetric matrix and $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. The **quadratic form** Q of the matrix A is defined as

$$Q = x^T A x$$

Problem 1.1 – Example

Consider the matrix $A = \begin{pmatrix} 5 & -5 \\ -5 & 1 \end{pmatrix}$. The quadratic form of A is

$$Q(x) = 5x_1^2 - 10x_1x_2 + x_2^2$$

Definition 1.2 – Classification of Quadratic Forms

Let Q be a quadratic form of a matrix A . Then Q is

1. positive definite if $Q(x) > 0$ for all non-zero vectors x , and $Q(x) = 0$ if and only if $x = 0$. Or all eigenvalues of A are positive.
2. positive semidefinite if $Q(x) \geq 0$ for all vectors x , with $Q(x) = 0$ occurring for some non-zero vectors x . Or all eigenvalues of A are non-negative.
3. negative definite if $Q(x) < 0$ for all non-zero vectors x , and $Q(x) = 0$ if and only if $x = 0$. Or all eigenvalues of A are negative.
4. negative semidefinite if $Q(x) \leq 0$ for all vectors x , with $Q(x) = 0$ occurring for some non-zero vectors x . Or all eigenvalues of A are non-positive.
5. indefinite if $Q(x)$ can be positive or negative. Or there are positive and negative eigenvalues for A .

Definition 1.3 – Big O and little o

Big O is basically the rate of growth of that function. A function $f(n)$ is of order 1, or $O(1)$ if there exists some non zero constant c such that

$$\frac{f(n)}{c} \rightarrow 1$$

as $n \rightarrow \infty$.

Little o is the upper bound of the rate of growth of that function. Therefore, a function $f(n)$ is of order 1, or $o(1)$ if for all constants $c > 0$,

$$\frac{f(n)}{c} \rightarrow 0$$

as $n \rightarrow \infty$.

Definition 1.4 – Differentiability Based on Big o and Little o

If f is differentiable at $x = a$, then

$$f(a + h) = f(a) + f'(a)h + o(h)$$

Conversely, if there exists constants A and B such that

$$f(a+h) = A + Bh + o(h)$$

then f is differentiable at $x = a$. Moreover, $A = f(a)$ and $B = f'(a)$.

Definition 1.5 – Product Rule

If f, g are differentiable at $x = a$, then

$$f(a+h) = f(a) + f'(a)h + o(h), \quad g(a+h) = g(a) + g'(a)h + o(h)$$

Then

$$\begin{aligned} p(a+h) &= f(a+h)g(a+h) \\ &= f(a)g(a) + [f(a)g'(a) + g(a)f'(a)]h + o(h) \end{aligned}$$

Then by above theorem, $p = fg$ is differentiable at $x = a$, and $p'(a) = f(a)g'(a) + g(a)f'(a)$.

Definition 1.6 – Chain Rule

WIP

Definition 1.7 – Inner Product Space

Let $x \in \mathbb{R}^n$, represented as:

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

The inner product space is defined as:

$$\langle x, y \rangle = x^T y = \sum_{i=1}^n x_i y_i \quad (\text{dot product})$$

The angle between vectors x and y is given by $\cos(\theta) = \frac{\langle x, y \rangle}{\|x\| \|y\|}$.

With corresponding norm to be the Euclidean Norm

Definition 1.8 – Open ball

Given $\delta > 0$, $\bar{x} \in \mathbb{R}^n$, the open ball $B_\delta(\bar{x}) = \{x \in \mathbb{R}^n \mid \|x - \bar{x}\| < \delta\}$

Definition 1.9 – map

Suppose the map $f : \mathbb{R}^n \rightarrow \mathbb{R}$.

Definition 1.10 – open set

Let $D \subset \mathbb{R}^n$, D open set. $\forall x \in D, \exists \delta > 0$, s.t $B_\delta(x) \subset D$

Definition 1.11 – differ

We define f to be in C^1, C^2 on an open set $D \subseteq \mathbb{R}^n$, denoted $f \in C^1(D), C^2(D)$, respectively, if the partial first $\frac{\partial f(x)}{\partial x_i}$ and second $\frac{\partial^2 f(x)}{\partial x_i \partial x_j}$ derivatives exist and are continuous for all i, j , respectively. We then get the gradient vector in \mathbb{R}^n and the $n \times n$ symmetric Hessian matrix, respectively denoted as:

$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_i} \right) \in \mathbb{R}^n, \quad \nabla^2 f(x) = \left[\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \right] \in \mathbb{S}^n.$$

Here, \mathbb{S}^n is the vector space of $n \times n$ symmetric matrices.

Definition 1.12 – General Nonlinear opt. function NLO

The general problem of nonlinear optimization, denoted NLO, is defined as follows: Given C^2 -smooth functions $f, g_i, h_j : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ for $i = 1, \dots, m$ and $j = 1, \dots, p$, where D is an open subset of \mathbb{R}^n , the objective is to find the optimal value p^* and an optimum x^* of NLO, represented as:

$$p^* := \min f(x) \text{ s.t. } g_i(x) \leq 0, \quad \forall i = 1, \dots, m, h_j(x) = 0, \quad \forall j = 1, \dots, p, x \in D$$

If f, g_i, h_i are all **affine** function and $D = \mathbb{R}^2$, then we have an LP

Definition 1.13 – affine

$$f(x) = Ax + b \tag{1}$$

where $b \neq 0$

Definition 1.14 – Types of Minimality

Consider $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $D \subset \mathbb{R}^n$. Then $\bar{x} \in D$ is:

- a *global minimizer* for f on D if $f(\bar{x}) \leq f(x)$ for all $x \in D$.
- a *strict global minimizer* for f on D if $f(\bar{x}) < f(x)$ for all $x \in D$ where $x \neq \bar{x}$.
- a *local minimizer* for f on D if there exists $\delta > 0$ such that $f(\bar{x}) \leq f(x)$ for all $x \in D \cap B_\delta(\bar{x})$.
- a *strict local minimizer* for f on D if there exists $\delta > 0$ such that $f(\bar{x}) < f(x)$ for all $x \in D \cap B_\delta(\bar{x})$ where $x \neq \bar{x}$.

Definition 1.15 – Linear Approximation

Suppose f is a function that is differentiable on an interval I containing the point a . The **linear approximation** to f at a is the linear function

$$L(x) = f(a) + f'(a)(x - a)$$

for $x \in I$.

Definition 1.16 – Quadratic Approximation

Similar as above, the **quadratic approximation** to f at a is the quadratic function

$$Q(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2$$

for $x \in I$.

Definition 1.17 – Formal Definition of Derivative

The **derivative** of f at a is defined as

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

if the limit exists.

An alternate definition is

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

1.2 Lecture 2

Definition 1.18 – General NLO/NLP

A **Non-linear Optimization Problem** (NLP) is of the following form:

$$\underbrace{p^*}_{\text{Optimal Value}} = \min \underbrace{f(x)}_{\text{Objective function}}$$

s.t.

$$\begin{aligned} g(x) = (g_i(x)) &\leq 0 \in \mathbb{R}^m \\ h(x) = (h_j(x)) &= 0 \in \mathbb{R}^p \end{aligned}$$

Problem 1.2 – Example

$$\min (x_1 - 2)^2 + (x_2 - 1)^2$$

s.t.

$$\begin{aligned} x_1^2 - x_2 &\leq 0 & (g_1(x) \leq 0) \\ x_1 + x_2 - 2 &\leq 0 & (g_2(x) \leq 0) \end{aligned}$$

Definition 1.19 – Contour

For $\alpha \in \mathbb{R}$, the **contour** of a function f is

$$C_\alpha = \{x \in \mathbb{R}^n : f(x) = \alpha\}$$

Definition 1.20 – Feasible Set

The **feasible set** is

$$F = \{x \in \mathbb{R}^n : g(x) \leq 0, h(x) = 0, x \in D\}$$

(Is D the domain??)

Definition 1.21 – Gradient

The **gradient** of f is

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f(x)}{\partial x_1} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{pmatrix}$$

For the optimal solution x^* , we have

$$\alpha \nabla f(x^*) = \lambda_1 \nabla g_1(x^*) + \lambda_2 \nabla g_2(x^*)$$

for some $\alpha, \lambda_1, \lambda_2 \in \mathbb{R}$.

We will see later that we can choose $\alpha = 1$ and we need $\lambda_1 \geq 0, \lambda_2 \geq 0$.

Problem 1.3 – Max-cut Problem

Given a weighted graph $G = (\underbrace{V}_{\text{vertices}}, \underbrace{E}_{\text{edges}}, \underbrace{w}_{\text{weight}})$, a **cut** is $U \subseteq V, U \neq \emptyset$. The objective function

$$\max \quad \frac{1}{2} \sum_{\substack{i \in U, j \notin U \\ (i,j) \in E}} w_{i,j}$$

maximizes the sum of edges in a cut.

Formulating as an NLP, we introduce variables $x_i \in \{\pm 1\}, i = 1, \dots, n$. Then the Max-cut problem (MC) is as follows:

$$\max \quad \frac{1}{2} \sum_{ij \in E} w_{ij} (1 - x_i x_j)$$

Why 1/2 s.t.

$$x_i \in \{\pm 1\} \quad (\text{equivalent to } x_i^2 = 1) \quad \forall i = 1, \dots, n$$

This works because

$$1 - x_i x_j = \begin{cases} 0 & \text{if } x_i = x_j \quad (i, j \text{ in the same set, } U \text{ or } U^c) \\ 2 & \text{otherwise} \end{cases}$$

MC is a **quadratically constrained quadratic program** (QOP) since each constraint $x_i \in \{-1, 1\}$ is equivalent to the quadratic constraint $x_i^2 = 1$. Note that MC is an NP-hard problem.

2 Unconstrained Optimization

2.1 Lecture 2

Problem 2.1 – Simplest Case - No Constraints

Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Assume f is sufficiently smooth (differentiable) then the NLP with no constraints is

$$\min_{x \in \Omega} f(x)$$

Theorem 2.1 – Taylor's Theorem on the real line

Let $f : (a, b) \rightarrow \mathbb{R}$, and $\bar{x}, x \in (a, b)$, then there exists z strictly between x, \bar{x} such that

$$f(x) = f(\bar{x}) + f'(\bar{x})(x - \bar{x}) + \frac{f''(z)}{2}(x - \bar{x})^2$$

or equivalently

$$f(\bar{x} + \delta x) = \underbrace{f(x) + f'(x)\delta x}_{\text{Linear approximation}} + o(|\delta x|) \text{ (little O)}$$

Lemma 2.1 – Directional Derivative

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $\bar{x}, d \in \mathbb{R}^n$ where d is the direction. We define

$$\phi(\epsilon) = f(\bar{x} + \epsilon d) : \mathbb{R} \rightarrow \mathbb{R}$$

Then the **directional derivative**, denoted $f'(x; d)$ of f at x at the direction d is

$$f'(x; d) = \phi'(0) = \lim_{\epsilon \rightarrow 0} \frac{f(x + \epsilon d) - f(x)}{\epsilon} = \nabla f(x)^T d$$

Problem 2.2

Let $f(x, y, z) = x^2z + y^3z^2 - xyz$ with $d = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$ Then the **directional derivative** in the direction d is

$$\nabla f(x, y, z)^T d = \begin{pmatrix} 2xz - yz \\ 3y^2z^2 - xz \\ x^2 + 2y^3z - xy \end{pmatrix}^T \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} = -2xz + yz + 3x^2 + 6y^3z - 3xy$$

Corollary 2.1

Let $f : (a, b) \rightarrow \mathbb{R}$

1. If \bar{x} is a **local minimizer** of f on (a, b) , then $f'(\bar{x}) = 0$ and $f''(\bar{x}) \geq 0$.
2. If $f'(\bar{x}) = 0$, $f''(\bar{x}) > 0$ then \bar{x} is a **strict local minimizer** of f .

Definition 2.1 – Hessian

The **Hessian** of f at $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is the matrix

$$\nabla^2 f(x) = \begin{pmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} & \cdots & \frac{\partial^2 f(x)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_n^2} \end{pmatrix}$$

Theorem 2.2 – Multivariate Taylor

Consider a C^2 -smooth function $f : U \rightarrow \mathbb{R}$ on an open set $U \subset \mathbb{R}^n$. If \bar{x} and x are such that the segment $[\bar{x}, x] := \{\bar{x} + t(x - \bar{x}) : t \in [0, 1]\}$ is contained in U , then there exists a point $z \in [\bar{x}, x]$ such that

$$f(x) = f(\bar{x}) + \langle \nabla f(\bar{x}), x - \bar{x} \rangle + \frac{1}{2} \langle \nabla^2 f(z)(x - \bar{x}), (x - \bar{x}) \rangle$$

Lemma 2.2

Let $v \in \mathbb{R}^n$. Then

$$v = 0 \iff \langle v, d \rangle = 0, \quad \forall d \in \mathbb{R}^n$$

2.2 Lecture 3**Definition 2.2 – Matrix Norm**

$$\|Q\| = \max_{\|x\|=1} \|Qx\| = \text{Largest singular value of } A$$

Definition 2.3

Define f, D, \bar{x} , D is an open set Then:

1. Nec: If \bar{x} is a local minimum for f on D , then $\nabla f(\bar{x}) = 0$ and $\nabla^2 f(\bar{x}) \succeq 0$ is positive semidefinite.
2. Suff: If $\nabla f(\bar{x}) = 0$, $\nabla^2 f(\bar{x}) \succ 0$ is positive definite then \bar{x} is a strict local minimum of f on D .

Proof. 1. updated later after I confirmed some details with the professor □

Definition 2.4 – Critical/Stationary Points

A point $\bar{x} \in U$ is a critical point of a function $f : U \rightarrow \mathbb{R}$ if $\nabla f(\bar{x})$ exists and satisfies $\nabla f(\bar{x}) = 0$.

Problem 2.3 – Algorithm to Find Local Minimizer

Given $f : \mathbb{R} \rightarrow \mathbb{R}$ and $f'(\bar{x}) \neq 0$, then $x_{new} = \bar{x} - (\text{step}) * f'(\bar{x})$.

The idea is that if $f'(\bar{x}) > 0$, then we know that the function is increasing at \bar{x} , so we want to move to the left to obtain the minimum. Similarly, if $f'(\bar{x}) < 0$, then we know that the function is decreasing at \bar{x} , so we want to move to the right to obtain the minimum.

Problem 2.4

Given $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $\phi(\epsilon) = f(\bar{x} + \epsilon d)$

using Taylor expansion $f(\bar{x} + \epsilon d) = f(\bar{x}) + \epsilon \nabla f(\bar{x})^T d + o(\|\epsilon\|)$ **shouldnt be ϵd ? or d is the unit vector**

let $d = -\nabla f(\bar{x}) / \|\nabla f(\bar{x})\|$, $f(\bar{x}) - \epsilon \|\nabla f(\bar{x})\|^2 + o(\epsilon) < f(\bar{x})$ (if $\nabla f(\bar{x}) \neq 0$)

i.e test nec condition

If $\nabla f(\bar{x}) \neq 0$, then $x_{new} = \bar{x} + \epsilon(-\nabla f(\bar{x}))$ Move to the deepest direction

Definition 2.5 – Cauchy's method of steepest descent

<https://www.math.usm.edu/lambers/mat419/lecture10.pdf> $x_0 \in \mathbb{R}^n$.

$$\|\nabla f(x_k)\| \approx 0? \text{ IF yes Stop}$$

O.W, find a $\alpha > 0$

$$x_{k+1} = x_k - \alpha \nabla f(x_k)$$

repeat

Problem 2.5

$$f(x, y) = x^3 - 12xy + 8y^3$$

$$\nabla f(x, y) = \begin{pmatrix} 3x^2 - 12y \\ -12x + 24y^2 \end{pmatrix}, \quad \nabla^2 f(x, y) = \begin{pmatrix} 6x & -12 \\ -12 & 48y \end{pmatrix}.$$

$$\nabla f = 0 \implies (x, y) = (0, 0) \text{ or } (2, 1)$$

$$\nabla^2 f(2, 1) = \begin{pmatrix} 12 & -2 \\ -12 & 48 \end{pmatrix}$$

All leading principal minors are positive \implies strict local minimum
while $(0, 0)$ are indefinite and this is called **inflection point**

2.3 Lecture 4**Definition 2.6 – Principal Submatrices**

Let

$$A = \begin{pmatrix} 1 & 1 & 2 & 7 \\ 1 & 1 & 4 & 6 \\ 2 & 4 & 7 & 8 \\ 7 & 6 & 8 & 1 \end{pmatrix}, \quad I = \{1, 3\}, \quad A[I] = \begin{pmatrix} 1 & 2 \\ 2 & 7 \end{pmatrix}$$

Then $A[I]$ is a **principal submatrix** of A .

Definition 2.7 – Principal Minors

Let $A \in \mathbb{S}^n$, where \mathbb{S}^n is the set of all symmetric $n \times n$ matrices.

1. $\det(A[I])$ is called the **principal minor** of A .
2. If $I = \{1, \dots, k\}$ then $\det(A[I])$ is called the **leading principal minor** of A .

Proposition 2.1 – Characterizing Positive Definiteness with Principal Minors

Let $A \in \mathbb{S}^n$. Then

1. $A \succeq 0 \iff \det(A[I]) \geq 0$ for all principal minors $\det(A[I])$.
2. $A \succ 0 \iff \det(A[I]) > 0$ for all **leading** principal minors $\det(A[I])$.

Definition 2.8 – Eigenvectors and Eigenvalues

$0 \neq v \in \mathbb{R}^n$ is an **eigenvector** of A if there exists $\lambda \in \mathbb{R}$ such that $Av = \lambda v$. The number λ is called an **eigenvalue** of A .

Theorem 2.3 – Finding Eigenvectors and Eigenvalues

Let A be a matrix.

1. Set up the characteristic equation. We find

$$\det(A - \lambda I) = 0$$

2. Solve for λ . These are the eigenvalues.

3. Plug eigenvalues $\lambda_1, \dots, \lambda_n$ into $(A - \lambda I)v = 0$ and solve for v . These are the eigenvectors.

Theorem 2.4 – Orthogonal Spectral Decomposition

Let $A \in \mathbb{S}^n$. Then A has an **orthogonal spectral decomposition**

$$A = \sum_i \lambda_i u_i u_i^T = U D U^T$$

where U is orthogonal with the orthogonal eigenvectors u_i as columns and D is a diagonal matrix with real eigenvalues on the diagonal.

Corollary 2.2

Let $A \in \mathbb{S}^n$. Then

1. $A \succeq 0$ (positive semidefinite) iff all eigenvalues of A are nonnegative.
2. $A \succ 0$ (positive definite) iff all eigenvalues of A are positive.

Proposition 2.2

Let $A \in \mathbb{S}^n$. The following are equivalent (Positive definite):

1. $A \succ 0$.
2. All the eigenvalues of A are in \mathbb{R}_{++}^n , the interior of the nonnegative orthant.
3. A has a real symmetric positive definite square root, $A = SS$, $S \in \mathbb{S}_{++}^n$.
4. A has a lower triangular factorization, a Cholesky factorization, $A = LL^T$ and L has positive diagonal elements.
5. All principal minors of A are positive.
6. All leading principal minors of A are positive.

And the following are equivalent (Positive semidefinite):

1. $A \succeq 0$.
2. All the eigenvalues of A are in \mathbb{R}_+^n , the nonnegative orthant.
3. A has a real symmetric square root, $A = SS$, $S \in \mathbb{S}^n$.
4. A has a lower triangular factorization, a Cholesky factorization, $A = LL^T$.
5. All principal minors of A are nonnegative.

Problem 2.6 – Motivation

When can we guarantee that global minimizers of $f : \mathbb{R}^n \rightarrow \mathbb{R}$ exist?

For example, the real valued function on \mathbb{R} $f(x) = e^x$ is bounded below by 0 but has no minimizers. The minimum value is 0 but is not attained.

Proposition 2.3 – Weierstrass Extreme Value Theorem

If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous, and if $D \subset \mathbb{R}^n$ is a closed and bounded set, then f is bounded below and the minimum value is attained on D .

Definition 2.9 – Coercive function

A continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is **coercive** if for any sequence x_i with $\|x_i\| \rightarrow \infty$, it must be the case that $f(x_i) \rightarrow +\infty$. In other words,

$$\lim_{\|x\| \rightarrow \infty} f(x) = +\infty$$

Here are some examples:

1. $f_1(x) = x^2$ is coercive.
2. $g(x) = x$ is not coercive (because as $x \rightarrow -\infty$, $g(x) \rightarrow -\infty \neq \infty$).
3. $h(x) = e^x$ is not coercive.

Proposition 2.4 – Coercive Functions and Minimizers

A coercive function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ has a global minimizer.