CO 367 Practice Problems for Final

December 6, 2023

1 Unconstrained Optimality Conditions

Find an example showing:

1. The second order necessary optimality conditions for unconstrained optimization are not sufficient.

   Solution. We find a function $f(x)$ such that its gradient at $\bar{x}$ is 0 and hessian at $\bar{x}$ is positive semidefinite, but it is not a local minimizer. Let $f(x,y) = x^3 + y^3$. Then

   $$\nabla f(x,y) = \begin{bmatrix} 3x^2 \\ 3y^2 \end{bmatrix}, \quad \nabla^2 f(x,y) = \begin{bmatrix} 6x & 0 \\ 0 & 6y \end{bmatrix}$$

   At (0,0), $\nabla f(0,0) = (0,0)^T$ and $\nabla^2 f(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. So this satisfies the necessary optimality conditions, but it is a saddle point.

2. The second order sufficient optimality conditions for unconstrained optimization are not necessary.

   Solution. We find a function $f(x)$ such that $\bar{x}$ is a strict local minimizer, but the hessian at $\bar{x}$ is not positive definite. Let $f(x) = x^4 + y^4$. Then

   $$\nabla f(x,y) = \begin{bmatrix} 4x^3 \\ 4y^3 \end{bmatrix}, \quad \nabla^2 f(x,y) = \begin{bmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{bmatrix}$$

   At (0,0), $\nabla f(0,0) = (0,0)^T$ and $\nabla^2 f(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \succeq 0$. So the hessian is not positive definite.

2 Constrained Optimality Conditions

Find an explicit optimal solution to the following problems:

1. $\min\{c^T x : Ax = b, x \in \mathbb{R}\}$

   Solution. We use second order necessary condition: Let x^* be a local minimizer and LICQ holds at x^*. Then x^* is a KTT point with unique lagrange multipliers λ^*, μ^* such that

   $$\langle \nabla^2_{x,x} L(x^*, \lambda^*, \mu^*) d, d \rangle \geq 0, \quad \forall d \in C(x^*, \lambda^*, \mu^*)$$

   Suppose we have a local minimizer $\bar{x}$ that is a KTT point and KTT pair $(\bar{x}, \mu)$. We derive it using the necessary conditions. The lagrangian is

   $$L(\bar{x}, \mu) = c^T \bar{x} + \mu^T (A\bar{x} - b)$$

   Then

   $$\nabla_x L(\bar{x}, \mu) = c + A^T \mu, \quad \nabla^2_{x,x} L(\bar{x}, \mu) = 0$$

The KTT conditions are

$$\nabla_x L(\bar{x}, \mu) = c + A^T \mu = 0$$
$$A\bar{x} - b = 0$$

$\langle \nabla^2_{x,x} L(x^*, \lambda^*, \mu^*)d, d\rangle = 0$ for all d, so we just need $\bar{x}$ that satisfies the KTT conditions. So the local minimizer is $\bar{x} = A^{-1}b$ along with $\mu = -A^{-T}c$. idk

2. $\min\{c^T x : e^T x = 1, x \geq 0\}$

   Solution.

3 Constrained Equivalent Optimality Conditions

Is the following claim true or false? Justify your answer.

Consider the following constrained optimization problem:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & c_i(x) = 0, \quad i \in \mathcal{E} \end{aligned}$$

Assume LICQ holds for this problem. We consider the following equivalent problem (equivalent geometrically)

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & c_i^2(x) = 0, \quad i \in \mathcal{E} \end{aligned}$$

Let x^* be a KTT point of the above problem, then x^* satisfies

$$\begin{aligned} & 0 = \nabla f(x^*) + 2\sum_{i\in\mathcal{E}} \lambda^* c_i(x^*)\nabla c_i(x^*) \\ & 0 = c_i(x^*) \quad \forall i \in \mathcal{E} \end{aligned}$$

where λ_i^* are the corresponding Lagrangian multipliers. By rearrangement, we have $\nabla f(x^*) = 0$. This implies for equality constrained optimization problems, we can get the equivalent first order necessary optimal condition as for the unconstrained optimization problem.
Solution.

4 Duality

Find the dual problem of the following problem and the dual of the dual problem.

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & x^T A x + 2b^T x \\ \text{s.t.} \quad & \|x\|_2 leq 1 \end{aligned}$$

where $A \in \mathcal{S}^n, b \in \mathbb{R}^n$ (Use the generalized inverse to derive the dual).
Solution.

5 Question 5

Given the following optimization problem

$$\begin{aligned} \min_{x\in\mathbb{R}, y>0} \quad & e^{-x} \\ \text{s.t.} \quad & \frac{x^2}{y} \leq 0 \end{aligned}$$

1. Show that this is a convex optimization problem, find its global minimizer, and verify if Slater condition holds.
   Solution.
2. Find the dual problem of this problem; find the optimal solution of the dual problem; compute the duality gap.
   Solution.