

# CO 367 Practice Problems for Final

December 6, 2023

## 1 Unconstrained Optimality Conditions

Find an example showing:

1. The second order necessary optimality conditions for unconstrained optimization are not sufficient.

**Solution.** We find a function  $f(x)$  such that its gradient at  $\bar{x}$  is 0 and hessian at  $\bar{x}$  is positive semidefinite, but it is not a local minimizer. Let  $f(x, y) = x^3 + y^3$ . Then

$$\nabla f(x, y) = \begin{bmatrix} 3x^2 \\ 3y^2 \end{bmatrix}, \quad \nabla^2 f(x, y) = \begin{bmatrix} 6x & 0 \\ 0 & 6y \end{bmatrix}$$

At  $(0,0)$ ,  $\nabla f(0,0) = (0,0)^T$  and  $\nabla^2 f(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ . So this satisfies the necessary optimality conditions, but it is a saddle point.

2. The second order sufficient optimality conditions for unconstrained optimization are not necessary.

**Solution.** We find a function  $f(x)$  such that  $\bar{x}$  is a strict local minimizer, but the hessian at  $\bar{x}$  is not positive definite. Let  $f(x) = x^4 + y^4$ . Then

$$\nabla f(x, y) = \begin{bmatrix} 4x^3 \\ 4y^3 \end{bmatrix}, \quad \nabla^2 f(x, y) = \begin{bmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{bmatrix}$$

At  $(0,0)$ ,  $\nabla f(0,0) = (0,0)^T$  and  $\nabla^2 f(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \succeq 0$ . So the hessian is not positive definite.

## 2 Constrained Optimality Conditions

Find an explicit optimal solution to the following problems:

1.  $\min\{c^T x : Ax = b, x \in \mathbb{R}\}$

**Solution.** We use second order necessary condition: Let  $x^*$  be a local minimizer and LICQ holds at  $x^*$ . Then  $x^*$  is a KKT point with unique lagrange multipliers  $\lambda^*, \mu^*$  such that

$$\langle \nabla_{x,x}^2 L(x^*, \lambda^*, \mu^*) d, d \rangle \geq 0, \quad \forall d \in C(x^*, \lambda^*, \mu^*)$$

Suppose we have a local minimizer  $\bar{x}$  that is a KKT point and KKT pair  $(\bar{x}, \mu)$ . We derive it using the necessary conditions. The lagrangian is

$$L(\bar{x}, \mu) = c^T \bar{x} + \mu^T (A\bar{x} - b)$$

Then

$$\nabla_x L(\bar{x}, \mu) = c + A^T \mu, \quad \nabla_{x,x}^2 L(\bar{x}, \mu) = 0$$

The KTT conditions are

$$\begin{aligned}\nabla_x L(\bar{x}, \mu) &= c + A^T \mu = 0 \\ A\bar{x} - b &= 0\end{aligned}$$

$\langle \nabla_{x,x}^2 L(x^*, \lambda^*, \mu^*) d, d \rangle = 0$  for all  $d$ , so we just need  $\bar{x}$  that satisfies the KTT conditions. So the local minimizer is  $\bar{x} = A^{-1}b$  along with  $\mu = -A^{-T}c$ . **idk**

2.  $\min\{c^T x : e^T x = 1, x \geq 0\}$

**Solution.**

### 3 Constrained Equivalent Optimality Conditions

Is the following claim true or false? Justify your answer.

Consider the following constrained optimization problem:

$$\begin{aligned}\min \quad & f(x) \\ \text{s.t.} \quad & c_i(x) = 0, \quad i \in \mathcal{E}\end{aligned}$$

Assume LICQ holds for this problem. We consider the following equivalent problem (equivalent geometrically)

$$\begin{aligned}\min \quad & f(x) \\ \text{s.t.} \quad & c_i^2(x) = 0, \quad i \in \mathcal{E}\end{aligned}$$

Let  $x^*$  be a KTT point of the above problem, then  $x^*$  satisfies

$$\begin{aligned}0 &= \nabla f(x^*) + 2 \sum_{i \in \mathcal{E}} \lambda_i^* c_i(x^*) \nabla c_i(x^*) \\ 0 &= c_i(x^*) \quad \forall i \in \mathcal{E}\end{aligned}$$

where  $\lambda_i^*$  are the corresponding Lagrangian multipliers. By rearrangement, we have  $\nabla f(x^*) = 0$ . This implies for equality constrained optimization problems, we can get the equivalent first order necessary optimal condition as for the unconstrained optimization problem.

**Solution.**

### 4 Duality

Find the dual problem of the following problem and the dual of the dual problem.

$$\begin{aligned}\min_{x \in \mathbb{R}^n} \quad & x^T A x + 2b^T x \\ \text{s.t.} \quad & \|x\|_2 \leq 1\end{aligned}$$

where  $A \in \mathcal{S}^n, b \in \mathbb{R}^n$  (Use the generalized inverse to derive the dual).

**Solution.**

### 5 Question 5

Given the following optimization problem

$$\begin{aligned}\min_{x \in \mathbb{R}, y > 0} \quad & e^{-x} \\ \text{s.t.} \quad & \frac{x^2}{y} \leq 0\end{aligned}$$

1. Show that this is a convex optimization problem, find its global minimizer, and verify if Slater condition holds.

**Solution.**

2. Find the dual problem of this problem; find the optimal solution of the dual problem; compute the duality gap.

**Solution.**