ACTSC 445/845: Final Project
Quantitative Analysis of Delta-Hedged Portfolio Risk: Monte Carlo Simulation Approach

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1 Introduction

Risk management through financial modelling is a cornerstone of contemporary finance mathematics. Inspired by Pochart et al.[8], this ACTSC 445 project explores the Monte Carlo Simulation on Risk Measures and Discrete Delta Hedging portfolio hedging strategies.

To begin with, we introduce concepts of no-arbitrage principles, European options, their valuation, and their role in hedging strategies. We also examined the Black-Scholes model, a fundamental framework in financial mathematics that revolutionized option pricing and risk management. Moreover, our analysis incorporates advanced risk measures like Value-at-Risk (VaR) and Expected Shortfall (ES). We then explain the basic idea of the Monte Carlo Simulation.

In the section of the Description of the Method, we examine the classical Black-Scholes model and its assumptions. Acknowledging the discrepancy between the Normal distribution assumption and the real-world data, we introduce a modified version of the Black-Scholes model, which integrates fat-tailed processes [8] to simulate price paths better and capture the leptokurtic feature prevalent in financial markets. We also explore the implementation of a delta-hedging strategy within a self-financing portfolio. This strategy is important for risk management of the price volatility of the underlying asset. We then construct risk measure functions and apply continuous delta adjustments to align the portfolio with the designated risk parameters. Our approach ensures the portfolio remains self-financing and delta-neural. Furthermore, we implement Monte Carlo Simulation, enabling us to quantify risk measures such as Value at Risk (VaR) and Expected Shortfall (ES). We analyze the asset price paths under various volatility conditions under different market volatilities.

Finally, based on the results generated by the Monte Carlo simulation, we find the properties that follow the concepts we learned in ACTSC445. However, there are limitations in our approaches, such as insufficient delta hedging, the choice of risk measurements, limited simulations, and less detailed statistical analysis.

2 Preliminary

2.1 Fundamentals in Portfolio Hedging

Before introducing the methods of this project, the concepts of mathematical finance should be clarified. In this chapter, we introduce the basic concepts in mathematical finance that are used later in the implementation.

2.1.1 Principle of No Arbitrage

There is no such thing as a free lunch. According to McDonald in 2013 [6], an arbitrage opportunity is taking advantage of buying the index at a low price and selling the index at a high cost, which generates a positive money flow.

The following equation portfolio price process $\{V_t\}_{t \geqslant 0}$ to demonstrate the arbitrage opportunity:

- $V_0 \leqslant 0$, and
- $V_T > 0$ for some time $T > 0$

The value process of the portfolio strategy [6] $\theta = \{\theta_t\}_{t=0,h...T}$ is a stochastic process $\{V_t\}_{t=0,h,...,T}$. The market value V_t^θ at time t of the portfolio θ is

$$V_t^\theta = \Delta_t S_t + b_t e^{rh} \tag{1}$$

In this project, we refer to the financial market as a perfect capital market. That is, there is an arbitrage-free market, and the Principle of No Arbitrage holds in this market.

Besides, the financial market is complete here. A market is complete [1] if it is arbitrage-free and all contingent claims attainable.

A portfolio is strategy $\theta = \{\theta_t\}_{t=0,h...T}$ is called self-financing if for all $t = h, 2h, ..., T - h$

$$\Delta_t S_t + b_t e^{rh} = \Delta_{t+h} S_t + b_{t+h} \tag{2}$$

Self-financing indicates that money-in equals money-out in the market.

2.1.2 The European Options

A derivative [6] is a financial contract that has been determined by the price of underlying (primitive instruments) or other derivative. It has an expiration date of T.

A standard option is a contract that gives an investor the right but not the obligation to buy or sell an asset at or before the expiration time or maturity time for a predetermined price.

The action of carrying out a transaction is referred to as "exercising the option." The expiration time (T) is the latest time that an option can exercised with the price at which the asset will be exchanged (K), which is called the strike price. Exercising on options can be specified as "call options" and "sell options."

A call option is a contract that gives investors rights but not the obligations to buy with the payoff

$$\max\{S_T - K, 0\} = (S_T - K)_+ \tag{3}$$

A put option is a contract that gives investors rights but not the obligations to sell with the payoff

$$\max\{K - S_T, 0\} = (K - S_T)_+ \tag{4}$$

The European option is one of the standard options or the plain vanilla option [11]. In the European option process, an option can only be exercised at time T (i.e., the expiration date or the maturity). In this project, we will focus on the European options.

Also, we denote the European call options as c_t and European put options as p_t

2.1.3 The Black-Scholes Model

Geometric Brownian Motion (GBM) [10] $\{X_t\}_{t\geqslant 0}$ is a commonly used model for valuing finances in option pricing schemes. GBM is a standard and one-dimensional stochastic process with continuous-time assumption with the following representation in the Langevin equation:

$$dX_t = X_t(\alpha dt + \sigma dB_t), \tag{5}$$

where X_0 is the particle position, α is an adapted process called the drift, $\sigma > 0$ is also a adapted process called volatility or diffusion. We prefer volatility here. B_t is a standard Brownian Motion with property $E[B_t] = 0$, $Var(B_t) = t$.

The solution to the GBM representation is

$$X_t = X_0 e^{(\alpha - \frac{\sigma^2}{2})t + \sigma B_t}, \qquad X_0 > 0 \tag{6}$$

With the GBM concepts, we can introduce the Black-Scholes (B-S) model. The Black-Scholes (B-S) model [2] is a groundbreaking method for valuing options and corporate liabilities. According to Fischer Black and Myron Scholes in 1973, The B-S model calculates the theoretical price of European-style options. Its main assumption is that the stock price follows a Geometric Brownian motion with constant volatility and interest rates.

The valuation formula of the B-S model is in "ideal conditions." [2] with

- The short-term interest rate is known.
- The stock price is in a random walk or Markov Chain stochastic process in continuous time. The distribution of stock price S_K follows the log-normal distributions, and the variance rate of the return on stock is constant.
- There are no dividends in stocks.
- It is using European options.
- There are no transaction costs.
- Investors can borrow securities with any fraction of the price and decide to hold it or buy it at the known short-term interest rate.
- There are no short-selling penalties.

With these conditions, a theoretical European call and put prices can be generated using a B-S framework.

Due to the property of fixed exercising time at maturity of the European Options, it can be conveniently priced using the B-S formula. In the B-S model, the arbitrage-free time t price of a European call option ct and put option p_t with strike K, stock price S_T maturity T is

$$c_t = S_t N(d_1) - Ke^{-r(T-t)} N(d_2),\ p_t = Ke^{-r(T-t)} N(-d_2) - S_t N(-d_1) \tag{7}$$

where

$$d_1 = \frac{log(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t},$$

and $N(\cdot)$ in the equation is the cumulative distribution function (CDF) of the standard normal distribution.

2.1.4 Delta Hedging

Option Greeks measure the option price change while changing the input of the function [6]. One of the key applications of Greeks is in the assessment of risk exposure. Greeks can be interpreted as partial derivatives of the B-S formula for European option prices with respect to specific parameters. Besides, the Greeks express the change by assuming only one input changing at a time.

Delta (Δ) is one of the options for Greeks. It represents the change in the option price when the stock price increases in one unit (usually $1). Mathematically, it is the first derivative of the option price with respect to the stock price with the following representations for Delta of the call $\Delta^{(c)}$ and Delta of the put $\Delta^{(p)}$.

$$\Delta^{(c)} = \frac{\partial c}{\partial S} \qquad \Delta^{(p)} = \frac{\partial p}{\partial S}$$

Under the B-S framework, the deltas for European calls and puts are

$$\Delta^{(c)} = e^{-\delta(T-t)}N(d_1) \qquad \Delta^{(p)} = -e^{-\delta(T-t)}N(-d_1) \tag{8}$$

Based on these equations, there are several properties regarding the deltas. For the European call option, the delta is between 0 and 1. This means that the call option price increases as the stock price increases. For the European put option, the delta is between -1 and 0. This can be interpreted as the put option price decreases as the sock price increases.

After giving an insight into the "Delta," we consider the "Hedging" here. Hedging [6] is a strategic financial position to offset possible losses or gains associated with a corresponding investment. A hedge can consist of combinations of financial instruments. A hedge is a portfolio θ to regulate the impact of changes in other model parameters on the value of the portfolio V_t^θ. Without hedging, there is likely to be an arbitrary portfolio with uncontrolled risks. [6].

Market-makers can control risk by delta-hedging. A portfolio is delta-hedged or delta-neutral if its delta is zero. Here, we prefer the term – delta hedgeing.

$$\Delta^\theta = \sum_{i=1}^{N} \theta_i \Delta^{(i)} = 0 \tag{9}$$

This equation indicates that the value of the portfolio will not change when small changes occur in the underlying asset S prices.

Delta Hedging strategy will be used in the following methods to minimize the risks.

2.2 Risk Measure

Financial Organizations deal with risk management all the time. Based on the Alexander et al. mention in Quantitative Risk Management, there are risks like market risks, credit risks, and more risk types. Potential losses caused risks. To determine risk measures, we need first to know the losses.

We have already defined the value of the portfolio to be V_{t+1}. The change of the portfolio across time period t to t+1 is $\Delta V_{t+1} = V_{t+1} - Vt$. Now, we define the loss as [7]

$$L_{t+1} := -\Delta V_{t+1} = -(V_{t+1} - V_t)$$

Loss L_{t+1} is a random variable with its loss distribution [7]. Denote the Cumulative Density Function as $F_L(l) = P(L \leqslant l)$

The V_t [7] consists of time t and a d-dimensional random vector $Z_t = (Z_{t,1}, ..., Z_{t,d})$ called risk factor. Hence, $V_t = f(t, Z_t)$ for $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$

There is a random vector called risk-factor changes [7] $X_{t+1} := Z_{t+1} - Zt$

Portfolio loss L_{t+1} over the time period $[t, t+1]$ [7] is given by

$$L_{t+1} = -\Delta V_{t+1} = -(V_{t+1} - V_t) = -(f(t+1, z_t + X_{t+1}) - f(t, z)) \tag{10}$$

Where t is the current time,

With a differentiable f, by using Taylor Expansion, the first-order approximation L^{Δ}_{t+1} of this portfolio loss is

$$L^{\Delta}_{t+1} := -\left(f(_t t, Z_t) + \sum_{i=1}^{d} f_{Z_i}(t, Z_i) X_{t+1,i} \right)$$

where f denotes the partial derivative, the L^{Δ} is the delta in the hedging of the derivative.

Hence, the loss can be linearized.

We here introduce the risk measures. Risk Measures can be used to determine the capital that needs to be reserved as a buffer against the potential losses on extreme events [7]. Mathematically, a function $\rho : \mathcal{M} \to \mathbb{R}$ is called a risk measure, where $\mathcal{M}$ is a random variable.

There are many types of risk measurements (notional-amount approach, measurement based on the loss distribution, and scenario-based risk measures). We will focus on the measurement based on the loss distribution in this project, specifically Value-at-Risk and Expected Shortfall (ES).

Value-at-Risk (VaR) [7] is an important risk measurement in financial regulations such as Basel and Solvency II. Define the VaR of a portfolio with loss L at a confidence level $\alpha \in (0, 1)$ given by the smallest number l such that the probability that the loss L exceeds l is no longer than $1 - \alpha$

$$VaR_{\alpha} = VaR_{\alpha}(L) = \inf\{x \in \mathbb{R} : F_L(x) \geqslant \alpha\} = F_L^{<-}(\alpha) \tag{11}$$

For a loss $L \sim F_L$ with $E(L_+) < \infty$, the expected shortfall (ES) is defined as [7]

$$ES_\alpha = ES_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(L)du \tag{12}$$

From ES_α expression above, the ES_α is the average of VaR_u over all $u \geqslant \alpha$ [7]

Hence, if F_L is continuous, the equation of ES_α can be written by

$$ES_\alpha(L) = E(L|L > VaR_\alpha(L)) \tag{13}$$

2.3 Monte Carlo Simulation

To obtain the solution to an established model of the system, we can approximate using a proper approximation procedure and generate a numerical solution to this model [9]. This is why we need the Monte Carlo (MC) Simulation with its inclusion of randomness in the underlying model [9]. According to Kroese et al. in 2014, there are many applications of MC simulations: sampling, estimation, and optimization. MC method is easy and efficient. Its algorithm is flexible and allows much more general models to be implemented [5]. We can also rely on MC to analyze large high-dimensional data sets.

We explore the idea of the MC Simulation. In 1996, based on the research of Cowles et al., the Monte Carlo is to be analyzed to establish a set number of iterations that will guarantee convergence within a specified tolerance of the true stationary distribution in terms of total variation distance [3]. Then, it applies diagnostic tools to the output produced by the algorithm.

Based on Homem-de-Mello in 2001, formally, we want to solve the problem [4]

$$\min_{x \in X}\left\{g(x) := E[G(x)] = \int_\Omega G(x,\omega)P(d\omega)\right\}$$

with a probability space $(\Omega, \mathcal{F}, P)$, a subset $X \subseteq \mathbb{R}^m$, and a (measurable) function $G : X \times \Omega \to \mathbb{R}$ The corresponding Monte Carlo approximation mentioned above is

$$\hat{g}_N(x) = \frac{1}{N}\sum_{i=1}^{N} G(x,\omega_i)$$

for $\omega_1, ...\omega N$ are identically independent distributed (i.i.d) samples.

3 Description of the method

For the sake of simplicity, we will assume the following structure:

- the market is complete and arbitrary-free, which means the Black-Scholes model holds, and we can use them to generate the path and obtain the proper price of the option.
- we only consider the European option, which can be priced by the Black-Scholes formula
- The delta-hedging portfolio consists of a long position of stock (owning a stock) and a short position of a call option (selling a call option)

There are three parts of the implementation: generating the price trajectories, calculating the risk measure of the price path based on a delta-hedging portfolio, and Monte-Carlo simulation.

For convention, let us first define

3.1 Generating Price Paths

In financial modelling, the generation of price paths often utilizes the Black-Scholes methodology, traditionally based on Geometric Brownian Motion. That is,

$$dS_t = \alpha S_t dt + \sigma S_t dW_t^P. \tag{14}$$

$$S_t \sim \log\mathcal{N}\left(\ln(S_0) + \left(\alpha - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right) \tag{15}$$

$$S_t = S_0 e^{\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W_t^P}. \tag{16}$$

where S_t represents the asset price at time t, α is the drift coefficient, σ the volatility, and dW_t^P denotes the increment of the Brownian Motion process, Here, S_t represents the stock price at time t, S_0 is the initial stock price, α is the drift coefficient, σ is the volatility coefficient, and W_t^P denotes a Brownian Motion process under the risk-neutral measure P with the increment following the normal distribution. The first method to generate a price path is described above.

However, this approach has limitations, particularly because it assumes the stock prices follow the log-normal distribution and the Brownian motion increment follows the normal distribution. But by the stylized facts, financial time series often have a heavy tail[4]. To address this, we replace the standard Brownian motion W_t with a process L_t, where L_t is a fat-tailed process with increments distributed according to a Student-t distribution.[8] This modification aligns better with the empirical evidence of the financial time series. The modified stochastic process is defined as:

$$dS_t = \alpha S_t dt + \sigma S_t dL_t^P, \tag{17}$$

$$S_t = S_0 e^{\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma L_t^P}. \tag{18}$$

where S_t represents the asset price at time t, α is the drift coefficient, σ the volatility, and dL_t denotes the increments of the fat-tailed process. Unlike the standard Brownian motion, the increments $\Delta L_t = L_t - L_{t-\Delta t}$ in this model follow the Student-t distribution, introducing a more accurate representation of the leptokurtic nature of financial markets. The flexibility of the Student-t distribution, particularly its degrees of freedom, allows for a more nuanced calibration of the model, enhancing its ability to simulate realistic price paths reflective of actual market behaviours.

3.2 Risk Measure functions

In this section, we explore the construction of a delta-hedging strategy for a self-financing portfolio, building upon the price paths generated by the traditional Black-Scholes model and a modified Black-Scholes model that incorporates fat-tailed distributions. The hedging strategy is designed to neutralize the portfolio's risk exposure to the underlying asset's price volatility through continuous delta adjustments. For our research, we use delta-hedging as simple as the following strategy: We start with the portfolio $\theta = (\theta_0^S, \theta_0^C)$, where θ_0 is the portfolio at t_0, and θ_0^S is the share for stock at t_0 and θ_0^C is the share for the call option. Thus, at time $T = t$, we re-balance the portfolio on two conditions: 1, the portfolio is self-financing. 2, the rebalanced portfolio is delta-hedged, which yields the following equations at each time $T = t + 1$:
Self-Financing:

$$\theta_t^S * S_{t+1} + \theta_t^C * C_{t+1} = \theta_{t+1}^S * S_{t+1} + \theta_{t+1}^C * C_{t+1} \tag{19}$$

Delta-hedging:

$$\theta_{t+1}^S * 1 + \theta_{t+1}^C * \Delta_{t+1} = 0 \tag{20}$$

where C_{t+1} is the price of the option, which is calculated by the Black-Scholes formula, and Δ_{t+1} (the delta of the call option at time t+1) is calculated by the Black-Scholes formula based on the stock price at time $t + 1$

$$\Delta_t = N(d_1), d_1 == \frac{\ln(\frac{S_t}{K}) + (r + \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}} \tag{21}$$

where T: Time to maturity. t: Current time. r: risk-free rate. N is the CDF of Normal(0,1)

After we generate the portfolio values by our delta-hedging strategy, We assess the strategy's performance by calculating the portfolio loss at each rebalancing instance,

$$L_{t+1} = -(V_{t+1} - V_t) \tag{22}$$

Where L is the loss and V_t is the delta-hedged portfolio value, which reflects the trading outcomes.

After we calculate the loss for each path, we need to decide the risk measures. Subsequently, we utilize two distinct risk measures: Value at Risk (VaR) and Expected Shortfall (ES). These metrics capture the risk profile from various angles, offering a comprehensive analysis of the hedging strategy's resilience to financial market anomalies.

3.3 Monte Carlo Simulation

The method we used to quantify the risk measure is Monte Carlo Simulation, with an emphasis on the generation of asset price paths and calculating the risk measure under each path. We perform these simulations N_{MC} times to obtain a statistically significant dataset for analysis. This methodology is pivotal for understanding the behaviour of asset prices under a spectrum of scenarios influenced by varied parameter configurations.

We first analyze the role of volatility, denoted as σ, on the dispersion of asset price paths. The variation in σ allows us to model conditions from low to high market volatility, each providing distinct perspectives on potential asset price movements under different market volatility.

Different path-generating methodologies are also scrutinized. Each methodology incorporates unique assumptions that affect the simulation results. For example, while some methods may employ a simplistic random walk approach, others might integrate complex processes such as mean-reversion or jump-diffusion to more accurately reflect market behaviours [4]. The selection of the path-generating algorithm is stated in our previous section.

Additionally, the application of diverse risk measures is investigated to evaluate the risk profiles of the generated asset paths. These measures include but are not limited to, VaR, ES, and other metrics that assess tail risk in the simulation studies.

The Monte Carlo simulations are an integral component of our analysis, presenting the versatility to assess numerous parameters and hypothetical market scenarios. Through these simulations, we try to understand the complex dynamics of financial markets and derive predictive insights that have both academic and practical relevance in the field of finance.

4 Numerical Result

Our simulation price data is built upon four different models: traditional Black-Scholes model with standard normal increments and fat-tailed models with the increments following Student-t distribution with degree of freedom $\nu = 4, 6, 8$ respectively. The initial stock price is fixed at 100, whereas the strike K is to be 100, 140, and 180. We set the option time to maturity to be one year and the interval to rebalance to be one day, which leads to rebalancing 365 times on each price trajectory. However, in reality, we only have 252 trading days, so we are trying to show an approximate result. In terms of the volatility, we consider only two cases $\sigma = 0.4, \sigma = 0.2$, which shows the low volatility market and high volatility market. For the fixed market parameters, we have a risk-free rate $r = 5\%$. The number of price trajectories is $N_{MC} = 2000$ (It is supposed to be higher, but we have some computational limitations).

4.1 VaR

Histograms of VaR under different parameters and price trajectories (See Appendix). Based on the result:

- Under the same strike price $k = 100$ and the σ, we are expecting the loss distribution to be a lighter tail and lower loss for lower ν on the t-distribution. However, for $k = 140, 180$. Brownian Motion and higher ν on t-distribution have a heavier tail and higher loss. (since we use $V_t - V_{t+1}$ to calculate the loss, the lower the value, the higher the loss.)
- In terms of volatility, a higher volatility market also leads to a higher loss.

4.2 Expected Shortfall

Histograms of ES under different parameters and price trajectories (See Appendix). Based on the result:

- Lower ν value on the t-distribution demonstrates the heavier tails on the Expected shortfall, and the higher ν is closer to Brownian Motion, which is within our expectation, the portfolio also has higher ES on the lower t (heavier) tail market.
- Higher volatility again implies a higher Expected Shortfall.

4.3 Average Risk Measures on Different Strike Price

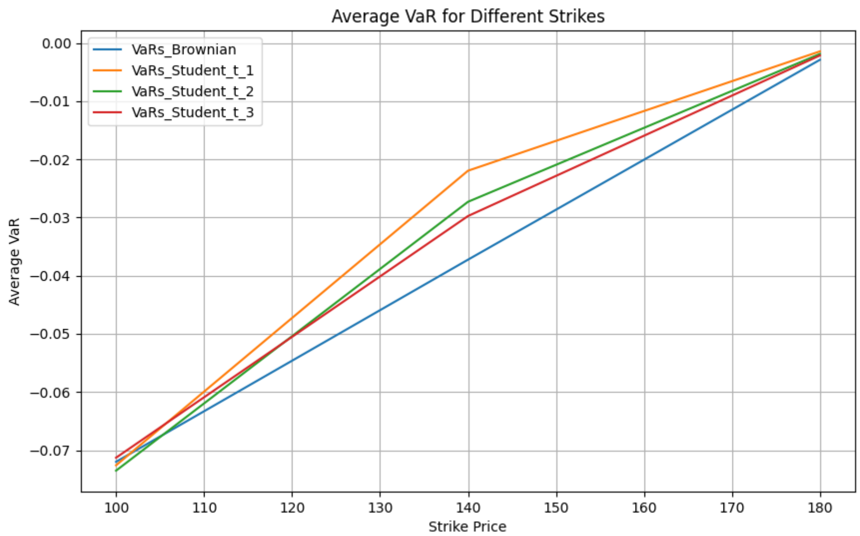

Figure 1: K=100,140,180 Sigma=0.2

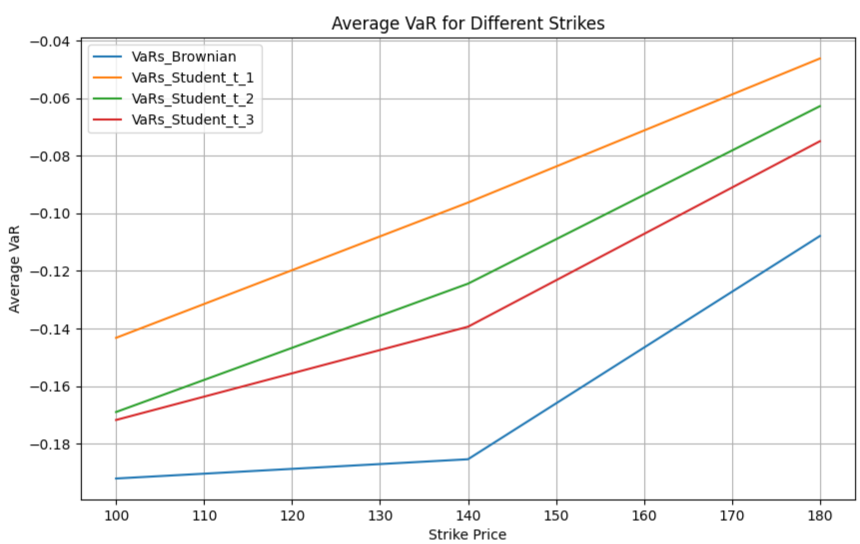

Figure 2: K=100,140,180 Sigma=0.4

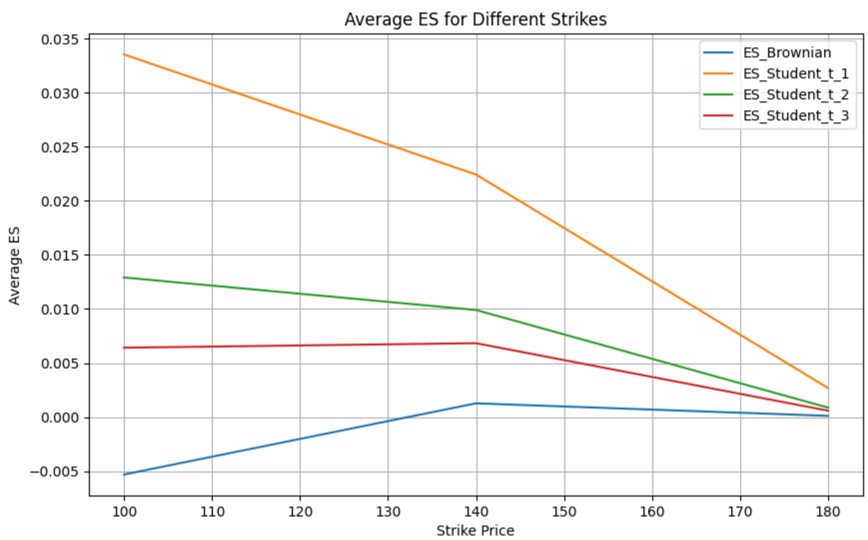

Figure 3: K=100,140,180 Sigma=0.2

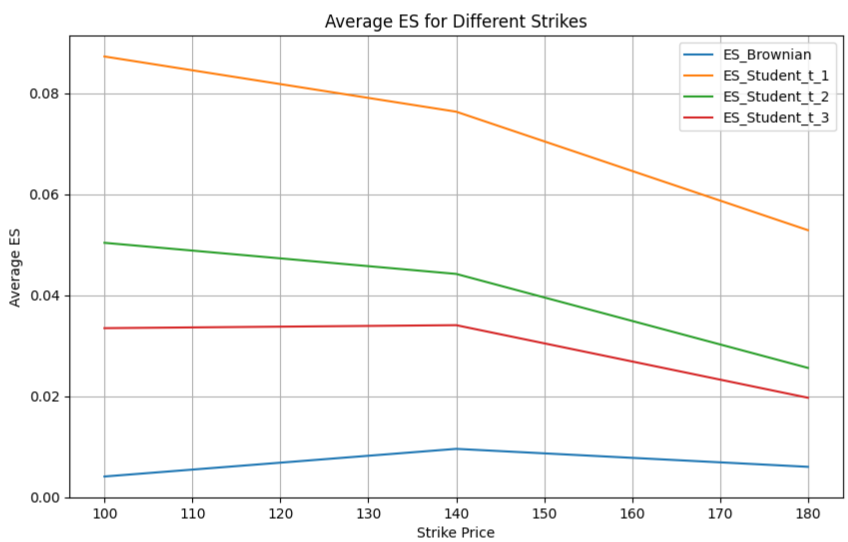

Figure 4: K=100,140,180 Sigma=0.4

5 Limitations

There are four main limitations in our research due to the lack of knowledge and time, which will cause bias for the study project.

Insufficient delta hedging: For the simplicity of the project, we hedge on a discrete time period, which means we adjust it by only two rules: self-financing and delta-hedging. But for the actual dynamic delta hedging, we need to consider the time value and the sunk cost when we adjust the portfolio, which will essentially involve the delta-gamma hedging. Thus, our study focuses more on the application of the risk measure in different market stages (price path and volatility).

Diversified Risk Measures: At the proposal stage, we consider replicating the unconditional risk measure [8], where we penalize the excess loss in the following way:

$$U(\Delta W_k) = (\Delta_0 - \Delta W_k)_+^q = |\Delta_0 - \Delta W_k|^q \mathbf{1}_{\Delta W_k < \Delta_0} \tag{23}$$

where ΔW_k denotes the change in the local wealth balance, and the indicator function $\mathbf{1}_{\Delta W_k < \Delta_0}$ ensures penalization occurs only for losses exceeding the threshold Δ_0. The exponent q is set to 1, which implies a linear relationship in penalizing losses beyond the threshold. And we can set different q for different penalization. However, since the wealth balance formula (loss) we calculate is rather simplified, we do not implement the continuous loss model to calculate the unconditional risk.

Simulation Numbers: Due to the limitation of the algorithm. It takes more than 10 hours to simulate 5000 trajectories and calculate the results. However, we are supposed to simulate 20000 trajectories for a better Monte Carlo simulation and a more trustful confidence range. Eventually, our result will be based on 2000 trajectory simulation price paths, which will have a larger variance in the data. Also, it would be more appropriate for us to simulate more strike prices if we want to look at the performance of the delta-hedged portfolio under different strikes. However, we are limited by the computational complexity.

Statistical Inference on the results: Due to the limited time, we are not able to conduct the statistical test to conclude the statistical significance of the results, but from an empirical conclusion on the data, which is also a potential limitation.

6 Conclusion

This project consolidates the knowledge we learned from ACTSC445 and furthers our interests in risk management. We will continue this passion in the following study.

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7 Appendix

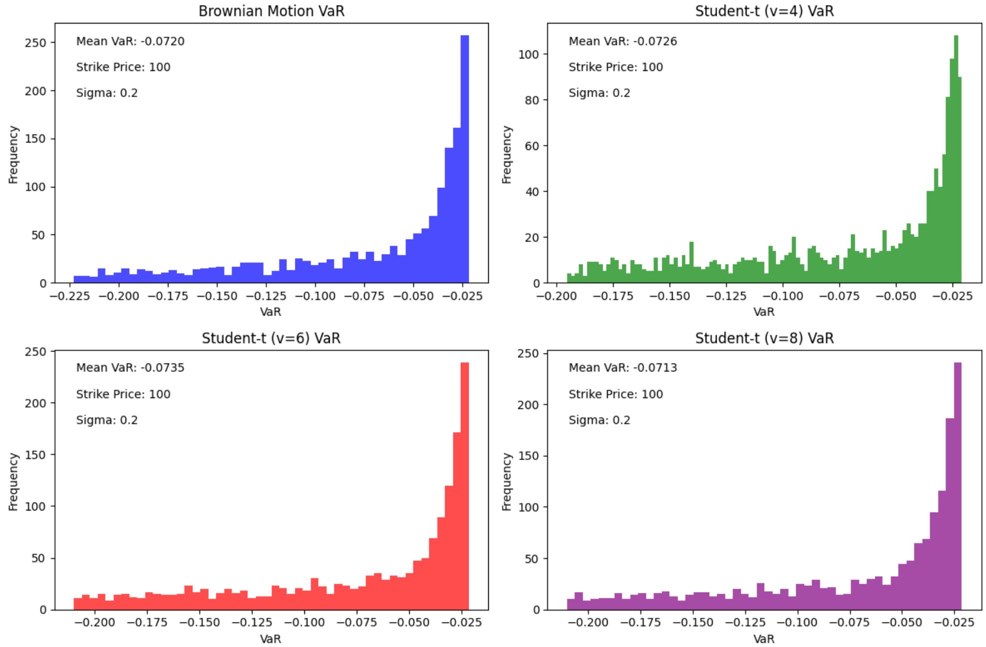

Figure 5: K=100 Sigma=0.2

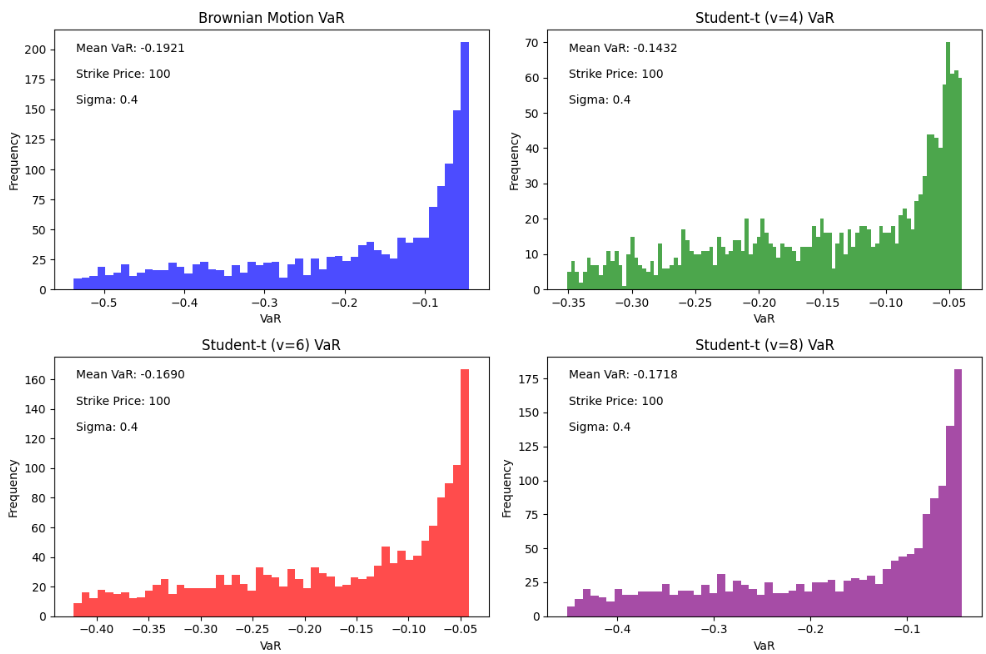

Figure 6: K=100 Sigma=0.4

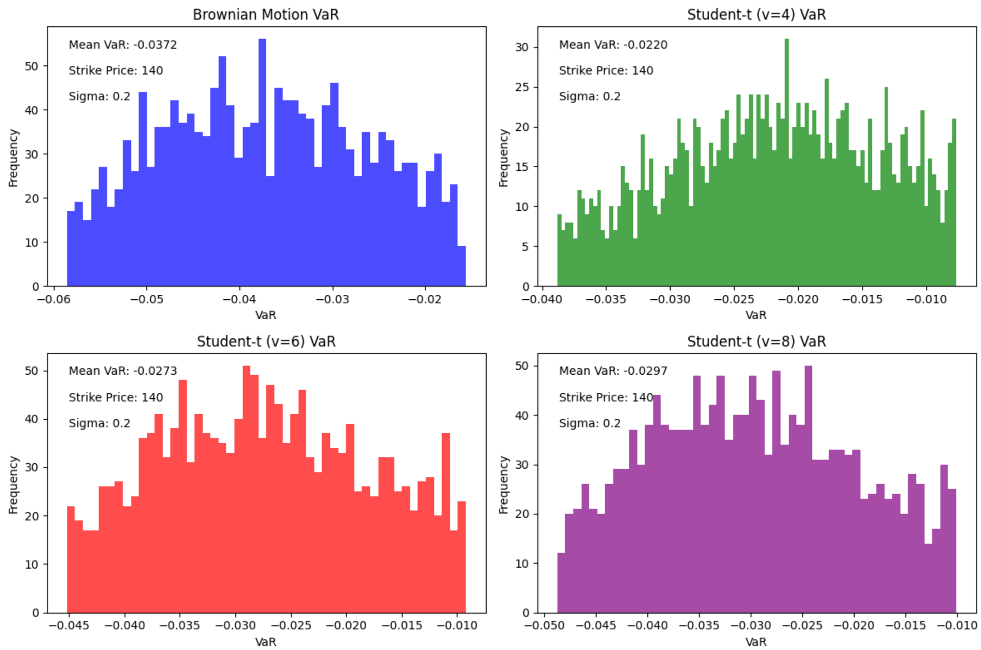

Figure 7: K=140 Sigma=0.2

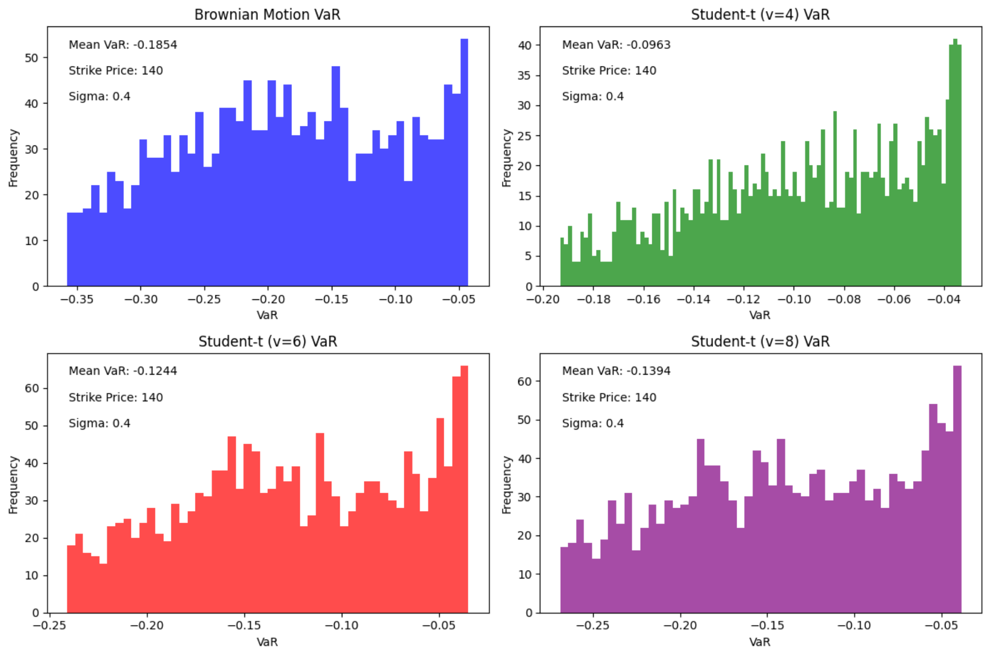

Figure 8: K=140 Sigma=0.4

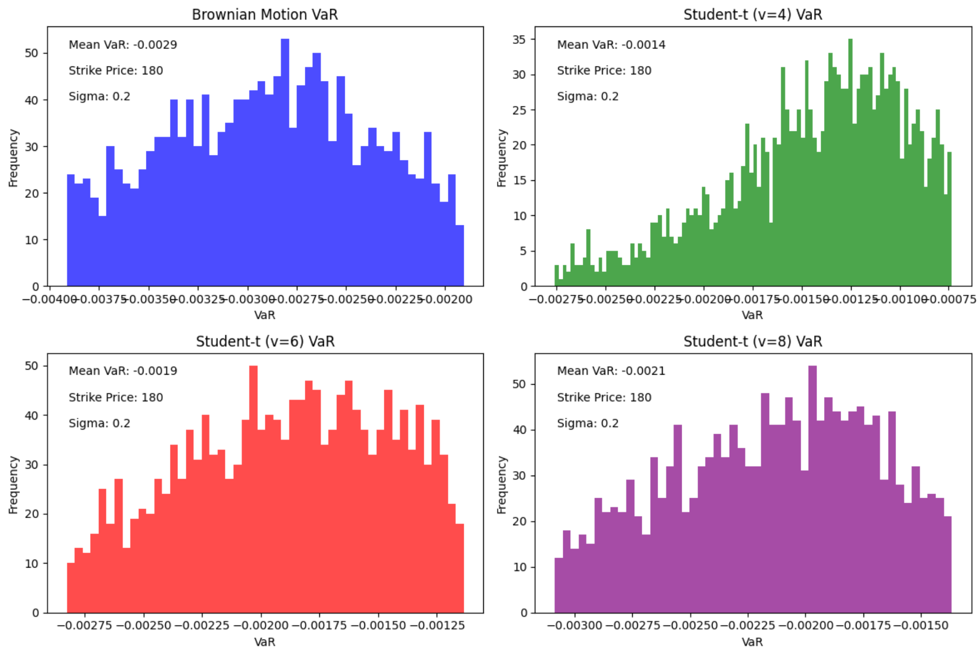

Figure 9: K=180 Sigma=0.2

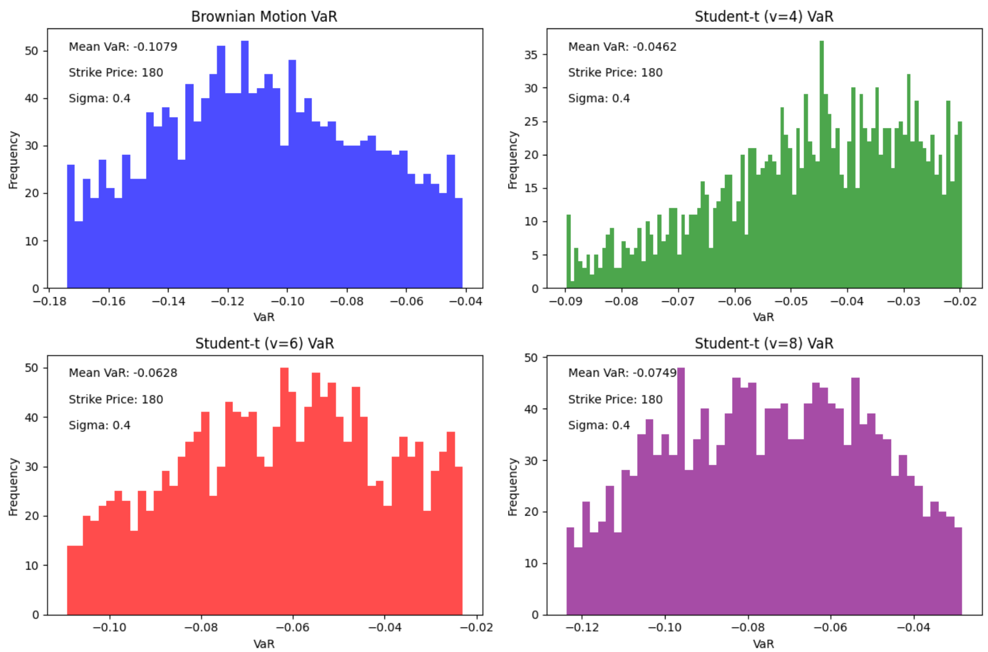

Figure 10: K=180 Sigma=0.4

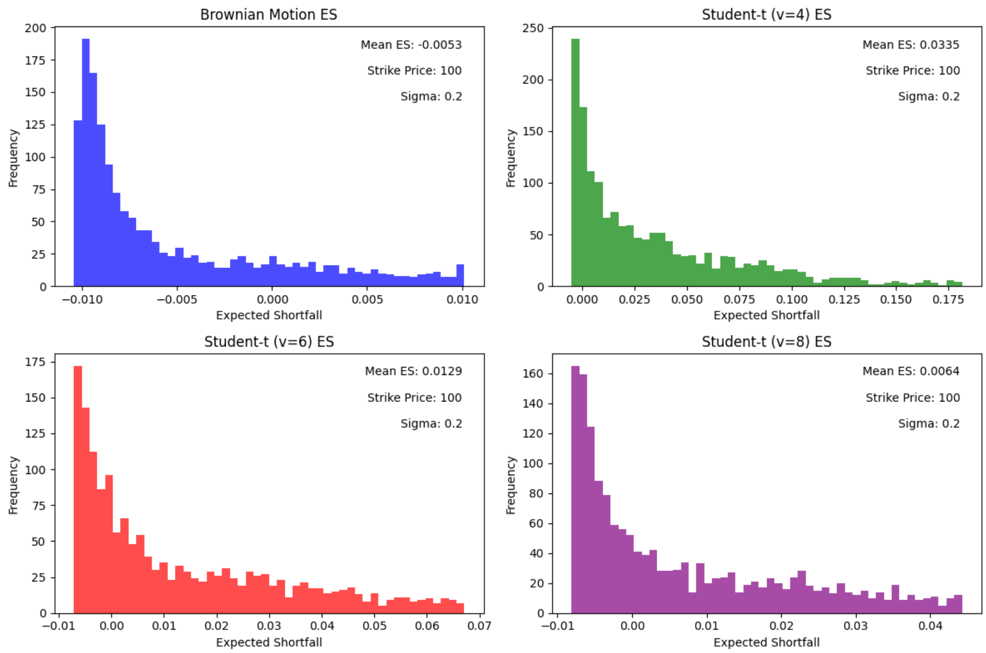

Figure 11: K=100 Sigma=0.2

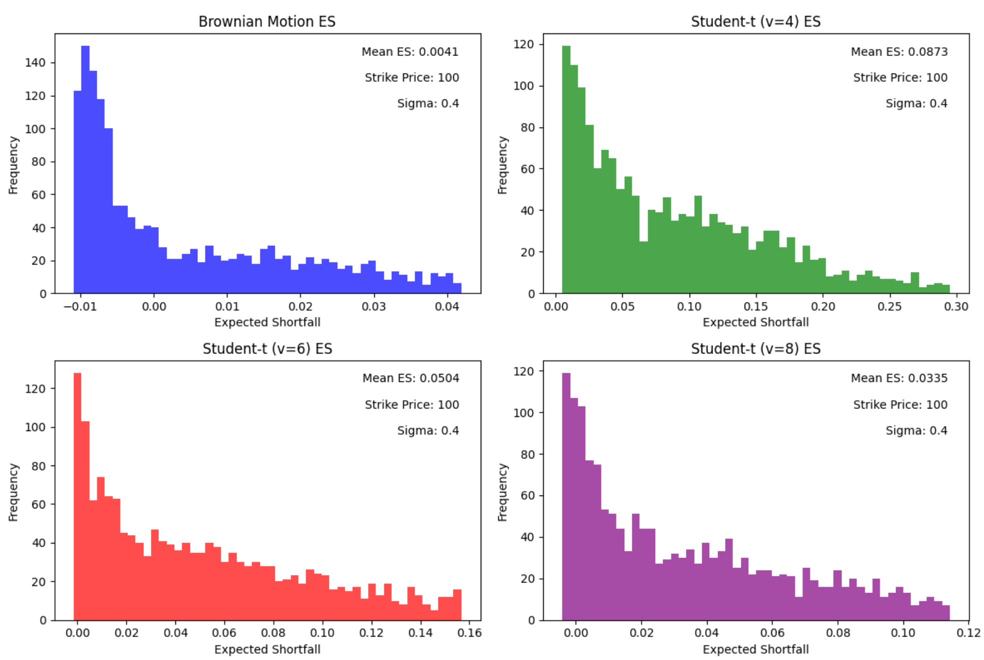

Figure 12: K=100 Sigma=0.4

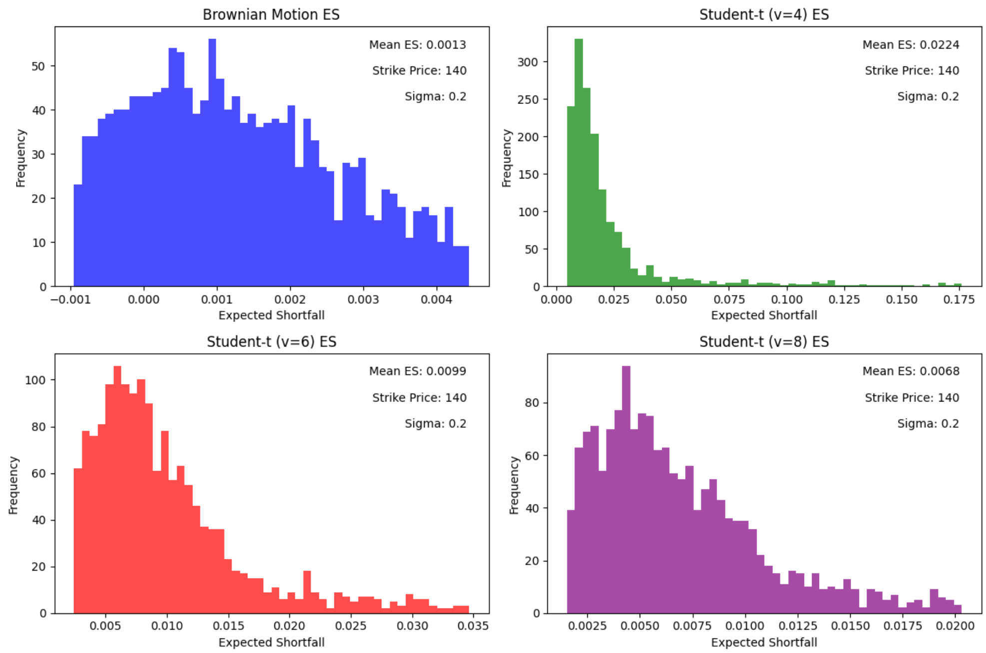

Figure 13: K=140 Sigma=0.2

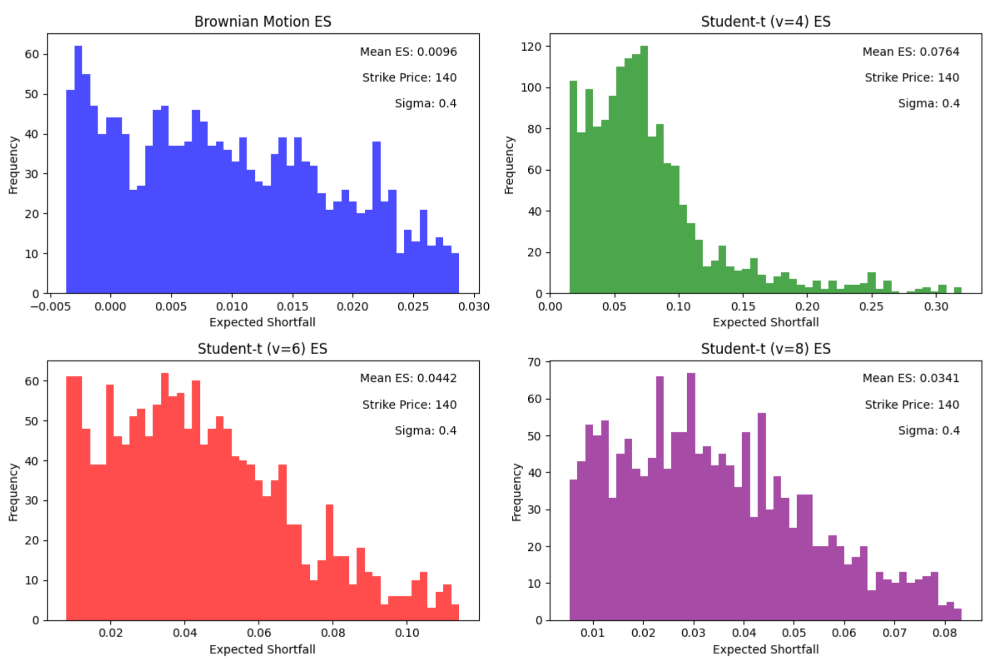

Figure 14: K=140 Sigma=0.4

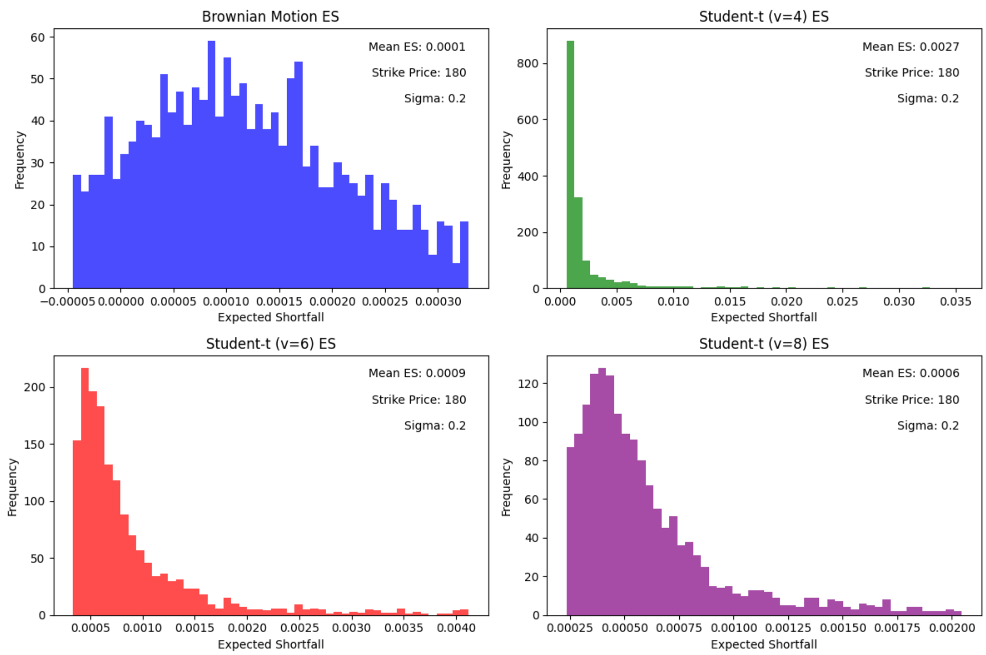

Figure 15: K=180 Sigma=0.2

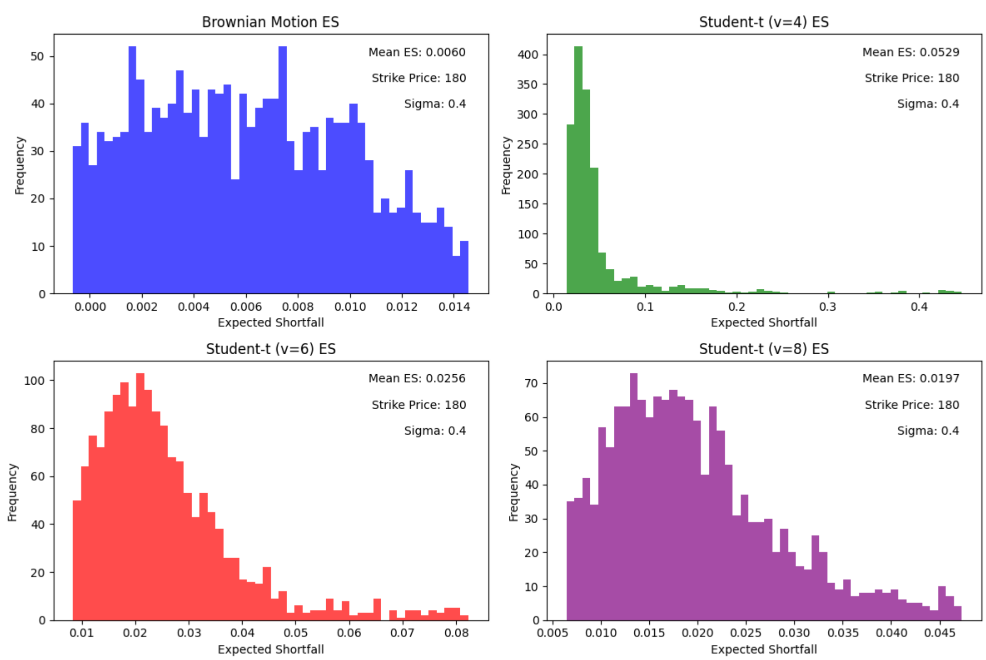

Figure 16: K=180 Sigma=0.4