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Option pricing and hedging with minimum local expected shortfall

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Abstract

We propose a versatile Monte-Carlo method for pricing and hedging options when the market is incomplete, for an arbitrary risk criterion (chosen here to be the expected shortfall), for a large class of stochastic processes, and in the presence of transaction costs. We illustrate the method on plain vanilla options when the price returns follow a Student-*t* distribution. We show that in the presence of fat-tails, our strategy allows us to significantly reduce extreme risks, and generically leads to low Gamma hedging. We also find that using an asymmetric risk function generates option skews, even when the underlying dynamics is unskewed. Finally, we show the proper accounting of transaction costs leads to an optimal strategy with reduced Gamma, which is found to outperform Leland's hedge.

1. Introduction

In their seminal 1973 article, Black and Scholes (BS) [7] have founded the very basis of modern financial mathematics. Their work has since been much studied and refined, and has become a rather abstract conceptual framework, deeply related with modern Probability Theory [19, 25]. The BS model is the paradigm of *complete markets*, where every contingent claims can be replicated by a portfolio of underlying assets. Using a no-arbitrage principle we can deduce that any option has a unique price, independent of the agent's risk preferences, which is given by the price of the replicating (or hedging) strategy. Mathematically speaking, these properties are equivalent to the existence of a unique equivalent martingale measure (also called risk-neutral measure) under which one should average the final pay-off of an option to obtain its price. This measure is in general different from the 'true' (or objective) real world probability measure (under which we observe the evolution of financial assets) [19, 25]. The knowledge of this true probability distribution is thus in principle of no use for the pricing of options, although, as discussed in detail

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in [6, 27], this message is in fact rather misleading. Within the BS framework, analytical formulae for the price and hedge exist for several cases, such as the simple European options. On the other hand, for more complex products like American or path dependent options, numerical procedures often have to be used [20, 36].

Faced with reality, the hypotheses of the BS model (Gaussian distribution and independence of log-returns, continuous time, absence of friction, etc.) have been widely questioned by practitioners and we can now observe a growing interest in the academic community for more general stochastic processes (Lévy processes [5, 6, 9, 10, 14], stochastic volatility [12, 18, 24, 28, 33], multi-fractal processes [8, 23, 26]) which usually result in much more complex *incomplete markets*.

In this case contingent claims cannot in general be replicated using underlying assets and the no-arbitrage principle is no longer enough to yield a unique price. In the presence of an incomplete market, it usually exists as an infinite set of equivalent martingale measure compatible with the no-arbitrage principle, each one leading to an arbitrage-free price for the contingent claim. In a wide class of models, the obtained range of prices has trivial bounds, resulting in a large spectrum of prices incompatible with the observed bid-ask spread and therefore cannot be used practically [4, 13]. For instance, in the case of a European call option, the super replication strategy, which ensures a succesful hedge under any martingal measure, is simply to buy and hold, leading to a price equal to the initial price of the stock. In order to keep a reasonable price, the trader has therefore to abandon the paradigm of perfect hedging. To select a strategy, he then has to introduce additional criteria related to its own risk preferences and accept the existence of an intrinsic residual risk [6].

Several approaches have been proposed to solve this problem. One is known as utility based derivative pricing [11]. The idea is, given a certain utility function, to find the 'indifference price' for the derivative security, that makes the agent indifferent between including or not the claim in his trading portfolio. In practice we need to specify a functional form for the utility. Although this should carefully reflect the preferences structure of the trader, computational purposes usually impose simplistic shape (like an exponential form) in order for the problem to remain tractable.

Another way is to select the strategy according to an optimal criteria. In this framework one simple and popular methodology is the variance-minimizing hedging strategy [6, 31]. It consists in finding the (self-financing) portfolio whose difference with the pay-off of the option at maturity has the smallest variance. One drawback of this approach is that the risk-function is quadratic and therefore penalizes both profits and losses. Also, a quadratic measure of risk does not strongly penalize extreme risks. Alternative criteria based on higher moments of the distribution or on the notion of Value at Risk (VaR) and its extensions [1] have been considered recently in the literature [6, 16, 29, 30]. In [16] the authors are interested in finding, given an initial investment, the portfolio strategy which maximizes the probability of a successful hedge. Treated in a general setting and in a rather abstract way, this problem is shown to reduce to the replication of a particular knockout option. Although very appealing, this solution can be hard to implement from a practical point of view. Indeed an explicit form of the option is only available in specific models and even in that case, as highlighted by the authors, the practical replication of such an option is not an easy task. The approach of [29] is much more concrete and numerically oriented. Nevertheless it is unsatisfactory from an optimization point of view: it is a static approach (the weights are determined at the initial time and remain constant) not well adapted to determine the full dynamical replication strategy. A similar observation can be made concerning the work presented in [30], where the optimal static strategy that minimizes the fourth moment of the profit and loss distribution is determined. The approximate dynamical strategy is then constructed by 'translating' (in time) the optimal static strategy, treating the current time as the new initial time. An interesting observation made in that work is that the hedging strategy varies less rapidly with the underlying than the quadratic hedge, implying lower transaction costs (see also [6]).

In practice, the trader usually specifies *a priori* a model with certain parameters for the underlying price process and then uses market data such as quotes of vanilla options to calibrate the model to consistently price other exotic products [2]. The main disadvantage of such an approach is that it focuses on the pricing part of the problem and says little about the hedging strategy to be followed.

The aim of this paper is to propose a general Monte-Carlo algorithm that allows to simultaneously price and hedge options using an arbitrary (but sufficiently smooth) risk criterion, and for a large class of stochastic processes describing the underlying asset. The algorithm, that generalizes the work of [27] (see also [37]), which only treats the case of a quadratic risk measure, is both easy to implement and versatile, and can be used for pricing different types of options. The paper is organized as follows: we first explain, following [21], the main ideas of our methodology. We then present some numerical results for different underlying processes, which show that extreme risks can be efficiently reduced compared to the standard BS hedge. As in [30], we find that these extreme hedges have a smaller 'Gamma' at and around the money. Finally, we show how transaction costs can be treated within our method, and discuss several other possible extensions of our scheme.

2. Description of the method

For simplicity we consider the case of a European option with one underlying asset of maturity $T = N\tau$, where N is the number of rebalancing dates and τ the time interval between two dates. We denote the price of the underlying asset at time $t_k = k\tau$ by x_k, the strike by K and the final pay-off is $(x_N - K)_+ \equiv \max(x_N - K, 0)$. We suppose that the price of the option only depends on the current price x_k of the asset and call it by $\mathcal{C}_k(x_k)$ at time t_k[1]. The interest rate is assumed to be constant and equal to r. Averaging (denoted by angled brackets $\langle \cdots \rangle$) will in the following always refer to the *objective* (real world) probability measure under which we observe the distribution of the asset returns, and *not* any abstract risk neutral measure.

2.1. Principles

The method we investigate here is an extension of the hedged Monte-Carlo strategy presented in [27]. Our aim is to construct a self-financing portfolio, whose wealth variation only depends on the variation of the asset price [25], that best minimizes the chosen (instantaneous) risk measure. We denote by $\phi_k(x_k)$ the fraction of the underlying asset in the portfolio at time k, when the asset price is x_k. Between time t_k and t_{k+1} the self-financing condition leads to a local wealth balance given by [27]:

$$\Delta W_k = \mathrm{e}^{r\tau}\mathcal{C}_k(x_k) - \mathcal{C}_{k+1}(x_{k+1}) + \phi_k(x_k)(x_{k+1} - \mathrm{e}^{r\tau}x_k) \tag{1}$$

The measure of the quality of the replication is given by a local risk function $\mathcal{U}(\Delta W_k)$. The average risk, over all paths of the real process, is thus given by:

$$\mathcal{R}_k = \langle \mathcal{U}(\Delta W_k) \rangle \tag{2}$$

For purposes of illustration, we have chosen in the following a function $\mathcal{U}(\Delta W_k)$ which penalizes losses that exceed a certain threshold $-\Delta_0$:

$$\mathcal{U}(\Delta W_k) = (\Delta_0 - \Delta W_k)_+^q = |\Delta_0 - \Delta W_k|^q \mathbf{1}_{\Delta W_k < \Delta_0} \tag{3}$$

where the exponent q was chosen to be $q = 1$, corresponding to what we called an unconditional expected shortfall. This kind of measure of risks has been introduced in finance by the works of Artzner and co-authors about coherent risk measures [1]. Since then, closely related concepts (Conditional VaR, Tail VaR or Mean Excess Loss for instance) have been used by many other authors [3, 29, 34]. The generalization to arbitrary q, or in fact to other functional forms for $\mathcal{U}(\Delta W_k)$, does not lead to any numerical difficulty ($q > 1$ penalizes more strongly extreme losses). The minimization of $\mathcal{R}_k$ is quite sensible from a financial point of view: we try to control in a marked to market way, during the whole life of the option, the occurrence of downside moves. Choosing a large negative Δ_0 means that we aim at controlling extreme losses.

From a practical point of view we solve the above optimization problem using Monte Carlo simulations, that allows us to use a rather general stochastic process for the price evolution. We generate N_{MC} trajectories of the asset price over which we will average. Following [22, 27], we decompose the functions $\mathcal{C}_k(x)$ and $\phi_k(x)$ on a set of p fixed basis functions[2]:

$$\mathcal{C}_k(x) = \sum_{a=1}^{p} \gamma_a^k C_a^k(x) \tag{4}$$

$$\phi_k(x) = \sum_{a=1}^{p} \varphi_a^k F_a^k(x) \tag{5}$$

Doing this we reduce the original functional optimization (find the functions ϕ_k and $\mathcal{C}_k$) to a numerical optimization: we now have a minimization problem in term of the parameters γ_a^k and φ_a^k. If p is large enough we expect to have a good approximation of the true functional solution (see the following section for numerical implementation). We solve the problem by working backward in time from maturity, where the option is worth its known final pay-off. For each time k, we decompose the problem into the following steps:

- If the time discretization mesh is sufficiently small, we can approximate $\mathcal{C}_k(x_k)$ by $\mathcal{C}_{k+1}(x_k)$ whose functional form is already known from the previous step[3]. We then find the coefficients φ_a^k which minimize the average risk over the N_{MC} paths:

$$\mathcal{R}_k^* = \sum_{\ell=1}^{N_{MC}} \Bigg(\Delta_0 - \big(\mathrm{e}^{r\tau} C_{k+1}(x_k^\ell) - C_{k+1}(x_{k+1}^\ell)\big) - (x_{k+1}^\ell - \mathrm{e}^{r\tau}x_k^\ell) \sum_{a=1}^{p} \varphi_a^k F_a^k(x_k^\ell) \Bigg)_+ \tag{6}$$

This can be done using a steepest gradient method. Indeed we can easily compute the partial derivative of $\mathcal{R}_k^*$ with respect to the coefficients φ_a^k:

$$\frac{\partial \mathcal{R}_k^*}{\partial \varphi_a^k} = -\sum_{\ell=1}^{N_{MC}} (x_{k+1}^\ell - \mathrm{e}^{r\tau}x_k^\ell) F_a^k(x_k^\ell) \mathbf{1}_{\Delta W_k^\ell < \Delta_0} \tag{7}$$

- Using the fact that, on average, the local wealth balance

[1] If the volatility was stochastic, we should assume that the option price also depends on the current level of volatility σ_k and rather write $\mathcal{C}_k(x_k, \sigma_k)$.

[2] A different possibility, that we use in section 4 below, is to write ϕ_k and $\mathcal{C}_k$ as simple functions (with correct asymptotic behaviours), parameterized by a few numbers that are determined by the optimization.

[3] A better approximation that takes into account the (already known) time derivative of $\mathcal{C}_k(x_k)$ is to write $\mathcal{C}_k(x_k) \simeq 2\mathcal{C}_{k+1}(x_k) - \mathcal{C}_{k+2}(x_k)$.

must equal zero, we now compute the coefficients γ_a^k by solving the least square problem:

$$\min_{\gamma} \sum_{\ell=1}^{N_{MC}} \left[\sum_{a=1}^{p} \gamma_a^k C_a^k(x_k^\ell) - \mathrm{e}^{-r\tau}(\mathcal{C}_{k+1}(x_{k+1}^\ell) - \phi_k(x_k^\ell)(x_{k+1}^\ell - e^{r\tau} x_k^\ell)) \right]^2 \tag{8}$$

which is done using standard procedures [35].

In summary, our method consist of two main steps. We first generate a set of trajectories. Averaging over these paths will allow us to compute the risk function at each time step. Although we focused on simulated series, we could also use, as in [27], purely historical datas to construct the trajectories.

Finding the optimal hedging policy and the option price is then an optimization problem, solved backward in time starting from the maturity date. This functional minimization is reduced to a finite-dimensional one by using a suitable parametrized form for the option price and the hedging strategy.

3. Numerical results

In this section we compare the results obtained following a standard Black and Scholes strategy (delta hedging) with those obtained following strategies with different values of the threshold Δ_0, as explained above.

3.1. Implementation issues

We price a European option, with a maturity of 1 year and an annualized volatility $\sigma = 20\%$. We choose a rather small number of time intervals when re-hedging is possible, $N = 10$. The initial stock price is $x_0 = 100$. We use $N_{MC} = 20\,000$ trajectories for averaging. We first consider realizations of a standard geometric Brownian motion with a constant drift $\mu = 0.05$:

$$\mathrm{d}x_t = x_t(\mu \mathrm{d}t + \sigma \mathrm{d}W_t)$$

It is well known that financial time series are very poorly represented by such a process and display much heavier tails. To qualitatively account for this fact we also use realizations of a fat-tailed process where we replaced the previous Brownian motion W_t by a fat-tailed process L_t whose increments are distributed according to a Student-t distribution, with $\nu = 4$ or $\nu = 6$ degrees of freedom.

The value of ν characterizes the power-law decay of the distribution for large arguments; $\nu = 4$ is in the range of reported value for this exponent for rather liquid markets. The value $\nu = 6$ corresponds to faster decaying tails; the limit $\nu \to \infty$ corresponds to the usual Black–Scholes model. Some markets (like emerging country markets, or emerging country currencies) would correspond to small values of ν (for example $\nu \approx 1.5$ for the Mexican Peso). We have in fact truncated the very far tail of the Student distribution in order to make the problem mathematically well defined, with no influence on the following numerical results. We use $p = 20$ basis functions, which we find to be accurate enough[4]. Following [27] we choose for F_a^k piecewise linear functions and for C_a^k piecewise quadratic functions with both the same adaptive breakpoints. These breakpoints are chosen so that at each stage the same number of simulated trajectories, $N_{MC}/(p+2)$, fall between two successive breakpoints. F_a^k is worth 0 below the ath breakpoint, 1 above the $(a+1)$th breakpoint and is linear between these two values. C_a^k is taken as the integral of F_a^k which is worth 0 below the ath breakpoint. Our numerical simulations were systematically conducted as follows: we first generate a set of trajectories and apply our algorithm to find the coefficients γ_a^k and φ_a^k, i.e., the price of the option and the optimal hedge. We then simulate a new large set of paths (with 10^6 trajectories) to compute and compare different statistical indicators of the performance of the proposed strategies. In other words, different paths are used for the optimization, and for back-testing the optimization in an 'out of sample' fashion.

3.2. Expected shortfall hedging in the Black–Scholes case

We first present the results obtained within the framework of the Black and Scholes model. Since perfect replication is theoretically possible in this case, we expect that the minimum short-fall strategy we find should be close to the Black and Scholes strategy. In the continuous time limit, the Black–Scholes strategy actually leads to a zero expected shortfall, for any value of $\Delta_0 < 0$. This is indeed what we observe in figure 1 where we plot, for a strike price equal to $K = 110$, the optimal solutions found with different values of Δ_0 and the BS strategy: they all look very similar, in particular for small values of Δ_0 (-0.5 and -1).

This observation is confirmed by the shape of the distribution of the final wealth at the maturity date of the option, for the different strategies (figure 2).

3.3. The case of a fat-tailed dynamics

We are now interested in a market whose dynamics is governed by a fat-tailed Lévy process, where the

[4] The aim of the present paper is not to discuss in detail the optimal choice of p and of the form of the basis functions, but rather to demonstrate the overall feasability of the method.

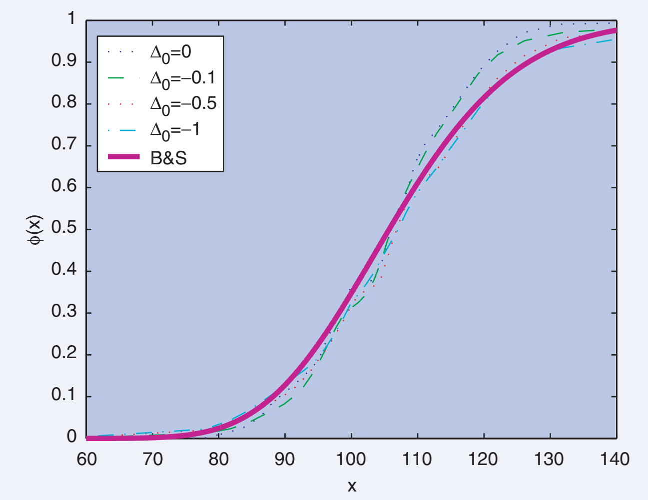

Figure 1. Optimal number of risky assets ϕ in the hedging portfolio, as a function of the level x of the underlying asset for different strategies within the Black–Scholes. As expected the different curves are very similar. The value of the strike is $K = 110$, and the time is half the maturity of the option $k = N/2$. The parameters of the model are $\mu = 5\%$, $r = 3\%$ and $\sigma = 20\%$.

relative price increments are independent identically distributed with jumps. In this case the market is no longer complete (existence of unhedgeable jumps) and perfect replication does not exist. The use of a subjective criteria is then needed for pricing and hedging purposes, we now expect to get different strategies depending on the value of Δ_0. Our expectations are numerically confirmed in figure 3 where we clearly see strong differences between the strategies. In particular, we observe that extreme losses hedging (corresponding to a large value of $|\Delta_0|$) leads to a flatter function $\phi(x)$ (i.e. a smaller Gamma). This was already emphasized in [6, 30] and can be quite interesting in the presence of transaction costs (see section 4).

Using an independent set of paths (10^6 trajectories), we can check that our method indeed leads to smaller local expected shortfalls when the corresponding optimal strategy is adopted, as can be seen in tables 1 and 2, where we give both the unconditional expected shortfall $\mathcal{R}$ (the quantity we used during the optimization part), and the conditional expected shortfall (noted ESF), defined as $\mathcal{R}/\mathcal{P}$, where $\mathcal{P}$ is the probability to exceed the threshold Δ_0, as well as 95%-confidence error bars computed using the variance of the Monte-Carlo results divided by the square-root of the number of relevant trajectories (i.e. those exceeding the threshold), which is here large enough to be confident in the statistical significance of our results. The expected shortfalls are computed between two re-hedging times k and $k+1$, where $k = N/2$ corresponds to half the life of the option.

We can also check that our method leads to satisfactory results for *global* quantities (i.e. concerning the wealth balance at the end of the option lifetime). We therefore determined the distribution of the final wealth (figure 4 and 5). We clearly see that the strategy proposed here can significantly reduce the value of the extreme losses (note that because of the power-tails of the return distribution, these extreme losses can still be large). Moreover we observe a remarkable change in the shape of this distribution (in particular, note the change in the scale of the y-axis): from relatively peaked for small $|\Delta_0|$ but with an appreciable number of extreme events, it becomes broader but with a smaller support when $|\Delta_0|$ increases. This fact can be qualitatively explained: when $|\Delta_0|$ is large, the constraint is easier to fulfill but losses of amplitude less than $|\Delta_0|$ are not penalized, leading to a broader looking but more sharply truncated final distribution.

We give in tables 3 and 4 the mean and the standard deviation of these final wealth distributions, as well as the initial price of the option and the associated value at risk and expected shortfalls (with error bars). Because of the non-Gaussian nature of our undelying process, we should really focus more on quantities like VaR to compare the different strategies. Note also that the option price is smaller than the Black–Scholes price, which is expected when the moneyness is small: non-zero kurtosis indeed leads to a *decrease* of the at-the-money volatility [6] as can be seen in figure 6. The option price in fact decreases when larger risks are hedged.

In figure 6, we interestingly find that the option smile develops a skewed form when the risk function used is asymmetric, even if (as is the case here) the dynamics of the underlying asset is symmetrical. We also note that this asymmetry increases with that of our risk function. Such skewed smiles are often found on stock options or index options and the results reported here suggest that at least part of this empirically observed skew could come from the asymmetric risk aversion of option sellers, a very reasonable assumption indeed.

4. Application to transaction costs

As mentioned above, hedging against extreme risks generically leads to a strategy that varies more slowly with the underlying asset price. This can be of great interest in the presence of transaction costs. These costs can in fact be endogenously taken into account within the present numerical scheme, which allows us to determine how both the price and the optimal hedge are impacted by transaction costs. Previous analytical work on this problem in the framework of the BS model can be found in [21, 36], and further discussions and extensions can be found in [15, 17, 32].

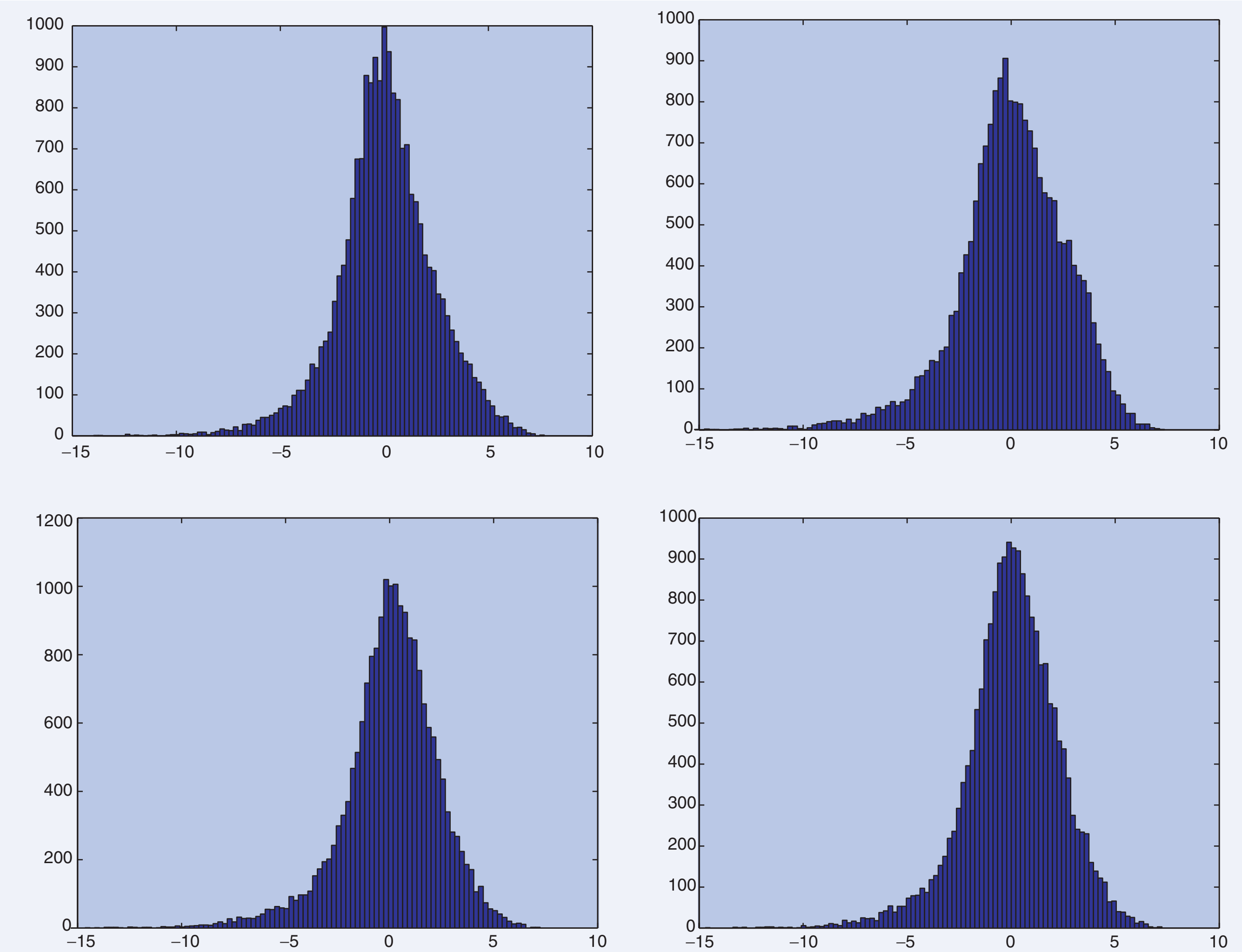

Figure 2. Distribution of the final wealth for different strategies for a BS market (upper left, BS, upper right, $\Delta_0 = 0$, lower left, $\Delta_0 = -0.5$, lower right, $\Delta_0 = -1$). Because of the existence of a theoretical perfect hedging strategy, the four curves don't display significant differences. The value of the strike is $K = 110$. The parameters of the model are $\mu = 5\%$, $r = 3\%$ and $\sigma = 20\%$.

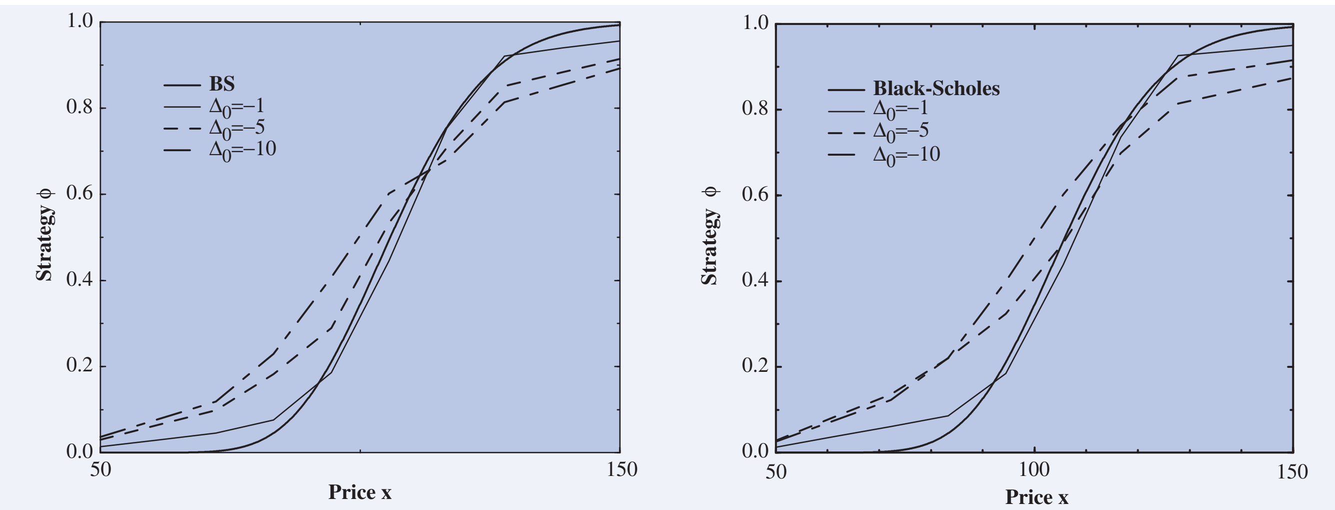

Figure 3. Optimal number of risky assets ϕ in the hedging portfolio, as a function of the level x of the underlying asset for different strategies in the case of a fat-tailed, incomplete, market (left: $\nu = 6$ and right: $\nu = 4$). The hedging strategies now present a very different dependence on the underlying, in particular when $|\Delta_0|$ increases. The value of the strike is $K = 110$ and the time is half the maturity of the option: $k = N/2$. The parameters of the model are $\mu = 5\%$, $r = 3\%$ and $\sigma = 20\%$.

Table 1. Conditional expected shortfall (ESF) and unconditional expected shortfall $\mathcal{R}$, for different threshold Δ_0 and different strategies, in the case $\nu = 6$. These *local* quantities are computed for a time equal to half the maturity of the option (here 1 year), for a strike K=110. The parameters of the model are $\mu = 5\%$, $r = 3\%$ and $\sigma = 20\%$. We added the 95%-confidence error bars, proportional to the variance of the Monte–Carlo results divided by the square-root of the number of relevant trajectories (i.e. those exceeding the corresponding threshold).

Strategy	BS	$\Delta_0 = 0$	$\Delta_0 = -5$	$\Delta_0 = -10$
ESF(0)	$-0.78 \pm \times 5 \times 10^{-3}$	$-0.89 \pm 5 \times 10^{-3}$	$-0.92 \pm 4 \times 10^{-3}$	$-1.20 \pm 4 \times 10^{-3}$
ESF(−5)	-2.94 ± 0.11	-2.96 ± 0.09	-2.52 ± 0.09	-2.07 ± 0.05
ESF(−10)	-4.54 ± 0.36	-4.32 ± 0.31	-3.95 ± 0.33	-3.19 ± 0.25
$\mathcal{R}(0)$	$-0.23 \pm 1.4 \times 10^{-3}$	$-0.25 \pm 1.6 \times 10^{-3}$	$-0.32 \pm 1.4 \times 10^{-3}$	$-0.45 \pm 1.7 \times 10^{-3}$
$\mathcal{R}(-5)$	$-0.014 \pm 5 \times 10^{-4}$	$-0.018 \pm 6 \times 10^{-4}$	$-0.013 \pm 5 \times 10^{-4}$	-0.017 ± 5.10^{-4}
$\mathcal{R}(-10)$	$-0.004 \pm 3 \times 10^{-4}$	$-0.004 \pm 3 \times 10^{-4}$	$-0.003 \pm 2 \times 10^{-4}$	-0.002 ± 2.10^{-4}

Table 2. Conditional expected shortfall (ESF) and unconditional expected shortfall $\mathcal{R}$, for different threshold Δ_0 and different strategies, in the case $\nu = 4$. These *local* quantities are computed for a time equal to half the maturity of the option (here 1 year), for a strike K = 110. The parameters of the model are $\mu = 5\%$, $r = 3\%$ and $\sigma = 20\%$. We added the 95%-confidence error bars, proportional to the variance of the Monte-Carlo results divided by the square-root of the number of relevant trajectories (i.e. those exceeding the threshold).

Strategy	BS	$\Delta_0 = 0$	$\Delta_0 = -5$	$\Delta_0 = -10$
ESF(0)	$-0.92 \pm 9 \times 10^{-3}$	-1.06 ± 0.01	$-1.06 \pm 7 \times 10^{-3}$	$-1.18 \pm 6 \times 10^{-3}$
ESF(−5)	-4.87 ± 0.24	-4.78 ± 0.20	-4.12 ± 0.18	-3.46 ± 0.13
ESF(−10)	-8.20 ± 0.73	-8.14 ± 0.65	-7.58 ± 0.68	-6.48 ± 0.57
$\mathcal{R}(0)$	$-0.24 \pm 2.3 \times 10^{-3}$	$-0.25 \pm 2.4 \times 10^{-3}$	$-0.35 \pm 2.3 \times 10^{-3}$	$-0.41 \pm 2.2 \times 10^{-4}$
$\mathcal{R}(-5)$	$-0.033 \pm 1.6 \times 10^{-3}$	$-0.038 \pm 1.7 \times 10^{-3}$	$-0.032 \pm 1.4 \times 10^{-3}$	$-0.033 \pm 1.3 \times 10^{-3}$
$\mathcal{R}(-10)$	$-0.016 \pm 1.4 \times 10^{-3}$	$-0.017 \pm 1.4 \times 10^{-3}$	$-0.013 \pm 1.2 \times 10^{-3}$	$-0.011 \pm 1 \times 10^{-3}$

We model these frictions by adding to the wealth balance a cost proportional to the price and number of bought or sold assets. Equation (1) now becomes

$$\Delta W_k = e^{r\tau}\mathcal{C}_k(x_k) - \mathcal{C}_{k+1}(x_{k+1}) + \phi_k(x_k)(x_{k+1} - e^{r\tau}x_k) - \beta x_k|\phi_k - \phi_{k-1}| \tag{9}$$

where βx_k represents the transaction costs per share. Following the same steps as above, we now want to minimize the risk function

$$\mathcal{R}^*{}_k = \sum_{\ell=1}^{N_{MC}} \Bigg(\Delta_0 - \left(e^{r\tau}C_{k+1}(x_k^\ell) - C_{k+1}(x_{k+1}^\ell)\right) - (x_{k+1}^\ell - e^{r\tau}x_k^\ell)\phi_k(x_k^\ell) + \beta x_k^\ell|x_k^\ell - x_{k-1}^\ell|\frac{\partial\phi}{\partial x}(x_k^\ell)\Bigg)_+ \tag{10}$$

where we approximated $|\phi_k - \phi_{k-1}|$ by $\partial\phi/\partial x|\Delta x|$. This is justified if the time step is sufficiently small, and is the key step to make the problem tractable. However, since this involves the derivative of ϕ, we have preferred to work with a smooth parameterization of the function ϕ_k with only two optimization parameters, rather than the full decomposition over a set of basis functions, as was used above. We have checked that the following choice gives very similar results to the ones obtained above in the absence of transaction costs. We thus take:

$$\phi_k(x) = \frac{1}{2}\left(1 + [\tanh|A_k\mathcal{M}_k|]^{\beta_k}\,\mathrm{sign}(\mathcal{M}_k)\right)$$

with a rescaled moneyness $\mathcal{M}_k$ given by:

$$\mathcal{M}_k = \frac{x - Ke^{-r(T-t_k)}}{\sigma\sqrt{T-t_k}}$$

By varying the two variational parameters A_k and β_k, we can, at each time step, optimize any risk measure. Using essentially the same numerical optimization procedure as above we are able to find the parameters A_k and β_k, and thus the optimal strategy ϕ_k^*. In order to obtain the option price, we have then to solve the following least square problem:

$$\min_\gamma \sum_{\ell=1}^{N_{MC}}\Bigg[\sum_{a=1}^{p}\gamma_a^k C_a^k(x_k^\ell) - e^{-r\tau}\Bigg(\mathcal{C}_{k+1}(x_{k+1}^\ell) - \phi_k(x_k^\ell)(x_{k+1}^\ell - e^{r\tau}x_k^\ell) + \beta x_k^\ell|x_k^\ell - x_{k-1}^\ell|\frac{\partial\phi}{\partial x}(x_k^\ell)\Bigg)\Bigg]^2 \tag{11}$$

We have numerically tested the above scheme in the case of a BS market with the same characteristics as in the previous section, and for different values of the friction parameter β. We compared our results with those

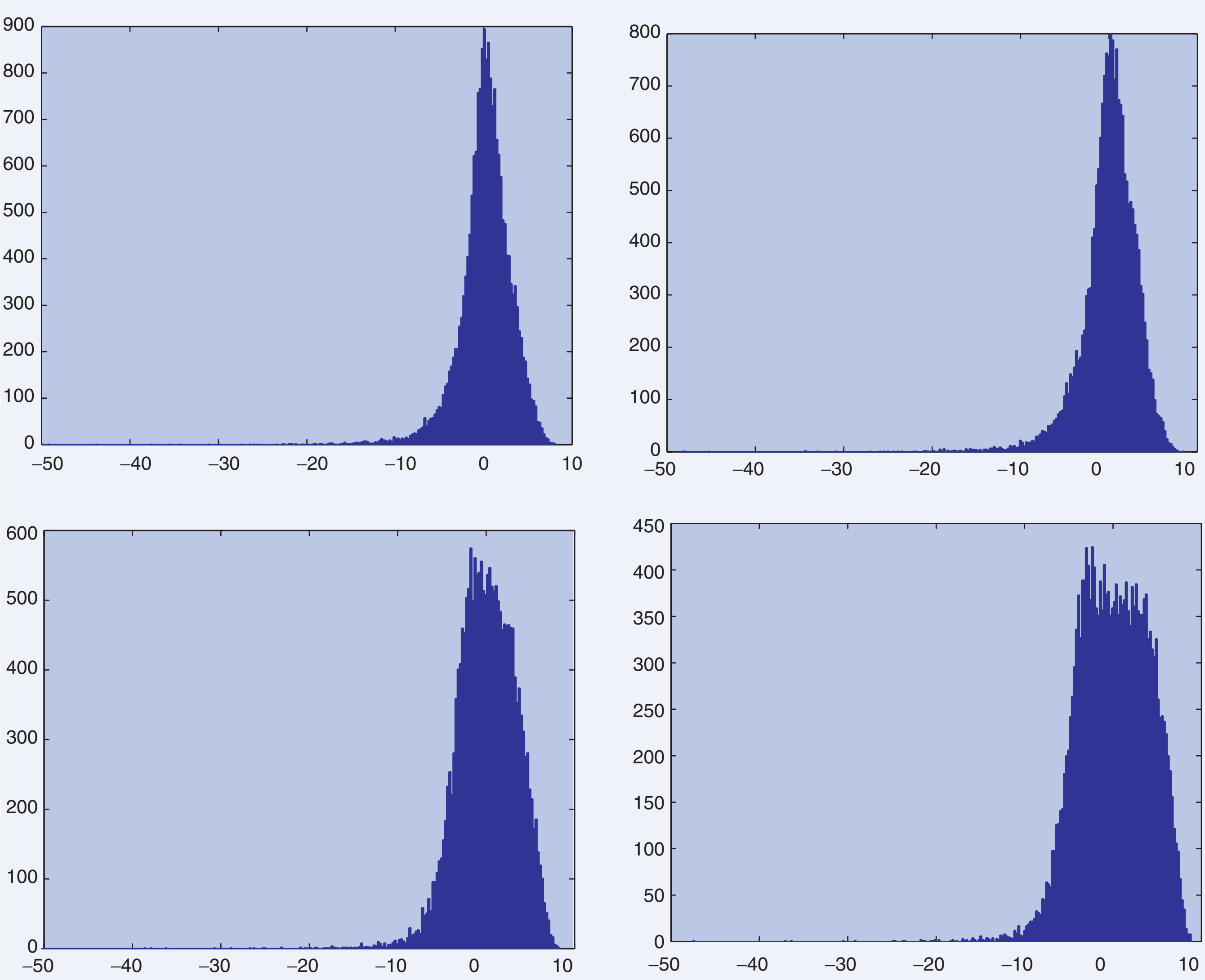

Figure 4. Distribution of the final wealth for different strategies in the case of a fat-tailed market with $\nu = 6$ (upper left, BS; upper right, $\Delta_0 = 0$; lower left, $\Delta_0 = -5$; lower right, $\Delta_0 = -10$). Due to the incompleteness of the market, the different strategies now display distinct characteristics which result in differences in the shape of the distributions of the final wealth. Note in particular the scales of the y-axis. The value of the strike is $K = 110$. The parameters of the model are $\mu = 5\%$, $r = 3\%$ and $\sigma = 20\%$.

obtained following a naive BS strategy and the more advanced Leland strategy. Using simple arguments, similar in spirit to the above approximation on $\Delta\phi$, Leland showed in [21] how the BS strategy can be modified to account for transaction costs. Indeed using a modified volatility

$$\sigma_L = \sigma\sqrt{1 + 2\beta\sqrt{\frac{2}{\pi}}\frac{1}{\sigma\sqrt{\Delta t}}}$$

instead of the real volatility σ leads to a strategy which, on average, approximately covers the transaction costs and hedges the risk. Since $\sigma_L > \sigma$, the option price is, as expected, higher than in the BS case. Figure 7 compares the obtained optimal strategies using our method with both the BS and Leland hedging schemes. As expected, the costs affects the BS strategy in such a way as to reduce its at-the-money Gamma. Tables 5, 6 and 7 give summary statistics of our results. As shown previously, we used $N_{MC} = 20\,000$ trajectories during the optimization part of our algorithm and we then simulated a new set of 10^6 trajectories for evaluating the performances of our strategy.

We note that even in the case where the stock price is log-normal, when the transaction costs are reasonable our strategy allows us to improve significantly over the Leland strategy if the threshold $|\Delta_0|$ is large enough (compare tables 5 and 6). Using the optimal strategy allows us to simultaneously reduce the occurrence of large risks (measures both by the VaR and the ESF) while keeping the option price lower than in the Leland scheme. For larger and quite extreme transaction costs, the comparison is not so obvious but our results, although not so good, are still competitive and present a much less expensive option price (table 7).

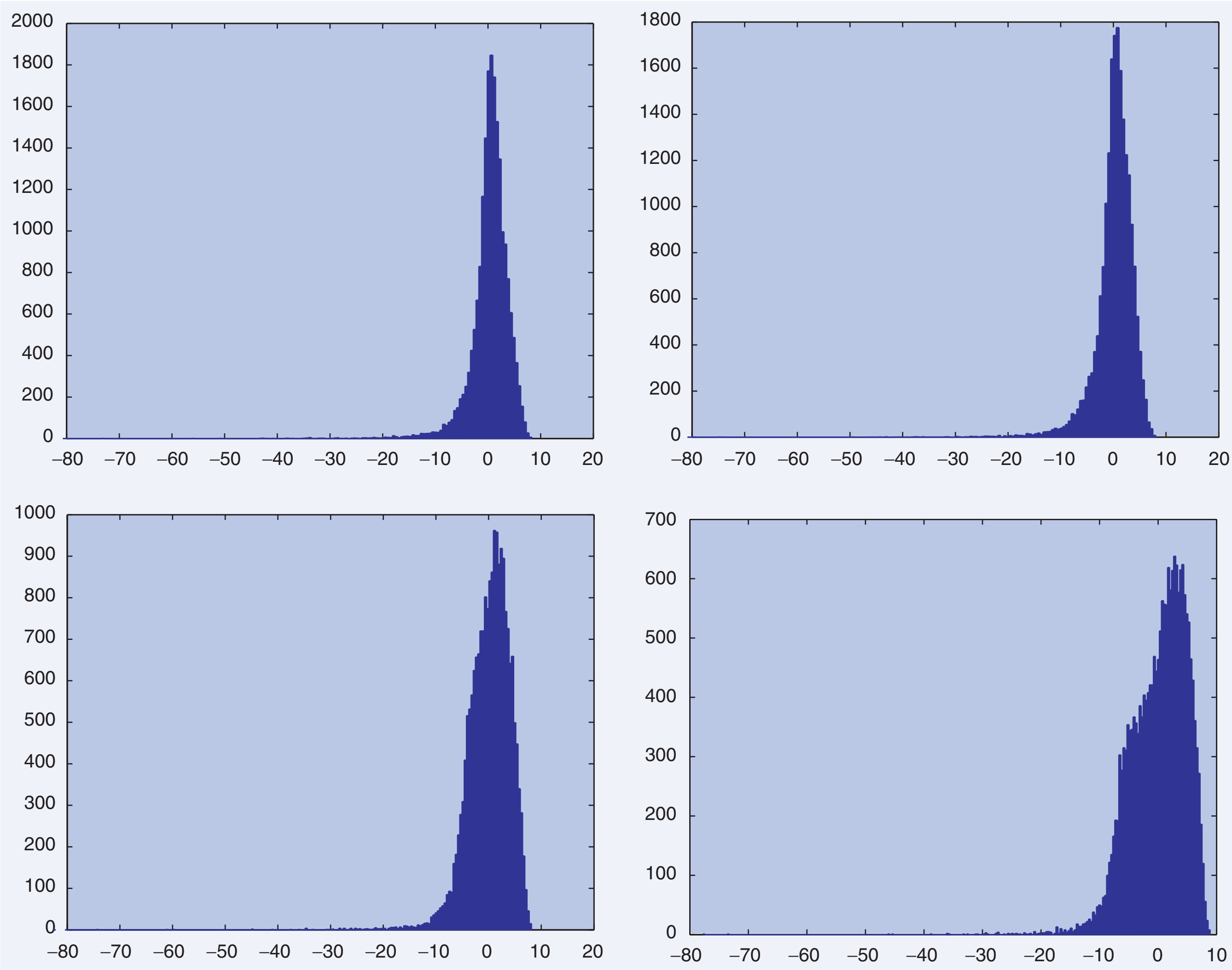

Figure 5. Distribution of the final wealth for different strategies in the case of a fat-tailed market with $\nu=4$ (upper left: BS, upper right: $\Delta_0=0$, lower left: $\Delta_0=-5$, lower right: $\Delta_0=-10$). As noted in the previous figure, the incompleteness of the market leads to differences in the shape of the final distribution of wealth. The strike is $K=110$ and the parameters of the model are $\mu=5\%$, $r=3\%$ and $\sigma=20\%$.

Table 3. Statistical characteristics of the global wealth balance for $\nu=6$. The strike is $K=110$ and the parameters of the model are $\mu=5\%$, $r=3\%$ and $\sigma=20\%$. Our strategy, depending on the choice of the threshold Δ_0, significantly overperforms the Black and Scholes strategy with both a smaller option price and smaller Value-at-Risk and Expected Shortfall. The 95%-confidence intervals confirm the statistical significance of our results.

Strategy	BS	$\Delta_0=0$	$\Delta_0=-5$	$\Delta_0=-10$
Option price	5.29	5.14	5.13	5.06
Mean of final wealth	0.11	−0.01	0.01	0.01
Std of final wealth	3.20	3.33	3.46	4.11
VaR 0.1%	−22.13	−22.73	−19.35	−18.14
	[−22.57; −21.79]	[−23.21; −22.33]	[−19.68; −18.97]	[−18.47; −17.90]
ESF 0.1%	−9.48 ± 0.90	−9.70 ± 0.91	−8.34 ± 0.78	−7.33 ± 0.68
VaR 1%	−10.32	−10.94	−9.36	−9.88
	[−10.39; −10.25]	[−11.01; −10.87]	[−9.41; −9.31]	[−9.94; −9.84]
ESF 1%	−5.08 ± 0.15	−5.12 ± 0.15	−4.27 ± 0.13	−3.60 ± 0.11
VaR 5%	−5.02	−5.53	−5.30	−6.37
	[−5.05; −5.00]	[−5.56; −5.50]	[−5.31; −5.28]	[−6.39; −6.36]
ESF 5%	−3.54 ± 0.04	−3.62 ± 0.04	−2.76 ± 0.03	−2.40 ± 0.03

Table 4. Statistical characteristics of the global wealth balance for $\nu = 4$. The strike is $K = 110$ and the parameters of the model are $\mu = 5\%$, $r = 3\%$ and $\sigma = 20\%$. The 95%-confidence intervals confirm the statistical significance of our results.

Strategy	BS	$\Delta_0 = 0$	$\Delta_0 = -5$	$\Delta_0 = -10$
Option price	5.29	5.07	4.98	4.89
Mean of final wealth	0.22	0.04	0.03	0.04
Std of final wealth	4.55	4.67	4.63	5.13
VaR 0.1%	−37.93	−39.20	−33.09	−29.63
	[−38.88; − 37.13]	[−40.25; − 38.31]	[−34.00; − 32.29]	[−30.41; − 28.83]
ESF 0.1%	-28.50 ± 3.22	-28.69 ± 3.25	-24.48 ± 2.78	-21.14 ± 2.47
VaR 1%	−14.00	−14.81	−12.76	−12.96
	[−14.14; − 13.88]	[−14.95; − 14.70]	[−12.88; − 12.66]	[−13.04; − 12.89]
ESF 1%	-10.95 ± 0.43	-11.10 ± 0.43	-9.24 ± 0.37	-7.63 ± 0.32
VaR 5%	−5.78	−6.33	−6.46	−7.86
	[−5.81; − 5.75]	[−6.36; − 6.30]	[−6.49; − 6.44]	[−7.88; − 7.84]
ESF 5%	-5.99 ± 0.11	-6.15 ± 0.10	-4.67 ± 0.09	-3.86 ± 0.07

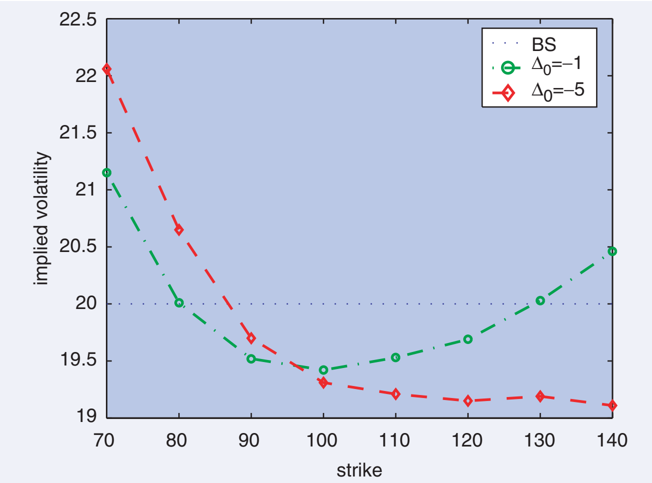

Figure 6. Implied volatility smile in a fat-tailed market ($\nu = 4$) at $t = 0$ for an initial price of the asset $S_0 = 100$. The parameters of the model are $\mu = 5\%$, $r = 3\%$ and $\sigma = 20\%$. Even if the underlying process remains symmetric, the obtained smile is skewed. Its skewness increases with the asymmetry of the risk function which is used. This suggests that at least part of the asymmetry observed in real volatility smile figures can be explained by the risk aversion of the agents.

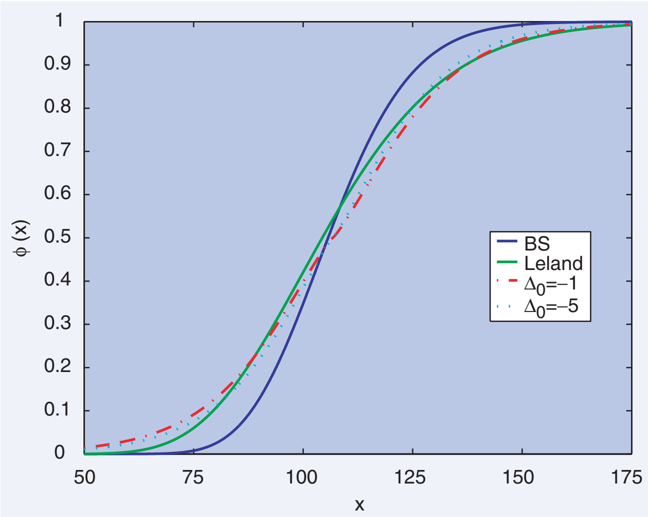

Figure 7. Optimal number of risky assets ϕ in the hedging portfolio, as a function of the level x of the underlying asset for different strategies: Black–Scholes, Leland and our strategy with $\Delta_0 = -1$ and $\Delta_0 = -5$. The market is Gaussian but imperfect with the existence of transaction costs. These are proportional to the current asset price with a multiplicative coefficient $\beta = 0.05$. The value of the strike is $K = 110$ and the parameters of the model are $\mu = 5\%$, $r = 3\%$ and $\sigma = 20\%$.

5. Conclusion

In this paper we have extended the work of [27] and proposed a general numerical (Monte-Carlo) methodology for the pricing and hedging of options when the market is incomplete, for an arbitrary risk criterion (chosen here to be the expected shortfall) and in the presence of transaction costs. We have shown that in the presence of fat-tails, our strategy allows us to significantly reduce extreme risks, and generically leads to low Gamma hedging, as anticipated in [6, 30]. Many other risk criteria could be considered, in particular functions that give more weights to extreme losses. In this work we focused on plain vanilla European options, but (as shown in [27]) the method is readily extended to a large family of exotic options. Finally, we showed how our method allows us to deal consistently with transaction costs. When compared to the standard Leland hedging scheme, our optimal strategy leads both to lower option prices and better hedging of large risks, even in the simplest case of a log-normal market.

There are many extensions of the above method that would be worth investigating, in particular the case where the underlying has a stochastic volatility with some persistence, such as, for example, the models studied in [12, 18, 23, 24, 26]. In this case, both the price and the optimal hedge should explicitly depend on the local value of the volatility, or of a noisy estimate of this volatility.

Table 5. Impact of the transaction costs for the different strategies. The parameters of the model are $\mu = 5\%$, $r = 3\%$, $\sigma = 20\%$ and $\beta = 0.005$. Our strategy with $\Delta_0 = -5$ significantly overperforms Leland's strategy with smaller Value-at-Risk and Expected Shortfall. The 95%-confidence intervals confirm the statistical significance of our results.

Strategy	BS	Leland	$\Delta_0 = -1$	$\Delta_0 = -5$
Option price	5.29	5.77	5.69	5.63
Mean of final wealth	−0.43	0.08	0.00	0.00
Std of final wealth	2.39	2.40	2.40	2.66
VaR 0.1%	−11.14	−10.16	−11.12	−8.93
	[−11.22; − 11.04]	[−10.25; − 10.06]	[−11.25; − 11.03]	[−9.00; − 8.86]
ESF 0.1%	-1.55 ± 0.09	-1.51 ± 0.09	-1.63 ± 0.09	-1.18 ± 0.07
VaR 1%	−7.30	−6.48	−7.04	−6.10
	[−7.34; − 7.27]	[−6.51; − 6.45]	[−7.09; − 7.01]	[−6.14; − 6.08]
ESF 1%	-1.66 ± 0.03	-1.59 ± 0.03	-1.77 ± 0.03	-1.23 ± 0.02
VaR 5%	−4.50	−3.84	−4.05	−4.05
	[−4.51; − 4.48]	[−3.85; − 3.83]	[−4.06; − 4.03]	[−4.07; − 4.04]
ESF 5%	-1.73 ± 0.01	-1.63 ± 0.01	-1.84 ± 0.01	-1.27 ± 0.01

Table 6. Impact of the transaction costs for different strategies. The parameters of the model are $\mu = 5\%$, $r = 3\%$, $\sigma = 20\%$ and $\beta = 0.01$. Our strategy with $\Delta_0 = -5$ still significantly overperforms Leland's strategy. Note also that the option price, although higher than the BS price, is less than Leland's price. The statistical significance of our results is confirmed by the 95%-confidence intervals.

Strategy	BS	Leland	$\Delta_0 = -1$	$\Delta_0 = -5$
Option price	5.29	6.22	6.07	5.98
Mean of final wealth	−0.92	0.11	−0.04	−0.05
Std of final wealth	2.46	2.45	2.44	2.74
VaR 0.1%	−12.00	−10.08	−11.08	−9.25
	[−12.11; − 11.90]	[−10.18; − 9.97]	[−11.20; − 10.97]	[−9.33; − 9.18]
ESF 0.1%	-1.61 ± 0.09	-1.51 ± 0.09	-1.59 ± 0.09	-1.20 ± 0.07
VaR 1%	−8.11	−6.48	−7.07	−6.36
	[−8.14; − 8.08]	[−6.50; − 6.44]	[−7.11; − 7.04]	[−6.39; − 6.34]
ESF 1%	-1.69 ± 0.03	-1.56 ± 0.03	-1.73 ± 0.03	-1.26 ± 0.02
VaR 5%	−5.19	−3.87	−4.13	−4.25
	[−5.21; − 5.17]	[−3.88; − 3.85]	[−4.15; − 4.12]	[−4.26; − 4.24]
ESF 5%	-1.80 ± 0.02	-1.61 ± 0.01	-1.80 ± 0.01	-1.30 ± 0.01

Table 7. Impact of the transaction costs for different strategies. The parameters of the model are $\mu = 5\%$, $r = 3\%$, $\sigma = 20\%$ and $\beta = 0.05$. Our strategy with $\Delta_0 = -1$ displays still competitive results with respect to Leland's strategy but with a much less expensive price. The 95%-confidence intervals confirm the statistical significance of our results.

Strategy	BS	Leland	$\Delta_0 = -1$	$\Delta_0 = -5$
Option price	5.29	9.27	8.29	8.44
Mean of final wealth	−4.80	0.30	−0.26	−0.76
Std of final wealth	3.30	2.99	3.09	3.40
VaR 0.1%	−19.65	−10.31	−10.93	−12.00
	[−19.78; − 19.51]	[−10.38; − 10.21]	[−11.02; − 10.84]	[−12.10; − 11.92]
ESF 0.1%	-1.88 ± 0.11	-1.39 ± 0.09	-1.41 ± 0.08	-1.35 ± 0.08
VaR 1%	−14.89	−6.84	−7.59	−8.66
	[−14.94; − 14.85]	[−6.87; − 6.81]	[−7.63; − 7.57]	[−8.69; − 8.63]
ESF 1%	-2.11 ± 0.04	-1.50 ± 0.03	-1.44 ± 0.03	-1.46 ± 0.02
VaR 5%	−11.06	−4.31	−5.09	−6.09
	[−11.08; − 11.04]	[−4.32; − 4.29]	[−5.11; − 5.08]	[−6.11; − 6.08]
ESF 5%	-2.36 ± 0.02	-1.56 ± 0.01	-1.54 ± 0.01	-1.57 ± 0.01

Other hedging instruments, like options of different maturities, could in this case be included in the local wealth balance to reduce the risk further. Another interesting path is to consider the problem of hedging a whole portfolio of options. Contrarily to the Black–Scholes case, where the optimal strategy for the whole portfolio is the linear sum of the individual hedges, extreme value hedges lead to a non-linear composition of the individual hedges.

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