MATH2099 PROBLEM SHEET 1
GAUSSIAN ELIMINATION

Any augmented matrix may be reduced to echelon form via the elementary row operations

$$R_i = R_i \pm \alpha R_j \qquad \text{and} \qquad R_i \leftrightarrow R_j$$

Once in echelon form the system may be solved via back-substitution.

An inconsistent equation at any stage of reduction indicates that there is no solution and you may stop. Else

If every column on the LHS of the echelon form is a leading column then the solution is unique. Else

The presence of a non-leading column on the LHS of the echelon form indicates infinite solutions with the non-leading variables then serving as parameters.

Harder problems are marked with a ★

1. Which of the following augmented matrices are in echelon form? For those in echelon form, circle all the leading elements, identify the leading columns and discuss the nature of solution. For echelon forms with infinite solutions identify the variables which need to be assigned as parameters. Note that you are not being asked to solve the systems.

a) $\left(\begin{array}{ccc|c} 5 & 4 & 7 & 4 \\ 0 & 3 & -1 & 5 \\ 0 & 8 & 8 & 2 \end{array}\right)$ b) $\left(\begin{array}{ccc|c} 5 & 0 & 0 & 14 \\ 0 & 2 & 1 & 6 \\ 0 & 0 & 8 & 7 \end{array}\right)$

c) $\left(\begin{array}{ccc|c} 9 & 5 & 1 & 4 \\ 0 & 3 & -1 & 6 \\ 0 & 0 & 0 & 5 \end{array}\right)$ d) $\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right)$

e) $\left(\begin{array}{ccccc|c} 3 & 5 & 1 & 0 & 2 & 4 \\ 0 & 0 & 0 & 8 & 1 & 6 \end{array}\right)$ f) $\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 4 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$

g) $\left(\begin{array}{ccc|c} 1 & 4 & 7 & 4 \\ 0 & 5 & -1 & 6 \\ 0 & 6 & 0 & 0 \end{array}\right)$ h) $\left(\begin{array}{ccc|c} 1 & 3 & 9 & 1 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 8 & 0 \end{array}\right)$

i) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 1 & 6 & 0 & 6 \\ 1 & 6 & 2 & 5 \end{array}\right)$ j) $\left(\begin{array}{ccc|c} 0 & 0 & 8 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

2. Solve each of the following systems of linear equations by reducing to echelon form and then back-substituting if a solution exists. Interpret each problem geometrically.

$$a) \quad \begin{aligned} x + y + z &= 7 \\ x + 2y + z &= 8 \\ x + 3y + 2z &= 13 \end{aligned}$$

$$b) \quad \begin{aligned} x + y + z &= 7 \\ x + 2y + 3z &= 16 \\ 2x + 3y + 4z &= 22 \end{aligned}$$

$$c) \quad \begin{aligned} x + y + z &= 7 \\ x + 2y + 3z &= 16 \\ 2x + 3y + 4z &= 23 \end{aligned}$$

3. a) By substituting directly into the equations show that $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$ satisfies the system of equations

$$\begin{aligned} x - y + z &= 7 \\ 2x + 2y + z &= 13 \\ x + 7y - z &= 5 \end{aligned}$$

b) Find the solution to the system.

ANSWERS

1. a) Not in echelon form. b) Echelon form, Unique solution. c) Echelon form, No solution. d) Echelon form, Infinite solutions, $x_2 = t$. e) Echelon form, Infinite solutions, $x_2 = \mu, x_3 = \lambda, x_5 = t$. f) Echelon form, Unique solution. g) Not in echelon form. h) Not in echelon form. i) Not in echelon form. j) Echelon form, Infinite solutions, $x_1 = \mu, x_2 = \lambda$.

2. a) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$. Three planes meeting at a unique point.

b) No Solution. Three planes with no common point of intersection.

c) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 9 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} t \; ; \; t \in \mathbb{R}$. Three planes meeting on a common line.

3. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{27}{4} \\ -\frac{1}{4} \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{3}{4} \\ \frac{1}{4} \\ 1 \end{pmatrix} t \; ; \; t \in \mathbb{R}$.

MATH2099 PROBLEM SHEET 2
MATRIX ANALYSIS

An $m \times n$ matrix is a rectangular array of real or complex numbers with m rows and n columns.

Matrices are added and subtracted using simple pointwise operations.

If A is $m \times n$ and B is $p \times q$ then the product AB exists if and only if $n = p$ and the resulting matrix is $m \times q$.

In general $AB \neq BA$

A^T has for its rows the columns of A.

Given a square $n \times n$ matrix A the inverse of A (denoted by A^{-1}) is another $n \times n$ matrix with the property that $AA^{-1} = A^{-1}A = I$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$ exists if and only if $ad - bc \neq 0$.

Harder problems are marked with a ★

1. For the following system of equations, find conditions on the RHS vector

$\mathbf{b}=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ which ensure that the system is consistent (that is has a solution).

$$\begin{aligned} x + 3y + z &= b_1 \\ 2x + 4y + 7z &= b_2 \\ 4x + 10y + 9z &= b_3 \end{aligned}$$

2. For the following system of equations determine which values of λ (if any) will yield:
a) no solution b) infinite solutions c) a unique solution.

$$\begin{aligned} x + y + z &= 3 \\ x + (\lambda + 1)y + 2z &= 5 \\ 2x + (\lambda + 2)y + (\lambda^2 - 6)z &= \lambda + 11 \end{aligned}$$

3. Write down a 2×2 matrix A with the property that $A^2 = 0$ but $A \neq 0$.

4. Is it always true that $(A+B)^2 = A^2+2AB+B^2$ where A and B are square matrices of the same size?

5. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 4 \\ 1 & 6 \\ -2 & 3 \end{pmatrix}$. Find (if possible):

a) AB
b) BA
c) A^2
d) B^2
e) $(AB)^2$
f) A^T.

6. For each of the following matrices evaluate A^{-1} if possible. Where A^{-1} exists check your answer by calculating the matrix product AA^{-1}:

a) $A = \begin{pmatrix} 3 & 4 \\ 6 & 7 \end{pmatrix}$ b) $A = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}$ c) $A = \begin{pmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{pmatrix}$

d) $A = \begin{pmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 1 & 1 & 1 \end{pmatrix}$ e) $A = \begin{pmatrix} 3 & -3 & -2 \\ 3 & -4 & -2 \\ -4 & 3 & 3 \end{pmatrix}$

7. Prove that $(A^{-1})^{-1} = A$.

8.★ Although the calculations are a little more difficult, our formulae and inversion techniques extend naturally to complex numbers. Find the inverse of $A = \begin{pmatrix} i & 3 \\ -1 & 2i \end{pmatrix}$ and check your answer via multiplication.

9.★ Two square matrices A and B are said to be similar (written $A \sim B$) if there exists an invertible matrix S such that $B = S^{-1}AS$. Prove that:
a) $A \sim A$
b) If $A \sim B$ then $B \sim A$
c) If $A \sim B$ and $B \sim C$ then $A \sim C$.

10.★ Let $\mathbf{v}$ and $\mathbf{w}$ be two $n \times 1$ column vectors with $\mathbf{w}^T\mathbf{v} = 2$. Suppose that A is the $n \times n$ matrix given by $A = I - \mathbf{v}\mathbf{w}^T$. Prove that $A = A^{-1}$.

11.★ If $\mathbf{w} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ show that the conditions of the previous question are satisfied, find A and verify that A is equal to its own inverse.

ANSWERS

1. $b_3 = 2b_1 + b_2$.

2. $a)$ No solution: $\lambda = 3, 0$ $\quad b)$ ∞ solutions: $\quad \lambda = -3,$ $\quad c)$ Unique Solution: All other λ.

3. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ will do. Many other solutions are possible!

4. NO! Since we do not have $AB = BA$ in general. The best we can say is that

$$(A+B)^2 = A^2 + AB + BA + B^2.$$

Note that the general Binomial theorem will also suffer the same consequences.

5. $AB = \begin{pmatrix} 2 & 16 \\ -7 & 29 \end{pmatrix}, \quad BA = \begin{pmatrix} -4 & 12 & 20 \\ -5 & 20 & 30 \\ -5 & 5 & 15 \end{pmatrix},$ A^2 and B^2 do not exist but

$(AB)^2 = \begin{pmatrix} -108 & 496 \\ -217 & 729 \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ 0 & 5 \end{pmatrix}.$

6. a) $A = -\frac{1}{3}\begin{pmatrix} 7 & -4 \\ -6 & 3 \end{pmatrix}$ b) Singular c) $A^{-1} = \begin{pmatrix} -24 & 20 & -5 \\ 18 & -15 & 4 \\ 5 & -4 & 1 \end{pmatrix}$

d) Singular e) $A^{-1} = \begin{pmatrix} 6 & -3 & 2 \\ 1 & -1 & 0 \\ 7 & -3 & 3 \end{pmatrix}.$

7. Proof.

8. $A^{-1} = \begin{pmatrix} 2i & -3 \\ 1 & i \end{pmatrix}.$

9. Proof.

10. Proof. Hint: Show that $A^2 = I$.

11. $A = \begin{pmatrix} -3 & -1 \\ 8 & 3 \end{pmatrix}$

MATH2099 PROBLEM SHEET 3
VECTOR SPACES

A subset S of a vector space V is a subspace of V if it is non-empty and closed under vector addition and scalar multiplication.

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V is said to be linearly independent if $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = 0 \to \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.
(linear independent $\leftrightarrow$ every column in the LHS of the echelon form is a leading column)

The span of a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V is the set of all possible linear combinations
$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n$ where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are scalars.
(spanning set for $V \leftrightarrow$ no zero rows in the echelon form)

Harder problems are marked with a ★

1. Let

$$S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 + z^2 = 1 \right\}$$

a) Describe S geometrically.
b) Prove that S is not closed under addition in $\mathbb{R}^3$.

2. Let

$$S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, 2x + y + 4z = 0 \right\}$$

a) Prove that S is a subspace of $\mathbb{R}^3$.
b) Describe S geometrically.

3. Let

$$W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, x^2 = y^2 \right\}$$

a) Prove that S is a closed under scalar multiplication.
b) Is S a subspace of $\mathbb{R}^2$?

4. Let W be the set of all vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ in $\mathbb{R}^4$ given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 5 \\ -3 \end{pmatrix} \lambda \text{ where } \lambda \in \mathbb{R}.$$

a) Show that W is a subspace of $\mathbb{R}^4$.
b) Describe W geometrically.

5. Let

$$S = \left\{ p \in P_2(\mathbb{R}) \;\middle|\; p(7) = 0 \right\}$$

Prove that S is a subspace of $P_2(\mathbb{R})$ (the vector space of all real polynomials of degree at most 2).

6. Let

$$S = \left\{ p \in P_3(\mathbb{R}) \;\middle|\; p(0) = 7 \right\}$$

Prove that S is not a subspace of $P_3(\mathbb{R})$.

7. Show that $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a linearly independent set in $\mathbb{R}^4$.

8. Show that $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 6 & 2 \\ 1 & 0 \end{pmatrix} \right\}$ is a dependent set in $M_{22}(\mathbb{R})$ and find a linear dependence between the four vectors.

9. Prove that a linearly independent set cannot contain the zero vector.

10. Show that $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ is not in span $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} \right\}$ in the vector space $\mathbb{R}^4$.

11. Show that $\mathbf{w} = 5 + 2x - x^2$ is in $\operatorname{span}(S)$ where $S = \{\mathbf{v}_1, \mathbf{v}_2\} = \{1 - x^2,\ 1 + x + x^2\}$. Express $\mathbf{w}$ as a linear combination of the vectors in S.

12.* For $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in a vector space V show that

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\}.$$

13.* Let W be a vector space with subspaces U and V. Show that $U \cap V$ is a subspace of W.

14.* Let W be a vector space with subspaces U and V. Show that $U \cup V$ is a subspace of W iff one of the two subspaces U and V is contained entirely within the other.

15.* Prove that a subset of a linearly independent set is also linearly independent.
16.* True or False $\rightsquigarrow$ A subset of a linearly dependent set is also linearly dependent.

ANSWERS

1. Sphere of radius 1, centre at the origin.
2. a) Proof. b) Plane passing through the origin.
3. a) Proof. b) No.
4. a) Proof. b) Line in $\mathbb{R}^4$ passing through the origin.
5. Proof.
6. Proof.
7. Proof.
8. $-4\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} - 2\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 6 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
9. Proof.
10. Proof.
11. $\mathbf{w} = 3\mathbf{v}_1 + 2\mathbf{v}_2$.
12. Proof.
13. Proof.
14. Proof. Hint Use contradiction. Let $\mathbf{v}_1$ be an element of U not in V, and $\mathbf{v}_2$ be an element of V not in U.
15. Proof.
16. False. Consider a subset consisting of a single non-zero vector.

MATH2099 PROBLEM SHEET 4
BASES AND DIMENSION

A basis for a vector space is a linearly independent spanning set.

The number of vectors in any basis is the dimension of the vector space.

If V is an n dimensional vector space with ordered basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $\mathbf{v} \in V$ is (uniquely) expressed as $\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n$ then the coordinate vector $[\mathbf{v}]_B$ of $\mathbf{v}$ with respect to B is given by $[\mathbf{v}]_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \\ \alpha_n \end{pmatrix}$

Every n dimensional vector space V is essentially just $\mathbb{R}^n$. Once an ordered basis B for V is chosen, the map $T : V \to \mathbb{R}^n$ given by $T(\mathbf{v}) = [\mathbf{v}]_B$ will serve as an isomorphism (structure preserving map) between V and $\mathbb{R}^n$.

Harder problems are marked with a ★

1. Write down the standard basis B for each of the following vector spaces V over $\mathbb{R}$. In each case state the dimension of the space:

a) $V = \mathbb{R}^4$.

b) $V = \mathbb{R}$.

c) $V = P_4(\mathbb{R})$ (the set of all real polynomials of degree at most 4).

d) $V = M_{32}(\mathbb{R})$ (the set of all real 3×2 matrices).

2. What is the dimension of each of the following real vector spaces V?

a) $V = P_{21}(\mathbb{R})$.

b) $V = M_{89}(\mathbb{R})$

c) $V = span\left\{ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix}, \begin{pmatrix} 10 & 20 \\ 30 & 40 \end{pmatrix}, \begin{pmatrix} 100 & 200 \\ 300 & 400 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ in $M_{22}(\mathbb{R})$.

3. Without carrying out any calculations explain why

$$B = \left\{ \begin{pmatrix} 1 \\ 6 \\ 0 \\ 7 \end{pmatrix}, \begin{pmatrix} 11 \\ 13 \\ 0 \\ 28 \end{pmatrix}, \begin{pmatrix} -113 \\ 217 \\ 0 \\ 4.7 \end{pmatrix}, \begin{pmatrix} \pi \\ \pi^2 \\ 0 \\ \pi^3 \end{pmatrix} \right\}$$

is not a basis for $\mathbb{R}^4$.

4. Suppose that a set S of m vectors in $\mathbb{R}^n$ have been assembled vertically into a matrix and then reduced to the following echelon forms. In each case determine:

i) The values of m and n.

ii) Whether the m vectors are linearly independent.

iii) Whether the m vectors span $\mathbb{R}^n$.

iv) Whether the m vectors from a basis for $\mathbb{R}^n$.

a) $\begin{pmatrix} 1 & 4 & 7 & 2 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 8 & 4 \\ 0 & 0 & 0 & 7 \end{pmatrix}$ b) $\begin{pmatrix} 6 & 3 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 8 & 3 \end{pmatrix}$

c) $\begin{pmatrix} 0 & 3 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ d) $\begin{pmatrix} 1 & 8 & 0 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{pmatrix}$

e) $\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & -7 & 9 \end{pmatrix}$ f) $\begin{pmatrix} 1 & 2 \\ 0 & 4 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

5. True or False $\leadsto$ A set of 2 linearly independent vectors in $\mathbb{R}^3$ could form a basis for $\mathbb{R}^3$.

6. True or False $\leadsto$ A set of 2 linearly independent vectors in $\mathbb{R}^3$ could form a basis for $\mathbb{R}^2$.

7. Find (by inspection) bases for each of the following vector subspaces. In each case state the dimension of S:

a) $S = \left\{ \begin{pmatrix} a & b & c \\ b & a & b \\ c & b & a \end{pmatrix} \in M_{33}(\mathbb{R}) \right\}$.

b) $S = \left\{ \begin{pmatrix} x+y \\ 0 \\ x-y \end{pmatrix} \in \mathbb{R}^3 \right\}$.

b) $S = \{a + (a+b)x + (3a+2b)x^2 \in P_2(\mathbb{R})\}$.

8. In the previous problem set you showed that

$$S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; 2x + y + 4z = 0 \right\}$$

is a subspace of $\mathbb{R}^3$. Find a basis for S.

9. In the previous problem set you showed that

$$S = \left\{ p \in P_2(\mathbb{R}) \;\middle|\; p(7) = 0 \right\}$$

is a subspace of $P_2(\mathbb{R})$ (the vector space of all real polynomials of degree at most 2). Find a basis for S.

10. Find the coordinate vector $[\mathbf{v}]_B$ of the vector $\mathbf{v} = \begin{pmatrix} 3 & 1 \\ 8 & 19 \end{pmatrix}$ with respect to the ordered basis

$$B = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix} \right\} \text{ of } M_{22}(\mathbb{R}).$$

11. The coordinate vector of vector $\mathbf{v} \in P_2(\mathbb{R})$ with respect to the ordered basis

$B = \{1 - x^2, 1 + x^2, x\}$ is $[\mathbf{v}]_B = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$. Find $\mathbf{v}$.

12.★ Let $V = M_{22}(\mathbb{C})$ be the vector space of all 2×2 complex matrices over $\mathbb{C}$. Find the dimension of V.

13.★ Let $W = M_{22}(\mathbb{C})$ be the vector space of all 2×2 complex matrices over $\mathbb{R}$. Find the dimension of W.

14.★ Suppose that a vector space V has a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Using the first harder problem from the previous problem set show that $\{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\}$ is also a basis for V.

15.★ A 3×3 real matrix is said to be a magic square if the sum of the three entries in each row, column and diagonal is the same.

a) Verify that $\begin{pmatrix} 3 & 0 & 0 \\ -2 & 1 & 4 \\ 2 & 2 & -1 \end{pmatrix}$ is a magic square.

b) Let M be the set of all possible 3×3 magic squares. Prove that M is a subspace of $M_{33}(\mathbb{R})$.

c) It can be shown that for $n \geq 3$, the dimension of the subspace of all magic squares in $M_{nn}(\mathbb{R})$ is given by $n^2 - 2n$. Use this fact to show that

$$B = \left\{ \begin{pmatrix} 3 & 0 & 0 \\ -2 & 1 & 4 \\ 2 & 2 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 3 & 0 \\ 1 & 1 & 1 \\ 2 & -1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 3 \\ 4 & 1 & -2 \\ -1 & 2 & 2 \end{pmatrix} \right\}$$

is a basis for M, the vector space of all possible 3×3 magic squares.

d) The ancient Lo Shu square (650BC) is given by $\mathbf{v} = \begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix}$. Find $[\mathbf{v}]_B$ where B is the ordered basis from part c).

e) Let D be the subspace of M consisting of 3×3 magic squares with a leading diagonal consisting entirely of zeros. Find a basis for and the dimension of D.

ANSWERS

1.

a) $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$, Dim($V$)=4.

b) $B = \{1\}$, Dim(V)=1.

c) $B = \{1, t, t^2, t^3, t^4\}$, Dim($V$)=5.

d) $B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, Dim($V$)=6.

2. a) 22 b) 72 c) 1.

3. Hint: There are some vectors which clearly can't be built.

4.

a) $i)m = 4, n = 4$ $ii)$ Lin. Ind.=Yes $iii)$ Spans=Yes $iv)$ Basis=Yes.
b) $i)m = 4, n = 3$ $ii)$ Lin. Ind.=No $iii)$ Spans=Yes $iv)$ Basis=No.
c) $i)m = 3, n = 3$ $ii)$ Lin. Ind.=No $iii)$ Spans=No $iv)$ Basis=No.
d) $i)m = 3, n = 3$ $ii)$ Lin. Ind.=No $iii)$ Spans=No $iv)$ Basis=No.
e) $i)m = 4, n = 2$ $ii)$ Lin. Ind.=No $iii)$ Spans=Yes $iv)$ Basis=No.
f) $i)m = 2, n = 4$ $ii)$ Lin. Ind.=Yes $iii)$ Spans=No $iv)$ Basis=No.

5. False.

6. False.

7.

a) $B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}$, Dim=3.

b) $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$, Dim=2.

c) $B = \{1 + x + 3x^2, x + 2x^2\}$, Dim=2.

8. $B = \left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right\}$.

9. $B = \{x - 7, x^2 - 14x + 49\}$

10. $[\mathbf{v}]_B = \begin{pmatrix} 2 \\ 1 \\ 4 \\ 3 \end{pmatrix}$.

11. $\mathbf{v} = 2 + 2x - 4x^2$.

12. 4.

13. 8.

14. Proof.

15. d) $[\mathbf{v}]_B = \begin{pmatrix} \frac{4}{3} \\ 3 \\ \frac{2}{3} \end{pmatrix}$ e) $B = \left\{ \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \right\}$, Dim(D)=1.

MATH2099 PROBLEM SHEET 5
DOT PRODUCTS AND PROJECTIONS

Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ . \\ . \\ u_n \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ . \\ . \\ v_n \end{pmatrix}$ be two vectors in $\mathbb{R}^n$: Then

The vector $\overrightarrow{\mathbf{uv}}$ from u to v is given by $\overrightarrow{\mathbf{uv}} = v - u$.

The dot product of $\mathbf{u}$ and $\mathbf{v}$ is given by $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$.

The magnitude of the vector $\mathbf{u}$ is $\parallel \mathbf{u} \parallel = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2} = \sqrt{\mathbf{u} \cdot \mathbf{u}}$.

The angle θ between $\mathbf{u}$ and $\mathbf{v}$ is given by $\cos(\theta) = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\parallel \mathbf{u} \parallel \parallel \mathbf{v} \parallel}$

$$\mathbf{u} \perp \mathbf{v} \Longleftrightarrow \mathbf{u} \cdot \mathbf{v} = 0.$$

The projection of $\mathbf{u}$ onto $\mathbf{v}$ is given by $\text{Proj}_{\mathbf{v}}(\mathbf{u}) = (\dfrac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}})\mathbf{v}$.

Harder problems are marked with a ★

1. Let $\begin{pmatrix} 2 \\ 3 \end{pmatrix} \in \mathbb{R}^2$. Sketch the point $u = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and the vector $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ in the plane. Find the magnitude and direction of $\mathbf{u}$.

2. Let $\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \in \mathbb{R}^3$. Sketch the point $u = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$ and the vector $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$ in space. Find the magnitude of $\mathbf{u}$.

3. Let $\mathbf{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$ be two vectors in $\mathbb{R}^2$. Calculate $\mathbf{u} + \mathbf{v}$ algebraically and display the process geometrically using the parallelogram law of addition.

4. Let $\mathbf{u} = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 7 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ 4 \\ 6 \\ 8 \end{pmatrix}$ be two vectors in $\mathbb{R}^4$. Calculate (if possible):

a) $3\mathbf{u} + 7\mathbf{v}$. b) $\mathbf{v} - \mathbf{u}$. c) $\parallel \mathbf{v} \parallel$ d) Two unit vectors parallel to $\mathbf{v}$.

e) $\mathbf{uv}$ f) $\dfrac{\mathbf{u}}{\mathbf{v}}$ g) $\mathbf{u} \cdot \mathbf{v}$ h) $\mathbf{v} \cdot \mathbf{u}$ i) $\mathbf{u} \cdot \mathbf{v} \cdot \mathbf{v}$

j) The angle between $\mathbf{u}$ and $\mathbf{v}$.

k) $\mathbf{u}^2$ l) $\mathbf{v}^{-1}$ m) Two vectors parallel to $\mathbf{u}$ with a magnitude of 10. n) $\sqrt{\mathbf{u}}$

5. Let $\mathbf{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$ be two vectors in $\mathbb{R}^2$. Calculate $\overrightarrow{\mathbf{uv}}$ algebraically and display the answer geometrically.

6. Let $\mathbf{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$ be two vectors in $\mathbb{R}^2$. Calculate $\mathrm{Proj}_{\mathbf{v}}(\mathbf{u})$ algebraically and display the answer geometrically.

7. Let $\mathbf{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$ be two vector in $\mathbb{R}^2$. Calculate $\mathrm{Proj}_{\mathbf{u}}(\mathbf{v})$ algebraically and display the answer geometrically.

8. Let $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5 \\ 50 \\ 7 \end{pmatrix}$. Verify that $\mathbf{a} \perp \mathbf{b}$.

9. Let $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ -5 \\ 7 \\ 4 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ 6 \\ -15 \\ \beta \\ 12 \end{pmatrix}$ in $\mathbb{R}^5$. Find β if $\mathbf{u}$ and $\mathbf{v}$ are:

a) Parallel. b) Perpendicular.

10. Let $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}$ in space. Find:

a) $\mathrm{Proj}_{\mathbf{v}}(\mathbf{u})$.

b) $\mathrm{Proj}_{\mathbf{u}}(\mathbf{v})$.

c) $\overrightarrow{\mathbf{uv}}$.

d) $\overrightarrow{\mathbf{vu}}$.

11. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be three non-zero vectors in $\mathbb{R}^2$.

True or False: $\mathbf{u} \perp \mathbf{v}$ and $\mathbf{v} \perp \mathbf{w} \to \mathbf{u} \parallel \mathbf{w}$.

12. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be three non-zero vectors in $\mathbb{R}^3$.

True or False: $\mathbf{u} \perp \mathbf{v}$ and $\mathbf{v} \perp \mathbf{w} \to \mathbf{u} \parallel \mathbf{w}$.

13. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be three distinct non-zero vectors in $\mathbb{R}^n$ satisfying

$$\mathrm{Proj}_{\mathbf{v}}(\mathbf{u}) = \mathrm{Proj}_{\mathbf{v}}(\mathbf{w})$$

Prove that $\overrightarrow{\mathbf{uw}} \perp \mathbf{v}$ and display the situation with a sketch.

14.* Let A, B, C, and D be the vertices of any quadrilateral in space with E, F, G, and H the midpoints of AB, BC, CD, and DA respectively. Use vector techniques to prove that EFGH is a parallelogram.

The rest of this problem set deals with complex vectors. This case will not be examined directly but is of interest. We need to slightly modify our definitions to accommodate complex vectors.

Given two vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{pmatrix}$ in $\mathbb{C}^n$ the dot product of $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\mathbf{a} \cdot \mathbf{b} = a_1\overline{b_1} + a_2\overline{b_2} + \cdots + a_n\overline{b_n}$$

The length (or norm or magnitude) of the vector $\mathbf{a}$ is given by

$$\| \mathbf{a} \| = \sqrt{a_1\overline{a}_1 + a_2\overline{a}_2 + \cdots + a_n\overline{a}_n} = \sqrt{|a_1|^2 + |a_2|^2 + \cdots + |a_n|^2} = \sqrt{\mathbf{a} \cdot \mathbf{a}}.$$

Observe that $\mathbf{a} \cdot \mathbf{a}$ is real and non-negative, even when $\mathbf{a} \in \mathbb{C}^n$.

Note also that in the complex case that $\mathbf{a} \cdot \mathbf{b} \neq \mathbf{b} \cdot \mathbf{a}$. In fact $\mathbf{a} \cdot \mathbf{b} = \overline{\mathbf{b} \cdot \mathbf{a}}$.

We still have $\mathbf{a} \perp \mathbf{b} \iff \mathbf{a} \cdot \mathbf{b} = 0$.

15.* Let $\mathbf{a} = \begin{pmatrix} i \\ 3-4i \\ 1+5i \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -8-14i \\ 1-i \\ 2+i \end{pmatrix}$.

a) Find $\| \mathbf{a} \|$.

b) Find two unit vectors parallel to $\mathbf{a}$.

c) Prove that $\mathbf{a} \perp \mathbf{b}$.

16.* Find the projection of $\mathbf{u} = \begin{pmatrix} 1+i \\ 2 \end{pmatrix}$ onto $\mathbf{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ in $\mathbb{C}^2$.

ANSWERS

1. $\| \mathbf{u} \| = \sqrt{13}$ $\quad \theta = 56°19'$.

2. $\| \mathbf{u} \| = \sqrt{30}$.

3. $\mathbf{u} + \mathbf{v} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$.

4. a) $\begin{pmatrix} 17 \\ 37 \\ 57 \\ 77 \end{pmatrix}$ b) $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ c) $\sqrt{120}$ d) $\pm\frac{1}{\sqrt{120}}\begin{pmatrix} 2 \\ 4 \\ 6 \\ 8 \end{pmatrix}$ e) Undefined f) Undefined

g) 100 h) 100 i) Undefined j) $5°7'$ k) Undefined l) Undefined m) $\pm\frac{10}{\sqrt{84}}\begin{pmatrix} 1 \\ 3 \\ 5 \\ 7 \end{pmatrix}$

n) Undefined.

5. $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

6. $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

7. $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$.

8. Proof.

9. a) 21 b) $-\frac{138}{7}$.

10. a) $\frac{1}{2}\begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}$ b) $\frac{13}{14}\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ c) $\begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix}$ d) $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

11. True.

12. False.

13. Proof.

14. Proof.

15. $a)\sqrt{52}$ $b)\pm\frac{1}{\sqrt{52}}\begin{pmatrix} i \\ 3-4i \\ 1+5i \end{pmatrix}$ $c)$ $Proof.$

16. $\frac{1}{2}\begin{pmatrix} 1-i \\ 1+i \end{pmatrix}$

MATH2099 PROBLEM SHEET 6
THE GRAM-SCHMIDT ALGORITHM

Given a linearly independent set $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ in $\mathbb{R}^n$: Let

$$\begin{aligned}
\mathbf{a}_1 &= \mathbf{v}_1 \\
\mathbf{a}_2 &= \mathbf{v}_2 - \mathrm{Proj}_{\mathbf{a}_1}(\mathbf{v}_2) \\
\mathbf{a}_3 &= \mathbf{v}_3 - \mathrm{Proj}_{\mathbf{a}_1}(\mathbf{v}_3) - \mathrm{Proj}_{\mathbf{a}_2}(\mathbf{v}_3) \\
&\vdots \\
\mathbf{a}_n &= \mathbf{v}_n - \mathrm{Proj}_{\mathbf{a}_1}(\mathbf{v}_n) - \mathrm{Proj}_{\mathbf{a}_2}(\mathbf{v}_n) - \mathrm{Proj}_{\mathbf{a}_3}(\mathbf{v}_n) - \cdots - \mathrm{Proj}_{\mathbf{a}_{n-1}}(\mathbf{v}_n)
\end{aligned}$$

Then $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\}$ is an orthogonal set with the same span as $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$.

To produce an orthonormal spanning set simply normalise the vectors to produce

$$\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\} = \left\{ \frac{\mathbf{a}_1}{\| \mathbf{a}_1 \|}, \frac{\mathbf{a}_2}{\| \mathbf{a}_2 \|}, \cdots, \frac{\mathbf{a}_n}{\| \mathbf{a}_n \|} \right\}.$$

Any matrix A with linearly independent columns can be decomposed as $A = QR$ where Q has orthonormal columns and R is upper triangular. Q is found by applying Gram-Schmidt to the columns of A and R is found via $R = Q^T A$.

Harder problems are marked with a ★

1. Prove that $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 6 \\ 3 \\ -6 \end{pmatrix} \right\}$ is orthogonal in $\mathbb{R}^3$ and construct an orthonormal set W from S by normalising the vectors.

2. Let $Q = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}$.

By evaluating $Q^T Q$, show that Q is orthogonal and hence find Q^{-1}.

3. Prove that the product of orthogonal matrices is also orthogonal.

4. a) Apply the Gram-Schmidt algorithm to the linearly independent set $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$ in $\mathbb{R}^2$.

b) Hence find the QR decomposition of $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

c) Check your answer in b) through matrix multiplication.

5. a) Apply the Gram-Schmidt algorithm to the linearly independent set $\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \right\}$ in $\mathbb{R}^3$.

b) Hence find the QR decomposition of $A = \begin{pmatrix} 0 & 3 \\ 1 & 2 \\ 1 & 0 \end{pmatrix}$.

c) Check your answer in b) through matrix multiplication.

6. a) Apply the Gram-Schmidt algorithm to the linearly independent set $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ in $\mathbb{R}^3$.

b) Hence find the QR decomposition of $A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$.

c) Check your answer in b) through matrix multiplication.

7. Find the QR decomposition of $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$.

8. a) Use the Gram-Schmidt algorithm to find an orthonormal basis for the span of the linearly independent set $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 2 \\ 0 \end{pmatrix} \right\}$ in $\mathbb{R}^4$.

9.★ Consider the basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$ for $\mathbb{R}^3$.

a) Use the Gram-Schmidt algorithm to convert B into an orthonormal basis for $\mathbb{R}^3$.

b) Hence find the QR decomposition of $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{pmatrix}$.

c) Explain why the solution of $A\mathbf{x} = \mathbf{b}$ is given by $R\mathbf{x} = Q^T\mathbf{b}$.

d) Hence use the QR decomposition to solve the system of equations:

$$
\begin{aligned}
x - y + z &= 7 \qquad (1)\\
x + y + z &= 9 \qquad (2)\\
x + 2y - z &= 0 \qquad (3)
\end{aligned}
$$

Although a little clunky by hand, this QR decomposition technique lends itself to the use of computers in solving systems of equations. Observe that since R is upper triangular no reduction is required!!

10.$^\bigstar$ If $B = \{\mathbf{e_1}, \mathbf{e_2}, \cdots, \mathbf{e_n}\}$ is an ordered orthonormal basis for $\mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$ show that the coordinate vector $[\mathbf{v}]_B$ of $\mathbf{v}$ with respect to B is given by $[\mathbf{v}]_B = \begin{pmatrix} \mathbf{v}\cdot\mathbf{e_1} \\ \mathbf{v}\cdot\mathbf{e_2} \\ \vdots \\ \mathbf{v}\cdot\mathbf{e_n} \end{pmatrix}$.

11.$^\bigstar$ We saw earlier that dot products and projections work fine for complex vectors. So too therefore, will the Gram-Schmidt algorithm!

Use Gram-Schmidt to find an orthonormal basis for span$\{\begin{pmatrix} 1+i \\ 1-i \end{pmatrix}, \begin{pmatrix} 1-2i \\ 5i \end{pmatrix}\}$.

ANSWERS

1. $W = \left\{ \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{pmatrix} \right\}$

2. $Q^{-1} = Q^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$.

3. Proof.

4. a) $\left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right\}$ b) $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$

5. a) $\left\{ \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{3}{\sqrt{11}} \\ \frac{1}{\sqrt{11}} \\ -\frac{1}{\sqrt{11}} \end{pmatrix} \right\}$

b) $\begin{pmatrix} 0 & 3 \\ 1 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{3}{\sqrt{11}} \\ \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{11}} \\ \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{11}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{11} \end{pmatrix}$

6. a) $\left\{ \begin{pmatrix} -\dfrac{1}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} -\dfrac{1}{\sqrt{6}} \\ -\dfrac{1}{\sqrt{6}} \\ \dfrac{2}{\sqrt{6}} \end{pmatrix} \right\}$

b) $\begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{6}} \\ \dfrac{1}{\sqrt{2}} & -\dfrac{1}{\sqrt{6}} \\ 0 & \dfrac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \dfrac{1}{\sqrt{2}} \\ 0 & \dfrac{3}{\sqrt{6}} \end{pmatrix}$

7. $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{2} & -\dfrac{1}{2} \\ \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{1}{2} & -\dfrac{1}{2} \\ \dfrac{1}{2} & \dfrac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}.$

8. $$\left\{ \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} \\ \frac{-1}{3\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{2\sqrt{3}} \\ \frac{-5}{6\sqrt{3}} \\ \frac{7}{6\sqrt{3}} \\ \frac{-5}{6\sqrt{3}} \end{pmatrix} \right\}$$

9. b) $$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{5}{\sqrt{42}} & -\frac{1}{\sqrt{14}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{42}} & \frac{3}{\sqrt{14}} \\ \frac{1}{\sqrt{3}} & \frac{4}{\sqrt{42}} & -\frac{2}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} \sqrt{3} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{\sqrt{42}}{3} & -\frac{8}{\sqrt{42}} \\ 0 & 0 & \frac{4}{\sqrt{14}} \end{pmatrix}$$

c) Proof.

d) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}.$

10. Proof.

11. $\{\mathbf{a}_1, \mathbf{a}_2\} = \left\{ \begin{pmatrix} 1+i \\ 1-i \end{pmatrix}, \begin{pmatrix} 3-i \\ 1+3i \end{pmatrix} \right\}$ is an orthogonal basis.

Check

$\mathbf{a}_1 \cdot \mathbf{a}_2 = (1+i)(3+i) + (1-i)(1-3i) = 3 + i + 3i - 1 + 1 - 3i - i - 3 = 0. \quad (\mathbf{a}_1 \perp \mathbf{a}_2).$

Through normalisation the orthonormal basis is $\{\mathbf{u}_1, \mathbf{u}_2\} = \left\{ \frac{1}{2} \begin{pmatrix} 1+i \\ 1-i \end{pmatrix}, \frac{1}{\sqrt{20}} \begin{pmatrix} 3-i \\ 1+3i \end{pmatrix} \right\}$

MATH2099 PROBLEM SHEET 7
LINEAR TRANSFORMATIONS

A transformation T mapping the vector space V into the vector space W is linear if:

$$\begin{aligned} T(\mathbf{v}_1 + \mathbf{v}_2) &= T(\mathbf{v}_1) + T(\mathbf{v}_2) \qquad \text{and} && (1) \\ T(\alpha \mathbf{v}_1) &= \alpha T(\mathbf{v}_1) && (2) \end{aligned}$$

for all vectors $\mathbf{v}_1, \mathbf{v}_2 \in V$ and scalars α.

$\mathrm{Ker}(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = 0\}$. (The kernel of T)

$\mathrm{Im}(T) = \{T(\mathbf{v}) \mid \mathbf{v} \in V\}$. (The image of T)

$\mathrm{Ker}(T)$ is a subspace of V and $\mathrm{Im}(T)$ is a subspace of W.

$\mathrm{Dim}(\mathrm{Ker}(T))$ is referred to as the nullity of T.

$\mathrm{Dim}(\mathrm{Im}(T))$ is referred to as the rank of T.

$\mathrm{Rank}(T)+\mathrm{Nullity}(T)=\mathrm{Dim}(V)$ (The Rank-Nullity Theorem)

For a linear transformation T induced by a matrix A:

$\mathrm{Ker}(T)$ is simply the solution of $A\mathbf{x} = \mathbf{0}$.

$\mathrm{Nullity}(T)$ is the number of non-leading columns in the Echelon form.

$\mathrm{Im}(T)$ is the span of all the leading columns of A.

$\mathrm{Rank}(T)$ is the number of leading columns in the Echelon form.

Harder problems are marked with a ★

1. Define $T : P_2(\mathbb{R}) \to M_{22}(\mathbb{R})$ by

$$T(a + bx + cx^2) = \begin{pmatrix} 0 & a+b \\ c & 0 \end{pmatrix}.$$

a) Find $T(3 + 7x - 8x^2)$.

b) Prove that T is a linear transformation.

2. Define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y^2 \\ 0 \\ x^2 \end{pmatrix}.$$

a) Find $T\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

b) Prove that T is not linear.

3. Suppose that $T : V \to W$ is linear. Prove that $T(\mathbf{0}) = \mathbf{0}$.

4. Define a transformation $T : \mathbb{R}^3 \to P_2(\mathbb{R})$ by

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (a+c)x^2 + bx.$$

a) Find $T\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$.

b) Prove that T is linear.

c) What is the dimension of $\mathbb{R}^3$ and $P_2(\mathbb{R})$?

d) Find a basis for $\mathrm{Ker}(T)$ and $\mathrm{Nullity}(T)$.

e) Find a basis for $\mathrm{Im}(T)$ and $\mathrm{Rank}(T)$.

f) Verify the Rank-Nullity theorem for this transformation.

5. Define a transformation $T : M_{22}(\mathbb{R}) \to \mathbb{R}^3$ by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b \\ 0 \\ c+d \end{pmatrix}$$

a) Find $T\begin{pmatrix} 9 & 8 \\ 6 & 5 \end{pmatrix}$.

b) Prove that T is linear.

c) What is the dimension of $M_{22}(\mathbb{R})$ and $\mathbb{R}^3$?

d) Find a basis for $\mathrm{Ker}(T)$ and $\mathrm{Nullity}(T)$.

e) Find a basis for $\mathrm{Im}(T)$ and $\mathrm{Rank}(T)$.

f) Verify the Rank-Nullity theorem for this transformation.

6. Consider the following matrices:

(I) $\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$ (II) $\begin{pmatrix} 1 & 3 & 0 & 1 & 5 \\ 2 & 6 & 0 & 0 & 8 \\ 3 & 9 & 0 & 1 & 13 \end{pmatrix}$ (III) $\begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 3 \\ 1 & 4 \end{pmatrix}$ (IV) $\begin{pmatrix} 1 & 3 & 5 & 7 \end{pmatrix}$

For each of the above matrices, define $T : \mathbb{R}^n \to \mathbb{R}^m$ to be the transformation induced by the matrix. For each matrix:

a) Find n and m and transform a random vector to see what T does.

b) Find a basis for $\mathrm{Ker}(T)$ and $\mathrm{Nullity}(T)$.

c) Find a basis for $\mathrm{Im}(T)$ and $\mathrm{Rank}(T)$.

d) Verify the Rank-Nullity theorem for T.

7.★ Suppose that $T : V \to W$ is linear. Prove that $\mathrm{Ker}(T)$ is a subspace of V.

8.★ Show that $\mathrm{Rank}(AB) \leq \mathrm{Rank}(A)$. Hint: Show that $\mathrm{Im}(AB) \subseteq \mathrm{Im}(A)$.

9.★ Show that $\mathrm{Nullity}(AB) \geq \mathrm{Nullity}(B)$. Hint: Show that $\mathrm{Ker}(AB) \supseteq \mathrm{Ker}(B)$.

10.★ Hence show that $\mathrm{Rank}(AB) \leq \mathrm{Rank}(B)$. Hint: Use the previous two results and the Rank-Nullity Theorem.

ANSWERS

1. a) $\begin{pmatrix} 0 & 10 \\ -8 & 0 \end{pmatrix}$ b) Proof.

2. $\begin{pmatrix} 16 \\ 0 \\ 9 \end{pmatrix}$ b) Proof.

3. Proof.

4. a) $3x^2+4x$ b) Proof. c) 3 and 3. d) $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$, 1 e) $\{x, x^2\}$, 2 f) $2+1 = 3$.

5. a) $\begin{pmatrix} 17 \\ 0 \\ 11 \end{pmatrix}$ b) Proof. c) 4 and 3. d) $\left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \right\}$, 2

e) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$, 2 f) $2 + 2 = 4$.

6.

(I) a) $n = m = 2$ b) $\left\{ \begin{pmatrix} -3 \\ 1 \end{pmatrix} \right\}$, 1 c) $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$, 1 d) $1 + 1 = 2$.

(II) a) $n = 5$, $m = 3$ b) $\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$, 3 c) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$, 2
d) $2 + 3 = 5$.

(III) a) $n = 2$, $m = 5$ b) $\mathrm{Ker}(T)=\{\mathbf{0}\}$, 0 c) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}$, 2 d) $2 + 0 = 2$.

(IV) a) $n = 4$, $m = 1$ b) $\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -7 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$, 3 c) $\{1\}$,1 d) $1 + 3 = 4$.

7. Proof.

8. Proof.

9. Proof.

10. Proof.

MATH2099 PROBLEM SHEET 8
MATRIX TRANSFORMATIONS ON $\mathbb{R}^2$

SOME STANDARD MATRIX TRANSFORMATIONS ON $\mathbb{R}^2$

Reflection in the x axis: $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Reflection in the y axis: $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Reflection across the origin: $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Rotation anticlockwise about the origin by θ: $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

For clockwise rotations use $-\theta$ in the above matrix.

Dilation by α in the x direction and β in the y direction: $A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$

$\alpha, \beta > 1$ yields expansions $\quad \alpha, \beta < 1$ yields contractions.

$k-$shear in the x direction: $A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$

$k-$shear in the y direction: $A = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$

Reflection in the line $y = mx$: $A = \dfrac{1}{1+m^2}\begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$

Projection onto the line $y = mx$: $A = \dfrac{1}{1+m^2}\begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix}$

In all of the above the inverse of the transformation (if it exists) is given by A^{-1}.

Harder problems are marked with a ★

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x + 2y + z \\ 11x - y - 5z \end{pmatrix}.$$

a) Write down the standard matrix A of T by inspection.

b) Find the standard matrix A of T by considering the action of T on the standard basis $B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ for $\mathbb{R}^3$.

2. Consider the linear transformation T which reflects all the vectors in $\mathbb{R}^2$ across the y axis.

a) Find the standard matrix C of T by considering the action of T on the standard

basis $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

b) Check your answer by using one of the formulae on page 1.

c) Display the action of the transformation by premultiplying each of of the vectors

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ \mathbf{Q} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \mathbf{R} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{S} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

by C to produce the transformed vectors $\mathbf{P}'$, $\mathbf{Q}'$, $\mathbf{R}'$ and $\mathbf{S}'$. Sketch the quadrilaterals $PQRS$ and $P'Q'R'S'$ on the same set of axes.

3. Consider the linear transformation T which rotates all the vectors in $\mathbb{R}^2$ anticlockwise by $90°$.

a) Find the standard matrix D of T by considering the action of T on the standard

basis $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

b) Check your answer by using one of the formulae on page 1.

c) Display the action of the transformation by premultiplying each of of the vectors

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ \mathbf{Q} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \mathbf{R} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{S} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

by D to produce the transformed vectors $\mathbf{P}'$, $\mathbf{Q}'$, $\mathbf{R}'$ and $\mathbf{S}'$. Sketch the quadrilaterals $PQRS$ and $P'Q'R'S'$ on the same set of axes.

4. Consider the linear transformation T on $\mathbb{R}^2$ which compresses the x direction by $\dfrac{1}{2}$ and stretches the y direction by 5.

a) Find the standard matrix E of T by considering the action of T on the standard

basis $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

b) Check your answer by using one of the formulae on page 1.

c) Display the action of the transformation by premultiplying each of of the vectors

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ \mathbf{Q} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \mathbf{R} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{S} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

by E to produce the transformed vectors $\mathbf{P}'$, $\mathbf{Q}'$, $\mathbf{R}'$ and $\mathbf{S}'$. Sketch the quadrilaterals $PQRS$ and $P'Q'R'S'$ on the same set of axes.

5. Consider the linear transformation T which projects all vectors in $\mathbb{R}^2$ onto the line $y = 3x$.

a) Find the standard matrix F of T by considering the action of T on the standard basis $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

b) Check your answer by using one of the formulae on page 1.

c) Display the action of the transformation by premultiplying each of of the vectors

$$\mathbf{P} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ \mathbf{Q} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \mathbf{R} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{S} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

by F to produce the transformed vectors $\mathbf{P}'$, $\mathbf{Q}'$, $\mathbf{R}'$ and $\mathbf{S}'$. Sketch the quadrilaterals $PQRS$ and $P'Q'R'S'$ on the same set of axes.

d) Evaluate F^2. Explain the significance of your answer.

e) By reducing F find $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$. Interpret your answers geometrically.

6. Find the inverse (if possible) of C,D,E and F above and interpret the inverse transformations geometrically.

7. Use the matrices C and D above to find the standard matrix of the transformation T on $\mathbb{R}^2$ which first rotates vectors anticlockwise by $90°$ and then reflects across the y axis.

8. Use the matrices C and D above to find the standard matrix of the transformation T on $\mathbb{R}^2$ which first reflects vectors across the y axis and then rotates anticlockwise by $90°$.

9.★ Find the standard matrix of the transformation T on $\mathbb{R}^2$ which first reflects vectors across the y axis, then rotates anticlockwise by $90°$, then compresses by $\dfrac{1}{2}$ along the x direction and stretches by 5 in the y direction and finally projects onto the line $y = 3x$.

10.★ The projection onto the line $y = mx$ has standard matrix $\quad P = \dfrac{1}{1+m^2} \begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix}$.

a) Show that $P^2 = P$ for all m and explain why this is so.

b) Show that P^{-1} fails to exist for all m and explain why this is so.

11.★ Reflection across the line $y = mx$ has standard matrix $R = \dfrac{1}{1+m^2}\begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$.

a) Show that $R^2 = I$ for all m and explain why this is so.

b) Hence show that $R^{-1} = R$ for all m and explain why this is so.

12.★ Let $A(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ be the standard rotation matrix by θ in $\mathbb{R}^2$. Show that

$$(A(\theta))^{-1} = A(-\theta).$$

Explain this property geometrically.

ANSWERS

1. $A = \begin{pmatrix} 3 & 2 & 1 \\ 11 & -1 & -5 \end{pmatrix}$

2. $C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$

3. $D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

4. $E = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 5 \end{pmatrix}$ $\quad \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 5 & 5 & 0 \end{pmatrix}$

5. $F = \begin{pmatrix} \frac{1}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{9}{10} \end{pmatrix}$ $\quad \begin{pmatrix} \frac{1}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{9}{10} \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & \frac{3}{10} & \frac{4}{10} & \frac{1}{10} \\ 0 & \frac{9}{10} & \frac{12}{10} & \frac{3}{10} \end{pmatrix}$

Note that $\mathbf{P}'$, $\mathbf{Q}'$, $\mathbf{R}'$ and $\mathbf{S}'$ all lie on $y = 3x$.

d) $F^2 = F$ since projecting twice is the same as projecting once.

e) Im(T)= *span* $\left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right\}$ which is simply the line $y = 3x$ upon which T projects. Also Ker(T)= *span* $\left\{\begin{pmatrix} -3 \\ 1 \end{pmatrix}\right\}$ which is $\perp$ to Im(T). That is, vectors perpendicular to the line project to $\mathbf{0}$.

6.
$C^{-1} = C$. Reflecting is the same forwards and backwards.

$D^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Rotation clockwise about the origin by 90°.

$E^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{5} \end{pmatrix}$. Stretching by 2 in the x direction and compression by $\frac{1}{5}$ in the y direction.

F^{-1} does not exist. Projections on to proper subspaces cannot be inverted. You cannot recreate space from a photo.

7. $A = CD = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

8. $A = DC = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

9. $A = FEDC = \begin{pmatrix} -\frac{3}{2} & -\frac{1}{20} \\ -\frac{9}{2} & -\frac{3}{20} \end{pmatrix}$.

10.
a) Projecting twice is the same as projecting once.

b) You cannot recover the entire space from a projection. The structure has been squished.

11.
a) Reflecting twice is the same as doing nothing.

b) Reflecting is the same forwards or backwards.

12. Proof. The inverse of rotation anticlockwise by θ is rotation clockwise by θ.

MATH2099 PROBLEM SHEET 9
MATRICES OF LINEAR TRANSFORMATIONS

> Let $T : V \to W$ be a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W. Suppose that $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an ordered basis for V and that $C = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ is an ordered basis for W. Then there is a unique matrix A referred to as the matrix of the linear transformation with the property that $[T(\mathbf{v})]_C = A[\mathbf{v}]_B$ for all $\mathbf{v} \in V$.
>
> The i^{th} column of A is the coordinate vector $[T(\mathbf{v}_i)]_C$.
>
> The matrix A depends only upon the transformation T and the two ordered bases B and C.

Harder problems are marked with a ★

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 3x + 6y \end{pmatrix}$ and let $\mathbf{v} = \begin{pmatrix} 5 \\ 7 \end{pmatrix} \in \mathbb{R}^2$.

a) Find the standard matrix A of T with respect to the standard ordered bases

$$B = \{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad C = \{\mathbf{w}_1, \mathbf{w}_2\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

for domain and codomain respectively.

b) Find $[\mathbf{v}]_B$.

c) Find $T(\mathbf{v})$ directly.

d) Find $T(\mathbf{v})$ by using $[\mathbf{v}]_B$ and A.

e) Find Nullity(T) and Ker(T) by considering A.

f) Find Rank(T) and Im(T) by considering A.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 3x + 6y \end{pmatrix}$ and let $\mathbf{v} = \begin{pmatrix} 5 \\ 7 \end{pmatrix} \in \mathbb{R}^2$.

a) Find the matrix F of T with respect to the ordered bases

$$D = \{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \quad \text{and} \quad E = \{\mathbf{w}_1, \mathbf{w}_2\} = \left\{ \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right\}$$

for domain and codomain respectively.

b) Find $[\mathbf{v}]_D$.

c) Find $T(\mathbf{v})$ directly.

d) Find $T(\mathbf{v})$ by using $[\mathbf{v}]_D$ and F.

3. Let $T : P_2(\mathbb{R}) \to P_1(\mathbb{R})$ be given by $T(p) = p'' + 6p'$ and let $\mathbf{v} = 7 - 5x + 2x^2 \in P_2(\mathbb{R})$.

a) Find the matrix A of T with respect to the standard ordered bases $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{1, x, x^2\}$ and $C = \{\mathbf{w}_1, \mathbf{w}_2\} = \{1, x\}$ for $P_2(\mathbb{R})$ and $P_1(\mathbb{R})$ respectively.

b) Find $[\mathbf{v}]_B$.

c) Find $T(\mathbf{v})$ directly.

d) Find $T(\mathbf{v})$ by using $[\mathbf{v}]_B$ and A.

e) Find Nullity(T) and Ker(T) by considering A.

4. Let $T : P_2(\mathbb{R}) \to P_1(\mathbb{R})$ be given by $T(p) = p'' + 6p'$ and let $\mathbf{v} = 7 - 5x + 2x^2 \in P_2(\mathbb{R})$.

a) Find the matrix F of T with respect to the ordered bases $D = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{1 - x, 1 + x, 4x^2\}$ and $E = \{\mathbf{w}_1, \mathbf{w}_2\} = \{1 - 2x, x\}$ for $P_2(\mathbb{R})$ and $P_1(\mathbb{R})$ respectively.

b) Find $[\mathbf{v}]_D$.

c) Find $T(\mathbf{v})$ directly.

d) Find $T(\mathbf{v})$ by using $[\mathbf{v}]_D$ and F.

e) Find Nullity(T) and Ker(T) by considering F.

5. Let $\mathbf{v} = 1 + 5x + 10x^2$ be an element of $P_2(\mathbb{R})$.

a) Find the coordinate vector $[\mathbf{v}]_B$ of $\mathbf{v}$ with respect to the ordered basis

$B = \{1, x, x^2\}$ for $P_2(\mathbb{R})$.

b) Find the coordinate vector $[\mathbf{v}]_D$ of $\mathbf{v}$ with respect to the ordered basis

$D = \{1 + x, 1 - x, 5x^2\}$ for $P_2(\mathbb{R})$.

c) Find the change of basis matrix P from D to B.

d) Verify that $P[\mathbf{v}]_D = [\mathbf{v}]_B$.

6. Let $\mathbf{v} = \begin{pmatrix} 1 \\ -7 \end{pmatrix}$ be an element of $\mathbb{R}^2$.

a) Find the coordinate vector $[\mathbf{v}]_C$ of $\mathbf{v}$ with respect to the standard ordered basis

$C = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ for $\mathbb{R}^2$.

b) Find the coordinate vector $[\mathbf{v}]_E$ of $\mathbf{v}$ with respect to the ordered basis

$E = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$ for $\mathbb{R}^2$.

c) Find the change of basis matrix Q from C to E.

d) Verify that $Q[\mathbf{v}]_C = [\mathbf{v}]_E$.

ANSWERS

1. a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ $b) \begin{pmatrix} 5 \\ 7 \end{pmatrix}$ $c) = d) \begin{pmatrix} 19 \\ 57 \end{pmatrix}$.

e) $\mathrm{Ker}(A) = \mathrm{span}\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$ $\quad \mathrm{Ker}(T) = \mathrm{span}\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$ $\quad \mathrm{Nullity}(T) = 1.$

f) $\mathrm{Im}(A) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$ $\quad \mathrm{Im}(T) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$ $\quad \mathrm{Rank}(T) = 1.$

2. a) $F = \begin{pmatrix} 1 & -\frac{1}{3} \\ 3 & -1 \end{pmatrix}$ $b) \begin{pmatrix} 6 \\ -1 \end{pmatrix}$ $c) = d) \begin{pmatrix} 19 \\ 57 \end{pmatrix}$.

3. a) $A = \begin{pmatrix} 0 & 6 & 2 \\ 0 & 0 & 12 \end{pmatrix}$ $b) \begin{pmatrix} 7 \\ -5 \\ 2 \end{pmatrix}$ $c) = d) - 26 + 24x.$

e) $\mathrm{Ker}(A) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ $\quad \mathrm{Ker}(T) = \mathrm{span}\{1\}$ $\quad \mathrm{Nullity}(T) = 1.$

4. a) $A = \begin{pmatrix} -6 & 6 & 8 \\ -12 & 12 & 64 \end{pmatrix}$ $b) \begin{pmatrix} 6 \\ 1 \\ \frac{1}{2} \end{pmatrix}$ $c) = d) - 26 + 24x.$

e) $\mathrm{Ker}(F) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ $\quad \mathrm{Ker}(T) = \mathrm{span}\{2\}$ $\quad \mathrm{Nullity}(T) = 1.$

5. a) $[\mathbf{v}]_B = \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix}$ b) $[\mathbf{v}]_D = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$ c) $P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ d) Proof.

6. a) $[\mathbf{v}]_C = \begin{pmatrix} 1 \\ -7 \end{pmatrix}$ b) $[\mathbf{v}]_E = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$ c) $Q = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$ d) Proof.

MATH2099 PROBLEM SHEET 10
COMMUTATIVE DIAGRAMS

$$
\begin{tikzcd}
V \arrow[r, "A"] & W \arrow[d, "Q"] \\
V \arrow[u, "P"] \arrow[r, "F"'] \arrow[ur, phantom, "\mathbf{QAP = F}"] & W
\end{tikzcd}
$$

Harder problems are marked with a ★

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 3x + 6y \end{pmatrix}$

In Q1 of the previous problem set you showed that the standard matrix A of T with respect to the standard ordered bases

$$B = \{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad C = \{\mathbf{w}_1, \mathbf{w}_2\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

is given by $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$.

In Q2 of the previous problem set you showed that the matrix F of T with respect to the ordered bases

$$D = \{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \quad \text{and} \quad E = \{\mathbf{w}_1, \mathbf{w}_2\} = \left\{ \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right\}$$

is given by $F = \begin{pmatrix} 1 & -\frac{1}{3} \\ 3 & -1 \end{pmatrix}$.

a) Find the matrix of change of basis P from D to B.

b) Find the matrix of change of basis Q from C to E.

c) Set up a commutative diagram displaying all of the information above.

d) Check your answer for F by chasing some matrices around your diagram in c)

2. Let $T : P_2(\mathbb{R}) \to P_1(\mathbb{R})$ be given by $T(p) = p'' + 6p'$

In Q3 of the previous problem set you showed that the matrix A of T with respect to the standard ordered bases $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{1, x, x^2\}$ and $C = \{\mathbf{w}_1, \mathbf{w}_2\} = \{1, x\}$ for

$P_2(\mathbb{R})$ and $P_1(\mathbb{R})$ respectively is given by $A = \begin{pmatrix} 0 & 6 & 2 \\ 0 & 0 & 12 \end{pmatrix}$.

In Q4 of the previous problem set you showed that the matrix F of T with respect to the ordered bases $D = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{1-x, 1+x, 4x^2\}$ and $E = \{\mathbf{w}_1, \mathbf{w}_2\} = \{1-2x, x\}$ for

$P_2(\mathbb{R})$ and $P_1(\mathbb{R})$ respectively is given by $F = \begin{pmatrix} -6 & 6 & 8 \\ -12 & 12 & 64 \end{pmatrix}$.

a) Find the matrix of change of basis P from D to B.

b) Find the matrix of change of basis Q from C to E.

c) Set up a commutative diagram displaying all of the information above.

d) Check your answer for F by chasing some matrices around your diagram in c).

3. A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ has the property that

$T\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 21 \end{pmatrix}$ and $T\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

a) Calculate the matrices of change of basis P and Q and the matrix F of T in the following commutative diagram.

$$\begin{tikzcd}
B = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} \quad \mathbb{R}^2 \arrow[r, "A"] & \mathbb{R}^2 \quad C = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} \arrow[d, "Q"] \\
D = \left\{\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right\} \quad \mathbb{R}^2 \arrow[u, "P"] \arrow[r, "F"'] \arrow[ur, phantom, "\mathbf{QAP = F}"] & \mathbb{R}^2 \quad E = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}
\end{tikzcd}$$

b) By chasing matrices around the diagram find the standard matrix A of T and a formula for $T\begin{pmatrix} x \\ y \end{pmatrix}$.

c) Check that your formula in part b) produces the known actions for T.

4.★ A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ has the property that

$$T\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \end{pmatrix}.$$

a) Calculate the matrices of change of basis P and Q and the matrix F of T in the following commutative diagram.

$$\begin{tikzcd}
B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \quad \mathbb{R}^2 \arrow[r, "A"] & \mathbb{R}^2 \quad C = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \arrow[d, "Q"] \\
D = \left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\} \quad \mathbb{R}^2 \arrow[u, "P"] \arrow[r, "F"'] \arrow[ur, phantom, "\mathbf{QAP = F}"] & \mathbb{R}^2 \quad E = \left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}
\end{tikzcd}$$

b) By chasing matrices around the diagram find the standard matrix A of T and a formula for $T\begin{pmatrix} x \\ y \end{pmatrix}$.

c) Check that your formula in part b) produces the known actions for T.

5.★★ Prove that the standard matrix of reflection in the line

$y = mx$ in $\mathbb{R}^2$ is given by $\quad A = \dfrac{1}{1+m^2}\begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$

ANSWERS

1.

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \quad \mathbb{R}^2 \arrow[r, "A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}"] \arrow[dr, phantom, "\mathbf{QAP} = \mathbf{F}"] & \mathbb{R}^2 \quad C = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \arrow[d, "Q = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}"] \\
D = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \quad \mathbb{R}^2 \arrow[u, "P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}"] \arrow[r, "F = \begin{pmatrix} 1 & -\frac{1}{3} \\ 3 & -1 \end{pmatrix}"'] & \mathbb{R}^2 \quad E = \left\{ \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right\}
\end{tikzcd}
$$

2.

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
B = \{1, x, x^2\} \quad P_2(\mathbb{R}) \arrow[r, "A = \begin{pmatrix} 0 & 6 & 2 \\ 0 & 0 & 12 \end{pmatrix}"] \arrow[dr, phantom, "\mathbf{QAP} = \mathbf{F}"] & P_1(\mathbb{R}) \quad C = \{1, x\} \arrow[d, "Q = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}"] \\
D = \{1 - x, 1 + x, 4x^2\} \quad P_2(\mathbb{R}) \arrow[u, "P = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}"] \arrow[r, "F = \begin{pmatrix} -6 & 6 & 8 \\ -12 & 12 & 64 \end{pmatrix}"'] & P_1(\mathbb{R}) \quad E = \{1 - 2x, x\}
\end{tikzcd}
$$

3. a) $P = \begin{pmatrix} 2 & 2 \\ 3 & -1 \end{pmatrix}$ $\qquad Q = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad F = \begin{pmatrix} 5 & 1 \\ 21 & 1 \end{pmatrix}$.

b) $A = Q^{-1}FP^{-1} = FP^{-1} = \begin{pmatrix} 1 & 1 \\ 3 & 5 \end{pmatrix}$ $\qquad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ 3x + 5y \end{pmatrix}$.

4. a) $P = \begin{pmatrix} 2 & 2 \\ 3 & -1 \end{pmatrix}$ $\qquad Q = P^{-1} = -\frac{1}{8}\begin{pmatrix} -1 & -2 \\ -3 & 2 \end{pmatrix}$ $\qquad F = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$.

b) $A = Q^{-1}FP^{-1} = PFP^{-1} = -\frac{1}{8}\begin{pmatrix} -26 & 12 \\ 9 & -14 \end{pmatrix}$ $\qquad T\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{8}\begin{pmatrix} -26x + 12y \\ 9x - 14y \end{pmatrix}$.

5. Proof. Hint set up a basis for $\mathbb{R}^2$ consisting of a vector parallel to the line and a vector perpendicular to the line and then use commutative diagrams and matrices of change of basis. (See the last example on this topic in lectures where projections are analysed).

MATH2099 PROBLEM SHEET 11
ORTHOGONAL COMPLEMENTS AND PROJECTIONS

$$W^\perp = \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v}\cdot\mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}$$

Let W be a k dimensional subspace of $\mathbb{R}^n$ with an orthogonal basis $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_k\}$. Then for any vector $\mathbf{v}$ in $\mathbb{R}^n$

$$\mathrm{Proj}_W(\mathbf{v}) = \mathrm{Proj}_{\mathbf{a}_1}(\mathbf{v}) + \mathrm{Proj}_{\mathbf{a}_2}(\mathbf{v}) + \cdots + \mathrm{Proj}_{\mathbf{a}_k}(\mathbf{v})$$

Harder problems are marked with a ★

1. Let $W = \text{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\3\end{pmatrix}\right\}$ in $\mathbb{R}^3$.

a) Find a basis B_1 for the orthogonal complement $W^\perp$ of W.

b) Check that your calculated basis vectors for $W^\perp$ are orthogonal to the basis vectors supplied for W.

c) Describe W and $W^\perp$ geometrically.

2. Let $U = \text{span}\left\{\begin{pmatrix}1\\2\\4\end{pmatrix}\right\}$ in $\mathbb{R}^3$.

a) Find a basis B_2 for the orthogonal complement $U^\perp$ of U.

b) Check that your calculated basis vectors for $U^\perp$ are orthogonal to the basis vector supplied for U.

c) Describe U and $U^\perp$ geometrically.

3. Let $W = \text{span}\left\{\begin{pmatrix}1\\1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\3\end{pmatrix}\right\}$ in $\mathbb{R}^4$.

a) Find a basis B_3 for the orthogonal complement $W^\perp$ of W.
b) Check that your two calculated basis vectors for $W^\perp$ are orthogonal to the two basis vectors supplied for W.

4. Would you expect that $(W^\perp)^\perp = W$ where W is the two dimensional subspace of $\mathbb{R}^4$ defined in Question 3 ?

5.★ Calculate a basis B_4 for the orthogonal complement $(W^\perp)^\perp$ of $W^\perp$ using your basis for $W^\perp$ calculated in Q3. Did you obtain the original basis $\left\{\begin{pmatrix}1\\1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\3\end{pmatrix}\right\}$? Explain.

6. Let $\mathbf{v} = \begin{pmatrix} 15 \\ -1 \\ -1 \end{pmatrix} \in \mathbb{R}^3$ and W be the subspace of $\mathbb{R}^3$ with orthogonal basis $\left\{ \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \right\}$.

a) Find $\text{Proj}_W(\mathbf{v})$. (Note that the basis vectors are already orthogonal so Gram-Schmidt is not needed).

b) Express $\mathbf{v}$ as a sum $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1 \in W$ and $\mathbf{w}_2 \in W^\perp$.

c) Hence find the shortest distance between the vector $\begin{pmatrix} 15 \\ -1 \\ -1 \end{pmatrix}$ in $\mathbb{R}^3$ and W.

d) Find a vector $\mathbf{u}_1 \in \mathbb{R}^3$ which is twice as far from W as $\mathbf{v}$ is from W, but with the same projection onto W as $\mathbf{v}$.

e) Find another vector $\mathbf{u}_2$ in $\mathbb{R}^3$ with the property specified in d).

f) What is the distance between $\mathbf{u}_1$ and $\mathbf{u}_2$?

7. Let $\mathbf{v} = \begin{pmatrix} 1 \\ 13 \end{pmatrix} \in \mathbb{R}^2$ and W be the subspace of $\mathbb{R}^2$ spanned by $\left\{ \begin{pmatrix} 3 \\ 5 \end{pmatrix} \right\}$.

a) Find $\text{Proj}_W(\mathbf{v})$.

b) Express $\mathbf{v}$ as a sum $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1 \in W$ and $\mathbf{w}_2 \in W^\perp$.

c) Hence find the shortest distance between the vector $\begin{pmatrix} 1 \\ 13 \end{pmatrix}$ in $\mathbb{R}^2$ and W.

d) By considering the decomposition in b) find the vector $\mathbf{u} \in \mathbb{R}^2$ which is the reflection of $\mathbf{v}$ in W.

e) Show that $\|\mathbf{u}\| = \|\mathbf{v}\|$. Explain with a sketch.

f) Show that $(\mathbf{u} - \mathbf{v}) \in W^\perp$. Explain with a sketch.

8. Find the projection of the vector $\mathbf{v} = \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix}$ onto the subspace W of $\mathbb{R}^3$ spanned by $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$. Hence find the shortest distance between $\mathbf{v}$ and W and the reflection of $\mathbf{v}$ in the plane W.

9.$\bigstar$ Let $\mathbf{v} = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 1 \end{pmatrix} \in \mathbb{R}^4$ and W be the subspace of $\mathbb{R}^4$ with basis $\left\{ \begin{pmatrix} 2 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 2 \\ -1 \end{pmatrix} \right\}$.

a) Find $\text{Proj}_W(\mathbf{v})$.

b) Express $\mathbf{v}$ as a sum $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1 \in W$ and $\mathbf{w}_2 \in W^\perp$.

c) Find the shortest distance between $\mathbf{v}$ and W.

10.** Let A be an $m \times n$ matrix. Prove that $\mathrm{Im}(A^T)^\perp = \mathrm{Ker}(A)$.

ANSWERS

1. a) $B_1 = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$. c) W is a plane in $\mathbb{R}^3$ and $W^\perp$ is a line in $\mathbb{R}^3$.

2. a) $B_2 = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} \right\}$. c) U is a line in $\mathbb{R}^3$ and $U^\perp$ is a plane in $\mathbb{R}^3$.

3. a) $B_3 = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$.

4. Yes.

5. a) $B_4 = \left\{ \begin{pmatrix} -2 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$. Bases for subspaces are highly non-unique.

6. a) $\mathrm{Proj}_W(\mathbf{v}) = \begin{pmatrix} 14 \\ 3 \\ -2 \end{pmatrix}$ b) $\mathbf{v} = \begin{pmatrix} 14 \\ 3 \\ -2 \end{pmatrix} \in W + \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} \in W^\perp$ c) $\sqrt{18}$

d) $\mathbf{u}_1 = \mathbf{w}_1 + 2\mathbf{w}_2 = \begin{pmatrix} 16 \\ -5 \\ 0 \end{pmatrix}$ e) $\mathbf{u}_2 = \begin{pmatrix} 12 \\ 11 \\ -4 \end{pmatrix}$ f) $4\sqrt{18}$.

7. a) $\begin{pmatrix} 6 \\ 10 \end{pmatrix}$ b) $\begin{pmatrix} 1 \\ 13 \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \end{pmatrix} + \begin{pmatrix} -5 \\ 3 \end{pmatrix}$ c) $\sqrt{34}$ d) $\begin{pmatrix} 11 \\ 7 \end{pmatrix}$

8. Applying G-S we have an orthogonal basis $\left\{ \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 13 \\ 2 \\ -14 \end{pmatrix} \right\}$ for W.

a) $\begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}$ b) $\sqrt{41}$ c) $\begin{pmatrix} 7 \\ -4 \\ 3 \end{pmatrix}$

9. a) $\frac{1}{3}\begin{pmatrix} 6 \\ 7 \\ 2 \\ 1 \end{pmatrix}$ b) $\begin{pmatrix} 3 \\ 1 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 6 \\ 7 \\ 2 \\ 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 3 \\ -4 \\ 4 \\ 2 \end{pmatrix}$

10. Proof. Hint: To show that two sets Z and Y are equal show that $x \in Z \to x \in Y$ **and** $x \in Y \to x \in Z$. Recall also that $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T\mathbf{b}$.

MATH2099 PROBLEM SHEET 12
MATRIX TRANSFORMATIONS IN SPACE

Some standard rotation matrices in $\mathbb{R}^3$ are:

Rotation anticlockwise by θ about the z-axis $\quad R_z = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Rotation anticlockwise by θ about the y-axis $\quad R_y = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$

Rotation anticlockwise by θ about the x-axis $\quad R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$

All rotations about the positive axis according to the right-hand rule

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Let G be the $n \times k$ matrix $G = (\mathbf{u}_1 | \mathbf{u}_2 | \cdots | \mathbf{u}_k)$ which has as its columns an orthonormal basis for a subspace W of $\mathbb{R}^n$ and let $S = GG^T$. Then the standard matrix S projects $\mathbb{R}^n$ onto W.

S is a projection matrix $\leftrightarrow$ S is symmetric and $S^2 = S$.

Harder problems are marked with a ★

1. In each of the following, establish the required standard matrix by considering the action of the transformation on the standard ordered basis. For the rotation matrices you may use the formulae above.

a) Find the standard matrix R_z of the transformation which rotates $\mathbb{R}^3$ anticlockwise by $30°$ about the z axis.

b) Find the standard matrix A_1 of the transformation which rotates $\mathbb{R}^3$ clockwise by $90°$ about the x axis.

c) Find the standard matrix A_2 of the transformation which reflects $\mathbb{R}^3$ in the $x-z$ plane.

d) Find the standard matrix A_3 of the transformation which projects $\mathbb{R}^3$ onto the $x-z$ plane.

e) Find the standard matrix A_4 of the transformation which 'projects' the $x-y$ plane in $\mathbb{R}^3$ onto $\mathbb{R}^2$.

f) Find the standard matrix A_5 of the transformation which 'embeds' $\mathbb{R}^2$ as the $x-y$ plane in $\mathbb{R}^3$.

2. a) Find the standard matrix of the transformation which first rotates $\mathbb{R}^3$ anticlockwise by $30°$ about the z axis and then reflects in the $x-z$ plane.

b) Find the standard matrix of the transformation which first rotates $\mathbb{R}^3$ clockwise by $90°$ about the x axis and then projects onto the $x-z$ plane.

3.★ Find the standard matrix of the linear transformation with rotates $\mathbb{R}^3$ anticlockwise by $90°$, first about the x axis, then the y axis and finally the z axis.

4.★ Let C be the standard matrix of the transformation which rotates $\mathbb{R}^3$ anti-clockwise by $10°$ about the x axis.

a) Evaluate C^{36}.

b) Evaluate C^{18}.

5. Let $\mathbf{v} = \begin{pmatrix} 15 \\ -1 \\ -1 \end{pmatrix} \in \mathbb{R}^3$ and W be the subspace of $\mathbb{R}^3$ with orthogonal basis $\left\{ \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \right\}$.

a) Find the matrix S which projects $\mathbb{R}^3$ onto W.

b) Check your answer for $\mathrm{Proj}_W(\mathbf{v})$ in the previous problem set by evaluating $S\mathbf{v}$.

c) Show that S is symmetric and idempotent.

6. Let $\mathbf{v} = \begin{pmatrix} 1 \\ 13 \end{pmatrix} \in \mathbb{R}^2$ and W be the subspace of $\mathbb{R}^2$ spanned by $\left\{ \begin{pmatrix} 3 \\ 5 \end{pmatrix} \right\}$.

a) Find the matrix S which projects $\mathbb{R}^2$ onto W.

b) Check your answer for $\mathrm{Proj}_W(\mathbf{v})$ in the previous problem set by evaluating $S\mathbf{v}$.

c) Show that S is symmetric and idempotent.

d) Show that $\mathrm{Im}(S) = W$ and $\mathrm{Ker}(S) = W^{\perp}$.

7. Let $\mathbf{v} = \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix}$ and W be the subspace of $\mathbb{R}^3$ spanned by $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$.

a) Find the matrix S which projects $\mathbb{R}^3$ onto W. (You may use your G-S calculations from the previous problem set).

b) Check your answer for $\mathrm{Proj}_W(\mathbf{v})$ in the previous problem set by evaluating $S\mathbf{v}$.

c) Show that S is symmetric and idempotent.

d)★ Show that $\mathrm{Ker}(S) = W^{\perp}$.

8.★ Suppose that the matrix S projects $\mathbb{R}^n$ onto a subspace W and let $T = I - S$.

a) Show that T is also a projection matrix.

b) Prove that T projects onto $W^\perp$.

c) Prove that $\mathrm{Ker}(T) = W$.

9.★★ Let $\mathbf{v} = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 1 \end{pmatrix} \in \mathbb{R}^4$ and W be the subspace of $\mathbb{R}^4$ with basis $\left\{ \begin{pmatrix} 2 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 2 \\ -1 \end{pmatrix} \right\}$.

a) Find the matrix S which projects $\mathbb{R}^3$ onto W. (You may use your G-S calculations from the previous problem set).

b) Check your answer for $\mathrm{Proj}_W(\mathbf{v})$ in the previous problem set by evaluating $S\mathbf{v}$.

c) Show that S is symmetric and idempotent.

d) Show that $\mathrm{Im}(S) = W$ and $\mathrm{Ker}(S) = W^\perp$.

ANSWERS

1. a) $R_z = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ b) $A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ c) $A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

d) $A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ e) $A_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ f) $A_5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

2. a) $A_2R_z = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ b) $A_3A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$

3. $R_zR_yR_x = R_y$.

4. a) $C^{36} = I$ b) $C^{18} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

5. a) $S = \frac{1}{18}\begin{pmatrix} 17 & 4 & -1 \\ 4 & 2 & 4 \\ -1 & 4 & 17 \end{pmatrix}$ 6. a) $S = \frac{1}{34}\begin{pmatrix} 9 & 15 \\ 15 & 25 \end{pmatrix}$

7. $S = \frac{1}{41}\begin{pmatrix} 37 & 12 & -2 \\ 12 & 5 & 6 \\ -2 & 6 & 40 \end{pmatrix}$

8. Proof.

9. a) $S = \frac{1}{9}\begin{pmatrix} 4 & 4 & 0 & 2 \\ 4 & 5 & 2 & 0 \\ 0 & 2 & 4 & -4 \\ 2 & 0 & -4 & 5 \end{pmatrix}$

MATH2099 PROBLEM SHEET 13
LINES OF BEST FIT AND REFLECTIONS

To find a curve of best fit, find the least squares solution to $A\mathbf{x} = \mathbf{y}$ by moving over to the normal equations $A^T A\mathbf{x} = A^T\mathbf{y}$.

Let W be an $(n-1)$ dimensional subspace (hyperplane) in $\mathbb{R}^n$ with $\mathbf{d} \in W^\perp$. Then the standard matrix which reflects $\mathbb{R}^n$ across W is given by

$$R = I - (\frac{2}{\mathbf{d} \cdot \mathbf{d}})\mathbf{d}\mathbf{d}^T$$

If R is a reflection matrix then R is symmetric and orthogonal and hence $R^2 = I$.

Harder problems are marked with a ★

1. Within each of the following 6 families of curves find the best fit to the set of points

$$\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \end{pmatrix} \right\} \quad \text{in} \quad \mathbb{R}^2.$$

In each of the 6 cases, sketch the curve of best fit and the data points on the same set of axes, and find an estimate for y when $x = 1$.

a) $y = mx + b$.

b) $y = mx$.

c) $y = \alpha x^2$.

d) $y = \alpha x^2 + \gamma$.

e) $y = \alpha x^2 + \beta x$.

f)* $y = \alpha x^2 + \beta x + \gamma$.

2.★ Find the plane of best fit of the form $z = ax + by + c$ passing through the points

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix} \right\} \quad \text{in} \quad \mathbb{R}^3.$$

3.★★ Suppose that $\left\{ \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \cdots, \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right\}$ are n data points in $\mathbb{R}^2$.

Let $\overline{x} = \dfrac{x_1 + x_2 + \cdots + x_n}{n}$ be the average of all the x values and $\overline{y} = \dfrac{y_1 + y_2 + \cdots + y_n}{n}$ be the average of all the y values. Prove that $(\overline{x}, \overline{y})$ (the centroid) lies on the line of best fit $y = mx + b$.

4. In our first example from lectures we showed that the line of best fit of the form $y = mx + b$ through the set of points

$$\left\{ \begin{pmatrix} 4 \\ 9 \end{pmatrix}, \begin{pmatrix} 5 \\ 8 \end{pmatrix}, \begin{pmatrix} 9 \\ 6 \end{pmatrix}, \begin{pmatrix} 12 \\ 3 \end{pmatrix} \right\}$$

is given by $y = -\dfrac{29}{41}x + \dfrac{484}{41}$. Check this answer using the result from Example 3.

5. Let R be the reflection matrix in the line $y = 3x$ in $\mathbb{R}^2$.

a) Noting that the vector $\mathbf{d} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ is orthogonal to the line $y = 3x$ use the formula $R = I - (\dfrac{2}{\mathbf{d} \cdot \mathbf{d}})\mathbf{d}\mathbf{d}^T$ to calculate R.

b) Check your answer using our earlier formula $R = \dfrac{1}{1+m^2} \begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$.

Note that this formula only works in $\mathbb{R}^2$.

c) Show that R is orthogonal.

d) Find the reflection $\mathbf{v}$ of $\mathbf{u} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$ in the line $y = 3x$.

e) Verify that $\|\mathbf{u}\| = \|\mathbf{v}\|$ and explain this result geometrically.

6. Let R be the matrix of reflection across the plane $2x - y + 3z = 0$ in $\mathbb{R}^3$.

a) Evaluate R.

b) Verify that $R^2 = I$.

c) Find the reflection $\mathbf{v}$ of $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and explain your result geometrically.

d) Find the reflection $\mathbf{v}$ of $\mathbf{w} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ and explain your result geometrically.

7. Let R be the matrix of reflection which exchanges the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ in $\mathbb{R}^3$.

a) Verify that $\|\mathbf{a}\| = \|\mathbf{b}\|$.

b) Evaluate R.

c) Check that $R\mathbf{a} = \mathbf{b}$.

d) Find the equation of the plane of reflection.

e) Show that $\mathbf{a} + \mathbf{b}$ lies on the plane of reflection. Explain.

8. Explain why there is no reflection which exchanges $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

9.★ Find the matrix of reflection which exchanges the vectors $\mathbf{a} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 7 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ 2 \\ -1 \\ 6 \end{pmatrix}$ in $\mathbb{R}^4$. Also find the hyperplane fixed by this reflection.

10.★ Let W be an $(n-1)$ dimensional subspace (hyperplane) in $\mathbb{R}^n$ with $\mathbf{d} \in W^{\perp}$. Then the standard matrix which reflects $\mathbb{R}^n$ across W is given by

$$R = I - (\frac{2}{\mathbf{d} \cdot \mathbf{d}})\mathbf{d}\mathbf{d}^T$$

a) Prove that R is orthogonal.

b) Prove that R is symmetric.

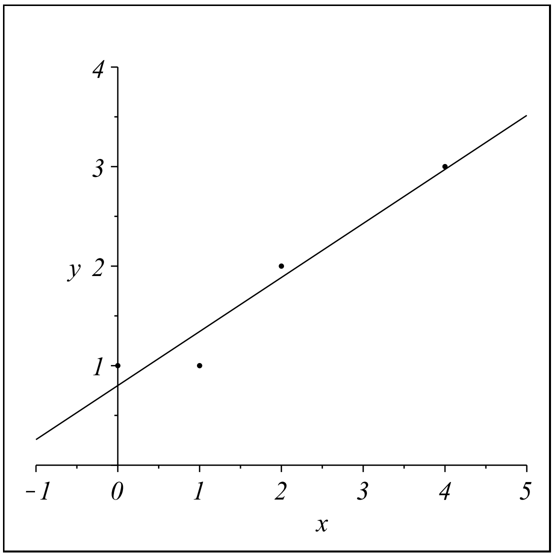
c) Prove that $R^2 = I$.

d) Prove that $R\mathbf{d} = -\mathbf{d}$.

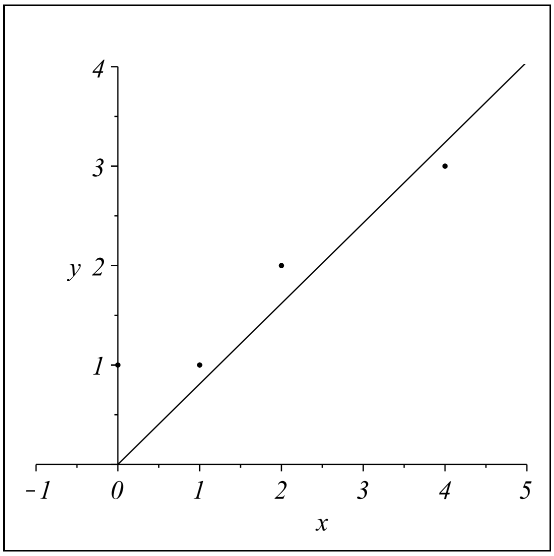
e) Prove that $R\mathbf{w} = \mathbf{w}$ for $\mathbf{w} \in W$.

ANSWERS

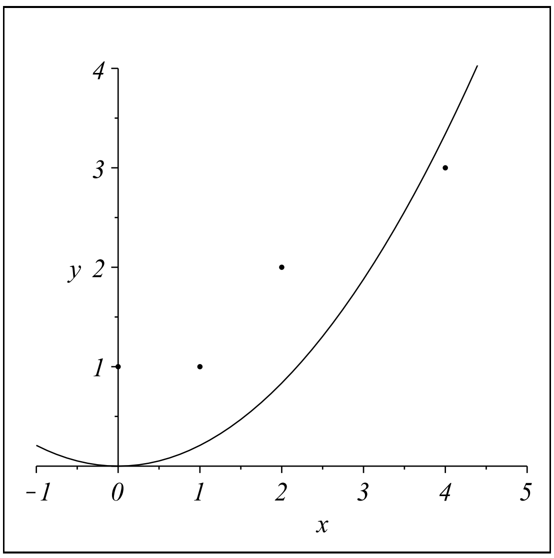
1. a) $y = \dfrac{19}{35}x + \dfrac{4}{5}$ $\qquad y(1) \approx 1.343$



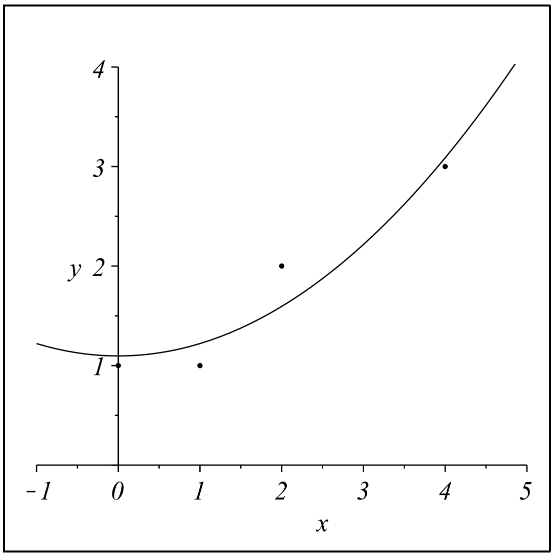
b) $y = \dfrac{17}{21}x$ $\qquad y(1) \approx .810$



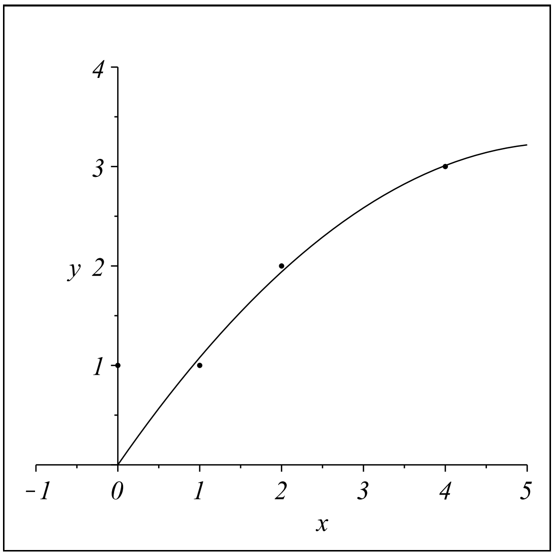
c) $y = \dfrac{19}{91}x^2$ $\qquad y(1) \approx .209$



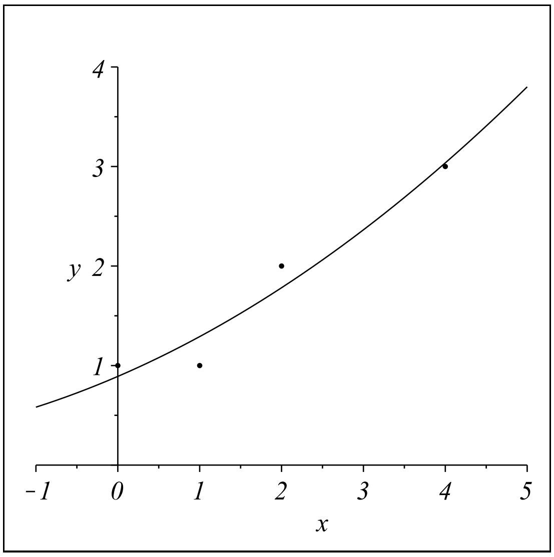
d) $y = \frac{27}{217}x^2 + \frac{34}{31}$ $y(1) \approx 1.221$



e) $y = -\frac{11}{101}x^2 + \frac{120}{101}x$ $y(1) \approx 1.709$



f) $y = \frac{1}{22}x^2 + \frac{39}{110}x + \frac{49}{55}$ $y(1) \approx 1.291$



2. $z = \frac{59}{35}x - \frac{38}{35}y + \frac{117}{35}$

3. Hint: Consider the second of the normal equations.

4. $(\overline{x}, \overline{y}) = \left(\frac{30}{4}, \frac{26}{4}\right)$ satisfies $y = -\frac{29}{41}x + \frac{484}{41}$.

5. a) $R = \frac{1}{5}\begin{pmatrix} -4 & 3 \\ 3 & 4 \end{pmatrix}$. c) Proof. d) $\mathbf{v} = \begin{pmatrix} 2 \\ 11 \end{pmatrix}$

e) Reflections do not change the size of objects.

6. $R = \frac{1}{7}\begin{pmatrix} 3 & 2 & -6 \\ 2 & 6 & 3 \\ -6 & 3 & -2 \end{pmatrix}$ b) Proof. c) $\mathbf{v} = -\mathbf{u}$, since $\mathbf{u} \perp$ to the plane.

d) $\mathbf{v} = \mathbf{w}$, since $\mathbf{w}$ is in the plane.

7. a) $\|\mathbf{a}\| = \|\mathbf{b}\| = 3$ b) $R = \frac{1}{3}\begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ 2 & -2 & -1 \end{pmatrix}$ c) Proof. d) $x - y - 2z = 0$

e) $R(\mathbf{a} + \mathbf{b}) = R(\mathbf{a}) + R(\mathbf{b}) = \mathbf{b} + \mathbf{a} = \mathbf{a} + \mathbf{b}$.

8. The vector are of different magnitude.

9. $R = \frac{1}{5}\begin{pmatrix} 1 & 4 & -2 & -2 \\ 4 & 1 & 2 & 2 \\ -2 & 2 & 4 & -1 \\ -2 & 2 & -1 & 4 \end{pmatrix}$ $2x_1 - 2x_2 + x_3 + x_4 = 0.$

10. Proofs.

MATH2099 PROBLEM SHEET 14
DETERMINANTS

Given a square $n \times n$ matrix A the inverse of A (denoted by A^{-1}) is another $n \times n$ matrix with the property that $AA^{-1} = I$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

More generally $\det(A) = a_{11}|A_{11}| - a_{12}|A_{12}| + a_{13}|A_{13}| - \cdots (-1)^n a_{1n}|A_{1n}|$

A^{-1} exists if and only if $\det(A) \neq 0$.

$\det(AB) = \det(A)\det(B)$.

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

In general $\det(A + B) \neq \det(A) + \det(B)$.

$\det(A) = \det(A^T)$.

If a square matrix has a zero row or column then its determinant is 0.

Swapping rows or columns multiplies the determinant by -1.

Multiplying a row or column by α multiplies the determinant by α.

If a row of a square matrix is a scalar multiple of any other row then the determinant of the matrix is 0 and hence the matrix is singular.

If a column of a square matrix is a scalar multiple of any other column then the determinant of the matrix is 0 and hence the matrix is singular.

The determinant of a square matrix in echelon form is the product of the diagonal elements of the echelon form.

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \qquad \begin{pmatrix} + & - \\ - & + \end{pmatrix}$$

Harder problems are marked with a ★

1. Find the determinant of each of the following matrices and hence determine which of the matrices are invertible.

a) $\begin{pmatrix} 1 & 4 \\ 2 & 9 \end{pmatrix}$ $\quad$ $b)$ $\begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}$ $\quad$ $c)$ $\begin{pmatrix} 1-i & 2i \\ 2 & -2+2i \end{pmatrix}$.

2. For each of the following matrices evaluate the determinant by expanding across the top row. Then check your answer by re-evaluating the determinant down the middle column. Which of the three matrices is singular ?

a) $\begin{pmatrix} 1 & 4 & 1 \\ 2 & 1 & 8 \\ 2 & 5 & 7 \end{pmatrix}$ $\quad$ b) $\begin{pmatrix} 1 & 4 & 1 \\ 2 & 1 & 8 \\ 3 & 5 & 9 \end{pmatrix}$ $\quad$ c) $\begin{pmatrix} -1 & 3 & -4 \\ 8 & -2 & -3 \\ 1 & 0 & 7 \end{pmatrix}$

3. Evaluate $\begin{vmatrix} 5 & 0 & 1 & -2 \\ 0 & 0 & 0 & 3 \\ 8 & 0 & 2 & 9 \\ 3 & 1 & 5 & -2 \end{vmatrix}$

4. Find $\det \begin{pmatrix} 1 & 4 & -2 & 1 & 3 & 8 & \pi \\ 0 & 3 & \sqrt{2} & \sqrt{3} & \sqrt{5} & \sqrt{6} & \sqrt{7} \\ 0 & 0 & 1 & 4 & -2 & e & e^2 \\ 0 & 0 & 0 & 4 & 0.1 & 0.01 & 0.001 \\ 0 & 0 & 0 & 0 & -1 & 10^6 & 10^7 \\ 0 & 0 & 0 & 0 & 0 & 2 & 10^8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}$.

5. A particular 5×5 matrix C has determinant equal to 3. Find the determinant of E if E is defined to be:

a) C^T.

b) C with the last two rows swapped.

c) C with the last two rows swapped and the first two columns swapped.

d) C with the second row multiplied by 2.

e) $2C$.

f) C^{-1}.

6. Prove that $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ is singular.

7. For which value of k is $\begin{pmatrix} 2 & 1 & 1 \\ 2 & k & 4 \\ 3 & 1 & 7 \end{pmatrix}$ not invertible ?

8. Suppose that A and B are two $n \times n$ matrices such that there exists an invertible matrix P with the property that $P^{-1}AP = B$. Prove that det(A)=det(B).

9.★ A square matrix Q is said to be orthogonal if $Q^TQ = I$. Show that the determinant of an orthogonal matrix is ± 1.

10.★ A matrix P is said to be idempotent if $P^2 = P$. Show that the determinant of an idempotent matrix is 0 or 1.

11.★ Suppose that A is a square matrix with QR decomposition $A = QR$. Show that $\det(A) = \pm$the product of the diagonal entries of R.

12.★ Suppose that A is a 2×3 matrix and that $B = A^TA$. Show that $\det(B) = 0$.

13.★ An $n \times n$ matrix A is said to be skew symmetric if $A^T = -A$. Suppose that A is skew-symmetric and that n is odd. Prove that $\det(A) = 0$.

ANSWERS

1. a) 1 b) 0 c) 0 Only a) is invertible.

2. a) -17 b) 0 c) -171. Only b) is singular.

3. -6

4. -120

5. a) 3 b) -3 c) 3 d) 6 e) 96 f) $\frac{1}{3}$

6. $R_1 = R_3 \to \det A = 0$. Many other reasons exist!

7. $k = \frac{8}{11}$

8. Proof.

9. Proof.

10. Proof.

11. Proof.

12. Hint: There is a non-zero vector $\mathbf{v} \in \mathbb{R}^3$ such that $A\mathbf{v} = \mathbf{0}$. Why?

13. Proof.

MATH2099 PROBLEM SHEET 15
EIGENVALUES, EIGENVECTORS AND DIAGONALISATION

Given a square matrix A, a non-zero vector $\mathbf{v}$ is said to be an eigenvector of A if $A\mathbf{v} = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{R}$. The number λ is referred to as the associated eigenvalue of A.

We first find eigenvalues through the characteristic equation $\det(A - \lambda I) = 0$. The eigenvectors are then found via row reduction and back substitution.

The zero vector is **never** an eigenvector but it is OK to have a zero eigenvalue.

If an $n \times n$ matrix A has n linearly independent eigenvectors and P is the matrix of eigenvectors aligned vertically then $P^{-1}AP = D$ where D is the diagonal matrix of eigenvalues. The order of the eigenvalues in D must match the order of the eigenvectors in P. This is referred to as the diagonalization of A.

A matrix can be non-diagonalisable by coming up short on eigenvectors. The only general way to find out if a matrix has a full set of eigenvectors is to find them all.

A useful check is the fact that $\Sigma(\text{eigenvalues}) = \text{Trace}(A)$.

Establishing the eigenanalysis of a particular matrix gives you a clear vision of the internal workings of that matrix, and through diagonalisation the matrix may be transformed into a more workable diagonal structure.

Eigenvectors from different eigenvalues are linearly independent.

Two $n \times n$ matrices A and B are said to be similar if there exists an $n \times n$ matrix P with the property that $P^{-1}AP = B$.

Similar matrices share the same eigenvalues.

Harder problems are marked with a ★

1. For each of the three following 2×2 matrices:

I) $A = \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}$ **II)** $A = \begin{pmatrix} 3 & 4 \\ 1 & 3 \end{pmatrix}$ **III)** $A = \begin{pmatrix} 7 & 4 \\ 4 & 1 \end{pmatrix}$

a) Find the eigenvalues and eigenvectors of A.

b) Verify that $\Sigma(\text{eigenvalues}) = \text{Trace}(A)$.

c) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

d) Check your answer by verifying that $AP = PD$.

e) Find a different invertible matrix P and a different diagonal matrix D such that $P^{-1}AP = D$.

f) Check your answer again.

g) For the smaller of the two eigenvalues find an eigenvector with an x component of 12.

h) For the larger of the two eigenvalues find a unit eigenvector.

2. Show that $C = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$ is not diagonalisable.

3.★ Consider $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

a) Find the eigenvalues of I.

b) Find the eigenvectors of I.

c) Is I diagonalisable?

4.★★ The concepts of eigenvalues and eigenvectors are workable for complex numbers but the algebra becomes difficult. Find all the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$$

.

5. Show that $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector of the matrix $\begin{pmatrix} 2 & 0 & 0 & 0 \\ -3 & 8 & -3 & 3 \\ -8 & 0 & 10 & 8 \\ -8 & 0 & 8 & 10 \end{pmatrix}$ and find the associated eigenvalue. Hint: This is less than a minute's work.

6. The matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ -8 & 4 & -6 \\ 8 & 1 & 9 \end{pmatrix}$ has an eigenvalue $\lambda = 1$ with associated eigenspace $E_1 = \text{span}\left\{\begin{pmatrix} 15 \\ 8 \\ -16 \end{pmatrix}\right\}$ and eigenvalue $\lambda = 6$ with associated eigenspace $E_6 = \text{span}\left\{\begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}\right\}$.

a) Find the third eigenvalue without calculating the characteristic polynomial.

b) Find the third eigenspace using Gaussian Elimination.

c) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

d) Find a different invertible matrix P and a different diagonal matrix D such that $P^{-1}AP = D$.

e) How many different possible diagonal matrices D exist so that $P^{-1}AP = D$?

f) How many different possible invertible matrices P exist so that $P^{-1}AP = D$?

g) Find an eigenvector in the eigenspace in b) with the property that the sum of the three components is 10.

7. Let $C = \begin{pmatrix} 5 & 6 & 0 \\ -1 & 0 & 0 \\ 1 & 2 & 2 \end{pmatrix}$.

a) Show that C has an eigenvalue $\lambda = 2$. (Hint: Try to find some eigenvectors for $\lambda = 2$)

b) Find the complete set of eigenvalues for C without calculating the characteristic polynomial.

c) Is C diagonalisable ?

8. Establish a diagonalisation of $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. This will take some time !

9. Let $A = \begin{pmatrix} 1 & 3 & 2 & 7 \\ -1 & 2 & 8 & 8 \\ 4 & 0 & 5 & 9 \\ 1 & 8 & 2 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 1 & 3 \\ 2 & 4 & 1 & 8 \\ 8 & 1 & 4 & 0 \\ -1 & 6 & -2 & 2 \end{pmatrix}$. Prove that A and B are not similar.

10. If A is a matrix with eigenvector $\mathbf{v}$ and corresponding eigenvalue λ prove that $\mathbf{v}$ is also an eigenvector of A^2 and find the corresponding eigenvalue.

11.$\star$ Let $A = \begin{pmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{pmatrix}$. Without calculating the characteristic polynomial find all the eigenvalues of A and their geometric multiplicities.

12.$\star$ Let W be a plane in $\mathbb{R}^3$. Suppose that S is the 3×3 projection matrix onto W and that R is the 3×3 reflection matrix in W. Considering the geometry of the situation only:

a) Find the eigenvalues of S and describe the eigenspaces.

b) Find the eigenvalues of R and describe the eigenspaces.

ANSWERS

Please note that the whole theory of eigenvalues and eigenvectors involves choices and options. Your answers may be quite different from the ones below and still be correct!

1.

I) a) $\lambda = 4, 5$ $\quad E_4 = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$ $\quad E_5 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

b) Check.

c) $P = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}$.

d) Check.

e) $P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix}$.

f) Check.

g) $\begin{pmatrix} 12 \\ 24 \end{pmatrix}$.

h) $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

II) a) $\lambda = 1, 5$ $\quad E_1 = \text{span}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$ $\quad E_5 = \text{span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$.

b) Check.

c) $P = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$.

d) Check.

e) $P = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$.

f) Check.

g) $\begin{pmatrix} 12 \\ -6 \end{pmatrix}$.

h) $\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$.

III) a) $\lambda = -1, 9$ $\quad E_{-1} = \text{span}\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}$ $\quad E_9 = \text{span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$

b) Check.

c) $P = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} -1 & 0 \\ 0 & 9 \end{pmatrix}$.

d) Check.

e) $P = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$ and $D = \begin{pmatrix} 9 & 0 \\ 0 & -1 \end{pmatrix}$.

f) Check.

g) $\begin{pmatrix} 12 \\ -24 \end{pmatrix}$.

h) $\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$.

2. $\lambda = 3, 3$ $\quad E_3 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$ $\quad$ Insufficient independent eigenvectors!

3. $\lambda = 1, 1$ $\quad E_1 = \mathbb{R}^2$.

Certainly diagonalisable! Every non-zero vector is an eigenvector and any invertible 2×2 matrix will serve as P.

4. $\lambda = -1 \pm i$ $\quad E_{-1+i} = \text{span}\left\{\begin{pmatrix} 1 \\ i \end{pmatrix}\right\}$ $\quad E_{-1-i} = \text{span}\left\{\begin{pmatrix} 1 \\ -i \end{pmatrix}\right\}$.

5. $\lambda = 2$.

6. a) $\lambda = 7$ b) $E_7 = \text{span}\left\{\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}\right\}$ c) $P = \begin{pmatrix} 15 & 0 & 0 \\ 8 & -3 & -2 \\ -16 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{pmatrix}$.

d) $P = \begin{pmatrix} 0 & 0 & 15 \\ -2 & -3 & 8 \\ 1 & 1 & -16 \end{pmatrix}$, $D = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. e) 6 f) ∞ g) $\begin{pmatrix} 0 \\ 20 \\ -10 \end{pmatrix}$.

7. b) $\lambda = 2, 2, 3$ $\quad$ c) Yes. Short on eigenvalues but has a complete set of eigenvectors.

8. $P = \begin{pmatrix} 2 & 0 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to P^{-1}AP = D$.

9. Proof.

10. New eigenvalue is λ^2.

11. Eigenvalues are $\lambda = a - 1$ with multiplicity 3 and $\lambda = a + 3$ with multiplicity 1.

12. a) $\lambda = 1, 1, 0$ with $E_1 = W$ and $E_0 = W^{\perp}$.

b) $\lambda = 1, 1, -1$ with $E_1 = W$ and $E_{-1} = W^{\perp}$.

MATH2099 PROBLEM SHEET 16
Applications of Eigenvectors

$$P^{-1}AP = D \to A = PDP^{-1} \to A^n = PD^nP^{-1}.$$

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots \frac{(At)^n}{n!} + \cdots$$

$$P^{-1}AP = D \to A = PDP^{-1} \to e^{At} = Pe^{Dt}P^{-1}.$$

If $AB = BA$ (that is A and B commute) then $e^{At}e^{Bt} = e^{(A+B)t}$.

$$(e^{At})^{-1} = e^{-At}.$$

$$(e^{At})' = Ae^{At}.$$

The solution of the system of differential equations $\mathbf{y}' = A\mathbf{y}$ with initial conditions $\mathbf{y}(0) = \mathbf{c}$ is given by $\mathbf{y} = e^{At}\mathbf{c}$.

Suppose that A is a 2×2 matrix with two linearly independent eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ and associated eigenvalues λ_1, λ_2. Then the solution to $\mathbf{y}' = A\mathbf{y}$ takes the form

$$\mathbf{y} = \alpha_1\mathbf{v}_1e^{\lambda_1 t} + \alpha_2\mathbf{v}_2e^{\lambda_2 t}$$

where α_1, and α_2 are arbitrary constants which may be determined by applying initial conditions. *Mutatis mutandis* the result also holds for larger matrices.

Harder problems are marked with a ★

1. Let $A = \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}$. In the previous problem set you showed that A has eigenvalues $\lambda = 4, 5$ and eigenspaces $E_4 = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$, $E_5 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$. Hence for $P = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}$ we have $P^{-1}AP = D$. Using this diagonalisation:

a) Find a formula for A^n, checking your answer with $n = 0$ and $n = 1$.

b) Find the matrix e^{At}, checking your answer with $t = 0$.

c) Determine the solution to the system of equations

$$\begin{aligned} y_1' &= 6y_1 - y_2 \\ y_2' &= 2y_1 + 3y_2 \end{aligned}$$

with initial conditions $y_1(0) = 1$ and $y_2(0) = 3$

by implementing $\mathbf{y} = e^{At}\mathbf{c}$ where $\mathbf{y}(0)$=$\mathbf{c}$.

d) Solve the system in c) again, this time using $\mathbf{y} = \alpha_1 \mathbf{v}_1 e^{\lambda_1 t} + \alpha_2 \mathbf{v}_2 e^{\lambda_2 t}$ where α_1 and α_2 are arbitrary constants which may be determined by applying initial conditions.

e) Check your solution to the system by verifying that $y_1' = 6y_1 - y_2$ and that $y_1(0) = 1$.

2. Let $A = \begin{pmatrix} 3 & 4 \\ 1 & 3 \end{pmatrix}$. In the previous problem set you showed that A has eigenvalues $\lambda = 1, 5$ and eigenspaces $E_1 = \text{span}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$, $E_5 = \text{span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$. Hence for $P = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$ we have $P^{-1}AP = D$. Using this diagonalisation:

a) Find a formula for A^n, checking your answer with $n = 0$ and $n = 1$.

b) Find the matrix e^{At}, checking your answer with $t = 0$.

c) Determine the solution to the system of equations

$$\begin{aligned} y_1' &= 3y_1 + 4y_2 \\ y_2' &= y_1 + 3y_2 \end{aligned}$$

with initial conditions $y_1(0) = 0$ and $y_2(0) = 2$

by implementing $\mathbf{y} = e^{At}\mathbf{c}$ where $\mathbf{y}(0)=\mathbf{c}$.

d) Solve the system in c) again, this time using $\mathbf{y} = \alpha_1 \mathbf{v}_1 e^{\lambda_1 t} + \alpha_2 \mathbf{v}_2 e^{\lambda_2 t}$ where α_1 and α_2 are arbitrary constants which may be determined by applying initial conditions.

e) Check your solution to the system by verifying that $y_2' = y_1 + 3y_2$ and that $y_2(0) = 2$.

3. Let $A = \begin{pmatrix} 7 & 4 \\ 4 & 1 \end{pmatrix}$. In the previous problem set you showed that A has eigenvalues $\lambda = -1, 9$ and eigenspaces $E_{-1} = \text{span}\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}$, $E_9 = \text{span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$. Hence for $P = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} -1 & 0 \\ 0 & 9 \end{pmatrix}$ we have $P^{-1}AP = D$. Using this diagonalisation:

a) Find a formula for A^n, checking your answer with $n = 0$ and $n = 1$.

b) Find the matrix e^{At}, checking your answer with $t = 0$.

c) Determine the solution to the system of equations

$$\begin{aligned} y_1' &= 7y_1 + 4y_2 \\ y_2' &= 4y_1 + y_2 \end{aligned}$$

with initial conditions $y_1(0) = 10$ and $y_2(0) = 5$.

by implementing $\mathbf{y} = e^{At}\mathbf{c}$ where $\mathbf{y}(0)=\mathbf{c}$.

d) Solve the system in c) again this time using $\mathbf{y} = \alpha_1 \mathbf{v}_1 e^{\lambda_1 t} + \alpha_2 \mathbf{v}_2 e^{\lambda_2 t}$ where α_1 and α_2 are arbitrary constants which may be determined by applying initial conditions.

e) Check your solution to the system by verifying that $y_1' = 7y_1 + 4y_2$ and that $y_1(0) = 10$.

4.★ Let $A = \begin{pmatrix} 1 & 0 & 0 \\ -8 & 4 & -6 \\ 8 & 1 & 9 \end{pmatrix}$. In the previous problem set you showed that A has eigenvalues $\lambda = 7, 6, 1$ and eigenspaces $E_7 = \text{span}\left\{\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}\right\}$, $E_6 = \text{span}\left\{\begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}\right\}$, and $E_1 = \text{span}\left\{\begin{pmatrix} 15 \\ 8 \\ -16 \end{pmatrix}\right\}$. Hence for $P = \begin{pmatrix} 0 & 0 & 15 \\ -2 & -3 & 8 \\ 1 & 1 & -16 \end{pmatrix}$ and $D = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ we have $P^{-1}AP = D$. Using this diagonalisation and the fact that $P^{-1} = \dfrac{1}{15}\begin{pmatrix} 40 & 15 & 45 \\ -24 & -15 & -30 \\ 1 & 0 & 0 \end{pmatrix}$:

a) Find a formula for A^n, checking your answer with $n = 0$ and $n = 1$.

b) Find the matrix e^{At}, checking your answer with $t = 0$.

c) Determine the solution to the system of equations

$$\begin{aligned} y_1' &= y_1 \\ y_2' &= -8y_1 + 4y_2 - 6y_3 \\ y_3' &= 8y_1 + y_2 + 9y_3 \end{aligned}$$

with initial conditions $y_1(0) = 15$, $y_2(0) = 3$ and $y_3(0) = 5$

by implementing $\mathbf{y} = e^{At}\mathbf{c}$ where $\mathbf{y}(0)=\mathbf{c}$.

d) Check your solution to the system in c) by verifying that $y_3' = 8y_1 + y_2 + 9y_3$ and that $y_3(0) = 5$.

5. A matrix A is said to be skew symmetric if $A^T = -A$. Show that if A is skew symmetric then e^A is orthogonal.

6. Suppose that A is an $n \times n$ matrix such that $A^2 = 0$.

a) Show that $e^A = I + A$.

b) Hence show that $(I + A)^{-1} = I - A$.

7.★ Let S be a projection matrix on $\mathbb{R}^n$. Prove that $e^S = I + (e - 1)S$.

8.★★ Let R be a reflection matrix on $\mathbb{R}^n$. Prove that $e^R = \dfrac{1}{2}\left\{\left(e + \dfrac{1}{e}\right) I + \left(e - \dfrac{1}{e}\right) R\right\}$.

ANSWERS

1. a) $A^n = \begin{pmatrix} 2(5^n) - 4^n & 4^n - 5^n \\ 2(5^n) - 2(4^n) & 2(4^n) - 5^n \end{pmatrix}$ b) $e^{At} = \begin{pmatrix} 2e^{5t} - e^{4t} & e^{4t} - e^{5t} \\ 2e^{5t} - 2e^{4t} & 2e^{4t} - e^{5t} \end{pmatrix}$.

c) and d) $y_1 = 2e^{4t} - e^{5t}$ and $y_2 = 4e^{4t} - e^{5t}$.

e) Check.

2. a) $A^n = \frac{1}{4}\begin{pmatrix} 2(5^n) + 2 & 4(5^n) - 4 \\ 5^n - 1 & 2(5^n) + 2 \end{pmatrix}$ b) $e^{At} = \frac{1}{4}\begin{pmatrix} 2e^{5t} + 2e^{t} & 4e^{5t} - 4e^{t} \\ e^{5t} - e^{t} & 2e^{5t} + 2e^{t} \end{pmatrix}$.

c) and d) $y_1 = 2e^{5t} - 2e^{t}$ and $y_2 = e^{5t} + e^{t}$.

e) Check.

3. a) $A^n = \frac{1}{5}\begin{pmatrix} 4(9^n) + (-1)^n & 2(9^n) - 2(-1)^n \\ 2(9^n) - 2(-1)^n & (9^n) + 4(-1)^n \end{pmatrix}$ b) $e^{At} = \frac{1}{5}\begin{pmatrix} 4e^{9t} + e^{-t} & 2e^{9t} - 2e^{-t} \\ 2e^{9t} - 2e^{-t} & e^{9t} + 4e^{-t} \end{pmatrix}$.

c) and d) $y_1 = 10e^{9t}$ and $y_2 = 5e^{9t}$.

e) Check.

4. a) $$A^n = \frac{1}{15}\begin{pmatrix} 15 & 0 & 0 \\ 72(6^n) - 80(7^n) + 8 & 45(6^n) - 30(7^n) & 90(6^n) - 90(7^n) \\ -24(6^n) + 40(7^n) - 16 & 15(7^n) - 15(6^n) & 45(7^n) - 30(6^n) \end{pmatrix}$$

b) $$e^{At} = \frac{1}{15}\begin{pmatrix} 15e^{t} & 0 & 0 \\ 72(e^{6t}) - 80e^{7t} + 8e^{t} & 45e^{6t} - 30e^{7t} & 90e^{6t} - 90e^{7t} \\ -24e^{6t} + 40e^{7t} - 16e^{t} & 15e^{7t} - 15e^{6t} & 45e^{7t} - 30e^{6t} \end{pmatrix}$$

c) $y_1 = 15e^{t}$, $y_2 = 111e^{6t} + 8e^{t} - 116e^{7t}$ and $y_3 = -37e^{6t} - 16e^{t} + 58e^{7t}$.

d) Check.

5. Proof.

6. Proof.

7. Hint $S^2 = S$.

8. Hint $R^2 = I$.

MATH2099 PROBLEM SHEET 17
SYMMETRIC MATRICES AND QUADRATIC CURVES

A matrix A is said to be symmetric if $A = A^T$.

The eigenvectors from different eigenvalues of a symmetric matrix are mutually perpendicular.

A square matrix Q is said to be orthogonal if $Q^T Q = I$ or equivalently $Q^{-1} = Q^T$.

The linear transformations induced by orthogonal matrices are rotations, reflections or a composition of both.

The columns of an orthogonal matrix are an orthonormal set.

Let A be a symmetric matrix and Q the orthogonal matrix made up of unit eigenvectors of A. Then $Q^T A Q = D$ is an orthogonal diagonalisation of A with the matrix D being the diagonal matrix of corresponding eigenvalues of A.

$$(AB)^{-1} = B^{-1}A^{-1}.$$

$$(AB)^T = B^T A^T.$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is an ellipse in } \mathbb{R}^2. \ (++)$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is a hyperbola in } \mathbb{R}^2 \ (+-).$$

The quadratic form $\mathbf{x}^T A\mathbf{x}$ in the plane, where A is a symmetric matrix, has principal axes given by the orthogonal eigenvectors of A. The associated quadratic curve $\mathbf{x}^T A\mathbf{x} = C$ may be orthogonally transformed through rotations and/or reflections into a standard ellipse or hyperbola with the eigenvalues appearing as coefficients.

Harder problems are marked with a ★

1. Consider the curve $5x^2 + 3y^2 = 45$.

a) Identify and sketch the curve.

b) Find the shortest distance between the origin and the curve and the closest points on the curve to the origin.

2. Consider the curve $-7x^2 + 4y^2 = 100$.

a) Identify and sketch the curve.

b) Find the shortest distance between the origin and the curve and the closest points on the curve to the origin.

3. Consider the curve with equation $2x^2 + 5y^2 - 4xy = 54$.

a) Write the equation as $\begin{pmatrix} x & y \end{pmatrix} A \begin{pmatrix} x \\ y \end{pmatrix} = 54$ where A is a symmetric matrix.

b) Find all the eigenvalues and eigenvectors of A.

c) Verify that the eigenvectors are orthogonal.

d) Establish an orthogonal digaonalisation of A.

e) Assign a pair of principal axes X and Y and write down the equation of the curve with respect to these new axes.

f) Identify the curve and find the shortest distance from the curve to the origin.

g) Sketch the curve $2x^2 + 5y^2 - 4xy = 54$ in the $x - y$ plane, clearly marking the position of the curve's principal axes X and Y.

h) Find the $\begin{pmatrix} x \\ y \end{pmatrix}$ coordinates of the closest point(s) on the curve to the origin.

4. Consider the curve with equation $x^2 + y^2 + 10xy = 150$.

a) Write the equation as $\begin{pmatrix} x & y \end{pmatrix} A \begin{pmatrix} x \\ y \end{pmatrix} = 150$ where A is a symmetric matrix.

b) Find all the eigenvalues and eigenvectors of A.

c) Verify that the eigenvectors are orthogonal.

d) Establish an orthogonal digaonalisation of A.

e) Assign a pair of principal axes X and Y and write down the equation of the curve with respect to these new axes.

f) Identify the curve and find the shortest distance from the curve to the origin.

g) Sketch the curve $x^2 + y^2 + 10xy = 150$ in the $x - y$ plane, clearly marking the position of the curve's principal axes X and Y.

h) Find the $\begin{pmatrix} x \\ y \end{pmatrix}$ coordinates of the closest point(s) on the curve to the origin.

5. Suppose that B is any matrix and that $A = BB^T$. Prove that A is symmetric.

6. Choose B to be a particular 3×2 matrix. Use the previous question to produce a symmetric matrix A.

7. Suppose that P and Q are orthogonal matrices and let $R = PQ$. Prove that R is also orthogonal.

8. Suppose that A, B and C are square matrices of the same size and that A is invertible and symmetric. Assume also that both B and C are orthogonal. Simplify $(ABC)^T(B^{-1}A)^{-1}C$.

9. Prove that $\left(P^T\right)^{-1} = (P^{-1})^T$ for all invertible matrices P.

10.* Let A be a square matrix with the property that $I + A$ is invertible and define a matrix B by $B = (I - A)(I + A)^{-1}$. Prove that A is skew symmetric if and only if B is orthogonal. (Recall that a matrix A is said to be skew symmetric if $A^T = -A$).

ANSWERS

1. a) Ellipse. b) Shortest distance of 3 at $(\pm 3, 0)$.

2. a) Hyperbola. b) Shortest distance of 5 at $(0, \pm 5)$.

3. a) $\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 54.$

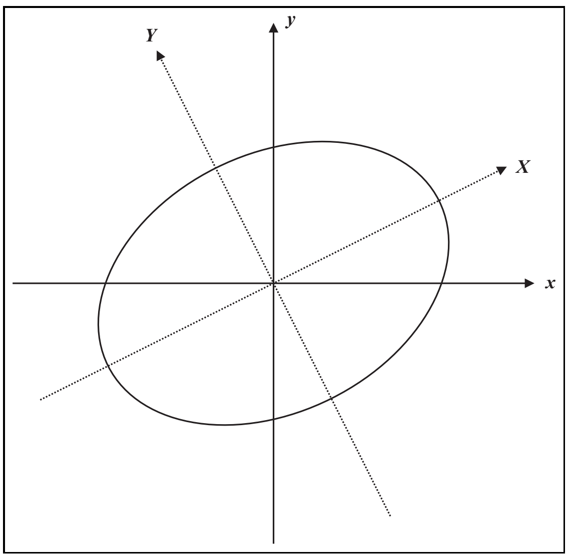
b) $\lambda = 1,\ 6 \qquad E_1 = \text{span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$ and $E_6 = \text{span}\left\{\begin{pmatrix} -1 \\ 2 \end{pmatrix}\right\}$.

c) $\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 0.$

d) $P = \dfrac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \to P^TAP = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} = D.$

e) $X = \dfrac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $Y = \dfrac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$. The new equation is $X^2 + 6Y^2 = 45$.

f) The curve is an ellipse. Shortest distance to the origin is 3 units (in the Y direction).

g) 

h) $\begin{pmatrix} x \\ y \end{pmatrix} = \pm\frac{3}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

4. a) $\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 150$.

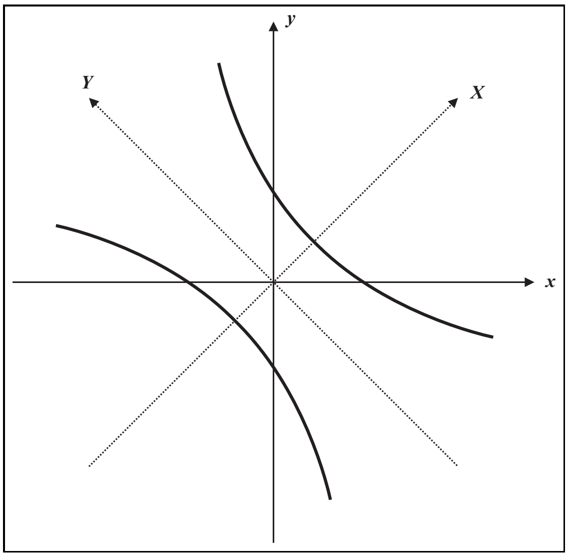
b) $\lambda = 6,\ -4 \quad E_6 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$ and $E_{-4} = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$.

c) $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0$.

d) $P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \to P^T A P = \begin{pmatrix} 6 & 0 \\ 0 & -4 \end{pmatrix} = D$.

e) $X = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $Y = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. The new equation is $6X^2 - 4Y^2 = 150$.

f) The curve is a hyperbola. Shortest distance to the origin is 5 units (in the X direction).

g) 

h) $\begin{pmatrix} x \\ y \end{pmatrix} = \pm\frac{5}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

5. Proof.

6. Construction.

7. Proof.

8. I.

9. Hint: Transpose the equation $P^{-1}P = I$.

10. Hint: Use the previous question and $(I - A)(I + A) = (I + A)(I - A)$. Note that the result claims **if and only if** so you must verify that A skew symmetric $\to$ B orthogonal **and** B orthogonal $\to$ A skew symmetric.

MATH2099 PROBLEM SHEET 18
SYMMETRIC MATRICES AND QUADRIC SURFACES IN SPACE

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ is an ellipsoid in $\mathbb{R}^3$ $(+ + +)$.

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$ is a hyperboloid of one sheet in $\mathbb{R}^3$ $(+ + -)$. (Axis on the negative)

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$ is a hyperboloid of 2 sheets in $\mathbb{R}^3$ $(+ - -)$. (Axis on the positive)

In $\mathbb{R}^3$ the number of sheets is the number of $(-)$'s.

The quadric surface $\mathbf{x}^T A\mathbf{x}$ in space, where A is a symmetric matrix, has principal axes given by the orthogonal eigenvectors of A. The associated quadratic surface $\mathbf{x}^T A\mathbf{x} = C$ may be orthogonally transformed through rotations and/or reflections into a standard hyperboloid or ellipsoid with the eigenvalues appearing as coefficients.

Harder problems are marked with a ★

1. Sketch each of the following quadric surfaces. Find the smallest distance from the surface to the origin in each case and determine the point(s) on the surface where this minimal distance is achieved.

a) $\dfrac{x^2}{9} + \dfrac{y^2}{16} + \dfrac{z^2}{25} = 1.$

b) $-\dfrac{x^2}{9} + \dfrac{y^2}{16} - \dfrac{z^2}{25} = 1.$

c) $\dfrac{x^2}{9} + \dfrac{y^2}{16} - \dfrac{z^2}{25} = 1.$

d) $-\dfrac{x^2}{9} - \dfrac{y^2}{16} - \dfrac{z^2}{25} = 1.$

2. a) Sketch the graph of $x^2 + y^2 = 1$ in $\mathbb{R}^2$.

b) Sketch the graph of $x^2 + y^2 = 1$ in $\mathbb{R}^3$.

3. Suppose that D is an $n \times n$ diagonal matrix and that Q is an $n \times n$ orthogonal matrix. Let $A = Q^T DQ$. Prove that A is symmetric.

4. Consider the surface with equation $8x^2 + 7y^2 + 8z^2 + 2xy + 2yz = 144$.

a) Write the equation in the form $\begin{pmatrix} x & y & z \end{pmatrix} A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 144$ where A is a symmetric matrix.

b) You are given that two of the eigenvalues of A are $\lambda = 9$ and $\lambda = 8$ with corresponding eigenspaces $E_9 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$ and $E_8 = \text{span}\left\{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$. Without resorting to the characteristic equation find the remaining eigenvalue and its eigenspace.

c) Verify that the eigenvectors are mutually orthogonal.

d) Assign principal axes X, Y and Z and express the equation of the surface in terms of these axes. Hence determine the nature of the surface.

e) Find an orthogonal matrix P implementing the transformation to the new variables through $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = P \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$.

f) Find the shortest distance from the surface to the origin and the $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ coordinates of the closest point(s).

g) Sketch the surface in space, showing both the original and the principal axes.

5. Consider the surface with equation $-6x^2 - y^2 + 2z^2 + 4xy - 2xz + 4yz = 12$.

a) Write the equation in the form $\begin{pmatrix} x & y & z \end{pmatrix} A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 12$ where A is a symmetric matrix.

b) You are given that two of the eigenvalues of A are $\lambda = -7$ and $\lambda = -1$ with corresponding eigenspaces $E_{-7} = \text{span}\left\{\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}\right\}$ and $E_{-1} = \text{span}\left\{\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}\right\}$. Without resorting to the characteristic equation find the remaining eigenvalue and its eigenspace.

c) Verify that the eigenvectors are mutually orthogonal.

d) Assign principal axes X, Y and Z and express the equation of the surface in terms of these axes. Hence determine the nature of the surface.

e) Find an orthogonal matrix P implementing the transformation to the new variables through $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = P \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$.

f) Find the shortest distance from the surface to the origin and the $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ coordinates of the closest point(s).

g) Sketch the surface in space, showing both the original and the principal axes.

6.★ Identify the surface $x^2 - 2y^2 + z^2 + 6xy - 2yz = 16$ and find the shortest distance between the surface and the origin. (Hint: You need the eigenvalues of the associated symmetric matrix A but not its eigenvectors).

7.★ Let A be the symmetric matrix $A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$.

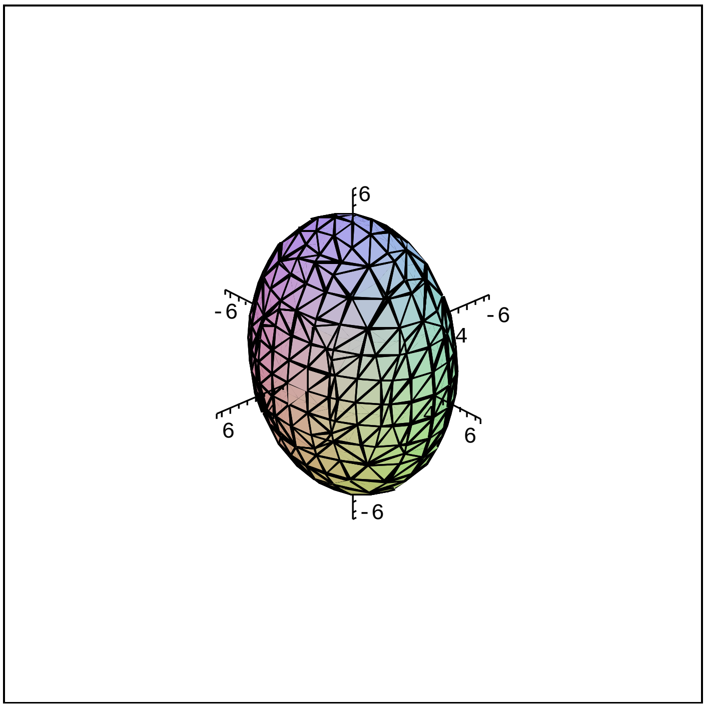
a) Without calculating the characteristic equation show that $\mathbf{u} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ are independent eigenvectors of A.

b) Show that $\mathbf{u}$ and $\mathbf{v}$ are NOT orthogonal. Explain.

c) Describe how you would establish an orthogonal diagonalisation of A.

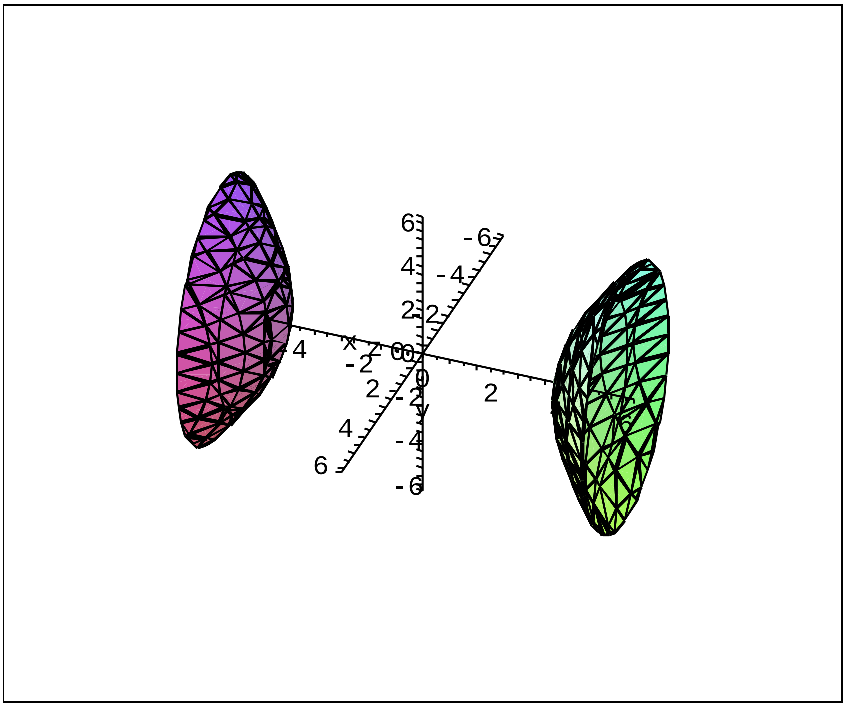
ANSWERS

1. a)



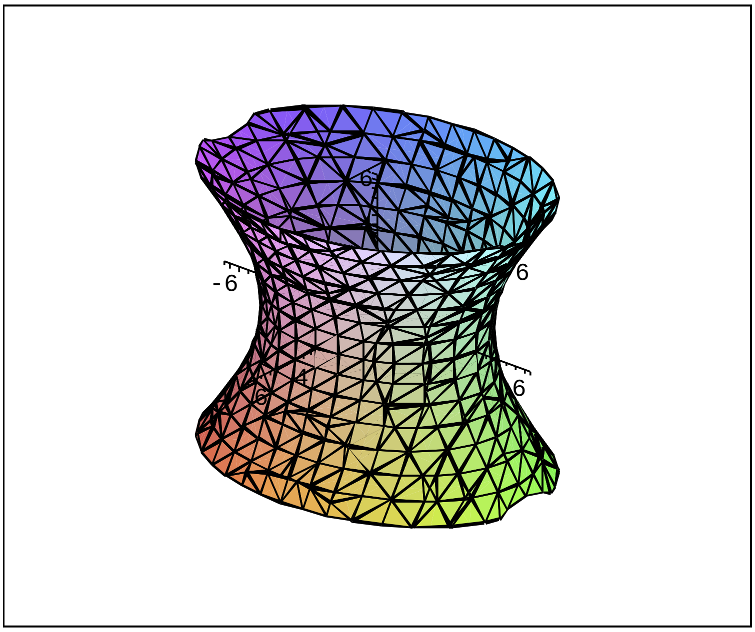
Ellipsoid.

Smallest distance to the origin of 3 achieved at $\begin{pmatrix} \pm 3 \\ 0 \\ 0 \end{pmatrix}$.

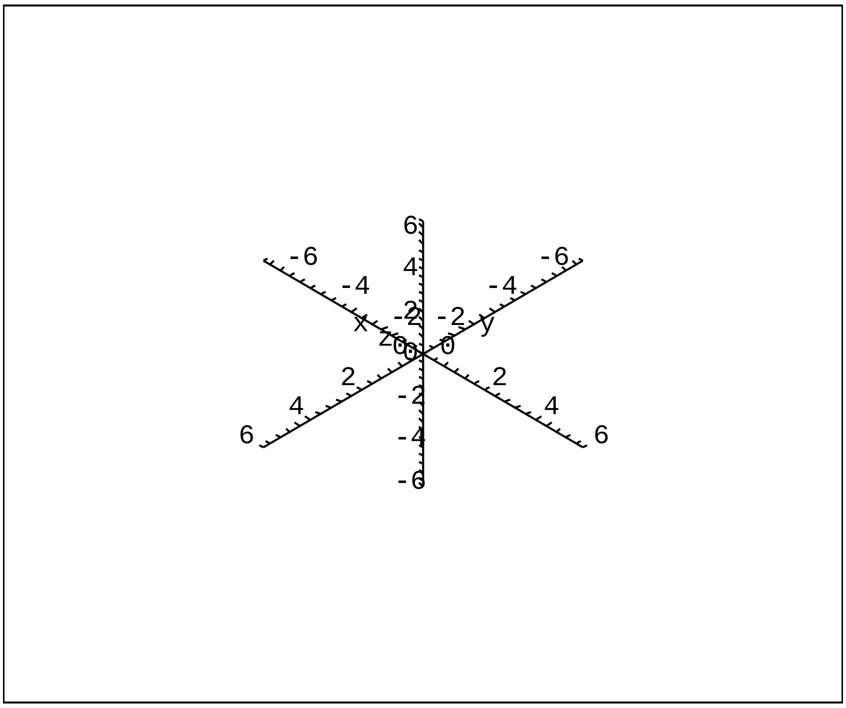
b)  Hyperboloid of two sheets.

Axis along the y-axis.

Smallest distance to the origin of 4 achieved at $\begin{pmatrix} 0 \\ \pm 4 \\ 0 \end{pmatrix}$.

c)  Hyperboloid of one sheet.

Axis along the z-axis. Smallest distance to the origin of 3 achieved at $\begin{pmatrix} \pm 3 \\ 0 \\ 0 \end{pmatrix}$.

d) 

2. a) Circle.

b) Vertical Cylinder. If a variable is missing do not assume that it is zero....in fact it is free!

3. Proof.

4. a) $\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 8 & 1 & 0 \\ 1 & 7 & 1 \\ 0 & 1 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 144.$

b) $\lambda = 6$ with corresponding eigenspaces $E_6 = \text{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$.

c) All dot products between the eigenvectors are 0.

d) $X = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $Y = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $Z = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

The new equation is $9X^2 + 8Y^2 + 6Z^2 = 144$. This is an ellipsoid $(+++)$.

e) $P = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$.

f) Closest points to the origin are $\pm\frac{4}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ at a distance of 4.

g) Sketch.

5. a) $\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} -6 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 12.$

b) $\lambda = 3$ with corresponding eigenspaces $E_3 = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\right\}$.

c) All dot products between the eigenvectors are 0.

d) $X = \frac{1}{\sqrt{30}}\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$, $Y = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ and $Z = \frac{1}{\sqrt{5}}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

The new equation is $-7X^2 - 1Y^2 + 3Z^2 = 12$. This is a hyperboloid of 2 sheets $(- - +)$ with axis in the Z direction.

e) $P = \begin{pmatrix} \frac{5}{\sqrt{30}} & -\frac{1}{\sqrt{6}} & 0 \\ -\frac{2}{\sqrt{30}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{30}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} \end{pmatrix}$.

f) Closest points to the origin are $\pm\frac{2}{\sqrt{5}}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ at a distance of 2.

g) Sketch.

6. $X^2 - 4Y^2 + 3Z^2 = 16$. Hyperboloid of one sheet. Shortest distance of $\frac{4}{\sqrt{3}}$.

7. a) Proof.

b) The problem is that the two eigenvectors come from the same eigenvalue. They come out from row reduction pretty much randomly.

c) Looks like a job for Gram-Schmidt!!

MATH2099 PROBLEM SHEET 19
JORDAN DECOMPOSITIONS

The Jordan block $J_n(\alpha)$ is an $n \times n$ matrix with α on the diagonal, 1's immediately above the diagonal and zero's everywhere else.

A Jordan matrix J is a direct sum of Jordan blocks.

Every square matrix A is similar to a Jordan matrix J.

The algebraic multiplicity of an eigenvalue is the number of times the eigenvalue appears in the characteristic equation while the geometric multiplicity is the number of eigenvectors that it eventually delivers.

If we are short on eigenvectors diagonalisation is impossible and we instead establish a Jordan decomposition which has similar applications.

To produce a Jordan decomposition inflate the eigenspace to produce a sequence of nested generalised eigenspaces terminating with a dimension equal to the algebraic multiplicity of the eigenvalue.

Using the stepping tool $\mathbf{v}_{n-1} = (A - \lambda I)\mathbf{v}_n$ form eigen-chains from the outer rings to the eigencore. Each eigen-chain produces a Jordan block with the size of the block equaling the length of the chain. The number of blocks=the number of chains=dimension of the eigenspace=geometric multiplicity of the eigenvalue.

Harder problems are marked with a ★

1. Express each of the following Jordan matrices using $J_n(\alpha)$ and $\oplus$ notation:

a) $\begin{pmatrix} 7 & 1 \\ 0 & 7 \end{pmatrix}$ b) $\begin{pmatrix} 7 & 1 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}$ c) $\begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & 7 \end{pmatrix}$ d) $\begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}$

e) $\begin{pmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 9 \end{pmatrix}$ f) $\begin{pmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}$ g) $\begin{pmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 8 & 1 \\ 0 & 0 & 0 & 0 & 8 \end{pmatrix}$

2. Write out the full version of each of the following Jordan matrices:

a) $J_2(-3)$ b) $J_1(5) \oplus J_2(-3)$ c) $J_2(7) \oplus J_1(4)$ d) $J_3(7) \oplus J_2(0) \oplus J_1(9)$

e) $J_2(-1) \oplus J_2(8)$ f) $J_3(4) \oplus J_1(-9)$ g) $J_3(-2) \oplus J_2(8)$

3. Determine which of the following matrices are Jordan matrices. For those that are Jordan matrices write down a different Jordan matrix similar to the given matrix:

a) $\begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix}$ b) $\begin{pmatrix} 7 & 1 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$ c) $\begin{pmatrix} 7 & 0 & 0 \\ 0 & 8 & 1 \\ 0 & 0 & 8 \end{pmatrix}$ d) $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ e) $\begin{pmatrix} 5 & 1 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{pmatrix}$

4. You are given that each of the following two 3×3 matrices has the characteristic equation $(\lambda - 5)^3 = 0$.

$$\text{(I) } A = \begin{pmatrix} 8 & 6 & 3 \\ -1 & 3 & -1 \\ -1 & -2 & 4 \end{pmatrix} \qquad \text{(II) } A = \begin{pmatrix} 7 & 5 & 1 \\ 0 & 4 & 1 \\ -1 & -2 & 4 \end{pmatrix}$$

For each matrix:

a) Write down the algebraic multiplicity of $\lambda = 5$.

b) Find the eigenspace E_5 and the geometric multiplicity of $\lambda = 5$.

c) By calculating the generalised eigenspaces set up a ring diagram for $\lambda = 5$ and hence find a possible Jordan matrix J similar to A.

d) Find an invertible matrix P such that $P^{-1}AP = J$.

e) Write down (if possible) a different Jordan decomposition to the one found in d).

5. In problem sheet 15 you showed that $C = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$ is not diagonalisable. Find a Jordan decomposition of C.

6. Write down two matrices which have the same eigenvalues but are not similar.

7.★ Let $P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} a & e & i & m \\ b & f & j & n \\ c & g & k & o \\ d & h & l & p \end{pmatrix}$

a) Calculate PA and describe in words the effect of premultiplication by P.

b) Calculate AP and describe in words the effect of postmultiplication by P.

c) Show that $P^2 = I$.

d) Explain why $P\big(J_2(\alpha) \oplus J_2(\beta)\big)P = J_2(\beta) \oplus J_2(\alpha)$.

e) Hence show that $J_2(\alpha) \oplus J_2(\beta)$ is similar to $J_2(\beta) \oplus J_2(\alpha)$.

8.★ Suppose that two matrices A and B have the same Jordan form J. Prove that A and B are similar.

ANSWERS

1. a) $J_2(7)$ b) $J_2(7)\oplus J_1(7)$ c) $J_1(7)\oplus J_2(7)$ d) $J_1(7)\oplus J_1(7)\oplus J_1(7)$

e) $J_2(5)\oplus J_2(9)$ f) $J_3(5)\oplus J_1(9)$ g) $J_2(4)\oplus J_1(-2)\oplus J_2(8)$

2. a) $\begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix}$ b) $\begin{pmatrix} 5 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{pmatrix}$ c) $\begin{pmatrix} 7 & 1 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ d) $\begin{pmatrix} 7 & 1 & 0 & 0 & 0 & 0 \\ 0 & 7 & 1 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{pmatrix}$

e) $\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 8 & 1 \\ 0 & 0 & 0 & 8 \end{pmatrix}$ f) $\begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -9 \end{pmatrix}$ g) $\begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 8 & 1 \\ 0 & 0 & 0 & 0 & 8 \end{pmatrix}$.

3. Only c)$=J_1(7)\oplus J_2(8)$ is Jordan. A similar Jordan matrix is $J_2(8)\oplus J_1(7)$.

Throughout the theory of Jordan decompositions be aware that there are many different possible correct answers due to various choices and options when implementing our algorithms. Your answers may not agree with those below yet still be correct!

4. (I) a) 3 b) $E_5=\text{span}\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$. Hence the geometric multiplicity is 2.

c) $J=J_2(5)\oplus J_1(5)$ d) $P=\begin{pmatrix} 3 & 0 & -1 \\ -1 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix}$ e) $J=J_1(5)\oplus J_2(5)$.

(II) a) 3 b) $E_5=\text{span}\left\{\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}\right\}$. Hence the geometric multiplicity is 1.

c) $J=J_3(5)$ d) $P=\begin{pmatrix} 3 & 2 & 1 \\ -1 & 0 & 0 \\ -1 & -1 & 0 \end{pmatrix}$ e) Only one Jordan form possible since there is a single block.

5. $P=\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\to P^{-1}CP=\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}=J_2(3)$.

6. How about $A=\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ and $B=\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$. Both have eigenvalues $\lambda=3,3$ but cannot be similar since A is diagonalisable but B is not.

7. a) Rows 1 and 2 are swapped with Rows 3 and 4 respectively.

b) Columns 1 and 2 are swapped with Columns 3 and 4 respectively. c,d,e) Proof.

8. Proof.

MATH2099 PROBLEM SHEET 20
JORDAN DECOMPOSITIONS PART 2

The Jordan block $J_n(\alpha)$ is an $n \times n$ matrix with α on the diagonal, 1's immediately above the diagonal and zero's everywhere else.

A Jordan matrix J is a direct sum of Jordan blocks.

Every square matrix A is similar to a Jordan matrix J.

The algebraic multiplicity of an eigenvalue is the number of times the eigenvalue appears in the characteristic equation while the geometric multiplicity is the number of eigenvectors that it eventually delivers.

If we are short on eigenvectors diagonalisation is impossible and we instead establish a Jordan decomposition which has similar applications.

To produce a Jordan decomposition inflate the eigenspace to produce a sequence of nested generalised eigenspaces terminating with a dimension equal to the algebraic multiplicity of the eigenvector.

Using the stepping tool $\mathbf{v}_{n-1} = (A - \lambda I)\mathbf{v}_n$ form eigen-chains from the outer rings to the eigencore. Each eigen-chain produces a Jordan block with the size of the block equaling the length of the chain. The number of blocks=the number of chains=dimension of the eigenspace=geometric multiplicity of the eigenvalue.

Harder problems are marked with a ★

1. Find a Jordan decomposition of $A = \begin{pmatrix} 9 & -1 \\ 4 & 5 \end{pmatrix}$.

2. Consider the matrix $A = \begin{pmatrix} 6 & 6 & 3 \\ -1 & 1 & -1 \\ -1 & -2 & 2 \end{pmatrix}$.

a) Show that $\lambda = 3$ is an eigenvalue of A.

b) Hence write down all of the eigenvalues of A.

c) Find a Jordan decomposition of A.

3. The matrix $B = \begin{pmatrix} 13 & 9 & 9 \\ -1 & 5 & -1 \\ -4 & -5 & 0 \end{pmatrix}$ has a repeated eigenvalue $\lambda = 7$.

a) Write down all of the eigenvalues of B.

b) Find a Jordan decomposition of B.

4. Find a Jordan Decomposition of $C = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{pmatrix}$.

5.$^\bigstar$ Prove that the Jordan form for $\begin{pmatrix} \alpha+2 & -1 \\ 4 & \alpha-2 \end{pmatrix}$ is $J_2(\alpha)$.

6. A 7×7 matrix B has a single eigenvalue $\lambda = 9$ with $\mathrm{Ker}(B - 9I) \equiv$ 3-dimensional, $\mathrm{Ker}(B - 9I)^2 \equiv$6 dimensional and $\mathrm{Ker}(B - 9I)^3 \equiv$7 dimensional. Set up an appropriate ring diagram for the eigenvalue and hence find the Jordan form of B.

7. A 7×7 matrix B has a single eigenvalue $\lambda = 9$ with $\mathrm{Ker}(B - 9I) \equiv$ 2-dimensional, $\mathrm{Ker}(B - 9I)^2 \equiv$4 dimensional, $\mathrm{Ker}(B - 9I)^3 \equiv$6 dimensional and $\mathrm{Ker}(B - 9I)^4 \equiv$7 dimensional. Set up an appropriate ring diagram for the eigenvalue and hence find the Jordan form of B.

8. A 7×7 matrix B has a single eigenvalue $\lambda = 9$ with $\mathrm{Ker}(B - 9I) \equiv$ 4-dimensional and $\mathrm{Ker}(B - 9I)^2 \equiv$7 dimensional. Set up an appropriate ring diagram for the eigenvalue and hence find the Jordan form of B.

9. A matrix C has Jordan form $J_1(8) \oplus J_2(8) \oplus J_2(8)$.

a) Write down the eigenvalue of C.

b) Write down the size and the characteristic equation of C.

c) What is the geometric multiplicity of the eigenvalue. Remember that the geometric multiplicity equals the number of blocks!

d) By considering an appropriate ring diagram find the dimension of each generalised eigenspace.

10. A matrix C has Jordan form $J_1(-9) \oplus J_2(-9) \oplus J_2(-9) \oplus J_3(-9)$.

a) Write down the eigenvalue of C.

b) Write down the size and the characteristic equation of C.

c) What is the geometric multiplicity of the eigenvalue.

d) By considering an appropriate ring diagram find the dimension of each generalised eigenspace.

11. A matrix C has Jordan form $J_5(1) \oplus J_1(1) \oplus J_4(-2) \oplus J_4(-2) \oplus J_2(-2)$.

a) Write down the eigenvalues of C.

b) Write down the size and the characteristic equation of C.

c) What is the geometric multiplicity of each of the eigenvalues.

d) By considering appropriate ring diagrams find the dimension of each generalised eigenspace.

12. An 8×8 matrix A has a single eigenvalue $\lambda = 12$ with $\mathrm{Ker}(A - 12I) \equiv$ 3-dimensional and $\mathrm{Ker}(A - 12I)^2 \equiv$ 5-dimensional. Find all possible Jordan forms of A.

13.$^\bigstar$ Let $A = \begin{pmatrix} \gamma & a & c \\ 0 & \gamma & b \\ 0 & 0 & \gamma \end{pmatrix}$. Show that A is similar to a single Jordan block if and only if $ab \neq 0$.

ANSWERS

Your answers for P may not agree with those below yet still be correct!

1. $P = \begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix} \to P^{-1}AP = \begin{pmatrix} 7 & 1 \\ 0 & 7 \end{pmatrix} = J_2(7).$

2. b) $\lambda = 3, 3, 3$ c) $P = \begin{pmatrix} 3 & 0 & -1 \\ -1 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix} \to P^{-1}AP = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = J_2(3) \oplus J_1(3).$

3. a) $\lambda = 7, 7, 4$ b) $P = \begin{pmatrix} -1 & 3 & 2 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{pmatrix} \to P^{-1}AP = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & 7 \end{pmatrix} = J_1(4) \oplus J_2(7).$

4. $P = \begin{pmatrix} -2 & -1 & 0 \\ 0 & -4 & 0 \\ 0 & -2 & 1 \end{pmatrix} \to P^{-1}AP = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} = J_3(-1).$

5. Proof.

6. $J_2(9) \oplus J_2(9) \oplus J_3(9)$.

7. $J_3(9) \oplus J_4(9)$.

8. $J_1(9) \oplus J_2(9) \oplus J_2(9) \oplus J_2(9)$.

9. a) $\lambda = 8$ b) 5×5 $(\lambda - 8)^5 = 0$ c) $\lambda = 8$ has geometric multiplicity 3.

d) $\mathrm{Ker}(A - 8I) \equiv$ 3-dimensional, $\mathrm{Ker}(A - 8I)^2 \equiv$ 5-dimensional.

10. a) $\lambda = -9$ b) 8×8 $(\lambda + 9)^8 = 0$ c) $\lambda = -9$ has geometric multiplicity 4.

d) $\mathrm{Ker}(A + 9I) \equiv$ 4-dimensional, $\mathrm{Ker}(A + 9I)^2 \equiv$ 7-dimensional,

$\mathrm{Ker}(A + 9I)^3 \equiv$ 8-dimensional.

11. a) $\lambda = 1, -2$ b) 16×16 $(\lambda - 1)^6(\lambda + 2)^{10} = 0$.

c) $\lambda = 1$ has geometric multiplicity 2 and $\lambda = -2$ has geometric multiplicity 3.

d) $\mathrm{Ker}(A - I) \equiv$ 2-dimensional, $\mathrm{Ker}(A - I)^2 \equiv$ 3-dimensional,

$\mathrm{Ker}(A - I)^3 \equiv$ 4-dimensional, $\mathrm{Ker}(A - I)^4 \equiv$ 5-dimensional, $\mathrm{Ker}(A - I)^5 \equiv$ 6-dimensional.

$\mathrm{Ker}(A + 2I) \equiv$ 3-dimensional, $\mathrm{Ker}(A + 2I)^2 \equiv$ 6-dimensional,

$\mathrm{Ker}(A + 2I)^3 \equiv$ 8-dimensional, $\mathrm{Ker}(A + 2I)^4 \equiv$ 10-dimensional.

12. $J_1(12) \oplus J_3(12) \oplus J_4(12)$ or $J_1(12) \oplus J_2(12) \oplus J_5(12)$.

13. Proof.

MATH2099 PROBLEM SHEETS 21 AND 22
MATRIX EXPONENTIALS AND SYSTEMS OF DE'S REVISITED

$$e^{J_2(\alpha)t} = e^{\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} t} = e^{\alpha t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$e^{J_3(\alpha)t} = e^{\begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{pmatrix} t} = e^{\alpha t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

$$e^{J_4(\alpha)t} = e^{\begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 1 & 0 \\ 0 & 0 & \alpha & 1 \\ 0 & 0 & 0 & \alpha \end{pmatrix} t} = e^{\alpha t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} \\ 0 & 1 & t & \frac{t^2}{2!} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Given a Jordan decomposition $P^{-1}AP = J$ we have $e^{At} = Pe^{Jt}P^{-1}$.

The exponential of a direct sum is the direct sum of the exponentials.

The solution of the system of linear differential equations $\mathbf{y}' = A\mathbf{y}$ with initial conditions $\mathbf{y}(0) = \mathbf{c}$ is given by $\mathbf{y} = e^{At}\mathbf{c}$.

Harder problems are marked with a ★

1. Write down e^{Jt} if:

a) $J = J_2(5)$.

b) $J = J_3(7)$.

c) $J = J_2(5) \oplus J_3(7)$.

d) $J = J_4(9)$.

2. Prove that $\mathbf{y} = e^{At}\mathbf{c}$ is a solution of the system of linear differential equations $\mathbf{y}' = A\mathbf{y}$ with initial conditions $\mathbf{y}(0) = \mathbf{c}$.

3. Consider the matrix $A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$. In Problem Sheet 19 Question 5, it was shown that

$$P = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \rightarrow P^{-1}AP = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} = J_2(3)$$

a) Using the above Jordan Decomposition of A find e^{At}.

b) Write down e^{-At}.

c) Solve the system of differential equations

$$\begin{aligned} y_1' &= 2y_1 + y_2 \\ y_2' &= -y_1 + 4y_2 \end{aligned}$$

with initial conditions $y_1(0) = 0$ and $y_2(0) = 1$.

d) Check that your solution satisfies the initial conditions.

e) Check that your solution satisfies the second equation $y_2' = -y_1 + 4y_2$.

4. Consider the matrix $A = \begin{pmatrix} 9 & -1 \\ 4 & 5 \end{pmatrix}$. In Problem Sheet 20 Question 1, it was shown that

$$P = \begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix} \rightarrow P^{-1}AP = \begin{pmatrix} 7 & 1 \\ 0 & 7 \end{pmatrix} = J_2(7)$$

a) Using the above Jordan Decomposition of A find e^{At}.

b) Write down e^{-At}.

c) Solve the system of differential equations

$$\begin{aligned} y_1' &= 9y_1 - y_2 \\ y_2' &= 4y_1 + 5y_2 \end{aligned}$$

with initial conditions $y_1(0) = 1$ and $y_2(0) = 2$.

d) Check that your solution satisfies the initial conditions.

e) Check that your solution satisfies the first equation $y_1' = 9y_1 - y_2$.

5. Consider the matrix $A = \begin{pmatrix} 12 & -4 \\ 16 & -4 \end{pmatrix}$.

a) Evaluate e^{At} by finding and then implementing a Jordan Decomposition of A.

b) Solve the system of differential equations

$$\begin{aligned} y_1' &= 12y_1 - 4y_2 \\ y_2' &= 16y_1 - 4y_2 \end{aligned}$$

with initial conditions $y_1(0) = -1$ and $y_2(0) = 1$.

c) Check that your solution satisfies all the required equations and conditions.

6. Consider the matrix $A = \begin{pmatrix} 6 & 6 & 3 \\ -1 & 1 & -1 \\ -1 & -2 & 2 \end{pmatrix}$. In Problem Sheet 20 Question 2, it was shown that

$$P = \begin{pmatrix} 3 & 0 & -1 \\ -1 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix} \to P^{-1}AP = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = J_2(3) \oplus J_1(3)$$

a) Using the above Jordan Decomposition of A find e^{At}. You are given that

$$P^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 2 & 1 \\ -1 & -3 & 0 \end{pmatrix}.$$

b) Solve the system of differential equations

$$\begin{aligned} y_1' &= 6y_1 + 6y_2 + 3y_3 && (1) \\ y_2' &= -y_1 + y_2 - y_3 && (2) \\ y_3' &= -y_1 - 2y_2 + 2y_3 && (3) \end{aligned}$$

satisfying the initial conditions $y_1(0) = 1$, $y_2(0) = 0$ and $y_3(0) = -1$.

c) Check that your solution satisfies all the required equations and conditions.

7.★ Let $A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 4 & -1 \\ 2 & 2 & 1 \end{pmatrix}$. Solve the system of differential equations $\mathbf{y}' = A\mathbf{y}$ satisfying the initial conditions $\mathbf{y}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

8.★ Let A be a square matrix with eigenvector $\mathbf{v}$ and corresponding eigenvalue λ. Show that the solution to the system of linear differential equations $\mathbf{y}' = A\mathbf{y}$ with initial conditions $\mathbf{y}(0) = \mathbf{v}$ is given by $\mathbf{y} = e^{\lambda t}\mathbf{v}$.

ANSWERS

1. a) $e^{J_2(5)t} = e^{\begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix} t} = e^{5t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$

b) $e^{J_3(7)t} = e^{\begin{pmatrix} 7 & 1 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & 7 \end{pmatrix} t} = e^{7t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$

c) $$e^{J_2(5)\oplus J_3(7)t} = \begin{pmatrix} e^{5t} & te^{5t} & 0 & 0 & 0 \\ 0 & e^{5t} & 0 & 0 & 0 \\ 0 & 0 & e^{7t} & te^{7t} & \frac{t^2}{2!}e^{7t} \\ 0 & 0 & 0 & e^{7t} & te^{7t} \\ 0 & 0 & 0 & 0 & e^{7t} \end{pmatrix}.$$

d) $e^{J_4(9)t} = e^{\begin{pmatrix} 9 & 1 & 0 & 0 \\ 0 & 9 & 1 & 0 \\ 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 9 \end{pmatrix} t} = e^{9t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} \\ 0 & 1 & t & \frac{t^2}{2!} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

2. Proof.

3. a) $e^{At} = e^{3t} \begin{pmatrix} 1-t & t \\ -t & 1+t \end{pmatrix}$ b) $e^{-At} = e^{-3t} \begin{pmatrix} 1+t & -t \\ t & 1-t \end{pmatrix}$

c) $y_1 = te^{3t}$ and $y_2 = te^{3t} + e^{3t}$ d) Check. e) Check.

4. a) $e^{At} = e^{7t} \begin{pmatrix} 1+2t & -t \\ 4t & 1-2t \end{pmatrix}$ b) $e^{-At} = e^{-7t} \begin{pmatrix} 1-2t & t \\ -4t & 1+2t \end{pmatrix}$

c) $y_1 = e^{7t}$ and $y_2 = 2e^{7t}$ d) Check. e) Check.

5. a) $e^{At} = e^{4t} \begin{pmatrix} 1+8t & -4t \\ 16t & 1-8t \end{pmatrix}$ b) $y_1 = -12te^{4t} - e^{4t}$ and $y_2 = -24te^{4t} + e^{4t}$

c) Check.

6. a) $e^{At} = e^{3t} \begin{pmatrix} 1+3t & 6t & 3t \\ -t & 1-2t & -t \\ -t & -2t & 1-t \end{pmatrix}$ b) $y_1 = e^{3t}$, $y_2 = 0$ and $y_3 = -e^{3t}$.

c) Check.

7. $y_1 = e^{3t}$ and $y_2 = e^{3t} - e^{2t}$ $y_3 = 2e^{3t} - 2e^{2t}$

8. Proof.

MATH2099 PROBLEM SHEET 23 and 24
NON-HOMOGENEOUS SYSTEMS OF DIFFERENTIAL EQUATIONS

The solution to $\mathbf{y}' = A\mathbf{y} + \mathbf{b}(t)$ satisfying the initial condition $\mathbf{y}(0) = \mathbf{c}$ is given by

$$\mathbf{y} = e^{At}\left\{\mathbf{c} + \int_0^t e^{-As}\mathbf{b}(s)ds\right\}$$

Harder problems are marked with a ★

1. Let

$$\mathbf{y} = e^{At}\left\{\mathbf{c} + \int_0^t e^{-As}\mathbf{b}(s)ds\right\}.$$

a) Verify that $\mathbf{y}(0) = \mathbf{c}$.

b) Using the product rule show that $\mathbf{y}$ satisfies the system of differential equations

$$\mathbf{y}' = A\mathbf{y} + \mathbf{b}(t).$$

2. Suppose that $\mathbf{v}$ is an eigenvector of a matrix A with corresponding eigenvalue λ. It was shown in lectures that $e^{At}\mathbf{v} = e^{\lambda t}\mathbf{v}$. Deduce that $e^{-As}\mathbf{v} = e^{-\lambda s}\mathbf{v}$.

3. Consider the system of differential equations

$$\begin{aligned} y_1' &= 2y_1 + y_2 + 4e^{5t} \\ y_2' &= -y_1 + 4y_2 + 4e^{5t} \end{aligned}$$

with initial conditions $y_1(0) = 0$ and $y_2(0) = 0$.

This system may be written as $\mathbf{y}' = A\mathbf{y} + \mathbf{b}(t)$ where $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$ and $\mathbf{b}(t) = \begin{pmatrix} 4e^{5t} \\ 4e^{5t} \end{pmatrix}$. The initial conditions are then $\mathbf{y}(0) = \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
In the previous problem set you showed that the matrix $A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$ has an exponential matrix given by $e^{At} = e^{3t}\begin{pmatrix} 1-t & t \\ -t & 1+t \end{pmatrix}$.

a) Write down e^{-As}.

b) Evaluate $e^{-As}\mathbf{b}(s)$ via matrix multiplication.

c) Noting that $\mathbf{b}(s) = 4e^{5s}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvalue of A with eigenvalue $\lambda = 3$, evaluate $e^{-As}\mathbf{b}(s)$ without matrix multiplication. (That's just luck!)

d) Evaluate $\int_0^t e^{-As}\mathbf{b}(s)ds$.

e) Find the solution to the system of differential equations.

f) Check that your solution satisfies the initial conditions.

g) Check that your solution satisfies the first differential equation $y_1' = 2y_1 + y_2 + 4e^{5t}$.

4. Consider the system of differential equations

$$\begin{aligned} y_1' &= 9y_1 - y_2 \\ y_2' &= 4y_1 + 5y_2 + 2e^{7t} \end{aligned}$$

with initial conditions $y_1(0) = 7$ and $y_2(0) = 14$.

This system may be written as $\mathbf{y}' = A\mathbf{y} + \mathbf{b}(t)$ where $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $A = \begin{pmatrix} 9 & -1 \\ 4 & 5 \end{pmatrix}$ and $\mathbf{b}(t) = \begin{pmatrix} 0 \\ 2e^{7t} \end{pmatrix}$. The initial conditions are then $\mathbf{y}(0) = \mathbf{c} = \begin{pmatrix} 7 \\ 14 \end{pmatrix}$.

In the previous problem set you showed that the matrix $A = \begin{pmatrix} 9 & -1 \\ 4 & 5 \end{pmatrix}$ has an exponential matrix given by $e^{At} = e^{7t}\begin{pmatrix} 1+2t & -t \\ 4t & 1-2t \end{pmatrix}$.

a) Write down e^{-As}.

b) Evaluate $e^{-As}\mathbf{b}(s)$ via matrix multiplication.

c) Evaluate $\int_0^t e^{-As}\mathbf{b}(s)ds$.

d) Find the solution to the system of differential equations.

e) Check that your solution satisfies the initial conditions.

f) Check that your solution satisfies the second differential equation $y_2' = 4y_1 + 5y_2 + 2e^{7t}$.

5. Consider the system of differential equations

$$\begin{aligned} y_1' &= 12y_1 - 4y_2 + 12e^{-8t} \\ y_2' &= 16y_1 - 4y_2 + 24e^{-8t} \end{aligned}$$

with initial conditions $y_1(0) = 8$ and $y_2(0) = 16$.

This system may be written as $\mathbf{y}' = A\mathbf{y} + \mathbf{b}(t)$ where $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $A = \begin{pmatrix} 12 & -4 \\ 16 & -4 \end{pmatrix}$ and $\mathbf{b}(t) = e^{-8t}\begin{pmatrix} 12 \\ 24 \end{pmatrix}$. The initial conditions are then $\mathbf{y}(0) = \mathbf{c} = \begin{pmatrix} 8 \\ 16 \end{pmatrix}$.

In the previous problem set you showed that the matrix $A = \begin{pmatrix} 12 & -4 \\ 16 & -4 \end{pmatrix}$ has an exponential matrix given by $e^{At} = e^{4t}\begin{pmatrix} 1+8t & -4t \\ 16t & 1-8t \end{pmatrix}$.

a) Find the solution to the system of differential equations.

b) Check that your solution satisfies the initial conditions.

c) Did you use the fact that $\begin{pmatrix} 8 \\ 16 \end{pmatrix}$ and $\begin{pmatrix} 12 \\ 24 \end{pmatrix}$ are both eigenvectors of A with eigenvalue $\lambda = 4$? If not, try the example again using this observation to streamline the calculations when multiplying by e^{-As} and e^{At}.

6.★ Solve the system

$$\begin{aligned} y_1' &= -3y_1 + y_2 + 3e^{-2t} \\ y_2' &= -y_1 - y_2 + 2e^{-2t} \end{aligned}$$

with initial conditions $y_1(0) = 1$ and $y_2(0) = 1$.

7.★★ Solve the system

$$\begin{aligned} y_1' &= 4y_1 + y_2 - y_3 + e^{2t} \\ y_2' &= 2y_1 + 3y_2 - y_3 \\ y_3' &= 2y_1 + 2y_2 + y_3 + 2e^{2t} \end{aligned}$$

with initial conditions $y_1(0) = 1$, $y_2(0) = 1$ and $y_3(0) = 1$.

ANSWERS

1. Proof.

2. Proof.

3. a) $e^{-As} = e^{-3s}\begin{pmatrix} 1+s & -s \\ s & 1-s \end{pmatrix}$ b) and c) $\begin{pmatrix} 4e^{2s} \\ 4e^{2s} \end{pmatrix}$. d) $\begin{pmatrix} 2e^{2t}-2 \\ 2e^{2t}-2 \end{pmatrix}$.

e) $y_1 = 2e^{5t} - 2e^{3t}, y_2 = 2e^{5t} - 2e^{3t}$. f) Check. g) Check.

4. a) $e^{-As} = e^{-7s}\begin{pmatrix} 1-2s & s \\ -4s & 1+2s \end{pmatrix}$ b) $\begin{pmatrix} 2s \\ 2+4s \end{pmatrix}$. c) $\begin{pmatrix} t^2 \\ 2t^2+2t \end{pmatrix}$.

d) $y_1 = 7e^{7t} - t^2e^{7t}, y_2 = -2t^2e^{7t} + 2te^{7t} + 14e^{7t}$. e) Check. f) Check.

5. a) $y_1 = 9e^{4t} - e^{-8t}$, $y_2 = 18e^{4t} - 2e^{-8t}$ b) Check.

6. $y_1 = e^{-2t}(1 + 3t - \frac{1}{2}t^2)$, $y_2 = e^{-2t}(1 + 2t - \frac{1}{2}t^2)$.

7. $y_1 = e^{3t}(1+t) + te^{2t}$, $y_2 = e^{3t}(1+t)$, $y_3 = e^{3t}(1+2t) + 2te^{2t}$.