Computation of the Far field operator

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Let $u_i(x) = e^{ikx.d}$ be a plane wave of direction $d \in \mathbb{S}^1$. Plane waves satisfies the following expansion [1, eq. 3.89]:

$$e^{ikx.d} = J_0(kr) + 2\sum_{p=1}^{\infty} i^p J_p(kr) \cos p(\widehat{x} - \widehat{d}) \tag{1}$$

Consider the following scattering problem:

$$\begin{array}{rcl} \Delta u + k^2 u & = & 0 \quad \text{in } \mathbb{R}^2 \setminus B(0,R) \\ \lambda u + \mu \partial_\nu u & = & 0 \quad \text{on } \partial B(0,R) \end{array} \tag{2}$$

and $u_s = u - u_i$ is outgoing. We search u_s in the form

$$u_s(r, \widehat{x}) = \sum_{p=0}^{\infty} \alpha_p H_p(kr) \cos p(\widehat{x} - \widehat{d}) \tag{3}$$

We use the boundary conditions of u to deduce α:

$$\begin{array}{rcl} \alpha_0 & = & -\frac{\lambda J_0(kR) + k\mu J_0'(kR)}{\lambda H_0(kR) + k\mu H_0'(kR)} \\ \alpha_p & = & -2i^p \frac{\lambda J_p(kR) + k\mu J_p'(kR)}{\lambda H_p(kR) + k\mu H_p'(kR)} \quad p \geq 1 \end{array} \tag{4}$$

We then compute the farfield of u_s by using the asymptotic behavior of $H_p(t)$ [1, eq. 3.82]:

$$H_p(t) = \sqrt{\frac{2}{\pi t}} e^{i(t - \frac{p\pi}{2} - \frac{\pi}{4})} \left\{ 1 + O\left(\frac{1}{t}\right) \right\}, \quad t \to \infty. \tag{5}$$

Consequently,

$$\begin{aligned} u_s(r, \widehat{x}) &= \sqrt{\frac{2}{\pi k r}} e^{i(kr - \frac{\pi}{4})} \sum_{p=0}^{\infty} \alpha_p (-i)^p \cos p(\widehat{x} - \widehat{d}) \left\{ 1 + O\left(\frac{1}{r}\right) \right\}, \quad r \to \infty \\ &= \frac{e^{ikr}}{\sqrt{r}} \left(\sqrt{\frac{2}{\pi k}} e^{-i\frac{\pi}{4}} \sum_{p=0}^{\infty} \alpha_p (-i)^p \cos p(\widehat{x} - \widehat{d}) \right) \left\{ 1 + O\left(\frac{1}{r}\right) \right\}, \quad r \to \infty \end{aligned} \tag{6}$$

The quantity under the parenthesis corresponds to [1, eq 3.86-3.87]

$$u_s^{\infty}(\widehat{x}) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int_{\partial B_R} \left\{ u(y) \partial_\nu e^{-ik\widehat{x}.y} - \partial_\nu u(y) e^{-ik\widehat{x}.y} \right\} ds(y). \tag{7}$$

We introduce the notation

$$x_p = -\frac{\lambda J_p(kR) + k\mu J_p'(kR)}{\lambda H_p(kR) + k\mu H_p'(kR)},\ p \geq 0. \tag{8}$$

Then

$$u_s^\infty(\widehat{x}) = \sqrt{\frac{2}{\pi k}} e^{-i\frac{\pi}{4}} \left(x_0 + 2 * \sum_{p=1}^{\infty} x_p \cos p(\widehat{x} - \widehat{d}) \right) \tag{9}$$

References

[1] David Colton and Rainer Kress. *Inverse acoustic and electromagnetic scattering theory*, volume 93. Springer Science & Business Media, 2012. (page 1.)